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Long-time behavior of solutions to the derivative nonlinear Schrödinger equation for soliton-free initial data ☆

Comportement aux temps longs des solutions de l'équation de Schrödinger nonlinéraire avec dérivée en l'absence de solitons

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Abstract

The large-time behavior of solutions to the derivative nonlinear Schrödinger equation is established for initial conditions in some weighted Sobolev spaces under the assumption that the initial conditions do not support solitons. Our approach uses the inverse scattering setting and the nonlinear steepest descent method of Deift and Zhou as recast by Dieng and McLaughlin.

Résumé

On établit le comportement au temps long des solutions de l'équation de Schrödinger nonlinéraire avec dérivée dans des espaces de Sobolev à poids, sous l'hypothèse que les conditions initiales ne supportent pas de solitons. Notre approche utilise l'inverse scattering et la méthode de la plus grande pente ("steepest descent") nonlinéaire de Deift et Zhou revisitée par Dieng et McLaughlin.

Keywords: Riemann–Hilbert problem; Inverse scattering method; Nonlinear steepest descent method

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1. Introduction

This paper is devoted to the large-time asymptotic behavior of solutions to the Derivative Nonlinear Schrödinger Equation (DNLS)

$$iu_t + u_{xx} = i\varepsilon(|u|^2 u)_x \qquad x \in \mathbb{R} \tag{1.1}$$

where $\varepsilon = \pm 1$. It follows our recent work [1] (referred to hereafter as Paper I) where we established global existence of solutions for initial conditions in weighted Sobolev spaces satisfying some additional spectral constraints. To make these assumptions more precise, let us first fix $\varepsilon = 1$ (since solutions of (1.1) with $\varepsilon = 1$ are mapped to solutions of (1.1) with $\varepsilon = -1$ by $u \mapsto u(-x, t)$). It is convenient to consider a gauge-equivalent form of (1.1). Under the transformation

$$q(x,t) = u(x,t)\exp\left(-i\varepsilon \int_{-\infty}^{x} |u(y,t)|^2 dy\right), \tag{1.2}$$

solutions of (1.1) are mapped into solutions of

$$iq_t + q_{xx} + iq^2\bar{q}_x + \frac{1}{2}|q|^4 q = 0. \tag{1.3}$$

This equation is sometimes referred to as the Gerjikov–Ivanov equation [2].

It is well-known since the seminal article of Kaup and Newell [3] that the DNLS equation is solvable by the inverse scattering method. In his doctoral thesis, Lee [4] studied in detail the spectral problem posed by Kaup and Newell, and the direct and inverse scattering maps for generic Schwartz class data.

In Paper I, we develop a rigorous analysis of the direct and inverse scattering transform for a class of initial conditions $q_0(x) = q(x, t = 0)$ belonging to the space $H^{2,2}(\mathbb{R})$ and obeying additional spectral constraints that rule out "bright" and algebraic solitons that led us to a global existence result in this setting. Here, $H^{2,2}(\mathbb{R})$ denotes the completion of $C_0^\infty(\mathbb{R})$ in the norm

$$\|u\|_{H^{2,2}(\mathbb{R})} = \left(\left\|(1+|x|^2)u\right\|_2^2 + \|u''\|_2^2\right)^{1/2}.$$

A recent work by Pelinovsky–Shimabukuro [5] addresses these questions in somewhat different spaces. In the present paper, we give a full description of the large-time behavior of solutions. Before stating our assumptions and results more precisely, we recall known results concerning the long-time behavior of DNLS solutions. The first results go back to the work of Hayashi, Naumkin and Uchida [6] where the authors consider a class of one-dimensional nonlinear Schrödinger equations with general nonlinearities containing first-order derivatives. They prove a global existence result for smooth initial conditions that are small in some weighted Sobolev spaces, as well as a time-decay rate. Their analysis gives the existence of asymptotic states and a logarithmic correction to the phase.

In the context of inverse scattering, the first work to provide explicit formulas (i.e., depending only on initial conditions) for large-time asymptotics of solutions is due to Zakharov and Manakov [7] in the context of the NLS equation. In this setting, the inverse scattering map and the reconstruction of the solution (potential) is formulated through an oscillatory Riemann Hilbert problem (RHP). The latter (in our case, Problem 1.1) consists of an oriented contour specifying the discontinuities of a piecewise analytic function, and jump matrices relating their limits from above and below. The solution to the original PDE is recovered from the asymptotics of solutions to the RHP (for our case, see the reconstruction formula (1.9)).

The now well-known steepest descent method of Deift and Zhou [8] provides a systematic method to reduce the original RHP to a canonical model RHP whose solution is calculated in terms of parabolic cylinder functions. This reduction is done through a sequence of transformations whose effects do not change the large-time behavior of the recovered solution at leading order. In this way, one obtains the asymptotic behavior of the solution in terms of the spectral data (thus in terms of the initial conditions) with a degree of precision that is not currently obtainable through direct PDE methods. This approach has been applied to a number of integrable systems including mKdV [9,8] and defocusing NLS [10].

A formal analysis of general oscillatory RHP with Schwartz class scattering data is presented in Varzugin [11]. More recently, Do [12] developed a version of the Deift–Zhou steepest descent method that emphasizes real-variable methods and extends to a much larger class of RHPs. A key step in the nonlinear steepest descent method consists in deforming the contour associated to the RHP in a way adapted to the structure of the phase function that defines the oscillatory dependence on parameters (for our case, see (1.7) for the jump matrix, (1.8) for the phase function, and Fig. 4.1 for the deformation). When the entries of the jump matrix are not analytic, they must be approximated by rational functions so that the deformation can be carried out, and the error in the recovered solution due to the approximation must be estimated.

Dieng and McLaughlin [13] proposed a variant of Deift–Zhou method combining steepest descent and $\bar{\partial}$-problem asymptotics. This approach allows a certain amount of non-analyticity in the RHP reductions, leading to a $\bar{\partial}$-problem to be solved in some sectors of the complex plane where analyticity of the jump matrix (and hence the solution to the RHP) fails. The new $\bar{\partial}$-problem can be recast into an integral equation and solved by Neumann series. These ideas were implemented by Miller and McLaughlin [14] to the study of asymptotic stability of orthogonal polynomials. In the context of NLS with soliton solutions, they were successfully applied to prove asymptotic stability of N-soliton solutions to defocusing NLS [15] and address the soliton resolution problem for focusing NLS [16].

In this paper, we adapt this analysis to the DNLS equation for initial conditions excluding solitons, building on our Paper I where we proved the Lipschitz continuity of the direct and inverse scattering map from $H^{2,2}(\mathbb{R})$ to itself. The presence of solitons will be addressed in a forthcoming article.

To describe our approach, we recall that (1.3) generates an isospectral flow for the problem

$$\frac{d}{dx}\Psi = -i\zeta^2\sigma_3\Psi + \zeta Q(x)\Psi + P(x)\Psi \tag{1.4}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q(x)} & 0 \end{pmatrix}, \quad P(x) = \frac{i}{2}\begin{pmatrix} -|q(x)|^2 & 0 \\ 0 & |q(x)|^2 \end{pmatrix}.$$

If $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, equation (1.4) admits bounded solutions for $\zeta \in \Sigma$ where

$$\Sigma = \left\{ \zeta \in \mathbb{C} : \operatorname{Im}(\zeta^2) = 0 \right\}.$$

For $\zeta \in \Sigma$ and $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, there exist unique solutions $\Psi^{\pm}$ of (1.4), obeying the respective asymptotic conditions

$$\lim_{x \to \pm\infty} \Psi^{\pm}(x,\zeta) e^{ix\zeta^2\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and there is a matrix $T(\zeta)$, the transition matrix, with $\Psi^+(x,\zeta) = \Psi^-(x,\zeta)T(\zeta)$. The functions $\Psi\pm$ are called Jost functions. The matrix $T(\zeta)$ takes the form

$$T(\zeta) = \begin{pmatrix} a(\zeta) & \breve{b}(\zeta) \\ b(\zeta) & \breve{a}(\zeta) \end{pmatrix} \tag{1.5}$$

where $a, b, \breve{a}, \breve{b}$ obey the determinant relation

$$a(\zeta)\breve{a}(\zeta) - b(\zeta)\breve{b}(\zeta) = 1$$

and the symmetry relations (see Paper I, eq. (1.20))

$$a(-\zeta) = a(\zeta), \quad b(-\zeta) = -b(\zeta), \quad \breve{a}(\zeta) = \overline{a(\overline{\zeta})}, \quad \breve{b}(\zeta) = \overline{b(\overline{\zeta})}. \tag{1.6}$$

In order to rule out algebraic and bright solitons, we assume that q_0 is so chosen that $a(\zeta)$ is nonvanishing on Σ (which rules out algebraic solitons) and admits a zero-free analytic continuation to $\operatorname{Im}(\zeta^2) < 0$ (which rules out bright solitons).

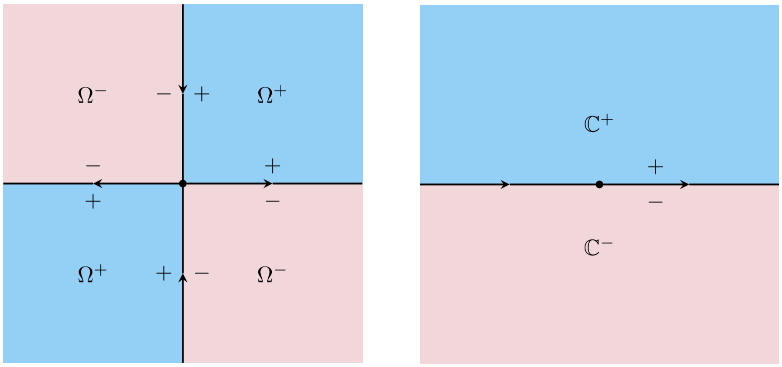

Fig. 1.1. The Contours Σ and ℝ.

As shown in Section 1.2 of Paper I, the scattering data and Jost solutions, which are naturally functions of $\zeta \in \Sigma$, may be transformed to functions on $\mathbb{R}$, with consequent simplifications of the direct and inverse scattering problems. Even functions f on Σ define functions g on the real line $\mathbb{R}$ via $g(\zeta^2) = f(\zeta)$ and the map $\zeta \to \zeta^2$ maps the contour Σ onto the contour $\mathbb{R}$. This fact, together with the symmetry relations (1.6), implies that, letting $z = \zeta^2$, the functions

$$\rho(z) = \zeta^{-1}\breve{b}(\zeta)/a(\zeta), \quad \breve{\rho}(z) = \zeta^{-1}\breve{b}(\zeta)/\breve{a}(\zeta)$$

are defined on the real line, and, under an appropriate change of variable (see Section 1.2 of Paper I), the Jost solutions may be regarded as functions of $z = \zeta^2$. The functions ρ and $\breve{\rho}$ are called the *scattering data* for q_0.

Fig. 1.1 displays the contours Σ and $\mathbb{R}$ with their orientation as well as the sectors $\Omega^\pm = \{\zeta \in \mathbb{C} \,:\, \pm \operatorname{Im}(\zeta^2) > 0\}$ and $\mathbb{C}^\pm = \{z \in \mathbb{C} \,:\, \pm \operatorname{Im}(z) > 0\}$. The map $\zeta \mapsto \zeta^2$ preserves the orientations shown there.

We note the important identity

$$a(\zeta)\breve{a}(\zeta) = (1 - z|\rho(z)|^2)^{-1} = (1 - z|\breve{\rho}(z)|^2)^{-1}, \quad z = \zeta^2.$$

Hence, $1 - z|\rho(z)|^2 > c > 0$ if $|a(\zeta)|$ is bounded from above. The latter is true when in particular $q \in H^{2,2}(\mathbb{R})$ (see Propositions 3.1 and 3.2 of Paper I).

In Paper I, we showed that the maps $q_0 \mapsto \rho$ and $q_0 \mapsto \breve{\rho}$ are Lipschitz continuous from the soliton-free $H^{2,2}(\mathbb{R})$ potentials q_0 into $H^{2,2}(\mathbb{R})$. We assume that the Cauchy data are soliton-free, thus only the reflection coefficient ρ is needed for the reconstruction of the solution.

The scattering data ρ and $\breve{\rho}$ are not independent; as showed in Section 6 of Paper I (see the remarks at the beginning of Section 6 and Lemma 6.14), $\breve{\rho}$ can be recovered from ρ by solving a scalar RHP. We proved in turn that, given ρ corresponding to the Cauchy data $q(x,0)$, we may recover the solution $q(x,t)$ of (1.3) through RHPs. There are two versions of the RHP, one to recover the solution for $x \geq 0$ and one for $x \leq 0$. For example, the following RHP provides the reconstruction formula when $x \geq 0$.

Problem 1.1. Given $\rho \in H^{2,2}(\mathbb{R})$ with $1 - z|\rho(z)|^2 > 0$ for all $z \in \mathbb{R}$, find a row vector-valued function $\mathbf{N}(z; x, t)$ on $\mathbb{C} \setminus \mathbb{R}$ with the following properties:

1. $\mathbf{N}(z; x, t) \to (1, 0) + \mathcal{O}(1/z)$ as $|z| \to \infty$,
2. $\mathbf{N}(z; , x, t)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ with continuous boundary values

$$\mathbf{N}_\pm(z; x, t) = \lim_{\varepsilon \downarrow 0} \mathbf{N}(z \pm i\varepsilon; x, t),$$

3. The jump relation $\mathbf{N}_+(z; x, t) = \mathbf{N}_-(z; x, t)V(z)$ holds, where

$$V(z) = \begin{pmatrix} 1 - z|\rho(z)|^2 & \rho(z)e^{2it\theta} \\ -z\overline{\rho(z)}e^{-2it\theta} & 1 \end{pmatrix} \tag{1.7}$$

and the real phase function θ is given by

$$\theta(z; x, t) = -\left(z\frac{x}{t} + 2z^2\right). \tag{1.8}$$

From the solution of Problem 1.1, we recover

$$q(x, t) = \lim_{z\to\infty} 2iz\mathbf{N}_{12}(z; x, t) \tag{1.9}$$

for $x \geq 0$, where the limit is taken in $\mathbb{C}\setminus\mathbb{R}$ along any direction not tangent to $\mathbb{R}$.

Remark 1.2. The jump matrix (1.7) satisfies $V \in L^\infty(\mathbb{R})$ and $\det V(z) = 1$. It follows from a standard result in RHP theory (see, for example, [10, Theorem 2.10]) that Problem 1.1 may have at most one solution.

Remark 1.3. The symmetry reduction from the contour Σ to the contour $\mathbb{R}$ significantly simplifies the analysis of the RHP because in this setting, the phase factor θ has only one stationary point (instead of two as is the case in [17]). The reason why we seek a row vector-valued solution rather than a matrix-valued solution is that the matrix-valued solution is not properly normalized; see Paper I, Section 1.2 for further discussion.

The central results of this paper are the following theorems that give the long-time behavior of the solutions q of (1.3) and u of (1.1) respectively.

Theorem 1.4. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ is a soliton-free potential. In particular, its reflection coefficient $\rho \in H^{2,2}(\mathbb{R})$ and $c = \inf_{z\in\mathbb{R}}\left(1 - z|\rho(z)|^2\right) > 0$. Denote by $\xi = -x/4t$ the stationary phase point of the phase function (1.8).*

(i) *As $t \to +\infty$,*

$$q(x,t) \sim \begin{cases} \dfrac{1}{\sqrt{t}}\alpha_1(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}\left(t^{-3/4}\right), & x > 0 \\ \dfrac{1}{\sqrt{t}}\alpha_2(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}\left(t^{-3/4}\right), & x < 0 \end{cases} \tag{1.10}$$

(ii) *As $t \to -\infty$,*

$$q(x,t) \sim \begin{cases} \dfrac{1}{\sqrt{-t}}\alpha_2(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}\left((-t)^{3/4}\right), & x > 0 \\ \dfrac{1}{\sqrt{-t}}\alpha_1(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}\left((-t)^{3/4}\right), & x < 0. \end{cases} \tag{1.11}$$

Here

$$\kappa(z) = -\frac{1}{2\pi}\log(1 - z|\rho(z)|^2), \tag{1.12}$$

$$|\alpha_1(\xi)|^2 = |\alpha_2(\xi)|^2 = \frac{\kappa(\xi)}{2\xi}. \tag{1.13}$$

For $t > 0$,

$$\arg\alpha_1(\xi) = \frac{\pi}{4} + \arg\Gamma(i\kappa(\xi)) + \arg\rho(\xi) + \frac{1}{\pi}\int_{-\infty}^{\xi}\log|s-\xi|\,d\log\left(1 - s|\rho(s)|^2\right),$$

$$\arg\alpha_2(\xi) = \arg\alpha_1(\xi) - \pi$$

while for $t < 0$,

$$\arg\alpha_1(\xi) = -\frac{\pi}{4} - \arg\Gamma(i\kappa(\xi)) + \arg\rho(\xi) + \frac{1}{\pi}\int_{\xi}^{\infty} \log|s-\xi|\, d\log\left(1 - s|\rho(s)|^2\right),$$

$$\arg\alpha_2(\xi) = \arg\alpha_1(\xi) + \pi.$$

In (1.10) and (1.11), the implied constants in the remainder terms depend only on $\|\rho\|_{H^{2,2}(\mathbb{R})}$ and $c > 0$.

As a consequence, we get the long-time behavior of the solution u to the original DNLS equation (1.1).

Theorem 1.5. *Suppose that $u_0 \in H^{2,2}(\mathbb{R})$ and let*

$$q_0(x) = u_0(x)\exp\left(-i\int_{-\infty}^{x} |u_0(y)|^2\, dy\right).$$

Let ρ be the reflection coefficient associated to q_0 by the direct scattering map and κ defined by (1.12). Assume also that $c = \inf_{z\in\mathbb{R}}\left(1 - z|\rho(z)|^2\right) > 0$. Denote by $\xi = -x/4t$ the stationary phase point of the phase function (1.8) and fix $\xi \neq 0$. Then:

(i) *As $t \to +\infty$,*

$$u(x,t) \sim \begin{cases} \frac{1}{\sqrt{t}}\alpha_3(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}_\xi\left(t^{-3/4}\right), & x > 0 \\ \frac{1}{\sqrt{t}}\alpha_4(\xi)e^{-i\kappa(\xi)\log(8t)+ix^2/(4t)} + \mathcal{O}_\xi\left(t^{-3/4}\right), & x < 0 \end{cases} \tag{1.14}$$

(ii) *As $t \to -\infty$,*

$$u(x,t) \sim \begin{cases} \frac{1}{\sqrt{-t}}\alpha_4(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}_\xi\left((-t)^{-3/4}\right) & x > 0 \\ \frac{1}{\sqrt{-t}}\alpha_3(\xi)e^{i\kappa(\xi)\log(-8t)+ix^2/(4t)} + \mathcal{O}_\xi\left((-t)^{-3/4}\right) & x < 0 \end{cases} \tag{1.15}$$

Here,

$$|\alpha_3(\xi)|^2 = |\alpha_4(\xi)|^2 = \frac{\kappa(\xi)}{2\xi} \tag{1.16}$$

For $t > 0$,

$$\arg\alpha_3(\xi) = \arg\alpha_1(\xi) - \frac{1}{\pi}\int_{\xi}^{\infty} \frac{\log(1 - s|\rho(s)|^2)}{s}\, ds \tag{1.17}$$

$$\arg\alpha_4(\xi) = \arg\alpha_2(\xi) - \frac{1}{\pi}\int_{\xi}^{\infty} \frac{\log(1 - s|\rho(s)|^2)}{s}\, ds, \tag{1.18}$$

while for $t < 0$,

$$\arg\alpha_3(\xi) = \arg\alpha_1(\xi) - \frac{1}{\pi}\int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s}\, ds \tag{1.19}$$

$$\arg\alpha_4(\xi) = \arg\alpha_2(\xi) - \frac{1}{\pi}\int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s}\, ds. \tag{1.20}$$

Theorem 1.5 is a direct consequence of Theorem 1.4 and Proposition 8.1.

Remark 1.6. Here we examine the continuity of our asymptotic formulas for $q(x,t)$ at $x=0$ by computing left- and right-hand limits as $x \to 0$ for the two cases in (1.10). A similar analysis can be made for the two cases in (1.11). First, notice that the Gamma function has the property that

$$\lim_{x\to 0^+} \arg \Gamma(ix) = -\frac{\pi}{2}, \qquad \lim_{x\to 0^-} \arg \Gamma(ix) = \frac{\pi}{2}.$$

Recalling that

$$\kappa(\xi) = -\frac{1}{2\pi} \log\left(1 - \xi|\rho(\xi)|^2\right),$$

we see that $\kappa(\xi) < 0$ for $\xi < 0$ while $\kappa(\xi) > 0$ for $\xi > 0$. Since $\xi = -x/4t$, for $x > 0$ and $t > 0$, $\xi < 0$, and therefore

$$\lim_{x\to 0^+} \arg\left(\Gamma(i\kappa(\xi))\right) = \frac{\pi}{2},$$

while for $x < 0$ and $t > 0$, $\xi < 0$ and therefore

$$\lim_{x\to 0^-} \arg\left(\Gamma(i\kappa(\xi))\right) = -\frac{\pi}{2}.$$

This observation, and the fact that $\arg \alpha_1(\xi)$ and $\arg \alpha_2(\xi)$ differ by π, shows that the asymptotic formulas for $q(x,t)$ in (1.10) agree in the respective limits $x \to 0^-$ and $x \to 0^+$. A similar argument shows that the asymptotic formulas for $q(x,t)$ when $t < 0$ and $x \to 0^+$ and $x \to 0^-$ also agree.

Remark 1.7. In contrast to Theorem 1.4, the remainder estimates in Theorem 1.5 depend on ξ as well as on $\|\rho\|_{H^{2,2}}$ and $c > 0$. This dependence arises from Proposition 8.1. The error estimate is well-behaved for $|\xi| > 1$ but poorly behaved as $|\xi| \to 0$.

Remark 1.8. Although we do not make any explicit "small data" assumption, we are have so far been unable to construct large initial data satisfying our hypotheses.

Kitaev and Vartanian [17] as well as more recently Xu and Fan [18], considered the same problem for Schwartz class initial data in the soliton-free sector and obtain in the asymptotic formula (1.10) an error term of order $(\log t)/t$. Our results apply to a larger class of initial data and, thanks to the $\overline{\partial}$-approach, arguably entail a simpler proof than earlier studies of the problem.

