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Ann. I. H. Poincaré – AN 26 (2009) 1–21
ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

An integral equation in conformal geometry

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Received 20 January 2007; accepted 27 March 2007

Available online 31 August 2007

Abstract

Motivated by Carleman's proof of the isoperimetric inequality in the plane, we study the problem of finding a metric with zero scalar curvature maximizing the isoperimetric ratio among all zero scalar curvature metrics in a fixed conformal class on a compact manifold with boundary. We derive a criterion for the existence and make a related conjecture.

Keywords: Isoperimetric inequalities; Poisson kernel; Yamabe type integral equations

1. Introduction

Among the many proofs of two dimensional isoperimetric inequalities, the one due to Carleman [3] is particularly interesting. Indeed by an application of Riemann mapping theorem we only need to show

$$\int_D e^{2u}\,dx \leqslant \frac{1}{4\pi}\left(\int_{S^1} e^{u}\,d\theta\right)^2 \tag{1.1}$$

for every harmonic function u on D. Here D is the unit disk in the plane. Carleman deduced (1.1) by showing

$$\int_D |f|^2\,dx \leqslant \frac{1}{4\pi}\left(\int_{S^1} |f|\,d\theta\right)^2$$

for every holomorphic function f on D. Along this line, in [8] Jacobs showed that for every bounded open subset Ω of $\mathbb{R}^2$ with smooth boundary, there exists a positive constant c_Ω such that for every holomorphic function f on Ω,

$$\int_\Omega |f|^2\,dx \leqslant c_\Omega\left(\int_{\partial\Omega} |f|\,ds\right)^2.$$

Moreover when Ω is not simply connected, the best constant $c_\Omega > \frac{1}{4\pi}$ and it is achieved by some particular holomorphic function f. Here we formulate a higher dimensional generalization of these statements.

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.03.006

Assume $n \geqslant 3$, (M^n, g) is a smooth compact Riemannian manifold with nonempty boundary $\Sigma = \partial M$, we write the isoperimetric ratio

$$I(M, g) = \frac{|M|^{\frac{1}{n}}}{|\Sigma|^{\frac{1}{n-1}}}. \tag{1.2}$$

Here $|M|$ is the volume of M with respect to g and $|\Sigma|$ is the area of Σ. Let $[g] = \{\rho^2 g\colon \rho \in C^\infty(M), \rho > 0\}$ be the conformal class of g. The set

$$\{\tilde{g} \in [g]\colon \text{the scalar curvature } \widetilde{R} = 0\}$$

is nonempty if and only if the first eigenvalue of the conformal Laplacian operator $L_g = -\frac{4(n-1)}{n-2}\Delta + R$ with respect to Dirichlet boundary condition, $\lambda_1(L_g)$ is strictly positive (see Section 2).

Assume $\lambda_1(L_g) > 0$, we denote

$$\Theta_{M,g} = \sup\{I(M, \tilde{g})\colon \tilde{g} \in [g] \text{ with } \widetilde{R} = 0\}. \tag{1.3}$$

Standard technique from harmonic analysis gives us $\Theta_{M,g} < \infty$ (see Proposition 2.1). But is $\Theta_{M,g}$ achieved? In another word, can we find a conformal metric with zero scalar curvature maximizing the isoperimetric ratio?

It follows from [7, Theorem 1.1] or Theorem 3.1 that

$$\Theta_{\overline{B}_1, g_{\mathbb{R}^n}} = I(\overline{B}_1, g_{\mathbb{R}^n}) = n^{-\frac{1}{n-1}} \omega_n^{-\frac{1}{n(n-1)}},$$

here ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $g_{\mathbb{R}^n}$ is the Euclidean metric on $\mathbb{R}^n$. This just says that $\Theta_{\overline{B}_1, g_{\mathbb{R}^n}}$ is achieved by the standard metric. In general we have the following

Theorem 1.1. *Assume $n \geqslant 3$, (M^n, g) is a smooth compact Riemannian manifold with nonempty boundary and $\lambda_1(L_g) > 0$, then*

$$n^{-\frac{1}{n-1}} \omega_n^{-\frac{1}{n(n-1)}} = \Theta_{\overline{B}_1, g_{\mathbb{R}^n}} \leqslant \Theta_{M,g} < \infty.$$

If in addition $\Theta_{\overline{B}_1, g_{\mathbb{R}^n}} < \Theta_{M,g}$, then $\Theta_{M,g}$ is achieved by some conformal metrics with zero scalar curvature.

The problem illustrates very similar behavior as the Yamabe problem of finding constant scalar curvature metrics in a fixed conformal class (cf. [9]) and its boundary versions (cf. [4,5]). On the other hand, it has more nonlocal features (e.g. the Euler–Lagrange equation is a nonlinear integral equation) than the two well studied problems. In analogy with the solution of the Yamabe problem, we make the following conjecture.

Conjecture 1.1. *Assume $n \geqslant 3$, (M^n, g) is a smooth compact Riemannian manifold with nonempty boundary and $\lambda_1(L_g) > 0$. If (M, g) is not conformally diffeomorphic to $(\overline{B}_1, g_{\mathbb{R}^n})$, then $\Theta_{M,g} > \Theta_{\overline{B}_1, g_{\mathbb{R}^n}}$.*

In Section 2 below, we will describe some basics related to the above problem and reformulate it as a maximization problem for harmonic extensions. We will also discuss some elementary estimates of the Poisson kernels and show $\Theta_{M,g}$ is always finite. In Section 3 we will show $\Theta_{\overline{B}_1, g_{\mathbb{R}^n}}$ is achieved by the standard metric itself and deduce some corollaries. This is a consequence of [7, Theorem 1.1]. However the approach we present here is different and of independent interest. In Section 4 we will prove the regularity of the solutions to the Euler–Lagrange equations of the maximization problem for harmonic extensions. In Section 5 we derive some asymptotic expansion formulas for the standard Poisson kernel and the Poisson kernel for the conformal Laplacian operators. These expansion formulas will be useful in the future study of Conjecture 1.1. In Section 6, we will derive the concentration compactness principle for the maximization problem and this will be used in the last section to deduce Theorem 1.1.

2. Some preparations

Assume $n \geqslant 3$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$. The conformal Laplacian operator is given by

$$L_g = -\frac{4(n-1)}{n-2}\Delta + R.$$

It satisfies the transformation law

$$L_{\rho^{\frac{4}{n-2}}g}\varphi = \rho^{-\frac{n+2}{n-2}} L_g(\rho\varphi) \quad \text{for } \rho, \varphi \in C^\infty(M),\ \rho > 0.$$

Let

$$E_g(\varphi, \psi) = \int_M \left[\frac{4(n-1)}{n-2} \nabla\varphi \cdot \nabla\psi + R\varphi\psi\right] d\mu, \qquad E_g(\varphi) = E_g(\varphi, \varphi),$$

here $d\mu$ is the measure generated by g, then it follows from the transformation law that

$$E_{\rho^{\frac{4}{n-2}}g}(\varphi) = E_g(\rho\varphi) \quad \text{for } \rho, \varphi \in C^\infty(M),\ \rho > 0,\ \varphi|_\Sigma = 0. \tag{2.1}$$

Let $\lambda_1(L_g)$ be the first eigenvalue of L_g with respect to the Dirichlet boundary condition, then

$$\lambda_1(L_g) = \inf_{\varphi \in H_0^1(M)\setminus\{0\}} \frac{E_g(\varphi)}{\int_M \varphi^2\, d\mu_g}.$$

Assume $\rho \in C^\infty(M)$, $\rho > 0$. It follows from (2.1) that $\lambda_1(L_g) < 0$ implies $\lambda_1(L_{\rho^{\frac{4}{n-2}}g}) < 0$. On the other hand, if $\lambda_1(L_g) \geqslant 0$, then $\lambda_1(L_{\rho^{\frac{4}{n-2}}g}) \geqslant (\max_M \rho)^{-\frac{4}{n-2}} \lambda_1(L_g)$. Hence the sign of the first eigenvalue of the conformal Laplacian operator does not depend on the choice of particular metric in a conformal class. This sign is useful because of the following fact: $\lambda_1(L_g) > 0$ if and only if we may find a scalar flat metric in the conformal class of g. The only thing we need to verify is we can find a scalar flat conformal metric when $\lambda_1(L_g) > 0$. To see this we may solve the Dirichlet problem

$$\begin{cases} L_g\rho = 0 & \text{on } M, \\ \rho|_\Sigma = 1. \end{cases}$$

We claim $\rho > 0$ on M. To see this, we let φ be the first eigenfunction of L_g with $\varphi > 0$ on $M\setminus\Sigma$ and $\varphi|_\Sigma = 0$. Let $w = \frac{\rho}{\varphi}$, then

$$-\frac{4(n-1)}{n-2}\Delta w - \frac{8(n-1)}{n-2}\frac{\nabla\varphi}{\varphi} \cdot \nabla w + \lambda_1 w = 0 \quad \text{on } M\setminus\Sigma.$$

Since $w(x) \to \infty$ as $x \to \Sigma$, it follows from strong maximum principle that $w > 0$ on $M\setminus\Sigma$, hence $\rho > 0$ on M. Note that $R_{\rho^{\frac{4}{n-2}}g} = \rho^{-\frac{n+2}{n-2}} L_g\rho = 0$, we find the needed metric.

