Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 59–80
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Asymptotic analysis, in a thin multidomain, of minimizing maps with values in S^2

Antonio Gaudiello [a], Rejeb Hadiji [b,*]

[a] DAEIMI, Università degli Studi di Cassino, via G. Di Biasio 43, 03043 Cassino (FR), Italia
[b] Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, UFR des Sciences et Technologie, 61, Avenue du Général de Gaulle, Bât. P3, 4e étage, 94010 Créteil Cedex, France

Received 29 January 2007; received in revised form 14 June 2007; accepted 21 June 2007

Available online 17 October 2007

Abstract

We consider a thin multidomain of $\mathbb{R}^3$ consisting of two vertical cylinders, one placed upon the other: the first one with given height and small cross section, the second one with small thickness and given cross section. The first part of this paper is devoted to analyze, in this thin multidomain, a "static Landau–Lifshitz equation", when the volumes of the two cylinders vanish. We derive the limit problem, which decomposes into two uncoupled problems, well posed on the limit cylinders (with dimensions 1 and 2, respectively). We precise how the limit problem depends on limit of the ratio between the volumes of the two cylinders. In the second part of this paper, we study the asymptotic behavior of the two limit problems, when the exterior limit fields increase. We show that in some cases, contrary to the initial problem, the energies of the limit problems diverge and we find the order of these energies.

Résumé

Nous considérons un multi-domaine mince de $\mathbb{R}^3$ se composant de deux cylindres verticaux, superposés l'un sur l'autre : le premier possède une taille donnée et une petite section transversale, le second a une petite épaisseur et une section transversale donnée. La première partie de cet article est consacrée à analyser, dans ce multi-domaine, une équation stationnaire de type Landau–Lifshitz, quand les volumes des deux cylindres tendent vers 0. Nous montrons que le problème limite, se décompose en deux probèmes découplés, bien posés sur le domaine limite. Ensuite, nous précisons comment le problème limite dépend de la limite du rapport des volumes des deux cylindres. Dans la deuxième partie de cet article, nous étudions le comportement asymptotique des deux problèmes limites, quand les champs extérieurs limites augmentent. Nous prouvons que dans certains cas, contrairement au problème initial, les énergies des problèmes limites divergent et nous précisons l'ordre de ces énergies.

MSC: 78A25; 74K05; 74K30; 74K35; 35B25

Keywords: Maps with values in S^2; Thin multidomains; Dimension reduction; Singular perturbations

* Corresponding author.
E-mail addresses: gaudiell@unina.it (A. Gaudiello), hadiji@univ-paris12.fr (R. Hadiji).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.06.002

1. Introduction

This paper is devoted to an asymptotic analysis, in a thin multidomain of $\mathbb{R}^3$, of minimizing maps with values in S^2. Precisely, let $\Omega_n \subset \mathbb{R}^3$, $n \in \mathbb{N}$, be a thin multidomain consisting of two vertical cylinders, one placed upon the other: the first one with constant height 1 and small cross section $r_n\Theta$, the second one with small thickness h_n and constant cross section Θ, where r_n and h_n are two small parameters converging to zero (see Fig. 1). By denoting $H^1(D, S^2) = \{v \in H^1(D, \mathbb{R}^3), |v| = 1 \text{ a.e. in } D\}$ for an open subset $D \subset \mathbb{R}^N$ ($N = 1, 2, 3$), we consider the following minimization problem:

$$\min\left\{ \int_{\Omega_n} \left[\left|DV(x_1, x_2, x_3)\right|^2 - 2V(x_1, x_2, x_3)F_n(x_1, x_2, x_3) \right] d(x_1, x_2, x_3):\ V \in H^1(\Omega_n, S^2) \right\}, \tag{1.1}$$

where $F_n \in L^2(\Omega_n, \mathbb{R}^3)$. Problem (1.1) describes the classical 3d system for the static isotropic Heisenberg model (see [25]), where V is the spin-density with finite magnitude and F_n an external magnetic field. The Euler system associated to problem (1.1) is

$$\Delta V + |DV|^2 V + F_n - \langle V, F_n\rangle V = 0,$$

which is equivalent to the time independent spin equation of motion (see [19]). The time dependent spin equation of motion was first derived by Landau and Lifshitz (see [22]) and it plays a fundamental role in the understanding of nonequilibrium magnetism. See [17] and [19] about links between harmonic maps and the Landau–Lifshitz equation of the spin chain.

The first part of our paper is devoted to study the asymptotic behavior of problem (1.1), when $r_n \to 0$ and $h_n \to 0$, as $n \to +\infty$ (see Section 2). After having reformulated the problem on a fixed domain through appropriate rescaling of the kind proposed by P.G. Ciarlet and P. Destuynder in [5] and having imposed appropriate convergence assumptions on the rescaled exterior fields, we derive the limit problem which depends on the limit of the ratio between the volumes of the two cylinders (see Subsection 2.1). More precisely, if these two volumes vanish with same rate, i.e. $h_n \simeq r_n^2$, the limit problem decomposes into two uncoupled problems, well posed on the limit cylinders, with dimensions 1 and 2, respectively:

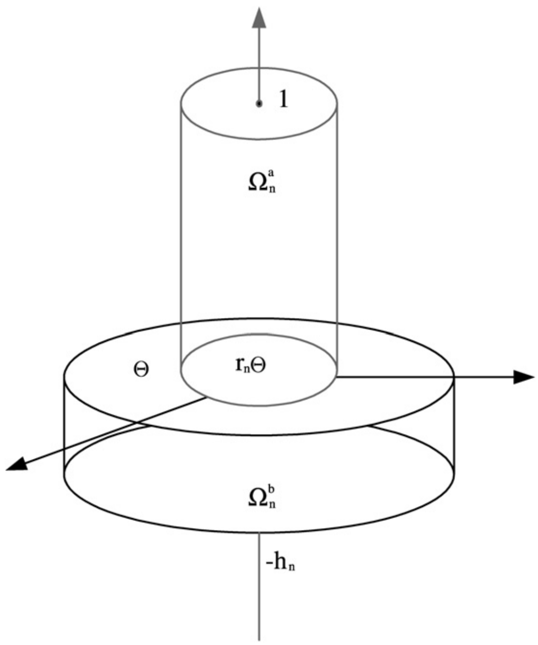

Fig. 1.

$$\min\left\{|\Theta|\int_0^1 \left|w'(x_3)\right|^2 dx_3 - 2\int_0^1\left(\int_\Theta f^a(x_1,x_2,x_3)\,d(x_1,x_2)\right)w(x_3)\,dx_3\colon\ w\in H^1\big(]0,1[,S^2\big)\right\},\tag{1.2}$$

$$\min\left\{\int_\Theta \left|D\zeta(x_1,x_2)\right|^2 d(x_1,x_2) - 2\int_\Theta\left(\int_{-1}^0 f^b(x_1,x_2,x_3)\,dx_3\right)\zeta(x_1,x_2)\,d(x_1,x_2)\colon\ \zeta\in H^1(\Theta,S^2)\right\},\tag{1.3}$$

where f^a and f^b are the L^2-weak limits of the rescaled exterior fields in the upper cylinder and in the lower cylinder, respectively (see (2.5) and (2.10) in Section 2); and w' stands for the derivative of w. If $h_n \ll r_n^2$, the limit problem reduces to problem (1.2). If $h_n \gg r_n^2$, the limit problem reduces to Problem (1.3). In all cases, strong convergences in H^1-norm are obtained for the rescaled minimizers.

The proofs of these results make use of the main ideas of Γ-convergence method introduced by E. De Giorgi (see [9]) and they develop in several steps: a priori estimates, construction of the recovery sequence, density results and l.s.c arguments (see Subsection 2.2). The main difficulty with respect to [10], where the asymptotic behavior of the Laplacian is studied when $h_n \simeq r_n^2$, arises from the fact that the set of the admissible vector valued functions of problem (1.1) is not a convex set, due to the constraint $|V((x_1,x_2,x_3))| = 1$. This difficulty is overcome by working with a projection from $\mathbb{R}^3$ into $S^2 = \{(x_1,x_2,x_3)\in\mathbb{R}^3\colon\ |(x_1,x_2,x_3)| = 1\}$, introduced in [3] (see also [1]), and by using the Sard's Lemma. Moreover, point out that the cases $h_n \ll r_n^2$ and $h_n \gg r_n^2$ are not treated in [10].

Remark that it is not necessary that the two cylinders are scaled to the same one or that the first cylinder has height 1. In fact, the results do not essentially change if one assumes $\Omega_n = (r_n\Theta_a \times [0,l[) \cup (\Theta_b \times]-h_n,0[)$, with $\Theta^a, \Theta^b \subset \mathbb{R}^2$, $0' \in \Theta^b$ and $l \in]0,+\infty[$.

In the second part of this paper (see Section 3), we consider the following problem:

$$\min\left\{\int_{\Omega_n}\left[|DV(x_1,x_2,x_3)|^2 + \lambda|V(x_1,x_2,x_3) - F_n(x_1,x_2,x_3)|^2\right]d(x_1,x_2,x_3)\colon\ V\in H^1(\Omega_n,S^2)\right\},\tag{1.4}$$

where $F_n : \Omega_n \to \mathbb{R}^3$ is a measurable function such that $|F_n((x_1,x_2,x_3))| = 1$ a.e. in Ω_n and $\lambda \geqslant 0$. Remark that problem (1.4) reduces to problem (1.1), up to the additive constant: $2|\Omega_n|\lambda$. Consequently, for λ fixed, by passing to the limit as $n \to +\infty$, one obtains limit problems (1.2) and (1.3), up to the additive constant: $2|\Theta|\lambda$. If we assume that $|f^a| = 1$, f^a is independent of (x_1,x_2), $|f^b| = 1$ and f^b is independent of x_3, then the limit problems can be rewritten as follows:

$$\min\left\{|\Theta|\int_0^1\left[\left|w'(x_3)\right|^2 + \lambda\left|w(x_3) - f^a(x_3)\right|^2\right]dx_3\colon\ w\in H^1\big(]0,1[,S^2\big)\right\},\tag{1.5}$$

$$\min\left\{\int_\Theta\left[\left|D\zeta(x_1,x_2)\right|^2 + \lambda\left|\zeta(x_1,x_2) - f^b(x_1,x_2)\right|^2\right]d(x_1,x_2)\colon\ \zeta\in H^1(\Theta,S^2)\right\}.\tag{1.6}$$

Note that, since smooth maps are dense in $H^1(\Theta,S^2)$ and in $H^1(]0,1[,S^2)$ (see [3]), the infimum in (1.5) (resp. (1.6)) does not change if we replace $H^1(\Theta,S^2)$ (resp. $H^1(]0,1[,S^2)$) by $C^1(\Theta,S^2)$ (resp. $C^1(]0,1[,S^2)$). This property does not hold true for initial Problem (1.4) (for instance, see [18]).

The second part of the paper is devoted to study the asymptotic behavior of problems (1.5) and (1.6), as $\lambda \to +\infty$, that is when the exterior limit field increases. The interesting cases occur when $f^a \notin H^1(]0,1[)$ or $f^b \notin H^1(\Theta)$, otherwise the asymptotic analysis is trivial. We examine some cases (see Subsection 3.1). For instance, if $F_n = \frac{x}{|x|}$ in (1.4), one obtains (1.5) and (1.6) with $f^a = (0,0,1)$ and $f^b = \frac{1}{|(x_1,x_2)|}(x_1,x_2,0)$, respectively (see (2.10) in Section 2). Remark that $f^b \notin H^1(\Theta)$, although $\frac{x}{|x|} \in H^1_{\mathrm{loc}}(\mathbb{R}^3,S^2)$. In this case, energy (1.6) diverges, as $\lambda \to +\infty$. By adapting some results proved by F. Bethuel, H. Brezis and F. Hélein in [2], we show that $\pi\log\lambda + c$ is an upper bound of energy (1.6), for λ large enough. It provides that every sequence of minimizers of problem (1.6) converges to $\frac{1}{|(x_1,x_2)|}(x_1,x_2,0)$ strongly in $L^2(\Theta)$, as $\lambda \to +\infty$. Moreover, with a technique introduced in [26] in the case of the Ginzburg–Landau energy, we prove that $\liminf_{\lambda\to+\infty}\int_\Theta \lambda|\zeta_\lambda(x_1,x_2) - f^b(x_1,x_2)|^2 d(x_1,x_2) < +\infty$, where ζ_λ solves (1.6). This result allows us to obtain, by an integration by parts, the existence of a diverging sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ for which corresponding energy (1.6) is bounded from below by $\pi\log\lambda_k - c$.