The proof of Theorem 1.4 addresses separately the four cases $x \lessgtr 0$, $t \to \pm\infty$. Indeed, to reconstruct the solution $q(x,t)$, we need to solve two different RHPs, one for $x > 0$ and one for $x < 0$. The sign of t is important in the phase factors of the entries of the jump matrix V of (1.7). Depending on the sign of t, one performs different factorizations of the jump matrix V in order to have the correct exponential decay on the deformed contour. Finally, a large-time estimate of the phase factor $\exp(-i\int_{-\infty}^{x} |q(y,t)|^2 dy)$ of (1.2) in terms of the scattering data, obtained in Section 8, is needed to obtain Theorem 1.5.

As discussed earlier, the proof of Theorem 1.4, following [16,13], consists of several steps corresponding to transformations of the initial RHP 1.1 implemented successively. For sake of clarity, we present in Section 2 a summary of the analysis of the various steps in each of the four cases, $x \lessgtr 0$, $t \to \pm\infty$ and we show how the RHPs and the respective factorizations are modified to take into account the signs of x and t. In the next Sections (Sections 3 to 7), we provide the details of each step in one case $x > 0$, $t \to \infty$ as follows.

The first step, carried out in Section 3, is the conjugation of the row vector $\mathbf{N}$ with a scalar function $\delta(z)$ that solves the scalar model RHP Problem 3.1 (see equation (3.1)). This operation is standard when performing the factorization of the jump matrix (3.3) as a product of a lower triangular and upper triangular matrix. The phase factors $e^{\pm it\theta}$ have to be placed so that they have the correct exponential decay when the contour deformation described in Section 4 is carried out. The conjugation with $\delta(z)$ allows us to remove the diagonal matrix $(1 - z|\rho(z)|^2)^{\pm 1}$ that would otherwise appear in between the two terms of one of the factorizations (in the case of (3.3) it would be for $z < \xi$).

The second step (Section 4) is a deformation of contour from $\mathbb{R}$ to a new contour $\Sigma^{(2)}$ defined in (4.1) (see Fig. 4.1), in such a way that the exponential factors $e^{\pm it\theta}$ have strong decay (in time) along the rays of the contour.

The solution has no jump along the real axis (this is important because there is no decay of the phase for large $z \in \mathbb{R}$). This transformation induces some 'small' deviation from analyticity in the sectors $\Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6$, and leads to a mixed $\overline{\partial}$-RHP-problem, Problem 4.3, for a new row-vector valued function denoted $\mathbf{N}^{(2)}$. This is where the approach of Dieng–McLaughlin [13] differs from the steepest descent of [10] which in contrast only deals with piecewise analytic solutions. In the approach of [10], the contour deformation is carried out by approximating the entries of the jump matrix by rational functions which admit a direct, analytic continuation.

The third step (Section 5) is a 'factorization' of $\mathbf{N}^{(2)}$ in the form $\mathbf{N}^{(2)} = \mathbf{N}^{(3)}\mathbf{N}^{\mathrm{PC}}$ where $\mathbf{N}^{\mathrm{PC}}$ is solution of a model RHP, Problem 5.2, and $\mathbf{N}^{(3)}$ a solution of a $\bar{\partial}$-problem, Problem 6.1.

The fourth step is the derivation of the explicit solution of the RHP for $\mathbf{N}^{\mathrm{PC}}$ by parabolic cylinder functions (Section 5); this procedure is standard but we give the key steps for the reader's convenience.

The fifth step is the solution of the $\overline{\partial}$-problem for $\mathbf{N}^{(3)}$ using integral equation methods. The $\overline{\partial}$ problem may be written as an integral equation (equation (6.2)) whose integral operator has small norm at large times (see equation (6.5)) allowing the use of Neumann series (Section 6).

At each step of the analysis, one needs to estimate how the reductions modify the long-time asymptotics of the solution and carefully keep track of the dependency of the constants (as functions of the stationary phase point ξ).

The sixth step, carried out in Section 7, consists in regrouping the transformations to find the behavior of the solution of DNLS for $x > 0$ as $t \to \infty$, using the large-z behavior of the RHP solutions.

Finally, the long-time behavior of the phase factor appearing in (1.2) necessary to obtain Theorem 1.5, is given in Section 8.

The paper ends with some technical appendices. Appendix A gives the asymptotics of the functions δ_ℓ and δ_r which solve scalar model RHPs and are used in the first step of the reduction. Appendix B outlines the solution of the appropriate RHP's for all four cases $\pm t > 0$, $\pm x > 0$. Appendix C records solution formulae important for the four model RHP's. Appendix D proves L^∞-bounds on the solution to the model RHP. Appendix E contains figures illustrating how the different jump matrices in the sequence of transformations of RHPs are modified according to the four cases $\pm t > 0$, $\pm x > 0$.

2. Summary of the proof

As discussed above, the large-time behavior of the solution to DNLS is obtained through a sequence of transformations of RHP's. Special attention has to be given to the signs of x and t as slightly different RHP's are involved depending on the signs under consideration. In Sections 3 to 7, we present the full calculations of the derivation in one case $x > 0, t > 0$. In this Section, we summarize the computations without details in the four cases $\pm t > 0$, $\pm x > 0$ as they are needed to get the final expressions of Theorems 1.4 and 1.5.

The initial normalized RHPs that provide the reconstruction formula for the potential have contour $\mathbb{R}$ and phase function

$$\theta(z; x, t) = -\left(z\frac{x}{t} + 2z^2\right).$$

If $x > 0$, the initial RHP is

$$\mathbf{N}_+(z; x, t) = \mathbf{N}_-(z; x, t)e^{it\theta \operatorname{ad}\sigma_3} V_0(z) \tag{2.1a}$$

$$V_0(z) = \begin{pmatrix} 1 - z|\rho(z)|^2 & \rho(z) \\ -z\overline{\rho(z)} & 1 \end{pmatrix} \tag{2.1b}$$

$$\mathbf{N}(z; x, t) = (1, 0) + \mathcal{O}\left(\frac{1}{z}\right) \tag{2.1c}$$

while if $x < 0$, the initial RHP is

$$\mathbf{N}_+(z; x, t) = \mathbf{N}_-(z; x, t)e^{it\theta \operatorname{ad}\sigma_3} \breve{V}_0(z) \tag{2.2a}$$

$$\breve{V}_0(z) = \begin{pmatrix} 1 & \breve{\rho}(z) \\ -z\overline{\breve{\rho}(z)} & 1 - z|\breve{\rho}(z)|^2 \end{pmatrix} \tag{2.2b}$$

$$\mathbf{N}(z;x,t) = (1,0) + \mathcal{O}\left(\frac{1}{z}\right) \tag{2.2c}$$

where $\breve{\rho}(z) = \rho(z)/\Delta(z)$ and

$$\Delta(\lambda) = \exp\left(\frac{1}{\pi i}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{\kappa(s)}{\lambda - s}\, ds\right).$$

In both of these cases, the solution $q(x,t)$ of (1.2) is recovered from the reconstruction formula

$$q(x,t) = \lim_{z\to\infty} \left[2iz\,(\mathbf{N}(z;x,t))_{12}\right]. \tag{2.3}$$

The derivation of the large-time behavior is obtained through several steps. The first steps

(1) Preparation for steepest descent
(2) Contour deformation from $\mathbb{R}$ to $\Sigma^{(2)}$ (see Fig. 4.1)
(3) Reduction to a model RHP
(4) Solution to the model RHP

have to be performed successively for each case $\pm t > 0$, $\pm x > 0$ as the calculations, although similar, are specific to each situation. They are followed by

(5) Analysis of $\bar{\partial}$ problem
(6) Regrouping of the transformations.

The latter are common to all cases and detailed in Sections 6 and 7 for $x > 0, t > 0$.

We now summarize steps 1–4.

Step 1: We change variables in the initial RHP using the analytic functions (with branch cut either on the left or right half-line with endpoint ξ)

$$\delta_\ell(z;\xi) := \exp\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s-z}\, ds\right), \quad z \in \mathbb{C} \setminus (-\infty, \xi] \tag{2.4}$$

and

$$\delta_r(z;\xi) := \exp\left(-i \int_{\xi}^{\infty} \frac{\kappa(s)}{s-z}\, ds\right), \quad z \in \mathbb{C} \setminus [\xi, \infty). \tag{2.5}$$

Here

$$\kappa(s) = -\frac{1}{2\pi} \log\left(1 - s|\rho(s)|^2\right) = -\frac{1}{2\pi} \log\left(1 - s|\breve{\rho}(s)|^2\right).$$

The functions δ_ℓ and δ_r are solutions of scalar model RHPs: δ_ℓ satisfies Problem 3.1 and δ_r satisfies a similar one with its branch cut at the right of the endpoint ξ. Their properties are recalled in Appendix A. In particular, they obey the bounds

$$e^{-\|\kappa\|_\infty/2} \le \left|\delta^*(z)\right| \le e^{\|\kappa\|_\infty/2}$$

where δ^* is $\delta_\ell^{\pm 1}$ or $\delta_r^{\pm 1}$, as easily follows from

$$\left|\operatorname{Im}\left(\int_{\pm\infty}^{\xi} \frac{\kappa(s)}{s-z}\, ds\right)\right| \le \frac{\|\kappa\|_\infty}{2}.$$

By defining

$$\mathbf{N}^{(1)}(z;x,t) = \mathbf{N}(z;x,t) \times \begin{cases} \delta_\ell^{-\sigma_3} & t>0,\ x>0 \\ \delta_r^{-\sigma_3} & t>0,\ x<0 \\ \delta_r^{\sigma_3} & t<0,\ x>0 \\ \delta_\ell^{\sigma_3} & t<0,\ x<0 \end{cases} \tag{2.6}$$

we obtain a RHP for $\mathbf{N}^{(1)}$ with a new jump matrix $e^{2it\theta \,\mathrm{ad}\,\sigma_3} V^{(1)}$. We give expressions for $V^{(1)}$ for each of the four cases $\pm t > 0$, $\pm x > 0$ in (B.2), (B.6), (B.10), and (B.14) respectively. The new RHP's are 'prepared' for the steepest descent method in the sense that contours can be deformed so that the exponential functions $e^{\pm it\theta}$ have maximum decay in $|z-\xi|$.

Step 2: We introduce a new unknown

$$\mathbf{N}^{(2)} = \mathbf{N}^{(1)}\mathcal{R} \tag{2.7}$$

where $\mathcal{R}$ is a piecewise continuous matrix-valued function taking the form shown in Fig. E.1 if $t>0$, and in Fig. E.2 if $t<0$. The purpose of the deformation is to remove the jumps along the real axis and introduce jumps on the contours Σ_1, Σ_2, Σ_3, and Σ_4 corresponding to the model problem. Thus the values of the R_i along $(-\infty,\xi)$ and (ξ,∞) are determined by the jump matrix $V^{(1)}$, while their values along the Σ_i are determined as follows:

(1) Scattering data are replaced by their values at $z=\xi$ ('freezing coefficients')
(2) Powers of δ are replaced by their asymptotic forms near $z=\xi$ (see Appendix A, equations (A.2), (A.3), (A.4), (A.5)).

The expressions of the matrix $\mathcal{R}$ in each of the four cases are given respectively in (B.3), (B.7), (B.11), and (B.15), noting that the symbols δ, δ_0, and $\delta_\pm$ are defined at the beginning of each subsection and *have different meanings in each of them* as indicated in (B.1), (B.5), (B.9), and (B.13).

The new unknown $\mathbf{N}^{(2)}$ has a jump matrix which is most easily described by introducing the scaled variable

$$\zeta(z) = \sqrt{8|t|}(z-\xi). \tag{2.8}$$

We then have

$$V^{(2)} = \begin{cases} \zeta^{i\kappa\,\mathrm{ad}\,\sigma_3} e^{-\frac{i}{4}\zeta^2\,\mathrm{ad}\,\sigma_3} V_0^{(2)}(\zeta;\xi) & \pm x>0,\ t>0, \\ \zeta^{-i\kappa\,\mathrm{ad}\,\sigma_3} e^{\frac{i}{4}\zeta^2\,\mathrm{ad}\,\sigma_3} V_0^{(2)}(\zeta;\xi) & \pm x>0,\ t<0. \end{cases} \tag{2.9}$$

In the above expression, the complex powers are defined by choosing the branch of the logarithm with $-\pi < \arg\zeta < \pi$ in the cases $t>0$, $x>0$ and $t<0$, $x<0$, and the branch of the logarithm with $0<\arg\zeta<2\pi$ in the cases $t>0$, $x<0$ and $t<0$, $x>0$. The matrices $V_0^{(2)}(\zeta;\xi)$ for each of the four cases are shown in Figs. E.3, E.4, E.5, and E.6. The branch cut for the logarithm is also indicated. Because $\mathcal{R}$ is not a holomorphic function, the new unknown $\mathbf{N}^{(2)}$ obeys a mixed $\overline{\partial}$-RHP.

Step 3: Suppose that $\mathbf{N}^{\mathrm{PC}}$ solves the pure RHP with jump matrix $V^{(2)}$. By factoring

$$\mathbf{N}^{(2)} = \mathbf{N}^{(3)}\mathbf{N}^{\mathrm{PC}}, \tag{2.10}$$

we see that $\mathbf{N}^{(3)}$ solves the $\overline{\partial}$ problem (in the z-variable)

$$\begin{aligned} \overline{\partial}\mathbf{N}^{(3)}(z;x,t) &= \mathbf{N}^{(3)}(z;x,t)W(z;x,t) \\ W(z;x,t) &= \mathbf{N}^{\mathrm{PC}}(\zeta;\xi)(\overline{\partial}\mathcal{R})(z;x,t)\mathbf{N}^{\mathrm{PC}}(\zeta;\xi)^{-1} \\ \mathbf{N}^{(3)} &= (1,0) + \mathcal{O}\left(\frac{1}{z}\right) \end{aligned}$$

which is equivalent to the integral equation

$$\mathbf{N}^{(3)}(z;x,t) = (1,0) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-z'}\mathbf{N}^{(3)}(z';x,t)W(z',x,t)\,dz'.$$

It can be shown (see Proposition 6.3) that

$$\mathbf{N}^{(3)}(z;x,t) = (1,0) + \frac{1}{z}\mathbf{N}_1^{(3)}(x,t) + o_{\xi,t}\left(\frac{1}{z}\right)$$

where

$$\left|\mathbf{N}_1^{(3)}(x,t)\right| \lesssim t^{-3/4}.$$

This estimate shows that the leading asymptotics of $q(x,t)$, as computed from (2.3), will be determined by the solution $\mathbf{N}^{\mathrm{PC}}$ of the model Riemann–Hilbert problem.

Step 4: It remains to solve the model RHP for $\mathbf{N}^{\mathrm{PC}}$. It has contour $\Sigma_0^{(2)}$ (centered at $\zeta = 0$ in the new variables) and the solution has the form

$$\mathbf{N}_+^{\mathrm{PC}}(\zeta;\xi) = \mathbf{N}_-^{\mathrm{PC}}(\zeta;\xi)V^{(2)}(\zeta;\xi)$$
$$\mathbf{N}^{\mathrm{PC}}(z;\xi) \sim I + \frac{m^{(0)}}{\zeta} + o\left(\frac{1}{\zeta}\right) \text{ in } \mathbb{C}\setminus\Sigma_0^{(2)}$$

where $V^{(2)}$ is given by (2.9). This problem can be solved in a standard way using parabolic cylinder functions (see, for example, [9,8,19,20]). We factor

$$\mathbf{N}^{\mathrm{PC}}(\zeta;\xi) = \begin{cases} \Phi(\zeta;\xi)P(\xi)e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3} & t>0 \\ \Phi(\zeta;\xi)P(\xi)e^{-\frac{i}{4}\zeta^2\sigma_3}\zeta^{i\kappa\sigma_3} & t<0. \end{cases} \tag{2.11}$$

The constant matrix $P(\xi)$ is derived from $V_0^{(2)}$ as shown in Fig. E.7; for $i = 1,2,3,4$, V_i denotes the restriction of $V^{(2)}$ to Σ_i. This factorization introduces a new unknown, $\Phi(\zeta;\xi)$, which obeys an RHP with contour $\mathbb{R}$ and constant jump matrix. In case $x>0$, we have

$$\begin{aligned} &\Phi_+(\zeta;\xi) = \Phi_-(\zeta;\xi)V^{(0)} \\ &\quad V^{(0)} = \begin{pmatrix} 1-\xi|r_\xi|^2 & r_\xi \\ -\xi\overline{r_\xi} & 1 \end{pmatrix} \\ &\Phi(\zeta;\xi) \sim e^{-\frac{i}{4}\zeta^2\sigma_3}\zeta^{i\kappa\sigma_3}\left(I + \frac{m^{(1)}}{\zeta} + o\left(\zeta^{-1}\right)\right), \end{aligned} \tag{2.12}$$

while for $x<0$, we have

$$\begin{aligned} &\Phi_+(\zeta;\xi) = \Phi_-(\zeta;\xi)\breve{V}^{(0)} \\ &\quad \breve{V}^{(0)} = \begin{pmatrix} 1 & \breve{r}_\xi \\ -\xi\overline{\breve{r}_\xi} & 1-\xi|\breve{r}_\xi|^2 \end{pmatrix} \\ &\Phi(\zeta;\xi) \sim e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3}\left(I + \frac{m^{(0)}}{\zeta} + o\left(\zeta^{-1}\right)\right). \end{aligned} \tag{2.13}$$

Note that the meaning of r_ξ or $\breve{r}_\xi$ is *different* depending on which of the four cases is under consideration (see equations (B.4), (B.8), (B.12), (B.16)).

The matrix function Φ is obtained as a solution of an ODE. Differentiating the jump relation in (2.12) or (2.13) with respect to ζ, one can show that

$$\frac{d\Phi}{d\zeta} \pm i\frac{i\zeta}{2}\sigma_3\Phi = \beta\Phi, \quad \pm t>0 \tag{2.14}$$

where

$$\beta = \frac{i}{2}\left[\sigma_3, m^{(0)}\right] \tag{2.15}$$

or equivalently

$$\beta_{12} = i\left(m^{(0)}\right)_{12}, \quad \beta_{21} = -i\left(m^{(0)}\right)_{21}$$

is unknown at this stage of the calculation. The difference in sign between the $t > 0$ and $t < 0$ cases comes from the difference in the prescribed factorization (2.11). The goal is to compute $m^{(0)}$ which will determine leading asymptotics of $q(x, t)$.

The solution of (2.14) is expressed explicitly in terms parabolic cylinder functions, treating β_{12} and β_{21} as (unknown) constants. The solution formulas are given in Appendix C. One then substitutes these solutions into the appropriate jump relation (2.12) or (2.13) in order to compute β_{12} and hence, by (2.15), $m_{12}^{(0)}$. Indeed, one may easily deduce from the jump relation (2.12) that

$$V_{21}^{(0)} = -\xi\overline{r_\xi} = \Phi_{11}^{-}\Phi_{21}^{+} - \Phi_{21}^{-}\Phi_{11}^{+} \tag{2.16}$$

for $t > 0$, and similarly from the jump relation (2.13), that

$$\breve{V}_{21}^{(0)} = -\xi\overline{\breve{r}_\xi} = \Phi_{11}^{-}\Phi_{21}^{+} - \Phi_{21}^{-}\Phi_{11}^{+} \tag{2.17}$$

for $t < 0$. These Wronskians are evaluated for each of the four cases $\pm t > 0$, $\pm x > 0$ in Appendix C, equations (C.7) and (C.8). Using these results in (2.16) and (2.17), we find

$$\beta_{12} = \begin{cases} \dfrac{\sqrt{2\pi}e^{-\pi\kappa/2}e^{i\pi/4}}{-\xi\overline{r_\xi}\,\Gamma(-i\kappa)} & t > 0,\ x > 0 \\[2ex] \dfrac{\sqrt{2\pi}e^{-\pi\kappa/2}e^{i\pi/4}}{-\xi\overline{\breve{r}_\xi}\,\Gamma(-i\kappa)}e^{-2\pi\kappa} & t > 0,\ x < 0 \end{cases} \tag{2.18}$$

and

$$\beta_{12} = \begin{cases} \dfrac{\sqrt{2\pi}e^{-\pi\kappa/2}e^{3\pi i/4}}{-\xi\overline{r_\xi}\,\Gamma(i\kappa)}e^{2\pi\kappa}, & t < 0,\ x > 0 \\[2ex] \dfrac{\sqrt{2\pi}e^{-\pi\kappa/2}e^{3\pi i/4}}{-\xi\overline{\breve{r}_\xi}\,\Gamma(i\kappa)}, & t < 0,\ x < 0 \end{cases} \tag{2.19}$$

We recall that the values of r_ξ and $\breve{r}_\xi$ differ from case to case.

We can now deduce the leading asymptotic behavior of $q(x, t)$ from the reconstruction formula

$$q_{\mathrm{as}}(x, t) = \lim_{z\to\infty} 2iz\frac{\left(m^{(0)}\right)_{12}}{\zeta} = 2\frac{\beta_{12}}{\sqrt{8|t|}}$$

where we used (2.8) and (2.15). For $\pm t > 0$ we find

$$q_{\mathrm{as}}(x, t) = \frac{1}{\sqrt{|t|}}\alpha(\xi)e^{\pm i\kappa(\xi)\log(8|t|)}e^{-i\frac{x^2}{4t}} \tag{2.20}$$

with

$$|\alpha(\xi)|^2 = \frac{1}{2}|\beta_{12}|^2 \tag{2.21}$$

$$\arg\alpha(\xi) = \arg\beta_{12} \mp \kappa(\xi)\log(8|t|) + x^2/4t \tag{2.22}$$

From (2.21)–(2.22), (2.18), (2.19), and (2.20), we can compute $q_{\mathrm{as}}(x, t)$ in each of the four cases. In Appendix B we summarize the key formulae leading to $q_{\mathrm{as}}(x, t)$.

In the next five sections, we present the details of the proof of Theorem 1.4 in the case $x > 0$, $t > 0$.

3. Preparation for steepest descent

In this section, we provide the detailed analysis of Step 1 (as described in Section 2), for the case $x > 0$, $t > 0$. In order to apply the method of steepest descent, we introduce a new unknown

$$\mathbf{N}^{(1)}(z; x, t) = \mathbf{N}(z; x, t)\delta(z)^{-\sigma_3} \tag{3.1}$$

where $\delta(z) = \delta_\ell(z)$ as defined in (2.4) and solves the scalar RHP Problem 3.1 below. To state the scalar RHP, recall that the phase function (1.8) satisfies

$$\theta_z(x, t, z) = -\left(\frac{x}{t} + 4z\right)$$

and has a single critical point at

$$\xi = -\frac{x}{4t}.$$

Problem 3.1. Given $\xi \in \mathbb{R}$ and $\rho \in H^{2,2}(\mathbb{R})$ with $1 - s|\rho(s)|^2 > 0$ for all $s \in \mathbb{R}$, find a scalar function $\delta(z) = \delta(z; \xi)$, analytic for $z \in \mathbb{C} \setminus (-\infty, \xi]$ with the following properties:

1. $\delta(z) \to 1$ as $z \to \infty$,
2. $\delta(z)$ has continuous boundary values $\delta_\pm(z) = \lim_{\varepsilon \downarrow 0} \delta(z \pm i\varepsilon)$ for $z \in (-\infty, \xi)$,
3. $\delta_\pm$ obey the jump relation

$$\delta_+(z) = \begin{cases} \delta_-(z)\left(1 - z\,|\rho(z)|^2\right), & z \in (-\infty, \xi) \\ \delta_-(z), & z \in (\xi, \infty) \end{cases}$$

The following lemma is "standard" (see, for example, [10, Proposition 2.12] or [12, Proposition 6.1 and Lemma 6.2]). Recall the definition (1.12) of κ.

