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Ann. I. H. Poincaré – AN 26 (2009) 191–222

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

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On the spectrum of a nonlinear planar problem ☆

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Received 28 March 2007; received in revised form 24 October 2007; accepted 25 October 2007

Available online 1 November 2007

Abstract

We consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda\mu e^{u_\lambda} v & \text{in } \Omega, \\ \|v\|_\infty = 1 \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^2$, $\lambda > 0$ is a real parameter and u_λ is a solution of

$$\begin{cases} -\Delta u_\lambda = \lambda e^{u_\lambda} & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases}$$

such that $\lambda \int_\Omega e^{u_\lambda} \to 8\pi$ as $\lambda \to 0$. In this paper we study the asymptotic behavior of the eigenvalues μ of (0.1) as $\lambda \to 0$. Moreover some explicit estimates for the four first eigenvalues and eigenfunctions are given.

Other related results as the Morse index of the solution u_λ will be proved.
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1. Introduction

Let us consider the Gelfand problem,

$$\begin{cases} -\Delta u = \lambda e^{u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$ and $\lambda > 0$ is a real parameter.

Eq. (1.1) has many applications. For example it arises in the contest of the statistical Mechanics as done in [5,6] (see also [17,9] and references therein).

Another interesting field where (1.1) appears is in the Chern–Simon–Higgs model (see for example [23] and the references therein).

☆ Supported by M.I.U.R., project "Variational methods and nonlinear differential equations".

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doi:10.1016/j.anihpc.2007.10.004

Throughout the paper we will consider a solution u_λ to (1.1) satisfying

$$\lambda \int_\Omega e^{u_\lambda} \to 8\pi \quad \text{as } \lambda \to 0. \tag{1.2}$$

The behavior of solutions of (1.1) satisfying (1.2) was largely studied by many authors. We can just mention here the papers [8,11,12,20,22,23] as well as many others.

Condition (1.2) corresponds to study the so-called one point blowing-up solution, i.e. solutions whose maximum is achieved exactly at one point where the solution goes to $+\infty$.

For this class of solutions, denoted by u_λ, we consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda \mu e^{u_\lambda} v & \text{in } \Omega, \\ \|v\|_\infty = 1, & \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

and we study some properties of the eigenvalues μ and of the corresponding eigenfunctions.

Problem (1.3) comes out from the eigenvalue problem related to the second derivative of the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \int_\Omega e^u$$

in the Hilbert space $H_0^1(\Omega)$. The study of the spectrum of F'' is crucial to calculate the Morse index of the solution u_λ. One of the result of this paper will be the computation of the Morse index of the solution u_λ in some special cases.

Another interesting problem linked to (1.3) is the classical problem of the nodal line of the second eigenfunction. It was proved by A. Melas [19] that if we consider the second eigenfunction of the Laplace operator in a planar convex domains then its nodal line touches the boundary. This result is largely open for eigenfunctions of higher order. In this paper we describe some properties of the nodal line of the eigenfunctions to (1.3). For example we show that, if Ω is convex, the nodal line of the second and third eigenfunction touches the boundary. On the other hand, without any assumption on Ω, we prove that the nodal line of the fourth eigenfunction does not touch the boundary of Ω. Moreover, the asymptotic behavior of these eigenfunctions is described.

A crucial tool in the study the eigenvalue problem (1.3) is given by the following "limit" problem,

$$\begin{cases} -\Delta v = \frac{\mu_\infty}{(1+|x|^2/8)^2} v & \text{in } \mathbb{R}^2, \\ v \in L^\infty(\mathbb{R}^2). \end{cases} \tag{1.4}$$

Roughly speaking, the eigenvalues μ_λ of (1.3) converge to the eigenvalues μ_∞ of the problem (1.4) as $\lambda \to 0$ and the same happens for the corresponding eigenfunctions (up to a scale argument). This will be stated precisely in Section 11.

The eigenfunctions of problem (1.4) can be explicitly computed using the *Legendre* function. In this way one can see that any eigenvalue has multiplicity greater than 1 and the corresponding eigenfunctions can be divided in two classes.

The first one is given by nonradial functions which go to zero at infinity and the second one is given by radial functions which converge to a nonzero constant at infinity. These two types of limiting eigenfunctions give rise to eigenfunctions of problem (1.3) which behave differently.

In this way we give some "global" results about the spectrum of (1.3) but the most important aim of this paper is to study with great attention the first fourth eigenvalues.

For example we will see that the second and the third eigenfunctions of (1.3) look like the nonradial eigenfunctions related to the eigenvalue $\mu_\infty = 1$ and the fourth eigenfunction of (1.3) looks like the radial eigenfunction related again to $\mu_\infty = 1$.

In this case we will compute asymptotic expansions for the eigenvalues and related eigenfunction. Note that, even in the case of the first eigenvalue, more work is needed.

The asymptotic estimate on the second and third eigenvalue enables to derive some results on the Morse index of the solution. The first one says that if Ω is a convex domain then the Morse index of the solution u_λ of (1.1) is exactly 1 (see Corollary 2.8). Moreover, for a general domain we will derive that the Morse index of the solution u_λ is at most 2. This last result appears differently from singular problems in higher dimensions (see [1] for example).

Finally, we observe that our results have some similarities with the corresponding in [16], where was considered a perturbed critical Sobolev exponent in $\mathbb{R}^N$ for $N \geqslant 3$. But since in $\mathbb{R}^2$ we do not have the Sobolev Embedding Theorem and some orthogonality properties of the eigenfunctions, here the problem seems harder.

The paper is organized as follows: in Section 2 we state our main results; in Section 3 we recall some known facts about problem (1.1); in Section 4 we consider the first eigenvalue and the first eigenfunction and in Section 5 we give an important estimate on the second eigenvalue; in Section 6 we study the behavior of the second eigenfunction; Section 7 is devoted to the third eigenvalue and the third eigenfunction; in Section 8 the asymptotic behavior of the second and third eigenvalues of (1.1) is proved and, using a result of [4], we have that, for a convex domain, the Morse index of the solution u_λ is 1; in Section 9 we consider the nodal region for the second and third eigenfunctions; in Section 10 we treat the case of the fourth eigenfunction; in Section 11 finally we get the asymptotic behavior of the spectrum of problem (1.1).

2. Statement of the results

Let $G(x,y)$ be the Green's function of $-\Delta$ in Ω with Dirichlet boundary conditions. Then

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + H(x,y), \tag{2.1}$$

where $H(x,y)$ is the regular part of the Green function. Let $R(x) = H(x,x)$ be the Robin function of Ω.

Let us consider the solution u_λ of (1.1) satisfying (1.2). For such a solution we consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda\mu e^{u_\lambda} v & \text{in } \Omega, \\ \|v\|_\infty = 1, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It is well known that problem (2.2) admits a sequence of eigenvalues $\mu_{1,\lambda} < \mu_{2,\lambda} \leqslant \mu_{3,\lambda} \leqslant \cdots$. Let $v_{i,\lambda}$ be the eigenfunction corresponding to the eigenvalue $\mu_{i,\lambda}$, i.e. $v_{i,\lambda}$ solves

$$\begin{cases} -\Delta v_{i,\lambda} = \lambda\mu_{i,\lambda} e^{u} v_{i,\lambda} & \text{in } \Omega, \\ \|v_{i,\lambda}\|_\infty = 1, \\ v_{i,\lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In order to state our results we recall that if x_λ is a maximum point of u_λ, i.e. a point such that $u_\lambda(x_\lambda) = \|u_\lambda\|_\infty$, we have that x_λ converges to a point $x_0 \in \Omega$ (see Section 3 for details). Let

$$\delta_\lambda = \left(\frac{1}{\lambda e^{\|u_\lambda\|_\infty}}\right)^{1/2} \tag{2.4}$$

and $\tilde v_{i,\lambda}(x) = v_{i,\lambda}(\delta_\lambda x + x_\lambda)$ be the rescaled eigenfunction defined in the domain $\Omega_\lambda = \frac{1}{\delta_\lambda}(\Omega - x_\lambda)$.

We start with some results concerning the eigenvalues and the eigenfunctions of (2.2).

Theorem 2.1. *Let u_λ be a solution of* (1.1) *which satisfies* (1.2)*, and let $\mu_{1,\lambda}$ be the first eigenvalue of* (2.3) *and $v_{1,\lambda}$ be the first eigenfunction. Then*

$$\mu_{1,\lambda} = -\frac{1}{2\log\lambda}\big(1 + o(1)\big); \tag{2.5}$$

$$\frac{v_{1,\lambda}}{\mu_{1,\lambda}} \to 8\pi G(x,x_0) \quad \text{in } C^1_{\mathrm{loc}}\big(\bar\Omega \setminus \{x_0\}\big)\ \lambda \to 0; \tag{2.6}$$

$$\tilde v_{1,\lambda} \to 1 \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2)\ \lambda \to 0. \tag{2.7}$$

Theorem 2.2. *In the same assumption of Theorem* 2.1*, we have*

$$\tilde v_{i,\lambda}(x) \to \tilde v_i = \frac{a_1^i x_1 + a_2^i x_2}{8 + |x|^2} \quad \text{as } \lambda \to 0 \tag{2.8}$$

in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ *for* $i=2,3$ *and some vectors* $a^i=(a^i_1,a^i_2)\neq 0$,

$$\frac{v_{i,\lambda}(x)}{\delta_\lambda}\to 2\pi\sum_{k=1}^{2}a^i_k\frac{\partial G(x,x_0)}{\partial y_k}\quad \text{as } \lambda\to 0 \tag{2.9}$$

in $C^1_{\mathrm{loc}}(\bar{\Omega}\setminus\{x_0\})$, *for* $i=2,3$, *where* (a^i_1,a^i_2) *are the same as in* (2.8).

Theorem 2.3. *Let* $c_1\leqslant c_2$ *be the eigenvalues of the Hessian matrix* $D^2R(x_0)$ *of the Robin function at* x_0. *Then*

$$\frac{1-\mu_{i,\lambda}}{\delta^2_\lambda}\to 24\pi\eta_i\quad \text{as } \lambda\to 0 \tag{2.10}$$

for $i=2,3$. *Moreover* $\eta_2=c_1$ *and* $\eta_3=c_2$, *and the vector* a^i *of* (2.8) *are the eigenvectors corresponding to* c_{i-1}.

Next we study some properties of the nodal line of the eigenfunctions. Let us recall that the nodal set of $v_{i,\lambda}$ is defined as

$$N_{i,\lambda}=\{x\in\Omega\colon\ v_{i,\lambda}(x)=0\}. \tag{2.11}$$

Theorem 2.4. *In the same assumptions of Theorem* 2.1, *we have*:

(i) *if* Ω *be convex, then* $\overline{N_{i,\lambda}}\cap\partial\Omega\neq\emptyset$, *for* $i=2,3$ *with* λ *small enough*;
(ii) *the eigenfunctions* $v_{i,\lambda}(x)$, $i=2,3$, *have only two nodal regions, if* λ *is small enough.*

Now we consider the fourth eigenvalue, getting the following results:

Theorem 2.5. *In the same hypothesis of Theorem* 2.1 *we have*

$$\tilde v_{4,\lambda}\to\tilde v_4=b\frac{8-|x|^2}{8+|x|^2}\quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2) \text{ as } \lambda\to 0; \tag{2.12}$$

$$v_{4,\lambda}(x)\log\lambda\to 4\pi bG(x,x_0)\quad \text{in } C^1_{\mathrm{loc}}\big(\bar{\Omega}\setminus\{x_0\}\big) \text{ as } \lambda\to 0; \tag{2.13}$$

$$1-\mu_{4,\lambda}=-\frac{1}{\log\lambda}\big(c_1+\mathrm{o}(1)\big), \tag{2.14}$$

where $b\in\mathbb{R}$, $b\neq 0$, $c_0=\frac{\pi}{6}$ *and* $c_1=\frac{2(1-4\pi)}{c_0}<0$.

Theorem 2.6. *The eigenvalue* $\mu_{4,\lambda}$ *is simple and the corresponding eigenfunction* $v_{4,\lambda}$ *has only two nodal regions if* λ *is small enough. Moreover the closure of the nodal set of* $v_{4,\lambda}$ *does not touch the boundary.*

Corollary 2.7. *Let* x_λ *be the maximum point of* u_λ *in* Ω, *and* $\lim x_\lambda=x_0\in\Omega$. *Then if* x_0 *is a nondegenerate critical point of the Robin function* $R(x)$ *of* Ω, *denoting by* $m(x_0)$ *the Morse index of* x_0 *as a critical point of* $R(x)$, *we find that the Morse index of* u_λ *is equal to* $m(x_0)+1$.

The previous result enable us to compute the Morse index of the solution u_λ. We recall that the Morse index of a solution u_λ is the number of eigenvalue μ less than 1.

Corollary 2.8. *Let* Ω *be a domain of* $\mathbb{R}^2$. *Then,*

(i) *the Morse index of* u_λ *in* Ω *is* 1 *or* 2;
(ii) *if* Ω *is a convex set then the Morse index of* u_λ *in* Ω *is exactly one.*

Our final result concerns the convergence of the *whole* spectrum. A crucial role is played by the limit problem (1.4). In order to state the precise result we need to introduce long and noisy notations. For this reason we prefer to state the results in Section 10. We just say that it will be proved that the whole spectrum converges to the corresponding one of the “limit problem” and an analogous convergence holds for the related eigenfunctions.

3. Preliminaries and known results

Let us recall the following known facts:

Theorem 3.1. *Let u_λ be a solution of* (1.1) *satisfying* (1.2). *Then if x_λ is a point such that $u_\lambda(x_\lambda)=\|u_\lambda\|_\infty$, we have that x_λ converges to a point $x_0\in\Omega$ such that*

$$\nabla R(x_0)=0, \tag{3.1}$$

$$u_\lambda(x)\to 8\pi G(x,x_0) \quad \text{in } C^1_{\mathrm{loc}}\big(\bar\Omega\setminus\{x_0\}\big), \tag{3.2}$$

$$\left|u_\lambda(x)-\log\frac{e^{u_\lambda(x_\lambda)}}{(1+\frac{1}{8}\lambda e^{u_\lambda(x_\lambda)}|x-x_\lambda|^2)^2}\right|\leqslant C \quad \text{in } \bar\Omega, \tag{3.3}$$

$$\|u_\lambda\|_\infty=-2\log\lambda+C_0-8\pi R(x_0)+\mathrm{o}(1) \quad \text{as } \lambda\to 0, \tag{3.4}$$

where $C_0=2\log 8$ and $R(x_0)=H(x_0,x_0)$.

Proof. Estimates (3.1) and (3.2) are proved in [20] while estimate (3.3) is proved in [15] using a result of [18]. In [15] it is also proved (3.4). □

Let δ_λ be as in (2.4). Then, from (3.4) it follows that $\delta_\lambda^2\to 0$. Considering the rescaled function

$$\tilde u_\lambda(x)=u(\delta_\lambda x+x_\lambda)-\|u_\lambda\|_\infty \tag{3.5}$$

for $x\in\Omega_\lambda=\frac{1}{\delta_\lambda}(\Omega-x_\lambda)$, the estimate (3.3) gives the following

$$\tilde u_\lambda(x)\leqslant C+\log\frac{1}{(1+|x|^2/8)^2} \quad \text{in } \Omega_\lambda, \tag{3.6}$$

where $C>0$.

Theorem 3.2. *Every solution $U\in C^2(\mathbb{R}^2)$ of the problem*

$$\begin{cases} -\Delta u=e^u & \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2}e^u<\infty, \end{cases} \tag{3.7}$$

is given by

$$U_{\delta,y}(x)=\log\frac{8\delta}{(\delta+|x-y|^2)^2} \tag{3.8}$$

for any $(\delta,y)\in\mathbb{R}^+\times\mathbb{R}^2$.

Proof. See Chen and Li [10]. □

In the sequel we write

$$U(x)=U_{8,0}=\log\frac{1}{(1+|x|^2/8)^2}.$$

Theorem 3.3. *Let $v\in C^2(\mathbb{R}^2)$ be a solution of the following problem*

$$\begin{cases} -\Delta v=\frac{1}{(1+|x|^2/8)^2}v & \text{in } \mathbb{R}^2,\\ v\in L^\infty(\mathbb{R}^2). \end{cases} \tag{3.9}$$

Then

$$v(x)=\sum_{i=1}^{2}\frac{a_ix_i}{8+|x|^2}+b\frac{8-|x|^2}{8+|x|^2} \tag{3.10}$$

for some $a_i,b\in\mathbb{R}$.

