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K3 surfaces with a symplectic automorphism of order 11

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Abstract. We classify possible finite groups of symplectic automorphisms of K3 surfaces of order divisible by 11. The characteristic of the ground field must be equal to 11. The complete list of such groups consists of five groups: the cyclic group C_{11} of order 11, $C_{11} \rtimes C_5$, $\mathrm{PSL}_2(\mathbb{F}_{11})$ and the Mathieu groups M_{11}, M_{22}. We also show that a surface X admitting an automorphism g of order 11 admits a g-invariant elliptic fibration with the Jacobian fibration isomorphic to one of explicitly given elliptic K3 surfaces.

1. Introduction

Let X be a K3 surface over an algebraically closed field k of characteristic $p \geq 0$. An automorphism g of X is called *symplectic* if it preserves a regular 2-form of X. In positive characteristic p, an automorphism of order a power of p is called *wild*. A wild automorphism is symplectic. A subgroup G of the automorphism group $\mathrm{Aut}(X)$ is called *symplectic* if all elements of G are symplectic, and *wild* if it contains a wild automorphism. No K3 surface in characteristic $p > 11$ can admit a wild automorphism ([4, Theorem 2.1]). The existence of such a bound is not a common property among all classes of surfaces, e.g. in every positive characteristic one can find a rational surface and an abelian surface admitting a wild automorphism.

It is a well-known result of V. Nikulin that the order of a symplectic automorphism of finite order of a complex K3 surface takes value in the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. This result is true over an algebraically closed field k of positive characteristic p if the order is coprime to p. The latter condition is automatically satisfied if $p > 11$ [4]. If $p = 11$, an elliptic K3 surface X_ε defined by the equation of degree 12 in $\mathbb{P}(1, 1, 4, 6)$

$$y^2 + x^3 + \varepsilon x^2 t_0^4 + t_0 t_1^{11} - t_0^{11} t_1 = 0, \quad \varepsilon \in k, \tag{1.1}$$

admits a symplectic automorphism of order 11

$$g_\varepsilon : (t_0, t_1, x, y) \mapsto (t_0, t_0 + t_1, x, y). \tag{1.2}$$

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The main result of the paper is the following.

Theorem 1.1. *Let G be a finite group of symplectic automorphisms of a K3 surface X over an algebraically closed field of characteristic $p \geq 0$. Assume that the order of G is divisible by 11. Then $p = 11$ and G is isomorphic to one of the following five groups:*

$$C_{11}, \quad C_{11} \rtimes C_5, \quad \mathrm{PSL}_2(\mathbb{F}_{11}), \quad M_{11}, \quad M_{22},$$

where C_n is the cyclic group of order n and M_n is one of the Mathieu groups. Moreover, the following assertions are true.

(i) *For any element $g \in G$ of order 11, X admits a g-invariant elliptic pencil $|F|$ and X is C_{11}-equivariantly isomorphic to a torsor of one of the surfaces X_ε equipped with its standard elliptic fibration.*
(ii) *If $X = X_\varepsilon$ and G contains an element of order 11 leaving invariant both the standard elliptic fibration and a section, then $G \cong C_{11}$ if $\varepsilon \neq 0$, and G is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_{11})$ if $\varepsilon = 0$.*

The surface X_0 is a supersingular K3 surface with Artin invariant 1 isomorphic to the Fermat surface

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

In a recent paper of Kondō [7] it is proven that both M_{11} and M_{22} appear as symplectic automorphism groups of X_0. Thus the surface X_0 admits three maximal finite simple symplectic groups of automorphisms, isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{11})$, M_{11} and M_{22}. An element g of order $p = 11$ in the latter two groups leaves invariant an elliptic pencil. We do not know whether the pencil has a section; all we know is that it does not have a g-invariant section.

In our earlier work [4] finite groups of symplectic automorphisms of K3 surfaces in positive characteristic p have been studied. If such a group is not wild, then it is isomorphic to a subgroup of the Mathieu group M_{23} with at least four orbits in its natural action on a set of 24 elements. In characteristic $p < 7$, there are examples of finite wild symplectic groups which cannot be embedded into M_{23}. In characteristic $p = 11$, Theorem 1.1 shows that every finite symplectic group, wild or not, can be embedded into M_{23}. We do not know whether the same holds true in characteristic $p = 7$.

Corollary 1.2. *A finite group G acts symplectically and wildly on a K3 surface over an algebraically closed field of characteristic 11 if and only if G is isomorphic to a subgroup of M_{23} of order divisible by 11 and having three or four orbits in its natural action on a set of 24 elements.*

Notation

For an automorphism group G or an automorphism g of X, we denote by X^g the *fixed locus* with reduced structure, i.e. the set of fixed points of g.

A subset T of X is *G-invariant* if $g(T) = T$ for all $g \in G$. In this case we say G leaves T invariant.

An elliptic pencil $|F|$ on X is *G-invariant* if $g(F) \in |F|$ for all $g \in G$. In this case we say G leaves $|F|$ invariant.

We also use the following notations for groups:

- C_n the cyclic group of order n, sometimes denoted by n,
- $m : n = m \rtimes n$ the semidirect product of cyclic groups C_m and C_n,
- M_n the Mathieu group of degree n,
- $L_n(q) = \mathrm{PSL}_n(\mathbb{F}_q)$,
- $\mathrm{GU}_n(q)$ the general unitary group over $\mathbb{F}_q$,
- $\#G$ the cardinality of a group G,
- V^g the subspace of g-invariant vectors of a vector space V.

2. Wild C_{11}-actions on K3 surfaces

In this section we improve the result of [3] and [4] in the case of wild C_{11}-actions on K3 surfaces.

Lemma 2.1. *Let X be a K3 surface over an algebraically closed field k of characteristic* 11. *Assume that X admits an automorphism g of order* 11. *Then X admits a g-invariant elliptic pencil $|F|$.*

Proof. By Theorem 2.2 and Propositions 2.3, 2.5 of [4], we see that the fixed locus X^g is either the support of a fibre of an elliptic fibration, or a point. Thus we need to prove the lemma in the second case.

Assume that X^g is a point. In this case, by Theorems 1 and 2.4 of [3] (or Proposition 2.4 of [4]), the quotient surface $X/\langle g\rangle$ is a rational surface with trivial canonical divisor and one isolated elliptic Gorenstein singularity. By Proposition 2.9 of [4], X admits a g-invariant elliptic pencil.

Lemma 2.2. *Let X be a K3 surface over an algebraically closed field k of characteristic* 11. *Assume that X admits an automorphism g of order* 11. *Let $\phi : X \to \mathbb{P}^1$ be a g-invariant elliptic fibration. Then the following assertions are true.*

(i) *ϕ has either* 12 *cuspidal fibres, or one cuspidal fibre and* 22 *nodal fibres.*
(ii) *g^* acts on the base $\mathbb{P}^1$ faithfully, leaving one cuspidal fibre F_0 invariant and has either one orbit or two orbits on the set of remaining singular fibres.*
(iii) *X^g is either the whole curve F_0 or the cusp of F_0.*

Proof. Assume that g^* acts as identity on the base $\mathbb{P}^1$. Then g becomes an automorphism of the elliptic curve, $X/\mathbb{P}^1$ over the function field of $\mathbb{P}^1$. On the Jacobian J of this elliptic curve, g induces an automorphism g' of order 11. Note that J is an elliptic K3 surface with a section (cf. [2, Theorem 5.7.2]). Since the order of g' is greater than 3, g' must be a translation by an 11-torsion section. This contradicts Theorem 2.13 of [4] stating that no jacobian elliptic K3 surface in characteristic p admits a nontrivial p-torsion if $p > 7$. Thus g^* acts on the base $\mathbb{P}^1$ faithfully, and hence fixes only one point. Let F_0 be the corresponding fibre. The fibre F_0 contains X^g. An elliptic K3 surface cannot have only one singular fibre. This follows from the equality $e(X) = 24$ for the Euler number and the bound $\rho(X) \leq 22$ for the Picard number. Singular fibres away from F_0 form orbits of fibres of the same type, and the number of such singular fibres is a multiple of 11, the order of g. Thus the Euler number of such a fibre must be ≤ 2. This leaves the following four possibilities:

(a) one orbit of singular fibres of type I_2,
(b) one orbit of singular fibres of type II,
(c) one orbit of singular fibres of type I_1,
(d) two orbits of singular fibres of type I_1.