By choosing $F_n(x_1,x_2,x_3) = \frac{1}{|(x_1,x_2,x_3-\gamma)|}(x_1,x_2,x_3-\gamma)$, with $\gamma \in]0,1[$, in (1.4), one obtains (1.5) and (1.6) with $f^a(x_3) = (0,0,\frac{x_3-\gamma}{|x_3-\gamma|})$ and $f^b(x_1,x_2) = \frac{1}{|(x_1,x_2,-\gamma)|}(x_1,x_2,-\gamma)$, respectively (see (2.10) in Section 2). Remark that $F_n \in H^1_{\mathrm{loc}}(\mathbb{R}^3,S^2)$, $f^b \in H^1(\Theta,S^2)$, but $f^a \notin H^1(]0,1[)$. In this case, by using suitable test functions, we derive the upper bound $|\Theta|2\sqrt{2\pi}\sqrt{\lambda}$ of energy (1.5). It provides that every sequence of minimizers of problem (1.5) converges to $(0,0,\frac{x_3-\gamma}{|x_3-\gamma|})$ strongly in $L^2(]0,1[)$, as $\lambda \to +\infty$. Moreover, by virtue of an auxiliary scalar problem, we obtain the lower bounds $|\Theta|(2-\varepsilon)\sqrt{\lambda}$ of energy (1.5), for $\lambda > \lambda_\varepsilon$. The proofs of this results will be developed in Subsection 3.2.

For the study of thin structures and multi-structures we refer to [4,6,8,20,21,23,27] and the references quoted therein. For a thin multi-structure as considered in this paper, we refer to [10–14] and [16]. Precisely, the model, described in [10] and [11] through its integral energy, and in [12] through the related constitutive equations, is a quasilinear Neumann second order scalar problem. A fourth order problem is examined in [16]. The case of the linearized elasticity system in $\mathbb{R}^3$ is studied in [14]. The spectrum of a Laplacian Problem is considered in [15].

For n fixed, problem (1.4) is studied in [7] and in [18]. The authors show that any minimizer of (1.4) is regular if λ is small enough; while, if λ is large and F_n is not a strong limit of smooth maps in $H^1(\Omega_n,S^2)$ (for instance, this is the case when $F_n(x) = \frac{x}{|x|}$), then any minimizer of (1.4) possesses singularities. In this case, a minimizer of (1.4) is of the type: $R(\frac{x-x_0}{|x-x_0|})$, where R is a rotation, near each singularity x_0. It is also shown that any minimizer for (1.4) tends to F_n weakly in H^1, as λ tends to $+\infty$.

2. First part: derivation of the limit model

In the sequel, $x=(x_1,x_2,x_3)=(x',x_3)$ denotes the generic point of $\mathbb{R}^3$ and, $D_{x'}$ and D_{x_3} stand for the gradient with respect to the first 2 variables x_1,x_2 and for the derivative with respect to the last variable x_3, respectively.

Let $\Theta \subset \mathbb{R}^2$ be a bounded open connected set with smooth boundary such that the origin in $\mathbb{R}^2$, denoted by $0'$, belongs to Θ, and $\{r_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ be two sequences such that

$$\lim_n h_n = 0 = \lim_n r_n. \tag{2.1}$$

For every $n \in \mathbb{N}$, let $\Omega_n^a = r_n\Theta \times [0,1[$, $\Omega_n^b = \Theta \times]-h_n,0[$ and $\Omega_n = \Omega_n^a \cup \Omega_n^b$ (see Fig. 1).

For every $n \in \mathbb{N}$, let $F_n \in L^2(\Omega_n,\mathbb{R}^3)$ and

$$J_n : V \in H^1(\Omega_n,S^2) \longrightarrow \int_{\Omega_n} \left|DV(x)\right|^2 dx - 2\int_{\Omega_n} V(x)F_n(x)\,dx. \tag{2.2}$$

By applying the Direct Method of Calculus of Variations, for every $n \in \mathbb{N}$ there exists a solution $U_n \in H^1(\Omega_n,S^2)$ of the following problem:

$$J_n(U_n) = \min\left\{J_n(V):\ V \in H^1(\Omega_n,S^2)\right\}. \tag{2.3}$$

As it is usual (see [5]), problem (2.3) can be reformulated on a fixed domain through an appropriate rescaling which maps Ω_n into $\Omega = \Theta \times]-1,1[$. Namely, for every $n \in \mathbb{N}$ by setting

$$u_n(x) = \begin{cases} u_n^a(x',x_3) = U_n(r_nx',x_3), & (x',x_3) \text{ a.e. in } \Omega^a = \Theta \times]0,1[, \\ u_n^b(x',x_3) = U_n(x',h_nx_3), & (x',x_3) \text{ a.e. in } \Omega^b = \Theta \times]-1,0[, \end{cases} \tag{2.4}$$

$$f_n(x) = \begin{cases} f_n^a(x',x_3) = F_n(r_nx',x_3), & (x',x_3) \text{ a.e. in } \Omega^a = \Theta \times]0,1[, \\ f_n^b(x',x_3) = F_n(x',h_nx_3), & (x',x_3) \text{ a.e. in } \Omega^b = \Theta \times]-1,0[, \end{cases} \tag{2.5}$$

$$V_n = \left\{(v^a,v^b) \in H^1(\Omega^a,S^2) \times H^1(\Omega^b,S^2):\ v^a(x',0) = v^b(r_nx',0),\ \text{for } x' \text{a.e. in } \Theta\right\}, \tag{2.6}$$

$$\begin{aligned} j_n : v = (v^a,v^b) \in V_n \longrightarrow & \int_{\Omega^a} \left|\left(\frac{1}{r_n}D_{x'}v^a, D_{x_3}v^a\right)\right|^2 - 2v^a f_n^a\,dx \\ & + \frac{h_n}{r_n^2}\int_{\Omega^b} \left|\left(D_{x'}v^b, \frac{1}{h_n}D_{x_3}v^b\right)\right|^2 - 2v^b f_n^b\,dx, \end{aligned} \tag{2.7}$$

it results that $u_n \in V_n$ solves the following problem:

$$j_n(u_n) = \min\{ j_n(v) \colon v \in V_n \}. \tag{2.8}$$

Remark that we have also multiplied the rescaled functional by $1/r^2$.

To study the asymptotic behavior of problem (2.8), as $n \to +\infty$, assume that

$$\lim_n \frac{h_n}{r_n^2} = q \in [0, +\infty], \tag{2.9}$$

and

$$f_n^a \rightharpoonup f^a \quad \text{weakly in } L^2(\Omega^a, \mathbb{R}^3), \qquad f_n^b \rightharpoonup f^b \quad \text{weakly in } L^2(\Omega^b, \mathbb{R}^3). \tag{2.10}$$

Moreover, set

$$j^a \colon w \in H^1(]0,1[, S^2) \longrightarrow |\Theta| \int_0^1 |w'(x_3)|^2 \, dx_3 - 2 \int_0^1 w(x_3) \left(\int_\Theta f^a(x', x_3)\, dx' \right) dx_3, \tag{2.11}$$

$$j^b \colon \zeta \in H^1(\Theta, S^2) \longrightarrow \int_\Theta |D\zeta(x')|^2 \, dx' - 2 \int_\Theta \zeta(x') \left(\int_{-1}^0 f^b(x', x_3)\, dx_3 \right) dx', \tag{2.12}$$

where w' stands for the derivative of w.

2.1. Convergence results when $n \to +\infty$

The main result of this section, describing the asymptotic behavior of problem (2.8) when $q \in]0, +\infty[$, is the following one:

Theorem 2.1. *For every $n \in \mathbb{N}$, let $u_n = (u_n^a, u_n^b)$ be a solution of problem* (2.6)–(2.8)*, under assumptions* (2.1), (2.9) *with $q \in]0, +\infty[$ and* (2.10).

Then, there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$, $u^a \in \{w \in H^1(\Omega^a, S^2)\colon w$ is independent of $x'\} \simeq H^1(]0,1[, S^2)$ and $u^b \in \{\zeta \in H^1(\Omega^b, S^2)\colon \zeta$ is independent of $x_3\} \simeq H^1(\Theta, S^2)$ (u^a and u^b depending on the selected subsequence) such that

$$u_{n_i}^a \to u^a \quad \text{strongly in } H^1(\Omega^a, S^2), \qquad u_{n_i}^b \to u^b \quad \text{strongly in } H^1(\Omega^b, S^2), \tag{2.13}$$

as $i \to +\infty$, and u^a, u^b solve the following problems:

$$j^a(u^a) = \min\{ j^a(w) \colon w \in H^1(]0,1[, S^2) \}, \tag{2.14}$$

$$j^b(u^b) = \min\{ j^b(\zeta) \colon \zeta \in H^1(\Theta, S^2) \}, \tag{2.15}$$

respectively, with j^a and j^b defined in (2.11) *and* (2.12)*, respectively. Moreover,*

$$\frac{1}{r_n} D_{x'} u_n^a \to 0 \quad \text{strongly in } L^2(\Omega^a, \mathbb{R}^6), \qquad \frac{1}{h_n} D_{x_3} u_n^b \to 0 \quad \text{strongly in } L^2(\Omega^b, \mathbb{R}^3), \tag{2.16}$$

as $n \to +\infty$. Furthermore, the energies converge in the sense that

$$\lim_n j_n(u_n) = j^a(u^a) + q j^b(u^b). \tag{2.17}$$

If $q = 0$, the following result holds true:

Theorem 2.2. *For every $n \in \mathbb{N}$, let $u_n = (u_n^a, u_n^b)$ be a solution of problem* (2.6)–(2.8)*, under assumptions* (2.1), (2.9) *with $q = 0$ and* (2.10).

Then, there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$ *and* $u^a \in \{w \in H^1(\Omega^a, S^2)$: w *is independent of* $x'\} \simeq H^1(]0,1[, S^2)$ (u^a *depending on the selected subsequence*) *such that*

$$u^a_{n_i} \to u^a \quad \text{strongly in } H^1(\Omega^a, S^2), \tag{2.18}$$

as $i \to +\infty$, *and* u^a *solves problem* (2.14). *Moreover,*

$$\begin{aligned} &\frac{1}{r_n} D_{x'} u^a_n \to 0 \quad \text{strongly in } L^2(\Omega^a, \mathbb{R}^6),\\ &\frac{\sqrt{h_n}}{r_n} u^b_n \to 0 \quad \text{strongly in } H^1(\Omega^b, \mathbb{R}^3),\\ &\frac{1}{\sqrt{h_n} r_n} D_{x_3} u^b_n \to 0 \quad \text{strongly in } L^2(\Omega^b, \mathbb{R}^3), \end{aligned} \tag{2.19}$$

as $n \to +\infty$. *Furthermore, the energies converge in the sense that*

$$\lim_n j_n(u_n) = j^a(u^a). \tag{2.20}$$

If $q = +\infty$, the following result holds true:

Theorem 2.3. *For every* $n \in \mathbb{N}$, *let* $u_n = (u^a_n, u^b_n)$ *be a solution of problem* (2.6)–(2.8), *under assumptions* (2.1), (2.9) *with* $q = +\infty$ *and* (2.10).

Then, there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$ *and* $u^b \in \{\zeta \in H^1(\Omega^b, S^2)$: ζ *is independent of* $x_3\} \simeq H^1(\Theta, S^2)$ (u^b *depending on the selected subsequence*) *such that*

$$u^b_{n_i} \to u^b \quad \text{strongly in } H^1(\Omega^b, S^2), \tag{2.21}$$

as $i \to +\infty$, *and* u^b *solves problem* (2.15). *Moreover,*

$$\begin{aligned} &\frac{1}{h_n} D_{x_3} u^b_n \to 0 \quad \text{strongly in } L^2(\Omega^b, \mathbb{R}^3),\\ &\frac{r_n}{\sqrt{h_n}} u^a_n \to 0 \quad \text{strongly in } H^1(\Omega^a, \mathbb{R}^3),\\ &\frac{1}{\sqrt{h_n}} D_{x'} u^a_n \to 0 \quad \text{strongly in } H^1(\Omega^a, \mathbb{R}^6), \end{aligned} \tag{2.22}$$

as $n \to +\infty$. *Furthermore, the energies converge in the sense that*

$$\lim_n \left(\frac{r_n^2}{h_n} j_n(u_n) \right) = j^b(u^b). \tag{2.23}$$

As regard as the asymptotic behavior of original problem (2.3), as $n \to +\infty$, from the rescaling (2.4)–(2.5) and Theorems 2.1, 2.2 and 2.3, the result below follows immediately.

Corollary 2.4. *For every* $n \in \mathbb{N}$, *let* U_n *be a solution of problem* (2.3), *under assumptions* (2.1) *and* (2.10) *with* $\{f_n\}_{n\in\mathbb{N}}$ *defined by* (2.5), *and let* q *be given by* (2.9).