Proof. See [11] or also [13] for a more detailed proof. □

Lemma 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. For any $y \in \Omega$ we have*

$$\int_{\partial\Omega} (x-y)\cdot\nu(x)\left(\frac{\partial G(x,y)}{\partial\nu_x}\right)^2 d\sigma_x = \frac{1}{2\pi}, \tag{3.11}$$

$$\int_{\partial\Omega} \nu_i(x)\left(\frac{\partial G(x,y)}{\partial\nu_x}\right)^2 d\sigma_x = -\frac{\partial R(y)}{\partial y_i}, \tag{3.12}$$

$$\frac{\partial^2 R(y)}{\partial y_i \partial y_j} = -2\int_{\partial\Omega} \frac{\partial G(x,y)}{\partial x_i}\frac{\partial}{\partial y_j}\left(\frac{\partial G(x,y)}{\partial\nu_x}\right) d\sigma_x, \tag{3.13}$$

$$2\int_{\partial\Omega} (x-y)\cdot\nu_x \frac{\partial G}{\partial\nu_x}(x,y)\frac{\partial^2 G}{\partial y_i\partial\nu_x}(x,y)\, d\sigma_x = -\frac{\partial R}{\partial y_i}(y), \tag{3.14}$$

$$\int_{\partial\Omega} \frac{\partial^2 G}{\partial\nu_x\partial y_i}(x,y)\, d\sigma_x = 0 \tag{3.15}$$

for $i, j = 1, 2$.

Proof. See [15] for the proof of (3.11)–(3.13). Let us prove (3.14). Differentiating (3.11) we obtain

$$2\int_{\partial\Omega} (x-y)\cdot\nu_x \frac{\partial G}{\partial\nu_x}(x,y)\frac{\partial^2 G}{\partial y_i\partial\nu_x}(x,y)\, d\sigma_x = \int_{\partial\Omega} \nu_i\left(\frac{\partial G}{\partial\nu_x}(x,y)\right)^2 d\sigma_x.$$

The claim follows from (3.12).

To prove (3.15) is sufficient to note that

$$\int_{\partial\Omega} \frac{\partial G}{\partial\nu_x}(x,y)\, d\sigma_x = \int_{\Omega} \Delta G(x,y)\, dx \equiv -1$$

and differentiate. □

Lemma 3.5. *Let u_λ be a solution of* (1.1). *Then the function $\tilde u_\lambda : \Omega_\lambda = (\Omega - x_\lambda)/\delta_\lambda \mapsto \mathbb{R}$,*

$$\tilde u_\lambda(x) = u_\lambda(\delta_\lambda x + x_\lambda) - \|u_\lambda\| \tag{3.16}$$

verifies

$$\tilde u_\lambda(x) \to \log\frac{1}{(1+|x|^2/8)^2} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2). \tag{3.17}$$

Proof. The result is standard (see [15] for example). □

4. On the first eigenvalue and the first eigenfunction

Let $\mu_{1,\lambda}$ be the first eigenvalue of problem (2.2) and $v_{1,\lambda}$ the corresponding eigenfunction which solves

$$\begin{cases} -\Delta v_{1,\lambda} = \lambda\mu_{1,\lambda}e^{u_\lambda}v_{1,\lambda} & \text{in } \Omega, \\ \|v_{1,\lambda}\|_\infty = 1, \\ v_{1,\lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The first eigenvalue is given by the classical (Rayleigh–Ritz) variational formula, namely

$$\mu_{1,\lambda} = \inf_{\substack{v\in H^1_0(\Omega)\\ v\not\equiv 0}} \frac{\int_\Omega |\nabla v|^2\, dx}{\lambda\int_\Omega e^{u_\lambda}v^2\, dx}. \tag{4.2}$$

Lemma 4.1. *Let $\mu_{1,\lambda}$ be as stated before. Then $\mu_{1,\lambda} \to 0$ as $\lambda \to 0$.*

Proof. We want to estimate $\mu_{1,\lambda}$ using formula (4.2). Consider the function $u_\lambda \in H_0^1(\Omega)$. Then, by (1.1)

$$\mu_{1,\lambda} \leqslant \frac{\int_\Omega |\nabla u_\lambda|^2}{\lambda \int_\Omega e^{u_\lambda} u_\lambda^2} = \frac{\int_\Omega e^{u_\lambda} u_\lambda}{\int_\Omega e^{u_\lambda} u_\lambda^2}. \tag{4.3}$$

Using the rescaled function $\tilde{u}_\lambda(y) = u_\lambda(\delta_\lambda y + x_\lambda) - \|u_\lambda\|_\infty$ we have

$$\lambda \int_\Omega e^{u_\lambda} u_\lambda = \lambda \int_\Omega e^{u_\lambda}\big(u_\lambda - \|u_\lambda\|_\infty\big) + \lambda \|u_\lambda\|_\infty \int_\Omega e^{u_\lambda} \tag{4.4}$$

while

$$\lambda \int_\Omega e^{u_\lambda} u_\lambda^2 = \lambda \int_\Omega e^{u_\lambda}\big(u_\lambda - \|u_\lambda\|_\infty\big)^2 + \lambda \|u_\lambda\|_\infty^2 \int_\Omega e^{u_\lambda} + 2\lambda \|u_\lambda\|_\infty \int_\Omega e^{u_\lambda}\big(u_\lambda - \|u_\lambda\|_\infty\big). \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3), and then rescaling, we get

$$\mu_{1,\lambda} \leqslant \frac{\int_{\Omega_\lambda} e^{\tilde{u}_\lambda} \tilde{u}_\lambda + \|u_\lambda\|_\infty \lambda \int_\Omega e^{u_\lambda}}{\int_{\Omega_\lambda} e^{\tilde{u}_\lambda} \tilde{u}_\lambda^2 + 2\|u_\lambda\|_\infty \int_{\Omega_\lambda} e^{\tilde{u}_\lambda} \tilde{u}_\lambda + \|u_\lambda\|_\infty^2 \lambda \int_\Omega e^{u_\lambda}}. \tag{4.6}$$

By the estimate (3.6), we see that

$$\big|e^{\tilde{u}_\lambda} \tilde{u}_\lambda\big| \leqslant C \frac{C - \log(1 + |y|^2/8)^2}{(8 + |y|^2)^2}; \tag{4.7}$$

$$\big|e^{\tilde{u}_\lambda} \tilde{u}_\lambda^2\big| \leqslant C \frac{(C - \log(1 + |y|^2/8)^2)^2}{(8 + |y|^2)^2}. \tag{4.8}$$

By (4.7) and (4.8), we can pass into the limit into (4.6) getting

$$\mu_{1,\lambda} \leqslant \frac{C_1 + o(1) + \|u_\lambda\|_\infty(8\pi + o(1))}{C_2 + o(1) + 2\|u_\lambda\|_\infty(C_1 + o(1)) + (8\pi + o(1))\|u_\lambda\|_\infty^2} = \frac{1}{\|u_\lambda\|_\infty}\big(1 + o(1)\big) \tag{4.9}$$

where $C_1 = \int_{\mathbb{R}^2} e^U U = -16\pi$ and $C_2 = \int_{\mathbb{R}^2} e^U U^2 = 64\pi$. The claim follows using (3.4). $\square$

Lemma 4.2. *Let $\mu_{1,\lambda}$ and $v_{1,\lambda}$ be as stated before and let $\tilde{v}_{1,\lambda}(x) = v_{1,\lambda}(\delta_\lambda x + x_\lambda)$. Then $\tilde{v}_{1,\lambda} \to c$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, where $c \neq 0$ is a constant.*

Proof. It is easy to see that $\tilde{v}_{1,\lambda}$ satisfies the equation

$$\begin{cases} -\Delta \tilde{v}_{1,\lambda} = \mu_{1,\lambda} e^{\tilde{u}_\lambda} \tilde{v}_{1,\lambda} & \text{in } \Omega_\lambda, \\ \|\tilde{v}_{1,\lambda}\|_\infty = 1, & \\ \tilde{v}_{1,\lambda} = 0 & \text{on } \partial\Omega_\lambda, \end{cases} \tag{4.10}$$

where $\Omega_\lambda = (\Omega - x_\lambda)/\delta_\lambda$. From (3.6) the right-hand side of equation (4.10) is bounded in L^∞. Moreover $\|\tilde{v}_{1,\lambda}\|_\infty = 1$ implies that $|\nabla \tilde{v}_{1,\lambda}|$ is bounded in $L^2(\mathbb{R}^2)$. Then using the standard elliptic regularity theory, $\tilde{v}_{1,\lambda} \to v_1$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, where v_1 satisfies

$$\Delta v_1 = 0 \quad \text{in } \mathbb{R}^2, \tag{4.11}$$

and since $\|\tilde{v}_1\|_\infty = 1$ we infer that v_1 is a constant. We want to show that $v_1 \not\equiv 0$. Let z_λ be the points of Ω_λ such that $\tilde{v}_{1,\lambda}(z_\lambda) = 1$. If $v_1 \equiv 0$ then z_λ should go to the infinity. So let us consider

$$\hat{v}_{1,\lambda} = \tilde{v}_{1,\lambda}\left(\frac{x}{|x|^2}\right),$$

and

$$\hat{u}_\lambda = \tilde{u}_\lambda\left(\frac{x}{|x|^2}\right).$$

Then $\hat v_{1,\lambda}$ satisfies the equation

$$-\Delta \hat v_{1,\lambda} = \frac{1}{|x|^4}\mu_{1,\lambda} e^{\hat u_\lambda(x)} \hat v_{1,\lambda}(x)$$

where (again by (3.3))

$$\frac{1}{|x|^4}\mu_{1,\lambda} e^{\hat u_\lambda(x)} \leqslant \mu_{1,\lambda}\frac{1}{|x|^4}\left(\frac{C}{1+1/(8|x|^2)}\right)$$
$$\leqslant C\mu_{1,\lambda}\frac{64|x|^4}{|x|^4(8|x|^2+1)^2} \leqslant 64C\mu_{1,\lambda} \to 0.$$

Moreover $\|\hat v_{1,\lambda}\|_\infty \leqslant 1$ and $\hat v_{1,\lambda} \to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$, so that $\hat v_{1,\lambda} \to 0$ in $L^2(B_1(0))$. Since the capacity of one point is zero, we can apply the regularity theory to $\hat v_\lambda$ (see Theorem 8.17 in [14]) observing that

$$\|\hat v_{1,\lambda}\|_{L^\infty(B_{\frac12}(0))} \leqslant C\|\hat v_{1,\lambda}\|_{L^2(B_1(0))} \to 0.$$

This gives a contradiction since $\|\hat v_\lambda\|_{L^\infty(B_{\frac12}(0))} = \hat v_\lambda(z_\lambda) = 1$. □

Remark 4.3. In Lemma 4.5 we will show that $c = 1$.

Lemma 4.4. *Let $\mu_{1,\lambda}$ be the first eigenvalue of problem* (4.1). *Then*

$$\mu_{1,\lambda} = -\frac{1}{2\log\lambda(1+\mathrm{o}(1))}.$$

Proof. Multiplying Eq. (4.1) for u_λ and integrating, we get

$$\int_\Omega \nabla v_{1,\lambda}\cdot\nabla u_\lambda\,dx = \lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}u_\lambda v_{1,\lambda}\,dx; \tag{4.12}$$

while using Eq. (1.1) we get

$$\int_\Omega \nabla u_\lambda\cdot\nabla v_{1,\lambda}\,dx = \lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}\,dx. \tag{4.13}$$

Then

$$\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}\,dx = \lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}u_\lambda v_{1,\lambda}\,dx.$$

Rescaling both sides we have

$$\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde v_{1,\lambda}\,dy = \mu_{1,\lambda}\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde u_\lambda\tilde v_{1,\lambda}\,dy + \mu_{1,\lambda}\|u_\lambda\|_\infty\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde v_{1,\lambda}\,dy. \tag{4.14}$$

Using estimate (3.6) and $\|\tilde v_{1,\lambda}\|_\infty = 1$, we can pass to the limit in (4.14) and then

$$\int_{\mathbb{R}^2} e^{U(y)}v_1(y)\,dy + \mathrm{o}(1) = \mu_{1,\lambda}\int_{\Omega_\lambda} e^{\tilde u_\lambda(y)}\tilde u_\lambda(y)\tilde v_{1,\lambda}(y)\,dy + \mu_{1,\lambda}\|u_\lambda\|_\infty\left(\int_{\mathbb{R}^2} e^{U(y)}v_1(y)\,dy + \mathrm{o}(1)\right).$$

Again by estimate (4.7) and the boundedness of $\|v_{1,\lambda}\|_\infty$ we have

$$8\pi c + \mathrm{o}(1) = \mu_{1,\lambda}\left(\int_{\mathbb{R}^2} e^{U(y)}U(y)v_1(y)\,dy + \mathrm{o}(1)\right) + \mu_{1,\lambda}\|u_\lambda\|_\infty\big(8\pi c + \mathrm{o}(1)\big)$$
$$= \mu_{1,\lambda}\big(-16\pi c + \mathrm{o}(1)\big) + \mu_{1,\lambda}\|u_\lambda\|_\infty\big(8\pi c + \mathrm{o}(1)\big).$$

Passing to the limit as $\lambda \to 0$, we infer that $\lim_{\lambda\to0}\mu_{1,\lambda}\|u_\lambda\|_\infty = 1$ and the lemma is proved using the estimate (3.4). □

Lemma 4.5. *Let $v_{1,\lambda}$ and $\mu_{1,\lambda}$ be as stated before. Then*

$$\frac{v_{1,\lambda}(x)}{\mu_{1,\lambda}} \to 8\pi G(x,x_0) \quad \text{in } C^1_{\text{loc}}\big(\bar{\Omega}\setminus\{x_0\}\big) \text{ as } \lambda \to 0. \tag{4.15}$$

Proof. Let $x \in \bar{\Omega}\setminus\{x_0\}$. Using the Green's identity formula we have from (4.1),

$$\begin{aligned}\frac{v_{1,\lambda}(x)}{\mu_{1,\lambda}} &= \lambda\int_\Omega G(x,y)e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy\\ &= \lambda G(x,x_0)\int_\Omega e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy + \lambda\int_\Omega \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy\\ &= G(x,x_0)\int_{\tilde{\Omega}_\lambda} e^{\tilde{u}_\lambda(y)}\tilde{v}_{1,\lambda}(y)\,dy + I_{1,\lambda}\\ &= 8\pi c G(x,x_0) + o(1) + I_{1,\lambda}.\end{aligned} \tag{4.16}$$

The passage into the limit is done by using the estimate (3.6) and $|v_{1,\lambda}| \leqslant 1$.

To prove (4.15), we have to show that $I_{1,\lambda} \to 0$. We can choose $\rho > 0$ such that $B_\rho(x_0) \subset \Omega$ and $x \notin B_\rho(x_0)$. Then

$$\begin{aligned}I_{1,\lambda} &= \lambda\int_\Omega \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy\\ &= \lambda\int_{\Omega\setminus B_\rho(x_0)} \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy + \lambda\int_{B_\rho(x_0)} \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy.\end{aligned}$$

By estimate (3.3) and (3.4) we have

$$\lambda e^{u_\lambda(y)}v_{1,\lambda}(y) \leqslant \frac{C\lambda}{(c\lambda + \frac{1}{8}|y-x_\lambda|^2)^2}.$$

Recalling that $x_\lambda \to x_0$ we have that $|y-x_\lambda| \geqslant \frac{\rho}{2}$ in $\Omega\setminus B_\rho(x_0)$, if λ is small enough. Hence we get

$$\begin{aligned}\lambda\int_{\Omega\setminus B_\rho(x_0)} \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy &\leqslant \frac{C\lambda}{(c\lambda+\rho^2/32)^2}\int_{\Omega\setminus B_\rho(x_0)}\big[G(x,y)-G(x,x_0)\big]\,dy\\ &\leqslant \frac{C'\lambda}{(c\lambda+\rho^2/32)^2}.\end{aligned} \tag{4.17}$$

Choosing $\rho = \lambda^k$ with $k < \frac{1}{4}$ we get

$$\lambda\int_{\Omega\setminus B_\rho(x_0)} \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy \to 0.$$

On the other hand we have

$$\lambda\int_{B_\rho(x_0)} \big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{1,\lambda}(y)\,dy \leqslant \sup_{y\in B_\rho(x_0)}\big|G(x,y)-G(x,x_0)\big|\lambda\int_{B_\rho(x_0)} e^{u_\lambda}v_{1,\lambda}\,dy \to 0$$

because $x \notin B_\rho(x_0)$ and $\lambda e^{u_\lambda(y)}v_{1,\lambda}(y) \in L^1(\Omega)$. In this way we have that $I_{1,\lambda} \to 0$ and from estimate (4.16) we get (4.15).

The same proof applies for the derivatives of $v_{1,\lambda}$.