Let Y be a minimal resolution of $X/\langle g\rangle$ and Z the relative minimal model of Y. Then Z is a rational elliptic surface with $e(Z) = 12$ and $\rho(Z) = 10$. The rationality follows from [3, Theorems 1 and 3.7].

In case (a), Z has two singular fibres: one fibre of type I_2 and the other fibre Z_0 coming from F_0. The fibre Z_0 has Euler number $e(Z_0) = 10$ and eight irreducible components. Such an elliptic fibre does not exist. Thus case (a) does not occur.

In case (b), Z has two singular fibres: one fibre of type II and the other fibre Z_0 coming from F_0. The fibre Z_0 has Euler number $e(Z_0) = 10$ and nine irreducible components, hence is of type $\tilde{D}_8$ or $\tilde{E}_8$, additive in both cases. It follows that F_0 cannot be of multiplicative type I_2, hence it is of type II.

In case (c), Z has two singular fibres; one fibre of type I_1 and the other fibre Z_0 coming from F_0. The fibre Z_0 has Euler number $e(Z_0) = 11$ and nine irreducible components. Such an elliptic fibre does not exist. Thus case (c) does not occur.

In case (d), Z has three singular fibres; two fibres of type I_1 and the fibre Z_0 coming from F_0. The fibre Z_0 has Euler number $e(Z_0) = 10$ and has nine irreducible components, hence is of type $\tilde{D}_8$ or $\tilde{E}_8$, additive in both cases. By the same reason as in case (b), F_0 is of type II. This completes the proof of (i) and (ii). The action of g on the rational curve F_0 gives (iii).

Lemma 2.3. *Let X be a K3 surface over an algebraically closed field k of characteristic* 11. *Assume that X admits an automorphism g of order* 11. *Then the following assertions are true.*

(i) $\operatorname{rank}\operatorname{Pic}(X/\langle g\rangle) = 2$.
(ii) *For any* $l \neq 11$, $\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^g = \operatorname{rank}\operatorname{Pic}(X)^g = 2$.
(iii) $\operatorname{rank}\operatorname{Pic}(X) = 2$, 12 *or* 22.

Proof. By Lemma 2.1, X admits a g-invariant elliptic fibration. The quotient surface $X/\langle g\rangle$ is rational (Theorems 1 and 3.7 of [3]), and has no reducible fibre (Lemma 2.2). Let Y be a minimal resolution of $X/\langle g\rangle$. Then Y is an extremal rational elliptic surface with only one reducible fibre, not necessarily relatively minimal. Thus the sublattice N of $\operatorname{Pic}(Y)$ spanned by the exceptional divisor classes has corank 2. This proves (i).

Recall that for a nonsingular projective variety Z in positive characteristic, there is an exact sequence of $\mathbb{Q}_l$-vector spaces

$$0 \to \operatorname{NS}(Z) \otimes \mathbb{Q}_l \to H^2_{\text{et}}(Z, \mathbb{Q}_l) \to T^2_l(Z) \to 0, \tag{2.1}$$

where $T_l^2(Z) = T_l(\mathrm{Br}(Z))$ in the standard notation in the theory of étale cohomology. The Brauer group $\mathrm{Br}(X)$ is known to be a birational invariant, in particular, it is trivial when Z is a rational variety. In fact, one can show that

$$\mathrm{NS}(Z) \otimes \mathbb{Q}_l = \mathrm{Ker}(H^2_{\mathrm{et}}(Z, \mathbb{Q}_l) \to H^2(k(Z), \mathbb{Q}_l)),$$
$$T_l^2(Z) = \mathrm{Im}(H^2_{\mathrm{et}}(Z, \mathbb{Q}_l) \to H^2(k(Z), \mathbb{Q}_l)).$$

Here $H^2(k(Z), \mathbb{Q}_l) = \varinjlim_U H^2(U, \mathbb{Q}_l)$, where U runs through the set of open subsets of Z (see [15]). It is known that the dimensions of all $\mathbb{Q}_l$-spaces above do not depend on l prime to the characteristic. Since Y is rational, the Lefschetz number $\lambda(Y)$ equals $\dim T_l^2(Y) = 0$. It follows from [15, Proposition 5] that

$$T_l^2(X)^g \cong T_l^2(Y) = 0.$$

Hence,

$$\dim H^2_{\mathrm{et}}(X, \mathbb{Q}_l)^g = \mathrm{rank}\, \mathrm{NS}(X)^g = \mathrm{rank}\, \mathrm{Pic}(X)^g = \mathrm{rank}\, \mathrm{Pic}(X/\langle g \rangle) = 2.$$

This proves (ii).

Considering the $\mathbb{Q}$-representation of the cyclic group $\langle g \rangle$ of order 11 on $\mathrm{Pic}(X) \otimes \mathbb{Q}$, we get (iii) from (ii).

3. The surfaces X_ε

Let $p = 11$ and X_ε be the K3 surface from (1.1). The surface X_ε has an elliptic pencil defined by the projection to the t_0, t_1 coordinates

$$f_\varepsilon : X_\varepsilon \to \mathbb{P}^1.$$

We will refer to it as the *standard elliptic fibration*. Its zero section, the section at infinity, will be denoted by S_ε. It is immediately checked that the surface X_ε is nonsingular. Computing the discriminant Δ_ε of the Weierstrass equation of the general fibre of the elliptic fibration on X_ε we find that

$$\Delta_\varepsilon = -t_0^2(t_1^{11} - t_0^{10} t_1)(5t_1^{11} - 5t_0^{10} t_1 + 4\varepsilon^3 t_0^{11}). \tag{3.1}$$

This shows that the set of singular fibres of the elliptic fibration on X_0 (resp. X_ε, $\varepsilon \neq 0$) consists of 12 irreducible cuspidal curves (resp. one cuspidal fibre and 22 nodal fibres). The automorphism g_ε given by (1.2) is symplectic and of order 11. It fixes pointwise the cuspidal fibres over the point $\infty = (0, 1)$ and has one orbit (resp. two orbits) on the set of remaining singular fibres. It leaves invariant the zero section S_ε. The quotient surface $X_\varepsilon / \langle g_\varepsilon \rangle$ is a rational elliptic surface with a double rational point of type E_8 equal to the image of the singular point of the fixed fibre. A minimal resolution of the surface has one reducible nonmultiple fibre of type $\tilde{E}_8$ and one irreducible singular cuspidal fibre (resp. two nodal fibres).