Then, there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$, $u^a \in \{w \in H^1(\Omega^a, S^2)$: w *is independent of* $x'\} \simeq H^1(]0,1[, S^2)$ *and* $u^b \in \{\zeta \in H^1(\Omega^b, S^2)$: ζ *is independent of* $x_3\} \simeq H^1(\Theta, S^2)$ (u^a *and* u^b *depending on the selected subsequence*) *such that*

(1) *if* $q \in]0, +\infty[$,

$$\lim_i \left(\frac{1}{r_{n_i}^2} \int_{r_{n_i}\Theta \times]0,1[} |U_{n_i} - u^a|^2 + |D_{x'} U_{n_i}|^2 + |D_{x_3} U_{n_i} - D_{x_3} u^a|^2 \, dx \right) = 0, \tag{2.24}$$

$$\lim_i \left(\frac{1}{h_{n_i}} \int_{\Theta\times]-h_{n_i},0[} |U_{n_i} - u^b|^2 + |D_{x'}U_{n_i} - D_{x'}u^b|^2 + |D_{x_3}U_{n_i}|^2 \, dx \right) = 0, \tag{2.25}$$

$$\lim_n \frac{J_n(U_n)}{r_n^2} = j^a(u^a) + qj^b(u^b);$$

(2) *if* $q = 0$,

$$\lim_i \left(\frac{1}{r_{n_i}^2} \int_{r_{n_i}\Theta\times]0,1[} |U_{n_i} - u^a|^2 + |D_{x'}U_{n_i}|^2 + |D_{x_3}U_{n_i} - D_{x_3}u^a|^2 \, dx \right) = 0, \tag{2.26}$$

$$\lim_n \left(\frac{1}{r_n^2} \int_{\Theta\times]-h_n,0[} |U_n|^2 + |D_{x'}U_n|^2 + |D_{x_3}U_n|^2 \, dx \right) = 0,$$

$$\lim_n \frac{J_n(U_n)}{r_n^2} = j^a(u^a);$$

(3) *if* $q = +\infty$,

$$\lim_n \left(\frac{1}{h_n} \int_{r_n\Theta\times]0,1[} |U_n|^2 + |D_{x'}U_n|^2 + |D_{x_3}U_n|^2 \, dx \right) = 0,$$

$$\lim_i \left(\frac{1}{h_{n_i}} \int_{\Theta\times]-h_{n_i},0[} |U_{n_i} - u^b|^2 + |D_{x'}U_{n_i} - D_{x'}u^b|^2 + |D_{x_3}U_{n_i}|^2 \, dx \right) = 0, \tag{2.27}$$

$$\lim_n \frac{J_n(U_n)}{h_n} = j^b(u^b);$$

and u^a *and* u^b *solve problems* (2.14) *and* (2.15), *respectively.*

Remark 2.5. If problem (2.14) (resp. (2.15)) admits a unique solution, then the first convergence in (2.13) and convergence (2.18), (2.24) and (2.26) (resp. the second convergence in (2.13) and convergences (2.21), (2.25) and (2.27)) hold true for the whole sequence.

2.2. Proof of Theorems 2.1, 2.2 and 2.3

Proof of Theorem 2.1. The proof of Theorem 2.1 will be performed in several steps. In the sequel, $|A|_i$, $i = 2, 3$, denotes the $\mathbb{R}^i$-Lebesgue measure of a measurable set $A \subset \mathbb{R}^i$.

1) *A priori estimates*. Being $((0,0,1),(0,0,1)) \in V_n$ for every $n \in \mathbb{N}$, by virtue of (2.9) with $q \in [0,+\infty[$ and (2.10), there exists a constant c, independent of n, such that

$$j_n(u_n) \leqslant -2 \int_{\Omega^a} (0,0,1) f_n^a \, dx - 2\frac{h_n}{r_n^2} \int_{\Omega^b} (0,0,1) f_n^b \, dx \leqslant c, \quad \forall n \in \mathbb{N}. \tag{2.28}$$

Consequently, by taking into account that $q \in]0,+\infty[$, $|u_n| = 1$ a.e. in Ω for every $n \in \mathbb{N}$ and (2.10), there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$, $u^a \in H^1(\Omega^a, S^2)$ independent of x', $u^b \in H^1(\Omega^b, S^2)$ independent of x_3, $\xi^a \in L^2(\Omega^a, \mathbb{R}^6)$ and $\xi^b \in L^2(\Omega^b, \mathbb{R}^3)$ such that

$$u_{n_i}^a \rightharpoonup u^a \quad \text{weakly in } H^1(\Omega^a, S^2), \qquad u_{n_i}^b \rightharpoonup u^b \quad \text{weakly in } H^1(\Omega^b, S^2), \tag{2.29}$$

$$\frac{1}{r_{n_i}} D_{x'}u_{n_i}^a \rightharpoonup \xi^a \quad \text{weakly in } L^2(\Omega^a, \mathbb{R}^6), \qquad \frac{1}{h_{n_i}} D_{x_3}u_{n_i}^b \rightharpoonup \xi^b \quad \text{weakly in } L^2(\Omega^b, \mathbb{R}^3), \tag{2.30}$$

as $i \to +\infty$, Remark that $u^a \in H^1(]0,1[, S^2)$ and $u^b \in H^1(\Theta, S^2)$.

2) *Recovery sequence*. Let $(w,\zeta)\in C^1([0,1],S^2)\times C^1(\overline{\Theta},S^2)$ such that and $w(0)=\zeta(0')$. This step is devoted to prove the existence of a sequence $\{v_n\}_{n\in\mathbb{N}}$, with $v_n\in V_n$, such that

$$\lim_n j_n(v_n)=j^a(w)+qj^b(\zeta). \tag{2.31}$$

For every $n\in\mathbb{N}$, set

$$g_n(x)=\begin{cases} w(x_3), & \text{if } x=(x',x_3)\in\Theta\times\,]r_n,1[,\\ w(r_n)\frac{x_3}{r_n}+\zeta(r_nx')\frac{r_n-x_3}{r_n}, & \text{if } x=(x',x_3)\in\Theta\times[0,r_n],\\ \zeta(x'), & \text{if } x=(x',x_3)\in\Omega^b. \end{cases} \tag{2.32}$$

Remark that, for every $n\in\mathbb{N}$, $g_{n|_{\Theta\times]0,r_n[}}\in C^1(\Theta\times\,]0,r_n[)$. Moreover, assumption (2.9) with $q\in\,]0,+\infty[$ and, in particular, the transmission condition $w(0)=\zeta(0')$ provide (for the proof, see (4.11) and (4.12) in [11]) that

$$\lim_n \int_{(\Theta\times]0,r_n[)} \left|\left(\frac{1}{r_n}D_{x'}g_n(x),D_{x_3}g_n(x)\right)\right|^2 dx=0. \tag{2.33}$$

Of course, $g_n^a\in H^1(\Omega^a)$, $g_n^b\in H^1(\Omega^b)$, and $g_n^a(x',0)=g_n^b(r_nx',0)$ for x' a.e. in Θ; but $|g_n(x)|\leqslant 1$ for every $x\in\Theta\times\,]0,r_n[$. Then, g_n is not an admissible test function for problem (2.6)–(2.8). To overcome this difficulty, for $y\in B_{\frac12}(0)=\{x\in\mathbb{R}^3\colon |x|\leqslant\frac12\}$, introduce the function

$$\pi_y: x\in B_1(0)\setminus\{y\}\to y+\frac{y(y-x)+\sqrt{(y(x-y))^2+|x-y|^2(1-|y|^2)}}{|x-y|^2}(x-y)\in S^2$$

projecting $x\in B_1(0)\setminus\{y\}=\{x\in\mathbb{R}^3\colon |x|\leqslant 1\}\setminus\{y\}$ on S^2 along the direction $x-y$ (see [3] and [1]). It is easily seen that

$$\pi_y(x)=x,\quad \forall x\in S^2, \tag{2.34}$$

and there exists a constant $c>0$ such that

$$|D\pi_y(x)|^2\leqslant\frac{c}{|x-y|^2},\quad \forall y\in B_{\frac12}(0),\ \forall x\in B_1(0)\setminus\{y\}. \tag{2.35}$$

The idea is to choose $y\in B_{\frac12}(0)$ opportunely, and to define $v_n=\pi_y\circ g_n$. To do that, one has to be careful that the set $\{x\colon g_n(x)=y\}$ is "sufficiently small".

By setting $G=\bigcup_{n\in\mathbb{N}}\{y\in B_{\frac12}(0)\colon \exists x\in\Theta\times\,]0,r_n[\text{ with } g_n(x)=y \text{ and } \operatorname{rank}((Dg_n)(x))<3\}$, Sard's Lemma assures that $\operatorname{meas}(G)=0$. Moreover, for every $n\in\mathbb{N}$ and for every $y\in B_{\frac12}(0)\setminus G$, the set $G_{n,y}=\{x\in\Theta\times\,]0,r_n[\colon g_n(x)=y\}$ has dimension 0 (see [24], ch. 13, par. 14). Consequently, for every $n\in\mathbb{N}$ and for every $y\in B_{\frac12}(0)\setminus G$, the function $\pi_y\circ(g_{n|_{\Omega\setminus G_{n,y}}})$ is well defined and, by virtue of (2.35) there exists a constant $c>0$ such that

$$\begin{aligned}
&\int_{B_{\frac12}(0)\setminus G}\int_{(\Theta\times]0,r_n[)\setminus G_{n,y}} \left|\left(\frac{1}{r_n}D_{x'}\big(\pi_y\big(g_n(x)\big)\big),D_{x_3}\big(\pi_y\big(g_n(x)\big)\big)\right)\right|^2 dx\,dy\\
&\quad\leqslant \int_{B_{\frac12}(0)\setminus G}\int_{(\Theta\times]0,r_n[)\setminus G_{n,y}} \big|\big(D\pi_y\big(g_n(x)\big)\big|^2\left|\left(\frac{1}{r_n}D_{x'}g_n(x),D_{x_3}g_n(x)\right)\right|^2 dx\,dy\\
&\quad\leqslant c\int_{B_{\frac12}(0)\setminus G}\int_{(\Theta\times]0,r_n[)\setminus G_{n,y}} \frac{1}{|g_n(x)-y|^2}\left|\left(\frac{1}{r_n}D_{x'}g_n(x),D_{x_3}g_n(x)\right)\right|^2 dx\,dy\\
&\quad= c\int_{B_{\frac12}(0)\setminus G}\int_{(\Theta\times]0,r_n[)} (1-\chi_{G_{n,y}})\frac{1}{|g_n(x)-y|^2}\left|\left(\frac{1}{r_n}D_{x'}g_n(x),D_{x_3}g_n(x)\right)\right|^2 dx\,dy
\end{aligned}$$

$$\leqslant c \int\limits_{(\Theta\times]0,r_n[)} \left(\int\limits_{B_{\frac{1}{2}}(0)\setminus G} \frac{1}{|g_n(x)-y|^2}\,dy \right) \left| \left(\frac{1}{r_n} D_{x'} g_n(x), D_{x_3} g_n(x) \right) \right|^2 dx$$

$$\leqslant c \int\limits_{(\Theta\times]0,r_n[)} \left(\int\limits_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2}\,dz \right) \left| \left(\frac{1}{r_n} D_{x'} g_n(x), D_{x_3} g_n(x) \right) \right|^2 dx$$

$$= c \int\limits_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2}\,dz \int\limits_{(\Theta\times]0,r_n[)} \left| \left(\frac{1}{r_n} D_{x'} g_n(x), D_{x_3} g_n(x) \right) \right|^2 dx, \quad \forall n \in \mathbb{N},$$

where $\int_{B_{\frac{3}{2}}(0)} |z|^{-2}\,dz < +\infty$. Consequently, there exist a constant $C > 0$ and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset B_{\frac{1}{2}}(0) \setminus G$ such that

$$\int\limits_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} \left| \left(\frac{1}{r_n} D_{x'}\big(\pi_{y_n}\big(g_n(x)\big)\big), D_{x_3}\big(\pi_{y_n}\big(g_n(x)\big)\big) \right) \right|^2 dx$$

$$\leqslant C \int\limits_{(\Theta\times]0,r_n[)} \left| \left(\frac{1}{r_n} D_{x'} g_n(x), D_{x_3} g_n(x) \right) \right|^2 dx, \quad \forall n \in \mathbb{N},$$

from which, by virtue of (2.33), it follows that

$$\lim_n \int\limits_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} \left| \left(\frac{1}{r_n} D_{x'}\big(\pi_{y_n}\big(g_n(x)\big)\big), D_{x_3}\big(\pi_{y_n}\big(g_n(x)\big)\big) \right) \right|^2 dx = 0. \tag{2.36}$$