Lemma 3.2. *Suppose $\rho \in H^{2,2}(\mathbb{R})$ and that $\kappa(s)$ is real for all $s \in \mathbb{R}$.*

(i) *(Existence, Uniqueness) Problem 3.1 has the unique solution*

$$\delta(z) = \exp\left(i \int_{-\infty}^{\xi} \frac{1}{s - z}\kappa(s)\, ds\right). \tag{3.2}$$

Moreover,

$$\delta(z)\overline{\delta(\overline{z})} = 1$$

holds.

(ii) *The function $\delta(z)$ satisfies the estimate*

$$e^{-\|\kappa\|_\infty/2} \le |\delta(z)| \le e^{\|\kappa\|_\infty/2}.$$

(iii) *(Large-z asymptotics) It admits a large-$|z|$ asymptotic expansion*

$$\delta(z) = 1 + \frac{i}{z}\int_{-\infty}^{\xi} \kappa(s)\, ds + \mathcal{O}\left(\frac{1}{z^2}\right).$$

(iv) *(Asymptotics as $z \to \xi$ along a ray in $\mathbb{C}\setminus\mathbb{R}$) Along any ray of the form $\xi + e^{i\phi}\mathbb{R}^+$ with $0 < \phi < \pi$ or $\pi < \phi < 2\pi$,*

$$\left|\delta(z) - \delta_0(\xi)(z - \xi)^{i\kappa(\xi)}\right| \lesssim_{\rho,\phi} -|z - \xi| \log|z - \xi|.$$

The implied constant depends on ρ through its $H^{2,2}(\mathbb{R})$-norm and is independent of $\xi \in \mathbb{R}$. Here $\delta_0(\xi) = e^{i\beta(\xi,\xi)}$ and

$$\beta(z,\xi) = -\kappa(\xi)\log(z-\xi+1) + \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s-z}\, ds,$$

where χ is the characteristic function of the interval $(\xi - 1, \xi)$. We choose the branch of the logarithm with $-\pi < \arg(z) < \pi$.

Proof. The proofs of these properties are similar, for example, to proofs given in [10, Section 2]. We provide some details for the reader's convenience.

(i) Existence follows from the explicit formula (3.2). Since ρ is C^1, uniqueness follows from Liouville's theorem.

(ii) These estimates are obtained from the observation that

$$\left| \operatorname{Re}\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s-z}\, ds \right) \right| \le \frac{\|\kappa\|_\infty}{2}.$$

(iii) and (iv) are proved in Appendix A. □

If $\mathbf{N}(z;x,t)$ solves Problem 1.1 and $\delta(z)$ solves Problem 3.1, then the row vector-valued function $\mathbf{N}^{(1)}(z;x,t)$ defined in (3.1) solves the following RHP.

Problem 3.3. Given $\rho \in H^{2,2}(\mathbb{R})$ with $1 - z|\rho(z)|^2 > 0$ for all $z \in \mathbb{R}$, find a row vector-valued function $\mathbf{N}^{(1)}(z;x,t)$ on $\mathbb{C} \setminus \mathbb{R}$ with the following properties:

1. $\mathbf{N}^{(1)}(z;x,t) \to (1,0)$ as $|z| \to \infty$,
2. $\mathbf{N}^{(1)}(z;x,t)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ with continuous boundary values

$$\mathbf{N}^{(1)}_{\pm}(z;x,t) = \lim_{\varepsilon \downarrow 0} \mathbf{N}^{(1)}(z + i\varepsilon; x,t)$$

3. The jump relation

$$\mathbf{N}^{(1)}_{+}(z;x,t) = \mathbf{N}^{(1)}_{-}(z;x,t) V^{(1)}(z)$$

holds, where

$$V^{(1)}(z) = \delta_-(z)^{\sigma_3} V(z) \delta_+(z)^{-\sigma_3}.$$

The jump matrix $V^{(1)}$ is factorized as

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\dfrac{\delta_-^{-2} z\overline{\rho}}{1 - z|\rho|^2} e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\delta_+^2 \rho}{1 - z|\rho|^2} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi), \\ \begin{pmatrix} 1 & \rho\delta^2 e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z\overline{\rho}\delta^{-2} e^{-2it\theta} & 1 \end{pmatrix}, & z \in (\xi, \infty). \end{cases} \tag{3.3}$$

Remark 3.4. The uniqueness of solutions to Problem 3.3 follows from the unique of solutions to the original RHP for $\mathbf{N}$ and the invertibility of the transformation $\mathbf{N} \to \mathbf{N}^{(1)}$.

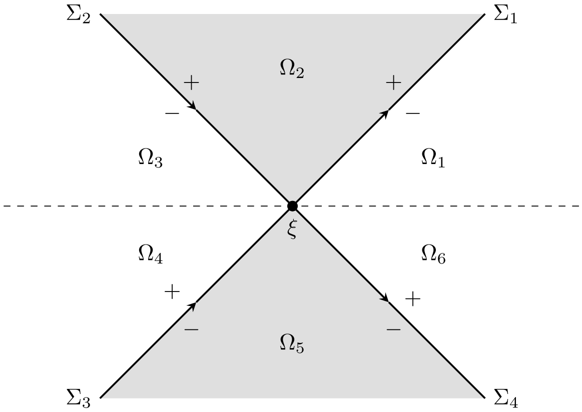

Fig. 4.1. Deformation from $\mathbb{R}$ to $\Sigma^{(2)}$.

4. Deformation to a mixed $\overline{\partial}$-Riemann–Hilbert problem

We now seek to deform Problem 3.3 by exploiting the method of Dieng and McLaughlin [13] and Borghese, Jenkins and McLaughlin [16]. The phase function (1.8) has a single critical point at $\xi = -x/4t$. The new contour

$$\Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \tag{4.1}$$

is shown in Fig. 4.1 and consists of oriented half-lines $\xi + e^{i\phi}\mathbb{R}^+$ where $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

In order to deform the contour $\mathbb{R}$ to the contour $\Sigma^{(2)}$, we introduce a new unknown $\mathbf{N}^{(2)}$ obtained from $\mathbf{N}^{(1)}$ as

$$\mathbf{N}^{(2)}(z) = \mathbf{N}^{(1)}(z)\mathcal{R}^{(2)}(z).$$

We choose $\mathcal{R}^{(2)}$ to remove the jump on the real axis and provide analytic jump matrices with the correct decay properties on the contour $\Sigma^{(2)}$. We have

$$\mathbf{N}_+^{(2)} = \mathbf{N}_+^{(1)}\mathcal{R}_+^{(2)} = \mathbf{N}_-^{(1)}V^{(1)}\mathcal{R}_+^{(2)} = \mathbf{N}_-^{(2)}\left(\mathcal{R}_-^{(2)}\right)^{-1}V^{(1)}\mathcal{R}_+^{(2)}$$

so the jump matrix will be the identity matrix on $\mathbb{R}$ provided

$$(\mathcal{R}_-^{(2)})^{-1}V^{(1)}\mathcal{R}_+^{(2)} = I$$

where $\mathcal{R}_\pm^{(2)}$ are the boundary values of $\mathcal{R}^{(2)}(z)$ as $\pm\,\mathrm{Im}(z) \downarrow 0$. On the other hand, the function $e^{2it\theta}$ is exponentially increasing on Σ_1 and Σ_3, and decreasing on Σ_2 and Σ_4, while the reverse is true of $e^{-2it\theta}$. Hence, we choose $\mathcal{R}^{(2)}$ as shown in Fig. E.1, where, letting

$$\eta(z;\xi) = (z-\xi)^{i\kappa(\xi)}, \tag{4.2}$$

the functions R_1, R_3, R_4, and R_6 satisfy

$$R_1(z) = \begin{cases} z\overline{\rho(z)}\delta^{-2}, & z \in (\xi,\infty) \\ \xi\overline{\rho(\xi)}\delta_0(\xi)^{-2}\eta(z;\xi)^{-2}, & z \in \Sigma_1 \end{cases} \tag{4.3}$$

$$R_3(z) = \begin{cases} -\dfrac{\delta_+^2(z)\rho(z)}{1-z|\rho(z)|^2}, & z \in (-\infty,\xi) \\ -\dfrac{\delta_0^2\eta(z;\xi)^2\rho(\xi)}{1-\xi|\rho(\xi)|^2}, & z \in \Sigma_2 \end{cases} \tag{4.4}$$

$$R_4(z) = \begin{cases} -\dfrac{z\overline{\rho(z)}\delta_-^{-2}}{1-z|\rho(z)|^2}, & z \in (-\infty,\xi) \\ -\dfrac{\delta_0^{-2}\eta(z;\xi)^{-2}\xi\overline{\rho(\xi)}}{1-\xi|\rho(\xi)|^2}, & z \in \Sigma_3 \end{cases} \tag{4.5}$$

$$R_6(z) = \begin{cases} \rho(z)\delta(z)^2 & z \in (\xi, \infty) \\ \rho(\xi)\delta_0(\xi)^2\eta(z;\xi)^2, & z \in \Sigma_4 \end{cases} \tag{4.6}$$

The idea is to construct $R_i(z)$ in Ω_i to have the prescribed boundary values and $\overline{\partial} R_i(z)$ small in the sector. This will allow us to reformulate Problem 3.3 as a mixed RHP-$\overline{\partial}$ problem. We will show how to remove the RHP component through an explicit model problem and then formulate a $\overline{\partial}$ problem for which the large-time contribution to the asymptotics of $q(x,t)$ is negligible. Note that the values of $R_i(z)$ on the contours Σ_i localize the scattering data to the stationary phase point ξ. This localization corresponds to the localization of the weights in the steepest descent method [10]. The latter requires a delicate analysis of modified Beals–Coifman resolvents that is greatly simplified in the current approach.

The following lemma and its proof are almost identical to [16, Lemma 4.1] or [13, Proposition 2.1]. It is useful in the estimates of the contribution of the solution of the $\overline{\partial}$-problem for large time (Section 6). To state it, we introduce the factors

$$p_1(z) = z\overline{\rho(z)}, \qquad p_3(z) = -\frac{\rho(z)}{1 - z|\rho(z)|^2},$$

$$p_4(z) = -\frac{z\overline{\rho(z)}}{1 - z|\rho(z)|^2}, \qquad p_6(z) = \rho(z),$$

that appear in (4.3)–(4.6).

Lemma 4.1. *Suppose $\rho \in H^{2,2}(\mathbb{R})$. There exist functions R_i on Ω_i, $i = 1, 3, 4, 6$ satisfying (4.3)–(4.6), so that*

$$|\overline{\partial} R_i(z)| \lesssim \begin{cases} \left(|p_i'(\mathrm{Re}(z))| - \log|z-\xi|\right), & z \in \Omega_i, \quad |z-\xi| \le 1 \\ \left(|p_i'(\mathrm{Re}(z)) + |z-\xi|^{-1}\right), & z \in \Omega_i, \quad |z-\xi| > 1, \end{cases}$$

where the implied constants are uniform in $\xi \in \mathbb{R}$ and ρ in a fixed bounded subset of $H^{2,2}(\mathbb{R})$ with $1 - z|\rho(z)|^2 \ge c > 0$ for a fixed constant c.

Remark 4.2. By adjusting numerical constants, we can rewrite the estimate on $\overline{\partial} R_i$ for $|z-\xi| > 1$ as

$$\left|\overline{\partial} R_i\right| \lesssim |p_i'(\mathrm{Re}(z))| + (1 + |z-\xi|^2)^{-1/2}.$$

Proof. We give the construction for R_1. Define $f_1(z)$ on Ω_1 by

$$f_1(z) = p_1(\xi)\delta_0^{-2}(\xi)\eta(z;\xi)^{-2}\delta(z)^2$$

and let

$$R_1(z) = \left(f_1(z) + \left[p_1(\mathrm{Re}(z)) - f_1(z)\right]\cos 2\phi\right)\delta(z)^{-2}$$

where $\phi = \arg(z-\xi)$. It is easy to see that R_1 as constructed has the boundary values (4.3). Writing $z - \xi = re^{i\phi}$ we have

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{2}e^{i\phi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right).$$

We therefore have

$$\overline{\partial} R_1(z) = \frac{1}{2}p_1'(\mathrm{Re}\, z)\cos 2\phi\; \delta(z)^{-2} - \left[p_1(\mathrm{Re}\, z) - f_1(z)\right]\delta(z)^{-2}\frac{ie^{i\phi}}{|z-\xi|}\sin 2\phi.$$

It follows from Lemma 3.2(iv) that

$$\left|\left(\overline{\partial} R_1\right)(z)\right| \lesssim_\rho \begin{cases} |p_1'(\mathrm{Re}\, z)| - \log|z-\xi|, & |z-\xi| \le 1, \\ |p_1'(\mathrm{Re}\, z)| + \dfrac{1}{|z-\xi|}, & |z-\xi| > 1, \end{cases}$$

where the implied constants depend on $\inf_{z\in\mathbb{R}}(1 - z|\rho(z)|^2)$ and $\|\rho\|_{H^{2,2}}$. The remaining constructions are similar. □

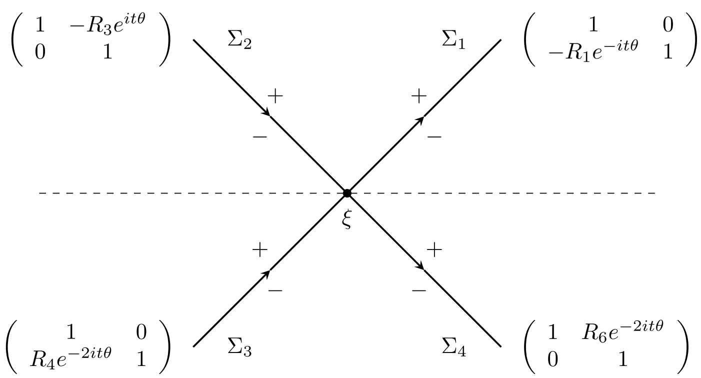

Fig. 4.2. Jump Matrices $V^{(2)}$ for $\mathbf{N}^{(2)}$.

The unknown $\mathbf{N}^{(2)}$ satisfies a mixed $\overline{\partial}$-RHP. We first compute the jumps of $\mathbf{N}^{(2)}$ along the contour $\Sigma^{(2)}$ with the given orientation, remembering that $\mathbf{N}^{(1)}$ is analytic there so that the jumps are determined entirely by the change of variables. Diagrammatically, the jump matrices are as in Fig. 4.2. Away from $\Sigma^{(2)}$ we have

$$\overline{\partial}\mathbf{N}^{(2)} = \mathbf{N}^{(2)}\left(\mathcal{R}^{(2)}\right)^{-1}\overline{\partial}\mathcal{R}^{(2)} = \mathbf{N}^{(2)}\overline{\partial}\mathcal{R}^{(2)} \tag{4.7}$$

where the last step follows by triangularity.

Problem 4.3. Given $\rho \in H^{2,2}(\mathbb{R})$ with $1 - z|\rho(z)|^2 > 0$ for all $z \in \mathbb{R}$, find a row vector-valued function $\mathbf{N}^{(2)}(z; x, t)$ on $\mathbb{C} \setminus \mathbb{R}$ with the following properties:

1. $\mathbf{N}^{(2)}(z; x, t) \to (1, 0)$ as $|z| \to \infty$ in $\mathbb{C} \setminus \Sigma^{(2)}$,
2. $\mathbf{N}^{(2)}(z; x, t)$ is continuous for $z \in \mathbb{C} \setminus \Sigma^{(2)}$ with continuous boundary values $\mathbf{N}^{(2)}_{\pm}(z; x, t)$ (where $\pm$ is defined by the orientation in Fig. 4.1),
3. The jump relation $\mathbf{N}^{(2)}_{+}(z; x, t) = \mathbf{N}^{(2)}_{-}(z; x, t)V^{(2)}(z)$ holds, where $V^{(2)}(z)$ is given in Fig. 4.2,
4. The equation

$$\overline{\partial}\mathbf{N}^{(2)} = \mathbf{N}^{(2)}\,\overline{\partial}\mathcal{R}^{(2)}$$

holds in $\mathbb{C} \setminus \Sigma^{(2)}$, where

$$\overline{\partial}\mathcal{R}^{(2)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ (\overline{\partial}R_1)e^{-2it\theta} & 0 \end{pmatrix}, \ z \in \Omega_1 & \begin{pmatrix} 0 & (\overline{\partial}R_3)e^{2it\theta} \\ 0 & 0 \end{pmatrix}, \ z \in \Omega_3 \\ \begin{pmatrix} 0 & 0 \\ (\overline{\partial}R_4)e^{-2it\theta} & 0 \end{pmatrix}, \ z \in \Omega_4 & \begin{pmatrix} 0 & (\overline{\partial}R_6)e^{2it\theta} \\ 0 & 0 \end{pmatrix}, \ z \in \Omega_6 \\ 0 \text{ otherwise.} \end{cases}$$

5. The model Riemann–Hilbert problem

The next step is to extract from $\mathbf{N}^{(2)}$ a contribution that is a pure RHP. We write

$$\mathbf{N}^{(2)} = \mathbf{N}^{(3)}\mathbf{N}^{\mathrm{PC}}$$

and we request that $\mathbf{N}^{(3)}$ has no jump. Thus we look for $\mathbf{N}^{\mathrm{PC}}$ solution of the model RHP 5.1 below with the jump matrix $V^{\mathrm{PC}} = V^{(2)}$. Unlike the previous RHP's, we seek a matrix-valued solution.

In the following RHPs (Problems 5.1, 5.2, 5.3), ξ is fixed, and we assume that $1 - \xi|\rho(\xi)|^2 > 0$. This is a spectral condition, automatically satisfied if $\xi > 0$ (i.e. if x and t have the same sign), but imposed on the spectral data ρ, to address the cases where x and t have opposite signs.

Problem 5.1. Find a 2×2 matrix-valued function $\mathbf{N}^{\mathrm{PC}}(z;\xi)$, analytic on $\mathbb{C} \setminus \Sigma^{(2)}$, with the following properties:

1. $\mathbf{N}^{\mathrm{PC}}(z;\xi) \to I$ as $|z| \to \infty$ in $\mathbb{C} \setminus \Sigma^{(2)}$, where I is the 2×2 identity matrix,
2. $\mathbf{N}^{\mathrm{PC}}(z;\xi)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^{(2)}$ with continuous boundary values $\mathbf{N}^{\mathrm{PC}}_{\pm}$ on $\Sigma^{(2)}$,
3. The jump relation $\mathbf{N}^{\mathrm{PC}}_{+}(z;\xi) = \mathbf{N}^{\mathrm{PC}}_{-}(z;\xi) V^{\mathrm{PC}}(z)$ holds on $\Sigma^{(2)}$, where

$$V^{\mathrm{PC}}(z) = V^{(2)}(z).$$

Now set

$$\zeta(z) = \sqrt{8t}(z - \xi) \tag{5.1}$$

and

$$r_\xi = \rho(\xi)\delta_0^2 e^{-2i\kappa(\xi)\log\sqrt{8t}} e^{4it\xi^2}. \tag{5.2}$$

Under the change of variables (5.1), the phase $e^{2it\theta}$ identifies to $e^{-i\zeta^2/2}e^{ix^2/4t}$. The factor $e^{-i\zeta^2/2}$ will be later important in the identification of parabolic cylinder functions.

By abuse of notation, set $\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi) = \mathbf{N}^{\mathrm{PC}}(z;\xi)$ where ζ is given by (5.1). We can then recast Problem 5.1 as follows.

Problem 5.2. Find a 2×2 matrix-valued function $\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi)$, analytic on $\mathbb{C} \setminus \Sigma^{(2)}$, with the following properties:

1. $\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi) \to I$ as $|z| \to \infty$ in $\mathbb{C} \setminus \Sigma^{(2)}$, where I is the 2×2 identity matrix,
2. $\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^{(2)}$ with continuous boundary values $\mathbf{N}^{\mathrm{PC}}_{\pm}$ on $\Sigma^{(2)}$,
3. The jump relation $\mathbf{N}^{\mathrm{PC}}_{+}(\zeta(z);\xi) = \mathbf{N}^{\mathrm{PC}}_{-}(\zeta(z);\xi) V^{\mathrm{PC}}(\zeta(z);\xi)$ holds on $\Sigma^{(2)}$, where

$$V^{\mathrm{PC}}(\zeta(z);\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\xi\overline{r_\xi}\,\zeta^{-2i\kappa(\xi)}e^{i\zeta^2/2} & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & \dfrac{r_\xi}{1-\xi|r_\xi|^2}\,\zeta^{2i\kappa(\xi)}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ \dfrac{-\xi\overline{r_\xi}}{1-\xi|r_\xi|^2}\,\zeta^{-2i\kappa(\xi)}e^{i\zeta^2/2} & 1 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 1 & r_\xi\,\zeta^{2i\kappa(\xi)}e^{-i\zeta^2/2} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_4. \end{cases}$$

It is possible to further reduce the RHP for $\mathbf{N}^{\mathrm{PC}}(\zeta;\xi)$ to a model RHP whose 2×2 matrix solution is piecewise analytic in the upper and lower complex plane. In each half-plane, the entries of the matrix satisfy ODEs that are obtained from analyticity properties as well as the large-ζ behavior. The solutions of the ODEs are explicitly calculated in terms of parabolic cylinder functions. This transformation is standard and has been performed for NLS and mKdV (see, for example, [9,8,19,20]). Let

$$\mathbf{N}^{\mathrm{PC}}(\zeta;\xi) = \Phi(\zeta;\xi)\mathcal{P}(\xi)e^{\frac{i}{4}\zeta^2\sigma_3}\zeta^{-i\kappa\sigma_3}, \tag{5.3}$$

where

$$\mathcal{P}(\xi)=\begin{cases}\begin{pmatrix}1 & 0\\ \xi\,\overline{r_\xi} & 1\end{pmatrix}, & z\in\Omega_1 \quad \begin{pmatrix}1 & \dfrac{-r_\xi}{1-\xi|r_\xi|^2}\\ 0 & 1\end{pmatrix},\ z\in\Omega_3,\\ \begin{pmatrix}1 & 0\\ \dfrac{-\xi\overline{r_\xi}}{1-\xi|r_\xi|^2} & 1\end{pmatrix}, & z\in\Omega_4, \quad \begin{pmatrix}1 & r_\xi\\ 0 & 1\end{pmatrix}, \quad z\in\Omega_6,\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & z\in\Omega_2\cup\Omega_5.\end{cases} \tag{5.4}$$

By construction, the matrix Φ is continuous along the rays of $\Sigma^{(2)}$. Let us set up the RHP it satisfies and compute its jumps along the real axis. We have along the real axis

$$\Phi_+ = \Phi_-\left(\mathcal{P}\,e^{i\sigma_3\zeta^2/4}\zeta^{-i\kappa(\xi)\sigma_3}\right)_-\left(e^{-i\sigma_3\zeta^2/4}\zeta^{i\kappa(\xi)\sigma_3}\mathcal{P}^{-1}\right)_+. \tag{5.5}$$

Due to the branch cut of the logarithmic function along $\mathbb{R}^-$, we have along the negative real axis,

$$(\zeta^{-i\kappa(\xi)\sigma_3})_-(\zeta^{i\kappa(\xi)\sigma_3})_+ = e^{-2\pi\kappa(\xi)\sigma_3} = e^{\log(1-\xi|r_\xi|^2)\sigma_3}$$

while along the positive real axis,

$$(\zeta^{-i\kappa(\xi)\sigma_3})_-(\zeta^{i\kappa(\xi)\sigma_3})_+ = \mathrm{I}.$$

This implies that the matrix Φ has the same (constant) jump matrix along the negative and positive real axis:

$$V^{(0)}=\begin{pmatrix}1-\xi\,|r_\xi|^2 & r_\xi\\ -\xi\,\overline{r}_\xi & 1\end{pmatrix}. \tag{5.6}$$

Note that the matrix V^0 is similar to the jump matrix $V^{(1)}$ of the original RHP 1.1 (see (1.7)). The effect of our sequence of transformations is that, in the large t limit, the entries have been replaced by their localized version at the stationary phase point ξ.