Proposition 3.1. *Let X be a K3 surface over an algebraically closed field k of characteristic 11. Assume that X admits an automorphism g of order 11. Let $f : X \to \mathbb{P}^1$ be a g-invariant elliptic fibration. Assume that f has a section S. Then there exists an isomorphism $\phi : X \to X_\varepsilon$ of elliptic surfaces such that $\phi g \phi^{-1} = \tau g_\varepsilon$ for some translation automorphism τ of X_ε. In particular, if $g(S) = S$ then $\phi g \phi^{-1} = g_\varepsilon$.*

Proof. Let

$$y^2 + x^3 + A(t_0, t_1)x + B(t_0, t_1) = 0$$

be the Weierstrass equation of the g-invariant elliptic pencil, where A (resp. B) is a binary form of degree 8 (resp. 12). By Lemma 2.2, g acts nontrivially on the base of the fibration. After a linear change of the coordinates (t_0, t_1) we may assume that g acts on the base by

$$g : (t_0, t_1) \mapsto (t_0, t_1 + t_0).$$

We know that a g-invariant elliptic fibration has one g-invariant irreducible cuspidal fibre F_0 and either 22 irreducible nodal fibres forming two orbits, or 11 irreducible cuspidal fibres forming one orbit (Lemma 2.2). Thus the discriminant polynomial $\Delta = -4A^3 - 27B^2$ must have one double root (corresponding to the fibre F_0) and either one orbit of double roots, or two orbits of simple roots. We know that the zeros of A correspond to either cuspidal fibres or nonsingular fibres with "complex multiplication" automorphism of order 6. Since this set is invariant with respect to our automorphism of order 11 acting on the base, we see that the only possibility is $A = ct_0^8$ for some constant $c \in k$. We obtain $\Delta = -4c^3t_0^{24} - 27B^2$. Again this uniquely determines B and hence the surface. Since B is of degree 12 and invariant under the action of g on the base, it must be of the form

$$B = a(t_1^{11} - t_0^{10}t_1)t_0 + bt_0^{12}$$

for some constants a, b. One can rewrite the above Weierstrass equation in the form

$$y^2 + x^3 + \varepsilon x^2 t_0^4 + a(t_0 t_1^{11} - t_0^{11} t_1) + b' t_0^{12} = 0.$$

A suitable linear change of variables $u_0 = t_0$, $u_1 = t_1 + dt_0$ makes $b' = 0$ without changing the action of g on the base. Thus $X \cong X_\varepsilon$ as an elliptic surface. Let $\phi : X \to X_\varepsilon$ be the isomorphism. The composite

$$\phi g \phi^{-1} g_\varepsilon^{-1} : X_\varepsilon \to X_\varepsilon$$

acts trivially on the base, hence must be a translation automorphism. Since ϕ maps the zero section S of $f : X \to \mathbb{P}^1$ to the zero section S_ε of $f_\varepsilon : X_\varepsilon \to \mathbb{P}^1$ and $g_\varepsilon(S_\varepsilon) = S_\varepsilon$, the last assertion follows. □

Lemma 3.2. *Let $\varepsilon = 0$. For any translation automorphism τ of X_0, the composite automorphisms τg_0 and $g_0 \tau$ are of order 11.*

Proof. Let $f : X \to B$ be any elliptic surface with a section S. Recall that its Mordell–Weil group $\mathrm{MW}(f)$ is isomorphic to the quotient of the Néron–Severi group by the subgroup generated by the divisor classes of S and the components of fibres. Thus any automorphism g of X which preserves the class of a fibre and the section S acts linearly on the group $\mathrm{MW}(f)$. Assume $\mathrm{MW}(f)$ is torsion free. Suppose g is of finite order n with $\operatorname{rank}\mathrm{MW}(f)^g = 0$ and let τ be a translation automorphism identified with an element of $\mathrm{MW}(f)$. Then for any $s \in \mathrm{MW}(f)$ we have

$$\tau g(s) = g(s) + \tau, \quad (\tau g)^n(s) = g^n(s) + g^{n-1}(\tau) + \cdots + g(\tau) + \tau = s.$$

The last equality follows from the fact that the linear action of $g - 1_X$ on $\mathrm{MW}(f)$ is invertible. This shows that $(\tau g)^n$ acts identically on $\mathrm{MW}(f)$. It also acts identically on the class of a fibre. Thus $(\tau g)^n$ acts identically on the Néron–Severi lattice.

Apply this to our case $\varepsilon = 0$, when $g = g_0$ is a symplectic automorphism of order 11 of X_0. We will see in the proof of Proposition 3.8 that $\mathrm{MW}(f_0)$ is torsion free. By Lemma 2.3(ii), $\operatorname{rank}\mathrm{MW}(f_0)^g = 0$. Since the surface X_0 is supersingular (see Remark 3.6), by a theorem of Ogus [10], an automorphism acting identically on the Picard group must be the identity. Thus τg_0 is a symplectic automorphism of order 11 for any section τ. □

An interesting question: Is there a τ such that the fixed locus $X_0^{\tau g_0}$ consists of an isolated point, the cusp of a cuspidal curve fixed pointwise by g_0? We do not know any example of a symplectic automorphism of order 11 with an isolated fixed point.

Corollary 3.3. *Let X be a K3 surface over an algebraically closed field k of characteristic* 11. *Assume that X admits an automorphism g of order* 11. *Then X is isomorphic to a torsor of one of the elliptic surfaces X_ε. The order of this torsor in the Shafarevich–Tate group of torsors is equal to* 1 *or* 11.

Proof. Let $f_J : J \to \mathbb{P}^1$ be the Jacobian fibration of the elliptic fibration $f : X \to \mathbb{P}^1$ defined by the g-invariant elliptic pencil. Let J^o be the open subset of J whose complement is the set of singular fibres of f_J. We know that the fibres of f are irreducible. By a result of M. Raynaud, this allows us to identify J^o with the component $\mathbf{Pic}^0_{X/\mathbb{P}^1}$ of the relative Picard scheme of invertible sheaves of degree 0 (see [2, Proposition 5.2.2]). The automorphism g acts naturally on the Picard functor and hence on J^o. Since J is minimal, it acts biregularly on J. This action preserves the elliptic fibration on J and defines an automorphism of order 11 on the base. This implies that there exists a C_{11}-equivariant isomorphism of elliptic surfaces J and X_ε.

The assertion about the order of the torsor follows from the existence of a section or an 11-section of f. In fact, let Y be a nonsingular relatively minimal model of the elliptic surface $X/\langle g\rangle$ with the elliptic fibration induced by f. It is a rational elliptic surface. Let F_0 be the g-invariant fibre of f over a point $s_0 \in \mathbb{P}^1$. The singular fibres of the elliptic fibration $f' : Y \to \mathbb{P}^1$ over $\mathbb{P}^1 \setminus \{s_0\}$ are either two irreducible nodal fibres ($\varepsilon \neq 0$) or one cuspidal irreducible fibre ($\varepsilon = 0$). The standard argument in the theory of elliptic surfaces shows that the fibre of f' over s_0 is either of type $\tilde{E}_8$ or $\tilde{D}_8$. This fibre is not multiple if and only if f' has a section. The pre-image of this section is a section of f making X the trivial torsor. A singular fibre of additive type can be multiple only if the characteristic is positive, and the multiplicity m must be equal to the characteristic (see [2, Proposition 5.1.5]). In this case an exceptional curve of the first kind on Y is an m-section. The pre-image of this multi-section on X is an m-section, in our case an 11-section. □

Remark 3.4. Note that, even in the case $X = X_\varepsilon$, the g-invariant fibration may be different from the standard elliptic fibration. In other words, a nontrivial torsor of an elliptic surface could be isomorphic to the same surface. This strange phenomenon could happen only in positive characteristic and only for torsors of order divisible by the characteristic. We do not know an example where this strange phenomenon really occurs. In Kondō's example, the g-invariant elliptic fibration for an element g of order 11 in $G = M_{11}$ or M_{22} may have a section (but no g-invariant section!). If this happens, it is isomorphic to the standard elliptic fibration and hence g is conjugate to τg_ε, as we have seen in Proposition 3.1.