Assume $\lambda_1(L_g) > 0$, the Green's function G_L of L_g satisfies

$$\begin{cases} (L_g)_x G_L(x, y) = \delta_y & \text{on } M, \\ G_L(x, y) = 0 & \text{for } x \in \Sigma. \end{cases}$$

The Poisson kernel of L_g is given by

$$P_L(x, \xi) = -\frac{4(n-1)}{n-2} \frac{\partial G_L(x, y)}{\partial_y \nu}\bigg|_{y=\xi},$$

here ν is the unit outer normal direction. The solution of

$$\begin{cases} L_g u = 0 & \text{on } M, \\ u|_\Sigma = f \end{cases}$$

is given by

$$u(x) = (P_L f)(x) = \int_\Sigma P_L(x, \xi) f(\xi)\, dS(\xi),$$

here dS is the measure generated by g on Σ. If ρ is a positive smooth function, then we have the following transformation laws,

$$G_{L,\rho^{\frac{4}{n-2}}g}(x, y) = \frac{G_{L,g}(x, y)}{\rho(x)\rho(y)}, \qquad P_{L,\rho^{\frac{4}{n-2}}g}(x, \xi) = \frac{P_{L,g}(x, \xi)}{\rho(x)\rho(\xi)^{\frac{n}{n-2}}},$$

and

$$P_{L,\rho^{\frac{4}{n-2}}g} f = \rho^{-1} P_{L,g}(\rho f).$$

If $\tilde{g} \in [g]$ has zero scalar curvature, then $\tilde{g} = u^{\frac{4}{n-2}} g$ for some positive smooth function u on M with $L_g u = 0$. Let $f = u|_\Sigma$, then $u = P_L f$ and

$$I(M, \tilde{g}) = \frac{|P_L f|^{\frac{2}{n-2}}_{L^{\frac{2n}{n-2}}(M)}}{|f|^{\frac{2}{n-2}}_{L^{\frac{2(n-1)}{n-2}}(\Sigma)}}.$$

Hence

$$\begin{aligned}
\Theta_{M,g} &= \sup\left\{\frac{|P_L f|^{\frac{2}{n-2}}_{L^{\frac{2n}{n-2}}(M)}}{|f|^{\frac{2}{n-2}}_{L^{\frac{2(n-1)}{n-2}}(\Sigma)}}:\ f \in C^\infty(\Sigma),\ f > 0\right\} \\
&= \sup\left\{\frac{|P_L f|^{\frac{2}{n-2}}_{L^{\frac{2n}{n-2}}(M)}}{|f|^{\frac{2}{n-2}}_{L^{\frac{2(n-1)}{n-2}}(\Sigma)}}:\ f \in L^{\frac{2(n-1)}{n-2}}(\Sigma),\ f \neq 0\right\} \\
&= \left[\sup\left\{|P_L f|_{L^{\frac{2n}{n-2}}(M)}:\ f \in L^{\frac{2(n-1)}{n-2}}(\Sigma), |f|_{L^{\frac{2(n-1)}{n-2}}(\Sigma)} = 1\right\}\right]^{\frac{2}{n-2}}. \qquad (2.2)
\end{aligned}$$

The second equality above follows from the fact P_L is positive and an approximation procedure.

It follows easily from the definition of $\Theta_{M,g}$ (see (1.3)) that $\Theta_{M,g}$ depends only on $[g]$. As a consequence we may choose the background metric g with zero scalar curvature. Under this assumption the conformal Laplacian operator reduces to the constant multiple of the Laplacian operator. To continue we will need some estimates of the Poisson kernels.

2.1. Basic estimates for Poisson kernel and harmonic extensions

Let us fix some notations. Throughout this subsection, we always assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$. For convenience we fix a smooth compact Riemannian manifold without boundary, $(\overline{M}^n, g)$ such that (M, g) is a smooth domain in $(\overline{M}, g)$. Denote d as the distance on $\overline{M}$ generated by g and d_Σ as the distance on Σ (when Σ is not connected and $\xi_1, \xi_2 \in \Sigma$ lie in different components, we set $d_\Sigma(\xi_1, \xi_2)$ equal to the maximal diameter of all the components of Σ). We write $t = t(x) = d(x, \Sigma)$ for $x \in \overline{M}$. Assume $\delta_0 > 0$ is small enough such that $V = \{x \in \overline{M}\colon t(x) < 2\delta_0\}$ is a tubular neighborhood of Σ and for $\xi, \zeta \in \Sigma$ with $d(\xi, \zeta) < 2\delta_0$, we have $d_\Sigma(\xi, \zeta) \leqslant 2d(\xi, \zeta)$. For $x \in V$, let $\pi(x) \in \Sigma$ be the unique nearest point on Σ to x. For $\delta > 0$, we write $M_\delta = \{x \in M\colon t(x) \leqslant \delta\}$. For $x \in \overline{M}$, $\delta > 0$, we use $B_\delta(x)$ to denote the ball with center at x, radius δ in $(\overline{M}, g)$.

The Green's function of the Laplace operator satisfies

$$\begin{cases} -\Delta_x G(x, y) = \delta_y & \text{on } M, \\ G(x, y) = 0 & \text{for } x \in \Sigma. \end{cases}$$

Note that $G(x_1, x_2) = G(x_2, x_1)$ for $x_1, x_2 \in M$.

- The solution of

$$\begin{cases} -\Delta u = h & \text{on } M, \\ u|_\Sigma = 0 \end{cases}$$

is given by

$$u(x)=\int_M G(x,y)h(y)\,d\mu(y).$$

- The solution of

$$\begin{cases}-\Delta u=0 & \text{on } M,\\ u|_\Sigma=f\end{cases}$$

is given by

$$u(x)=-\int_\Sigma \frac{\partial G(x,y)}{\partial_y\nu}\bigg|_{y=\xi} f(\xi)\,dS(\xi).$$

Here ν is the unit outer normal direction on Σ. In particular the Poisson kernel is given by

$$P(x,\xi)=-\frac{\partial G(x,y)}{\partial_y\nu}\bigg|_{y=\xi}.$$

- If

$$\begin{cases}-\Delta u=h & \text{on } M,\\ u|_\Sigma=0,\end{cases}$$

then $\frac{\partial u}{\partial\nu}(\xi)=-\int_M P(x,\xi)h(x)\,d\mu(x)$. In the future we will denote

$$(Th)(\xi)=\int_M P(x,\xi)h(x)\,d\mu(x).$$

Hence $\frac{\partial u}{\partial\nu}=-Th$.

For f defined on Σ, we write

$$(Pf)(x)=\int_\Sigma P(x,\xi)f(\xi)\,dS(\xi).$$

Pf is the harmonic extension of f.

Lemma 2.1. *For $0\leqslant\delta<\delta_0$, denote $\Sigma_\delta=\{x\in M\colon d(x,\Sigma)=\delta\}$. If $u\in C^\infty(M)$ is a nonnegative harmonic function, then*

$$\int_{\Sigma_\delta} u\,dS\leqslant c(M,g)\int_\Sigma u\,dS.$$

Proof. Denote ν as the unit outer normal direction. Since δ_0 is small, for $0\leqslant\delta<\delta_0$, the map $\psi_\delta:\Sigma\to\Sigma_\delta$ given by $\psi_\delta(\xi)=\exp_\xi(-\delta\nu(\xi))$ is a diffeomorphism and $\int_{\Sigma_\delta}u\,dS=\int_\Sigma u\circ\psi_\delta\cdot J_{\psi_\delta}\,dS$. Hence

$$\begin{aligned}\frac{d}{d\delta}\int_{\Sigma_\delta}u\,dS&=\int_{\Sigma_\delta}\frac{\partial u}{\partial t}\,dS+\int_\Sigma u\circ\psi_\delta\cdot\frac{dJ_{\psi_\delta}}{d\delta}\,dS\\&\leqslant c(M,g)\int_\Sigma u\circ\psi_\delta\cdot J_{\psi_\delta}\,dS\\&=c(M,g)\int_{\Sigma_\delta}u\,dS.\end{aligned}$$

Here we have used the equation $\int_{\Sigma_\delta} \frac{\partial u}{\partial t}\, dS = 0$ which follows from the divergence theorem and the fact u is harmonic. It follows that $\int_{\Sigma_\delta} u\, dS \leqslant c(M,g) \int_\Sigma u\, dS$. $\square$

To avoid confusion we emphasize that the constants $c(M,g)$'s are different in different formulas. This convention applies throughout the article. We will need the following classical estimate for Poisson kernels.

Lemma 2.2. *The Poisson kernel $P(x,\xi)$ satisfies*

$$0 \leqslant P(x,\xi) \leqslant c(M,g) \frac{t(x)}{[t(x)^2 + d_\Sigma(\pi(x),\xi)^2]^{\frac{n}{2}}}$$

for $x \in M_{\delta_0}$ and $\xi \in \Sigma$.

Proof. It follows from Lemma 2.1 and an approximation procedure that for $0 < \delta \leqslant \delta_0$,

$$\int\limits_{M_\delta} P(x,\xi)\, d\mu(x) \leqslant c(M,g)\delta.$$

Since $P(x,\xi)$ is nonnegative, harmonic in x and $P(x,\xi) = 0$ for $x \in \Sigma \backslash \{\xi\}$, it follows from the elliptic estimates of harmonic function that we only need to consider the case $t(x) + d_\Sigma(\pi(x),\xi)$ is small. Let $t(x) + d_\Sigma(\pi(x),\xi) = \delta$. If $t(x) \geqslant \frac{\delta}{7}$, by mean value inequality

$$\begin{aligned} P(x,\xi) &\leqslant \frac{c(M,g)}{\delta^n} \int\limits_{B_{\frac{\delta}{7}}(x)} P(y,\xi)\, d\mu(y) \leqslant \frac{c(M,g)}{\delta^{n-1}} \\ &\leqslant c(M,g) \frac{t(x)}{[t(x)^2 + d_\Sigma(\pi(x),\xi)^2]^{\frac{n}{2}}}. \end{aligned}$$

Assume $t(x) < \frac{\delta}{7}$, then $d(\pi(x),\xi) > \frac{3\delta}{7}$. By the gradient estimate of harmonic functions we know

$$\left|\nabla P(\cdot,\xi)\right|_{L^\infty(B_{2\delta/7}(\pi(x))\cap M)} \leqslant \frac{c(M,g)}{\delta^{n+1}} \int\limits_{B_{3\delta/7}(\pi(x))\cap M} P(y,\xi)\, d\mu(y) \leqslant \frac{c(M,g)}{\delta^n},$$

hence $P(x,\xi) \leqslant c(M,g) \frac{t(x)}{\delta^n}$. The lemma follows. $\square$

As an application of Lemma 2.2 we may derive the following inequality for harmonic extensions. Recall if X is a measure space, $p > 0$ and u is a measurable function on X, then

$$|u|_{L^p_W(X)} = \sup_{t>0} t \left||u| > t\right|^{\frac{1}{p}}.$$

Here $||u| > t|$ is the measure of the set $\{|u| > t\}$.

Proposition 2.1. *The harmonic extension operator P satisfies*

$$|Pf|_{L_W^{\frac{n}{n-1}}(M)} \leqslant c(M,g)|f|_{L^1(\Sigma)}$$

and

$$|Pf|_{L^{\frac{np}{n-1}}(M)} \leqslant c(M,g,p)|f|_{L^p(\Sigma)}$$

for $1 < p \leqslant \infty$.