Finally, for every $n \in \mathbb{N}$ set $v_n = \pi_{y_n} \circ (g_{n|_{\Omega\setminus G_{n,y_n}}})$. Then, by virtue of (2.32) and (2.34), it results that

$$v_n(x) = \begin{cases} w(x_3), & \text{if } x = (x', x_3) \in \Theta \times]r_n, 1[, \\ \pi_{y_n}(w(r_n)\frac{x_3}{r_n} + \zeta(r_n x')\frac{r_n - x_3}{r_n}), & \text{if } x = (x', x_3) \in (\Theta \times [0, r_n]) \setminus G_{n,y_n}, \\ \zeta(x'), & \text{if } x = (x', x_3) \in \Omega^b. \end{cases} \tag{2.37}$$

At first, remark that $v_n^a \in H^1(\Omega^b, S^2)$. Indeed, $v_n^a \in H^1(\Theta \times]r_n, 1[, S^2)$. Moreover, since $v_n^a \in L^2(\Theta \times]0, r_n[, S^2)$ and $Dv_n^a \in (L^2(\Theta \times]0, r_n[))^9$ (see (2.36)), it results that $v_n^a \in H^1(\Theta \times]0, r_n[, S^2)$. Furthermore, since $v_n^a \in C((\Theta \times [0, r_n]) \setminus G_{n,y_n})$ and G_{n,y_n} has dimension 0, the trace of $v^a_{n\,|_{\Theta\times]0,r_n[}}$ on $\Theta \times \{r_n\}$ is equal to $w(r_n)$. Consequently, these properties provide that $v_n^a \in H^1(\Omega^b, S^2)$. On the other hand, it is evident that $v_n^b \in H^1(\Omega^b, S^2)$, and $v_n^a(x', 0) = v_n^b(r_n x', 0)$ for x' a.e. in Θ. In conclusion, for every $n \in \mathbb{N}$, $v_n \in V_n$. Now, it remains to prove that $\{v_n\}_{n\in\mathbb{N}}$ satisfies (2.31).

By virtue of (2.37), it results that

$$\begin{aligned} j_n(v_n) = &\int\limits_{\Omega^a} \left(\left|(D_{x_3} w)\right|^2 - 2w f_n^a\right) dx - \int\limits_{\Theta\times]0,r_n[} \left(\left|(D_{x_3} w)\right|^2 - 2w f_n^a\right) dx \\ &+ \int\limits_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} \left[\left| \left(\frac{1}{r_n} D_{x'}(\pi_{y_n} \circ g_n), D_{x_3}(\pi_{y_n} \circ g_n) \right) \right|^2 - 2(\pi_{y_n} \circ g_n) f_n^a \right] dx \\ &+ \frac{h_n}{r_n^2} \int\limits_{\Omega^b} \left(\left|(D_{x'} \zeta)\right|^2 - 2\zeta f_n^b\right) dx. \end{aligned} \tag{2.38}$$

On the other side, convergence (2.10) provides that

$$\lim_n \int\limits_{\Omega^a} w f_n^a\,dx = \int\limits_{\Omega^a} w f^a\,dx, \qquad \lim_n \int\limits_{\Omega^b} \zeta f_n^b\,dx = \int\limits_{\Omega^b} \zeta f^b\,dx, \tag{2.39}$$

$$\lim_n \int_{\Theta\times]0,r_n[} \left(\left|(D_{x_3}w)\right|^2 - 2wf_n^a\right)dx = 0, \qquad \lim_n \int_{(\Theta\times]0,r_n[)\setminus G_{n,y_n}} (\pi_{y_n}\circ g_n) f_n^a\,dx = 0. \tag{2.40}$$

Then, by passing to the limit, as n diverges, in (2.38) and by taking into account (2.39), (2.40), (2.36) and (2.9) with $q \in]0,+\infty[$, one obtains that

$$\lim_n j_n(v_n) = \int_{\Omega^a} \left(\left|(D_{x_3}w)\right|^2 - 2wf^a\right)dx + q\int_{\Omega^b}\left(\left|(D_{x'}\zeta)\right|^2 - 2\zeta f^b\right)dx = j^a(w) + qj^b(\zeta).$$

3) *Density result*. Let $(w,\zeta) \in C^1([0,1],S^2)\times C^1(\overline{\Theta},S^2)$. This step is devoted to prove the existence of a sequence $\{(w_k,\zeta_k)\}_{k\in\mathbb{N}} \subset C^1([0,1],S^2)\times C^1(\overline{\Theta},S^2)$, with $w_k(0)=\zeta_k(0')$ for every $k\in\mathbb{N}$, such that

$$(w_k,\zeta_k)\to(w,\zeta) \quad \text{strongly in } H^1\big(]0,1[,S^2\big)\times H^1(\Theta,S^2).$$

For every $k\in\mathbb{N}$, set

$$\theta_k = w(0)\varphi_k + (1-\varphi_k)\zeta,$$

where φ_k is the solution of the following problem:

$$\min\left\{\int_{B_{\frac{1}{k}}(0')} \left|D\varphi_k(x')\right|^2 dx' \colon \varphi_k \in C_0^1\big(B_{\frac{1}{k}}(0')\big),\ \varphi_k = 1 \text{ in } B_{\frac{1}{k^2}}(0'),\ 0\leqslant\varphi_k\leqslant 1\right\},$$

with $B_{\frac{1}{k}}(0') = \{x'\in R^2\colon |x'|<\frac{1}{k}\}$ and $B_{\frac{1}{k^2}}(0') = \{x'\in R^2\colon |x'|<\frac{1}{k^2}\}$. Remark that (for instance, see (3.4) in [11])

$$\begin{aligned}\lim_k &\int_{B_{\frac{1}{k}}(0')\setminus B_{\frac{1}{k^2}}(0')} \left|D\theta_k(x')\right|^2 dx' \\ &\leqslant 2\lim_k\left(\|D\zeta\|^2_{L^\infty(\Theta)}\left|B_{\frac{1}{k}}(0')\right|_2 + \left(\left|w(0)\right| + \|\zeta\|_{L^\infty(\Theta)}\right)^2 \int_{B_{\frac{1}{k}}\setminus B_{\frac{1}{k^2}}} \left|D\varphi_k(x')\right|^2 dx'\right) = 0.\end{aligned} \tag{2.41}$$

Since $\theta_k : \overline{\Theta}\subset\mathbb{R}^2\to\mathbb{R}^3$ is a C^1 function, it results that $|\bigcup_{k\in\mathbb{N}}\theta_k(\overline{\Theta})|_3 = 0$. Consequently, now it is easier than in the previous step to apply the projection π_y for obtaining S^2-value functions. Indeed, for every $k\in\mathbb{N}$ and for every $y\in B_{\frac{1}{2}}(0)\setminus\bigcup_{k\in\mathbb{N}}\theta_k(\overline{\Theta})$, the function $\pi_y\circ\theta_k\in C^1(\overline{\Theta},S^2)$ and, by virtue of (2.35), there exists a constant $c>0$ such that

$$\begin{aligned}&\int_{B_{\frac{1}{2}}(0)\setminus\bigcup_{k\in\mathbb{N}}\theta_k(\overline{\Theta})} \int_{B_{\frac{1}{k}}(0')\setminus B_{\frac{1}{k^2}}(0')} \left|D\big(\pi_y\big(\theta_k(x')\big)\big)\right|^2 dx'dy \\ &\quad\leqslant c\int_{B_{\frac{3}{2}}(0)} \frac{1}{|z|^2}\,dz \int_{B_{\frac{1}{k}}(0')\setminus B_{\frac{1}{k^2}}(0')} \left|D\theta_k(x')\right|^2 dx', \quad \forall k\in\mathbb{N}.\end{aligned}$$

Consequently, by taking into account (2.41), there exists a subsequence, still denoted by $\{k\}$, and

$$\bar{y}\in B_{\frac{1}{2}}(0)\setminus\bigcup_{k\in\mathbb{N}}\theta_k(\overline{\Theta})$$

such that

$$\lim_k \int_{B_{\frac{1}{k}}(0')\setminus B_{\frac{1}{k^2}}(0')} \left|D\big(\pi_{\bar{y}}\big(\theta_k(x')\big)\big)\right|^2 dx' = 0. \tag{2.42}$$

Now, for every $k \in \mathbb{N}$ set $w_k = w$ and $\zeta_k = \pi_{\bar{y}} \circ \theta_k$. Then, it is evident that $\{(w_k, \zeta_k)\}_{k\in\mathbb{N}} \subset C^1([0,1], S^2) \times C^1(\overline{\Theta}, S^2)$, with $\zeta_k(0') = w_k(0)$, and $w_k \to w$ strongly in $H^1(\Theta, S^2)$. Moreover, it results that $\zeta_k \to \zeta$ strongly in $H^1(\Theta, S^2)$. In fact, by taking into account that (see (2.34))

$$\zeta_k(x') = \begin{cases} w(0'), & \text{if } x' \in B_{\frac{1}{k^2}}(0), \\ \pi_{\bar{y}}(w(0)\varphi_k(x') + (1-\varphi_k(x'))\zeta), & \text{if } x' \in B_{\frac{1}{k}}(0') \setminus B_{\frac{1}{k^2}}(0'), \\ \zeta(x'), & \text{if } x' \in \overline{\Theta} \setminus B_{\frac{1}{k}}(0'), \end{cases} \quad \forall k \in \mathbb{N}, \tag{2.43}$$

and (2.42), it results that

$$\lim_k \int_{\Theta} |\zeta_k - \zeta|^2 \, dx' = \lim_k \int_{B_{\frac{1}{k}}(0')} \left|(\pi_{\bar{y}} \circ \theta_k) - \zeta\right|^2 dx' = 0,$$

and

$$\lim_k \int_{\Theta} |D\zeta_k - D\zeta|^2 \, dx' = \lim_k \left(\int_{B_{\frac{1}{k}}(0') \setminus B_{\frac{1}{k^2}}(0')} \left|D(\pi_{\bar{y}} \circ \theta_k) - D\zeta\right|^2 dx' + \int_{B_{\frac{1}{k^2}}(0')} |D\zeta|^2 \, dx' \right)$$
$$\leqslant \lim_k \left(2 \int_{B_{\frac{1}{k}}(0') \setminus B_{\frac{1}{k^2}}(0')} \left|D(\pi_{\bar{y}} \circ \theta_k)\right|^2 dx' + 3\|D\zeta\|^2_{L^\infty(\Theta)} \left|B_{\frac{1}{k}}(0')\right|_2 \right) = 0.$$

4) *Conclusion*. By using a l.s.c. argument, from (2.9) with $q \in]0, +\infty[$, (2.10), (2.29) and (2.30) it follows that

$$\int_{\Omega^a} |\xi^a|^2 \, dx + j^a(u^a) + q\left(j^b(u^b) + \int_{\Omega^b} |\xi^b|^2 \, dx \right) \leqslant \liminf_i j_{n_i}(u_{n_i}). \tag{2.44}$$

On the other hand, by virtue of step 2, for every $(w, \zeta) \in C^1([0,1], S^2) \times C^1(\overline{\Theta}, S^2)$ with $w(0) = \zeta(0')$, there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$, with $v_n \in V_n$, such that

$$\limsup_i j_{n_i}(u_{n_i}) \leqslant \limsup_i j_{n_i}(v_{n_i}) = \lim_n j_n(v_n) = j^a(w) + q j^b(\zeta). \tag{2.45}$$

Then, by combining (2.44) with (2.45), one obtains that

$$\int_{\Omega^a} |\xi^a|^2 \, dx + j^a(u^a) + q\left(j^b(u^b) + \int_{\Omega^b} |\xi^b|^2 \, dx \right) \leqslant \liminf_i j_{n_i}(u_{n_i})$$
$$\leqslant \limsup_i j_{n_i}(u_{n_i}) \leqslant j^a(w) + q j^b(\zeta), \tag{2.46}$$

for every $(w, \zeta) \in C^1([0,1], S^2) \times C^1(\overline{\Theta}, S^2)$ such that $w(0) = \zeta(0')$.

Step 3 provides that inequality (2.46) holds true for every $(w, \zeta) \in C^1([0,1], S^2) \times C^1(\overline{\Theta}, S^2)$. Moreover, since $C^1([0,1], S^2) \times C^1(\overline{\Theta}, S^2)$ is dense in $H^1(]0,1[, S^2) \times H^1(\Theta, S^2)$ (see [3]), inequality (2.46) holds true also for every $(w, \zeta) \in H^1(]0,1[, S^2) \times H^1(\Theta, S^2)$. Consequently, it results that

$$\xi^a = 0, \qquad \xi^b = 0, \tag{2.47}$$

u^a and u^b solve problems (2.14) and (2.15), respectively, and

$$\lim_i j_{n_i}(u_{n_i}) = j^a(u^a) + q j^b(u^b). \tag{2.48}$$

Really, convergence (2.48) holds true for the whole sequence (so (2.17) is proved), since $j^a(u^a)$ and $j^b(u^b)$ are independent of the selected subsequence, being the minimum of problems (2.14) and (2.15), respectively.