The 2×2 matrix Φ satisfies the following model RHP.

Problem 5.3. Find a 2×2 matrix-valued function $\Phi(z;\xi)$, analytic on $\mathbb{C}\setminus\mathbb{R}$, with the following properties:

1. $\Phi(\zeta;\xi)\sim e^{-\frac{i}{4}\zeta^2\sigma_3}\zeta^{i\kappa\sigma_3}$ as $|\zeta|\to\infty$ in $\mathbb{C}\setminus\mathbb{R}$.
2. $\Phi(\zeta;\xi)$ is analytic for $z\in\mathbb{C}\setminus\mathbb{R}$ with continuous boundary values $\Phi_\pm$ on $\mathbb{R}$.
3. The jump relation along the real axis is

$$\Phi_+(\zeta;\xi)=\Phi_-(\zeta;\xi)V^{(0)}. \tag{5.7}$$

To solve this problem, we need to be more precise about the behavior of $\Phi(z)$ as $\zeta\to\infty$. We write the large-ζ behavior of Φ in the form

$$\Phi(\zeta)\sim\left(1+\frac{m^0}{\zeta}\right)\zeta^{i\kappa\sigma_3}e^{-i\sigma_3\zeta^2/4},\quad \zeta\to\infty. \tag{5.8}$$

At this step of the calculation, m^0 is unknown. It will be determined later when enforcing the jump conditions of the matrix Φ along the real axis.

We now compute the solution Φ in terms of parabolic cylinder functions by deriving differential equations for the entries of Φ and exploiting the required asymptotics.

Lemma 5.4. *The entries of Φ obey the differential equations*

$$\Phi_{11}'' + \left(\frac{\zeta^2}{4} - \beta_{12}\beta_{21} + \frac{i}{2}\right)\Phi_{11} = 0 \tag{5.9}$$

$$\Phi_{21}'' + \left(\frac{\zeta^2}{4} - \beta_{12}\beta_{21} - \frac{i}{2}\right)\Phi_{21} = 0 \tag{5.10}$$

$$\Phi_{12}'' + \left(\frac{\zeta^2}{4} - \beta_{12}\beta_{21} + \frac{i}{2}\right)\Phi_{12} = 0 \tag{5.11}$$

$$\Phi_{22}'' + \left(\frac{\zeta^2}{4} - \beta_{12}\beta_{21} - \frac{i}{2}\right)\Phi_{22} = 0 \tag{5.12}$$

The proof of this lemma is given in Appendix C, Section C.1.

The next step is to complement the ODEs with additional conditions taking into account the conditions at infinity as well as the jump conditions of Φ. This will determine Φ uniquely and will identify the coefficients β_{12}, β_{21}.

The parabolic cylinder equation is

$$y'' + \left(-\frac{z^2}{4} + a + \frac{1}{2}\right) y = 0 \tag{5.13}$$

The parabolic cylinder functions $D_a(z)$, $D_a(-z)$, $D_{-a-1}(iz)$, $D_{-a-1}(-iz)$ all satisfy (5.13) and are entire for any value a.

The large-z behavior of $D_a(z)$ is given by the following formulas.[1]

$$D_a(z) \sim \begin{cases} z^a e^{-z^2/4}, & |\arg(z)| < \frac{3\pi}{4} \\ z^a e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{ia\pi} z^{-a-1} e^{z^2/4}, & \frac{\pi}{4} < \arg(z) < \frac{5\pi}{4} \\ z^a e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-a)} e^{-ia\pi} z^{-a-1} e^{z^2/4}, & -\frac{5\pi}{4} < \arg(z) < -\frac{\pi}{4}. \end{cases} \tag{5.14}$$

Proposition 5.5. *The unique solution to Problem 5.3 is given by*

$$\Phi(\zeta;\xi) = \begin{pmatrix} e^{-\frac{3\pi}{4}\kappa} D_{i\kappa}(\zeta e^{-3i\pi/4}) & \frac{e^{\frac{\pi}{4}(\kappa-i)}}{\beta_{21}}(-i\kappa) D_{-i\kappa-1}(\zeta e^{-\pi i/4}) \\ \frac{e^{-\frac{3\pi}{4}(\kappa+i)}}{\beta_{12}} i\kappa D_{i\kappa-1}(\zeta e^{-3i\pi/4}) & e^{\pi\kappa/4} D_{-i\kappa}(\zeta e^{-i\pi/4}) \end{pmatrix} \tag{5.15}$$

for $\mathrm{Im}(\zeta) > 0$ *and*

$$\Phi(\zeta;\xi) = \begin{pmatrix} e^{\pi\kappa/4} D_{i\kappa}(\zeta e^{\pi i/4}) & -\frac{i\kappa}{\beta_{21}} e^{-\frac{3\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{3i\pi/4}) \\ \frac{(i\kappa)}{\beta_{12}} e^{\frac{\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{\pi i/4}) & e^{-3\pi\kappa/4} D_{-i\kappa}(\zeta e^{3i\pi/4}) \end{pmatrix} \tag{5.16}$$

if $\mathrm{Im}(\zeta) < 0$.

Proof. We set $\nu = \beta_{12}\beta_{21}$. For Φ_{11}, we introduce the new variable $\zeta_1 = \zeta\, e^{-3i\pi/4}$, and equation (5.9) becomes

$$\Phi_{11}'' + \left(-\frac{\zeta_1^2}{4} + i\nu + \frac{1}{2}\right)\Phi_{11} = 0.$$

In the upper half plane, $0 < \mathrm{Arg}\,\zeta < \pi$, thus $-3\pi/4 < \mathrm{Arg}\,\zeta_1 < \pi/4$. Choosing $\nu = \kappa$ (by comparing (5.8) and (5.14)) and identifying the large-ζ behavior gives

[1] Writing $D_a(z) = U_{-a-1/2}(z)$ (see http://dlmf.nist.gov/12.1), these formulae follow from http://dlmf.nist.gov/12.9.E1 and http://dlmf.nist.gov/12.9.E3.

$$\Phi_{11}(\zeta) = e^{-\frac{3\pi}{4}\kappa} D_{i\kappa}(\zeta e^{-3i\pi/4}), \qquad \zeta \in \mathbb{C}^+. \tag{5.17}$$

Using equation (C.3), we calculate

$$\Phi_{21} = \frac{1}{\beta_{12}} e^{-\frac{3\pi}{4}\kappa} \left(\partial_\zeta (D_{i\kappa}(\zeta e^{-3i\pi/4})) + \frac{i\zeta}{2} D_{i\kappa}(\zeta e^{-3i\pi/4}) \right). \tag{5.18}$$

We proceed in the same way for Φ_{12} and Φ_{22}. In term of $\zeta_1 = e^{-\pi i/4}\zeta$, equation (5.12) is

$$\Phi_{22}'' + \left(-\frac{\zeta_1^2}{4} - i\nu + \frac{1}{2} \right) \Phi_{22} = 0.$$

To correctly match the large-ζ behavior $\Phi_{22}(\zeta) \sim \zeta^{-i\kappa} e^{i\zeta^2/4}$, we choose the solution

$$\Phi_{22}(\zeta) = e^{\frac{\pi\kappa}{4}} D_{-i\kappa}(e^{-i\pi/4}\zeta) \qquad \zeta \in \mathbb{C}^+. \tag{5.19}$$

Finally, using equation (C.4)

$$\Phi_{12}(\zeta) = \frac{1}{\beta_{21}} e^{\frac{\pi}{4}\kappa} \left(\partial_\zeta (D_{-i\kappa}(\zeta e^{-i\pi/4})) - \frac{i\zeta}{2} D_{-i\kappa}(\zeta e^{-i\pi/4}) \right). \tag{5.20}$$

We repeat this calculation to compute $\Phi(\zeta)$ in the lower complex plane.

Let $\zeta_2 = \zeta e^{i\pi/4}$. In terms of ζ_2, Φ_{11} satisfies

$$\Phi_{11}'' + \left(-\frac{\zeta_2^2}{4} + i\nu + \frac{1}{2} \right) \Phi_{11} = 0. \tag{5.21}$$

For $-\pi < \operatorname{Arg} \zeta < 0$, $-3\pi/4 < \operatorname{Arg}(\zeta_2) < \pi/4$, thus we choose to identify Φ_{11} to a multiple of $D_{i\nu}(\zeta_2)$. We find that for $\zeta \in \mathbb{C}^-$

$$\Phi_{11}(\zeta) = e^{\frac{\pi}{4}\kappa} D_{i\kappa}(e^{i\pi/4}\zeta).$$

Similarly,

$$\Phi_{21}(\zeta) = \frac{1}{\beta_{12}} e^{\frac{\pi}{4}\kappa} \left(\partial_\zeta (D_{i\kappa}(\zeta e^{i\pi/4})) + \frac{i\zeta}{2} D_{i\kappa}(\zeta e^{i\pi/4}) \right) \tag{5.22}$$

We now turn to Φ_{22} and Φ_{12}. To match the large-ζ behavior $\Phi_{22}(\zeta) \sim \zeta^{-i\kappa} e^{i\zeta^2/4}$ we choose to identify Φ_{22} as

$$\Phi_{22}(\zeta) = e^{-3\pi\kappa/4} D_{-i\kappa}(e^{3i\pi/4}\zeta) \tag{5.23}$$

and

$$\Phi_{12}(\zeta) = \frac{1}{\beta_{21}} e^{-\frac{3\pi}{4}\kappa} \left(\partial_\zeta (D_{i\kappa}(\zeta e^{3i\pi/4})) - \frac{i\zeta}{2} D_{i\kappa}(\zeta e^{3i\pi/4}) \right). \tag{5.24}$$

Using (5.17), (5.18), (5.19), and (5.20) together with the identity

$$D_a'(z) + \frac{z}{2} D_a(z) = a D_{a-1}(z), \tag{5.25}$$

we can now write $\Phi(\zeta;\xi)$ for $\operatorname{Im}(\zeta) > 0$ in the form (5.15). Similarly, it follows from (5.21), (5.22), (5.23), and (5.24) that $\Phi(\zeta;\xi)$ is given by (5.16) for $\operatorname{Im}(\zeta) < 0$. □

We now impose the jump conditions to find the coefficients β_{12} and β_{21}. We will later use this computation of β_{12} to compute the asymptotic behavior of $q(x,t)$.

Lemma 5.6. *Suppose that $\rho \in H^{2,2}(\mathbb{R})$ with $\inf_{z\in\mathbb{R}}(1 - z|\rho(z)|^2) > 0$. Then:*

$$|\beta_{12}|^2 = \frac{\kappa}{\xi} = -\frac{1}{2\pi\xi} \log\left(1 - \xi|\rho(\xi)|^2\right) \tag{5.26}$$

and

$$\arg\beta_{12} = \frac{\pi}{4} - \kappa\log(8t) + 4t\xi^2 + \arg(\Gamma(i\kappa)) + \arg\rho(\xi) + \frac{1}{\pi}\int_{-\infty}^{\xi} \log|s-\xi|\, d\log(1-s|\rho(s)|^2). \tag{5.27}$$

Remark 5.7. Note that the amplitude (5.26) has a removable discontinuity at $\xi = 0$ as

$$\begin{aligned}\lim_{\xi\to 0}\frac{\log\left(1-\xi|\rho(\xi)|^2\right)}{\xi} &= -\lim_{\xi\to 0} -\frac{|\rho(\xi)|^2 + \xi\left[\rho'(\xi)\overline{\rho(\xi)} + \overline{\rho(\xi)}\rho'(\xi)\right]}{1-\xi|\rho(\xi)|^2} \\ &= -|\rho(0)|^2.\end{aligned}$$

The proof of this lemma is given in Appendix C, Section C.2.

6. The $\overline{\partial}$-problem

We now define the row vector-valued matrix

$$\mathbf{N}^{(3)}(z;x,t) = \mathbf{N}^{(2)}(z;x,t)\mathbf{N}^{\mathrm{PC}}(z;\xi)^{-1}. \tag{6.1}$$

It is clear that $\mathbf{N}^{\mathrm{PC}}$ needs to be an invertible matrix-valued function in order to carry out this reduction. An argument similar to that given in [16] shows that $\mathbf{N}^{(3)}$ satisfies a pure $\overline{\partial}$-problem; we will use this fact to prove that $\mathbf{N}^{(3)}$ is close to $(1,0)$ as $t\to\infty$ with an explicit rate of decay.

Since $\mathbf{N}^{\mathrm{PC}}(z;\xi)$ is holomorphic in $\mathbb{C}\setminus\Sigma^{(2)}$, we may compute

$$\begin{aligned}\overline{\partial}\mathbf{N}^{(3)}(z;x,t) &= \overline{\partial}\mathbf{N}^{(2)}(z;x,t)\mathbf{N}^{\mathrm{PC}}(z;\xi)^{-1} \\ &= \mathbf{N}^{(2)}(z;x,t)\,\overline{\partial}\mathcal{R}^{(2)}(z)\mathbf{N}^{\mathrm{PC}}(z;\xi)^{-1} && \text{(by (4.7))} \\ &= \mathbf{N}^{(3)}(z;x,t)\mathbf{N}^{\mathrm{PC}}(z;\xi)\,\overline{\partial}\mathcal{R}^{(2)}(z)\mathbf{N}^{\mathrm{PC}}(z;\xi)^{-1} && \text{(by (6.1))} \\ &= \mathbf{N}^{(3)}(z;x,t)W(z;x,t)\end{aligned}$$

where

$$W(z;x,t) = \mathbf{N}^{\mathrm{PC}}(z;\xi)\,\overline{\partial}\mathcal{R}^{(2)}(z)\mathbf{N}^{\mathrm{PC}}(z;\xi)^{-1}.$$

We thus arrive at the following pure $\overline{\partial}$-problem.

Problem 6.1. Given $x,t\in\mathbb{R}$ and $\rho\in H^{2,2}(\mathbb{R})$ with $1-z|\rho(z)|^2>0$ for all $z\in\mathbb{R}$, find a continuous, row vector-valued function $\mathbf{N}^{(3)}(z;x,t)$ on $\mathbb{C}$ with the following properties:

1. $\mathbf{N}^{(3)}(z;x,t)\to(1,0)$ as $|z|\to\infty$,
2. $\overline{\partial}\mathbf{N}^{(3)}(z;x,t) = \mathbf{N}^{(3)}(z;x,t)W(z;x,t)$.

We can recast this problem as a Fredholm-type integral equation using the solid Cauchy transform

$$(Pf)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-\zeta} f(\zeta)\, dm(\zeta)$$

where dm denotes Lebesgue measure on $\mathbb{C}$. The following lemma is standard.

Lemma 6.2. *A continuous, bounded row vector-valued function $\mathbf{N}^{(3)}(z;x,t)$ solves Problem 6.1 if and only if*

$$\mathbf{N}^{(3)}(z;x,t) = (1,0) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-\zeta}\mathbf{N}^{(3)}(\zeta;x,t)W(\zeta;x,t)\, dm(\zeta). \tag{6.2}$$

Using the formulation (6.2), we will prove:

Proposition 6.3. *Suppose that* $\rho \in H^{2,2}(\mathbb{R})$ *and* $c := \inf_{z\in\mathbb{R}} \left(1 - z|\rho(z)|^2\right) > 0$ *strictly. Then, for sufficiently large times* $t > 0$*, there exists a unique solution* $\mathbf{N}^{(3)}(z; x, t)$ *for Problem 6.1 with the property that*

$$\mathbf{N}^{(3)}(z; x, t) = I + \frac{1}{z}\mathbf{N}_1^{(3)}(x, t) + o_{\xi,t}\left(\frac{1}{z}\right) \tag{6.3}$$

for $z = i\sigma$ *with* $\sigma \to +\infty$*. Here*

$$\left|\mathbf{N}_1^{(3)}(x, t)\right| \lesssim t^{-3/4} \tag{6.4}$$

where the implied constant in (6.4) is independent of ξ *and* t *and uniform for* ρ *in a bounded subset of* $H^{2,2}(\mathbb{R})$ *with* $\inf_{z\in\mathbb{R}}(1 - z|\rho(z)|^2) \geq c > 0$ *for a fixed* $c > 0$*.*

Remark 6.4. The remainder estimate in (6.3) need not be (and is not) uniform in ξ and t; what matters for the proof of Theorem 1.4 is that the implied constant in the estimate (6.4) for $\mathbf{N}_1^{(3)}(x, t)$ is independent of ξ and t.

Proof of Proposition 6.3, given Lemmas 6.5–6.9. As in [16] and [13], we first show that, for large times, the integral operator K_W defined by

$$(K_W f)(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-\zeta} f(\zeta)W(\zeta)\,dm(\zeta)$$

(suppressing the parameters x and t) obeys the estimate

$$\|K_W\|_{L^\infty\to L^\infty} \lesssim t^{-1/4} \tag{6.5}$$

where the implied constants depend only on $\|\rho\|_{H^{2,2}}$ and $c := \inf_{z\in\mathbb{R}} \left(1 - z|\rho(z)|^2\right)$ and, in particular, are independent of ξ and t. This is the object of Lemma 6.7. It shows in particular that the solution formula

$$\mathbf{N}^{(3)} = (I - K_W)^{-1}(1, 0) \tag{6.6}$$

makes sense and defines an L^∞ solution of (6.2) bounded uniformly in $\xi \in \mathbb{R}$ and ρ in a bounded subset of $H^{2,2}(\mathbb{R})$ with $c > 0$.

We then prove that the solution $\mathbf{N}^{(3)}(z; x, t)$ has a large-z asymptotic expansion of the form (6.3) where $z \to \infty$ along the *positive imaginary axis* (Lemma 6.8). Note that, for such z, we can bound $|z - \zeta|$ below by a constant times $|z| + |\zeta|$. The remainder need not be bounded uniformly in ξ. Finally, we prove estimate (6.4) where the constants are uniform in ξ and in ρ belonging to a bounded subset of $H^{2,2}(\mathbb{R})$ with $\inf\left(1 - z|\rho(z)|^2\right)$ bounded below by a strictly positive fixed constant (Lemma 6.9). □

Estimates (6.3), (6.4), and (6.5) rest on the bounds stated in the next four lemmas.

Lemma 6.5. *Let* $z = (u + \xi) + iv$*. We have*

$$\left|\bar{\partial}\mathcal{R}^{(2)}(z; \xi)\right| \lesssim \begin{cases} \left(|p_i'(\mathrm{Re}(z))| - \log|z-\xi|\right) e^{-8t|u||v|}, & |z-\xi| \leq 1, \\ \left(|p_i'(\mathrm{Re}(z))| + \dfrac{1}{\left(1+|z-\xi|^2\right)}\right) e^{-8t|v||u|}, & |z-\xi| > 1, \end{cases} \tag{6.7}$$

where all implied constants are uniform in $\xi \in \mathbb{R}$ *and* $t > 1$*.*

Proof. Estimate (6.7) follows from Lemma 4.1 and Remark 4.2. The quantities $p_i'(\mathrm{Re}\,z)$ are all bounded uniformly for ρ in a bounded subset of $H^{2,2}(\mathbb{R})$ and $\inf_{z\in\mathbb{R}} \left(1 - z|\rho(z)|^2\right) \geq c > 0$ for a fixed c. □

Lemma 6.6.

$$\left\|\mathbf{N}^{\mathrm{PC}}(\,\cdot\,;\xi)\right\|_{\infty} \lesssim 1 \tag{6.8}$$

$$\left\|\mathbf{N}^{\mathrm{PC}}(\,\cdot\,;\xi)^{-1}\right\|_{\infty} \lesssim 1 \tag{6.9}$$

Again, all implied constants are uniform in $\xi \in \mathbb{R}$ and $t > 1$.

The proof of this Lemma is given in Appendix D.