Lemma 3.5. *Suppose* $p = 11$. *Then there is a finite subgroup* K_ε *of* $\mathrm{Aut}(X_\varepsilon)$ *with the following property:*

(i) K_ε *leaves invariant both the standard elliptic fibration of* X_ε *and the zero section* S_ε *which is the section at infinity.*
(ii) $K_0 \cong \mathrm{GU}_2(11)/(\pm I) \cong L_2(11) : 12$ *and* $K_1 \cong 11 : 4$, *where the first factor in the semidirect product is a symplectic subgroup and the second factor a nonsymplectic subgroup.*
(iii) *The image of* K_ε *in* $\mathrm{Aut}(\mathbb{P}^1)$ *is equal to the subgroup* $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$ *which leaves the set* $V(\Delta_\varepsilon)$ *invariant.*
(iv) $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_0)) \cong \mathrm{PGU}_2(11) \cong L_2(11).2$ *and* $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon)) \cong 11 : 2$ *if* $\varepsilon \neq 0$.

Proof. Assume $\varepsilon = 0$. After a linear change of variables

$$t_0 = \alpha^{11} t_0' + \alpha t_1', \quad t_1 = t_0' + t_1',$$

where $\alpha \in \mathbb{F}_{11^2} \setminus \mathbb{F}_{11} \subset k^*$, we can transform the polynomial $t_0 t_1^{11} - t_0^{11} t_1$ to the form $\lambda t_0^{12} + \mu t_1^{12}$. After scaling, we may assume that $f = t_0^{12} + t_1^{12}$. Now notice that f represents a hermitian form over the field $\mathbb{F}_{11^2}$, hence the finite unitary group $\mathrm{GU}_2(11)$ leaves the polynomial f invariant. The group $\mathrm{GU}_2(11)$ acts on the surface

$$X_0 \cong V(y^2 + x^3 + t_0^{12} + t_1^{12}) \tag{3.2}$$

in an obvious way, by acting on the variables t_0, t_1 and identically on the variables x, y. Note that

$$(t_0, t_1, x, y) = (\lambda t_0, \lambda t_1, \lambda^4 x, \lambda^6 y)$$

in $\mathbb{P}(1, 1, 4, 6)$ for all $\lambda \in k^*$. In particular, $(t_0, t_1, x, y) = (-t_0, -t_1, x, y)$, so $-I \in \mathrm{GU}_2(11)$ acts trivially on X_0. Note also that $\mathrm{SU}_2(11)$ and hence $\mathrm{PSU}_2(11)$ acts symplectically on X_0. The action of $\mathrm{PSU}_2(11)$ is faithful because it is a simple group. Take $K_0 = \mathrm{GU}_2(11)/(\pm I)$ and consider the homomorphism

$$\det : K_0 \to (\mathbb{F}_{11^2})^*.$$

It is known that

$$\mathrm{PSU}_2(11) \cong \mathrm{PSL}_2(\mathbb{F}_{11}) = L_2(11).$$

If $A \in \mathrm{GU}_2(11)$, then $(\det A)^{12} = (\det A)(\overline{\det A}) = \det A^t\overline{A} = \det I = 1$, so the image of det is a cyclic group of order dividing 12. On the other hand, if $\zeta \in \mathbb{F}_{11^2}$ is a 12-th root of unity, the unitary matrix $\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$ generates an order 12 subgroup of K_0, which acts on X_0 nonsymplectically. This proves (i) and (ii).

We know that the group $\mathrm{GU}_2(11)$ leaves the polynomial f invariant. Thus its image $\mathrm{PGU}_2(11)$ in $\mathrm{Aut}(\mathbb{P}^1)$ must coincide with $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_0))$. It is known that $\mathrm{PGU}_2(11)$ is a maximal subgroup in the permutation group $\mathfrak{S}_{12}$ and $\mathrm{PGU}_2(11) \cong \mathrm{PGL}_2(\mathbb{F}_{11}) \cong L_2(11).2$, a nonsplit extension. The quotient group is generated by the image of the automorphism $(t_0, t_1) \mapsto (t_0, \zeta t_1)$, where $\zeta \in \mathbb{F}_{11^2}$ is a 12-th root of unity. This proves (iii) and (iv).

Assume $\varepsilon \neq 0$. An element of $\mathrm{PGL}_2(k)$ leaving $V(\Delta_\varepsilon)$ invariant must either leave all factors of Δ_ε from (3.1) invariant or interchange the second and the third factors. It can be seen by computation that the group $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$ is generated by the following two automorphisms:

$$e(t_0, t_1) = (t_0, t_1 + t_0), \quad i(t_0, t_1) = (t_0, -t_1 + bt_0),$$

where b is a root of $b^{11} - b + 3\varepsilon^3 = 0$. The order of e (resp. i) is 11 (resp. 2) and i normalizes e. We see that they lift to automorphisms of X_ε

$$\tilde{e}(t_0, t_1, x, y) = (t_0, t_1 + t_0, x, y),$$
$$\tilde{i}(t_0, t_1, x, y) = (t_0, -t_1 + bt_0, -x + 3\varepsilon t_0^4, \sqrt{-1}y),$$

and we take $K_\varepsilon = (\tilde{e}, \tilde{i})$. Clearly $\tilde{i}$ is nonsymplectic of order 4 and normalizes $\tilde{e}$ which is symplectic of order 11, and both leave invariant the zero section S_ε. □

Remark 3.6. The equation (3.2) makes X_0 a weighted Delsarte surface according to the definition in [5]. It follows that X_0 is a supersingular surface with Artin invariant $\sigma = 1$ (loc.cit.). It is known that all such surfaces are isomorphic [10]. Thus X_0 is isomorphic to either the Fermat quartic

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

or the Kummer surface associated to the product of supersingular elliptic curves, or the modular elliptic surface of level 4 (see [14]). We do not know whether the surface X_ε, $\varepsilon \neq 0$, is supersingular. By Lemma 2.3, we know that $\mathrm{rank\,Pic}(X_\varepsilon) = 2$, 12 or 22.

Definition 3.7. *The subgroup $K_\varepsilon \subset \mathrm{Aut}(X_\varepsilon)$ from Lemma* 3.5 *contains a symplectic subgroup leaving invariant the standard elliptic fibration of X_ε, isomorphic to $L_2(11)$ if $\varepsilon = 0$ and to C_{11} if $\varepsilon = 1$. Denote this subgroup by H_ε. It leaves invariant the zero section S_ε of the elliptic fibration.*

The group H_ε acts on the base curve $\mathbb{P}^1$ and we have a homomorphism

$$\pi : H_\varepsilon \to \mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon)),$$

which is an embedding. The image $\pi(H_\varepsilon)$ is equal to the unique index 2 subgroup of $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$.

Proposition 3.8. *Let G be a finite group of symplectic automorphisms of the surface X_ε leaving invariant the standard elliptic fibration of X_ε. Let*

$$\psi : G \to \mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$$

be the natural homomorphism. Then ψ is an embedding. If in addition G is wild and leaves invariant the zero section S_ε, then G is contained in H_ε.

Proof. Let $\alpha \in \mathrm{Ker}(\psi)$. Then α acts trivially on the base curve. Since $p > 3$, α being symplectic must be a translation by a torsion section. It is known that there is no p-torsion in the Mordell–Weil group of an elliptic K3 surface if the characteristic p is greater than 7 ([4, Theorem 2.13]), and there are no other torsion sections because no symplectic automorphism of order coprime to p can have more than eight fixed points ([4], Theorem 3.3), while the fibration has 12 or 23 singular fibres. Hence α must be the identity automorphism. This proves that ψ is an embedding.