Finally, by combining (2.9) with $q \in]0, +\infty[$, (2.10), (2.29), (2.30) and (2.47) with (2.48), and by using the Rellich–Kondrachov compact embedding theorem and the uniform convexity of the space L^2, it is easy to see that convergences (2.29) and (2.30) occur in the strong sense, i.e. (2.13) and (2.16). □

Proof of Theorem 2.2. A priori estimates (2.28) hold true also if $q = 0$ in (2.9). Consequently, by taking into account that $q = 0$, $|u_n| = 1$ a.e. in Ω for every $n \in \mathbb{N}$ and (2.10), there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$, $u^a \in H^1(\Omega^a, S^2)$ independent of x', $\xi^a \in L^2(\Omega^a, \mathbb{R}^6)$ and $z^b \in L^2(\Omega^b, \mathbb{R}^3)$ such that convergence (2.19) holds true, and

$$u^a_{n_i} \rightharpoonup u^a \quad \text{weakly in } H^1(\Omega^a, S^2), \tag{2.49}$$

$$\frac{1}{r_{n_i}} D_{x'} u^a_{n_i} \rightharpoonup \xi^a \quad \text{weakly in } L^2(\Omega^a, \mathbb{R}^6), \tag{2.50}$$

$$\frac{1}{\sqrt{h_{n_i}} r_{n_i}} D_{x_3} u^b_{n_i} \rightharpoonup z^b \quad \text{weakly in } L^2(\Omega^b, \mathbb{R}^3), \tag{2.51}$$

as $i \to +\infty$. Remark that $u^a \in H^1(]0,1[, S^2)$.

By using a l.s.c. argument, from (2.10), (2.49), (2.50) and (2.51) it follows that

$$\begin{aligned}
&\int_{\Omega^a} |\xi^a|^2\,dx + j^a(u^a) + \int_{\Omega^b} |z^b|^2\,dx \\
&\quad \leqslant \liminf_i \left(\int_{\Omega^a} \left| \left(\frac{1}{r_{n_i}} D_{x'} u^a_{n_i}, D_{x_3} u^a_{n_i} \right) \right|^2 - 2u^a_{n_i} f^a_{n_i}\,dx + \int_{\Omega^b} \left| \frac{D_{x_3} u^b_{n_i}}{\sqrt{h_{n_i}} r_{n_i}} \right|^2 - \frac{2h_{n_i}}{r^2_{n_i}} u^b_{n_i} f^b_{n_i}\,dx \right) \\
&\quad \leqslant \liminf_i j_{n_i}(u_{n_i}).
\end{aligned} \tag{2.52}$$

On the other hand, for every $w \in C^1([0,1], S^2)$ the sequence $\{v_n\}_{n\in\mathbb{N}}$, defined by $v^a_n = w$ and $v^b_n = w(0)$, belongs to V_n and satisfies

$$\limsup_i j_{n_i}(u_{n_i}) \leqslant \limsup_i j_{n_i}(v_{n_i}) = \lim_n j_n(v_n) = j^a(w). \tag{2.53}$$

By combining (2.52) with (2.53), and by taking into account that $C^1([0,1], S^2)$ is dense in $H^1(]0,1[, S^2)$ (see [3]), one obtains that $\xi^a = 0$, $z^b = 0$, that u^a solves problem (2.14), and the convergence of the energies (2.20). One achieves the proof of Theorem 2.2, by arguing as in the last part of the proof of Theorem 2.1. □

Proof of Theorem 2.3. Being $((0,0,1),(0,0,1)) \in V_n$ for every $n \in \mathbb{N}$, by virtue of (2.9) with $q = +\infty$ and (2.10), there exists a constant c, independent of n, such that

$$\frac{r_n^2}{h_n} j_n(u_n) \leqslant -\frac{r_n^2}{h_n} \int_{\Omega^a} 2(0,0,1) f^a_n\,dx - \int_{\Omega^b} 2(0,0,1) f^b_n\,dx \leqslant c, \quad \forall n \in \mathbb{N}. \tag{2.54}$$

Consequently, by taking into account that $q = +\infty$, $|u_n| = 1$ a.e. in Ω for every $n \in \mathbb{N}$ and (2.10), there exist an increasing sequence of positive integer number $\{n_i\}_{i\in\mathbb{N}}$, $u^b \in H^1(\Omega^b, S^2)$ independent of x_3, $\xi^b \in L^2(\Omega^b, \mathbb{R}^3)$ and $z^a \in L^2(\Omega^b, \mathbb{R}^6)$ such that convergence (2.22) holds true, and

$$\frac{1}{\sqrt{h_n}} D_{x'} u^a_n \rightharpoonup z^a \quad \text{weakly in } H^1(\Omega^a, \mathbb{R}^6), \tag{2.55}$$

$$u^b_{n_i} \to u^b \quad \text{strongly in } H^1(\Omega^b, S^2), \tag{2.56}$$

$$\frac{1}{h_{n_i}} D_{x_3} u^b_{n_i} \rightharpoonup \xi^b \quad \text{weakly in } L^2(\Omega^b, \mathbb{R}^3), \tag{2.57}$$

as $i \to +\infty$. Remark that $u^b \in H^1(\Theta, S^2)$.

By using a l.s.c. argument, from (2.10), (2.56), (2.57) and (2.55) it follows that

$$\int_{\Omega^a} |z^a|^2\,dx + j^b(u^b) + \int_{\Omega^b} |\xi^b|^2\,dx$$

$$\leqslant \liminf_i \left(\int_{\Omega^a} \left| \frac{1}{\sqrt{h_{n_i}}} D_{x'} u^a_{n_i} \right|^2 - \frac{2r^2_{n_i}}{h_{n_i}} u^a_{n_i} f^a_{n_i} \, dx + \int_{\Omega^b} \left| \left(D_{x'} u^b_{n_i}, \frac{D_{x_3} u^b_{n_i}}{h_{n_i}} \right) \right|^2 - 2u^b_{n_i} f^b_{n_i} \, dx \right)$$
$$\leqslant \liminf_i \left(\frac{r^2_{n_i}}{h_{n_i}} j_{n_i}(u_{n_i}) \right). \tag{2.58}$$

On the other hand, for every $\zeta \in C^1(\overline{\Theta}, S^2)$, such that ζ is constant in a neighbourhood of $0'$, the sequence $\{v_n\}_{n\in\mathbb{N}}$, defined by $v^a_n = \zeta(0')$ and $v^b_n = \zeta$, belongs to V_n (for n sufficiently large) and satisfies

$$\limsup_i \left(\frac{r^2_{n_i}}{h_{n_i}} j_{n_i}(u_{n_i}) \right) \leqslant \limsup_i \left(\frac{r^2_{n_i}}{h_{n_i}} j_{n_i}(v_{n_i}) \right) = \lim_n \left(\frac{r^2_{n_i}}{h_{n_i}} j_n(v_n) \right) = j^b(\zeta). \tag{2.59}$$

Obviously, step 3 of the proof of Theorem 2.1 is independent of $q \in [0, +\infty]$. Moreover, a careful reading of this step (in particular, see (2.43)) shows that the space $\{\zeta \in C^1(\overline{\Theta}, S^2)\colon\ \zeta$ is constant in a neighbourhood of $0'\}$ is dense in $C^1(\overline{\Theta}, S^2)$ with respect to the H^1-norm. Consequently, by combining (2.58) with (2.59), it results that

$$\int_{\Omega^a} |z^a|^2 \, dx + j^b(u^b) + \int_{\Omega^b} |\xi^b|^2 \, dx \leqslant \liminf_i \left(\frac{r^2_{n_i}}{h_{n_i}} j_{n_i}(u_{n_i}) \right)$$
$$\leqslant \limsup_i \left(\frac{r^2_{n_i}}{h_{n_i}} j_{n_i}(u_{n_i}) \right) \leqslant j^b(\zeta), \quad \forall \zeta \in C^1(\overline{\Theta}, S^2)$$

from which, by taking into account that $C^1([0,1], S^2)$ is dense in $H^1(]0,1[, S^2)$ (see [3]), one obtains that $z^a = 0$, $\xi^b = 0$, that u^b solves problem (2.15), and the convergence of the energies (2.23). One achieves the proof of Theorem 2.3, by arguing as in the last part of the proof of Theorem 2.1. □

3. Second part: analysis of the limit model

For every $n \in \mathbb{N}$ and $\lambda \in [0, +\infty[$, consider the following problem:

$$J_{n,\lambda} : V \in H^1(\Omega_n, S^2) \longrightarrow \int_{\Omega_n} \left| DV(x) \right|^2 dx + \lambda \int_{\Omega_n} \left| V(x) - F_n(x) \right|^2 dx, \tag{3.1}$$

where $F_n : \Omega_n \to S^2$ is a measurable function.

Remark that $J_{n,\lambda}$ has the same minimum points of the functional:

$$\tilde{J}_{n,\lambda} : V \in H^1(\Omega_n, S^2) \longrightarrow \int_{\Omega_n} \left| DV(x) \right|^2 dx - 2\lambda \int_{\Omega_n} V(x) F_n(x) \, dx,$$

since $J_{n,\lambda}(V) = \tilde{J}_{n,\lambda}(V) + 2\lambda|\Omega_n|$, for every $V \in H^1(\Omega_n, S^2)$. Consequently, after a rescaling as in Section 2, by passing to the limit as $n \to +\infty$, one obtains all the results of Subsection 2.1 with

$$j^a_\lambda(w) = |\Theta| \int_0^1 \left| w'(x_3) \right|^2 dx_3 - 2\lambda \int_0^1 w(x_3) \left(\int_\Theta f^a(x', x_3) \, dx' \right) dx_3 + 2\lambda|\Theta|, \quad \forall w \in H^1\big(]0,1[, S^2\big), \tag{3.2}$$

$$j^b_\lambda(\zeta) = \int_\Theta \left| D\zeta(x') \right|^2 dx' - 2\lambda \int_\Theta \zeta(x') \left(\int_{-1}^0 f^b(x', x_3) \, dx_3 \right) dx' + 2\lambda|\Theta|, \quad \forall \zeta \in H^1(\Theta, S^2), \tag{3.3}$$

where f^a and f^b are given by (2.10), and w' stands for the derivative of w. Remark that, since $|f^a_n(x)| = 1$ a.e. in Ω^a and $|f^b_n(x)| = 1$ a.e. in Ω^b for every $n \in \mathbb{N}$, weak convergences in (2.10) are always satisfied for a subsequence.

If $|f^a(x)| = 1$ a.e. in Ω^a, f^a is independent of x', $|f^b(x)| = 1$ a.e. in Ω^b and f^b is independent of x_3, then functionals (3.2) an (3.3) can be rewritten as follows:

$$j_\lambda^a(w) = |\Theta| \int_0^1 \left|w'(x_3)\right|^2 + \lambda\left|w(x_3) - f^a(x_3)\right|^2 dx_3, \quad \forall w \in H^1\bigl(]0,1[, S^2\bigr), \tag{3.4}$$

$$j_\lambda^b(\zeta) = \int_\Theta \left|D\zeta(x')\right|^2 + \lambda\left|\zeta(x') - f^b(x')\right|^2 dx', \quad \forall \zeta \in H^1(\Theta, S^2). \tag{3.5}$$

In the sequel, w_λ and ζ_λ denote solutions of the following problems:

$$j_\lambda^a(w_\lambda) = \min\left\{ |\Theta| \int_0^1 \left|w'(x_3)\right|^2 + \lambda\left|w(x_3) - f^a(x_3)\right|^2 dx_3\colon\ w \in H^1\bigl(]0,1[, S^2\bigr) \right\}, \tag{3.6}$$

$$j_\lambda^b(\zeta_\lambda) = \min\left\{ \int_\Theta \left|D\zeta(x')\right|^2 + \lambda\left|\zeta(x') - f^b(x')\right|^2 dx'\colon\ \zeta \in H^1(\Theta, S^2) \right\}, \tag{3.7}$$

respectively.

This section is devoted to study the asymptotic behavior, as $\lambda \to +\infty$, of problem (3.6) and problem (3.7). Remark that, if $\lambda = 0$, the solutions of problem (3.6) and problem (3.7) are the constants of S^2.