Lemma 6.7. *Suppose that $\rho \in H^{2,2}(\mathbb{R})$ and $c : \inf_{z\in\mathbb{R}} \left(1 - z|\rho(z)|^2\right) > 0$ strictly. Then, the estimate (6.5) holds, where the implied constants depend on $\|\rho\|_{H^{2,2}}$ and c.*

Proof. To prove (6.5), first note that

$$\|K_W f\|_\infty \le \|f\|_\infty \int_{\mathbb{C}} \frac{1}{|z-\zeta|} |W(\zeta)|\, dm(\zeta) \tag{6.10}$$

so that we need only estimate the right-hand integral. We will prove the estimate in the region $z \in \Omega_1$ since estimates for Ω_3, Ω_4, and Ω_6 are similar. In the region Ω_1, we may estimate

$$|W(\zeta)| \le \left\|\mathbf{N}^{\mathrm{PC}}\right\|_\infty \left\|(\mathbf{N}^{\mathrm{PC}})^{-1}\right\|_\infty \left|\overline{\partial} R_1\right| |e^{2it\theta}|.$$

Setting $z = \alpha + i\beta$ and $\zeta = (u+\xi) + iv$, the region Ω_1 corresponds to $v \ge 0$, $u \ge v$. We then have from (6.7), (6.8), and (6.9) that

$$\int_{\Omega_1} \frac{1}{|z-\zeta|} |W(\zeta)|\, dm(\zeta) \lesssim I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^\infty \int_v^\infty \frac{1}{|z-\zeta|} |p_1'(u)| e^{-8tuv}\, du\, dv$$

$$I_2 = \int_0^1 \int_v^1 \frac{1}{|z-\zeta|} \left|\log(u^2+v^2)\right| e^{-8tuv}\, du\, dv$$

$$I_3 = \int_0^\infty \int_v^\infty \frac{1}{|z-\zeta|} \frac{1}{1+|\zeta-\xi|} e^{-8tuv}\, du\, dv.$$

We recall from [16, proof of Proposition C.1] the bound

$$\left\| \frac{1}{|z-\zeta|} \right\|_{L^2(v,\infty)} \le \frac{\pi^{1/2}}{|v-\beta|^{1/2}}$$

where $\zeta = u + \xi + iv$ and $z = \alpha + i\beta$ (our parameterization of ζ differs slightly from theirs). Using this bound and Schwarz's inequality on the u-integration we may bound I_1 by constants times

$$(1 + \|p_1'\|_2) \int_0^\infty \frac{1}{|v-\beta|^{1/2}} e^{-tv^2}\, dv \lesssim t^{-1/4}$$

(see for example [16, proof of Proposition C.1] for the estimate). For I_2, we remark that $|\log(u^2+v^2)| \lesssim 1 + |\log(u^2)|$ and that $1 + |\log(u^2)|$ is square-integrable on $[0,1]$. We can then argue as before to conclude that $I_2 \lesssim t^{-1/4}$. Finally, the inequality

$$\frac{1}{1+|\zeta-\xi|} \leq \frac{1}{1+u}$$

shows that we can bound I_3 in a similar way. It now follows that

$$\int_{\Omega_1} \frac{1}{|z-\zeta|} |W(\zeta)|\, dm(\zeta) \lesssim t^{-1/4}$$

which, together with similar estimates for the integrations over Ω_3, Ω_4, and Ω_6, proves (6.5). □

Lemma 6.8. *For $z = i\sigma$ with $\sigma \to +\infty$, the expansion (6.3) holds with*

$$\mathbf{N}_1^{(3)}(x,t) = \frac{1}{\pi} \int_{\mathbb{C}} \mathbf{N}^{(3)}(\zeta;x,t) W(\zeta;x,t)\, dm(\zeta). \tag{6.11}$$

Proof. We write (6.2) as

$$\mathbf{N}^{(3)}(z;x,t) = (1,0) + \frac{1}{z}\mathbf{N}_1^{(3)}(x,t) + \frac{1}{\pi z} \int_{\mathbb{C}} \frac{\zeta}{z-\zeta} \mathbf{N}^{(3)}(\zeta;x,t) W(\zeta;x,t)\, dm(\zeta)$$

where $\mathbf{N}_1^{(3)}$ is given by (6.11). If $z = i\sigma$ and $\zeta \in \Omega_1 \cup \Omega_3 \cup \Omega_4 \cup \Omega_6$, it is easy to see that $|\zeta|/|z-\zeta|$ is bounded above by a fixed constant independent of z, while $|\mathbf{N}^{(3)}(\zeta;x,t)| \lesssim 1$ by the remarks following (6.6). If we can show that $\int_{\mathbb{C}} |W(\zeta;x,t)|\, dm(\zeta)$ is finite, it will follow from the Dominated Convergence Theorem that

$$\lim_{\sigma\to\infty} \int_{\mathbb{C}} \frac{\zeta}{i\sigma-\zeta} \mathbf{N}^{(3)}(\zeta;x,t) W(\zeta;x,t)\, dm(\zeta) = 0$$

which implies the required asymptotic estimate. We will estimate the integral $\int_{\Omega_1} |W(\zeta)|\, dm(\zeta)$ since the other estimates are similar. We have

$$\Omega_1 = \{(u+\xi, v) : v \geq 0,\ v \leq u < \infty\}.$$

Using (6.7), (6.8), and (6.9), we may then estimate

$$\int_{\Omega_1} |W(\zeta;x,t)|\, dm(\zeta) \lesssim I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^\infty \int_v^\infty \left|p_1'(\xi+u)\right| e^{-8tuv}\, du\, dv$$

$$I_2 = \int_0^1 \int_v^1 \left|\log(u^2+v^2)\right| e^{-8tuv}\, du\, dv$$

$$I_3 = \int_0^\infty \int_v^\infty \frac{1}{\sqrt{1+u^2+v^2}} e^{-8tuv}\, du\, dv.$$

To estimate I_1, we use the Schwarz inequality on the u-integration to obtain

$$I_1 \leq \|p_1'\|_2 \frac{1}{4\sqrt{t}} \int_0^\infty \frac{1}{\sqrt{v}} e^{-8tv^2}\, dv = \|p_1'\|_2 \frac{\Gamma(1/4)}{8^{5/4} t^{3/4}}.$$

Similarly, since $\log(u^2+v^2) \le \log(2u^2)$ for $v \le u \le 1$, we may similarly bound

$$I_2 \le \left\| \log(2u^2) \right\|_{L^2(0,1)} \frac{\Gamma(1/4)}{8^{5/4} t^{3/4}}.$$

Finally, to estimate I_3, we note that $1+u^2+v^2 \ge 1+u^2$ and $(1+u^2)^{-1/2} \in L^2(\mathbb{R}^+)$, so we may similarly conclude that

$$I_3 \le \left\| (1+u^2)^{-1/2} \right\|_2 \frac{\Gamma(1/4)}{8^{5/4} t^{3/4}}.$$

These estimates together show that

$$\int_{\Omega_1} |W(\zeta; x, t)| \, dm(\zeta) \lesssim t^{-3/4} \tag{6.12}$$

and that the implied constant depends only on $\|\rho\|_{H^{2,2}}$. In particular, the integral (6.12) is bounded uniformly as $t \to \infty$. □

The estimate (6.12) is also strong enough to prove (6.4).

Lemma 6.9. *The estimate (6.4) holds with constants uniform in ρ in a bounded subset of $H^{2,2}(\mathbb{R})$ and $\inf_{z\in\mathbb{R}} \left(1 - z|\rho(z)|^2\right) > 0$ strictly.*

Proof. From the representation formula (6.11), Lemma 6.7, and the remarks following, we have

$$\left| \mathbf{N}_1^{(3)}(x,t) \right| \lesssim \int_{\mathbb{C}} |W(\zeta; x, t)| \, dm(\zeta).$$

In the proof of Lemma 6.8, we bounded this integral by $t^{-3/4}$ modulo constants with the required uniformities. □

7. Large-time asymptotics

We now use estimates on the RHPs to compute $q(x,t)$ via the reconstruction formula (1.9) in the case $x > 0$, and $t \to +\infty$. Working through the various changes of variables, we have

$$\mathbf{N}(z; x, t) = \mathbf{N}^{(3)}(z; x, t) \mathbf{N}^{\mathrm{PC}}(z; \xi) \mathcal{R}^{(2)}(z)^{-1} \delta(z)^{\sigma_3} \tag{7.1}$$

Recalling (1.9), we need to compute the coefficient of z^{-1} in the large-z expansion for $\mathbf{N}(z; x, t)$.

Lemma 7.1. *For $z = i\sigma$ and $\sigma \to +\infty$, the asymptotic relations*

$$\mathbf{N}(z; x, t) = (1, 0) + \frac{1}{z} \mathbf{N}_1(x,t) + o\left(\frac{1}{z}\right) \tag{7.2}$$

$$\mathbf{N}^{\mathrm{PC}}(z; x, t) = I + \frac{1}{z} \mathbf{N}_1^{\mathrm{PC}}(x,t) + o\left(\frac{1}{z}\right) \tag{7.3}$$

hold. Moreover,

$$(\mathbf{N}_1(x,t))_{12} = \left(\mathbf{N}_1^{\mathrm{PC}}(x,t) \right)_{12} + \mathcal{O}\left(t^{-3/4}\right) \tag{7.4}$$

and the implied constants are uniform in ξ and $t > 0$.

Proof. By Lemma 3.2(iii), the expansion

$$\delta(z)^{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1^{-1} \end{pmatrix} + \mathcal{O}\left(z^{-2}\right) \tag{7.5}$$

holds, with the remainder in (7.5) uniform in ρ in a bounded subset of $H^{2,2}$. The form of the asymptotic expansion (7.3) follows by construction, while (7.2) follows from (7.1), (7.3), the fact that $\mathcal{R}^{(2)} \equiv I$ in Ω_2, and (7.5).

To prove (7.4), we notice that the diagonal matrix in (7.5) does not affect the 12-component of $\mathbf{N}$. Hence, for $z = i\sigma$,

$$(\mathbf{N}(z;x,t))_{12} = \frac{1}{z}\left(\mathbf{N}_1^{(3)}(x,t)\right)_{12} + \frac{1}{z}\left(\mathbf{N}_1^{\mathrm{PC}}(x,t)\right)_{12} + o\left(\frac{1}{z}\right)$$

and result now follows from (6.4). □

We now evaluate the leading asymptotic term using large-z asymptotics of the model RHP.

Proposition 7.2. *The function*

$$q(x,t) = 2i \lim_{z\to\infty} z\mathbf{N}_{12}(z;x,t) \tag{7.6}$$

takes the form

$$q(x,t) = q_{as}(x,t) + \mathcal{O}\left(t^{-3/4}\right)$$

where $q_{as}(x,t)$ is given by (1.10) and the remainder is uniform in $\xi \in \mathbb{R}$.

Proof. By Lemma 7.1 and (7.6),

$$q_{as}(x,t) = \lim_{z\to\infty} \frac{2izm_{12}^{(0)}}{\zeta}.$$

Recalling that $m_{12}^{(0)} = -i\beta_{12}$, with β_{12} given in (5.26)–(5.27) of Lemma 5.6, and that z and ζ are related through (5.1), we get

$$\begin{aligned} q_{as}(x,t) &= \lim_{z\to\infty} \frac{2z\,\beta_{12}}{\sqrt{8t}(z-\xi)} \\ &= \frac{1}{\sqrt{t}}\alpha_1(\xi)e^{-i\kappa(\xi)\log 8t + ix^2/(4t)} \end{aligned}$$

where

$$\kappa(z) = -\frac{1}{2\pi}\log(1 - z|\rho(z)|^2), \quad |\alpha_1(\xi)|^2 = \frac{|\kappa(\xi)|}{2|\xi|}$$

and

$$\arg\alpha_1(\xi) = \frac{\pi}{4} + \arg\Gamma(i\kappa) + \arg\rho(\xi) + \frac{1}{\pi}\int_{-\infty}^{\xi} \log|s-\xi|\, d\log(1 - s|\rho(s)|^2). \quad \Box$$

Theorem 1.4 in the case $x > 0$, $t > 0$ is an immediate consequence of Proposition 7.2. We discuss the remaining three cases in Appendix B.

8. Gauge transformation

Given initial data u_0 for (1.1), we define gauge-transformed initial data for (1.3)

$$q_0(x) = u_0(x)\exp\left(-i\int_{-\infty}^{x} |u_0(y)|^2\, dy\right)$$

and the associated scattering data ρ for q_0. From these scattering data, we compute the solution to (1.3), and thus obtain the solution to the Cauchy problem for (1.1) with Cauchy data u_0 by the inverse gauge transformation

$$u(x,t) = q(x,t) \exp\left(i \int_{-\infty}^{x} |q(y,t)|^2 \, dy \right). \tag{8.1}$$

To find the large-time behavior for $u(x,t)$ purely in terms of spectral data, it suffices to evaluate large-time asymptotics for the expression

$$\exp\left(i \int_{-\infty}^{x} |q(y,t)|^2 \, dy \right).$$

We will prove:

Proposition 8.1. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and that $q(x,t)$ solves the Cauchy problem (1.3) with initial data q_0. Let ρ be the right-hand scattering data associated to q_0 and fix $\xi = -x/(4t)$ with $\xi \neq 0$. We have the asymptotic formulae:*

(i) *For $t > 0$,*

$$\exp\Big(i \int_{-\infty}^{x} |q(y,t)|^2 \, dy \Big) = \exp\Big(-\frac{i}{\pi} \int_{\xi}^{\infty} \frac{\log(1 - s|\rho(s)|^2)}{s} \, ds \Big) + \mathcal{O}_\xi\left(\frac{1}{\sqrt{t}} \right).$$

(ii) *Similarly, for $t < 0$,*

$$\exp\Big(i \int_{-\infty}^{x} |q(y,t)|^2 \, dy \Big) = \exp\Big(-\frac{i}{\pi} \int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s} \, ds \Big) + \mathcal{O}_\xi\left(\frac{1}{\sqrt{t}} \right).$$

8.1. Beals–Coifman solutions

Our analysis uses the Beals–Coifman solutions discussed in Paper I, Section 4. We recall a few key facts and refer the reader to Sections 1.2 and 4 of that paper for further details. Our Beals–Coifman solutions also depend on t since the potential $q(x,t)$ and its scattering data evolve in time.

In the ζ variables, the Beals–Coifman solutions $M_\ell(\zeta; x, t)$ and $M_r(\zeta; x, t)$[2] are 2×2 matrix-valued functions defined for $\zeta \in \mathbb{C} \setminus \Sigma$, are analytic in ζ and have the respective spatial normalizations

$$\lim_{x \to +\infty} M_r(\zeta; x, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lim_{x \to -\infty} M_\ell(\zeta; x, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.2}$$

The functions $M_\ell(\zeta; x, t) e^{-ix\zeta^2 \sigma_3}$ and $M_r(\zeta; x, t) e^{-ix\zeta^2 \sigma_3}$ solve (1.4).

By exploiting the symmetry reduction described in Section 1.2 of Paper I, we can form Beals–Coifman solutions $N_\ell(z; x, t)$ and $N_r(z; x, t)$ with the same respective spatial normalizations but analytic for $z \in \mathbb{C} \setminus \mathbb{R}$. The function $\mathbf{N}(z; x, t)$ that solves Problem 1.1 (the "right" Riemann–Hilbert problem) is the first row of $N_r(z; x, t)$. Analogously, the first row of $N_\ell(z; x, t)$ solves the corresponding "left" Riemann–Hilbert problem.

If $\zeta = 0$, (1.4) becomes $d\Psi/dx = P(x)\Psi$ and we can use the normalizations (8.2) to compute

$$M_{11}^{\pm}(0; x, t)_r = \exp\left(-\frac{i}{2} \int_{+\infty}^{x} |q(y)|^2 \, dy \right) \tag{8.3}$$

and

[2] The ordering of the variables for the Beals–Coifman solutions as defined in Paper I has been changed to be consistent with the notations of the present paper.

$$M_{11}^{\pm}(0;x,t)_{\ell} = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |q(y)|^2\,dy\right). \tag{8.4}$$

According to Proposition 2.9, Proposition 5.7 and equation (2.13) of Paper I, if N_r is the solution to the RHP Problem 5.2 of Paper I, then $M_{11}^{\pm}(0;x,t)_r = N_{11}^{\pm}(0;x,t)_r$. Following a similar argument, we have $M_{11}^{\pm}(0;x,t)_{\ell} = N_{11}^{\pm}(0;x,t)_{\ell}$. One can also directly read off from (6.1) and (6.2) of Paper I that

$$N_{11}^{+}(0;x,t)_r = N_{11}^{-}(0;x,t)_r\ ,\ \ N_{11}^{+}(0;x,t)_{\ell} = N_{11}^{-}(0;x,t)_{\ell}.$$

We conclude that

$$N_{11}^{\pm}(0;x,t)_r = \exp\left(-\frac{i}{2}\int_{+\infty}^{x} |q(y,t)|^2\,dy\right) \tag{8.5}$$

$$N_{11}^{\pm}(0:x,t)_{\ell} = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |q(y,t)|^2\,dy\right). \tag{8.6}$$

As we will see, we can also compute the large-ξ asymptotics of $N_{11}^{\pm}(0;x,t)_{\ell}$ and $N_{11}^{\pm}(0;x,t)_r$ since these functions are the first entry in the respective solutions of the "left" and "right" Riemann–Hilbert problems for $\mathbf{N}(z;x,t)$ evaluated at $z=0$. We will obtain asymptotic formulas in terms of scattering data alone which prove Proposition 8.1.

8.2. A weak Plancherel identity

The following lemma that can be seen as a weak version of a nonlinear Plancherel identity.

Lemma 8.2. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and let ρ be the scattering data. Then, the identity*

$$\exp\Big(i\int_{-\infty}^{+\infty} |q_0(y)|^2\,dy\Big) = \exp\Big(-\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\log\left(1-s|\rho(s)|^2\right)}{s}\,ds\Big)$$

holds.

Proof. The proof consists in computing the scattering coefficient $a(0)$ (defined in (1.5)) in two ways using the construction of left and right Beals–Coifman solutions M_{ℓ}, M_r, at $\zeta=0$ and $t=0$.

First, it follows from Lemma 5.6 of Paper I and the identity $\alpha(\zeta^2)=a(\zeta)$ that, for $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\alpha(z) = \exp\left(\int_{\mathbb{R}} \frac{\log(1-\lambda|\rho(\lambda)|^2)}{\lambda - z}\,\frac{d\lambda}{2\pi i}\right).$$

Since $\rho\in H^{2,2}(\mathbb{R})$, the function $\log(1-\lambda|\rho(\lambda)|^2)$ has a first-order zero at $\lambda=0$, so that $\alpha(0)=\lim_{z\to 0, z\in\mathbb{C}^-}\alpha(z)$ is given by

$$\alpha(0) = \exp\left(\int_{\mathbb{R}} \frac{\log(1-\lambda|\rho(\lambda)|^2)}{\lambda}\,\frac{d\lambda}{2\pi i}\right)$$

(although these identities are proved in Section 5 of Paper I for $\rho\in\mathcal{S}(\mathbb{R})$, their proof readily extends to $\rho\in H^{2,2}(\mathbb{R})$). On the other hand, from eq. (4.20) of Paper I, we have

$$\alpha(0) = \lim_{x\to-\infty}(M_{11}^{-}(0;x,0))_r.$$

It follows from (8.3) that

$$\lim_{x\to-\infty}(M_{11}^{-}(0;x,0))_r = \exp\left(\frac{i}{2}\int_{-\infty}^{+\infty}|q(y)|^2\,dy\right).$$

This concludes the proof of the lemma. □

Remark 8.3. When $x<0$ we reconstruct $q(x,t)$ using the left RHP, which, as shown in Proposition 6.2 of Paper I, gives a Lipschitz continuous map from soliton-free $H^{2,2}$ scattering data to $H^{2,2}(-\infty,a)$ for any fixed $a\in\mathbb{R}$. When we use the right RHP to recover q for $x>0$, the reconstruction map is only continuous into $H^{2,2}(a,\infty)$ (see Proposition 6.1 of Paper I) but need not be stable as $x\to-\infty$. In this case, the gauge transformation (8.1) is still valid for the following reason:

$$\begin{aligned}\exp\Big(i\int_{-\infty}^{x}|q(y,t)|^2\,dy\Big) &= \exp\Big(i\int_{-\infty}^{+\infty}|q(y,t)|^2\,dy - i\int_{x}^{+\infty}|q(y,t)|^2\,dy\Big)\\ &= \exp\Big(i\int_{-\infty}^{+\infty}|q_0(y)|^2\,dy\Big)\exp\Big(-i\int_{x}^{+\infty}|q(y,t)|^2\,dy\Big)\\ &= \exp\Big(-\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\,ds\Big)\exp\Big(i\int_{+\infty}^{x}|q(y,t)|^2\,dy\Big).\end{aligned} \tag{8.7}$$

The first term of (8.7) only depends on the initial data and the second term is stable.

8.3. *Proof of Proposition 8.1*

Proof. The proof is a consequence of (8.13), (8.15), (8.19) and (8.17) below. It suffices to evaluate $N_{11}^{\pm}(0;x,t)$ for large t from the spectral data via the RHP. We compute an asymptotic expression for the first row of $N^{\pm}(0;x,t)$ using the solution formula

$$\mathbf{N}(z;x,t)=\mathbf{N}^{(3)}(z;x,t)\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi)\mathcal{R}^{(2)}(z)^{-1}\delta^{\sharp}(z;\xi)^{\sigma_3} \tag{8.8}$$

using equations (2.6), (2.7), (2.10) of Section 2, where (see (2.4) and (2.5) for the definitions of δ_ℓ and δ_r)

$$\delta^{\sharp}(z;\xi)=\begin{cases}\delta_\ell(z;\xi) & t>0,\ x>0\\ \delta_r(z;\xi) & t>0,\ x<0\\ \delta_r(z;\xi)^{-1} & t<0,\ x>0\\ \delta_\ell(z;\xi)^{-1} & t<0,\ x<0\end{cases} \tag{8.9}$$

and the respective formulas

$$\mathbf{N}(0;x,t)=\begin{cases}\lim_{z\to0,z\in\Omega_1}\mathbf{N}(z;x,t) & t>0,\ x>0\\ \lim_{z\to0,z\in\Omega_4}\mathbf{N}(z;x,t) & t>0,\ x<0\\ \lim_{z\to0,z\in\Omega_3}\mathbf{N}(z;x,t) & t<0,\ x>0\\ \lim_{z\to0,z\in\Omega_6}\mathbf{N}(z;x,t) & t<0,\ x<0.\end{cases} \tag{8.10}$$

Let us examine each right-hand factor of (8.8) in turn. Since

$$\mathbf{N}^{(3)}(z;x,t)=(1,0)+\mathcal{O}\left(t^{-3/4}\right),$$

we need to consider only the last three factors.

Since $\mathbf{N}^{\mathrm{PC}}(z;x,t)$ is continuous at $z=0$ (if $\xi\neq0$), we may evaluate

$$\begin{aligned}\lim_{z\to0}\mathbf{N}^{\mathrm{PC}}(\zeta(z);\xi) &= \mathbf{N}^{\mathrm{PC}}(\sqrt{8|t|}\xi;\xi)\\ &= \begin{pmatrix}1&0\\0&1\end{pmatrix}+\mathcal{O}\left(\frac{1}{\sqrt{8|t|}\xi}\right)\end{aligned}$$

We show that, in each case of (8.10), $\lim_{z\to 0}\mathcal{R}^{(2)}(z;x,t)^{-1}$ is the identity matrix when the limit is taken in the prescribed sector.