Proof. We only need to prove the weak type estimate. The strong estimate follows from Marcinkiewicz interpolation theorem [11, p. 197] and the basic fact $|Pf|_{L^\infty(M)} \leqslant |f|_{L^\infty(\Sigma)}$. To prove the weak type estimate we may assume $f \geqslant 0$ and $|f|_{L^1(\Sigma)} = 1$. It follows from Lemma 2.2 that

$$0 \leqslant (Pf)(x) \leqslant \frac{c(M,g)}{t(x)^{n-1}}.$$

For $\delta_0 = \delta_0(M, g) > 0$ small, it follows from Lemma 2.1 that

$$\int_{M_\delta} P(x,\xi)\,d\mu(x) \leqslant c(M,g)\delta \quad \text{for } \xi \in \Sigma \text{ and } 0 < \delta < \delta_0.$$

Hence for $\delta \in (0, \delta_0)$,

$$\int_{M_\delta} (Pf)(x)\,d\mu(x) = \int_\Sigma dS(\xi)\left[f(\xi)\int_{M_\delta} P(x,\xi)\,d\mu(x)\right] \leqslant c(M,g)\delta.$$

For $\lambda \geqslant c(M,g)$, we have

$$\begin{aligned}|Pf > \lambda| &= \left|\left\{x \in M\colon t(x) < c(M,g)\lambda^{-\frac{1}{n-1}}, (Pf)(x) > \lambda\right\}\right| \\ &\leqslant \frac{1}{\lambda}\int_{M_{c(M,g)\lambda^{-\frac{1}{n-1}}}} (Pf)(x)\,d\mu(x) \\ &\leqslant c(M,g)\lambda^{-\frac{n}{n-1}}.\end{aligned}$$

The proposition follows. □

For $1 < p < \infty$, if we write

$$c_{M,g,p} = \sup\left\{|Pf|_{L^{\frac{np}{n-1}}(M)}\colon f \in L^p(\Sigma), |f|_{L^p(\Sigma)} = 1\right\}, \tag{2.3}$$

then $c_{M,g,p} < \infty$. In view of (2.2), when the background metric g has zero scalar curvature,

$$\Theta_{M,g} = c^{\frac{2}{n-2}}_{M,g,\frac{2(n-1)}{n-2}} < \infty. \tag{2.4}$$

In the future we will also need the following compactness property.

Corollary 2.1. *For $1 \leqslant p < \infty$, $1 \leqslant q < \frac{np}{n-1}$, the operator $P : L^p(\Sigma) \to L^q(M)$ is compact.*

Proof. First assume $1 < p < \infty$. If $f_i \in L^p(\Sigma)$ such that $|f_i|_{L^p(\Sigma)} \leqslant 1$, it follows from Lemma 2.2 that

$$\left|(Pf_i)(x)\right| \leqslant \frac{c(M,g)}{t(x)^{n-1}} \quad \text{for } x \in M\backslash\Sigma.$$

Using elliptic estimates of harmonic functions we know after passing to a subsequence we may find a $u \in C^\infty(M\backslash\Sigma)$ such that $Pf_i \to u$ in $C^\infty_{\mathrm{loc}}(M\backslash\Sigma)$. For $\delta > 0$ small, we have

$$\begin{aligned}|Pf_i - Pf_j|_{L^q(M)} &\leqslant |Pf_i - Pf_j|_{L^q(M\backslash M_\delta)} + |Pf_i - Pf_j|_{L^q(M_\delta)} \\ &\leqslant |Pf_i - Pf_j|_{L^q(M\backslash M_\delta)} + |Pf_i - Pf_j|_{L^{\frac{np}{n-1}}(M_\delta)}|M_\delta|^{\frac{1}{q}-\frac{n-1}{np}} \\ &\leqslant |Pf_i - Pf_j|_{L^q(M\backslash M_\delta)} + c(M,g,p)|M_\delta|^{\frac{1}{q}-\frac{n-1}{np}}.\end{aligned}$$

Hence

$$\limsup_{i,j\to\infty} |Pf_i - Pf_j|_{L^q(M)} \leqslant c(M,g,p)|M_\delta|^{\frac{1}{q}-\frac{n-1}{np}}.$$

Letting $\delta \to 0^+$, we see Pf_i is a Cauchy sequence in $L^q(M)$. In another word, $P : L^p(\Sigma) \to L^q(M)$ is compact.

When $p = 1$, the argument is similar. We only need to observe that for any $1 \leqslant q < \tilde{q} < \frac{n}{n-1}$, $P : L^1(\Sigma) \to L^{\tilde{q}}(M)$ is bounded. □

Let h be a function on M, recall $(Th)(\xi) = \int_M P(x,\xi)h(x)\,d\mu(x)$. We have the following dual statement to Proposition 2.1.

Proposition 2.2. *For $1 \leqslant p < n$ and $h \in L^p(M)$,*

$$|Th|_{L^{\frac{(n-1)p}{n-p}}(\Sigma)} \leqslant c(M,g,p)|h|_{L^p(M)}.$$

Proof. We may prove the inequality by a duality argument. Indeed for any nonnegative functions h on M and f on Σ, we have

$$\begin{aligned}0 &\leqslant \int_\Sigma (Th)(\xi) f(\xi)\, dS(\xi) = \int_\Sigma dS(\xi) \int_M P(x,\xi) h(x) f(\xi)\, d\mu(x)\\ &= \int_M (Pf)(x) h(x)\, d\mu(x) \leqslant |Pf|_{L^{\frac{p}{p-1}}(M)} |h|_{L^p(M)}\\ &\leqslant c(M,g,p)|h|_{L^p(M)} |f|_{L^{\frac{(n-1)p}{n(p-1)}}(\Sigma)},\end{aligned}$$

the proposition follows. One may also prove the inequality directly. Indeed it follows from Lemma 2.2 that $|P(\cdot,\xi)|_{L^{\frac{n}{n-1},\infty}(M)} \leqslant c(M,g) < \infty$ for $\xi \in \Sigma$. Hence $T : L^{n,1}(M) \to L^\infty(\Sigma)$ is a bounded linear map. On the other hand for $h \in L^1(M)$,

$$\int_\Sigma \left|(Th)(\xi)\right| dS(\xi) \leqslant \int_\Sigma dS(\xi) \int_M P(x,\xi) \left|h(x)\right| d\mu(x) = \int_M \left|h(x)\right| d\mu(x).$$

Hence $T : L^1(M) \to L^1(\Sigma)$ is also bounded. The proposition follows from the Marcinkiewicz interpolation theorem. Finally we point out for $1 < p < n$, we may solve

$$\begin{cases} -\Delta u = h & \text{on } M,\\ u|_\Sigma = 0 \end{cases}$$

and $(Th)(\xi) = -\frac{\partial u}{\partial \nu}(\xi)$. By the L^p theory we know $|u|_{W^{2,p}(M)} \leqslant c(M,g,p)|h|_{L^p(M)}$. It follows from boundary trace embedding theorem [1, p. 164] that

$$|Th|_{L^{\frac{(n-1)p}{n-p}}(\Sigma)} = \left|\frac{\partial u}{\partial \nu}\right|_{L^{\frac{(n-1)p}{n-p}}(\Sigma)} \leqslant c(M,g,p)|u|_{W^{2,p}(M)} \leqslant c(M,g,p)|h|_{L^p(M)}. \qquad \square$$

2.2. Miscellaneous

Later on we will need the following Hausdorff–Young type inequality to estimate some nonmajor terms.

Lemma 2.3. *Let X and Y be measure spaces, $1 \leqslant p, q_0, q_1, r \leqslant \infty$, $p \leqslant r$, $q_0 \leqslant r$ and*

$$\frac{1}{p} + \frac{1}{q_1} = \frac{q_0}{q_1 r} + 1.$$

Assume K is defined on $X \times Y$ such that

$$\left(\int_X \left|K(x,y)\right|^{q_0} dx\right)^{\frac{1}{q_0}} \leqslant A, \qquad \left(\int_Y \left|K(x,y)\right|^{q_1} dy\right)^{\frac{1}{q_1}} \leqslant A.$$

For a function f defined on Y, we let $(Kf)(x) = \int_Y K(x,y) f(y)\, dy$, then

$$|Kf|_{L^r(X)} \leqslant A|f|_{L^p(Y)}.$$

Proof. Without losing of generality we may assume $K \geqslant 0$ and $f \geqslant 0$, then

$$
\begin{aligned}
(Kf)(x) &= \int_Y K(x,y)^{\frac{q_0}{r}} f(y)^{\frac{p}{r}} K(x,y)^{\frac{r-q_0}{r}} f(y)^{\frac{r-p}{r}}\,dy \\
&\leqslant \Bigl(\int_Y K(x,y)^{q_0} f(y)^p\,dy\Bigr)^{\frac{1}{r}} \Bigl(\int_Y K(x,y)^{q_1}\,dy\Bigr)^{\frac{r-q_0}{q_1 r}} \Bigl(\int_Y f(y)^p\,dy\Bigr)^{\frac{r-p}{pr}} \\
&\leqslant A^{\frac{r-q_0}{r}} |f|_{L^p(Y)}^{\frac{r-p}{r}} \Bigl(\int_Y K(x,y)^{q_0} f(y)^p\,dy\Bigr)^{\frac{1}{r}}.
\end{aligned}
$$

Here we have used the Holder's inequality and the fact $\frac{1}{r} + \frac{1}{q_1 r/(r-q_0)} + \frac{1}{pr/(r-p)} = 1$. Hence

$$
(Kf)(x)^r \leqslant A^{r-q_0} |f|_{L^p(Y)}^{r-p} \int_Y K(x,y)^{q_0} f(y)^p\,dy.
$$

Integrating both sides, we get the needed inequality. □

3. Sharp inequalities on the unit ball

The aim of this section is to show $\Theta_{\overline{B}_1, g_{\mathbb{R}^n}} = c_{\overline{B}_1, g_{\mathbb{R}^n}, \frac{2(n-1)}{n-2}}^{\frac{2}{n-2}} = I(\overline{B}_1, g_{\mathbb{R}^n}) = n^{-\frac{1}{n-1}} \omega_n^{-\frac{1}{n(n-1)}}$ (see (2.3), (2.4)).

Theorem 3.1. *Assume $n \geqslant 3$, then for every $f \in L^{\frac{2(n-1)}{n-2}}(\partial B_1^n)$,*

$$
|Pf|_{L^{\frac{2n}{n-2}}(B_1)} \leqslant n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} |f|_{L^{\frac{2(n-1)}{n-2}}(\partial B_1)}.
$$

Here Pf is the harmonic extension of f, ω_n is the volume of the unit ball in $\mathbb{R}^n$. Equality holds if and only if $f(\xi) = c(1 + \lambda \xi \cdot \zeta)^{-\frac{n-2}{2}}$ for some constant c, $\zeta \in \partial B_1$ and $0 \leqslant \lambda < 1$.

Note that this theorem is a consequence of [7, Theorem 1.1] (see the discussions before [7, Theorem 1.1]). Below we will present a different argument which has its own interest. Before discussing the approach, we describe some corollaries of the theorem. Note that in Proposition 2.1 the strong inequality is not true for $p = 1$. Instead we have the following

Corollary 3.1. *Assume $n \geqslant 3$, then for $f \in L^\infty(\partial B_1^n)$,*

$$
\bigl|e^{Pf}\bigr|_{L^{\frac{n}{n-1}}(B_1^n)} \leqslant n^{-1} \omega_n^{-\frac{1}{n}} \bigl|e^f\bigr|_{L^1(\partial B_1)}.
$$

Moreover equality holds if and only if f is constant.