Now we prove that $c = 1$. We already know from Lemma 4.2 that $c \neq 0$. Let $z_\lambda \in \Omega$ such that $v_{1,\lambda}(z_\lambda) = 1$. This can be done since, by the definition of $v_{1,\lambda}$, we have $v_{1,\lambda} > 0$ in Ω. Up to a subsequence $z_\lambda \to z \in \bar{\Omega}$. If $z \neq x_0$ using equation (4.15), we have

$$\frac{1}{\mu_{1,\lambda}} = \frac{v_{1,\lambda}(z_\lambda)}{\mu_{1,\lambda}} \to 8\pi c G(z,x_0) \tag{4.18}$$

and this is not possible since the left-hand side goes to infinity while the right-hand side is bounded.

Hence we have that $z = x_0$. We have the following alternative: either $|z_\lambda - x_\lambda| > \delta_\lambda R$ for any $R > 0$ or $z_\lambda \in B_{\delta_\lambda R}(x_\lambda)$ for $R > 0$ and λ sufficiently small.

Case 1. We consider first the case where $|z_\lambda - x_\lambda| > \delta_\lambda R$ for any $R > 0$. Set $w_\lambda(x) = v_{1,\lambda}(r_\lambda x + x_\lambda) - \gamma_\lambda$ where $\gamma_\lambda = \frac{1}{2\pi}\mu_{1,\lambda} \log \frac{1}{r_\lambda} \int_{\Omega_\lambda} e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy$ and $r_\lambda = |z_\lambda - x_\lambda|$. The function $w_\lambda(x)$ is defined in the set $\hat{\Omega}_\lambda = (\Omega - x_\lambda)/r_\lambda$. Since $z_\lambda \to x_0$ we have that $r_\lambda \to 0$ as $\lambda \to 0$ and so $\hat{\Omega}_\lambda \to \mathbb{R}^2$.

Using the Green's representation formula we can write

$$w_\lambda(x) = \mu_{1,\lambda} \int_{\Omega_\lambda} G(r_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy - \gamma_\lambda$$

and by the standard decomposition of the Green's function we get

$$\begin{aligned} w_\lambda(x) &= \mu_{1,\lambda} \int_{\Omega_\lambda} \frac{1}{2\pi} \log \frac{1}{|r_\lambda x - \delta_\lambda y|} e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy - \gamma_\lambda + \mu_{1,\lambda} \int_{\Omega_\lambda} H(r_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy \\ &= \frac{1}{2\pi}\mu_{1,\lambda} \int_{\Omega_\lambda} \log \frac{1}{|x - (\delta_\lambda/r_\lambda)y|} e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy + \mu_{1,\lambda} \int_{\Omega_\lambda} H(r_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)} \tilde{v}_{1,\lambda}(y)\,dy \\ &= \mu_{1,\lambda} 8\pi c \left(\log \frac{1}{|x|} + H(x_0, x_0) + o(1) \right), \end{aligned} \tag{4.19}$$

where we used (3.6), and the boundedness of $H(r_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda)$ because $r_\lambda x + x_\lambda$ is an interior point of Ω_λ. Hence we obtain

$$\frac{w_\lambda(x)}{\mu_{1,\lambda}} \to w(x) = 8\pi c \left(\log \frac{1}{|x|} + R(x_0) \right) \quad \text{in } C_{\text{loc}}\big(R^2 \setminus \{0\}\big). \tag{4.20}$$

The same can be shown for the derivatives of w_λ to derive the convergence in $C^1_{\text{loc}}(R^2 \setminus \{0\})$.

This gives us a contradiction since we have a sequence of points $\hat{z}_n = (z_n - x_n)/r_n$ such that $\nabla w_n(\hat{z}_n) = 0$, which converge to a point $\hat{z}$ such that $|\hat{z}| = 1$ and $\nabla w(\hat{z}) = 0$. This is a contradiction with (4.20).

Case 2. Here we assume that $z_\lambda \in B_{\delta_\lambda R}(x_\lambda)$ for some $R > 0$. Let $\tilde{z}_\lambda = (z_\lambda - x_\lambda)/\delta_\lambda$. Then $\tilde{z}_\lambda \in B_R(0)$ and $\tilde{v}_{1,\lambda}(\tilde{z}_\lambda) = 1$. Reasoning as in the proof of Lemma 4.2 we have that $\tilde{v}_{1,\lambda} \to c$ uniformly in $B_{2R}(0)$ and since $\tilde{v}_{1,\lambda}(\tilde{z}_\lambda) = 1$ this implies that $c = 1$. □

Remark 4.6. We observe here that the proof of Lemma 4.5 implies that the maximum points of $v_{1,\lambda}$ are inside the ball $B_{\delta_\lambda R}(x_\lambda)$ for some $R > 0$ and hence they converge to x_0.

Proof of Theorem 2.1. This is derived from Lemmas 4.2, 4.4 and 4.5. □

5. Estimates for the second eigenvalue

Lemma 5.1. *For any eigenfunction $v_{i,\lambda}$ we have the following integral identity*

$$\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial x_j} \frac{\partial v_{i,\lambda}}{\partial \nu}\,d\sigma_x = \lambda(1 - \mu_{i,\lambda}) \int_{\Omega} e^{u_\lambda} v_{i,\lambda} \frac{\partial u_\lambda}{\partial x_j}\,dx \tag{5.1}$$

for $j = 1, 2$.

Proof. Differentiating Eq. (1.1) with respect to x_j we get

$$-\Delta \frac{\partial u_\lambda}{\partial x_j} = \lambda e^{u_\lambda} \frac{\partial u_\lambda}{\partial x_j} \quad \text{in } \Omega \text{ for } j = 1, 2. \tag{5.2}$$

Multiplying (5.2) by $v_{i,\lambda}$ and integrating we get

$$\int_\Omega \nabla\left(\frac{\partial u_\lambda}{\partial x_j}\right)\cdot\nabla v_{i,\lambda}\,dx=\lambda\int_\Omega e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_j}v_{i,\lambda}\,dx \tag{5.3}$$

while multiplying Eq. (2.3) by $\frac{\partial u_\lambda}{\partial x_j}$ we have

$$\int_\Omega \nabla v_{i,\lambda}\cdot\nabla\left(\frac{\partial u_\lambda}{\partial x_j}\right)dx-\int_{\partial\Omega}\frac{\partial u_\lambda}{\partial x_j}\frac{\partial v_{i,\lambda}}{\partial \nu}\,d\sigma_x=\lambda\mu_{i,\lambda}\int_\Omega e^{u_\lambda}v_{i,\lambda}\frac{\partial u_\lambda}{\partial x_j}. \tag{5.4}$$

Using (5.3) and (5.4) we get (5.1). □

Let u_λ be a solution of (1.1) satisfying (1.2), and let $x_\lambda\in\Omega$ such that $u_\lambda(x_\lambda)=\|u_\lambda\|_\infty$. By Theorem 3.1 $x_\lambda\to x_0\in\Omega$, hence there exists $\rho>0$ such that $B(x_\lambda,2\rho)\subset\Omega$. Let $\tilde\Phi\in C_0^\infty(B(0,2\rho))$ such that $\tilde\Phi=1$ in $B(0,\rho)$; $0\leqslant\tilde\Phi\leqslant1$ in $B(0,2\rho)$ and let

$$\Phi(x)=\tilde\Phi(x-x_\lambda). \tag{5.5}$$

Proposition 5.2. *We have*

$$\mu_{2,\lambda}\leqslant 1+C\delta_\lambda^2, \tag{5.6}$$

$$\mu_{2,\lambda}\to 1. \tag{5.7}$$

Proof. We estimate the second eigenvalue using again the variational formula

$$\mu_{2,\lambda}=\inf_{v\in H_0^1(\Omega),\ v\neq0,\ v\perp v_{1,\lambda}}\frac{\int_\Omega|\nabla v|^2}{\lambda\int_\Omega e^{u_\lambda}v^2}. \tag{5.8}$$

To this end let $\psi_1(x)=\frac{\partial u_\lambda}{\partial x_1}(x)$ and $v=\Phi\psi_1+a_{1,\lambda}v_{1,\lambda}$. We take

$$a_{1,\lambda}=-\frac{\lambda\int_\Omega e^{u_\lambda}\Phi\psi_1 v_{1,\lambda}}{\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2}=-\frac{N_{1,\lambda}}{D_{1,\lambda}} \tag{5.9}$$

so that $v\perp v_{1,\lambda}$ in $H_0^1(\Omega)$.

Step 1: Here we show that $a_{1,\lambda}=o(1)$. We estimate $D_{1,\lambda}$ in (5.9) as follows

$$D_{1,\lambda}=\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2=\int_{\Omega_\lambda}e^{\tilde u_\lambda}\tilde v_{1,\lambda}^2=8\pi+o(1).$$

This implies that $D_{1,\lambda}\geqslant 7\pi>0$ if λ is small enough. We only have to prove that $N_{1,\lambda}=o(1)$. To do this, we observe that, since $\Phi=1$ on $B_\rho(x_\lambda)$,

$$N_{1,\lambda}=\lambda\int_\Omega e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda}\Phi\,dx=\lambda\int_{\Omega\setminus B_\rho(x_\lambda)}e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda}\Phi\,dx+\lambda\int_{B_\rho(x_\lambda)}e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda}\,dx=I_1+I_2. \tag{5.10}$$

By the convergence results (3.2) and (4.15) it is easy to see that

$$I_1=\lambda\int_{\Omega\setminus B_\rho(x_\lambda)}e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda}\Phi\,dx=O(\lambda\mu_{1,\lambda})=o(1). \tag{5.11}$$

To estimate I_2 we use Eq. (5.1) where $i=1$, i.e.

$$(1-\mu_{1,\lambda})\lambda\int_\Omega e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda}=\int_{\partial\Omega}\frac{\partial u_\lambda}{\partial x_1}\frac{\partial v_{1,\lambda}}{\partial \nu_x}\,d\sigma_x. \tag{5.12}$$

Using convergence (3.2), (4.15) and (3.12) it is easy to see that

$$\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial x_1}\frac{\partial v_{1,\lambda}}{\partial \nu_x}\,d\sigma_x = \mu_{1,\lambda}\left((8\pi)^2\int_{\partial\Omega} \nu_1\left(\frac{\partial G}{\partial \nu_x}(x,x_0)\right)^2 d\sigma_x + \mathrm{o}(1)\right) = \mathrm{o}(\mu_{1,\lambda}).$$

By (2.5) we get that (5.12) implies

$$\lambda\int_\Omega e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda} = \lambda\int_{\Omega\setminus B_\rho(x_\lambda)} e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda} + \lambda\int_{B_\rho(x_\lambda)} e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda} = \mathrm{o}(\mu_{1,\lambda}).$$

Arguing as in (1.3) we get $\lambda\int_{\Omega\setminus B_\rho(x_\lambda)} e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda} = \mathrm{O}(\lambda\mu_{1,\lambda})$ and this implies that $\lambda\int_{B_\rho(x_\lambda)} e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{1,\lambda} = \mathrm{o}(\mu_{1,\lambda})$. This last estimate together with (5.10) and (5.11) implies that $N_{1,\lambda} = \mathrm{o}(\mu_{1,\lambda})$.

Step 2: Here we prove estimate (5.6). Recalling the definition of v we have

$$\begin{aligned}\int_\Omega |\nabla v|^2 &= \int_\Omega |\nabla\Phi\psi_1|^2 + 2a_{1,\lambda}\int_\Omega \nabla(\Phi\psi_1)\cdot\nabla v_{1,\lambda} + (a_{1,\lambda})^2\int_\Omega |\nabla v_{1,\lambda}|^2\\ &= \int_\Omega |\nabla\Phi\psi_1|^2 + 2a_{1,\lambda}\mu_{1,\lambda}\lambda\int_\Omega e^{u_\lambda}\Phi\psi_1 v_{1,\lambda} + (a_{1,\lambda})^2\lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}v_{1,\lambda}^2\\ &= \int_\Omega |\nabla\Phi\psi_1|^2 - (a_{1,\lambda})^2\lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}v_{1,\lambda}^2. \end{aligned}\tag{5.13}$$

Since ψ_1 solves Eq. (5.2) we have

$$\int_\Omega |\nabla\Phi\psi_1|^2 = \int_\Omega \psi_1^2|\nabla\Phi|^2 + \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2.$$

Inserting this in (5.13) and forgetting the lower order term we can write

$$\int_\Omega |\nabla v|^2 = \int_\Omega \psi_1^2|\nabla\Phi|^2 + \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + \mathrm{o}(1).$$

In a similar way we have

$$\begin{aligned}\lambda\int_\Omega e^{u_\lambda}v^2 &= \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + 2a_{1,\lambda}\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}\Phi\psi_1 + (a_{1,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2\\ &= \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + \mathrm{o}(1).\end{aligned}\tag{5.14}$$

Inserting (5.13) and (5.14) into (5.8) we get

$$\mu_{2,\lambda} \leqslant \frac{\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + \int_\Omega \psi_1^2|\nabla\Phi|^2 + \mathrm{o}(1)}{\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + \mathrm{o}(1)} = 1 + \frac{\int_\Omega \psi_1^2|\nabla\Phi|^2 + \mathrm{o}(1)}{\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2 + \mathrm{o}(1)}.\tag{5.15}$$

We only have to estimate both integrals in (5.15). For the first one we use the convergence of u_λ to $8\pi G(x,x_0)$ in $C^1_{\mathrm{loc}}(\bar\Omega\setminus\{x_0\})$, getting

$$\begin{aligned}\int_\Omega |\nabla\Phi|^2\psi_1^2 &= \int_{\Omega\setminus B_\rho(x_\lambda)} |\nabla\Phi|^2\psi_1^2\\ &= (8\pi)^2\int_{\Omega\setminus B_\rho(x_0)} |\nabla\Phi|^2\left(\frac{\partial G(x,x_0)}{\partial x_1}\right)^2 + \mathrm{o}(1) = c_0 + \mathrm{o}(1),\end{aligned}\tag{5.16}$$

where $c_0 > 0$; while, for the second one, a simple rescaling argument leads to

$$\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_1^2=\frac{1}{\delta_\lambda^2}\int_{\Omega_\lambda} e^{\tilde u_\lambda(y)}\Phi^2(\delta_\lambda y+x_\lambda)\left(\frac{\partial\tilde u_\lambda}{\partial y_1}(y)\right)^2dy$$
$$=\frac{1}{\delta_\lambda^2}\left(\Phi^2(x_0)\int_{\mathbb{R}^2} e^{U(y)}\left(\frac{\partial U}{\partial y_1}\right)^2dy+\mathrm{o}(1)\right)=\frac{1}{\delta_\lambda^2}\left(\frac{4}{3}\pi+\mathrm{o}(1)\right). \tag{5.17}$$

Here we use estimates (A.1) and (3.6) to pass to the limit. Finally we have

$$\mu_{2,\lambda}\leqslant 1+\frac{c_0+\mathrm{o}(1)}{(1/\delta_\lambda^2)(c_1+\mathrm{o}(1))+\mathrm{o}(1)}\leqslant 1+2\frac{c_0}{c_1}\delta_\lambda^2$$

where $c_1=\frac{4}{3}\pi$ and $c_0/c_1>0$. So (5.6) is proved.