If ψ is surjective, then $\#G = 2 \cdot \#L_2(11)$ or $2 \cdot 11$, which cannot be the order of a wild symplectic group in characteristic 11, by Proposition 4.2 and Lemma 5.2. Thus ψ is not surjective. From this we see that if G is wild, then $\psi(G)$ is contained in the unique index 2 subgroup $\pi(H_\varepsilon)$ of $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$. If an element $\alpha \in G$ and an element $h \in H_\varepsilon$ have the same image in $\mathrm{Aut}(\mathbb{P}^1, V(\Delta_\varepsilon))$, then αh^{-1} is a translation by a section. If α leaves invariant the zero section S_ε, so does αh^{-1}, hence αh^{-1} is the identity automorphism. This proves the second assertion. □

4. A Mathieu representation

From now on X is a K3 surface over an algebraically closed field of characteristic $p = 11$ and G a group of symplectic automorphisms of X of order divisible by 11.

Lemma 4.1. *Let G be a finite group of symplectic automorphisms of a K3 surface X over an algebraically closed field of characteristic $p = 11$. Then*

$$\mathrm{ord}(g) \in \{1, 2, 3, 4, 5, 6, 7, 8, 11\} \quad \textit{for all } g \in G.$$

Proof. If the order $\mathrm{ord}(g)$ of $g \in G$ is coprime to the characteristic $p = 11$, then $\mathrm{ord}(g) \in \{1, \dots, 8\}$ by Theorem 3.3 of [4]. It remains to show that G cannot contain any element of order $11r$, $r > 1$. Assume the contrary, and let $h \in G$ be an element of order $11r$. We may assume that r is a prime and hence $r = 2, 3, 5, 7$, or 11. Let $g = h^r$. Then g is of order 11. By Lemmas 2.1 and 2.3, X admits a g-invariant elliptic pencil $|F|$ and $\mathrm{rank}\,\mathrm{Pic}(X)^g = 2$.

Case 1: $r = 11$. In this case, let

$$M := \mathrm{Pic}(X)^g \cong \mathbb{Z}^2.$$

Then h acts on M. This action must be trivial, since a free $\mathbb{Z}$-module of rank 2 does not admit an automorphism of order 11. Thus $M = \mathrm{Pic}(X)^h$. Since the divisor class $[F]$ is contained in M, we see that the elliptic pencil $|F|$ is h-invariant. Note that by Lemma 2.2 the automorphism g leaves only one fibre F_0 invariant. This implies that h acts on the base curve $\mathbb{P}^1$ of the pencil $|F|$ faithfully. However, using the Jordan canonical form we see that $\mathbb{P}^1$ does not admit an automorphism of order 11^2. A contradiction.

Case 2: $r = 2, 3, 5, 7$. Let $F_0 \in |F|$ be the g-invariant fibre. Then F_0 is a cuspidal curve and X^g is either the cusp of F_0 or the whole curve F_0 (Lemma 2.2). Let

$$f = h^{11}.$$

Then f is of order r. By Theorem 3.3 of [4],

$$3 \le \#X^f \le 8.$$

Since r is prime to 11,

$$X^h = X^f \cap X^g.$$

Clearly g acts on the finite set X^f, and this action cannot be of order 11. This means that g acts trivially on X^f, i.e. $X^f \subset X^g$. Thus X^g cannot be a point and

$$X^h = X^f \subset X^g = F_0.$$

Now h acts on $X^g = F_0$ with $\#X^f$ fixed points. But no nontrivial action on a rational curve can fix more than two points. A contradiction. □

A Mathieu representation of a finite group G is a 24-dimensional representation on a vector space V over a field of characteristic zero with character

$$\chi(g) = \epsilon(\mathrm{ord}(g)),$$

where

$$\epsilon(n) = 24\left(n\prod_{p|n}\left(1+\frac{1}{p}\right)\right)^{-1}, \quad \epsilon(1) = 24. \tag{4.1}$$

The number

$$\mu(G) = \frac{1}{\#G}\sum_{g\in G}\epsilon(\mathrm{ord}(g)) \tag{4.2}$$

is equal to the dimension of the subspace V^G of V. Here V^G is the linear subspace of vectors fixed by G. The natural action of a finite group G of symplectic automorphisms of a complex K3 surface on the singular cohomology

$$H^*(X,\mathbb{Q}) = \bigoplus_{i=0}^{4} H^i(X,\mathbb{Q}) \cong \mathbb{Q}^{24}$$

is a Mathieu representation with

$$\mu(G) = \dim H^*(X, \mathbb{Q})^G \geq 5.$$

From this Mukai deduces that G is isomorphic to a subgroup of M_{23} with at least five orbits. In positive characteristic, if G is wild, then the formula for the number of fixed points is no longer true and the representation of G on the l-adic cohomology, $l \neq p$,

$$H^*_{\text{et}}(X, \mathbb{Q}_l) = \bigoplus_{i=0}^{4} H^i_{\text{et}}(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{24},$$

is not Mathieu in general. However, in our case we have the following.

Proposition 4.2. *Let G be a finite group acting symplectically on a K3 surface X over a field of characteristic* 11*. Then the representation of G on $H^*_{\text{et}}(X, \mathbb{Q}_l)$, $l \neq 11$, is a Mathieu representation with* $\dim H^*_{\text{et}}(X, \mathbb{Q}_l)^G \geq 3$.

Proof. Note that $\operatorname{rank} \operatorname{Pic}(X)^G \geq 1$, and the second assertion follows. It remains to prove that the representation is Mathieu. By Lemma 4.1, it is enough to show this for automorphisms of order 11. Let $g \in G$ be an element of order 11. We need to show that the character $\chi(g)$ of the representation on the l-adic cohomology $H^*_{\text{et}}(X, \mathbb{Q}_l)$ is equal to $\epsilon(11) = 2$. Since

$$\chi(g) = \operatorname{Tr}(g^*|H^*_{\text{et}}(X, \mathbb{Q}_l)),$$

it suffices to show that $\operatorname{Tr}(g^*|H^2_{\text{et}}(X, \mathbb{Q}_l)) = 0$. In order to show this, we recall a result of Illusie ([6, 3.7.3]) that the characteristic polynomial of $g^*|H^*_{\text{et}}(X, \mathbb{Q}_l)$ has integer coefficients and is independent of $l \neq p = 11$. Thus, $\operatorname{Tr}(g^*|H^2_{\text{et}}(X, \mathbb{Q}_l)) = 0$ if and only if $\dim H^2_{\text{et}}(X, \mathbb{Q}_l)^g = 2$. Now the result follows from Lemma 2.3(ii). □

5. Determination of the groups

In this section we determine all possible finite groups which may act symplectically and wildly on a K3 surface in characteristic 11. We use only purely group-theoretic arguments.

Proposition 5.1. *Let G be a finite group having a Mathieu representation over $\mathbb{Q}$ or over $\mathbb{Q}_l$ for all prime $l \neq 11$. Then*

$$\#G = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 23^f$$

for some $a \leq 7$, $b \leq 2$, $c \leq 1$, $d \leq 1$, $e \leq 1$, $f \leq 1$.

Proof. If the representation is over $\mathbb{Q}$, this is the theorem of Mukai [8, Theorem (3.22)]. In his proof, Mukai uses at several places the fact that the representation is over $\mathbb{Q}$. The only essential place where he uses this is Proposition (3.21), where G is assumed to be a 2-group containing a maximal normal abelian subgroup A and the case of $A = (\mathbb{Z}/4)^2$ with $\#(G/A) \geq 2^4$ is excluded by using the fact that a certain 2-dimensional representation of the quaternion group Q_8 cannot be defined over $\mathbb{Q}$. We use the fact that G also admits a Mathieu representation on the 2-adic cohomology, and it is easy to see that the representation of Q_8 cannot be defined over $\mathbb{Q}_2$. □

The following lemma is of purely group-theoretical nature and its proof follows an argument employed by S. Mukai [8].