3.1. Convergence results when $\lambda \to +\infty$

If $f^a \in H^1(]0,1[, S^2)$, from (3.6) it follows that

$$\left\|(w_\lambda)'\right\|^2_{(L^2(]0,1[))^3} + \lambda\|w_\lambda - f^a\|^2_{(L^2(]0,1[))^3} \leqslant \int_0^1 \left|(f^a)'(x_3)\right|^2 dx_3, \quad \forall \lambda \in]0,+\infty[,$$

($(w_\lambda)'$ and $(f^a)'$ stand for the derivative of w_λ and f^a, respectively) which provides that

$$w_{\lambda_\iota} \rightharpoonup f^a \quad \text{weakly in } H^1\bigl(]0,1[, S^2\bigr),$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$. Moreover, by using a l.s.c. argument, it results that

$$\begin{aligned} |\Theta|\left\|(f^a)'\right\|^2_{(L^2(]0,1[))^3} &\leqslant |\Theta| \liminf_\iota \left\|(w_{\lambda_\iota})'\right\|^2_{(L^2(]0,1[))^3} \leqslant \liminf_\iota j^a_{\lambda_\iota}(w_{\lambda_\iota}) \\ &\leqslant \limsup_\iota j^a_{\lambda_\iota}(w_{\lambda_\iota}) \leqslant \limsup_\iota j^a_{\lambda_\iota}(f^a) = |\Theta|\left\|(f^a)'\right\|^2_{(L^2(]0,1[))^3}, \end{aligned}$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$, from which it follows that

$$\lim_{\lambda\to+\infty} j^a_\lambda(w_\lambda) = |\Theta|\left\|(f^a)'\right\|^2_{(L^2(]0,1[))^3}.$$

Similarly, if $f^b \in H^1(\Theta, S^2)$, one has that

$$\zeta_{\lambda_\iota} \rightharpoonup f^b \quad \text{weakly in } H^1(\Theta, S^2),$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$, and

$$\lim_{\lambda\to+\infty} j^b_\lambda(\zeta_\lambda) = \|Df^b\|^2_{(L^2(\Theta))^6}.$$

Then, interesting situations occur when $f^a \notin H^1(]0,1[)$, or $f^b \notin H^1(\Theta)$.

At first, consider the case: $f^b = \frac{1}{|x'|}(x',0)$. Remark that $\frac{1}{|x'|}(x',0) \notin H^1(\Theta)$ (although $\frac{x}{|x|} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, S^2)$). Consequently, it results that

$$\lim_{\lambda\to+\infty} j^b_\lambda(\zeta_\lambda) = +\infty. \tag{3.8}$$

In fact, by arguing by contradiction, if (3.8) does not hold true, then there exists $c \in]0,+\infty[$ and a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $j^b_{\lambda_k}(\zeta_{\lambda_k}) \leqslant c$ for every k. Consequently, $\zeta_{\lambda_k} \rightharpoonup f^b$weakly in $H^1(\Theta, S^2)$, which is false, since $f^b \notin H^1(\Theta)$.

On the other hand, the following *a priori* estimates hold true (the proof will be performed in Subsection 3.2):

Proposition 3.1. *For every* $\lambda \in [0, +\infty[$*, let* ζ_λ *be a solution of problem* (3.7) *with* $f^b = (\frac{x'}{|x'|}, 0)$.
Then, there exist c_1 *and* $\lambda_1 \in]0, +\infty[$ *such that*

$$j^b_\lambda(\zeta_\lambda) \leqslant \pi \log \lambda + c_1, \quad \forall \lambda \in]\lambda_1, +\infty[. \tag{3.9}$$

Moreover, there exist a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ *and* $c_2, c_3 \in]0, +\infty[$ *such that*

$$\int_\Theta \left|\zeta_{\lambda_k}(x') - f^b(x')\right|^2 dx' \leqslant \frac{c_2}{\lambda_k}, \quad \forall k \in \mathbb{N}, \tag{3.10}$$

$$\pi \log \lambda_k - c_3 \leqslant j^b_{\lambda_k}(\zeta_{\lambda_k}), \quad \textit{for } k \in \mathbb{N} \textit{ large enough}. \tag{3.11}$$

Remark 3.2. If one can prove estimate (3.10) for λ large enough, the proof of Proposition 3.1 shows that also estimate (3.11) holds true for λ large enough.

Proposition 3.1 immediately provides the following convergence result:

Corollary 3.3. *For every* $\lambda \in [0, +\infty[$*, let* ζ_λ *be a solution of problem* (3.7) *with* $f^b = (\frac{x'}{|x'|}, 0)$.
Then, it results that

$$\zeta_{\lambda_\iota} \to f^b \quad \textit{strongly in } L^2(\Theta, S^2),$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$.
There exists a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ *and* $c \in]0, +\infty[$ *such that*

$$\int_\Theta \left|\zeta_{\lambda_k}(x') - f^b(x')\right|^2 dx' \leqslant \frac{c}{\lambda_k}, \quad \forall k \in \mathbb{N},$$

and

$$\lim_{k\to+\infty} \frac{j^b_{\lambda_k}(\zeta_{\lambda_k})}{\log \lambda_k} = \pi.$$

Obviously, $\{\zeta_{\lambda_\iota}\}_{\iota\in\mathbb{N}}$ does not converge weakly in $H^1(\Theta, S^2)$, since $f^b = (\frac{x'}{|x'|}, 0) \notin H^1(\Theta)$.

One obtains the same results, if $f^b = \frac{1}{|(x_1-\alpha, x_2-\beta)|}(x_1 - \alpha, x_2 - \beta, 0)$, where (α, β) is a fixed point in Θ.

Consider, now, the case: $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|})$, where γ is a fixed number in $]0, 1[$. Obviously, $f^a \notin H^1(]0, 1[)$ (remark that $f^a \in H^1(]0, 1[, S^2)$ if $\gamma \in \{0, 1\}$, and $w_\lambda = (0, 0, 1)$ if $\gamma = 0$, $w_\lambda = (0, 0, -1)$ if $\gamma = 1$). Consequently, it results that

$$\lim_{\lambda\to+\infty} j^a_\lambda(w_\lambda) = +\infty. \tag{3.12}$$

In fact, by arguing by contradiction, if (3.12) does not hold true, then there exists $c \in]0, +\infty[$ and a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $j^a_{\lambda_k}(w_{\lambda_k}) \leqslant c$ for every k. Consequently, $w_{\lambda_k} \rightharpoonup f^a$weakly in $H^1(]0, 1[, S^2)$, which is false, since $f^a \notin H^1(]0, 1[)$.

When $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|})$, the following *a priori* estimates hold true (the proof will be performed in Subsection 3.2):

Proposition 3.4. *For every* $\lambda \in [0, +\infty[$*, let* w_λ *be a solution of problem* (3.6) *with* $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|})$ *and* $\gamma \in]0, 1[$.
Then, it results that

$$j^a_\lambda(w_\lambda) \leqslant |\Theta| 2\sqrt{2\pi}\sqrt{\lambda}, \quad \forall \lambda \in]0, +\infty[. \tag{3.13}$$

Moreover, for every $\varepsilon \in]0, 2[$ *there exists* $\lambda_\varepsilon \in]0, +\infty[$ *such that*

$$|\Theta|(2 - \varepsilon)\sqrt{\lambda} \leqslant j^a_\lambda(w_\lambda), \quad \forall \lambda \in]\lambda_\varepsilon, +\infty[. \tag{3.14}$$

Estimate (3.13) immediately provides the following convergence result:

Corollary 3.5. *For every* $\lambda \in [0, +\infty[$, *let* w_λ *be a solution of problem* (3.6) *with* $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|})$ *and* $\gamma \in]0, 1[$. *Then, it results that*

$$w_{\lambda_\iota} \to f^a \quad \text{strongly in } L^2\big(]0,1[, S^2\big),$$

for any diverging sequence of positive numbers $\{\lambda_\iota\}_{\iota\in\mathbb{N}}$.

Obviously, $\{w_{\lambda_\iota}\}_{\iota\in\mathbb{N}}$ does not converge weakly in $H^1(]0,1[, S^2)$, since $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|}) \notin H^1(]0,1[)$. By making use of estimate (3.13) and by arguing as in the proof of estimate (3.10) it is easy to prove the following result:

Proposition 3.6. *For every* $\lambda \in [0, +\infty[$, *let* w_λ *be a solution of problem* (3.6) *with* $f^a = (0, 0, \frac{x_3-\gamma}{|x_3-\gamma|})$ *and* $\gamma \in]0, 1[$. *There exist a diverging sequence of positive numbers* $\{\lambda_k\}_{k\in\mathbb{N}}$ *and* $c \in]0, +\infty[$ *such that*

$$\int_0^1 \big|w_{\lambda_k}(x_3) - f^a(x_3)\big|^2\,dx_3 \leqslant \frac{c}{\sqrt{\lambda_k}}, \quad \forall k \in \mathbb{N}.$$

This subsection ends by showing some situations when the considered cases:

$$f^a = \left(0, 0, \frac{x_3-\gamma}{|x_3-\gamma|}\right), \qquad f^b = \frac{1}{|(x_1-\alpha, x_2-\beta)|}(x_1-\alpha, x_2-\beta, 0),$$

appear in the limit problem.

In the sequel, (α, β) denotes a fixed point in $\mathbb{R}^2$ and γ in $\mathbb{R}$.

For instance, by choosing in (3.1)

$$F_n(x_1,x_2,x_3) = \frac{1}{|(x_1-r_n\alpha, x_2-r_n\beta, x_3-\gamma)|}(x_1-r_n\alpha, x_2-r_n\beta, x_3-\gamma),$$

convergence (2.10) gives

$$f^a(x_3) = \left(0, 0, \frac{x_3-\gamma}{|x_3-\gamma|}\right), \qquad f^b(x_1,x_2) = \frac{1}{|(x_1,x_2,-\gamma)|}(x_1,x_2,-\gamma).$$

Remark that $F_n \in H^1_{\mathrm{loc}}(\mathbb{R}^3, S^2)$, $f^a \notin H^1(]0,1[) \Leftrightarrow \gamma \in]0,1[$, $f^b \notin H^1(\Theta) \Leftrightarrow \gamma = 0$.

By choosing in (3.1)

$$F_n(x_1,x_2,x_3) = \frac{1}{|(x_1-\alpha, x_2-\beta, x_3-h_n\gamma)|}(x_1-\alpha, x_2-\beta, x_3-h_n\gamma),$$

convergence (2.10) gives

$$f^a(x_3) = \frac{1}{|(-\alpha,-\beta,x_3)|}(-\alpha,-\beta,x_3), \qquad f^b(x_1,x_2) = \frac{1}{|(x_1-\alpha, x_2-\beta)|}(x_1-\alpha, x_2-\beta, 0).$$

Remark that $F_n \in H^1_{\mathrm{loc}}(\mathbb{R}^3, S^2)$, $f^a \in H^1(]0,1[, S^2)$, $f^b \notin H^1(\Theta) \Leftrightarrow (\alpha,\beta) \in \overline{\Theta}$.

By choosing in (3.1)

$$F_n(x_1,x_2,x_3) = \frac{1}{|(x_1-r_n\alpha, x_2-r_n\beta, x_3-h_n\gamma)|}(x_1-r_n\alpha, x_2-r_n\beta, x_3-h_n\gamma),$$

convergence (2.10) gives

$$f^a(x_3) = (0,0,1), \qquad f^b(x_1,x_2) = \frac{1}{|(x_1,x_2)|}(x_1,x_2,0).$$

Remark that $F_n \in H^1_{\mathrm{loc}}(\mathbb{R}^3, S^2)$, $f^a \in H^1(]0,1[, S^2)$, $f^b \notin H^1(\Theta)$.

3.2. Proof of Proposition 3.1 and Proposition 3.4

Proof of Proposition 3.1. To obtain estimate (3.9), for $r, \lambda \in]0, +\infty[$ introduce the functionals:

$$j^b_{\lambda,r} : \zeta \in H^1\big(C_r(0'), S^2\big) \to \int\limits_{C_r(0')} \big|D\zeta(x')\big|^2 + \lambda\big|\zeta(x') - f^b(x')\big|^2\, dx',$$

and denote with $\zeta_{\lambda,r}$ a solution of the following problem:

$$j^b_{\lambda,r}(\zeta_{\lambda,r}) = \min\big\{ j^b_{\lambda,r}(\zeta):\ \zeta \in H^1\big(C_r(0'), S^2\big),\ \zeta = f^b \text{ on } \partial C_r(0')\big\},$$

where $C_r(0') = \{x' \in \mathbb{R}^2:\ |x'| < r\}$.