Step 3: In this step we prove (5.7). Let $\mu_2=\lim_{\lambda\to 0}\mu_{2,\lambda}$. By (5.6) $\mu_2\in[0,1]$. We assume $\mu_2<1$ and we reach a contradiction. Let us consider a second eigenfunction $v_{2,\lambda}$ related to $\mu_{2,\lambda}$. Then $v_{2,\lambda}$ solves

$$\begin{cases}-\Delta v_{2,\lambda}=\lambda\mu_{2,\lambda}e^u v_{2,\lambda} & \text{in } \Omega,\\ \|v_{2,\lambda}\|_\infty=1, & \\ v_{2,\lambda}=0 & \text{on } \partial\Omega.\end{cases} \tag{5.18}$$

Let $\tilde v_{2,\lambda}=v_{2,\lambda}(\delta_\lambda x+x_\lambda)$ be the rescaled function. Now $\tilde v_{2,\lambda}$ solves

$$\begin{cases}-\Delta \tilde v_{2,\lambda}=\mu_{2,\lambda}e^{\tilde u_\lambda} v_{2,\lambda} & \text{in } \Omega_\lambda,\\ \|\tilde v_{2,\lambda}\|_\infty=1, & \\ \tilde v_{2,\lambda}=0 & \text{on } \partial\Omega_\lambda.\end{cases}$$

Let us show that $\nabla\tilde v_{2,\lambda}$ is uniformly bounded in $L^2(\mathbb{R}^2)$. Indeed

$$\int_{\mathbb{R}^2}|\nabla\tilde v_{2,\lambda}|^2=\mu_{2,\lambda}\int_{\Omega_\lambda}e^{\tilde u_\lambda}\tilde v_{2,\lambda}^2\leqslant\mu_{2,\lambda}\int_{\Omega_\lambda}e^{\tilde u_\lambda}\to 8\pi\mu_2.$$

So by the standard elliptic regularity theory we obtain that $\tilde v_{2,\lambda}\to\tilde v_2$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where $\tilde v_2$ is a solution of

$$\begin{cases}-\Delta\tilde v_2=\mu_2\frac{1}{(1+|x|^2/8)^2}\tilde v_2 & \text{in } \mathbb{R}^2,\\ \|\tilde v_2\|_\infty\leqslant 1.\end{cases} \tag{5.19}$$

Let us prove that $\tilde v_2\neq 0$. To do this let z_λ be the point of Ω_λ such that $\tilde v_{2,\lambda}(z_\lambda)=1$. If $\tilde v_2\equiv 0$, then z_λ should go to infinity. Let us consider the function

$$\hat v_{2,\lambda}=\tilde v_{2,\lambda}\left(\frac{x}{|x|^2}\right).$$

Then $\hat v_{2,\lambda}$ solves

$$-\Delta\hat v_{2,\lambda}=\frac{1}{|x|^4}e^{\hat u_\lambda}\hat v_{2,\lambda}. \tag{5.20}$$

As in the proof of Lemma 4.2, we can show that the right-hand side of (5.20) is bounded in $L^\infty(\mathbb{R}^2)$ so that $|\nabla\hat v_{2,\lambda}|$ is uniformly bounded in $L^2(\mathbb{R}^2)$ and $\hat v_{2,\lambda}\to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2\setminus\{0\})$. Using the regularity theory we reach a contradiction as in the proof of Lemma 4.2.

At this point we note that μ_2 is an eigenvalue for problem (1.4) an $\tilde v_2$ the corresponding eigenfunction. If $\mu_2<1$ then $\mu_2=0$ and $\tilde v_2=1$. But we get a contradiction since $v_{2,\lambda}$ is orthogonal to $v_{1,\lambda}$ and then

$$\int\nabla v_{1,\lambda}\cdot\nabla v_{2,\lambda}=0\quad\Longrightarrow\quad\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}v_{2,\lambda}=0\quad\Longrightarrow\quad\int_{\Omega_\lambda}e^{\tilde u_\lambda}\tilde v_{1,\lambda}\tilde v_{2,\lambda}=0.$$

Passing to the limit we obtain

$$\int_{\mathbb{R}^2}e^{U(y)}\,dy=0,$$

a contradiction. Hence $\mu_{2,\lambda}\to 1$. $\square$

Lemma 5.3. *We have*

$$\tilde v_{2,\lambda} \to \frac{a_1^{(2)}x_1 + a_2^{(2)}x_2}{8+|x|^2} + b^{(2)}\frac{8-|x|^2}{8+|x|^2} \tag{5.21}$$

in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, *with* $(a_1^{(2)}, a_2^{(2)}, b^{(2)}) \neq (0,0,0) \in \mathbb{R}^3$.

Proof. Arguing as in the last part of the proof of Proposition 5.2, we see that $\tilde v_{2,\lambda} \to \tilde v_2$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ where $\tilde v_2$ solves

$$\begin{cases} -\Delta \tilde v_2 = \frac{1}{(1+|x|^2/8)^2}\tilde v_2 & \text{in } \mathbb{R}^2, \\ \|\tilde v_2\|_\infty = 1. \end{cases} \tag{5.22}$$

Recalling Theorem 3.3 we have

$$\tilde v_2(x_1,x_2) = \frac{a_1^{(2)}x_1 + a_2^{(2)}x_2}{8+|x|^2} + b^{(2)}\frac{8-|x|^2}{8+|x|^2}.$$

But $\tilde v_2 \equiv 0$ is not possible by Step 3 of Proposition 5.2, so $(a_1^{(2)}, a_2^{(2)}, b^{(2)}) \neq 0$. □

6. Asymptotic behavior of the second eigenfunction

Lemma 6.1. *If the number* $b^{(2)}$ *of Lemma* 5.3 *is different from zero then*

$$v_{2,\lambda}(x)\log\lambda \to 4\pi b^{(2)} G(x,x_0) \quad \text{in } C^1(\omega) \tag{6.1}$$

where ω *is any compact set in* $\overline{\Omega} \setminus \{x_0\}$.

Proof. Multiplying Eq. (1.1) and (5.18) by $v_{2,\lambda}$ and u_λ respectively we get

$$\lambda \int_\Omega e^{u_\lambda} v_{2,\lambda} = \mu_{2,\lambda}\lambda \int_\Omega e^{u_\lambda} u_\lambda v_{2,\lambda}. \tag{6.2}$$

Then

$$\begin{aligned} \lambda \int_\Omega e^{u_\lambda} v_{2,\lambda} &= \lambda\mu_{2,\lambda} \int_\Omega e^{u_\lambda}\big(u_\lambda - \|u_\lambda\|_\infty\big)v_{2,\lambda} + \lambda\|u_\lambda\|_\infty \mu_{2,\lambda}\int_\Omega e^{u_\lambda} v_{2,\lambda} \\ &= \mu_{2,\lambda}\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde u_\lambda \tilde v_{2,\lambda} + \lambda\|u_\lambda\|_\infty \mu_{2,\lambda}\int_\Omega e^{u_\lambda} v_{2,\lambda} \\ &= -\int_{\mathbb{R}^2} \frac{2}{(1+|x|^2|/8)^2}\log\left(1+\frac{|x|^2}{8}\right)\left(\frac{a_1^{(2)}x_1 + a_2^{(2)}x_2}{8+|x|^2} + b^{(2)}\frac{8-|x|^2}{8+|x|^2}\right)dx + \mathrm{o}(1) \\ &\quad + \lambda\|u_\lambda\|_\infty \mu_{2,\lambda}\int_\Omega e^{u_\lambda} v_{2,\lambda}. \end{aligned}$$

In the last estimate we can pass to the limit since

$$\big|e^{\tilde u_\lambda}\tilde u_\lambda \tilde v_{2,\lambda}\big| \leqslant \left|\frac{2}{(1+|x|^2/8)^2}\log\left(1+\frac{|x|^2}{8}\right)\right| \in L^1(\mathbb{R}^2).$$

So we obtain the following estimate:

$$\lambda \int_\Omega e^{u_\lambda} v_{2,\lambda} = 8\pi b^{(2)} + \mathrm{o}(1) + \lambda\|u_\lambda\|_\infty \mu_{2,\lambda}\int_\Omega e^{u_\lambda} v_{2,\lambda}. \tag{6.3}$$

A simple rescaling argument shows us that $\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}\to 0$ as $\lambda\to 0$. Inserting this into the (6.3) and recalling (3.4), we find that

$$\lambda\int\limits_\Omega e^{u_\lambda}v_{2,\lambda}\,dx=\frac{4\pi b^{(2)}}{\log\lambda}\big(1+o(1)\big).\tag{6.4}$$

Using the Green's representation formula we can write

$$\begin{aligned}v_{2,\lambda}(x)\log\lambda&=\lambda\log\lambda\mu_{2,\lambda}\int\limits_\Omega G(x,y)e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy\\&=\mu_{2,\lambda}G(x,x_0)\lambda\log\lambda\int\limits_\Omega e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy+\mu_{2,\lambda}\lambda\log\lambda\int\limits_\Omega\big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy\\&=4\pi b^{(2)}G(x,x_0)+o(1)+I_{2,\lambda}.\end{aligned}\tag{6.5}$$

where we used (6.4). To prove (6.1) we only have to show that

$$I_{2,\lambda}=\mu_{2,\lambda}\lambda\log\lambda\int\limits_\Omega\big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy=o(1).\tag{6.6}$$

As in the proof of Lemma 4.5 $\forall x\in\Omega\setminus\{x_0\}$, we can choose $\rho>0$ such that $x\notin B_{2\rho}(x_0)\subset\Omega$ and we can split Ω in two pieces $\Omega\setminus B_\rho(x_0)$ and $B_\rho(x_0)$. Proceeding again as in the proof of Lemma 4.5 and using estimate (3.3), we get

$$\mu_{2,\lambda}\lambda\log\lambda\int\limits_{\Omega\setminus B_\rho(x_0)}\big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy\leqslant\frac{C'\lambda\log\lambda}{(c\lambda+\rho^2/32)^2}.\tag{6.7}$$

Now we can consider the integral inside the ball $B_\rho(x_0)$. Then

$$\begin{aligned}&\mu_{2,\lambda}\lambda\log\lambda\int\limits_{B_\rho(x_0)}\big[G(x,y)-G(x,x_0)\big]e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy\\&\quad\leqslant\lambda\log\lambda\big(1+o(1)\big)\int\limits_{B_\rho(x_0)}\big|G(x,y)-G(x,x_0)\big|e^{u_\lambda(y)}\,dy\\&\quad\leqslant C\lambda\log\lambda\int\limits_{B_\rho(x_0)}\big|\nabla G(x,\xi)\big||y-x_0|e^{u_\lambda(y)}\,dy\\&\quad\leqslant\rho\log\lambda\sup_{y\in B_\rho(x_0)}\big|\nabla G(x,y)\big|\lambda\int\limits_\Omega e^{u_\lambda(y)}\,dy\leqslant C\rho\log\lambda.\end{aligned}\tag{6.8}$$

Here we used that $x\notin B_{2\rho}(x_0)$ and $|\nabla G(x,y)|$ is uniformly bounded for $y\in B_\rho(x_0)$. Now we can let ρ go to zero in such a way that both (6.7) and (6.8) go to zero. We can take, for example, $\rho=\lambda^k$ for any $k<\frac14$. We proved so far (6.6) and then (6.1) follows from (6.5). □

Lemma 6.2. *For any eigenfunction $v_{i,\lambda}$ we have the following integral identity*

$$\int\limits_{\partial\Omega}\frac{\partial v_{i,\lambda}}{\partial\nu}\big((x-y)\cdot\nabla u_\lambda(x)+2\big)\,d\sigma=\lambda(1-\mu_{i,\lambda})\int\limits_\Omega e^{u_\lambda}v_{i,\lambda}\big((x-y)\cdot\nabla u_\lambda+2\big).\tag{6.9}$$

Proof. Let $w_\lambda(x)=(x-y)\cdot\nabla u_\lambda(x)+2$ for any $y\in\mathbb{R}^2$. Then it is easy to see that

$$\begin{cases}-\Delta w_\lambda=\lambda e^{u_\lambda}w_\lambda & \text{in }\Omega,\\ w_\lambda(x)=(x-y)\cdot\nu\frac{\partial u_\lambda}{\partial\nu}+2 & \text{on }\partial\Omega.\end{cases}\tag{6.10}$$

Using $v_{i,\lambda}$ as a test function we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla v_{i,\lambda} = \lambda \int_\Omega e^{u_\lambda} w_\lambda v_{i,\lambda} \tag{6.11}$$

while from Eq. (2.3) we get

$$\int_\Omega \nabla v_{i,\lambda} \cdot \nabla w_\lambda - \int_{\partial\Omega} \frac{\partial v_{i,\lambda}}{\partial \nu} w_\lambda(x)\, d\sigma = \lambda \mu_{i,\lambda} \int_\Omega e^{u_\lambda} v_{i,\lambda} w_\lambda. \tag{6.12}$$

Hence (6.9) follows. □

Proposition 6.3. *We have*

$$\tilde v_{2,\lambda}(x) \to \tilde v_2 = \frac{a_1^{(2)} x_1 + a_2^{(2)} x_2}{8+|x|^2} \quad \text{as } \lambda \to 0 \tag{6.13}$$

in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, *for some* $(a_1^{(2)}, a_2^{(2)}) \neq 0$.

Proof. By Lemma 5.3 we only have to prove that $b^{(2)} = 0$. To do this we use the identity (6.9) for $i = 2$. If by contradiction $b^{(2)} \neq 0$ using Lemma 6.1 and (3.2) we can pass to the limit in the left-hand side of (6.9) getting

$$\log\lambda \int_{\partial\Omega} \frac{\partial v_{2,\lambda}}{\partial \nu} w_\lambda(x)\, d\sigma = 4\pi b^{(2)} \int_{\partial\Omega} \left(8\pi (x-y)\cdot \nu \frac{\partial G(x,x_0)}{\partial \nu} + 2\right) \frac{\partial G(x,x_0)}{\partial \nu}\, d\sigma + o(1). \tag{6.14}$$

Using Eq. (3.11) with $y = x_0$ we have

$$\int_{\partial\Omega} (x-x_0)\cdot \nu \left(\frac{\partial G(x,x_0)}{\partial \nu}\right)^2 d\sigma = \frac{1}{2\pi}, \tag{6.15}$$

while using that $G(x,x_0)$ is harmonic in $\Omega \setminus \{x_0\}$, one can prove that

$$\int_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial \nu}\, d\sigma = -1. \tag{6.16}$$

Now we consider the right-hand side of (6.9). A simple rescaling argument give us

$$\begin{aligned} \lambda \int_\Omega e^{u_\lambda} v_{2,\lambda} w_\lambda\, dx &= \int_{\Omega_\lambda} e^{\tilde u_\lambda(y)} \tilde v_{2,\lambda}(y)\big(y\cdot \nabla \tilde u_\lambda(y) + 2\big)\, dy \\ &= \int_{\mathbb{R}^2} \frac{128(8-|y|^2)}{(8+|y|^2)^3} \left(\sum_{j=1}^2 a_j^{(2)} \frac{y_j}{8+|y|^2} + b^{(2)} \frac{8-|y|^2}{8+|y|^2}\right) dy + o(1) \\ &= c_0 b^{(2)}\big(1+o(1)\big), \end{aligned} \tag{6.17}$$

where the passage into the limit is done using estimate (3.6) and the boundedness of $y\cdot\nabla\tilde u_\lambda(y) + 2$ (see (A.1)). We observe here that $c_0 = \frac{16}{3}\pi > 0$. Inserting (6.15), (6.16) and (6.17) into (6.9), we have

$$(1-\mu_{2,\lambda})\big(c_0 b^{(2)} + o(1)\big) = \frac{4\pi b^{(2)}}{\log\lambda}\big(1+o(1)\big). \tag{6.18}$$

Hence if $b^{(2)} \neq 0$, we find

$$1-\mu_{2,\lambda} = -\frac{1}{\log\lambda} c_1\big(1+o(1)\big) \quad \text{where } c_1 = -\frac{8\pi}{c_0} = -\frac{3}{2} < 0. \tag{6.19}$$

This implies that for λ small enough

$$1-\mu_{2,\lambda}\leqslant -\frac{c_1}{2\log\lambda}. \tag{6.20}$$

But from (5.6), we get

$$1-\mu_{2,\lambda}\geqslant -C\delta_\lambda^2, \tag{6.21}$$

which implies with (6.20) that

$$\frac{c_1}{2}\geqslant C\delta_\lambda^2\log\lambda,$$

giving a contradiction since $c_1<0$ while the r.h.s. goes to zero. So $b^{(2)}=0$ and the claim follows. □

Proposition 6.4. *We have*

$$\frac{v_{2,\lambda}(x)}{\delta_\lambda}\to 2\pi\sum_{k=1}^{2}a_k^{(2)}\frac{\partial G(x,x_0)}{\partial y_k}\quad \text{as } \lambda\to 0 \tag{6.22}$$

in $C^1_{\mathrm{loc}}(\bar\Omega\setminus\{x_0\})$*, where* $(a_1^{(2)},a_2^{(2)})$ *is the same as in* (6.13)*.*

Proof. Using the Green's representation formula we have

$$v_{2,\lambda}(x)=\lambda\mu_{2,\lambda}\int_\Omega G(x,y)e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy.$$

Note that $\forall x\in\Omega\setminus\{x_0\}$ we can choose $\rho>0$, $\rho\in\mathbb{R}$ such that $B_{2\rho}(x_0)\subset\Omega$ and $x\notin B_{2\rho}(x_0)$. For such a value of ρ we can write

$$v_{2,\lambda}(x)=\lambda\mu_{2,\lambda}\int_{\Omega\setminus B_\rho(x_0)}G(x,y)e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy+\lambda\mu_{2,\lambda}\int_{B_\rho(x_0)}G(x,y)e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy=I_1+I_2. \tag{6.23}$$

First we study the behavior of I_1. Using the estimate (3.3) and since $G(x,y)\in L^1_y(\Omega)$ we have

$$|I_1|\leqslant\frac{C\lambda}{(c\lambda+\rho^2)^2}\int_\Omega G(x,y)\,dy\leqslant\frac{C\lambda}{(c\lambda+\rho^2)^2}. \tag{6.24}$$

We can let $\rho\to 0$ in such a way that

$$|I_1|=o(\delta_\lambda),$$

for example choosing $\rho=\lambda^k$ for $\lambda<\frac14$.