Proof. If u is a harmonic function, then $\Delta e^u = e^u |\nabla u|^2$. Hence e^u is subharmonic and not harmonic except when u is a constant function. It follows from Theorem 3.1 that

$$
\begin{aligned}
\bigl|e^{\frac{n-2}{2(n-1)} Pf}\bigr|_{L^{\frac{2n}{n-2}}(B_1)} &\leqslant \bigl|P\bigl(e^{\frac{n-2}{2(n-1)} f}\bigr)\bigr|_{L^{\frac{2n}{n-2}}(B_1)} \\
&\leqslant n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \bigl|e^{\frac{n-2}{2(n-1)} f}\bigr|_{L^{\frac{2(n-1)}{n-2}}(\partial B_1)}.
\end{aligned}
$$

Hence

$$
\bigl|e^{Pf}\bigr|_{L^{\frac{n}{n-1}}(B_1)} \leqslant n^{-1} \omega_n^{-\frac{1}{n}} \bigl|e^f\bigr|_{L^1(\partial B_1)}.
$$

If equality holds, then $e^{\frac{n-2}{2(n-1)} Pf} = P(e^{\frac{n-2}{2(n-1)} f})$ and $e^{\frac{n-2}{2(n-1)} Pf}$ must be a harmonic function, hence Pf is equal to constant and so is f. □

Corollary 3.2. *Assume $n \geqslant 3$, then for $\frac{2(n-1)}{n-2} < p < \infty$, $f \in L^p(\partial B_1^n)$,*

$$|Pf|_{L^{\frac{np}{n-1}}(B_1)} \leqslant n^{-\frac{1}{p}} \omega_n^{-\frac{1}{np}} |f|_{L^p(\partial B_1)}.$$

Equality holds if and only if f is constant.

Proof. Denote $r = \frac{p}{2(n-1)/(n-2)} > 1$. If u is a harmonic function on B_1, then $|u|^r$ is a subharmonic function and it is not harmonic except when u is a constant function. If $f \in L^p(\partial B_1)$, then by Theorem 3.1,

$$\left||Pf|^r\right|_{L^{\frac{2n}{n-2}}(B_1)} \leqslant \left|P\left(|f|^r\right)\right|_{L^{\frac{2n}{n-2}}(B_1)} \leqslant n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \left||f|^r\right|_{L^{\frac{2(n-1)}{n-2}}(\partial B_1)}.$$

Hence

$$|Pf|_{L^{\frac{np}{n-1}}(B_1)} \leqslant n^{-\frac{1}{p}} \omega_n^{-\frac{1}{np}} |f|_{L^p(\partial B_1)}.$$

If equality holds then $|Pf|^r = P(|f|^r)$. In particular $|Pf|^r$ is a harmonic function and hence Pf is a constant function, so is f. □

Remark 3.1. For $1 < p < \frac{2(n-1)}{n-2}$, 1 is still a critical point for the functional $\frac{|Pf|_{L^{\frac{np}{n-1}}(B_1)}}{|f|_{L^p(\partial B_1)}}$, but calculation shows for $f_\varepsilon(\xi) = 1 + \varepsilon\xi_1$,

$$\frac{|Pf_\varepsilon|_{L^{\frac{np}{n-1}}(B_1)}}{|f_\varepsilon|_{L^p(\partial B_1)}} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{np}} \left[1 + \frac{n-2}{2n(n-1)(n+2)}\left(\frac{2(n-1)}{n-2} - p\right)\varepsilon^2 + \mathrm{O}\left(\varepsilon^4\right)\right].$$

Hence 1 is not a local maximizer. It remains an interesting question to calculate

$$\sup\left\{|Pf|_{L^{\frac{np}{n-1}}(B_1)}:\ f \in L^p(\partial B_1), |f|_{L^p(\partial B_1)} = 1\right\}$$

for these p's.

The new approach to Theorem 3.1 needs an interesting Kazdan–Warner type condition. To formulate the condition, we introduce the weighted isoperimetric ratio.

Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$. Let K be a positive smooth function on Σ, then we write the weighted isoperimetric ratio

$$I(M, g, K) = \frac{\mu(M)^{\frac{1}{n}}}{(\int_\Sigma K\, dS)^{\frac{1}{n-1}}}.$$

Here $d\mu$ is the measure associated with g and dS is the measure on Σ. If $n \geqslant 3$ and (M^n, g) satisfies $\lambda_1(L_g) > 0$, for $\tilde{g} \in [g]$ with zero scalar curvature, we write $\tilde{g} = u^{\frac{4}{n-2}} g$, $u|_\Sigma = f$, then

$$I(M, \tilde{g}, K) = \frac{(\int_M (P_L f)^{\frac{2n}{n-2}}\, d\mu)^{\frac{1}{n}}}{(\int_\Sigma K f^{\frac{2(n-1)}{n-2}}\, dS)^{\frac{1}{n-1}}}.$$

The Euler–Lagrange equation of this functional reads as

$$\int_M P_L(x, \xi)(P_L f)(x)^{\frac{n+2}{n-2}}\, d\mu(x) = \mathrm{const} \cdot K(\xi) f(\xi)^{\frac{n}{n-2}}.$$

Lemma 3.1 *(Kazdan–Warner type condition). Assume $n \geqslant 3$, (M^n, g) is a smooth compact Riemannian manifold with boundary and $\lambda_1(L_g) > 0$, K and f are positive smooth functions on Σ such that*

$$\int_M P_L(x, \xi)(P_L f)(x)^{\frac{n+2}{n-2}}\, d\mu(x) = K(\xi) f(\xi)^{\frac{n}{n-2}}.$$

Let X be a conformal vector field on M (note X must be tangent to Σ), then

$$\int_{\Sigma} XK \cdot f^{\frac{2(n-1)}{n-2}}\, dS = 0.$$

Proof. Denote $u = P_L f$. Let ϕ_t be the smooth 1-parameter group generated by X, then

$$\left.\frac{d}{dt}\right|_{t=0} I\big(M, \phi_t^*\big(u^{\frac{4}{n-2}} g\big), K\big) = 0.$$

On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} I\big(M, \phi_t^*\big(u^{\frac{4}{n-2}} g\big), K\big) = \left.\frac{d}{dt}\right|_{t=0} I\big(M, u^{\frac{4}{n-2}} g, K \circ \phi_{-t}\big) = \frac{I(M, u^{\frac{4}{n-2}} g, K)}{n-1} \frac{\int_{\Sigma} XK \cdot f^{\frac{2(n-1)}{n-2}}\, dS}{\int_{\Sigma} K f^{\frac{2(n-1)}{n-2}}\, dS}.$$

This implies $\int_{\Sigma} XK \cdot f^{\frac{2(n-1)}{n-2}}\, dS = 0$. □

Corollary 3.3. *Assume $n \geqslant 3$, K and f are positive smooth functions on ∂B_1^n such that*

$$\int_{B_1} P(x,\xi)(Pf)(x)^{\frac{n+2}{n-2}}\, dx = K(\xi) f(\xi)^{\frac{n}{n-2}},$$

then $\int_{\partial B_1} \langle \nabla K(\xi), \nabla \xi_i \rangle f(\xi)^{\frac{2(n-1)}{n-2}}\, dS(\xi) = 0$ for $1 \leqslant i \leqslant n$.

This is because $\nabla \xi_i$ is the restriction to ∂B_1 of a conformal vector field on $(\overline{B}_1, g_{\mathbb{R}^n})$.

We will also need some rearrangement inequality on ∂B_1 which was proven in [2]. We say a function f on ∂B_1 is radially symmetric if $f(\xi)$ is a function of ξ_n. Let f be a measurable function on ∂B_1, then the symmetric rearrangement of f is a radial decreasing function f^* which has the same distribution as f. The following rearrangement inequality was proven in [2, Theorem 2]. Namely, if K is a nondecreasing bounded function on $[-1, 1]$, then for all $f, g \in L^1(\partial B_1)$,

$$\int_{\partial B_1 \times \partial B_1} f(\xi) g(\eta) K(\xi \cdot \eta)\, dS(\xi)\, dS(\eta) \leqslant \int_{\partial B_1 \times \partial B_1} f^*(\xi) g^*(\eta) K(\xi \cdot \eta)\, dS(\xi)\, dS(\eta).$$

It follows that if K is a bounded nonnegative nondecreasing function on $[-1, 1]$, f is nonnegative function on ∂B_1 and

$$(K * f)(\xi) = \int_{\partial B_1} K(\xi \cdot \eta) f(\eta)\, dS(\eta),$$

then for $1 \leqslant p < \infty$, $|K * f|_{L^p(\partial B_1)} \leqslant |K * f^*|_{L^p(\partial B_1)}$.

Recall the Poisson kernel on $(\overline{B}_1, g_{\mathbb{R}^n})$ is given by

$$P(x,\xi) = \frac{1 - |x|^2}{n\omega_n |x - \xi|^n}.$$

For $0 < r < 1$, $\xi, \zeta \in \partial B_1$,

$$P(r\zeta, \xi) = \frac{1 - r^2}{n\omega_n (r^2 + 1 - 2r\zeta \cdot \xi)^{\frac{n}{2}}} = K_r(\zeta \cdot \xi).$$

Hence for $1 \leqslant p < \infty$ and $f \geqslant 0$,

$$|Pf|^p_{L^p(B_1)} = \int_0^1 |K_r * f|^p_{L^p(\partial B_1)} r^{n-1}\, dr \leqslant \int_0^1 |K_r * f^*|^p_{L^p(\partial B_1)} r^{n-1}\, dr = |Pf^*|^p_{L^p(B_1)}.$$

It follows that $|Pf|_{L^p(B_1)} \leqslant |Pf^*|_{L^p(B_1)}$.