Lemma 5.2. *Let G be a finite group admitting a Mathieu representation over* $\mathbb{Q}$ *or over* $\mathbb{Q}_l$ *for all prime* $l \neq 11$. *Assume* $\mu(G) \geq 3$. *Assume that G contains an element of order* 11*, but no elements of order* > 11. *Then the order of G is equal to one of the following:*

$$11, \quad 5 \cdot 11, \quad 2^2 \cdot 3 \cdot 5 \cdot 11, \quad 2^4 \cdot 3^2 \cdot 5 \cdot 11, \quad 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11.$$

Proof. Since G has no elements of order 23, by Proposition 5.1, we have

$$\#G = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11 \quad \text{for } a \leq 7,\ b \leq 2,\ c \leq 1,\ d \leq 1.$$

Let S_q be a q-Sylow subgroup of G for $q = 5, 7$ or 11. Then S_q is cyclic and its centralizer coincides with S_q. Let N_q be the normalizer of S_q. Since G does not contain elements of order $5k, 7k, 11k$ with $k > 1$, the index

$$m_q := [N_q : S_q]$$

is a divisor of $\phi(q) = q - 1$, where ϕ is the Euler function. Since it is known that the dihedral groups D_{14} and D_{22} do not admit a Mathieu representation, we have

$$m_7 = 1 \text{ or } 3 \quad \text{and} \quad m_{11} = 1 \text{ or } 5.$$

Let a_n be the number of elements of order n in G. Then

$$a_q = \frac{\#G(q-1)}{q m_q} \quad \text{for } q = 5, 7, 11.$$

As in [8], we have

$$\mu(G) = \frac{1}{\#G} \sum \epsilon(n) a_n = 8 + \frac{1}{\#G}(16 - 2a_3 - 4a_4 - 4a_5 - 6a_6 - 5a_7 - 6a_8 - 6a_{11}). \quad (5.1)$$

Case 1: The order of G is divisible by 7, i.e. $\#G = 2^a \cdot 3^b \cdot 5^c \cdot 7 \cdot 11$. The formula (5.1) gives

$$\mu(G) \leq 8 + \frac{16}{\#G} - \frac{30}{7m_7} - \frac{60}{11 m_{11}}. \quad (5.2)$$

Since $\mu(G) \geq 3$, both m_7 and m_{11} must be greater than 1. Hence

$$m_7 = 3, \quad m_{11} = 5.$$

This implies that $\#G$ is divisible by 5, and the formula (5.1) gives

$$\mu(G) \leq 8 + \frac{16}{\#G} - \frac{16}{5m_5} - \frac{10}{7} - \frac{12}{11}. \quad (5.3)$$

If $m_5 = 1$, then this inequality gives $\mu(G) < 3$, a contradiction.

If $m_5 = 2$, then the number of q-Sylow subgroups is equal to

$$\frac{\#G}{qm_q} = 2^{a-1} \cdot 3^b \cdot 7 \cdot 11 \ (q = 5), \ 2^a \cdot 3^{b-1} \cdot 5 \cdot 11 \ (q = 7), \ 2^a \cdot 3^b \cdot 7 \ (q = 11).$$

Taking $q = 5$ and applying Sylow's theorem, we get

$$2^{a-1} \cdot 3^b \cdot 7 \cdot 11 \equiv 1 \bmod 5, \quad \text{i.e.} \quad a \equiv b \bmod 4.$$

Since $1 \leq a \leq 7$ and $1 \leq b \leq 2$, the only solutions are $(a, b) = (5, 1), (6, 2)$. However, neither $2^5 \cdot 5 \cdot 11$ nor $2^6 \cdot 3 \cdot 5 \cdot 11$ is congruent to 1 modulo 7.

If $m_5 = 4$, then the number of q-Sylow subgroups is equal to

$$\frac{\#G}{qm_q} = 2^{a-2} \cdot 3^b \cdot 7 \cdot 11 \ (q = 5), \ 2^a \cdot 3^{b-1} \cdot 5 \cdot 11 \ (q = 7), \ 2^a \cdot 3^b \cdot 7 \ (q = 11).$$

A similar argument shows that $a - b \equiv 1 \bmod 4$ and the possible order is

$$\#G = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11.$$

Case 2: The order of G is divisible by 5, but not by 7, i.e. $\#G = 2^a \cdot 3^b \cdot 5 \cdot 11$. The formula (5.1) gives

$$\mu(G) \leq 8 + \frac{16}{\#G} - \frac{16}{5m_5} - \frac{60}{11m_{11}}. \tag{5.4}$$

If $m_{11} = 1$, then this inequality gives $\mu(G) < 3$, a contradiction. Hence

$$m_{11} = 5.$$

If $m_5 = 1$, then the number of q-Sylow subgroups is equal to $2^a \cdot 3^b \cdot 11$ ($q = 5$), $2^a \cdot 3^b$ ($q = 11$). By Sylow's theorem, $a - b \equiv 0 \bmod 4$ and $a + 8b \equiv 0 \bmod 10$. This system of congruences has only one solution $a = b = 0$ in the range $a \leq 7, b \leq 2$. This gives the possible order

$$\#G = 5 \cdot 11.$$

If $m_5 = 2$, then the number of q-Sylow subgroups is equal to $2^{a-1} \cdot 3^b \cdot 11$ ($q = 5$), $2^a \cdot 3^b$ ($q = 11$). By Sylow's theorem, $a - b \equiv 1 \bmod 4$ and $a + 8b \equiv 0 \bmod 10$. Hence $a = 2, b = 1$, which gives the possible order

$$\#G = 2^2 \cdot 3 \cdot 5 \cdot 11.$$

If $m_5 = 4$, then the number of q-Sylow subgroups is equal to $2^{a-2} \cdot 3^b \cdot 11$ ($q = 5$), $2^a \cdot 3^b$ ($q = 11$). By Sylow's theorem, $a - b \equiv 2 \bmod 4$ and $a + 8b \equiv 0 \bmod 10$. Hence $a = 4, b = 2$, giving the possible order

$$\#G = 2^4 \cdot 3^2 \cdot 5 \cdot 11.$$

Case 3: The order of G is divisible by neither 5 nor 7, i.e. $\#G = 2^a \cdot 3^b \cdot 11$. In this case $m_{11} \neq 5$, and hence $m_{11} = 1$. Thus the formula (5.1) gives

$$\mu(G) \leq 8 + \frac{16}{\#G} - \frac{60}{11}. \tag{5.5}$$

The number of 11-Sylow subgroups is equal to $2^a \cdot 3^b$. By Sylow's theorem, $a + 8b \equiv 0 \mod 10$. This congruence has three solutions $(a, b) = (0, 0), (2, 1), (4, 2)$ in the range $a \leq 7, b \leq 2$. In the second and the third case, the inequality (5.5) gives $\mu(G) < 3$. The first gives the possible order

$$\#G = 11.$$

□

Proposition 5.3. *In the situation of the previous lemma, G is isomorphic to one of the following groups:*

$$C_{11}, \quad 11 : 5, \quad L_2(11), \quad M_{11}, \quad M_{22}.$$

Proof. By Lemma 5.2, there are five possible orders for G

$$11, \quad 5 \cdot 11, \quad 2^2 \cdot 3 \cdot 5 \cdot 11, \quad 2^4 \cdot 3^2 \cdot 5 \cdot 11, \quad 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11. \tag{5.6}$$

In the first two cases, the assertion is obvious. The remaining possible three orders are exactly the orders of the three simple groups given in the assertion. The theory of finite simple groups shows that there is only one simple group of the given order in each of these cases.