For $y\in B_\rho(x_0)$ and $x\notin B_{2\rho}(x_0)$ the function $G(x,y)$ is regular and we can expand it in Taylor series

$$G(x,y)=G(x,x_\lambda)+\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_\lambda)(y-x_\lambda)_j+\frac12\sum_{j,k=1}^{2}\frac{\partial^2 G}{\partial y_j\partial y_k}(x,\eta_\lambda)(y-x_\lambda)_j(y-x_\lambda)_k, \tag{6.25}$$

where η_λ is a point between y and x_λ which are both contained in $B_\rho(x_0)$. To study the behavior of I_2 we write

$$I_2=\lambda\mu_{2,\lambda}\int_{B_\rho(x_0)}\left(G(x,x_\lambda)+\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_\lambda)(y-x_\lambda)_j\right)e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy$$
$$+\lambda\mu_{2,\lambda}\int_{B_\rho(x_0)}\frac12\sum_{j,k=1}^{2}\frac{\partial^2 G}{\partial y_j\partial y_k}(x,\eta_\lambda)(y-x_\lambda)_j(y-x_\lambda)_k e^{u_\lambda(y)}v_{2,\lambda}(y)\,dy$$

$$= G(x,x_0)\big(1+\mathrm{o}(1)\big)\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)\,dy$$
$$+\,\delta_\lambda\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\big(1+\mathrm{o}(1)\big)\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)y_j\,dy + R_\lambda, \tag{6.26}$$

where $R_\lambda = \frac{1}{2}\lambda\mu_{2,\lambda}\sum_{j,k=1}^{2}\int_{B_\rho(x_0)}\frac{\partial^2 G}{\partial y_j\partial y_k}(x,\eta_\lambda)e^{u_\lambda(y)}v_{2,\lambda}(y)(y-x_\lambda)_j(y-x_\lambda)_k\,dy$ and $B_\lambda = (B_\rho(x_0)-x_\lambda)/\delta_\lambda$.

Now we want to show that

$$R_\lambda = \mathrm{o}(\delta_\lambda).$$

We observe that since y and $x_\lambda \in B_\rho(x_0)$ then also $\eta_\lambda \in B_\rho(x_0)$. Moreover since $x \notin B_{2\rho}(x_0)$ we have

$$\left|\frac{\partial^2 G}{\partial y_j\partial y_k}(x,\eta_\lambda)\right| \leqslant \sup_{z\in B_\rho(x_0),\ j,k=1,2}\left|\frac{\partial^2 G}{\partial y_j\partial y_k}(x,z)\right| = C.$$

Using that $|v_{2,\lambda}(y)| \leqslant 1$ and $\mu_{2,\lambda} = 1+\mathrm{o}(1)$ we get

$$|R_\lambda| \leqslant C\lambda\int\limits_{B_\rho(x_0)} e^{u_\lambda}|y-x_\lambda|^2\,dy \leqslant C\rho\delta_\lambda\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}|y|\,dy. \tag{6.27}$$

Letting ρ going to zero we obtain that $R_\lambda = \mathrm{o}(\delta_\lambda)$.

Gluing together (6.23), (6.24), (6.26), and (6.27) we derive

$$v_{2,\lambda}(x) = G(x,x_0)\big(1+\mathrm{o}(1)\big)\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)\,dy$$
$$+\,\delta_\lambda\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\big(1+\mathrm{o}(1)\big)\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)y_j\,dy + \mathrm{o}(\delta_\lambda). \tag{6.28}$$

In order to prove (2.9) we have to estimate $\gamma_\lambda = \int_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)\,dy$. We will show that

$$\gamma_\lambda = \mathrm{o}(\delta_\lambda).$$

We prove this by contradiction. So let us suppose that $\lim_{\lambda\to 0}\frac{\delta_\lambda}{\gamma_\lambda} = c$ with $c \in \mathbb{R}$, $c < \infty$. From (6.28) we derive

$$\frac{v_{2,\lambda}(x)}{\gamma_\lambda} = G(x,x_0) + c\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\int\limits_{B_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)y_j\,dy + \mathrm{o}(1). \tag{6.29}$$

Note that $B_\lambda \to \mathbb{R}^2$ and we can pass into the limit in (6.29) using (3.6). We also observe that $\tilde v_2(y) = (a_1^{(2)}y_1 + a_2^{(2)}y_2)/(8+|y|^2) = -\frac{1}{4}\sum_{j=1}^{2}a_k^{(2)}\frac{\partial U(y)}{\partial y_k}$. Then (6.29) becomes

$$\begin{aligned}\frac{v_{2,\lambda}(x)}{\gamma_\lambda} &= G(x,x_0) - \frac{c}{4}\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\int\limits_{\mathbb{R}^2} e^{U(y)}\sum_{k=1}^{2}a_k^{(2)}\frac{\partial U}{\partial y_k}y_j\,dy + \mathrm{o}(1)\\
&= G(x,x_0) - \frac{c}{4}\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\int\limits_{\mathbb{R}^2}\sum_{k=1}^{2}a_k^{(2)}\frac{\partial e^{U(y)}}{\partial y_k}y_j\,dy + \mathrm{o}(1)\\
&= G(x,x_0) + \frac{1}{4}c\sum_{j=1}^{2}\frac{\partial G}{\partial y_j}(x,x_0)\sum_{k=1}^{2}a_k^{(2)}\int\limits_{\mathbb{R}^2} e^{U(y)}\delta_k^j\,dy + \mathrm{o}(1)\\
&= G(x,x_0) + 2\pi c\sum_{j=1}^{2}a_j^{(2)}\frac{\partial G}{\partial y_j}(x,x_0) + \mathrm{o}(1).\end{aligned} \tag{6.30}$$

In a similar way one can prove the convergence in $C^1(\bar{\Omega} \setminus \{x_0\})$.

Now to estimate $1 - \mu_{2,\lambda}$ we use identity (5.1) evaluated in $v_{2,\lambda}$ i.e.

$$\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial x_j} \frac{\partial v_{2,\lambda}}{\partial \nu}\, d\sigma_x = \lambda(1 - \mu_{2,\lambda}) \int_{\Omega} e^{u_\lambda} v_{2,\lambda} \frac{\partial u_\lambda}{\partial x_j}\, dx. \tag{6.31}$$

Let us consider first the l.h.s. of (6.31). Passing to the limit we get

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial x_j} \frac{\partial v_{2,\lambda}}{\partial \nu_x}\, d\sigma_x &= \gamma_\lambda\left(8\pi \int_{\partial\Omega} \frac{\partial G}{\partial x_j}(x,x_0) \frac{\partial G}{\partial \nu_x}(x,x_0)\, d\sigma_x\right) \\
&\quad + \gamma_\lambda\left(8\pi \int_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial x_j} \frac{\partial}{\partial \nu}\left(2\pi c \sum_{k=1}^{2} a_k^{(2)} \frac{\partial G(x,x_0)}{\partial y_k}\right) d\sigma_x\right) + \mathrm{o}(\gamma_\lambda) \\
&= \gamma_\lambda\left(8\pi \int_{\partial\Omega} \nu_j \left(\frac{\partial G}{\partial \nu_x}(x,x_0)\right)^2 d\sigma_x\right) \\
&\quad + \gamma_\lambda\left(16\pi^2 c \sum_{k=1}^{2} a_k^{(2)} \int_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial x_j} \frac{\partial}{\partial y_k} \frac{\partial G(x,x_0)}{\partial \nu}\, d\sigma_x\right) + \mathrm{o}(\gamma_\lambda) \\
&= \gamma_\lambda\left(-8\pi \frac{\partial R}{\partial y_j}(x_0) - 8\pi^2 c \sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j} + \mathrm{o}(1)\right) \\
&= \gamma_\lambda\left(-8\pi^2 c \sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j} + \mathrm{o}(1)\right),
\end{aligned} \tag{6.32}$$

where we use the identities (3.12), (3.13) and that x_0 is a critical point of $R(y)$. For the right-hand side of (6.31) we have

$$\begin{aligned}
\lambda \int_{\Omega} e^{u_\lambda} v_{2,\lambda} \frac{\partial u_\lambda}{\partial x_j}\, dx &= \frac{1}{\delta_\lambda} \int_{\Omega_\lambda} e^{\tilde{u}_\lambda} \tilde{v}_{2,\lambda} \frac{\partial \tilde{u}_\lambda}{\partial y_j}\, dy \\
&= -\frac{4}{\delta_\lambda} \int_{\mathbb{R}^2} \frac{64 y_j}{(8+|y|^2)^3} \frac{a_1^{(2)} y_1 + a_2^{(2)} y_2}{8+|y|^2}\, dy\big(1+\mathrm{o}(1)\big);
\end{aligned} \tag{6.33}$$

where the passage into the limit is done using the estimate (see (A.1)

$$\left| e^{\tilde{u}_\lambda} \tilde{v}_{2,\lambda} \frac{\partial \tilde{u}_\lambda}{\partial y_j} \right| \leqslant C \frac{64}{(8+|y|^2)^2} \|\tilde{v}_{2,\lambda}\|_\infty.$$

Hence (6.33) becomes

$$\begin{aligned}
\lambda \int_{\Omega} e^{u_\lambda} v_{2,\lambda} \frac{\partial u_\lambda}{\partial x_j}\, dx &= \frac{1}{\delta_\lambda}\left(-256 \sum_{k=1}^{2} a_k^{(2)} \int_{\mathbb{R}^2} \frac{y_k y_j}{(8+|y|^2)^4}\, dy\big(1+\mathrm{o}(1)\big)\right) \\
&= \frac{1}{\delta_\lambda}\left(-\frac{\pi}{3} a_j^{(2)} + \mathrm{o}(1)\right).
\end{aligned} \tag{6.34}$$

Putting together (6.31), (6.32) and (6.34) we obtain

$$\gamma_\lambda\left(-8\pi^2 c \sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j} + \mathrm{o}(1)\right) = (1-\mu_{2,\lambda}) \frac{1}{\delta_\lambda}\left(-\frac{\pi}{3} a_j^{(2)} + \mathrm{o}(1)\right), \tag{6.35}$$

and finally

$$(1-\mu_{2,\lambda}) = 24\pi c \eta \delta_\lambda \gamma_\lambda \big(1+\mathrm{o}(1)\big), \tag{6.36}$$

where $\eta = (\sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j})/a_j^{(2)}$ for j such that $a_j^{(2)} \neq 0$.

Now we consider the Pohozaev identity (6.9) computed at the point x_λ. Using (6.30) and passing into the limit in the l.h.s. of (6.9) we find

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\partial v_{2,\lambda}}{\partial \nu_x}\big((x-x_\lambda)\cdot\nabla u_\lambda(x)+2\big)\,d\sigma_x \\
&\quad= \gamma_\lambda \int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x,x_0)\big(8\pi(x-x_0)\cdot\nabla G(x,x_0)+2\big)\,d\sigma_x \\
&\qquad+ \gamma_\lambda 2\pi c\sum_{j=1}^{2} a_j^{(2)} \int_{\partial\Omega} \frac{\partial^2 G}{\partial \nu_x y_j}(x,x_0)\big(8\pi(x-x_0)\cdot\nabla G(x,x_0)+2\big) + o(\gamma_\lambda) \\
&\quad= 2\gamma_\lambda\big(1+o(1)\big).
\end{aligned} \tag{6.37}$$

In the last passage we used (3.11), (3.12) and (3.15). Concerning the r.h.s. of (6.9) a simple scaling argument gives

$$\lambda\int_{\Omega} e^{u_\lambda} v_{2,\lambda}\big((x-x_\lambda)\cdot\nabla u_\lambda+2\big)\,dx = \int_{\Omega_\lambda} e^{\tilde u_\lambda(y)}\tilde v_{2,\lambda}(y)(y\cdot\nabla\tilde u+2)\,dy = o(1). \tag{6.38}$$

In (6.38) we can pass to the limit using estimate (3.6), (A.1) and the boundedness of $v_{2,\lambda}$. Putting together (6.37) and (6.38) we obtain

$$2\gamma_\lambda\big(1+o(1)\big) = (1-\mu_{2,\lambda})o(1). \tag{6.39}$$

Comparing (6.36) and (6.39) we have

$$2\gamma_\lambda\big(1+o(1)\big) = 24\pi c\gamma_\lambda\eta\delta_\lambda o(1)\big(1+o(1)\big)$$

which is impossible.

We have shown so far that $\gamma_\lambda/\delta_\lambda \to 0$. Then from (6.28) and (6.30) we have

$$\frac{v_{2,\lambda}(x)}{\delta_\lambda} = 2\pi\sum_{j=1}^{2} a_j^{(2)}\frac{\partial G}{\partial y_j}(x,x_0)+o(1). \qquad \square$$

Lemma 6.5. *We have*

$$1-\mu_{2,\lambda} = 24\pi\eta\delta_\lambda^2\big(1+o(1)\big), \tag{6.40}$$

where $\eta = (\sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j})/a_j^{(2)}$ *for* j *such that* $a_j^{(2)} \neq 0$.

Proof. We estimate the behavior of $(1-\mu_{2,\lambda})$ using (6.31). The proof is the same as before using (6.22) instead of (6.30). Then we obtain

$$1-\mu_{2,\lambda} = 24\pi\eta\delta_\lambda^2\big(1+o(1)\big) \tag{6.41}$$

where $\eta = (\sum_{k=1}^{2} a_k^{(2)} \frac{\partial^2 R(x_0)}{\partial x_k x_j})/a_j^{(2)}$ for j such that $a_j^{(2)} \neq 0$. $\square$

7. Estimates for the third eigenvalue and the third eigenfunction

Proposition 7.1. *We have*

$$\mu_{3,\lambda} \leqslant 1 + C\delta_\lambda^2, \tag{7.1}$$

$$\mu_{3,\lambda} \to 1. \tag{7.2}$$

Proof. To estimate the third eigenvalue we use the analogous of formula (5.8), i.e.

$$\mu_{3,\lambda} = \inf_{v\in H_0^1(\Omega),\ v\neq 0,\ v\perp\{v_{1,\lambda},v_{2,\lambda}\}} \frac{\int_\Omega |\nabla v|^2}{\lambda\int_\Omega e^{u_\lambda}v^2}. \tag{7.3}$$

Let $\psi_2(x) = b_1\frac{\partial u_\lambda}{\partial x_1}(x) + b_2\frac{\partial u_\lambda}{\partial x_2}(x)$ where $b=(b_1,b_2)\in\mathbb{R}^2$, $b\neq 0$ and $b\perp a^{(2)}$ in $\mathbb{R}^2$, where $a^{(2)}$ is the same as Proposition 6.3. To simplify the notation we suppose here $a_1^{(2)}=0$ and we let $\psi_2(x)=\frac{\partial u_\lambda}{\partial x_1}(x)$. The general case follows in the same way.

To estimate the third eigenvalue we chose $v = \Phi\psi_2 + a_{1,\lambda}v_{1,\lambda} + a_{2,\lambda}v_{2,\lambda}$, where

$$a_{i,\lambda} = -\frac{\lambda\int_\Omega e^{u_\lambda}\Phi\psi_2 v_{i,\lambda}}{\lambda\int_\Omega e^{u_\lambda}v_{i,\lambda}^2} = -\frac{N_{i,\lambda}}{D_{i,\lambda}} \quad \text{for } i=1,2$$

in such a way that $v\perp v_{1,\lambda}$ and $v\perp v_{2,\lambda}$ in $H_0^1(\Omega)$. We already know that $a_{1,\lambda}=\mathrm{o}(1)$ (see the proof of Proposition 5.2). To estimate $a_{2,\lambda}$ we observe, for the moment, that

$$D_{2,\lambda} = \int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde v_{2,\lambda}^2 \to \int_{\mathbb{R}^2} e^U\left(\frac{a_2^{(2)}x_2}{8+|x|^2}\right)^2 = \frac{1}{12}\big(a_2^{(2)}\big)^2\pi.$$

For what concerns $N_{2,\lambda}$ we can write down it in the following way

$$N_{2,\lambda} = \lambda\int_\Omega e^{u_\lambda}\Phi\frac{\partial u_\lambda}{\partial x_1}v_{2,\lambda} = \lambda\int_{\Omega\setminus B_\rho(x_\lambda)} e^{u_\lambda}\Phi\frac{\partial u_\lambda}{\partial x_1}v_{2,\lambda} + \lambda\int_{B_\rho(x_\lambda)} e^{u_\lambda}\frac{\partial u_\lambda}{\partial x_1}v_{2,\lambda} = I_1 + I_2. \tag{7.4}$$

As in the proof of Proposition 5.2 it is easy to see that $I_1 = \mathrm{O}(\lambda\delta_\lambda)$. Rescaling we can write

$$I_2 = \frac{1}{\delta_\lambda}\int_{B_{\frac{\rho}{\delta_\lambda}}(0)} e^{\tilde u_\lambda}\frac{\partial\tilde u_\lambda}{\partial x_1}\tilde v_{2,\lambda} = \frac{1}{\delta_\lambda}\left(-4a_2^{(2)}\int_{\mathbb{R}^2}\frac{y_1y_2}{(8+|y|^2)^4}\,dy + \mathrm{o}(1)\right). \tag{7.5}$$

This proves that $\delta_\lambda(a_{2,\lambda})^2 = \mathrm{o}(1)$.