By arguing as in Lemma III.1 of [2], it is easy to prove that

$$j^b_{\lambda,r}(\zeta_{\lambda,r}) \leqslant \pi \log\lambda + 2\pi \log r + j^b_{1,1}(\zeta_{1,1}), \quad \forall r \in]0, +\infty[,\ \forall \lambda \geqslant \frac{1}{r^2}. \tag{3.15}$$

Let $\bar r \in]0, +\infty[$ be such that $\Theta \subset C_{\bar r}(0')$. Then, by virtue of (3.15), it results that

$$j^b_{\lambda}(\zeta_\lambda) \leqslant j^b_{\lambda,\bar r}(\zeta_{\lambda,\bar r}) \leqslant \pi \log\lambda + 2\pi \log\bar r + j^b_{1,1}(\zeta_{1,1}), \quad \forall \lambda \geqslant \frac{1}{\bar r^2},$$

which provides estimate (3.9) with $\lambda_1 = \frac{1}{\bar r^2}$ and $c_1 = 2\pi \log\bar r + j^b_{1,1}(\zeta_{1,1})$.

The next step is devoted to prove that

$$\liminf_{\lambda\to+\infty} \int\limits_{\Theta} \lambda\big|\zeta_\lambda(x') - f^b(x')\big|^2\, dx' < +\infty. \tag{3.16}$$

The proof of (3.16) makes use of a technique introduced in [26] in the case of the Ginzburg–Landau energy.

Since, for $\bar\lambda \geqslant \underline{\lambda}$, it results

$$j^b_{\bar\lambda}(\zeta_{\underline{\lambda}}) \geqslant j^b_{\bar\lambda}(\zeta_{\bar\lambda}) \geqslant j^b_{\underline{\lambda}}(\zeta_{\bar\lambda}) \geqslant j^b_{\underline{\lambda}}(\zeta_{\underline{\lambda}}),$$

one derives that the function $\lambda \in]0, +\infty[\to j^b_\lambda(\zeta_\lambda) \in]0, +\infty[$ is increasing, and therefore derivable a.e. in $]0, +\infty[$, and that

$$\frac{j^b_{\bar\lambda}(\zeta_{\underline{\lambda}}) - j^b_{\underline{\lambda}}(\zeta_{\underline{\lambda}})}{\bar\lambda - \underline{\lambda}} \geqslant \frac{j^b_{\bar\lambda}(\zeta_{\bar\lambda}) - j^b_{\underline{\lambda}}(\zeta_{\underline{\lambda}})}{\bar\lambda - \underline{\lambda}}, \quad \forall \underline{\lambda} \in]0, +\infty[,\ \forall \bar\lambda \in]\underline{\lambda}, +\infty[, \tag{3.17}$$

$$\frac{j^b_{\bar\lambda}(\zeta_{\bar\lambda}) - j^b_{\underline{\lambda}}(\zeta_{\underline{\lambda}})}{\bar\lambda - \underline{\lambda}} \geqslant \frac{j^b_{\bar\lambda}(\zeta_{\bar\lambda}) - j^b_{\underline{\lambda}}(\zeta_{\bar\lambda})}{\bar\lambda - \underline{\lambda}}, \quad \forall \underline{\lambda} \in]0, +\infty[,\ \forall \bar\lambda \in]\underline{\lambda}, +\infty[. \tag{3.18}$$

Consequently, by passing to the limit in (3.17) with $\bar\lambda \to \underline{\lambda}^+$ and in (3.18) with $\underline{\lambda} \to \bar\lambda^-$, one obtains that

$$\frac{dj^b_\lambda(\zeta_\lambda)}{d\lambda} = \int\limits_{\Theta} \big|\zeta_\lambda(x') - f^b(x')\big|^2\, dx', \quad \text{for } \lambda \text{ a.e. in }]0, +\infty[,$$

i.e.

$$j^b_\lambda(\zeta_\lambda) = j^b_1(\zeta_1) + \int\limits_1^\lambda \bigg(\int\limits_\Theta \big|\zeta_\mu(x') - f^b(x')\big|^2\, dx' \bigg)\, d\mu, \quad \forall \lambda \in]0, +\infty[. \tag{3.19}$$

To prove (3.16), by arguing by contradiction, assume that

$$\lim_{\lambda\to+\infty} \int\limits_{\Theta} \lambda\big|\zeta_\lambda(x') - f^b(x')\big|^2\, dx' = +\infty.$$

Consequently, there exists $\lambda_2 \in]1, +\infty[$ such that

$$\lambda \int_\Theta \left|\zeta_\lambda(x') - f^b(x')\right|^2 dx' > \pi + 1, \quad \forall \lambda \in]\lambda_2, +\infty[. \tag{3.20}$$

By combining (3.9) with (3.19) and (3.20), one obtains

$$\begin{aligned}
\pi \log\lambda + c_1 \geqslant j_\lambda^b(\zeta_\lambda) &= j_1^b(\zeta_1) + \int_1^\lambda \left(\int_\Theta \left|\zeta_\mu(x') - f^b(x')\right|^2 dx' \right) d\mu \\
&= j_1^b(\zeta_1) + \int_1^{\lambda_1} \left(\int_\Theta \left|\zeta_\mu(x') - f^b(x')\right|^2 dx' \right) d\mu + \int_{\lambda_1}^\lambda \frac{1}{\mu} \left(\int_\Theta \mu \left|\zeta_\mu(x') - f^b(x')\right|^2 dx' \right) d\mu \\
&\geqslant j_{\lambda_1}^b(\zeta_{\lambda_1}) + (\pi + 1)(\log\lambda - \log\lambda_1), \quad \forall \lambda \in \big]\max\{\lambda_1, \lambda_2\}, +\infty\big[,
\end{aligned}$$

which gives

$$\pi \geqslant \pi + 1.$$

So estimate (3.16) holds true. In particular, (3.16) provides the existence of a constant $c_2 \in]0, +\infty[$ and of a diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ satisfying (3.10).

The next step is devoted to prove estimate (3.11).

Let $\tilde{r} \in]0, 1[$ be such that $C_{\tilde{r}}(0') \subset \Theta$. Then, it results that

$$\begin{aligned}
j_{\lambda_k}^b(\zeta_{\lambda_k}) &\geqslant \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \left|D\zeta_{\lambda_k}(x')\right|^2 dx' \\
&= \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \left|D\big(f^b(x') + \zeta_{\lambda_k}(x') - f^b(x')\big)\right|^2 dx' \\
&\geqslant \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \left|Df^b(x')\right|^2 dx' + 2 \int_{C_{\tilde{r}}(0')\setminus C_r(0')} Df^b(x') D\big(\zeta_{\lambda_k}(x') - f^b(x')\big)\, dx', \\
&\forall r \in]0, \tilde{r}[,\ \forall k \in \mathbb{N}.
\end{aligned} \tag{3.21}$$

Consequently, by integrating by parts the last integral in (3.21) and by recalling that $f^b = (\frac{x'}{|x'|}, 0)$, it follows that

$$\begin{aligned}
j_{\lambda_k}^b(\zeta_{\lambda_k}) \geqslant{}& \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \left|D\frac{x'}{|x'|}\right|^2 dx' + 2\sum_{\alpha=1}^2 \int_{\partial(C_{\tilde{r}}(0')\setminus C_r(0'))} \left((\zeta_{\lambda_k})_\alpha(x') - \frac{x_\alpha}{|x'|}\right) D\frac{x_\alpha}{|x'|} \cdot \nu\, dx' \\
&- 2\sum_{\alpha=1}^2 \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \big((\zeta_{\lambda_k})_\alpha(x') - f_\alpha^b(x')\big) \Delta \frac{x_\alpha}{|x'|}\, dx', \quad \forall r \in]0, \tilde{r}[,\ \forall k \in \mathbb{N},
\end{aligned} \tag{3.22}$$

where ν denotes the exterior unit normal to $C_{\tilde{r}}(0') \setminus C_r(0')$, and $\zeta_{\lambda_k} = ((\zeta_{\lambda_k})_1, (\zeta_{\lambda_k})_2, (\zeta_{\lambda_k})_3)$.

On the other hand, it is evident that

$$\int_{C_{\tilde{r}}(0')\setminus C_r(0')} \left|D\frac{x'}{|x'|}\right|^2 dx' = \int_{C_{\tilde{r}}(0')\setminus C_r(0')} \frac{1}{|x'|^2}\, dx' = 2\pi\left(\log\tilde{r} + \log\frac{1}{r}\right), \quad \forall r \in]0, \tilde{r}[, \tag{3.23}$$

and

$$2\sum_{\alpha=1}^2 \int_{\partial(C_{\tilde{r}}(0')\setminus C_r(0'))} \left((\zeta_{\lambda_k})_\alpha(x') - \frac{x_\alpha}{|x'|}\right) D\frac{x_\alpha}{|x'|} \cdot \nu\, dx' = 0, \quad \forall r \in]0, \tilde{r}[,\ \forall k \in \mathbb{N}, \tag{3.24}$$

since $D\frac{x_\alpha}{|x'|}\cdot\nu = 0$ on $\partial(C_{\tilde r}(0')\setminus C_r(0'))$.

In what concerns the last integral in (3.22), by recalling that $\Delta\frac{x_\alpha}{|x'|} = \frac{x_\alpha}{|x'|^3}$ and by applying the Hölder inequality, it results that

$$\begin{aligned}&\left|2\int_{C_{\tilde r}(0')\setminus C_r(0')}\sum_{\alpha=1}^{2}\big((\zeta_{\lambda_k})_\alpha(x') - f_\alpha^b(x')\big)\Delta\frac{x_\alpha}{|x'|}\,dx'\right|\\ &\quad\leqslant 2\Bigg(\int_{C_{\tilde r}(0')\setminus C_r(0')}\big|\zeta_{\lambda_k}(x') - f^b(x')\big|^2\,dx'\Bigg)^{\frac12}\Bigg(\int_{C_{\tilde r}(0')\setminus C_r(0')}\frac{1}{|x'|^4}\,dx'\Bigg)^{\frac12},\quad \forall r\in\,]0,\tilde r[,\ \forall k\in\mathbb{N}.\end{aligned}$$

Consequently, by taking into account estimate (3.10) and that

$$\Bigg(\int_{C_{\tilde r}(0')\setminus C_r(0')}\frac{1}{|x'|^4}\,dx'\Bigg)^{\frac12} = \sqrt{\pi}\bigg(-\frac{1}{\tilde r^2}+\frac{1}{r^2}\bigg)^{\frac12}\leqslant\frac{\sqrt{\pi}}{r},\quad \forall r\in\,]0,\tilde r[,\ \forall k\in\mathbb{N},$$

it follows that

$$\left|2\int_{C_{\tilde r}(0')\setminus C_r(0')}\sum_{\alpha=1}^{2}\big((\zeta_{\lambda_k})_\alpha(x') - f_\alpha^b(x')\big)\Delta\frac{x_\alpha}{|x'|}\,dx'\right|\leqslant\frac{2\sqrt{c_2\pi}}{\sqrt{\lambda_k}}\frac{1}{r},\quad \forall r\in\,]0,\tilde r[,\ \forall k\in\mathbb{N}. \tag{3.25}$$

Finally, by combining (3.22) with (3.23), (3.24) and (3.25), one derives that

$$j_{\lambda_k}^b(\zeta_{\lambda_k})\geqslant 2\pi\log\frac{1}{r}+2\pi\log\tilde r-\frac{2\sqrt{c_2\pi}}{\sqrt{\lambda_k}}\frac{1}{r}\quad \forall r\in\,]0,\tilde r[,\ \forall k\in\mathbb{N},$$

from which, by choosing $r = 1/\sqrt{\lambda_k}$ with $k\in\mathbb{N}$ large enough, one obtains (3.11) with $c_3 = -2\pi\log\tilde r + 2\sqrt{c_2\pi}$. □

Proof of Proposition 3.4. To prove estimate (3.13), for every $t\in\,]0,+\infty[$ introduce the function:

$$z_t : x_3\in\,]0,1[\,\to\frac{1}{\sqrt{t^2+(x_3-\gamma)^2}}(t,0,x_3-\gamma)\in S^2.$$

Since $z_t\in H^1(]0,1[,S^2)$, from (3.6) it follows that

$$\begin{aligned}&j_\lambda^a(w_\lambda)\leqslant|\Theta|\int_0^1\big|(z_t)'(x_3)\big|^2\,dx_3+|\Theta|\lambda t\frac{1}{t}\int_0^1\bigg|z_t(x_3)-\bigg(0,0,\frac{x_3-\gamma}{|x_3-\gamma|}\bigg)\bigg|^2\,dx_3,\\ &\quad\forall t\in\,]0,+\infty[,\ \forall\lambda\in\,]0,+\infty[,\end{aligned}\tag{3.26}$$

where $(z_t)'$ stands for the derivative of z_t.