Assume the last three cases. It suffices to show that G is simple.

Let K be a proper normal subgroup of G such that G/K is simple. If $\#K$ is not divisible by 11, then an order 11 element of G acts on the set $\mathrm{Syl}_q(K)$ of q-Sylow subgroups of K. Since $\#\mathrm{Syl}_q(K)$ is not divisible by 11 for any prime q dividing $\#K$, the order 11 element g must normalize a q-Sylow subgroup of K. If one of the numbers $q = 3, 5$, or 7 divides $\#K$, then g centralizes an element of one of these orders. This contradicts the assumption that G does not contain an element of order > 11. If $q = 2$ divides $\#K$, then a 2-Sylow subgroup of K is of order $\leq 2^7$, and hence g centralizes an element of order 2, again a contradiction. So, we may assume that $11|\#K$. If $\#K = 11$, then an order 2 element of G normalizes K. Neither a cyclic group of order 22 nor a dihedral group of order 22 has a Mathieu representation, so $\#K > 11$. If $K \cong 11 : 5$, then an order 2 element of G normalizes the unique 11-Sylow subgroup of K, again a contradiction. If $\#K$ is one of the remaining three possibilities, then the group G/K is of order $2^5 \cdot 3 \cdot 7$ or $2^3 \cdot 7$ or $2^2 \cdot 3$. In the first case an order 7 element of G normalizes, hence centralizes a Sylow 11-subgroup of K, again a contradiction. Obviously in the other two cases G/K cannot be simple. This proves that G is simple. □

Corollary 5.4. *Let G be a finite group acting symplectically and wildly on a K3 surface X over a field of characteristic* 11. *Let g be an element of order* 11 *in G. Then the normalizer of* $\langle g \rangle$ *in G must be isomorphic to* $11 : 5$ *if* $\#G > 11$.

6. Proof of the main theorem

In this section we complete the proof of Theorem 1.1 announced in Introduction. It remains to prove assertion (ii).

Lemma 6.1. *Assume $\varepsilon \neq 0$. Let $G \subset \mathrm{Aut}(X_\varepsilon)$ be a finite wild symplectic subgroup. If an element $g \in G$ of order 11 leaves invariant the standard elliptic fibration with a g-invariant section, then $G = \langle g \rangle \cong C_{11}$ and G is conjugate to $H_\varepsilon = \langle g_\varepsilon \rangle$. In particular, H_ε is a maximal finite wild symplectic subgroup of $\mathrm{Aut}(X_\varepsilon)$.*

Proof. Since g leaves a section invariant, it must be a conjugate to g_ε. So up to conjugation, we may assume that g leaves the zero section S_ε invariant. Thus $g = g_\varepsilon$ by Proposition 3.8.

Suppose $\langle g \rangle \neq G$. Let N be the normalizer of $\langle g \rangle$ in G. Then $N \cong 11 : 5$ by Corollary 5.4.

We claim that N leaves invariant the standard elliptic pencil $|F|$. It is enough to show that $h(F_0) = F_0$ for any $h \in N$, where $F_0 = X^g$ is a cuspidal curve in $|F|$. In fact, for any $x \in F_0$, we have $h(x) = hg(x) = g^i h(x)$ for some i, so $h(x) \in X^g = F_0$, which proves the claim.

Next, we claim that N leaves invariant the zero section S_ε. In fact, $h(S_\varepsilon) = hg(S_\varepsilon) = g^i h(S_\varepsilon)$, so g leaves invariant $h(S_\varepsilon)$, and hence $h(S_\varepsilon) = S_\varepsilon$ as g cannot leave invariant two distinct sections by Lemma 2.3(ii).

Now Proposition 3.8 gives a contradiction. Hence, $G = \langle g \rangle$. □

Lemma 6.2. *Let $G \subset \mathrm{Aut}(X_0)$ be a finite wild symplectic subgroup, isomorphic to $L_2(11)$. If an element $g \in G$ of order 11 leaves invariant both the standard elliptic fibration and a section, then G is conjugate to H_0. In particular, if G contains g_0 then $G = H_0$.*

Proof. Replacing G by a conjugate subgroup in $\mathrm{Aut}(X_0)$, we may assume that g leaves invariant both the standard elliptic fibration and the zero section S_0, i.e. $g = g_0$. We need to prove that $G = H_0$.

Let $|F|$ be the standard elliptic fibration. Then $g(S_0) = S_0$ and $X^g = F_0$, a cuspidal curve in $|F|$.

Let N be the normalizer of $\langle g \rangle$ in G. Then $N \cong 11 : 5$. The same argument as in the proof of Lemma 6.1 shows that N leaves invariant both the standard elliptic pencil $|F|$ and the zero section S_0. By Proposition 3.8, $N \subset H_0$.

We have $N \subset G \cap H_0$. Suppose $G \cap H_0 = N$. Consider the G-orbit of the divisor class $[F] \in \mathrm{Pic}(X_0)$,

$$G([F]) = \{h([F]) \in \mathrm{Pic}(X_0) \mid h \in G\}.$$

Clearly N acts on it. Note

$$\#G([F]) = [G : N] = 12.$$

Thus $G([F])$ is the set of 12 different elliptic fibrations with a section. The automorphism g cannot leave invariant an elliptic fibration other than $|F|$, hence fixes $[F]$ and has one orbit on the remaining 11 elliptic fibrations, which we denote by $[F_1], \dots, [F_{11}]$.

Recall that H_0 leaves invariant the zero section S_0. The three divisor classes

$$[F], \quad \sum_{j=1}^{11} [F_j], \quad [S_0]$$

are N-invariant, and their intersection matrix is

$$\begin{pmatrix} 0 & 11m & 1 \\ 11m & 110m & 11b \\ 1 & 11b & -2 \end{pmatrix}$$

where $m = F \cdot F_i,\ b = S_0 \cdot F_i, i \geq 1$. Its determinant is equal to

$$242(m^2 + bm) - 110m,$$

which cannot be 0 for any positive integers m and b. This implies that

$$\mu(N) = 2 + \operatorname{rank} \operatorname{Pic}(X_0)^N \geq 5,$$

a contradiction to the equality $\mu(N) = 4$. This proves that N is a proper subgroup of $G \cap H_0$. Since N is a maximal subgroup of G, we have $G = H_0$. □

Note that $\mu(M_{11}) = \mu(M_{22}) = 3$ and $\mu(L_2(11)) = 4$. Note also that $L_2(11)$ is isomorphic to a maximal subgroup of both M_{11} and M_{22}.

The following proposition completes the proof of Theorem 1.1(ii).

Proposition 6.3. *Let $G \subset \operatorname{Aut}(X_0)$ be a finite wild symplectic subgroup. Assume that $G \cong M_{11}$ or M_{22}. Then no conjugate of G in $\operatorname{Aut}(X_0)$ contains the automorphism g_0 given by* (1.2). *In other words, no element of G of order* 11 *can leave invariant both the standard elliptic fibration and a section. In particular, H_0 is a maximal finite wild symplectic subgroup of* $\operatorname{Aut}(X_0)$.

Proof. Suppose that a conjugate of G contains g_0. Replacing G by the conjugate, we may assume that $g_0 \in G$.