- $t>0$, $x>0$: The function $R_1(z)$ is continuous near $z=0$ and $R_1(0)=0$ (Fig. E.1 and equation (B.3)).
- $t>0$, $x<0$: The function $R_4(z)$ is continuous near $z=0$ and $R_4(0)=0$ (Fig. E.1 and equation (B.7)).
- $t<0$, $x>0$: The function $R_3(z)$ is continuous near $z=0$ and $R_3(0)=0$ (Fig. E.2 and equation (B.11)).
- $t<0$, $x<0$: The function $R_6(z)$ is continuous near $z=0$ and $R_6(0)=0$ (Fig. E.2 and equation (B.15)).

Finally, we evaluate $\lim_{z\to 0}\delta(z,\xi)$ for the appropriate choice of δ.

- $t>0$, $x>0$: $\xi<0$ and $z=0$ lies to the right of the branch cut (Fig. E.3).
- $t>0$, $x<0$: $\xi>0$ and $z=0$ lies to the left of the branch cut (Fig. E.4).
- $t<0$, $x>0$: $\xi>0$ and $z=0$ lies to the left of the branch cut (Fig. E.5).
- $t<0$, $x<0$: $\xi<0$ and $z=0$ lies to the right of the branch cut (Fig. E.6).

In all cases, δ is continuous at $z=0$ and $\lim_{z\to 0}\delta(z;\xi)^{\sigma_3}=\delta^\sharp(0;\xi)^{\sigma_3}$. Finally we arrive at

$$\mathbf{N}(0;x,t)=(\delta^\sharp(0;\xi),0)+\mathcal{O}_\xi\left(t^{-1/2}\right) \tag{8.11}$$

where $\delta^\sharp$ is given by (8.9). We now use (8.6) and (8.5) together with (8.11) to prove Proposition 8.1 in four cases.

In the following, we assume ξ is fixed, thus letting x and t to infinity.

The case $t>0$**,** $x>0$: We solve the right RHP (see (2.1) and the summary in Appendix B.1). Using (8.11), we have

$$N_{11}^+(0;x,t)_r=\delta_\ell(0)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right).$$

On the other hand,

$$\delta_\ell(z)=\exp\left(\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s-z}\frac{ds}{2\pi i}\right)$$

Hence,

$$\begin{aligned}\delta(0)&=\lim_{z\to 0,\, z\in\mathbb{C}^+}\exp\left(\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s-z}\frac{ds}{2\pi i}\right)\\&=\exp\left(\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right)\end{aligned}$$

and

$$N_{11}^+(0;x,t)_r=\exp\left(\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.12}$$

Using (8.3) and (8.12) we conclude that

$$\exp\Big(-\frac{i}{2}\int_{+\infty}^{x}|q(y,t)|^2\,dy\Big)=\exp\Big(\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\Big)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right),$$

which leads to

$$\exp\Big(i\int_{-\infty}^{x}|q(y,t)|^2\,dy\Big)=\exp\Big(-i\int_{\xi}^{+\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{\pi}\Big)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.13}$$

The case $t>0$, $x<0$: We use the left-hand RHP (see (2.2) and the summary in Appendix B.2). From (8.11) we conclude that

$$N_{11}^{-}(0;x,t)_\ell=\delta_r(0)+\mathcal{O}_\xi\left(t^{-1/2}\right)$$

Now

$$\delta_r(0)=\lim_{z\to 0,\,z\in\Omega_4}\exp\left(-\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s-z}\frac{ds}{2\pi i}\right)$$
$$=\exp\left(-\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right)$$

This gives

$$N_{11}^{-}(0;x,t)_\ell=\exp\left(-\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.14}$$

We deduce from (8.14) and (8.4) that

$$\exp\Big(i\int_{-\infty}^{x}|q(y,t)|^2\,dy\Big)=\exp\Big(-i\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{\pi}\Big)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.15}$$

The case $t<0$, $x>0$: We use the asymptotic formulas for the right-hand RHP (2.1) of Appendix B.3. From (8.11) we conclude that

$$N_{11}^{+}(0;x,t)_r=\delta_r(0)^{-1}+\mathcal{O}_\xi\left(t^{-1/2}\right).$$

Now

$$\delta_r(0)^{-1}=\exp\left(\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right).$$

This gives

$$N_{11}^{+}(0;x,t)_r=\exp\left(\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\right)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.16}$$

From (8.16) and (8.3), we get

$$\exp\Big(-\frac{i}{2}\int_{+\infty}^{x}|q(y,t)|^2\,dy\Big)=\exp\Big(\int_{\xi}^{\infty}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{2\pi i}\Big)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right)$$

which leads to

$$\exp\Big(i\int_{-\infty}^{x}|q(y,t)|^2\,dy\Big)=\exp\Big(-i\int_{-\infty}^{\xi}\frac{\log(1-s|\rho(s)|^2)}{s}\frac{ds}{\pi}\Big)+\mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.17}$$

The case $t < 0$, $x < 0$: Using the asymptotic formula for the left-hand RHP of Appendix B.4 and (8.11) we have

$$N_{11}^{-}(0; x, t)_\ell = \delta_r(0)^{-1} + \mathcal{O}_\xi\left(t^{-1/2}\right)$$

From

$$\delta_r(0)^{-1} = \exp\left(-\int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s} \frac{ds}{2\pi i}\right),$$

we have

$$N_{11}^{-}(0; x, t)_\ell = \exp\left(-\int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s} \frac{ds}{2\pi i}\right) + \mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \tag{8.18}$$

Finally from (8.18) and (8.4),

$$\exp\Big(\int_{-\infty}^{x} |q(y, t)|^2\, dy\Big) = \exp\Big(-i\int_{-\infty}^{\xi} \frac{\log(1 - s|\rho(s)|^2)}{s} \frac{ds}{\pi}\Big) + \mathcal{O}_\xi\left(\frac{1}{\sqrt{t}}\right). \quad \square \tag{8.19}$$

Conflict of interest statement

There are no conflicts of interests.

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Appendix A. Solutions to model scalar RHPs

A.1. Large-z asymptotics

Since $\kappa \in H^{2,2}(\mathbb{R})$, it follows that $s\kappa(s) \in L^1(\mathbb{R})$ and we may expand

$$\begin{aligned}\int_{-\infty}^{\xi} \frac{\kappa(s)}{s - z}\, ds &= -\frac{1}{z}\int_{-\infty}^{\xi} \kappa(s)\, ds - \frac{1}{z}\int_{-\infty}^{\xi} \frac{s}{s - z}\kappa(s)\, ds \\ &= -\frac{1}{z}\int_{-\infty}^{\xi} \kappa(s)\, ds + \mathcal{O}\left(\frac{1}{z^2}\right),\end{aligned} \tag{A.1}$$

where the implied constant is uniform in z with $-\pi + \varepsilon < \arg(z - \xi) < \pi - \varepsilon$ for a fixed $\varepsilon > 0$. Using (A.1) in (2.4) we conclude that

$$\delta_\ell(z) \sim 1 - \frac{i}{z}\int_{-\infty}^{\xi} \kappa(s)\, ds + \mathcal{O}\left(\frac{1}{z^2}\right),$$

and, by a similar argument

$$\delta_r(z) \sim 1 + \frac{i}{z}\int_{\xi}^{\infty} \kappa(s)\, ds + \mathcal{O}\left(\frac{1}{z^2}\right).$$

A.2. Asymptotics near the stationary phase point

The following asymptotic relations for δ_ℓ, δ_r, δ_r^{-1}, and δ_ℓ^{-1} are used to compute leading asymptotics near the critical point ξ and determine the model RHPs. Define complex powers of $(z-\xi)$ using the appropriate branch of the logarithm ($-\pi < \arg(\zeta-\xi) < \pi$ for $\delta_\ell^{\pm 1}$, and $0 < \arg(\zeta-\xi) < 2\pi$ for $\delta_r^{\pm 1}$). As $z \to \xi$ in the respective domains of δ_ℓ and δ_r,

$$\left|\delta_\ell(\zeta) - \delta_{0\ell}(z-\xi)^{i\kappa(\xi)}\right| \lesssim -|z-\xi| \log|z-\xi| \tag{A.2}$$

$$\left|\delta_r(\zeta) - \delta_{0r}(z-\xi)^{i\kappa(\xi)}\right| \lesssim -|z-\xi| \log|z-\xi| \tag{A.3}$$

$$\left|\delta_r(\zeta)^{-1} - \delta_{0r}^{-1}(z-\xi)^{-i\kappa(\xi)}\right| \lesssim -|z-\xi| \log|z-\xi| \tag{A.4}$$

$$\left|\delta_\ell(\zeta)^{-1} - \delta_{0\ell}^{-1}(z-\xi)^{-i\kappa(\xi)}\right| \lesssim -|z-\xi| \log|z-\xi|, \tag{A.5}$$

where the implied constants depend on $\|\kappa\|_{H^{2,2}}$ and a fixed $\varepsilon > 0$. The constants are uniform in z with $-\pi+\varepsilon < \arg(z-\xi) < \pi-\varepsilon$ (for $\delta_\ell^{\pm 1}$) or $\varepsilon < \arg(z-\xi) < 2\pi-\varepsilon$ (for $\delta_r^{\pm 1}$).

The constants $\delta_{0\ell}$ and δ_{0r} are defined as follows. Let χ_- be the characteristic function of $(\xi-1,\xi)$, and let χ_+ be the characteristic function of $(\xi,\xi+1)$. Then:

$$\delta_{0\ell} = \exp\left(i\int_{-\infty}^{\xi} \frac{\kappa(s)-\chi_-(s)\kappa(\xi)}{s-\xi}\, ds\right),$$

$$\delta_{0r} = e^{\pi\kappa(\xi)} \exp\left(-i\int_{\xi}^{\infty} \frac{\kappa(s)-\chi_+(s)\kappa(\xi)}{s-\xi}\, ds\right).$$

These asymptotics are easily deduced from the integral formulas (2.4) and (2.5). We illustrate the ideas for δ_ℓ; these computations are standard but we include them for the reader's convenience.

Using (2.4), we compute, for $z \in \mathbb{C} \setminus (-\infty,\xi]$,

$$\begin{aligned}\delta_\ell(z) &= \exp\left(i\int_{\xi-1}^{\xi} \frac{\kappa(\xi)}{s-z}\, ds\right) \cdot \exp\left(i\int_{-\infty}^{\xi} \frac{\kappa(s)-\chi_-(s)\kappa(\xi)}{s-z}\, ds\right)\\ &= (z-\xi)^{i\kappa(\xi)} e^{i\beta(z;\xi)}\end{aligned}$$

where

$$\beta(z;\xi) = -\kappa(\xi)\log(z-\xi+1) + \int_{-\infty}^{\xi} \frac{\kappa(s)-\chi_-(s)\kappa(\xi)}{s-z}\, ds.$$

We will show that $\beta(z,\xi)$ is continuous at $z=\xi$ and we set $\delta_{0\ell}(\xi) = \exp(i\beta(\xi,\xi))$. We wish to prove that

$$\left|\delta(z) - \delta_0(\xi)(\xi-z)^{-i\kappa(\xi)}\right| \lesssim_{\rho,\phi} -|z-\xi| \log|z-\xi| \tag{A.6}$$

as $z \to \xi$ for $z-\xi = re^{i\phi}$ with $-\pi < \phi < \pi$ and implied constants independent of $\xi \in \mathbb{R}$. To do this, it suffices to show that

$$\left|\beta(\xi+re^{i\phi};\xi) - \beta(\xi;\xi)\right| \lesssim_{\rho,\phi} -r\log r$$

where the implied constants have the same uniformity. But

$$\beta(\xi + re^{i\phi}; \xi) - \beta(\xi; \xi) = \kappa(\xi)\log(1 + re^{i\phi}) + \int_{-\infty}^{\xi} \left(\frac{1}{s-z} - \frac{1}{s-\xi}\right)(\kappa(s) - \chi(s)\kappa(\xi))\, ds = \int_{\xi-1}^{\xi} \left(\frac{1}{s-z} - \frac{1}{s-\xi}\right)(\kappa(s) - \kappa(\xi))\, ds + \mathcal{O}(r) \tag{A.7}$$

where the implied constants in $\mathcal{O}(r)$ depend on $\|\kappa\|_\infty$ and are independent of $\xi \in \mathbb{R}$. The first right-hand integral in the last line of (A.7) may be written

$$\begin{aligned} I(r;\xi) &= re^{i\phi} \int_{\xi-1}^{\xi} \frac{1}{s-\xi-re^{i\phi}} \frac{\kappa(s)-\kappa(\xi)}{s-\xi}\, ds \\ &= re^{i\phi} \int_{\xi-1}^{\xi} \frac{1}{s-\xi-re^{i\phi}} \kappa'(\xi)\, ds \\ &\quad + re^{i\phi} \int_{\xi-1}^{\xi} \frac{1}{s-\xi-re^{i\phi}} \frac{\int_\xi^s (s-y)\kappa''(y)\, dy}{s-\xi}\, ds \\ &= I_1(r;\xi) + I_2(r;\xi) \end{aligned}$$

By explicit computation,

$$I_1(r;\xi) = re^{i\phi}\kappa'(\xi)\left(\log(-re^{i\phi}) - \log(-1-re^{i\phi})\right) \lesssim -r\log r \tag{A.8}$$

with constants depending on κ through $\|\kappa'\|_\infty$ and otherwise independent of ξ. On the other hand, since $|s-y|/|s-\xi| \leq 1$ we may estimate

$$|I_2(r;\xi)| \leq r\left\|\kappa''\right\|_2 \int_{\xi-1}^{\xi} \frac{1}{|s-\xi-re^{i\phi}|}\, ds$$

The right-hand integral is easily seen to equal

$$\int_{-\frac{1}{r}-\cot\phi}^{-\cot\phi} \frac{1}{\sqrt{\mu^2+1}}\, d\mu$$

which is $\mathcal{O}(\log r)$ as $r \downarrow 0$ with constants depending on ϕ. These constants are bounded if $\varepsilon < \phi < \pi - \varepsilon$ or $-\pi + \varepsilon < \phi < -\varepsilon$ for some fixed $\varepsilon > 0$. For such ϕ we have

$$|I_2(r;\xi)| \lesssim_{\rho,\phi} -r\log r \tag{A.9}$$

with constants independent of $\xi \in \mathbb{R}$ and depending on ρ through $\|\rho\|_{H^{2,2}}$ since $\|\rho\|_{H^{2,2}}$ controls $\left\|\kappa''\right\|_2$.

Since $\|\rho\|_{H^{2,2}}$ also controls $\|\kappa\|_\infty$ and $\left\|\kappa'\right\|_\infty$, we conclude from (A.7), (A.8), and (A.9) that (A.6) holds.

Appendix B. Four model RHPs

We summarize the key formulas leading to $q_{\mathrm{as}}(x,t)$. We will write κ for $\kappa(\xi)$ when it appears in formulas. We denote by $\eta(z;\xi)$ or simply η the function

$$\eta(\zeta;\xi) = (z-\xi).$$

Thus $\eta^{\pm i\kappa}$ is shorthand for $(z-\xi)^{\pm i\kappa(\xi)}$, etc. We will make use of the identities

$$\overline{\Gamma(z)} = \Gamma(\overline{z}), \quad |\Gamma(i\kappa)|^2 = \frac{\pi}{\kappa \sinh(\pi\kappa)}$$

as well as

$$e^{-2\pi\kappa} = 1 - \xi|\rho(\xi)|^2 = 1 - \xi|\breve{\rho}(\xi)|^2$$

in the computations. Recall that the symbols δ, δ_0, and $\delta_\pm$ are defined at the beginning of each subsection and *have different meanings in each of them* as indicated in (B.1), (B.5), (B.9), and (B.13).

B.1. The case $t>0$, $x>0$

To prepare the initial RHP for steepest descent we set $\mathbf{N}^{(1)} = \mathbf{N}\delta_\ell^{-\sigma_3}$. Throughout this subsection,

$$\delta = \delta_\ell, \quad \delta_\pm = (\delta_\ell)_\pm, \quad \delta_0 = \delta_{0\ell}. \tag{B.1}$$

From (2.1) we get a new RHP for $\mathbf{N}^{(1)}$ with jump matrix $V^{(1)}$ where

$$V^{(1)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\dfrac{\delta_-^{-2} z\overline{\rho}}{1-z|\rho|^2} e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\delta_+^2 \rho}{1-z|\rho|^2} e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi) \\ \begin{pmatrix} 1 & \rho\delta^2 e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z\overline{\rho}\delta^{-2} e^{-2it\theta} & 1 \end{pmatrix}, & z \in (\xi, \infty) \end{cases} \tag{B.2}$$

$\mathbf{N}^{(1)}$ is then ready for steepest descent. We reduce to a mixed $\overline{\partial}$-RHP in the new variable $\mathbf{N}^{(2)} = \mathbf{N}^{(1)}\mathcal{R}$ where $\mathcal{R}$ is a piecewise continuous matrix-valued as shown in Fig. E.1. Here

$$\begin{aligned} R_1|_{(\xi,\infty)} &= z\overline{\rho(z)}\delta(z)^{-2} & R_1|_{\Sigma_1} &= \xi\overline{\rho(\xi)}\delta_0^{-2}\eta^{-2i\kappa} \\ R_3|_{(-\infty,\xi)} &= -\frac{\rho(z)\delta_+^2(z)}{1-z|\rho(z)|^2} & R_3|_{\Sigma_2} &= -\frac{\rho(\xi)\delta_0^2}{1-\xi|\rho(\xi)|^2}\eta^{2i\kappa} \\ R_4|_{(-\infty,\xi)} &= -\frac{z\overline{\rho(z)}\delta_-^{-2}}{1-z|\rho(z)|^2} & R_4|_{\Sigma_3} &= -\frac{\xi\overline{\rho(\xi)}\delta_0^{-2}}{1-\xi|\rho(\xi)|^2}\eta^{-2i\kappa} \\ R_6|_{(\xi,\infty)} &= \rho(z)\delta(z)^2 & R_6|_{\Sigma_4} &= \rho(\xi)\delta_0\eta^{2i\kappa} \end{aligned} \tag{B.3}$$

The resulting unknown $\mathbf{N}^{(2)}$ satisfies a mixed $\overline{\partial}$-RHP with jump matrix $V^{(2)}$ defined on the oriented contours of $\Sigma_\xi^{(2)}$.

As discussed above we reduce to a model RHP with contour Σ and jump matrix (2.9) where $V_0^{(2)}$ is shown in Fig. E.3 and

$$r_\xi = \rho(\xi)\delta_0^2 e^{-2i\kappa} e^{-2i\kappa\log(\sqrt{8t})} e^{4it\xi^2} \tag{B.4}$$

Using (2.18), (2.21), (2.22), and (B.4), we conclude that

$$\begin{aligned} |\alpha(\xi)|^2 &= \frac{\kappa(\xi)}{2\xi}, \\ \arg\alpha(\xi) &= \frac{\pi}{4} + \arg\Gamma(i\kappa) + \arg\rho(\xi) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\xi} \log|s-\xi|\, d\log(1-s|\rho(s)|^2). \end{aligned}$$

B.2. The case $t > 0$, $x < 0$

To prepare for steepest descent we set $\mathbf{N}^{(1)} = \mathbf{N}\delta_r^{-\sigma_3}$. Throughout this subsection,

$$\delta = \delta_r, \quad \delta_\pm = (\delta_r)_\pm, \quad \delta_0 = \delta_{0r}. \tag{B.5}$$

The new RHP for $\mathbf{N}^{(1)}$ has jump matrix $\breve{V}^{(1)}$ where

$$\breve{V}^{(1)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ -z\overline{\breve{\rho}}\delta^{-2}e^{-2it\theta} & 1 \end{pmatrix}\begin{pmatrix} 1 & \breve{\rho}\delta^2 e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi) \\ \begin{pmatrix} 1 & \dfrac{\breve{\rho}\delta_-^2}{1-z|\breve{\rho}|^2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{-z\overline{\breve{\rho}}\delta_+^{-2}}{1-z|\breve{\rho}|^2}e^{-2it\theta} & 1 \end{pmatrix}, & z \in (\xi, \infty) \end{cases} \tag{B.6}$$

$\mathbf{N}^{(1)}$ is then ready for steepest descent. As before we reduce to a mixed $\overline{\partial}$-RHP problem n the new variable $\mathbf{N}^{(2)} = \mathbf{N}^{(1)}\mathcal{R}$, where $\mathcal{R}$ is the piecewise continuous matrix-valued function as shown in Fig. E.1. We have the following formulas for R_1, R_3, R_4, and R_6:

$$\begin{aligned} R_1|_{(\xi,\infty)} &= \frac{z\overline{\breve{\rho}(z)}\delta_+(z)^{-2}}{1-z|\rho(z)|^2} & R_1|_{\Sigma_1} &= \frac{\xi\breve{\rho}(\xi)\delta_0^{-2}}{1-\xi|\rho(\xi)|^2}\eta^{-2i\kappa} \\ R_3|_{(-\infty,\xi)} &= -\breve{\rho}(z)\delta_+(z)^2 & R_3|_{\Sigma_2} &= -\breve{\rho}(\xi)\delta_0^2\eta^{2i\kappa} \\ R_4|_{(-\infty,\xi)} &= -z\overline{\breve{\rho}(z)}\delta_-(z)^{-2} & R_4|_{\Sigma_3} &= -\xi\overline{\breve{\rho}(\xi)}\delta_0^{-2}\eta^{-2i\kappa} \\ R_6|_{(\xi,\infty)} &= \frac{\breve{\rho}(z)}{1-z|\breve{\rho}(z)|^2}\delta_-(z)^2 & R_6|_{\Sigma_4} &= \frac{\breve{\rho}(\xi)}{1-\xi|\breve{\rho}(\xi)|^2}\delta_0^2\eta^{2i\kappa} \end{aligned} \tag{B.7}$$

The new unknown $\mathbf{N}^{(2)}$ satisfies a mixed $\overline{\partial}$-RHP in $\mathbf{N}^{(2)}$ with jump matrix $V^{(2)}$ on $\Sigma_\xi^{(2)}$.