Proof of Theorem 3.1. For $p > \frac{2(n-1)}{n-2}$, we consider the variational problem

$$\sup\left\{|Pf|_{L^{\frac{2n}{n-2}}(B_1)}:\ f \in L^p(\partial B_1), |f|_{L^p(\partial B_1)} = 1\right\}. \tag{3.1}$$

By Corollary 2.1 the operator $P : L^p(\partial B_1) \to L^{\frac{2n}{n-2}}(B_1)$ is compact, hence the supreme is achieved at some $f_p \geqslant 0$. Replacing f_p by f_p^* we may assume f_p is radial symmetric and decreasing. After scaling f_p satisfies

$$f_p(\xi)^{p-1} = \int_{B_1} P(x,\xi)(Pf_p)(x)^{\frac{n+2}{n-2}}\,dx.$$

Standard bootstrap using Propositions 2.1 and 2.2 shows $f_p \in C^\infty(\partial B_1)$ and $f_p > 0$. Rewrite the equation as

$$\int_{B_1} P(x,\xi)(Pf_p)(x)^{\frac{n+2}{n-2}}\,dx = f_p(\xi)^{\frac{n}{n-2}} f_p(\xi)^{p-\frac{2(n-1)}{n-2}}.$$

It follows from Corollary 3.3 that

$$\int_{\partial B_1} \left\langle \nabla f_p(\xi)^{p-\frac{2(n-1)}{n-2}}, \nabla \xi_n\right\rangle f_p(\xi)^{\frac{2(n-1)}{n-2}}\,dS(\xi) = 0.$$

We may write $g_p(r) = f_p(0,\dots,0,\sin r,\cos r)$ for $0 \leqslant r \leqslant \pi$. Then the equality becomes

$$\int_0^\pi g_p'(r) g_p(r)^{p-1} \sin^{n-1} r\,dr = 0.$$

Since $g_p' \leqslant 0$ and $g_p > 0$, we get $g_p' = 0$ and hence $f_p \equiv \text{const}$. This implies

$$|Pf|_{L^{\frac{2n}{n-2}}(B_1)} \leqslant \frac{\omega_n^{\frac{n-2}{2n}}}{(n\omega_n)^{\frac{1}{p}}} |f|_{L^p(\partial B_1)}.$$

Let $p \to \frac{2(n-1)}{n-2}$, we get the needed inequality. At last we may apply [7, Theorem 1.2] to identify all the functions which achieves the equality. $\square$

4. Regularity of solutions to some nonlinear integral equations

Assume $1 < p < \infty$. If $f \in L^p(\Sigma)$ is a maximizer for the variational problem

$$c_{M,g,p} = \sup\left\{|Pf|_{L^{\frac{np}{n-1}}(M)}:\ f \in L^p(\Sigma), |f|_{L^p(\Sigma)} = 1\right\},$$

then we may assume $f \geqslant 0$, moreover after suitable scaling it satisfies the nonlinear integral equation

$$f(\xi)^{p-1} = \int_M P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\,d\mu(x).$$

This section is aiming at proving all these solutions are in fact smooth.

Proposition 4.1. *Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$. If $1 < p < \infty$, $f \in L^p(\Sigma)$ is nonnegative, not identically zero and it satisfies*

$$f(\xi)^{p-1} = \int_M P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\,d\mu(x),$$

then $f \in C^\infty(\Sigma)$.

Proof. Let $p_0 = \frac{1}{p-1}$, $f_0(\xi) = f(\xi)^{p-1}$, $u_0(x) = (Pf)(x)$, then $0 < p_0 < \infty$, $f_0 \in L^{p_0+1}(\Sigma)$, $u_0 \in L^{\frac{n(p_0+1)}{(n-1)p_0}}(M)$ and

$$u_0(x) = \int_\Sigma P(x,\xi) f_0(\xi)^{p_0}\, dS(\xi), \qquad f_0(\xi) = \int_M P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\, d\mu(x).$$

Let $(\overline{M}, g)$ be the same as in Section 2.1. Given $\xi_0 \in \Sigma$, by choosing a local coordinate $\phi : U(\xi_0) \to \{x \in \mathbb{R}^n\colon |x| < 2\}$ with $\phi(\xi_0) = 0$ and $\phi(U(\xi_0) \cap M) = \{x \in \mathbb{R}^n\colon |x| < 2, x_n \geqslant 0\}$, we may identify $U(\xi_0)$ with $\{x \in \mathbb{R}^n\colon |x| < 2\}$. For $0 < R < 1$, we write

$$B_R^+ = \left\{x \in \mathbb{R}^n\colon |x| < R, x_n > 0\right\},$$
$$B_R = B_R^{n-1} = \left\{\xi \in \mathbb{R}^{n-1}\colon |\xi| < R\right\}$$

and

$$u_R(x) = \int_{\Sigma \setminus B_R} P(x,\xi) f_0(\xi)^{p_0}\, dS(\xi),$$
$$f_R(\xi) = \int_{M \setminus B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\, d\mu(x).$$

Then $u_R \in C^\infty(\{x \in \mathbb{R}^n\colon |x| < R, x_n \geqslant 0\})$, $f_R \in C^\infty(B_R)$. To prove the regularity of f, we discuss two cases.

Case 4.1. $0 < p_0 \leqslant \frac{n}{n-1}$.

In this case, we have $\frac{p_0+n}{(n-1)p_0} > 1$. Fix a number r such that

$$1 \leqslant r < \frac{p_0+n}{(n-1)p_0} \quad \text{and} \quad r > \frac{1}{p_0},$$

then

$$f_0(\xi)^{1/r} \leqslant \left[\int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\, d\mu(x)\right]^{1/r} + f_R(\xi)^{1/r}.$$

Hence using Lemma 2.2 we have

$$\begin{aligned}
u_0(x) &= \int_{B_R} P(x,\xi) f_0(\xi)^{p_0 - r^{-1}} f_0(\xi)^{1/r}\, dS(\xi) + u_R(x)\\
&\leqslant \int_{B_R} P(x,\xi) f_0(\xi)^{p_0 - r^{-1}} \left[\int_{B_R^+} P(y,\xi) u_0(y)^{\frac{p_0+n}{(n-1)p_0} - r} u_0(y)^r\, d\mu(y)\right]^{1/r} dS(\xi) + v_R(x)\\
&\leqslant c(M,g,p,r) \int_{B_R} \frac{x_n}{(|x'-\xi|^2 + x_n^2)^{n/2}} f_0(\xi)^{p_0 - r^{-1}}\\
&\quad \times \left[\int_{B_R^+} \frac{y_n}{(|y'-\xi|^2 + y_n^2)^{n/2}} u_0(y)^{\frac{p_0+n}{(n-1)p_0} - r} u_0(y)^r\, dy\right]^{1/r} d\xi + v_R(x)
\end{aligned}$$

here dx and $d\xi$ means the standard Lebesgue measure and

$$v_R(x) = \int_{B_R} P(x,\xi) f_0(\xi)^{p_0 - r^{-1}} f_R(\xi)^{1/r}\, dS(\xi) + u_R(x).$$

We have $v_R \in L^{\frac{n(p_0+1)}{(n-1)p_0}}(B_R^+) \cap L_{\mathrm{loc}}^{\frac{n(p_0+1)}{(n-1)(p_0-r^{-1})}}(B_R^+ \cup B_R^{n-1})$. Let

$$a = \frac{n(p_0+1)}{p_0+n-(n-1)p_0 r}, \qquad b = \frac{(p_0+1)r}{p_0 r - 1}.$$

Then $\frac{n}{ra} + \frac{n-1}{b} = \frac{1}{r}$ and

$$\frac{r}{n(p_0+1)/((n-1)p_0)} + \frac{1}{a} = \frac{p_0+n}{n(p_0+1)} < 1.$$

For $\frac{n(p_0+1)}{(n-1)p_0} < q < \frac{n(p_0+1)}{(n-1)(p_0-r^{-1})}$, we have $\frac{r}{q} + \frac{1}{a} > \frac{1}{n}$. It follows from [7, Proposition 5.2] that when R is small enough, $u_0|_{B_{R/4}^+} \in L^q(B_{R/4}^+)$. This implies

$$f_0(\xi) = \int_{B_{R/4}^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}}\,d\mu(x) + f_{R/4}(\xi)$$
$$\leqslant c(M,g,q)|u_0|_{L^q(B_{R/4}^+)}^{\frac{p_0+n}{(n-1)p_0}} + f_{R/4}(\xi)$$

when $q > \frac{n(p_0+n)}{(n-1)p_0}$. Such a choice of q is possible since $\frac{n(p_0+1)}{(n-1)(p_0-r^{-1})} > \frac{n(p_0+n)}{(n-1)p_0}$. In particular, we see $f_0|_{B_{R/8}} \in L^\infty(B_{R/8})$. Since ξ_0 is arbitrary, we see $f_0 \in L^\infty(\Sigma)$ and hence $u_0 \in L^\infty(M)$. Observing that $f_0 = T(u_0^{\frac{p_0+n}{(n-1)p_0}})$, here T is defined in Section 2.1, it follows from L^p theory [6, Chapter 9] and the Sobolev embedding theorem that $f_0 \in C^\alpha(\Sigma)$ for $0 < \alpha < 1$. In particular, $f_0(\xi) > 0$ for any $\xi \in \Sigma$. This implies $u_0 \in C^\beta(M)$ for some $0 < \beta < 1$ [6, Chapter 8]. It follows from Schauder theory [6, Chapter 6] that $f_0 \in C^{1,\beta}(\Sigma)$. Iterating this procedure we see $f_0 \in C^\infty(\Sigma)$ and so is f.

Case 4.2. $\frac{n}{n-1} \leqslant p_0 < \infty$.

In this case, we fix a number r such that

$$1 \leqslant r \leqslant p_0 \quad \text{and} \quad r \geqslant \frac{(n-1)p_0}{p_0+n},$$

then

$$u_0(x)^{1/r} \leqslant \left[\int_{B_R} P(x,\xi) f_0(\xi)^{p_0}\,dS(\xi)\right]^{1/r} + u_R(x)^{1/r}.$$

Hence

$$f_0(\xi) \leqslant \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}-r^{-1}} \left[\int_{B_R} P(x,\zeta) f_0(\zeta)^{p_0-r} f_0(\zeta)^r\,dS(\zeta)\right]^{1/r} d\mu(x) + g_R(\xi)$$
$$\leqslant c(M,g,p,r) \int_{B_R^+} \frac{x_n}{(|x'-\xi|^2 + x_n^2)^{n/2}} u_0(x)^{\frac{p_0+n}{(n-1)p_0}-r^{-1}}$$
$$\times \left[\int_{B_R} \frac{x_n}{(|x'-\zeta|^2 + x_n^2)^{n/2}} f_0(\zeta)^{p_0-r} f_0(\zeta)^r\,d\zeta\right]^{1/r} dx + g_R(\xi),$$

here

$$g_R(\xi) = \int_{B_R^+} P(x,\xi) u_0(x)^{\frac{p_0+n}{(n-1)p_0}-r^{-1}} u_R(x)^{1/r}\,d\mu(x) + f_R(\xi).$$

We have $g_R \in L^{p_0+1}(B_R) \cap L^q_{\mathrm{loc}}(B_R)$ for any $q < \infty$. Let

$$a = \frac{p_0+1}{p_0-r}, \qquad b = \frac{n(p_0+1)r}{(p_0+n)r-(n-1)p_0},$$

then $\frac{n-1}{ra} + \frac{n}{b} = 1$, $\frac{r}{p_0+1} + \frac{1}{a} = \frac{p_0}{p_0+1} \in (0,1)$. For any $p_0+1 < q < \infty$, it follows from [7, Proposition 5.3] that when R is small enough, we have $f_0 \in L^q(B_{R/4})$. Since ξ_0 is arbitrary, we see $f_0 \in L^q(\Sigma)$ and hence $u_0 \in L^{\frac{nq}{(n-1)p_0}}(M)$. Using the equations of f_0 and u_0, we see $f_0 \in L^\infty(\Sigma)$ and $u_0 \in L^\infty(M)$. The arguments in Case 4.1 tell us $f \in C^\infty(\Sigma)$. □

5. An asymptotic expansion formula of the Poisson kernel

Later on we will need more accurate information about the Poisson kernel than Lemma 2.2. For that purpose we need an asymptotic expansion formula for this kernel.

Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$, $\delta > 0$ is a small number such that $M_\delta = \{x \in M\colon d(x,\Sigma) \leqslant \delta\}$ is a tubular neighborhood of Σ and $\pi : M_\delta \to \Sigma$ denotes the nearest point projection. For $\xi \in \Sigma$, choose a normal coordinate for Σ at ξ, namely $\tau_1, \dots, \tau_{n-1}$. Let $C_\delta = \{x \in M_\delta\colon d_\Sigma(\pi(x), \xi) \leqslant \delta\}$. For δ small, we have a coordinate near ξ for M as

$$\phi : C_\delta \to \overline{B}^{n-1}_\delta \times [0,\delta] : x \mapsto \big(\tau\big(\pi(x)\big), t(x)\big).$$

It is usually called the Fermi coordinate at ξ. We will identify C_δ with $\overline{B}^{n-1}_\delta \times [0,\delta]$ through ϕ. Denote $r = |x|$ and $\theta = \frac{x}{|x|}$.

Theorem 5.1. *Under the above set up, we may find $a_i \in C^\infty(S^{n-1}_+)$ with $a_i|_{\partial S^{n-1}_+} = 0$ for $0 \leqslant i \leqslant n-1$ and a $\psi \in C^{1,1-\varepsilon}(M)$ (for all $\varepsilon > 0$) such that*

$$P(x,0) = \frac{2}{n\omega_n} r^{1-n} \sum_{i=0}^{n-1} r^i a_i(\theta) + \psi(x) \quad \text{for } x \text{ near } 0.$$

Here ω_n is the volume of the unit ball in $\mathbb{R}^n$. Moreover $a_0(\theta) = \theta_n = \frac{x_n}{|x|}$ and a_1 is determined by

$$\begin{cases} -\Delta_{S^{n-1}} a_1 = -H(0) - nH(0)\theta_n^2 + 2n(n+2)h_{ij}(0)\theta_i\theta_j\theta_n^2 & \text{on } S^{n-1}_+, \\ a_1|_{\partial S^{n-1}_+} = 0. \end{cases}$$

Here i, j runs from 1 to $n-1$, h_{ij} is the second fundamental form with respect to inner normal direction and H is the mean curvature.

To derive the asymptotic formula, we note that $g = g_{ij}\, dx_i \otimes dx_j + dx_n \otimes dx_n$. We will use i, j, k, l etc. to denote indices running from 1 to $n-1$. Calculation shows

$$g_{ij} = \delta_{ij} - 2h_{ij}(0)x_n - \frac{1}{3}(R_\Sigma)_{ikjl}(0)x_k x_l - 2h_{ij,k}(0)x_k x_n + \big(-R_{injn}(0) + h_{ik}(0)h_{jk}(0)\big)x_n^2 + O\big(r^3\big); \tag{5.1}$$

$$g^{ij} = \delta_{ij} + 2h_{ij}(0)x_n + \frac{1}{3}(R_\Sigma)_{ikjl}(0)x_k x_l + 2h_{ij,k}(0)x_k x_n + \big(R_{injn}(0) + 3h_{ik}(0)h_{jk}(0)\big)x_n^2 + O\big(r^3\big); \tag{5.2}$$

and

$$\sqrt{G} = 1 - H(0)x_n - \frac{1}{6}(Rc_\Sigma)_{ij}(0)x_i x_j - H_i(0)x_i x_n + \frac{1}{2}\big(H(0)^2 - \big|h(0)\big|^2 - Rc_{nn}(0)\big)x_n^2 + O\big(r^3\big). \tag{5.3}$$

Note that

$$\begin{aligned} \Delta_g u &= \frac{1}{\sqrt{G}} \partial_i\big(g^{ij}\sqrt{G}\partial_j u\big) + \frac{1}{\sqrt{G}}\partial_n\big(\sqrt{G}\partial_n u\big) \\ &= g^{ij}\partial_{ij}u + \partial_{nn}u + \frac{1}{\sqrt{G}}\partial_i\big(g^{ij}\sqrt{G}\big)\partial_j u + \frac{1}{\sqrt{G}}\partial_n\big(\sqrt{G}\big)\partial_n u. \end{aligned}$$

This and (5.2), (5.3) imply that for $\alpha \in \mathbb{R}$ and $b \in C^\infty(S_+^{n-1})$,

$$\Delta_g\big(r^\alpha b(\theta)\big) = r^{\alpha-2}\big[\Delta_{S^{n-1}} b(\theta) + \alpha(\alpha+n-2)b(\theta)\big] + \mathrm{O}\big(r^{\alpha-1}\big).$$

Let $a_0(\theta) = \theta_n$, then using (5.2), (5.3) we get

$$\Delta_g\big(r^{1-n} a_0(\theta)\big) = r^{-n}\big[-H(0) - nH(0)\theta_n^2 + 2n(n+2)h_{ij}(0)\theta_i\theta_j\theta_n^2\big] + \mathrm{O}\big(r^{-n+1}\big).$$

Assume for $1 \leqslant k \leqslant n-1$, we have found $a_i \in C^\infty(S_+^{n-1})$, vanishing on ∂S_+^{n-1} for $0 \leqslant i \leqslant k-1$ with

$$\Delta_g\left(r^{1-n}\sum_{i=0}^{k-1} a_i(\theta) r^i\right) = r^{k-1-n} b_{k-1}(\theta) + \mathrm{O}\big(r^{k-n}\big),$$

then may solve the Dirichlet problem

$$\begin{cases} -\Delta_{S^{n-1}} a_k + (k-1)(n-k-1)a_k(\theta) = b_{k-1}(\theta) & \text{on } S_+^{n-1}, \\ a_k|_{\partial S_+^{n-1}} = 0. \end{cases}$$

This is possible because $(k-1)(n-k-1) \geqslant 0$. Then

$$\Delta_g\left(r^{1-n}\sum_{i=0}^{k} a_i(\theta) r^i\right) = \mathrm{O}\big(r^{k-n}\big) = r^{k-n} b_k(\theta) + \mathrm{O}\big(r^{k+1-n}\big).$$

Hence by induction we may find a_i for $0 \leqslant i \leqslant n-1$ such that

$$\Delta_g\left(r^{1-n}\sum_{i=0}^{n-1} a_i(\theta) r^i\right) = \mathrm{O}\big(r^{-1}\big).$$

Fix a $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $|x| \leqslant \frac{\delta}{4}$ and $\eta(x) = 0$ for $|x| \geqslant \frac{\delta}{2}$. Let $u = \frac{2}{n\omega_n}\eta \cdot r^{1-n}\sum_{i=0}^{n-1} a_i(\theta) r^i$, then $\Delta_g u = \mathrm{O}(r^{-1})$. We solve

$$\begin{cases} -\Delta_g \psi = \Delta_g u & \text{on } M, \\ \psi|_{\partial M} = 0 \end{cases}$$

to find $\psi \in W^{2,n-\varepsilon}(M)$ for all $\varepsilon > 0$. In particular, $\psi \in C^{1,1-\varepsilon}(M)$ for all $\varepsilon > 0$ and the Poisson kernel $P(x,0) = \frac{2}{n\omega_n}\eta \cdot r^{1-n}\sum_{i=0}^{n-1} a_i(\theta) r^i + \psi(x)$.

An almost identical argument gives us similar results for the Poisson kernel of the conformal Laplacian operator.

Proposition 5.1. *Under the same set up as in Theorem* 5.1. *If $n \geqslant 3$ and $\lambda_1(L_g) > 0$, we may find $a_i \in C^\infty(S_+^{n-1})$ with $a_i|_{\partial S_+^{n-1}} = 0$ for $0 \leqslant i \leqslant n-1$ and a $\psi \in C^{1,1-\varepsilon}(M)$ (for all $\varepsilon > 0$) such that*

$$P_L(x,0) = \frac{2}{n\omega_n} r^{1-n}\sum_{i=0}^{n-1} r^i a_i(\theta) + \psi(x) \quad \textit{for } x \textit{ near } 0.$$

Moreover $a_0(\theta) = \theta_n$ and a_1 is determined by

$$\begin{cases} -\Delta_{S^{n-1}} a_1 = -H(0) - nH(0)\theta_n^2 + 2n(n+2)h_{ij}(0)\theta_i\theta_j\theta_n^2 & \textit{on } S_+^{n-1}, \\ a_1|_{\partial S_+^{n-1}} = 0. \end{cases}$$

6. A criterion for the existence of maximizers

We first recall some notations from [7]. For $x \in \mathbb{R}_+^n$, $\xi \in \mathbb{R}^{n-1}$, the Poisson kernel of the upper half space is

$$P(x,\xi) = \frac{2}{n\omega_n}\frac{x_n}{(|x'-\xi|^2 + x_n^2)^{n/2}}.$$

Here $x = (x', x_n)$. For a function f defined on $\mathbb{R}^{n-1}$, $(Pf)(x) = \int_{\mathbb{R}^{n-1}} P(x,\xi) f(\xi)\, d\xi$. For $1 < p < \infty$, $|Pf|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)} \leqslant c_{n,p} |f|_{L^p(\mathbb{R}^{n-1})}$, here

$$c_{n,p} = \sup\bigl\{|Pf|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}:\ f \in L^p(\mathbb{R}^{n-1}),\ |f|_{L^p(\mathbb{R}^{n-1})} = 1\bigr\}.$$

Theorem 6.1. *Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$, $1 < p < \infty$. Denote*

$$c_{M,g,p} = \sup\bigl\{|Pf|_{L^{\frac{np}{n-1}}(M)}:\ f \in L^p(\Sigma),\ |f|_{L^p(\Sigma)} = 1\bigr\}.$$

Then $c_{M,g,p} \geqslant c_{n,p}$. Any maximizer of the problem must be smooth and either strictly positive or strictly negative. Strictly positive maximizers satisfy the equation

$$\int_M P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\, d\mu(x) = c_{M,g,p}^{\frac{np}{n-1}} f(\xi)^{p-1}.$$

Moreover if $c_{M,g,p} > c_{n,p}$, then $c_{M,g,p}$ is achieved. Indeed any maximizing sequence has a convergent subsequence in $L^p(\Sigma)$.

We use the same notations as in Section 2.1. An ingredient in proving Theorem 6.1 is the following ε-version inequality.