Let K be a subgroup of G such that $g_0 \in K \subset G$ and $K \cong L_2(11)$. Then by Lemma 6.2, $K = H_0$. Thus $g_0 \in H_0 \subset G$. Since $H_0 \cong L_2(11)$ is a maximal subgroup of G, its normalizer subgroup $N_G(H_0)$ coincides with H_0.

Let $|F|$ be the standard elliptic fibration on X_0, and S_0 the zero section. Then $g(S_0) = S_0$ and $X^g = F_0$, a cuspidal curve in $|F|$. Furthermore, both the section S_0 and the elliptic pencil $|F|$ are H_0-invariant (see Definition 3.7).

Consider the G-orbit of the divisor class $[F]$,

$$G([F]) = \{h([F]) \in \operatorname{Pic}(X_0) \mid h \in G\}.$$

Consider the action of H_0 on it. By Proposition 3.8, the stabilizer subgroup $G_{[F]}$ of $[F]$ coincides with H_0. The automorphism g_0 cannot leave invariant two different elliptic fibrations, hence fixes $[F]$ and has orbits on the set $G([F]) \setminus \{[F]\}$ of cardinality divisible by 11. This implies that H_0 fixes $[F]$ and has orbits on the set $G([F]) \setminus \{[F]\}$ of cardinality divisible by 11. Write

$$G([F]) = \{[F] = [F_0], [F_1], \ldots, [F_{r-1}]\}$$

where $r = \#G([F]) = [G : H_0]$. Let $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_s$ be the orbit decomposition of the index set $\{1, \ldots, r-1\}$ by the action of H_0. Since H_0 fixes $[F]$ and acts transitively on each $\mathcal{O}_i$, the intersection number $F \cdot F_t$ is constant on the orbit $\mathcal{O}_i$ containing t, i.e. $F \cdot F_t = m_i$ for all $t \in \mathcal{O}_i$. Note that the divisor

$$\mathcal{F} = \sum_{j=0}^{r-1} F_j$$

is G-invariant, and

$$\mathcal{F}^2 = \Big(\sum_{j=0}^{r-1} F_j\Big)^2 = rF_0 \cdot \sum_{j=0}^{r-1} F_j = r\sum_{i=1}^{s} m_i \#\mathcal{O}_i. \tag{6.1}$$

Next recall that H_0 leaves invariant the zero section S_0. Similarly we consider the G-orbit of the divisor class $[S_0]$,

$$G([S_0]) = \{h([S_0]) \in \mathrm{Pic}(X_0) \mid h \in G\}.$$

Let G_0 be the stabilizer subgroup of $[S_0]$. Since it contains H_0 and H_0 is maximal in G, we see that $G_0 = H_0$ or $G_0 = G$.

Assume $G_0 = H_0$. Then all stabilizers are conjugate to H_0. Similarly to the above we claim that $g_0 \in H_0$ fixes no elements of $G([S_0])$ other than $[S_0]$. If $g_0h(S_0) = h(S_0)$ for some $h \in G$, then $g_0 \in hH_0h^{-1}$ and since all cyclic subgroups of order 11 in H_0 are conjugate inside H_0 we can write $(g_0) = hh'(g_0)h'^{-1}h^{-1}$ for some $h' \in H_0$. This implies $hh' \in N_G((g_0))$. Since $\#N_G((g_0)) = \#N_{H_0}((g_0)) = 55$ (see the proof of Lemma 5.2), we obtain $N_G((g_0)) = N_{H_0}((g_0)) \subset H_0$, hence $h \in H_0$ and $h(S_0) = S_0$. This proves the claim and shows that H_0 has orbits on the set $G(S_0) \setminus \{S_0\}$ of cardinality divisible by 11. Write

$$G([S_0]) = \{[S_0], [S_1], \ldots, [S_{r-1}]\}.$$

It is clear that the divisor

$$\mathcal{S} = \sum_{j=0}^{r-1} S_j$$

is G-invariant. Let $S_0 \cdot F_t = b_i$ for $t \in \mathcal{O}_i$. Then we have

$$\mathcal{F} \cdot \mathcal{S} = \Big(\sum_{j=0}^{r-1} F_j\Big) \cdot \Big(\sum_{j=0}^{r-1} S_j\Big) = rS_0 \cdot \sum_{j=0}^{r-1} F_j = r\Big(1 + \sum_{i=1}^{s} b_i \#\mathcal{O}_i\Big). \tag{6.2}$$

In either case $G \cong M_{11}$ or M_{22}, we know $\mu(G) = 3$ and hence the two divisors $\mathcal{F}$ and $\mathcal{S}$ are linearly dependent in $\mathrm{Pic}(X_0)$. This implies

$$\mathcal{F}^2\mathcal{S}^2 = (\mathcal{F} \cdot \mathcal{S})^2.$$

Substituting from (6.1), (6.2), we get

$$r\Big(\sum_{i=1}^{s} m_i \#\mathcal{O}_i\Big)\mathcal{S}^2 = r^2\Big(1 + \sum_{i=1}^{s} b_i \#\mathcal{O}_i\Big)^2. \tag{6.3}$$

Since $\#\mathcal{O}_i \equiv 0 \bmod 11$ for all i and $r \equiv 1 \bmod 11$, we have LHS $\equiv 0 \bmod 11$, but RHS $\equiv 1 \bmod 11$, a contradiction.

Assume $G_0 = G$. Then the divisor $\mathcal{S} = S_0$ is G-invariant, and we have a simpler equality

$$r\Big(\sum_{i=1}^{s} m_i \#\mathcal{O}_i\Big)\mathcal{S}^2 = \Big(1 + \sum_{i=1}^{s} b_i \#\mathcal{O}_i\Big)^2, \tag{6.4}$$

again a contradiction. □

Remark 6.4. In [7] Kondō proves that the unique supersingular K3 surface X with Artin invariant 1 admits symplectic automorphism groups $G \cong M_{11}$ or $G \cong M_{22}$. It follows from the previous results that any element $g \in G$ of order 11 leaves invariant an elliptic pencil without a g-invariant section. In fact, according to his construction of G on X, one can show that $\mathrm{Pic}(X)^g \cong U(11)$, hence a g-invariant elliptic pencil has only an 11-section.

It is known that the Brauer group of a supersingular K3 surface is isomorphic to the additive group of the field k [1]. It is well-known that the group of torsors of an elliptic fibration with a section is isomorphic to the Brauer group. We do not know which torsors admit a nontrivial automorphism of order p (maybe all?). Nor do we know whether they define elliptic fibrations on the same surface X_0. Note that the latter could happen only for torsors of order divisible by $p = \mathrm{char}(k)$. It would be very interesting to see how the three groups $L_2(11)$, M_{11} and M_{22} sit inside the infinite group $\mathrm{Aut}(X_0)$.

Remark 6.5. It follows from Lemma 3.5 that our surface X_0 admits a nonsymplectic automorphism of order 12. By Remark 3.6, X_0 is supersingular with Artin invariant $\sigma = 1$. It follows from [9] that the maximal order of a nonsymplectic isomorphism of a supersingular surface with Artin invariant σ divides $1 + p^\sigma$. Thus 12 is the maximum possible order. What is the maximum possible nonsymplectic extension of M_{11} or M_{22}?

Remark 6.6. A K3 surface may admit a nonsymplectic automorphism of order 11 over any field of characteristic 0 or $p \neq 2, 3, 11$. The well-known example is the surface $V(x^2 + y^3 + z^{11} + w^{66})$ in $\mathbb{P}(1, 6, 22, 33)$. It would be interesting to know whether there exists a supersingular K3 surface X which admits a nonsymplectic automorphism of order 11. It follows from [9] that, if $p \neq 2$, then 11 must divide $1 + p^\sigma$, where σ is the Artin invariant of X.

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