Reasoning as we did in (5.13), we can find

$$\begin{aligned}\int_\Omega |\nabla v|^2 &= \int_\Omega |\nabla\Phi\psi_2|^2 - (a_{1,\lambda})^2\lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}v_{1,\lambda}^2 - (a_{2,\lambda})^2\lambda\mu_{2,\lambda}\int_\Omega e^{u_\lambda}v_{2,\lambda}^2\\ &= \int_\Omega \psi_2^2|\nabla\Phi|^2 + \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_2^2 - (a_{1,\lambda})^2\lambda\mu_{1,\lambda}\int_\Omega e^{u_\lambda}v_{1,\lambda}^2 - (a_{2,\lambda})^2\lambda\mu_{2,\lambda}\int_\Omega e^{u_\lambda}v_{2,\lambda}^2.\end{aligned} \tag{7.6}$$

In the same way,

$$\lambda\int_\Omega e^{u_\lambda}v^2 = \lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_2^2 - (a_{1,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2 - (a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}^2. \tag{7.7}$$

Inserting (7.6) and (7.7) in (7.3) we obtain

$$\mu_{3,\lambda} \leqslant 1 + \frac{\int_\Omega \psi_2^2|\nabla\Phi|^2 + (1-\mu_{1,\lambda})(a_{1,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2 + (1-\mu_{2,\lambda})(a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}^2}{\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_2^2 - (a_{1,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{1,\lambda}^2 - (a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}^2} \tag{7.8}$$

and forgetting lower order terms we have

$$\mu_{3,\lambda} \leqslant 1 + \frac{\int_\Omega \psi_2^2|\nabla\Phi|^2 + (1-\mu_{2,\lambda})(a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}^2 + \mathrm{o}(1)}{\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi_2^2 - (a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}^2 + \mathrm{o}(1)}. \tag{7.9}$$

Using that $\delta_\lambda(a_{2,\lambda})^2 = \mathrm{o}(1)$ and estimate (6.40) we have

$$(1-\mu_{2,\lambda})(a_{2,\lambda})^2\lambda\int_\Omega e^{u_\lambda}(v_{2,\lambda})^2 = 24\pi\eta\delta_\lambda^2(a_{2,\lambda})^2\frac{1}{12}\big(a_2^{(2)}\big)^2\pi\big(1+\mathrm{o}(1)\big) = \mathrm{o}(1).$$

Finally using (5.16) and (5.17) we have

$$\mu_{3,\lambda} \leqslant 1+\delta_\lambda^2 \frac{c_0+\mathrm{o}(1)}{c_1+\delta_\lambda^2(a_{2,\lambda})^2+\mathrm{o}(1)} < 1+2\frac{c_0}{c_1}\delta_\lambda^2 \tag{7.10}$$

where $c_1=\frac{4}{3}\pi$ and $c_0=(8\pi)^2\int_{\Omega\setminus B_\rho(x_0)}|\nabla\Phi|^2(\frac{\partial G(x,x_0)}{\partial x_1})^2$. Moreover it is easy to see that $1\geqslant \lim\mu_{3,\lambda}\geqslant \lim\mu_{2,\lambda}=1$, so that $\mu_{3,\lambda}\to 1$. □

Proof of Theorem 2.2. It follows from Propositions 6.3 and 6.4 for $i=2$.

Having estimate (7.1) for the third eigenvalue, it is possible to repeat the proofs of Lemmas 5.3 and 6.1 and Propositions 5.2 and 6.3 substituting $v_{2,\lambda}$ with $v_{3,\lambda}$ and $\mu_{2,\lambda}$ with $\mu_{3,\lambda}$. This gives us the results of Propositions 6.3 and 6.4 for the third eigenfunction and hence the claim follows for $i=3$. □

8. Asymptotic behavior of the eigenvalues

Lemma 8.1. *If $v_{2,\lambda}$ and $v_{3,\lambda}$ are two eigenfunctions of* (2.2) *corresponding to $\mu_{2,\lambda}$ and $\mu_{3,\lambda}$, then the corresponding vector $a^{(2)}$ and $a^{(3)}$ defined in* (2.8) *are orthogonal in $\mathbb{R}^2$.*

Proof. By assumption $\int_\Omega \nabla v_{2,\lambda}\cdot\nabla v_{3,\lambda}\,dx=0$. Using Eq. (2.3) we get

$$\mu_{i,\lambda}\lambda\int_\Omega e^{u_\lambda}v_{2,\lambda}v_{3,\lambda}\,dx=0$$

and rescaling

$$\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde v_{2,\lambda}\tilde v_{3,\lambda}\,dx=0.$$

Passing to the limit, using estimate (3.6), we get

$$\int_{\mathbb{R}^2}\frac{1}{(1+|y|^2/8)^2}\frac{a_1^{(2)}y_1+a_2^{(2)}y_2}{8+|y|^2}\frac{a_1^{(3)}y_1+a_2^{(3)}y_2}{8+|y|^2}\,dy=0 \quad\Rightarrow\quad 64\sum_{h,l=1}^{2}a_h^{(2)}a_l^{(3)}\int_{\mathbb{R}^2}\frac{y_hy_l}{(8+|y|^2)^4}\,dy=0;$$

and this implies

$$\sum_{h=1}^{2}a_h^{(2)}a_h^{(3)}=0.$$

Hence the vector $a^{(2)}$ and $a^{(3)}$ are orthogonal in $\mathbb{R}^2$. □

Proof of Theorem 2.3. As in the proof of Proposition 6.4 we estimate the rate of $(1-\mu_{i,\lambda})$ using the identity (5.1). The proof is the same as before using (2.9) instead of (6.30). Then we obtain

$$1-\mu_{i,\lambda}=24\pi\eta_i\delta_\lambda^2\big(1+\mathrm{o}(1)\big) \tag{8.1}$$

where $\eta_i=(\sum_{k=1}^2 a_k^{(i)}\frac{\partial^2 R(x_0)}{\partial x_k x_j})/a_j^{(i)}$ for j such that $a_j^{(i)}\neq 0$.

Consequently we have

$$\sum_{k=1}^{2}a_k^{(i)}\frac{\partial^2 R(x_0)}{\partial x_k x_j}=\eta_i a_j^{(i)} \tag{8.2}$$

which holds both if $a_j^{(i)}\neq 0$ or $a_j^{(i)}=0$ by (8.1). From (8.2) we get that η_i is an eigenvalue of $D^2R(x_0)$ where D^2 denotes the Hessian matrix of $R(x)$ at the point x_0, and $a^{(i)}$ the corresponding eigenvector. Since by Lemma 8.1 the eigenvectors $a^{(i)}$ are orthogonal, the numbers η_i are the 2 eigenvalues c_1, c_2 of $D^2R(x_0)$. In particular $\eta_2=c_1$ and $\eta_3=c_2$ from the fact that $\mu_{2,\lambda}\leqslant\mu_{3,\lambda}$. □

9. Qualitative properties of the eigenfunctions

Lemma 9.1. *Let $\{z_n\} \in C^1(\mathbb{R}^2)$ be a sequence of functions such that*

$$z_n \to \sum_{k=1}^{2} d_k \frac{x_k}{(8+|x|^2)} \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2)$$

where $d=(d_1,d_2) \in \mathbb{R}^2$ with $d \neq 0$. Then, denoting by $Z_n^+ = \{x \in \mathbb{R}^2\colon z_n > 0\}$ and $Z_n^- = \{x \in \mathbb{R}^2\colon z_n < 0\}$, we have that for any $R>0$, the sets $Z_n^+ \cap B_R$ and $Z_n^- \cap B_R$ are both connected and nonempty for n sufficiently large and $B_R = \{x \in \mathbb{R}^2\colon |x| < R\}$.

Proof. See [16], Lemma 6.1 for an analogous proof in dimension $N \geqslant 3$. □

Proof of Theorem 2.4. The proof of part (i) of the theorem is very similar to the one of Theorem 1.4 in [16] and we do not report it. So let us concentrate on part (ii).

Using the convergence of $\tilde{v}_{i,\lambda}$ as in (2.8), and Lemma 9.1, we can state that there exist only two nodal sets for $\tilde{v}_{i,\lambda}$ inside the ball $B(0,R)$ for any $R>0$ if λ is small enough. This implies that $v_{i,\lambda}$ has two nodal regions inside the ball $B(x_\lambda, \delta_\lambda R)$ for any $R>0$ if λ is small enough and $i=2,3$.

By the Courant Nodal Line Theorem we can infer that $v_{2,\lambda}$ has only two nodal regions in Ω. It remains to consider $v_{3,\lambda}$.

We argue by contradiction. So let us suppose there exists a third nodal region $D_\lambda \subset \Omega$. D_λ should be an open, connected set. We can suppose $v_{3,\lambda} > 0$ in D_λ, and by continuity of $v_{3,\lambda}$ we know that $v_{3,\lambda} = 0$ on $\partial D_\lambda \subset \bar{\Omega}$. Choosing $R>0$ such that $\int_{\mathbb{R}^2 \setminus B_R(0)} e^{U(y)}\,dy < \frac{4\pi}{C}$ (where C is the constant in (3.3)), we have that $\lambda \int_{D_\lambda} e^{u_\lambda} = \int_{\tilde{D}_\lambda} e^{\tilde{u}_\lambda} \leqslant C \int_{\mathbb{R}^2 \setminus B_R(0)} e^U < 4\pi$, where $\tilde{D}_\lambda = (D_\lambda - x_\lambda)/\delta_\lambda$. If D_λ is simply connected, we can apply a result of [2] finding that the maximum principle holds in D_λ for the operator $-\Delta - \lambda e^{u_\lambda}$. Even if we do not know anything about the regularity of ∂D_λ, we can infer, using for example [3], that the first eigenvalue for the operator $-\Delta - \lambda e^{u_\lambda}$ in D_λ is strictly positive. But $v_{3,\lambda}$ has only one sign in D_λ so it should be the first eigenfunction for $-\Delta - \lambda e^{u_\lambda}$ in D_λ , and then the first eigenvalue should be zero, contradicting what we previously got.

It remains to consider the case where D_λ is not simply connected. By definition, a simply connected set contains the inside of each Jordan curve in it. Then we can find a Jordan curve, say γ_λ, contained in D_λ and some points inside of it not belonging to D_λ. Let us call $Z_\lambda \subset \Omega$ the inside of γ_λ and $V_\lambda := \{x \in Z_\lambda \text{ s.t. } v_{3,\lambda} \leqslant 0\} \subset Z_\lambda$. Now we can consider two different cases.

Firstly we suppose that there exists at least one point in V_λ such that $v_{3,\lambda} < 0$. This is not possible because it would imply that there exists another nodal region inside Z_λ, contradicting the Courant Nodal Line Theorem.

Secondly, if $v_{3,\lambda} \equiv 0$ in V_λ, then $v_{3,\lambda}$ solves $-\Delta v_{3,\lambda} = \lambda e^{u_\lambda} v_{3,\lambda} \geqslant 0$ in Z_λ, $v_{3,\lambda} \geqslant 0$ in Z_λ and $v_{3,\lambda} > 0$ on $\partial Z_\lambda = \gamma_\lambda$. This is not possible from the Strong Maximum Principle (see [21], for example). □

Remark 9.2. Following the proof of Lemma 4.5 one can prove even for $i=2,3$ that the maximum and minimum points of $v_{i,\lambda}$ in Ω lie inside the ball $B(x_\lambda, \delta_\lambda R)$ for some $R>0$ and they both converge to x_0.

10. On the fourth eigenvalue and the fourth eigenfunction

Proof of Theorem 2.5. Using the variational characterization of the eigenvalues we have

$$\mu_{4,\lambda} = \inf_{v \in H_0^1(\Omega),\ v \neq 0,\ v \perp \{v_{1,\lambda}, v_{2,\lambda}, v_{3,\lambda}\}} \frac{\int_\Omega |\nabla v|^2\,dx}{\lambda \int_\Omega e^{u_\lambda} v^2\,dx}. \tag{10.1}$$

Suppose $B(x_0,1) \subset \Omega$ and let us define the function

$$\hat{\Phi}(x) = \begin{cases} 1 & \text{if } |x-x_\lambda| < \delta_\lambda, \\ \frac{1}{\log \delta_\lambda} \log|x-x_\lambda| & \text{if } \delta_\lambda < |x-x_\lambda| < 1, \\ 0 & \text{if } |x-x_\lambda| > 1, \end{cases} \tag{10.2}$$

$0 \leqslant \hat{\Phi}(x) \leqslant 1$, and $\hat{\Phi} \in C^0(\Omega)$.

Let $\psi_{4,\lambda} = (x - x_\lambda) \cdot \nabla u_\lambda + 2$ and $v = \hat{\Phi}\psi_{4,\lambda} + a_{1,\lambda}v_{1,\lambda} + a_{2,\lambda}v_{2,\lambda} + a_{3,\lambda}v_{3,\lambda}$. We take

$$a_{i,\lambda} = -\frac{\int_\Omega e^{u_\lambda}\hat{\Phi}\psi_4 v_{i,\lambda}}{\int_\Omega e^{u_\lambda} v_{i,\lambda}^2} = -\frac{N_{i,\lambda}}{D_{i,\lambda}} \tag{10.3}$$

in such a way that $v \perp \{v_{1,\lambda}, v_{2,\lambda}, v_{3,\lambda}\}$ in $H_0^1(\Omega)$. Moreover $9\pi > D_{i,\lambda} > c_i > 0$ for $i = 1, 2, 3$, while

$$N_{i,\lambda} = \lambda \int_\Omega e^{u_\lambda} v_{i,\lambda}\hat{\Phi}\psi_{4,\lambda} = \int_{\Omega_\lambda} e^{\tilde{u}_\lambda(y)}\hat{\Phi}(\delta_\lambda y + x_\lambda)\tilde{v}_{i,\lambda}(y)\big(y \cdot \nabla\tilde{u}_\lambda(y) + 2\big)\,dy$$
$$= 2\int_{\mathbb{R}^2} e^{U(y)}\tilde{v}_i \frac{8 - |y|^2}{8 + |y|^2}\,dy + \mathrm{o}(1).$$

Set $\tilde{v}_i = \lim_{\lambda\to 0}\tilde{v}_{i,\lambda}$. From Theorem 2.1 and Theorem 2.2 it is easy to see that

$$\int_{\mathbb{R}^2} e^{U(y)}\tilde{v}_i \frac{8 - |y|^2}{8 + |y|^2}\,dy = 0$$

for $i = 1, 2, 3$ so that $a_{i,\lambda} = \mathrm{o}(1)$ for $i = 1, 2, 3$.