An easy computation shows that

$$\begin{aligned}\int_0^1\big|(z_t)'(x_3)\big|^2\,dx_3 &= \int_0^1\frac{t^2}{(t^2+(x_3-\gamma)^2)^2}\,dx_3 = \frac{1}{t^2}\int_0^1\frac{1}{(1+(\frac{x_3-\gamma}{t})^2)^2}\,dx_3\\ &= \frac1t\int_{-\frac{\gamma}{t}}^{\frac{1-\gamma}{t}}\frac{1}{(1+y^2)^2}\,dy = \frac1t\frac12\bigg[\frac{y}{1+y^2}+\arctan y\bigg]_{-\frac{\gamma}{t}}^{\frac{1-\gamma}{t}}\\ &= \frac1t\frac12\bigg(\frac{t(1-\gamma)}{t^2+(1-\gamma)^2}+\frac{t\gamma}{t^2+\gamma^2}+\arctan\bigg(\frac{1-\gamma}{t}\bigg)+\arctan\bigg(\frac{\gamma}{t}\bigg)\bigg).\end{aligned}$$

Consequently, since

$$\lim_{t\to0^+}\bigg[\frac12\bigg(\frac{t(1-\gamma)}{t^2+(1-\gamma)^2}+\frac{t\gamma}{t^2+\gamma^2}+\arctan\bigg(\frac{1-\gamma}{t}\bigg)+\arctan\bigg(\frac{\gamma}{t}\bigg)\bigg)\bigg]=\frac{\pi}{2},$$

and

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{t(1-\gamma)}{t^2+(1-\gamma)^2}+\frac{t\gamma}{t^2+\gamma^2}+\arctan\left(\frac{1-\gamma}{t}\right)+\arctan\left(\frac{\gamma}{t}\right)\right)\right]$$
$$=-\frac{t^2(1-\gamma)}{(t^2+(1-\gamma)^2)^2}-\frac{t^2\gamma}{(t^2+\gamma^2)^2}<0,\quad \forall t\in\,]0,+\infty[,$$

it results that

$$\int_0^1 \left|(z_t)'(x_3)\right|^2 dx_3 \leqslant \frac{\pi}{2}\frac{1}{t},\quad \forall t\in\,]0,+\infty[. \tag{3.27}$$

On the other hand, an easy computation shows that

$$\frac{1}{t}\int_0^1 \left|z_t(x_3)-\left(0,0,\frac{x_3-\gamma}{|x_3-\gamma|}\right)\right|^2 dx_3=\frac{2}{t}-\frac{2}{t}\int_0^1 \frac{|x_3-\gamma|}{\sqrt{(x_3-\gamma)^2+t^2}}\,dx_3$$
$$=\frac{2}{t}\left(1+2t-\sqrt{t^2+\gamma^2}-\sqrt{t^2+(1-\gamma)^2}\right),\quad \forall t\in\,]0,+\infty[.$$

Consequently, since

$$\lim_{t\to 0^+}\left[\frac{2}{t}\left(1+2t-\sqrt{t^2+\gamma^2}-\sqrt{t^2+(1-\gamma)^2}\right)\right]=4$$

and

$$\frac{d}{dt}\left[\frac{2}{t}\left(1+2t-\sqrt{t^2+\gamma^2}-\sqrt{t^2+(1-\gamma)^2}\right)\right]$$
$$=-\frac{2}{t^2}\left(\gamma-\frac{\gamma^2}{\sqrt{t^2+\gamma^2}}\right)-\frac{2}{t^2}\left((1-\gamma)-\frac{(1-\gamma)^2}{\sqrt{t^2+(1-\gamma)^2}}\right)<0,\quad \forall t\in\,]0,+\infty[,$$

it results that

$$\frac{1}{t}\int_0^1 \left|z_t(x_3)-\left(0,0,\frac{x_3-\gamma}{|x_3-\gamma|}\right)\right|^2 dx_3\leqslant 4,\quad \forall t\in\,]0,+\infty[. \tag{3.28}$$

By combining (3.26) with (3.27) and (3.28), it follows that

$$j_\lambda^a(w_\lambda)\leqslant|\Theta|\left(\frac{\pi}{2}\frac{1}{t}+4\lambda t\right),\quad \forall t\in\,]0,+\infty[,\ \forall\lambda\in\,]0,+\infty[,$$

from which, by choosing $t=\sqrt{\pi}/\sqrt{8\lambda}$, one obtains estimate (3.13).

To prove estimate (3.14), at first remark that

$$j_\lambda^a(w_\lambda)\geqslant|\Theta|\min\left\{\int_0^1\left|v'(x_3)\right|^2dx_3+\lambda\int_0^1\left|v(x_3)-\frac{x_3-\gamma}{|x_3-\gamma|}\right|^2dx_3\colon\ v\in H^1\big(]0,1[,\mathbb{R}\big)\right\},$$
$$\forall\lambda\in\,]0,+\infty[. \tag{3.29}$$

For every $\lambda\in\,]0,+\infty[$, the last minimum is attained in the solution $v_\lambda\in C^1(]0,1[)$ of the following problem:

$$\begin{cases} v_\lambda''-\lambda v_\lambda=-\lambda\frac{x_3-\gamma}{|x_3-\gamma|}, & \text{in }]0,1[\setminus\{\gamma\},\\ v_\lambda'(0)=0=v_\lambda'(1), & \end{cases}$$

that is in

$$v_\lambda(x_3) = \begin{cases} \frac{e^{-(2-\gamma)\sqrt{\lambda}} - e^{-\gamma\sqrt{\lambda}}}{e^{-2\sqrt{\lambda}}-1}(e^{x_3\sqrt{\lambda}} + e^{-x_3\sqrt{\lambda}}) - 1, & \text{in }]0,\gamma], \\ \frac{e^{-(2-\gamma)\sqrt{\lambda}} - e^{-(\gamma+2)\sqrt{\lambda}}}{e^{-2\sqrt{\lambda}}-1} e^{x_3\sqrt{\lambda}} + \frac{e^{\gamma\sqrt{\lambda}} - e^{-\gamma\sqrt{\lambda}}}{e^{-2\sqrt{\lambda}}-1} e^{-x_3\sqrt{\lambda}} + 1, & \text{in }]\gamma, 1[. \end{cases} \tag{3.30}$$

By combining (3.29) with (3.30), it follows that

$$\begin{aligned} j_\lambda^a(w_\lambda) &\geqslant |\Theta| \min\left\{ \int_0^1 \left|v'(x_3)\right|^2 dx_3 + \lambda \int_0^1 \left| v(x_3) - \frac{x_3-\gamma}{|x_3-\gamma|}\right|^2 dx_3 \colon\ v \in H^1\big(]0,1[,\mathbb{R}\big)\right\} \\ &= |\Theta| \frac{1 + e^{-2(\gamma+1)\sqrt{\lambda}} + e^{-2(2-\gamma)\sqrt{\lambda}} - e^{-4\sqrt{\lambda}} - e^{-2(1-\gamma)\sqrt{\lambda}} - e^{-2\gamma\sqrt{\lambda}}}{(e^{-2\sqrt{\lambda}}-1)^2} 2\sqrt{\lambda}, \quad \forall \lambda \in]0,+\infty[, \end{aligned}$$

from which, since

$$\lim_{\lambda\to+\infty} \frac{1 + e^{-2(\gamma+1)\sqrt{\lambda}} + e^{-2(2-\gamma)\sqrt{\lambda}} - e^{-4\sqrt{\lambda}} - e^{-2(1-\gamma)\sqrt{\lambda}} - e^{-2\gamma\sqrt{\lambda}}}{(e^{-2\sqrt{\lambda}}-1)^2} = 1,$$

one obtains estimate (3.14). □

Acknowledgements

This paper is part of the project: "Strutture sottili" of the program 2004–2006: "Collaborazioni interuniversitarie internazionali" of the Italian Ministry of Education, University and Research.

The authors are grateful to Catalin Lefter for suggesting them the proof of estimate (3.10).

References

[1] R. Alicandro, A. Corbo Esposito, C. Leone, Relaxation in BV of integral functionals defined on Sobolev functions with values in the unit sphere, J. Convex Anal. 14 (1) (2007) 69–98.
[2] F. Bethuel, H. Brezis, F. Hélein, Ginzburg–Landau Vortices, Birkhäuser, Boston, 1994.
[3] F. Bethuel, X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (2) (1988) 60–75.
[4] P.G. Ciarlet, Plates and Junctions in Elastic Multistructures: An Asymptotic Analysis, Research in Applied Mathematics, vol. 14, Masson, Springer-Verlag, Paris, Berlin, 1990.
[5] P.G. Ciarlet, P. Destuynder, A justification of the two-dimensional linear plate model, J. Mécanique 18 (2) (1979) 315–344.
[6] D. Cioranescu, J. Saint Jean Paulin, Homogenization of Reticulated Structures, Applied Mathematical Sciences, vol. 139, Springer-Verlag, New York, 1999.
[7] P. Courilleau, R. Hadiji, S. Dumont, Regularity of minimizing maps with values in S^2 and some numerical simulations, Adv. Math. Sci. Appl. 10 (2) (2000) 711–733.
[8] A. Damlamian, M. Vogelius, Homogenization limits of the equations of elasticity in thin domains, SIAM J. Math. Anal. 18 (2) (1987) 435–451.
[9] E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 58 (8) (6) (1975) 842–850.
[10] A. Gaudiello, B. Gustafsson, C. Lefter, J. Mossino, Coupled and uncoupled limits for a N-dimensional multidomain Neumann problem, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 985–990.
[11] A. Gaudiello, B. Gustafsson, C. Lefter, J. Mossino, Asymptotic analysis of a class of minimization problems in a thin multidomain, Calc. Var. Partial Differential Equations 15 (2) (2002) 181–201.
[12] A. Gaudiello, B. Gustafsson, C. Lefter, J. Mossino, Asymptotic analysis for monotone quasilinear problems in thin multidomains, Differential Integral Equations 15 (5) (2002) 623–640.
[13] A. Gaudiello, R. Monneau, J. Mossino, F. Murat, A. Sili, On the junction of elastic plates and beams, C. R. Acad. Sci. Paris Sér. I Math. 335 (8) (2002) 717–722.
[14] A. Gaudiello, R. Monneau, J. Mossino, F. Murat, A. Sili, Junction of elastic plates and beams, ESAIM Control Optim. Calc. Var. 13 (3) (2007) 419–457.
[15] A. Gaudiello, A. Sili, Asymptotic analysis of the eigenvalues of a Laplacian problem in a thin multidomain, Indiana Univ. Math. J. 56 (4) (2007) 1675–1710.
[16] A. Gaudiello, E. Zappale, Junction in a thin multidomain for a fourth order problem, M^3AS: Math. Models Methods Appl. Sci. 16 (12) (2006) 1887–1918.
[17] B.H. Guo, M.C. Hong, The Landau–Lifshitz equation of the ferromagnetic spin chain and harmonic maps, Calc. Var. Partial Differential Equations 1 (3) (1993) 311–334.

[18] R. Hadiji, F. Zhou, Regularity of $\int_\Omega |\nabla u|^2 + \lambda \int_\Omega |u - f|^2$ and some gap phenomenon, Potential Anal. 1 (1992) 385–400.
[19] M.C. Hong, The Landau–Lifshitz equation with the external field – a new extension or harmonic maps with values in S^2, Math. Z. 220 (2) (1995) 171–188.
[20] I. Gruais, Modélisation de la jonction entre une plaque et une poutre en élasticité linéarisée, RAIRO Modél. Math. Anal. Numér. 27 (1) (1993) 77–105.
[21] V.A. Kozlov, V.G. Ma'zya, A.B. Movchan, Asymptotic Analysis of Fields in a Multi-Structure, Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1999.
[22] L.D. Landau, E.M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, in: D. ter Haar (Ed.), Phys. Z. Sowjetunion 8 (1935) 153. Reproduced in: Collected Papers of L.D. Landau, Pergamon Press, New York, 1965, pp. 101–114.
[23] H. Le Dret, Problèmes variationnels dans le multi-domaines: modélisation des jonctions et applications, Research in Applied Mathematics, vol. 19, Masson, Paris, 1991.
[24] K.T. Smith, Primer of Modern Analysis, Springer-Verlag, New York, 1983.
[25] H.E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, New York and Oxford, 1971.
[26] M. Struwe, Une estimation asymptotique pour le modèle de Ginzburg–Landau, C. R. Acad. Sci. Paris Sér. I Math. 317 (7) (1993) 677–680.
[27] L. Trabucho, J.M. Viano, Mathematical Modelling of Rods, Handbook of Numerical Analysis, vol. 4, North-Holland, Amsterdam, 1996.