Lemma 6.1. *Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$, $1 < p < \infty$. Then for any $\varepsilon > 0$ small, there exists a $\delta = \delta(M, g, p, \varepsilon) > 0$ such that for every $f \in L^p(\Sigma)$,*

$$|Pf|_{L^{\frac{np}{n-1}}(M_\delta)} \leqslant (c_{n,p} + \varepsilon)|f|_{L^p(\Sigma)}.$$

To prove the lemma, we will need the following estimates.

Lemma 6.2. *Assume $0 \leqslant \alpha < n-1$, $1 < p < \infty$, then*

$$\left|\int_\Sigma \frac{f(\xi)}{d(x,\xi)^\alpha}\, dS(\xi)\right|_{L^{\frac{np}{\alpha}}(M)} \leqslant c(M, g, \alpha, p)|f|_{L^p(\Sigma)}.$$

Proof. We may assume $\alpha > 0$. For $\varepsilon > 0$ small enough, we let $q_0 = \frac{n}{\alpha}(1-\varepsilon)$, $q_1 = 1 + \frac{\varepsilon}{p-1}$, then $\frac{1}{p} + \frac{1}{q_1} = \frac{q_0}{q_1 \cdot np/\alpha} + 1$. The needed inequality follows from Lemma 2.3. □

Corollary 6.1. *Assume $\eta \in \mathrm{Lip}(\Sigma)$, $1 < p < \infty$, then*

$$\bigl|\eta \circ \pi \cdot Pf - P(\eta f)\bigr|_{L^{\frac{np}{n-2}}(M_{\delta_0})} \leqslant c(M, g, p)|\nabla_\Sigma \eta|_{L^\infty(\Sigma)}|f|_{L^p(\Sigma)}.$$

Proof. It follows from Lemma 2.2 that

$$\begin{aligned}\bigl|\eta\bigl(\pi(x)\bigr)(Pf)(x) - P(\eta f)(x)\bigr| &= \left|\int_\Sigma \bigl(\eta\bigl(\pi(x)\bigr) - \eta(\xi)\bigr)P(x,\xi) f(\xi)\, dS(\xi)\right| \\ &\leqslant c(M,g)|\nabla_\Sigma \eta|_{L^\infty(\Sigma)} \int_\Sigma \frac{|f(\xi)|}{d(x,\xi)^{n-2}}\, dS(\xi).\end{aligned}$$

Then the conclusion follows from Lemma 6.2. □

Corollary 6.2. *Let $K(x,\xi) = \frac{2}{n\omega_n} \frac{t(x)}{[t(x)^2 + d_\Sigma(\pi(x),\xi)^2]^{\frac{n}{2}}}$ for $x \in M_{\delta_0}$ and $\xi \in \Sigma$, $(Kf)(x) = \int_\Sigma K(x,\xi) f(\xi)\, dS(\xi)$, $1 < p < \infty$, then*

$$|Pf - Kf|_{L^{\frac{np}{n-2}}(M_{\delta_0})} \leqslant c(M, g, p)|f|_{L^p(\Sigma)}.$$

This follows from Theorem 5.1 and Lemma 6.2.

Proof of Lemma 6.1. Without losing of generality we may assume $f \geqslant 0$. For $\delta_1 > 0$ small, we may find $\eta_i \in C^\infty(\Sigma, \mathbb{R})$ for $1 \leqslant i \leqslant m$ such that $0 \leqslant \eta_i \leqslant 1$, $\sum_{i=1}^m \eta_i = 1$, $\eta_i^{1/p} \in C^\infty(\Sigma, \mathbb{R})$ and for each i, there exists a point $\xi_i \in \Sigma$ such that $\eta_i(\xi) = 0$ for $\xi \in \Sigma$ with $d_\Sigma(\xi, \xi_i) \geqslant \delta_1$. For $0 < \delta < \delta_1$, we denote

$$C_{i,\delta} = \bigl\{x \in M_\delta\colon\ d_\Sigma\bigl(\pi(x), \xi_i\bigr) \leqslant \delta_1\bigr\}.$$

Then

$$\begin{aligned}
|Pf|^p_{L^{\frac{np}{n-1}}(M_\delta)} &= \bigl|(Pf)^p\bigr|_{L^{\frac{n}{n-1}}(M_\delta)} = \left|\sum_{i=1}^m \eta_i \circ \pi \cdot (Pf)^p\right|_{L^{\frac{n}{n-1}}(M_\delta)} \\
&\leqslant \sum_{i=1}^m \bigl|\eta_i \circ \pi \cdot (Pf)^p\bigr|_{L^{\frac{n}{n-1}}(C_{i,\delta})} = \sum_{i=1}^m \bigl|\eta_i^{1/p} \circ \pi \cdot Pf\bigr|^p_{L^{\frac{np}{n-1}}(C_{i,\delta})}.
\end{aligned}$$

On the other hand, using Corollary 6.1 we see

$$\begin{aligned}
\bigl|\eta_i^{1/p} \circ \pi \cdot Pf\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} &\leqslant \bigl|P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} + \bigl|\eta_i^{1/p} \circ \pi \cdot Pf - P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} \\
&\leqslant \bigl|P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} + \bigl|\eta_i^{1/p} \circ \pi \cdot Pf - P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-2}}(C_{i,\delta})} |C_{i,\delta}|^{\frac{1}{np}} \\
&\leqslant \bigl|P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} + c(M, g, p, \delta_1)\delta^{\frac{1}{np}} |f|_{L^p(\Sigma)}.
\end{aligned}$$

Similarly, by Corollary 6.2 we have

$$\begin{aligned}
\bigl|P\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} &\leqslant \bigl|K\bigl(\eta_i^{1/p} f\bigr)\bigr|_{L^{\frac{np}{n-1}}(C_{i,\delta})} + c(M, g, p)\delta^{\frac{1}{np}} |f|_{L^p(\Sigma)} \\
&\leqslant c_{n,p}(1 + \varepsilon_1)\bigl|\eta_i^{1/p} f\bigr|_{L^p(\Sigma)} + c(M, g, p)\delta^{\frac{1}{np}} |f|_{L^p(\Sigma)}.
\end{aligned}$$

Here $\varepsilon_1 = \varepsilon_1(M, g, p, \delta_1)$ is a small number which tends to 0 when δ_1 tends to 0. Hence

$$\begin{aligned}
|Pf|^p_{L^{\frac{np}{n-1}}(M_\delta)} &\leqslant \sum_{i=1}^m \bigl[c_{n,p}(1 + \varepsilon_1)\bigl|\eta_i^{1/p} f\bigr|_{L^p(\Sigma)} + c(M, g, p, \delta_1)\delta^{\frac{1}{np}} |f|_{L^p(\Sigma)}\bigr]^p \\
&\leqslant \sum_{i=1}^m c^p_{n,p}(1 + 2\varepsilon_1)^p \int_\Sigma \eta_i f^p\, dS + c(M, g, p, \delta_1)\delta^{1/n} |f|^p_{L^p(\Sigma)} \\
&\leqslant c^p_{n,p}\left(1 + \frac{\varepsilon}{2}\right)^p |f|^p_{L^p(\Sigma)} + c(M, g, p, \varepsilon)\delta^{1/n} |f|^p_{L^p(\Sigma)} \\
&\leqslant c^p_{n,p}(1 + \varepsilon)^p |f|^p_{L^p(\Sigma)}
\end{aligned}$$

if we first fix $\delta_1 = \delta_1(M, g, p, \varepsilon)$ small enough and then $\delta = \delta(M, g, p, \varepsilon)$ small enough. This implies $|Pf|_{L^{\frac{np}{n-1}}(M_\delta)} \leqslant c_{n,p}(1 + \varepsilon)|f|_{L^p(\Sigma)}$. $\square$

Next we prove the following concentration compactness lemma (compare with [10, Lemma 2.1] and [7, Proposition 3.1]).

Proposition 6.1 *(Concentration compactness lemma). Assume $n \geqslant 2$, (M^n, g) is a smooth compact Riemannian manifold with boundary $\Sigma = \partial M$, $1 < p < \infty$, $f_i \in L^p(\Sigma)$ such that $f_i \rightharpoonup f$ in $L^p(\Sigma)$. After passing to a subsequence assume*

$$|f_i|^p\, dS \rightharpoonup \sigma \quad \text{in } \mathcal{M}(\Sigma), \qquad |Pf_i|^{\frac{np}{n-1}}\, d\mu \rightharpoonup \nu \quad \text{in } \mathcal{M}(M).$$

Here $\mathcal{M}(\Sigma)$ is the space of all Radon measures on Σ. Then we have

- *$\nu|_{M\backslash\Sigma} = |Pf|^{\frac{np}{n-1}}\,d\mu$. Moreover for every Borel set $E \subset \Sigma$, $\nu(E)^{\frac{n-1}{np}} \leqslant c_{n,p}\sigma(E)^{\frac{1}{p}}$.*
- *There exists a countable set of points $\zeta_j \in \Sigma$ such that $\nu = |Pf|^{\frac{np}{n-1}}\,d\mu + \sum_j \nu_j\delta_{\zeta_j}$, $\sigma \geqslant |f|^p\,dS + \sum_j \sigma_j\delta_{\zeta_j}$, here $\sigma_j = \sigma(\{\zeta_j\})$ and $\nu_j^{\frac{n-1}{np}} \leqslant c_{n,p}\sigma_j^{\frac{1}{p}}$.*

Proof. Without losing of generality we may assume $|f_i|_{L^p(\Sigma)} \leqslant 1$. Since $|(Pf_i)(x)| \leqslant c(M,g,p)t(x)^{-\frac{n-1}{p}}$ for $x \in M\backslash\Sigma$, it follows from the elliptic estimates of harmonic functions that $Pf_i \to Pf$ in $C^\infty_{\mathrm{loc}}(M\backslash\Sigma)$. In particular, $\nu|_{M\backslash\Sigma} = |Pf|^{\frac{np}{n-1}}\,d\mu$. For $\varepsilon > 0$ small, it follows from Lemma 6.1 and Corollary 6.1 that for $\varphi \in C^\infty(\Sigma)$ and $\delta > 0$ small enough,

$$\begin{aligned}|\varphi\circ\pi\cdot Pf_i|_{L^{\frac{np}{n-1}}(M_\delta)} &\leqslant \left|P(\varphi f_i)\right|_{L^{\frac{np}{n-1}}(M_\delta)} + \left|\varphi\circ\pi\cdot Pf_i - P(\varphi f_i)\right|_{L^{\frac{np}{n-1}}(M_\delta)}\\ &\leqslant (c_{n,p}+\varepsilon)|\varphi f_i|_{L^p(\Sigma)} + c(M,g,p)\delta^{\frac{1}{np}}|\nabla_\Sigma\varphi|_{L^\infty(\Sigma)}.\end{aligned}$$

Let $i \to \infty$ we see

$$\left(\int_\Sigma |\varphi|^{\frac{np}{n-1}}\,d\nu\right)^{\frac{n-1}{np}} \leqslant (c_{n,p}+\varepsilon)\left(\int_\Sigma |\varphi|^p\,d\sigma\right)^{\frac{1}{p}} + c(M,g,p)\delta^{\frac{1}{np}}|\nabla_\Sigma\varphi|_{L^\infty(\Sigma)}.$$

Let $\delta \to 0^+$ and then $\varepsilon \to 0^+$, we get

$$\left(\int_\Sigma |\varphi|^{\frac{np}{n-1}}\,d\nu\right)^{\frac{n-1}{np}} \leqslant c_{n,p}\left(\int_\Sigma |\varphi|^p\,d\sigma\right)^{\frac{1}{p}}.$$

A limit process shows for every nonnegative Borel function h on Σ,

$$\left(\int_\Sigma h^{\frac{np}{n-1}}\,d\nu\right)^{\frac{n-1}{np}} \leqslant c_{n,p}\left(\int_\Sigma h^p\,d\sigma\right)^{\frac{1}{p}}.$$

In particular, for every Borel set $E \subset \Sigma$, $\nu(E)^{\frac{n-1}{np}} \leqslant c_{n,p}\sigma(E)^{\frac{1}{p}}$. Based on this inequality we may proceed as in the proof of [7, Proposition 3.1] to get the second conclusion. □

Now we are ready to derive Theorem 6.1.