Using that $v_{i,\lambda}$ and $v_{j,\lambda}$ are orthogonal in $H_0^1(\Omega)$ if $i \neq j$, we have

$$\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla\hat{\Phi}\psi_{4,\lambda}|^2 + 2\lambda\sum_{i=1}^3 a_{i,\lambda}\mu_{i,\lambda}\int_\Omega e^{u_\lambda}v_{i,\lambda}\hat{\Phi}\psi_{4,\lambda} + \lambda\sum_{i=1}^3 (a_{i,\lambda})^2\mu_{i,\lambda}\int_\Omega e^{u_\lambda}v_{i,\lambda}^2.$$

Noting that $\lambda\int_\Omega e^{u_\lambda}v_{i,\lambda}^2 \leqslant 9\pi$ if λ is small enough, while $\lambda\int_\Omega e^{u_\lambda}v_{i,\lambda}\hat{\Phi}\psi_{4,\lambda} = \mathrm{o}(1)$, we can write

$$\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla\hat{\Phi}\psi_{4,\lambda}|^2 + \mathrm{o}(1).$$

Finally since $\psi_{4,\lambda}$ solves the linearized equation in Ω, we have

$$\int_\Omega |\nabla v|^2 = \lambda\int_\Omega e^{u_\lambda}\hat{\Phi}^2\psi_{4,\lambda}^2 + \int_\Omega \psi_{4,\lambda}^2|\nabla\hat{\Phi}|^2 + \mathrm{o}(1). \tag{10.4}$$

In a similar way we get

$$\lambda\int_\Omega e^{u_\lambda}v^2 = \lambda\int_\Omega e^{u_\lambda}\hat{\Phi}^2\psi_{4,\lambda}^2 + \mathrm{o}(1). \tag{10.5}$$

Inserting (10.4) and (10.5) into (10.1) we have

$$\mu_{4,\lambda} \leqslant 1 + \frac{\int_\Omega \psi_{4,\lambda}^2|\nabla\hat{\Phi}|^2 + \mathrm{o}(1)}{\lambda\int_\Omega e^{u_\lambda}\hat{\Phi}^2\psi_{4,\lambda}^2 + \mathrm{o}(1)}.$$

Let us estimate the last two integrals.

$$\lambda\int_\Omega e^{u_\lambda}\hat{\Phi}^2\psi_{4,\lambda}^2 = \int_{\Omega_\lambda} e^{\tilde{u}_\lambda(y)}\big(\hat{\Phi}(\delta_\lambda y + x_\lambda)\big)^2\big(y \cdot \nabla\tilde{u}_\lambda(y) + 2\big)^2\,dy$$
$$= \int_{\mathbb{R}^2} e^{U(y)}\big(y \cdot \nabla U(y) + 2\big)^2\,dy + \mathrm{o}(1) = c_1 + \mathrm{o}(1),$$

where $c_1 = \frac{32}{3}\pi$. To pass to the limit we use the fact that $|y \cdot \nabla\tilde{u}_\lambda(y)| \leqslant C$ in Ω_λ, (see (A.1)) and the estimate (3.6). Finally

$$\int_{\Omega} \psi_{4,\lambda}^2 |\nabla \hat{\Phi}|^2 = \frac{1}{(\log \delta_\lambda)^2} \int_{\Omega \cap \{1 \geqslant |x-x_\lambda| > \delta_\lambda\}} \frac{1}{|x-x_\lambda|^2} \big((x-x_\lambda)\cdot \nabla u_\lambda + 2\big)^2 \, dx$$
$$= \frac{1}{(\log \delta_\lambda)^2} \int_{\hat{\Omega}_\lambda \setminus B_1(0)} \frac{1}{|y|^2} \big(y \cdot \nabla \tilde{u}_\lambda(y) + 2\big)^2 \, dy, \tag{10.6}$$

where $\hat{\Omega}_\lambda = \{y \in \Omega_\lambda \text{ s.t. } |y| < \frac{1}{\delta \lambda}\}$. Then we have

$$\int_{\Omega} \psi_{4,\lambda}^2 |\nabla \hat{\Phi}|^2 \leqslant \frac{1}{(\log \delta_\lambda)^2} 2\pi C \int_1^{\frac{1}{\delta \lambda}} \frac{1}{r} \, dr \tag{10.7}$$
$$= \frac{2\pi C}{(\log \delta_\lambda)^2} \log \frac{1}{\delta \lambda} = -\frac{2\pi C}{\log \delta_\lambda} \to 0. \tag{10.8}$$

We have shown so far that $\mu_{4,\lambda} \leqslant 1 + o(1)$ and hence $\mu_{4,\lambda} \to 1$ as $\lambda \to 0$.

Arguing as in the proof of Lemma 5.3 we observe that $\tilde{v}_{4,\lambda} \to \frac{a_1 x_1 + a_2 x_2}{8+|x|^2} + b\frac{8-|x|^2}{8+|x|^2}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, with $(a_1, a_2, b) \neq (0,0,0) \in \mathbb{R}^3$. Here we want to show that $(a_1, a_2) = 0$ so that $b \neq 0$, proving (2.12). Let for $i = 2,3$ $a^{(i)} = (a_1^{(i)}, a_2^{(i)})$ be as in (2.8). The eigenfunctions $v_{4,\lambda}$ and $v_{i,\lambda}$ are orthogonal in $H_0^1(\Omega)$ for $i = 2,3$. Then by using Lemma 8.1 we have that the vector (a_1, a_2) is orthogonal to $a^{(2)} = (a_1^{(2)}, a_2^{(2)})$ and $a^{(3)} = (a_1^{(3)}, a_2^{(3)})$, which are both different from zero and orthogonal. This implies $(a_1, a_2) = (0,0)$.

We proved that $b \neq 0$. Then we can apply Lemma 6.1 getting

$$v_{4,\lambda}(x) \log \lambda \to 4\pi b G(x, x_0) \quad \text{in } C^1(\omega) \tag{10.9}$$

for any compact set ω in $\bar{\Omega} \setminus \{x_0\}$. As in the last part of the proof of Proposition 6.4 we have, from (6.19)

$$1 - \mu_{4,\lambda} = -\frac{1}{\log \lambda}\big(c_1 + o(1)\big) \tag{10.10}$$

where $c_1 = \frac{2(1-4\pi)}{c_0} < 0$ and $c_0 = \frac{\pi}{6}$. □

Proof of Theorem 2.6. We argue by contradiction. Let us suppose there exist at least two solutions $v_{4,\lambda}^1$ and $v_{4,\lambda}^2$ corresponding to the eigenvalue $\mu_{4,\lambda}$. Then $v_{4,\lambda}^1$ and $v_{4,\lambda}^2$ are orthogonal in $H_0^1(\Omega)$. Hence the rescaled functions $\tilde{v}_{4,\lambda}^i \to b^i \frac{8-|y|^2}{8+|y|^2}$ and $b^i \neq 0$ for $i = 1,2$. So we have

$$\int_{\Omega} \nabla v_{4,\lambda}^1 \cdot \nabla v_{4,\lambda}^2 = 0 \implies \int_{\Omega} e^{u_\lambda} v_{4,\lambda}^1 v_{4,\lambda}^2 = 0 \implies b^1 b^2 \int_{\mathbb{R}^2} e^{U(y)} \frac{(8-|y|^2)^2}{(8+|y|^2)^2} \, dy = 0 \tag{10.11}$$

and this gives a contradiction since $b^i \neq 0$. Then $\mu_{4,\lambda}$ is simple.

To prove that $v_{4,\lambda}$ has only two nodal regions, if λ is small enough, we observe that the function $\frac{8-|y|^2}{8+|y|^2}$ is positive in $B_8(0)$ and negative in $\mathbb{R}^2 \setminus \overline{B}_8(0)$. Hence assuming $b > 0$ (the same argument applies if $b < 0$), by the C^1_{loc} convergence we have

$$\tilde{v}_{4,\lambda} > 0 \quad \text{in } B_4(0) \text{ if } \lambda \text{ is small enough,}$$

and hence

$$v_{4,\lambda}(x) > 0 \quad \text{in } B(x_\lambda, 4\delta_\lambda) \subset \Omega.$$

In the same way

$$v_{4,\lambda}(x) < 0 \quad \text{on } \partial B(x_\lambda, 16\delta_\lambda) \subset \Omega.$$

To show that $v_{4,\lambda}(x) < 0$ in $\Omega \setminus B(x_\lambda, 16\delta_\lambda)$ we argue by contradiction. So let us suppose there exists a third nodal region D_λ inside this domain. Then $v_{4,\lambda} > 0$ in D_λ and $v_{4,\lambda} = 0$ on ∂D_λ.

Let us consider the points $z_\lambda \in D_\lambda$ such that $v_{4,\lambda}(z_\lambda) = \max_{D_\lambda} v_{4,\lambda}$ and $\nabla v_{4,\lambda}(z_\lambda) = 0$ for all λ. Up to a subsequence the points z_λ converge to a point $z \in \bar{\Omega}$. We have the two following cases.

Case 1. We suppose $z \neq x_0$. Then we can use the convergence in (6.1) to get a contradiction. We have $v_{4,\lambda}(z_\lambda)\log\lambda \leqslant 0$ for $\lambda < 1$. Passing to the limit we have $0 \geqslant v_{4,\lambda}(z_\lambda)\log\lambda \to 4\pi b G(z, x_0) \geqslant 0$. This would imply that $G(z, x_0) = 0$ and therefore $z \in \partial\Omega$. But this is not possible since $0 = |\nabla v_{4,\lambda}(z_\lambda)\log\lambda| \to 4\pi b|\nabla G(z, x_0)| \neq 0$ from the Hopf Maximum Principle.

Case 2. Let us suppose that $z = x_0$. We already know that $z_\lambda \in \Omega \setminus B(x_\lambda, 16\delta_\lambda)$. In this case we consider the function ζ_λ defined in the proof of Lemma 4.5, where $r_\lambda = |z_\lambda - x_\lambda|$. Reasoning as in the proof of Lemma 4.5, using the Green's representation formula we have

$$\begin{aligned}\zeta_\lambda(x) &= \frac{1}{2\pi}\mu_{4,\lambda}\int_{\Omega_\lambda} \log\frac{1}{|x - (\delta_\lambda/r_\lambda)y|} e^{\tilde{u}_\lambda(y)}\tilde{v}_{4,\lambda}(y)\,dy \\ &\quad + \mu_{4,\lambda}\int_{\Omega_\lambda} H(r_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)}\tilde{v}_{4,\lambda}(y)\,dy. \end{aligned}\tag{10.12}$$

Reasoning as in the proof of Lemma 6.1 we have that

$$\int_{\Omega_\lambda} e^{\tilde{u}_\lambda(y)}\tilde{v}_{4,\lambda}(y)\,dy = \frac{4\pi b}{\log\lambda}\big(1 + \mathrm{o}(1)\big)$$

and hence we can multiply (10.12) by $\log\lambda$ getting

$$-\zeta_\lambda(x)\log\lambda = \frac{1}{2\pi}\log\frac{1}{|x|}4\pi b + 4\pi b R(x_0) + \mathrm{o}(1).\tag{10.13}$$

Repeating the same argument for the first derivatives we can show that the convergence is $C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. This gives us a contradiction since we have a sequence of points $\hat{z}_\lambda = (z_\lambda - x_\lambda)/r_\lambda$ such that $\nabla\zeta_\lambda(\hat{z}_\lambda) = 0$ and which converges to a point $\hat{z}$ with $|\hat{z}| = 1$. This is inconsistent with formula (10.13).

This proves also that the nodal line of $v_{4,\lambda}$ does not touch the boundary of Ω.

Hence if $v_{4,\lambda}$ has more than two nodal regions, there should be in the annulus $A_\lambda = B(x_\lambda, 4\delta_\lambda) \setminus B(x_\lambda, 16\delta_\lambda)$ a nodal region D_λ such that $\bar{D}_\lambda \subset A_\lambda$.

If in D_λ the function $v_{4,\lambda}$ is positive, we take the maximum points z_λ of $v_{4,\lambda}$ in D_λ such that

$$v_{4,\lambda}(z_\lambda) > 0, \quad \nabla v_{4,\lambda}(z_\lambda) = 0, \quad z_\lambda \in A_\lambda.$$

Rescaling and passing to the limit, we have that the points $\tilde{z}_\lambda = (z_\lambda - x_\lambda)/\delta_\lambda$ converge to a point $\tilde{z} \in \mathbb{R}^2$ such that

$$\frac{8 - |\tilde{z}|^2}{8 + |\tilde{z}|^2} \geqslant 0, \quad |\tilde{z}| > 4, \quad \text{and} \quad \nabla\left(\frac{8 - |\tilde{z}|^2}{8 + |\tilde{z}|^2}\right) = 0$$

which is impossible.

If in D_λ the function $v_{4,\lambda}$ is negative considering the minimum points of $v_{4,\lambda}$ in D_λ, rescaling as before and passing to the limit, we would get a point $\tilde{z} \in \mathbb{R}^2$ such that

$$\frac{8 - |\tilde{z}|^2}{8 + |\tilde{z}|^2} \leqslant 0, \quad |\tilde{z}| \leqslant 16, \quad \text{and} \quad \nabla\left(\frac{8 - |\tilde{z}|^2}{8 + |\tilde{z}|^2}\right) = 0$$

which is again a contradiction. □

Remark 10.1. A consequence of Theorem 2.6 is that the maximum and the minimum point of $v_{4,\lambda}$ both converge to x_0.

Proof of Corollary 2.7. If x_0 is a nondegenerate critical point of $R(x)$ in Ω then both the eigenvalues c_1 and c_2 of the Hessian matrix $D^2R(x_0)$ are different from zero. By (2.10) each eigenvalue $c_i < 0$ implies that $\mu_{i,\lambda} > 1$, for λ small enough. Moreover, (2.14) implies that $\mu_{4,\lambda} > 1$, for λ small enough. Hence, denoting by $m(x_0)$ the Morse index of x_0 as a critical point of the Robin function $R(x)$, we find that the Morse index of u_λ is exactly $m(x_0) + 1$. □

Proof of Corollary 2.8. Using a result of [4], we know that the Robin function $R(x)$ satisfies the equation $\Delta R = -4e^{-2R}$ in Ω and $R(x) \to -\infty$ for $x \to \partial\Omega$. This implies that at least one eigenvalue of the Hessian matrix of R in x_0 is strictly negative. Then the Morse index of u_λ is less or equal to 2 and this proves (i).

Using a result of [4] we know that in a convex domain of $\mathbb{R}^2$ both the eigenvalues of the Hessian matrix of the Robin function in an interior point are negative. This implies that the Morse index of u_λ in a convex domain is 1 and so (ii) is proved. □

11. About the spectrum of the operator $-\frac{\Delta}{\lambda e^{u_\lambda}}$

In this section we want to give an explicit description of the eigenvalue of

$$\begin{cases} -\Delta v = \lambda\mu e^{u_\lambda} v & \text{in } \Omega, \\ \|v\|_\infty = 1, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11.1}$$

where $\lambda \to 0$. First we want to observe that due to the compactness of the operator $-\Delta^{-1}$ into the weighted space $L^2_{w_\lambda}(\Omega)$ (with $w_\lambda = e^{u_\lambda}$), the spectrum of the operator is a sequence of nonnegative values which go to the infinity.

A crucial tool in our study is the spectrum of the problem

$$\begin{cases} -\Delta v = c\frac{64}{(8+|x|^2)^2} v & \text{in } \mathbb{R}^2, \\ v \in L^\infty(\mathbb{R}^2). \end{cases} \tag{11.2}$$

We look for solutions in the space $W = \{v\colon\ |\nabla v| \in L^2(\mathbb{R}^2),\ v \in L^\infty(\mathbb{R}^2)\}$. Here again we have that due to the compactness and the autoadjointness of the operator $-\Delta^{-1}$ into the space $L^2_W(\mathbb{R}^2)$ where $W = e^U$, the spectrum is a sequence of nonnegative values that goes to infinity. We can state the following:

Theorem 11.1. *The eigenvalues of problem* (11.2) *are the numbers* $c_n = \frac{n(n+1)}{2}$ *for* $n = 0, 1, 2, \ldots$. *Each eigenvalue* c_n *has multiplicity* $2n + 1$ *and the eigenfunctions are* $Y_m(\theta)P_n^m(\frac{8-r^2}{8+r^2})$ *for* $m \in N$, $m \leqslant n$, *where* $Y_m(\theta) = A\sin(m\theta) + B\cos(m\theta)$ *and* P_n^m *are the associated Legendre functions.*

Proof. We look for solutions of (11.2) of the type $v = R(r)Y(\theta)$. Eq. (11.2) in radial coordinates becomes

$$-\frac{1}{r}\frac{\partial}{\partial r}\big(rR'(r)Y(\theta)\big) - \frac{1}{r^2}R(r)Y''(\theta) = \frac{64c}{(8+r^2)^2}R(r)Y(\theta) \tag{11.3}$$

with the conditions

$$\begin{cases} R'(0) = 0 & \text{i.e. regular in the origin,} \\ R(r) = O(1) \text{ at infinity} & \text{i.e. regular at infinity,} \\ Y(0) = Y(2k\pi) & \text{periodicity conditions.} \end{cases} \tag{11.4}$$

Separating variables, letting $v \neq 0$, we get the two following equations

$$\begin{cases} Y''(\theta) + kY(\theta) = 0, \quad Y(\theta) = Y(\theta + 2k\pi), \\ \frac{1}{r}(rR'(r))' + \frac{64c}{(8+r^2)^2}R(r) - \frac{k}{r^2}R(r) = 0, \\ R'(0) = 0, \quad R(r) = O(1) \quad \text{at infinity.} \end{cases}$$

The first equation (in θ), has the solutions, for $k \geqslant 0$:

$$Y(\theta) = A\cos(\omega\theta) + B\sin(\omega\theta)$$

with $\omega^2 = k$. The periodicity conditions then imply that $\omega = m$ for $m \in \mathbb{Z}$. Thus the eigenvalues are $k = m^2$ for $m \in \mathbb{Z} \setminus \{0\}$, and the eigenspace is two dimensional, spanned by the functions $\sin(m\theta), \cos(m\theta)$.