Following the procedure outlined at the beginning of this section we arrive at a model RHP with contour $\Sigma_0^{(2)}$ and jump matrix (2.9) where $V_0^{(2)}$ is shown in Fig. E.4 and

$$\begin{aligned} \breve{r}_\xi &= \breve{\rho}(\xi)\delta_0^2 e^{-2i\kappa(\xi)\log\sqrt{8t}}e^{4it\xi^2}. \\ &= \breve{\rho}(\xi)e^{2\kappa\pi}\exp\left(-2i\int_\xi^\infty \frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-\xi}\,ds\right)e^{-2i\kappa(\xi)\log\sqrt{8t}}e^{4it\xi^2} \end{aligned} \tag{B.8}$$

From (2.18), (2.21), (2.22), and (B.8), we conclude that

$$\begin{aligned} |\alpha(\xi)|^2 &= \frac{\kappa(\xi)}{2\xi} \\ \arg\alpha(\xi) &= -\frac{3\pi}{4} + \arg\Gamma(i\kappa) + \arg\rho(\xi) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\xi} \log(\xi - s)\,d\log(1 - s|\rho(s)|^2). \end{aligned}$$

B.3. The case $t < 0$, $x > 0$

In what follows we will set $t' = -t$ so that $|t| = t'$ and

$$\theta(z; x, t) = -\left(-z\frac{x}{t'} + 2z^2\right).$$

To prepare the initial RHP for steepest descent we take $\mathbf{N}^{(1)} = \mathbf{N}\delta_r^{\sigma_3}$. Throughout this subsection

$$\delta = \delta_r^{-1}, \quad \delta_\pm = \left(\delta_r^{-1}\right)_\pm, \quad \delta_0 = \delta_{0r}^{-1}. \tag{B.9}$$

The resulting RHP for $\mathbf{N}^{(1)}$ has jump matrix $V^{(1)}$ where

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & \rho\delta^2 e^{-2it'\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z\overline{\rho} e^{2it'\theta} & 1 \end{pmatrix}, & z \in (-\infty, \xi) \\ \begin{pmatrix} 1 & 0 \\ \dfrac{-z\overline{\rho}\delta_-^{-2}}{1-z|\rho|^2} e^{2it'\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\rho\delta_+^2}{1-z|\rho|^2} e^{-2it'\theta} \\ 0 & 1 \end{pmatrix}, & z \in (\xi, \infty) \end{cases} \tag{B.10}$$

We write $\mathbf{N}^{(1)} = \mathbf{N}^{(2)}\mathcal{R}$ where the piecewise continuous matrix-valued function $\mathcal{R}$ is shown in Fig. E.2, and the functions R_i are described as follows:

$$\begin{aligned} R_1|_{(\xi,\infty)} &= -\frac{\rho(z)\delta_+(z)^2}{1-z|\rho(z)|^2} & R_1|_{\Sigma_1} &= -\frac{\rho(\xi)\delta_0^2}{1-\xi|\rho(\xi)|^2}\eta^{2i\kappa} \\ R_3|_{(-\infty,\xi)} &= z\overline{\rho(z)}\delta_+(z)^{-2} & R_3|_{\Sigma_2} &= \xi\overline{\rho(\xi)}\delta_0^{-2}\eta^{-2i\kappa} \\ R_4|_{(-\infty,\xi)} &= \rho(z)\delta_-(z)^2 & R_4|_{\Sigma_3} &= \rho(\xi)\delta_0^2\eta^{2i\kappa} \\ R_6|_{(\xi,\infty)} &= \frac{-z\overline{\rho(z)}\delta_-(z)^{-2}}{1-z|\rho(z)|^2} & R_6|_{\Sigma_4} &= \frac{-\xi\overline{\rho(\xi)}\delta_0^{-2}}{1-\xi|\rho(\xi)|^2}\eta^{-2i\kappa} \end{aligned} \tag{B.11}$$

The function $\mathbf{N}^{(2)}$ obeys a mixed $\overline{\partial}$-RHP with jump matrix $V^{(2)}$ that we describe below.

Following the standard procedure we arrive at a model RHP with contour $\Sigma_0^{(2)}$ and jump matrix (2.9) where $V_0^{(2)}$ is shown in Fig. E.5 and

$$\begin{aligned} r_\xi &= \rho(\xi)\delta_{0r}^{-2} e^{2i\kappa(\xi)\log\sqrt{8t'}} e^{-4it'\xi^2} \\ &= \rho(\xi) e^{-2\kappa\pi} \exp\left(2i\int_\xi^\infty \frac{\kappa(s)-\chi(s)\kappa(\xi)}{s-\xi}\,ds\right) e^{2i\kappa(\xi)\log\sqrt{8t'}} e^{-4it'\xi^2}. \end{aligned} \tag{B.12}$$

From (2.19), (2.21), and (2.22), and (B.12), we conclude that

$$\begin{aligned} |\alpha(\xi)|^2 &= \frac{\kappa(\xi)}{2\xi} \\ \arg\alpha(\xi) &= \frac{3\pi}{4} - \arg\Gamma(i\kappa) + \arg\rho(\xi) \\ &\quad + \frac{1}{\pi}\int_\xi^\infty \log|s-\xi|\, d\log(1-s|\rho(s)|^2). \end{aligned}$$

B.4. The case $t<0$, $x<0$

To prepare for steepest descent we set $\mathbf{N}^{(1)} = \mathbf{N}\delta_\ell^{\sigma_3}$. Throughout this subsection,

$$\delta = \delta_\ell^{-1}, \quad \delta_\pm = (\delta_\ell^{-1})_\pm, \quad \delta_0 = \delta_{0\ell}^{-1}. \tag{B.13}$$

The resulting RHP for $\mathbf{N}^{(1)}$ has jump matrix $\breve{V}^{(1)}$ where

$$\breve{V}^{(1)} = \begin{cases} \begin{pmatrix} 1 & \dfrac{\breve{\rho}(z)}{1-z|\breve{\rho}(z)|^2}\breve{\delta}_-^2 e^{-2it'\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \dfrac{-z\overline{\breve{\rho}(z)}\breve{\delta}_+^{-2}}{1-z|\breve{\rho}(z)|^2} e^{2it'\theta} & 1 \end{pmatrix}, & z \in (-\infty, \xi) \\ \begin{pmatrix} 1 & 0 \\ -z\overline{\breve{\rho}(z)}\breve{\delta}^{-2} e^{2it'\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \breve{\rho}(z)\breve{\delta}^2 e^{-2it'\theta} \\ 0 & 1 \end{pmatrix}, & z \in (\xi, \infty) \end{cases} \tag{B.14}$$

We can now deform to a mixed $\overline{\partial}$-RHP by passing to $\mathbf{N}^{(2)} = \mathbf{N}^{(1)}\mathcal{R}$ where $\mathcal{R}$ is the piecewise continuous matrix-valued function shown in Fig. E.2, and the functions R_i have the boundary values:

$$\begin{aligned} R_1|_{(\xi,\infty)} &= -\breve{\rho}(z)\delta_+(z)^2 & R_1|_{\Sigma_1} &= -\breve{\rho}(\xi)\delta_0^2\eta^{2ik} \\ R_3|_{(-\infty,\xi)} &= \frac{z\breve{\rho}(z)}{1-z|\rho(z)|^2}\delta_+^{-2} & R_3|_{\Sigma_2} &= \frac{\xi\overline{\breve{\rho}(\xi)}}{1-\xi|\rho(\xi)|^2}\delta_0^{-2}\eta^{-2i\kappa} \\ R_4|_{(-\infty,\xi)} &= \frac{\breve{\rho}(z)}{1-z|\breve{\rho}(z)|^2}\delta_-^2 & R_4|_{\Sigma_3} &= \frac{\breve{\rho}(\xi)}{1-\xi|\breve{\rho}(\xi)|^2}\delta_0^2\eta^{2i\kappa} \\ R_6|_{(\xi,\infty)} &= -z\overline{\breve{\rho}(z)}\delta(z)^{-2} & R_6|_{\Sigma_4} &= -\xi\overline{\breve{\rho}(\xi)}\delta_0^{-2}\eta^{-2i\kappa} \end{aligned} \tag{B.15}$$

The new unknown $\mathbf{N}^{(2)}$ satisfies a mixed $\overline{\partial}$-RHP with jump matrix $\breve{V}^{(2)}$ on $\Sigma_\xi^{(2)}$.

Following the procedure outlined at the beginning of the section we arrive at a model RHP with contour $\Sigma_0^{(2)}$ and jump matrix (2.9), where $V_0^{(2)}$ is shown in Fig. E.6 and

$$\breve{r}_\xi = \breve{\rho}(\xi)\delta_{0\ell}^2 e^{2i\kappa(\xi)\log\sqrt{8t'}} e^{-4it'\xi^2}. \tag{B.16}$$

From (2.19), (2.21), (2.22), and (B.16), we conclude that

$$\begin{aligned} |\alpha(\xi)|^2 &= \frac{\kappa(\xi)}{2\xi} \\ \arg\alpha(\xi) &= -\frac{\pi}{4} - \arg\Gamma(i\kappa) + \arg\rho(\xi) \\ &\quad + \frac{1}{\pi}\int_\xi^\infty \log|s-\xi|\, d\log(1-s|\rho(s)|^2). \end{aligned}$$

Appendix C. Formulae and Wronskian for parabolic cylinder functions

For sake of completeness, we provide in this appendix details of proofs of various results used in Section 5 about parabolic cylinder functions.

C.1. *Proof of Lemma 5.4*

Differentiating (5.7) with respect to ζ, we obtain

$$\left(\frac{d\Phi}{d\zeta} + \frac{1}{2}i\zeta\sigma_3\Phi\right)_+ = \left(\frac{d\Phi}{d\zeta} + \frac{1}{2}i\sigma_3\zeta\Phi\right)_- V^{(0)}.$$

We know that $\det V^{(0)} = 1$, thus $\det\Phi_+ = \det\Phi_-$ and $\det\Phi$ is analytic in the whole complex plane. It is equal to one at infinity, thus by Liouville theorem, $\det\Phi = 1$. It follows that $(\Phi)^{-1}$ exists and is bounded. The matrix $\left(\frac{d\Phi}{d\zeta} + \frac{i}{2}\sigma_3\zeta\Phi\right)\Phi^{-1}$ has no jump along the real line and is therefore an entire function of ζ. Let us compute its behavior at infinity. Returning to (5.3), we have that

$$\left(\frac{d\Phi}{d\zeta}+\frac{i\zeta}{2}\sigma_3\Phi\right)\Phi^{-1}=\left(\frac{d\mathbf{N}^{\mathrm{PC}}}{d\zeta}+\mathbf{N}^{\mathrm{PC}}\frac{i\kappa\sigma_3}{\zeta}\right)(\mathbf{N}^{\mathrm{PC}})^{-1} + \frac{i\zeta}{2}\left[\sigma_3,\mathbf{N}^{\mathrm{PC}}\right](\mathbf{N}^{\mathrm{PC}})^{-1}. \tag{C.1}$$

The first term in the right-hand side of (C.1) tends to 0 as $\zeta\to\infty$, while the second term behaves like $O(1/\zeta)$. For the last term in the right-hand side of (C.1), we use that

$$\mathbf{N}^{\mathrm{PC}}(\zeta)\sim\left(1+\frac{m^{(0)}}{\zeta}\right).$$

Defining

$$\beta\equiv\frac{i}{2}[\sigma_3,\mathbf{N}^{\mathrm{PC}}_{(1)}]=\begin{pmatrix}0 & im^{(0)}_{12}\\ -im^{(0)}_{21} & 0\end{pmatrix}$$

Equivalently, $\beta_{12}=im^{(0)}_{12}$ and $\beta_{21}=-im^{(0)}_{21}$. Again applying Liouville's theorem, the 2×2 matrix Φ satisfies the ODE:

$$\frac{d\Phi}{d\zeta}+\frac{i\zeta}{2}\sigma_3\Phi=\beta\Phi \tag{C.2}$$

where β is an off-diagonal matrix.

The system (C.2) decouples into two first-order systems for (Φ_{11},Φ_{21}) and (Φ_{12},Φ_{22}),

$$\begin{cases}\dfrac{d\Phi_{11}}{d\zeta}+\frac{1}{2}i\zeta\Phi_{11} &=\beta_{12}\Phi_{21}\\ \dfrac{d\Phi_{21}}{d\zeta}-\frac{1}{2}i\zeta\Phi_{21} &=\beta_{21}\Phi_{11}\end{cases} \tag{C.3}$$

and

$$\begin{cases}\dfrac{d\Phi_{12}}{d\zeta}+\frac{1}{2}i\zeta\Phi_{12} &=\beta_{12}\Phi_{22}\\ \dfrac{d\Phi_{22}}{d\zeta}-\frac{1}{2}i\zeta\Phi_{22} &=\beta_{21}\Phi_{12}.\end{cases} \tag{C.4}$$

Combining the above equations, one obtains that the entries of Φ satisfy (5.9)–(5.12).

C.2. Proof of Lemma 5.6

We know that $\beta_{12}\beta_{21}=\kappa(\xi)$ and

$$(\Phi_-)^{-1}\Phi_+=V^{(0)}=\begin{pmatrix}1-\xi\,|r_\xi|^2 & r_\xi\\ -\xi\,\overline{r}_\xi & 1\end{pmatrix}.$$

Combining (5.17), (5.18), (5.21), and (5.22), we obtain

$$\begin{aligned}-\xi\,\overline{r}_\xi &=\Phi^-_{11}\Phi^+_{21}-\Phi^-_{21}\Phi^+_{11}\\
&=e^{\frac{\pi}{4}\kappa}D_{i\kappa}(e^{i\pi/4}\zeta)\frac{1}{\beta_{12}}e^{-\frac{3\pi}{4}\kappa}\left(\partial_\zeta(D_{i\kappa}(\zeta e^{-3i\pi/4}))+\frac{i\zeta}{2}D_{i\kappa}(\zeta e^{-3i\pi/4})\right)\\
&\quad-\frac{1}{\beta_{12}}e^{\frac{\pi}{4}\kappa}\left(\partial_\zeta(D_{i\kappa}(\zeta e^{i\pi/4}))+\frac{i\zeta}{2}D_{i\kappa}(\zeta e^{i\pi/4})\right)e^{-\frac{3\pi}{4}\kappa}D_{i\kappa}(\zeta e^{-3i\pi/4})\\
&=\frac{e^{-\frac{\pi\kappa}{2}}}{\beta_{12}}W\left(D_{i\kappa}(e^{i\pi/4}\zeta),\,D_{i\kappa}(\zeta e^{-3i\pi/4})\right)\\
&=\frac{\sqrt{2\pi}e^{-\frac{\pi\kappa}{2}}e^{i\pi/4}}{\beta_{12}\Gamma(-i\kappa)}\end{aligned}$$

where we have used the Wronskian (C.6) in the last equality. It follows from the above computations that

$$\beta_{12} = \frac{\sqrt{2\pi}\, e^{-\pi\kappa/2} e^{\pi i/4}}{-\xi\, \overline{r_\xi}\, \Gamma(-i\kappa)}.$$

From the identities

$$\Gamma(\overline{z}) = \overline{\Gamma(z)}, \quad |\Gamma(i\kappa)|^2 = \frac{\pi}{\kappa \sinh \pi\kappa}$$

we see that

$$|\beta_{12}|^2 = \left| \sqrt{2\pi} \frac{e^{-\pi\kappa/2}}{\xi\, \overline{r_\xi}\, \Gamma(i\kappa)} \right|^2 = 2 \frac{\kappa e^{-\pi\kappa} \sinh \pi\kappa}{\xi^2 |r_\xi|^2} = \kappa \frac{1}{\xi^2 |r_\xi|^2} \left(1 - e^{-2\pi\kappa}\right).$$

Recalling that

$$\kappa(\xi) = -\frac{1}{2\pi} \log\left(1 - \xi |\rho(\xi)|^2\right)$$

we compute

$$1 - e^{-2\pi\kappa} = \xi |\rho(\xi)|^2$$

so that (5.26) holds.

On the other hand, since $\xi < 0$

$$\arg \beta_{12} = \frac{\pi}{4} + \arg r_\xi + \arg(\Gamma(i\kappa)).$$

Substituting the definition of r_ξ given in (5.2)

$$\arg r_\xi = \arg \rho(\xi) + \arg \delta_0^2 - \kappa(\xi) \log(8t) + 4t\xi^2.$$

We also have, by integration by parts

$$\begin{aligned} \delta_0^2(\xi) &= \exp\left(2i \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - \xi} ds \right) \\ &= \exp\left(-2i \int_{-\infty}^{\xi} \log|s - \xi|\, d\kappa(s) \right) \\ &= \exp\left(\frac{i}{\pi} \int_{-\infty}^{\xi} \log|s - \xi|\, d \log(1 - s|\rho(s)|^2) \right) \end{aligned}$$

thus (5.27) holds.

C.3. Wronskians

We record the solution formulae for $\Phi(\zeta, \xi)$ arising in the factorization of the model RHP in each of the four cases $\pm t > 0$, $\pm x > 0$; see Step 4 of Section 2 and especially (2.11) for the set-up; see also (2.14) and the comments following for the solution method. These formulae together with the Wronskian identity for parabolic cylinder functions, allow the evaluations of (2.16) and (2.17) that in turn provide β_{12} in terms of frozen-coefficient scattering data.

We give explicit formulae for the solutions of the equations (2.14) with asymptotic behavior

$$\Phi(\zeta; \xi) \sim e^{\mp \frac{i}{4}\zeta^2 \sigma_3} \zeta^{\pm i\kappa\sigma_3} \left(I + \frac{m^{(1)}}{\zeta} + o\left(\zeta^{-1}\right) \right).$$

We denote by $D_a(z)$ the usual parabolic cylinder function, i.e., the solution to (5.13) with asymptotics prescribed in (5.14). The identity (5.25) is easily be derived from the relation

$$U(a,z) = D_{-a-\frac{1}{2}}(z) \tag{C.5}$$

(see [21, §12.1][3]) and [21, 12.8.2].[4] We also record the Wronskian identity

$$W(D_a(z), D_a(-z)) = \frac{\sqrt{2\pi}}{\Gamma(-a)} \tag{C.6}$$

which is a consequence of (C.5) and [21, eq. (12.2.11)].[5] We use this identity to compute β_{12} (see proof of Lemma 5.6).

For the + case of (2.14), taking $-\pi < \arg\zeta < \pi$ corresponding to $t > 0$, $x > 0$, the solution $\Phi(\zeta;\xi)$ is given by expressions (5.15) and (5.16) of Proposition 5.5.

For the + case of (2.14), taking $0 < \arg\zeta < 2\pi$ corresponding to $t > 0$, $x < 0$, the solution $\Phi(\zeta;\xi)$ is given by

$$\begin{cases} \begin{pmatrix} e^{-\frac{3\pi}{4}\kappa} D_{i\kappa}(\zeta e^{-\frac{3i\pi}{4}}) & -\frac{i\kappa}{\beta_{21}} e^{\frac{\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{-\frac{\pi i}{4}}) \\ \frac{i\kappa}{\beta_{12}} e^{-\frac{3\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{-\frac{3i\pi}{4}}) & e^{\frac{\pi\kappa}{4}} D_{-i\kappa}(e^{-i\pi/4}\zeta) \end{pmatrix} & \zeta \in \mathbb{C}^+, \\ \begin{pmatrix} e^{-\frac{7\pi\kappa}{4}} D_{i\kappa}(\zeta e^{-\frac{7\pi i}{4}}) & \frac{-i\kappa}{\beta_{21}} e^{\frac{5\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{-\frac{5\pi i}{4}}) \\ \frac{i\kappa}{\beta_{12}} e^{-\frac{7\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{-\frac{7\pi i}{4}}) & e^{\frac{5\pi\kappa}{4}} D_{-i\kappa}(\zeta e^{-\frac{5\pi i}{4}}) \end{pmatrix} & \zeta \in \mathbb{C}^- \end{cases}$$

For the − case of (2.14), taking $0 < \arg\zeta < 2\pi$ corresponding to $t < 0$, $x > 0$, the solution $\Phi(\zeta;\xi)$ is

$$\begin{cases} \begin{pmatrix} e^{\frac{\pi}{4}\kappa} D_{-i\kappa}(\zeta e^{-\frac{\pi i}{4}}) & \frac{i\kappa}{\beta_{21}} e^{-\frac{3\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{-\frac{3\pi i}{4}}) \\ \frac{-i\kappa}{\beta_{12}} e^{\frac{\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{-\frac{\pi i}{4}}) & e^{-\frac{3\pi\kappa}{4}} D_{i\kappa}(\zeta e^{-\frac{3\pi i}{4}}) \end{pmatrix} & \zeta \in \mathbb{C}^+, \\ \begin{pmatrix} e^{\frac{5\pi}{4}\kappa} D_{-i\kappa}(e^{-\frac{5\pi i}{4}}\zeta) & \frac{i\kappa}{\beta_{21}} e^{-\frac{7\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{-\frac{7\pi i}{4}}) \\ \frac{-i\kappa}{\beta_{12}} e^{\frac{5\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{-5\pi i/4}) & e^{-\frac{7\pi}{4}\kappa} D_{i\kappa}(e^{-\frac{7\pi i}{4}}\zeta) \end{pmatrix} & \zeta \in \mathbb{C}^-. \end{cases}$$

Finally, for the − case of (2.14), taking $-\pi < \arg\zeta < \pi$ corresponding to $t < 0$, $x < 0$, the solution $\Phi(\zeta;\xi)$ is

$$\begin{cases} \begin{pmatrix} e^{\frac{\pi\kappa}{4}} D_{-i\kappa}(\zeta e^{-\frac{\pi i}{4}}) & \frac{i\kappa}{\beta_{21}} e^{-\frac{3\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{-\frac{3\pi i}{4}}) \\ \frac{-i\kappa}{\beta_{12}} e^{\frac{\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{-\frac{\pi i}{4}}) & e^{\frac{-3\pi\kappa}{4}} D_{i\kappa}(\zeta e^{-\frac{3\pi i}{4}}) \end{pmatrix} & \zeta \in \mathbb{C}^+, \\ \begin{pmatrix} e^{-\frac{3\pi\kappa}{4}} D_{-i\kappa}(\zeta e^{\frac{3\pi i}{4}}) & \frac{i\kappa}{\beta_{21}} e^{\frac{\pi}{4}(\kappa+i)} D_{i\kappa-1}(\zeta e^{\frac{\pi i}{4}}) \\ \frac{-i\kappa}{\beta_{12}} e^{-\frac{3\pi}{4}(\kappa-i)} D_{-i\kappa-1}(\zeta e^{\frac{3\pi i}{4}}) & e^{\frac{\pi\kappa}{4}} D_{i\kappa}(\zeta e^{\frac{i\pi}{4}}) \end{pmatrix} & \zeta \in \mathbb{C}^- \end{cases}$$

From these formulae and the identities (5.25) and (C.6), we can compute (cf. (2.16)–(2.17))

$$\Phi_{11}^-\Phi_{21}^+ - \Phi_{21}^-\Phi_{11}^+ = \begin{cases} \frac{1}{\beta_{12}} e^{-\pi\kappa/2} e^{\pi i/4} \frac{\sqrt{2\pi}}{\Gamma(-i\kappa)} & t > 0,\ x > 0 \\ \frac{1}{\beta_{12}} e^{-\pi\kappa/2} e^{i\pi/4} \frac{\sqrt{2\pi}}{\Gamma(-i\kappa)} e^{-2\pi\kappa} & t > 0,\ x < 0 \end{cases} \tag{C.7}$$

and

[3] http://dlmf.nist.gov/12.1.
[4] http://dlmf.nist.gov/12.8.E2.
[5] http://dlmf.nist.gov/12.2.E11.

$$\Phi_{11}^{-}\Phi_{21}^{+} - \Phi_{21}^{-}\Phi_{11}^{+} = \begin{cases} \dfrac{1}{\beta_{12}} e^{-\pi\kappa/2} e^{3i\pi/4} \dfrac{\sqrt{2\pi}}{\Gamma(i\kappa)} e^{2\pi\kappa}, & t<0,\ x>0 \\ \dfrac{1}{\beta_{12}} e^{-\pi\kappa/2} e^{3\pi i/4} \dfrac{\sqrt{2\pi}}{\Gamma(i\kappa)}, & t<0,\ x<0 \end{cases} \tag{C.8}$$

Appendix D. L^∞-bounds for the model RHP

We prove the bounds (6.8) and (6.9). It suffices to prove (6.8) since the bound (6.9) follows from (6.8) and the fact that $\mathbf{N}^{\mathrm{PC}}(\zeta;\xi)$ takes values in $SL(2,\mathbb{C})$. Together with explicit estimates on the parabolic cylinder functions $D_a(\zeta)$, following a similar discussion in [22, §3.1.1, Lemma 3.5].