Proof of Theorem 6.1. First we want to show $c_{M,g,p} \geqslant c_{n,p}$ is always true. To see this we may fix a point $\xi_0 \in \Sigma$, choose a normal coordinate for Σ at ξ_0, namely $\tau_1, \ldots, \tau_{n-1}$. For $\delta > 0$ small, we denote $C_\delta = \{x \in M_\delta: d_\Sigma(\pi(x), \xi_0) \leqslant \delta\}$, then we have a natural coordinate near ξ_0 for M as

$$\phi : C_\delta \to \overline{B}^{n-1}_\delta \times [0,\delta] : x \mapsto \left(\tau\left(\pi(x)\right), t(x)\right).$$

We will identify C_δ with $\overline{B}^{n-1}_\delta \times [0,\delta]$ through ϕ. On C_δ we have the Euclidean metric $g_0 = \sum_{i=1}^n dx_i \otimes dx_i$. If $\bar f \in L^p(\Sigma)\backslash\{0\}$ and $\bar f$ vanishes outside $\overline{B}^{n-1}_\delta$, then it follows from Corollary 6.2 that

$$|K\bar f|_{L^{\frac{np}{n-1}}(C_\delta,g)} \leqslant |P\bar f|_{L^{\frac{np}{n-1}}(C_\delta,g)} + c(M,g,p)\delta^{\frac{1}{p}}|\bar f|_{L^p(\Sigma)}.$$

Let $f(\xi) = \bar f(\xi)$ for $|\xi| \leqslant \delta$ and $f(\xi) = 0$ for $|\xi| > \delta$, $\xi \in \mathbb{R}^{n-1}$, and u be the harmonic extension of f to $\mathbb{R}^n_+$, then

$$\begin{aligned}|u|_{L^{\frac{np}{n-1}}(C_\delta,g_0)} &\leqslant (1+\varepsilon_1)|K\bar f|_{L^{\frac{np}{n-1}}(C_\delta,g)}\\ &\leqslant (1+\varepsilon_1)|P\bar f|_{L^{\frac{np}{n-1}}(C_\delta,g)} + c(M,g,p)\delta^{\frac{1}{p}}|\bar f|_{L^p(\Sigma)}.\end{aligned}$$

Here $\varepsilon_1 = \varepsilon_1(M,g,p,\delta)$ and $\varepsilon_1 \to 0^+$ as $\delta \to 0^+$. Hence

$$c_{M,g,p} \geqslant \frac{|P\bar{f}|_{L^{\frac{np}{n-1}}(M)}}{|\bar{f}|_{L^p(\Sigma)}} \geqslant \frac{|P\bar{f}|_{L^{\frac{np}{n-1}}(C_\delta,g)}}{|\bar{f}|_{L^p(B_\delta^{n-1},g)}}$$
$$\geqslant \frac{1}{(1+\varepsilon_1)^2} \frac{|u|_{L^{\frac{np}{n-1}}(C_\delta,g_0)}}{|f|_{L^p(B_\delta^{n-1},g_0)}} - c(M,g,p)\delta^{\frac{1}{p}}.$$

Assume $f \in L^p(\mathbb{R}^{n-1})\backslash\{0\}$ and $f = 0$ outside a ball, u is the harmonic extension of f to $\mathbb{R}^n_+$, then for $\varepsilon > 0$ small enough, we write $f_\varepsilon(\xi) = \varepsilon^{-\frac{n-1}{p}} f(\frac{\xi}{\varepsilon})$ and $u_\varepsilon(x) = \varepsilon^{-\frac{n-1}{p}} u(\frac{x}{\varepsilon})$. Let $\bar{f} = f_\varepsilon$ on B_δ^{n-1} and 0 on $\Sigma\backslash B_\delta^{n-1}$, then we get

$$c_{M,g,p} \geqslant \frac{1}{(1+\varepsilon_1)^2} \frac{|u_\varepsilon|_{L^{\frac{np}{n-1}}(C_\delta,g_0)}}{|f_\varepsilon|_{L^p(B_\delta^{n-1},g_0)}} - c(M,g,p)\delta^{\frac{1}{p}}.$$

Let $\varepsilon \to 0^+$ then $\delta \to 0^+$, we see

$$c_{M,g,p} \geqslant \frac{|u|_{L^{\frac{np}{n-1}}(\mathbb{R}^n_+)}}{|f|_{L^p(\mathbb{R}^{n-1})}}.$$

By approximation we know the inequality remains true for all $f \in L^p(\mathbb{R}^{n-1})\backslash\{0\}$ and this implies $c_{M,g,p} \geqslant c_{n,p}$.

If f is a maximizer, then it is clear that f will be either nonnegative or nonpositive. Assume $f \geqslant 0$, then it satisfies the Euler–Lagrange equation

$$\int_M P(x,\xi)(Pf)(x)^{\frac{np}{n-1}-1}\, d\mu(x) = c_{M,g,p}^{\frac{np}{n-1}} f(\xi)^{p-1}.$$

It follows from Proposition 4.1 that f must be smooth and hence it is strictly positive.

Assume $c_{M,g,p} > c_{n,p}$. Let $f_i \in L^p(\Sigma)$ be a sequence of functions with $|f_i|_{L^p(\Sigma)} = 1$ and $|Pf_i|_{L^{\frac{np}{n-1}}(M)} \to c_{M,g,p}$. After passing to a subsequence we may assume $f_i \rightharpoonup f$ in $L^p(\Sigma)$, $|f_i|^p\, dS \rightharpoonup \sigma$ in $\mathcal{M}(\Sigma)$ and $|Pf_i|^{\frac{np}{n-1}}\, d\mu \rightharpoonup \nu$ in $\mathcal{M}(M)$. It follows from Proposition 6.1 that we may find a countable set of points $\zeta_j \in \Sigma$ such that $\nu = |Pf|^{\frac{np}{n-1}}\, d\mu + \sum_j \nu_j \delta_{\zeta_j}$ and $\sigma \geqslant |f|^p\, dS + \sum_j \sigma_j \delta_{\zeta_j}$. Here $\sigma_j = \sigma(\{\zeta_j\})$ and $\nu_j^{\frac{n-1}{n}} \leqslant c_{n,p}^p \sigma_j$. In particular $1 = \sigma(\Sigma) \geqslant |f|^p_{L^p(\Sigma)} + \sum_j \sigma_j$. We claim $\nu_j = 0$ for all j. If this is not the case, then

$$c_{M,g,p}^{\frac{np}{n-1}} = \nu(M) = |Pf|^{\frac{np}{n-1}}_{L^{\frac{np}{n-1}}(M)} + \sum_j \nu_j \leqslant c_{M,g,p}^{\frac{np}{n-1}} |f|^{\frac{np}{n-1}}_{L^p(\Sigma)} + \sum_j \nu_j.$$

Hence

$$c_{M,g,p}^p \leqslant c_{M,g,p}^p |f|^p_{L^p(\Sigma)} + \sum_j \nu_j^{\frac{n-1}{n}} \leqslant c_{M,g,p}^p |f|^p_{L^p(\Sigma)} + c_{n,p}^p \sum_j \sigma_j$$
$$< c_{M,g,p}^p |f|^p_{L^p(\Sigma)} + c_{M,g,p}^p \sum_j \sigma_j.$$

This implies $1 < |f|^p_{L^p(\Sigma)} + \sum_j \sigma_j$, a contradiction. Since $\nu_j = 0$ for all j, we see $|Pf|_{L^{\frac{np}{n-1}}(M)} = c_{M,g,p}$. Hence $|f|_{L^p(\Sigma)} \geqslant 1$. This implies $f_i \to f$ in $L^p(\Sigma)$. That is every maximizing sequence has a convergent subsequence in $L^p(\Sigma)$ and $c_{M,g,p}$ is achieved. □

7. Proof of the Theorem 1.1

In this section we finish the proof of Theorem 1.1. Without losing of generality we may assume $R = 0$. It follows from Theorems 3.1, 6.1 and [7, Theorem 1.1] that

$$\Theta_{M,g} = c_{M,g,\frac{2(n-1)}{n-2}}^{\frac{2}{n-2}} \geqslant c_{n,\frac{2(n-1)}{n-2}}^{\frac{2}{n-2}} = n^{-\frac{1}{n-1}} \omega_n^{-\frac{1}{n(n-1)}} = \Theta_{\overline{B}_1, g_{\mathbb{R}^n}}.$$

On the other hand, if $\Theta_{M,g} > \Theta_{\overline{B}_1, g_{\mathbb{R}^n}}$, then $c_{M,g,\frac{2(n-1)}{n-2}} > c_{n,\frac{2(n-1)}{n-2}}$. It follows from Theorem 6.1 that we may find a $f \in C^\infty(\Sigma)$ with $f > 0$ such that $|f|_{L^{\frac{2(n-1)}{n-2}}(\Sigma)} = 1$ and $c_{M,g,\frac{2(n-1)}{n-2}} = |Pf|_{L^{\frac{2n}{n-2}}(M)}$. Let $\tilde{g} = (Pf)^{\frac{4}{n-2}} g$, then clearly $\widetilde{R} = 0$ and $I(M, \tilde{g}) = \Theta_{M,g}$.

Acknowledgements

The research of F. Hang is supported by National Science Foundation Grant DMS-0647010 and a Sloan Research Fellowship. The research of X. Wang is supported by National Science Foundation Grant DMS-0505645. The research of X. Yan is supported by National Science Foundation Grant DMS-0401048. We would like to thank Professor R. Mazzeo and Professor E. Stein for some helpful discussions. We also thank the referee for his/her careful reading of the original manuscript.

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