Lemma D.1. *Let c_1 and c_2 be strictly positive constants, and suppose that $\rho \in H^{2,2}(\mathbb{R})$ with ρ with $\|\rho\|_{H^{2,2}} \le c_1$, $\inf_{z\in\mathbb{R}}(1 - z|\rho(z)|^2) \ge c_2$. Then, the estimate*

$$\left|\mathbf{N}^{\mathrm{PC}}(\zeta;\xi)\right| \lesssim 1$$

holds, where the implied constant depend only on c_1 and c_2.

Proof. We give the bound for the region Ω_1 since estimates for the other regions are similar. Using (5.3), (5.4), (5.15), (5.16) and writing

$$p_1(\xi) = r_\xi/(1-\xi|r_\xi|^2),$$

we have that, for ζ with $0 < \arg(\zeta) < \pi/4$, the entries N_{ij} of $\mathbf{N}^{\mathrm{PC}}$ are given by

$$\begin{aligned} N_{11}(\zeta;\xi) &= e^{\pi\kappa/4} e^{-\frac{i}{4}\zeta^2} \zeta^{i\kappa} D_{-i\kappa}(\zeta e^{-i\pi/4}) \\ N_{12}(\zeta;\xi) &= p_1(\xi) e^{\frac{i}{4}\zeta^2} \zeta^{-i\kappa} e^{\pi\kappa/4} D_{-i\kappa}(\zeta e^{-i\pi/4}) \\ &\quad + \frac{1}{\beta_{21}} e^{-3\pi\kappa/4} e^{-3\pi i/4} (i\kappa) e^{\frac{i}{4}\zeta^2} \zeta^{-i\kappa} D_{i\kappa-1}(\zeta e^{-3\pi i/4}) \\ N_{21}(\zeta;\xi) &= e^{\pi\kappa/4} e^{-\pi i/4} (-i\kappa) e^{-\frac{i}{4}\zeta^2} \zeta^{i\kappa} D_{-i\kappa-1}(\zeta e^{-i\pi/4}) \\ N_{22}(\zeta;\xi) &= \frac{1}{\beta_{12}} p_1(\xi) e^{\frac{i}{4}\zeta^2} \zeta^{-i\kappa} e^{\frac{\pi}{4}\kappa} e^{-i\pi/4} (-i\kappa) D_{-i\kappa-1}(\zeta e^{-i\pi/4}) \\ &\quad + e^{-3\pi\kappa/4} e^{\frac{i}{4}\zeta^2} \zeta^{-i\kappa} D_{i\kappa}(\zeta e^{-3i\pi/4}). \end{aligned}$$

Since

$$D_{-i\kappa}(\zeta e^{-i\pi/4}) \sim e^{-\pi\kappa/4} \zeta^{-i\kappa} e^{\frac{i}{4}\zeta^2}, \quad D_{i\kappa}(\zeta e^{-3i\pi/4}) \sim e^{3\pi\kappa/4} e^{\frac{i}{4}\zeta^2}$$

it is clear that $\mathbf{N}^{\mathrm{PC}}(\zeta;\xi) \to I$ as $\zeta \to \infty$ in Ω_1. To prove the uniform L^∞-estimate, we need a quantitative version of the asymptotics. We claim that, uniformly in a, in compacts of $\mathbb{C}$ and z with $|z| \ge 1$ and $|\arg(z)| < 3\pi/4$, the estimate

$$\left| e^{z^2/4} z^{-a} D_a(z) \right| \lesssim 1 \tag{D.1}$$

holds. The uniform L^∞-estimate will follow from the boundedness of κ, the fact that $\left|e^{\frac{i}{4}\zeta^2}\right| \le 1$ for $\zeta \in \Omega_1$, and the estimates

$$\begin{aligned} \left| e^{-\frac{i}{4}\zeta^2} \zeta^{i\kappa} D_{-i\kappa}(\zeta e^{-i/\pi/4}) \right| &\lesssim 1 \\ \left| e^{-\frac{i}{4}\zeta^2} \zeta^{-i\kappa} D_{i\kappa}(\zeta e^{-3i\pi/4}) \right| &\lesssim 1 \\ \left| e^{-\frac{i}{4}\zeta^2} \zeta^{-i\kappa} D_{-i\kappa-1}(\zeta e^{-i\pi/4}) \right| &\lesssim 1 \\ \left| e^{-\frac{i}{4}\zeta^2} \zeta^{-i\kappa} D_{i\kappa-1}(\zeta e^{-3\pi i/4}) \right| &\lesssim 1 \end{aligned}$$

which are a consequence of (D.1).

To complete the proof, we recall from [22] the proof of (D.1). The parabolic cylinder function $D_a(z)$ can be expressed in terms of the Whittaker function $W_{k,\mu}(z)$ [23] (see Lemma D.2 below) via the formula

$$D_a(\zeta) = 2^{\frac{1}{4}+\frac{a}{2}}\zeta^{-1/2}W_{\frac{1}{4}+\frac{a}{2},-1/4}(\zeta^2/2) \tag{D.2}$$

while, for $|\arg(z)| < 3\pi/2$, the Whittaker function admits the integral representation

$$W_{\frac{1}{4}+\frac{a}{2},-1/4}(z) = e^{-z/2}z^{\frac{1}{4}+\frac{a}{2}}\left[1 - \frac{\Gamma(\frac{3}{2}-a)\Gamma(1-\frac{a}{2})}{\Gamma(\frac{1}{2}-\frac{a}{2})\Gamma(-\frac{a}{2})}\frac{1}{z} + R(a,z)\right] \tag{D.3}$$

where

$$R(a,z) = \frac{1}{\Gamma(\frac{1}{2}-\frac{a}{2})\Gamma(-\frac{a}{2})}\int_{-i\infty-\frac{3}{2}}^{+i\infty-\frac{3}{2}} z^{\zeta}\Gamma(\zeta)\Gamma(-\zeta+\frac{1}{2}-\frac{a}{2})\Gamma(-\zeta-\frac{a}{2})\,d\zeta \tag{D.4}$$

The computations in [22, proof of Lemma 3.5] show that

$$|R(a,z)| \lesssim |z|^{-3/2}\left(\frac{3}{2}\pi - |\arg(z)|\right)^{-3/2}, \tag{D.5}$$

where the implied constant depends only on c_1 and c_2, if $a = \pm i\kappa$ or $a = \pm i\kappa - 1$. This estimate, (D.2), (D.3), and (D.5) imply (D.1). □

Lemma D.2. *The integral representation* (D.3) *holds.*

Proof. We begin with the following representation formula [21, (13.16.11)][6]:

$$W_{k,\mu}(z) = \frac{e^{-\frac{1}{2}z}}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\frac{1}{2}+\mu+t)\Gamma(\frac{1}{2}-\mu+t)\Gamma(-k-t)}{\Gamma(\frac{1}{2}+\mu-k)\Gamma(\frac{1}{2}-\mu-k)}z^{-t}\,dt$$

where the contour separates the poles of $\Gamma(\frac{1}{2}+\mu+t)\Gamma(\frac{1}{2}-\mu+t)$ from those of $\Gamma(-k-t)$, and $|\arg(z)| < 3\pi/2$. Thus, taking $k = \frac{1}{2}+\frac{a}{2}$ and $\mu = -\frac{1}{4}$, we obtain

$$W_{\frac{a}{2}+\frac{1}{4},-\frac{1}{4}}(z) = \frac{e^{-\frac{1}{2}z}}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(\frac{1}{4}+t)\Gamma(\frac{3}{4}+t)\Gamma(-\frac{a}{2}-\frac{1}{4}-t)}{\Gamma(-\frac{a}{2})\Gamma(\frac{1}{2}-\frac{a}{2})}z^{-t}\,dt$$

We wish to set $t = \zeta - \left(\frac{1}{4}+\frac{a}{2}\right)$. If $a = \pm i\kappa$ this contour shift can be made without picking up contributions from poles. We recover

$$W_{\frac{a}{2}+\frac{1}{4},-\frac{1}{4}}(z) = \frac{e^{-\frac{1}{2}z}z^{\frac{1}{4}+\frac{a}{2}}}{2\pi i}\times\frac{1}{\Gamma(-\frac{a}{2})\Gamma(\frac{1}{2}-\frac{a}{2})}\int_{-i\infty}^{+i\infty}\Gamma\left(\zeta-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\zeta-\frac{a}{2}\right)\Gamma(-\zeta)z^{-\zeta}\,d\zeta$$

We can now obtain a large-z expansion by shifting the contour to the right. We will pick up poles at $\zeta = 0, 1, \cdots$ depending on how far we shift. It is easy to compute the residues of the integrand at $\zeta = 0$ and $\zeta = 1$ using the facts that $\Gamma(-\zeta) = \Gamma(1-\zeta)/(-\zeta) = \Gamma(2-\zeta)/(-\zeta(1-\zeta))$. Note that the residues get multiplied by -1 in the computations since we shift the contour to the right. We then obtain

[6] http://dlmf.nist.gov/13.16.E11.

$$W_{\frac{a}{2}+\frac{1}{4},-\frac{1}{4}}(z) = \frac{e^{-\frac{1}{2}z}z^{\frac{1}{4}+\frac{a}{2}}}{2\pi i} \times \left(1 - \frac{\Gamma(1-\frac{a}{2})\Gamma(\frac{3}{2}-\frac{a}{2})}{\Gamma(-\frac{a}{2})\Gamma(\frac{1}{2}-\frac{a}{2})}\frac{1}{z} - \frac{1}{\Gamma(-\frac{a}{2})\Gamma(\frac{1}{2}-\frac{a}{2})}\int_{-i\infty+\frac{3}{2}}^{+i\infty+\frac{3}{2}} \Gamma\left(\zeta-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\zeta-\frac{a}{2}\right)\Gamma(-\zeta)z^{-\zeta}\,d\zeta\right)$$

A trivial change of variable gives (D.4). □

Appendix E. Figures

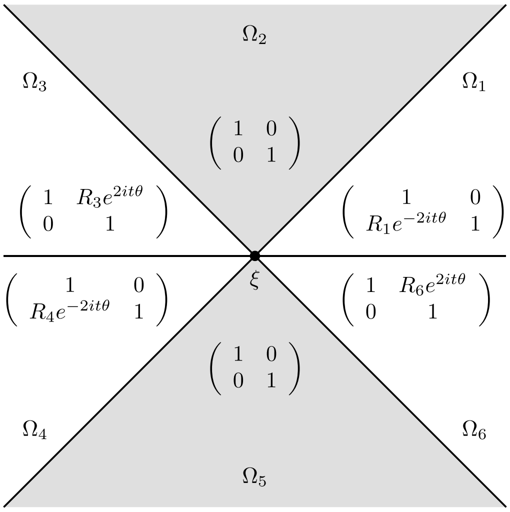

Fig. E.1. The Matrix $\mathcal{R}^{(2)}$ for $t > 0$, $\pm x > 0$.

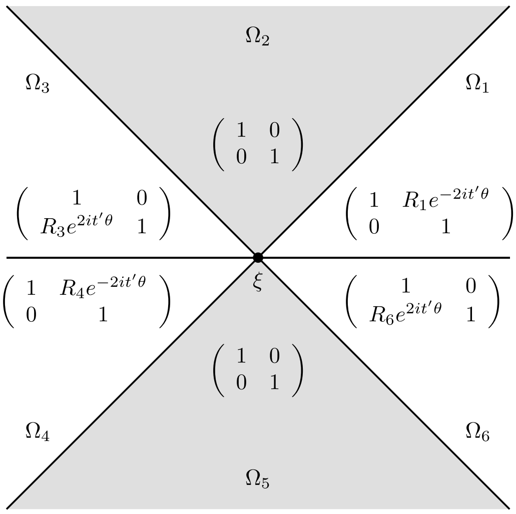

Fig. E.2. The Matrix $\mathcal{R}^{(2)}$ for $t < 0$, $\pm x > 0$ (note that $t' = -t$).

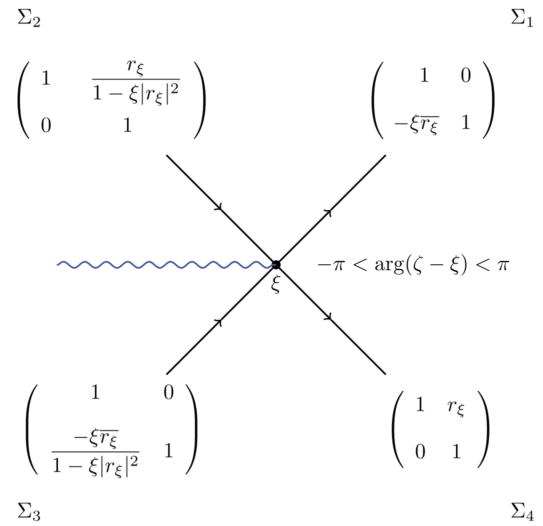

Fig. E.3. The Jump Matrix $V_0^{(2)}$ for $t > 0$, $x > 0$.

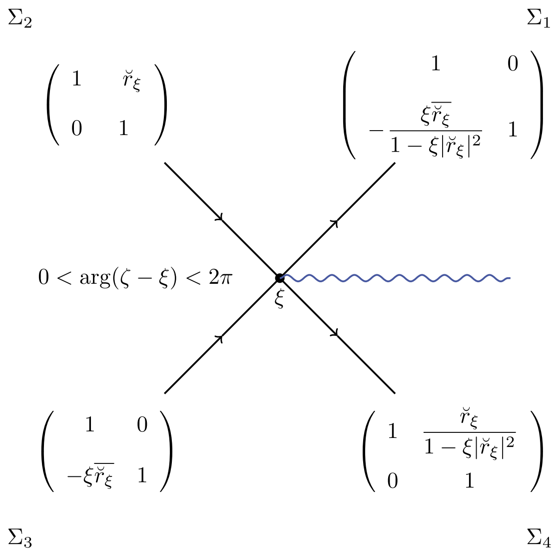

Fig. E.4. The Jump Matrix $V_0^{(2)}$ for $t > 0$, $x < 0$.

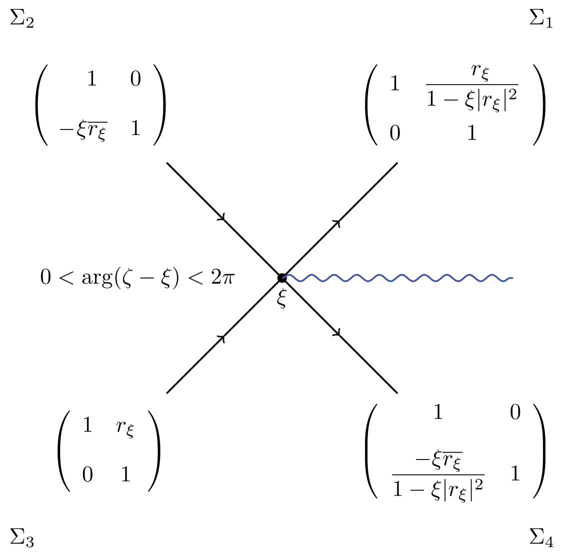

Fig. E.5. The Jump Matrix $V_0^{(2)}$ for $t < 0$, $x > 0$.

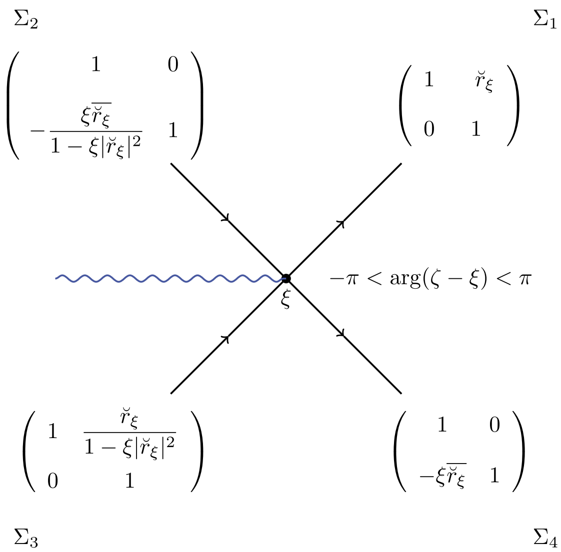

Fig. E.6. The Jump Matrix $V_0^{(2)}$ for $t < 0$, $x < 0$.

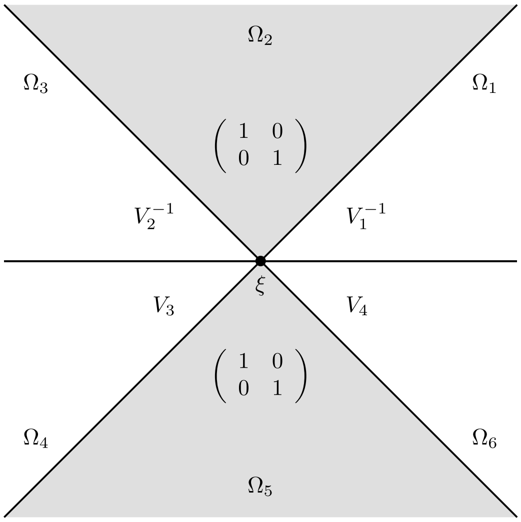

Fig. E.7. The Matrix P in terms of the Jump Matrix $V_0^{(2)}$, where $V_i = V_0^{(2)}|_{\Sigma_i}$.

References

[1] J. Liu, P. Perry, C. Sulem, Global existence for the derivative nonlinear Schrodinger equation by the method of inverse scattering, ArXiv e-prints, arXiv:1511.01173.
[2] E. Fan, Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation, J. Phys. A 33 (39) (2000) 6925–6933, http://dx.doi.org/10.1088/0305-4470/33/39/308.
[3] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (4) (1978) 798–801.
[4] J.-H. Lee, Analytic Properties of Zakharov–Shabat Inverse Scattering Problem with a Polynomial Spectral Dependence of Degree 1 in the Potential, ProQuest LLC, Ann Arbor, MI, thesis (Ph.D.), Yale University, 1983, http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:8329296.
[5] D.E. Pelinovsky, Y. Shimabukuro, Existence of global solutions to the derivative NLS equation with the inverse scattering transform method, Int. Math. Res. Not. (2017), http://dx.doi.org/10.1093/imrn/rnx051.
[6] N. Hayashi, P.I. Naumkin, H. Uchida, Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations, Publ. Res. Inst. Math. Sci. 35 (3) (1999) 501–513, http://dx.doi.org/10.2977/prims/1195143611.
[7] V.E. Zakharov, S.V. Manakov, Asymptotic behavior of non-linear wave systems integrated by the inverse scattering method, Zh. Èksp. Teor. Fiz. 71 (1) (1976) 203–215.
[8] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation, Ann. Math. 137 (2) (1993) 295–368, http://dx.doi.org/10.2307/2946540.
[9] P.A. Deift, A.R. Its, X. Zhou, Long-time asymptotics for integrable nonlinear wave equations, in: Important Developments in Soliton Theory, in: Springer Ser. Nonlinear Dynam., Springer, Berlin, 1993, pp. 181–204.
[10] P. Deift, X. Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space, Commun. Pure Appl. Math. 56 (8) (2003) 1029–1077, http://dx.doi.org/10.1002/cpa.3034, dedicated to the memory of Jürgen K. Moser.
[11] G.G. Varzugin, Asymptotics of oscillatory Riemann–Hilbert problems, J. Math. Phys. 37 (11) (1996) 5869–5892, http://dx.doi.org/10.1063/1.531706.
[12] Y. Do, A nonlinear stationary phase method for oscillatory Riemann–Hilbert problems, Int. Math. Res. Not. 2011 (12) (2011) 2650–2765, http://dx.doi.org/10.1093/imrn/rnq179.
[13] M. Dieng, K.D.T. McLaughlin, Long-time asymptotics for the NLS equation via dbar methods, ArXiv e-prints, arXiv:0805.2807.
[14] K.T.-R. McLaughlin, P.D. Miller, The $\overline{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights, Int. Math. Res. Pap. (2006) 1–77.
[15] S. Cuccagna, R. Jenkins, On asymptotic stability of N-solitons of the defocusing nonlinear Schrodinger equation, ArXiv e-prints, arXiv:1410.6887.
[16] M. Borghese, R. Jenkins, K.D.T.-R. McLaughlin, Long time asymptotic behavior of the focusing nonlinear Schrodinger equation, ArXiv e-prints, arXiv:1604.07436.
[17] A.V. Kitaev, A.H. Vartanian, Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: solitonless sector, Inverse Probl. 13 (5) (1997) 1311–1339, http://dx.doi.org/10.1088/0266-5611/13/5/014.
[18] J. Xu, E. Fan, Long-time asymptotic for the derivative nonlinear Schrödinger equation with decaying initial value, arXiv:1209.4245, http://dx.doi.org/10.1007/s11040-013-9132-3.

[19] P.A. Deift, X. Zhou, Long-time asymptotics for integrable systems. Higher order theory, Commun. Math. Phys. 165 (1) (1994) 175–191, http://projecteuclid.org/euclid.cmp/1104271038.
[20] A.R. Its, Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations, Dokl. Akad. Nauk SSSR 261 (1) (1981) 14–18.
[21] N.I.S.T. Digital, Library of mathematical functions, release 1.0.11 of 2016-06-08, online companion to [24], http://dlmf.nist.gov/.
[22] S. Cuccagna, D.E. Pelinovsky, The asymptotic stability of solitons in the cubic NLS equation on the line, Appl. Anal. 93 (4) (2014) 791–822, http://dx.doi.org/10.1080/00036811.2013.866227.
[23] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996, an introduction to the general theory of infinite processes and of analytic functions, with an account of the principal transcendental functions, reprint of the fourth (1927) edition.
[24] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, New York, NY, 2010, print companion to [21].