If $k < 0$, we do not have any periodic solutions.

So we have shown that Eq. (11.3) has solutions $Y_m(\theta) = A\cos(m\theta) + B\sin(m\theta)$ for $k = m^2$ and $m \in \mathbb{N}_0$.

We are left with the study of the radial part of Eq. (11.3) with $k = m^2$, i.e.

$$\frac{1}{r}\big(rR'(r)\big)' + \frac{64c}{(8+r^2)^2}R(r) - \frac{m^2}{r^2}R(r) = 0. \tag{11.5}$$

Setting $\xi = \frac{8-r^2}{8+r^2}$, Eq. (11.5) becomes

$$\frac{\partial}{\partial \xi}\left(\left(1-\xi^2\right)\frac{\partial R}{\partial \xi}\right) - \frac{m^2}{1-\xi^2}R(\xi) + 2cR(\xi) = 0 \tag{11.6}$$

for $-1 \leqslant \xi \leqslant 1$. The boundary condition reads as $R(1)$ and $R(-1)$ bounded.

Eq. (11.6) is the classical *Legendre equation*. By well-known results (see for example [7]), the eigenvalue problem (11.6) with the condition that $R(\xi)$ is bounded in $\xi = -1$ and $\xi = 1$ has solutions only for $2c = n(n+1)$, $n \in \mathbb{N}_0$. These solutions are exactly the Legendre associated functions

$$P_n^m(\xi) = (-1)^m (1-\xi^2)^{m/2} P_n^{(m)}(\xi), \tag{11.7}$$

where (m) denotes the m derivative. To avoid the trivial solution $R \equiv 0$, we let $m \leqslant n$. In (11.7) P_n are the Legendre polynomials

$$P_n(\xi) = \frac{1}{2^n n!}\frac{\partial^n}{\partial \xi^n}\left(\xi^2 - 1\right)^n. \tag{11.8}$$

It is easy to see that each eigenvalue $c_n = \frac{n(n+1)}{2}$ has $n+1$ multiplicity.

Moreover we recall that the Legendre associated functions span the space of polynomials in $[-1, 1]$. Hence one can prove that Eq. (11.6) cannot have any other eigenvalues different from c_n. We proved so far that problem (3.9) has eigenvalues $c_n = \frac{n(n+1)}{2}$ with multiplicity $2n+1$ for $n \in \mathbb{N}$ and each eigenvalue c_n has eigenfunctions $Y_m(\theta)P_n^m(\xi)$ for $m = 0, \dots, n$. □

For reader's convenience we write down some eigenvalues c_n and the corresponding eigenfunctions V_l^n:

$$c_0 = 0 \quad V^0 = 1,$$

$$c_1 = 1 \quad V_1^1 = \frac{x_1}{8+|x|^2}, \quad V_2^1 = \frac{x_2}{8+|x|^2}, \quad V_3^1 = \frac{8-|x|^2}{8+|x|^2},$$

$$c_2 = 3 \quad V_1^2 = \frac{x_1(8-|x|^2)}{(8+|x|^2)^2}, \quad V_2^2 = \frac{x_2(8-|x|^2)}{(8+|x|^2)^2},$$

$$V_3^2 = \frac{x_1^2 - x_2^2}{(8+|x|^2)^2}, \quad V_4^2 = \frac{x_1 x_2}{(8+|x|^2)^2},$$

$$V_5^2 = \frac{3}{2}\left(\frac{(8-|x|^2)}{(8+|x|^2)}\right)^2 - \frac{1}{2}.$$

Following this kind of notation we call c_n the eigenvalue and V_l^n with $l = 1, \dots, 2n+1$, the associated eigenfunctions. We observe that for each value of n there is one radial eigenfunction (it corresponds to the case $m = 0$ in the previous notation), and that the eigenfunctions go to zero for $|x| \to \infty$ except for the radial one which is bounded. We call V_{2n+1}^n the n-related radial eigenfunction.

Now we come back to problem (2.2). Let μ_i be the eigenvalue and v_i the associated eigenfunction. Set $\tilde{v}_i(y) = v_i(\delta_\lambda y + x_\lambda)$. We want to show that $\tilde{v}_i$ converge to the ith eigenfunction of problem (11.2) (considered with its multiplicity). In this way any eigenfunction of the limiting problem will be the limit of a rescaled eigenfunction of problem (2.2), and vice-versa. In order to prove this we write any $i \in \mathbb{N}$ in the following way

$$i = \sum_{k=0}^{n}(2k+1) + l_n, \tag{11.9}$$

for some $n \in \mathbb{N}$ and $l_n \in \mathbb{N}$ with $0 \leqslant l_n < 2n+3$. It is easy to check that for any number i there exist a unique n and l_n for which (11.9) holds;

Theorem 11.2. *Let $i \in \mathbb{N}$. Writing i as in* (11.9)*, we find that any eigenvalue μ_i of* (2.2) *converges to the eigenvalue c_{n+1} of* (11.2)*. Moreover $\tilde{v}_i(y)$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ to the sum $\sum_{l=1}^{2n+3} a_l V_l^{n+1}$ where $a_l \in \mathbb{R}$ are not all zero, and V_k^{n+1} are the eigenfunctions of* (11.2) *related to the eigenvalue c_{n+1}.*

Proof. To sake of simplicity we prove the result just for c_2, in the cases $i=5$ and $i=9$. Indeed any other eigenvalue can be handled in the same way and the proof of cases $i=6,7,8$ in c_2 is the same as in $i=5$.

Case $i=5$ with $n=2$ and $l_2=1$ in (11.9). Let V^2 be any eigenfunction of (11.2) related to the eigenvalue c_2 different from V_5^2. Hence V^2 is not radial and $V^2\to 0$ for $|x|\to\infty$. Let $\psi(x)=V^2((x-x_\lambda)/\delta_\lambda)$ and Φ be a cut off function centered in x_λ as defined in Section 5.5. We want to estimate the eigenvalue μ_5 using the well known formula

$$\mu_{5,\lambda}=\inf_{v\in H_0^1(\Omega),\ v\perp\{v_{1,\lambda},\dots,v_{4,\lambda}\}}\frac{\int_\Omega|\nabla v|^2}{\lambda\int_\Omega e^{u_\lambda}v^2}. \tag{11.10}$$

To this end let us consider the function $v=\Phi\psi+\sum_{j=1}^4 a_jv_{j,\lambda}$, where $a_j\in\mathbb{R}$ will be chosen so that v is orthogonal to $\{v_{1,\lambda},\dots,v_{4,\lambda}\}$. Using the orthogonality of the eigenfunctions $v_{i,\lambda}$, $v_{j,\lambda}$ for $i\neq j$ in $H_0^1(\Omega)$, we derive from (2.2) that

$$a_k=-\frac{\lambda\int_\Omega e^{u_\lambda}v_{k,\lambda}\Phi\psi\,dx}{\lambda\int_\Omega e^{u_\lambda}v_{k,\lambda}^2\,dx} \tag{11.11}$$

for $k=1,\dots,4$. Since $v_{k,\lambda}\not\equiv 0$ we know that $\int_\Omega e^{u_\lambda}v_{k,\lambda}^2\to b_k>0$, while rescaling the numerator we get

$$\begin{aligned}\lambda\int_\Omega e^{u_\lambda}\Phi\psi v_{k,\lambda}(x)\,dx&=\int_{\Omega_\lambda}e^{\tilde u_\lambda(y)}V^2(y)\Phi(\delta_\lambda y+x_\lambda)\tilde v_{k,\lambda}(y)\,dy\\&=\Phi(x_0)\int_{\mathbb{R}^2}e^{U(y)}V^2(y)\tilde v_k(y)\,dy+o(1),\end{aligned}$$

where $\tilde v_k=\lim_{\lambda\to 0}\tilde v_{k,\lambda}$. Here we note that the passage into the limit is done using estimate (3.6) and that $v_{k,\lambda}$, V^2 and Φ are uniformly bounded in $\mathbb{R}^2$. Finally by the orthogonality of the functions V^2 and $\tilde v_k$ with $k\leqslant 4$ we get

$$a_k=o(1). \tag{11.12}$$

With this choice of v we have

$$\int_\Omega|\nabla v|^2\,dx=\int_\Omega\left|\nabla(\psi\Phi)\right|^2+2\sum_{j=1}^4 a_j\int_\Omega\nabla v_{j,\lambda}\cdot\nabla(\psi\Phi)+\sum_{j,k=1}^4 a_ja_k\int_\Omega\nabla v_{j,\lambda}\cdot\nabla v_{k,\lambda}.$$

Using Eq. (2.2) and the orthogonality of $v_{j,\lambda}$ and $v_{k,\lambda}$ if $j\neq k$ we have

$$\int_\Omega|\nabla v|^2\,dx=\int_\Omega\psi^2|\nabla\Phi|^2+\int_\Omega(-\Delta\psi)\Phi^2\psi+2\sum_{j=1}^4 a_j\lambda\mu_{j,\lambda}\int_\Omega e^{u_\lambda}v_{j,\lambda}\psi\Phi+\sum_{j=1}^4 a_j^2\lambda\mu_{j,\lambda}\int_\Omega e^{u_\lambda}v_{j,\lambda}^2.$$

Since $\lambda\int_\Omega e^{u_\lambda}v_{j,\lambda}\Phi\psi=-a_j(\lambda\int_\Omega e^{u_\lambda}v_{j,\lambda}^2)=o(1)$ we get

$$\int_\Omega|\nabla v|^2\,dx=\int_\Omega\psi^2|\nabla\Phi|^2+\int_\Omega(-\Delta\psi)\Phi^2\psi+o(1).$$

In the same way we have

$$\begin{aligned}\lambda\int_\Omega e^{u_\lambda}v^2&=\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi^2+2\sum_{j=1}^4 a_j\lambda\int_\Omega e^{u_\lambda}v_{j,\lambda}\psi\Phi+\sum_{j=1}^4 a_j^2\lambda\int_\Omega e^{u_\lambda}v_{j,\lambda}^2\\&=\lambda\int_\Omega e^{u_\lambda}\Phi^2\psi^2+o(1).\end{aligned}$$

Some computation proves that ψ solves

$$-\Delta\psi=\frac{1}{\delta_\lambda^2}c_2e^{U((x-x_\lambda)/\delta_\lambda)}\psi.$$

So using $\Phi^2\psi$ as a test function we get

$$\int_\Omega |\nabla v|^2 = \int_\Omega \psi^2|\nabla\Phi|^2 + \frac{1}{\delta_\lambda^2}c_2\int_\Omega e^{U((x-x_\lambda)/\delta_\lambda)}\psi^2\Phi^2\,dx + \mathrm{o}(1).$$

Rescaling and passing to the limit we get

$$\begin{aligned}\int_\Omega |\nabla v|^2 &= \int_\Omega \psi^2|\nabla\Phi|^2 + c_2\int_{\Omega_\lambda} e^{U(y)}\big(V^2(y)\big)^2\Phi^2(\delta_\lambda y + x_\lambda)\,dx + \mathrm{o}(1)\\ &= \int_\Omega \psi^2|\nabla\Phi|^2 + c_2\int_{\mathbb{R}^2} e^{U(y)}\big(V^2(y)\big)^2 + \mathrm{o}(1),\end{aligned} \tag{11.13}$$

where the passage into the limit is done using that $e^U \in L^1(\mathbb{R}^2)$ and V^2 and Φ are uniformly bounded.

Similarly, rescaling and passing to the limit we get

$$\begin{aligned}\lambda\int_\Omega e^{u_\lambda}v^2 &= \int_{\Omega_\lambda} e^{\tilde u_\lambda(y)}\big(V^2(y)\big)^2\Phi^2(\delta_\lambda y + x_\lambda)\,dy + \mathrm{o}(1)\\ &= \int_{\mathbb{R}^2} e^{U(y)}\big(V^2(y)\big)^2\,dy + \mathrm{o}(1).\end{aligned} \tag{11.14}$$

Finally we estimate

$$\int_\Omega |\nabla\Phi|^2\psi^2\,dx. \tag{11.15}$$

Recalling that ψ is bounded, $|\nabla\Phi|^2 \in L^1(\Omega)$ and $V^2 \to 0$ as $|x| \to \infty$, we can pass to the limit in (11.15) getting $\int_\Omega |\nabla\Phi|^2\psi^2\,dx = \mathrm{o}(1)$. We are now ready to estimate the eigenvalue $\mu_{i,\lambda}$. From (11.10), (11.13) and (11.14) we get

$$\mu_{5,\lambda} \leqslant \frac{\int_\Omega |\nabla v|^2}{\lambda\int_\Omega e^{u_\lambda}v^2} = c_2 + \frac{\int_\Omega \psi^2|\nabla\Phi|^2 + \mathrm{o}(1)}{\int_{\mathbb{R}^2} e^{U(y)}(V^2(y))^2 + \mathrm{o}(1)} = 3 + \mathrm{o}(1). \tag{11.16}$$

Setting $\mu_5 = \lim_{\lambda\to 0}\mu_{5,\lambda}$ we have proved that $\mu_5 \leqslant c_2 = 3$.

Let us prove that $\mu_5 = c_2 = 3$. Let $v_{5,\lambda}$ be the eigenfunction related to the eigenvalue $\mu_{5,\lambda}$ and let $\tilde v_{5,\lambda} = v_{5,\lambda}(\delta_\lambda y + x_\lambda)$. Then $\tilde v_{5,\lambda}$ solves

$$\begin{cases} -\Delta\tilde v_{5,\lambda} = \mu_{5,\lambda}e^{\tilde u_\lambda}\tilde v_{5,\lambda} & \text{in } \Omega_\lambda,\\ \|\tilde v_{5,\lambda}\|_\infty = 1,\\ \tilde v_{5,\lambda} = 0 & \text{on } \partial\Omega_\lambda.\end{cases} \tag{11.17}$$

Let us show that $\tilde v_{5,\lambda}$ is uniformly bounded in the space W. Indeed

$$\int_{\mathbb{R}^2} |\nabla\tilde v_{5,\lambda}|^2 = \mu_{5,\lambda}\int_{\Omega_\lambda} e^{\tilde u_\lambda}\tilde v_{5,\lambda}^2 \leqslant \mu_{5,\lambda}\int_{\Omega_\lambda} e^{\tilde u_\lambda} \to 8\pi\mu_5.$$

So by standard elliptic regularity theory we have that $\tilde v_{5,\lambda} \to \tilde v_5$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where $\tilde v_5 \in W$ is a solution of

$$\begin{cases} -\Delta\tilde v_5 = \mu_5\frac{1}{(1+|x|^2/8)^2}\tilde v_5 & \text{in } \mathbb{R}^2,\\ \|\tilde v_5\|_\infty \leqslant 1.\end{cases} \tag{11.18}$$

Let us prove that $\tilde v_5 \neq 0$. Let z_λ be the point of Ω_λ such that $\tilde v_{5,\lambda}(z_\lambda) = 1$. If $\tilde v_5 \equiv 0$ then z_λ should go to infinity. Let us consider the function

$$\hat v_{5,\lambda} = \tilde v_{5,\lambda}\left(\frac{x}{|x|^2}\right).$$

Then $\hat{v}_{5,\lambda}$ solves

$$-\Delta\hat{v}_{5,\lambda} = \frac{1}{|x|^4} e^{\hat{u}_\lambda} \hat{v}_{5,\lambda}$$

and, as in the proof of Lemma 4.2, we have $\int_{\mathbb{R}^2} |\nabla\hat{v}_{5,\lambda}|^2$ is uniformly bounded and $\hat{v}_{5,\lambda} \to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. By the regularity theory we reach a contradiction as in the proof of Lemma 4.2. From Eq. (11.18) one has that μ_5 is an eigenvalue of problem (3.9) and $\tilde{v}_5$ is a related eigenfunction. However, since $v_{5,\lambda}$ is orthogonal in $H^1_0(\Omega)$ to $v_{j,\lambda}$ for each $j < 5$, we have that $\mu_5 > \mu_4$ and then $\mu_5 = 3$. As we remarked cases $i = 6, 7, 8$ can be handled in the same way.

Case $i = 9$ with $n = 2$ and $l_2 = 5$ in (11.9). By the previous step we have that $\tilde{v}_{i,\lambda} \to \sum_{l=1}^5 a^i_l V^2_l$ for $i = 5, 6, 7, 8$ and some coefficient $a^i \in \mathbb{R}^5$. Note that the coefficient a^j are orthogonal to the previous ones in $\mathbb{R}^5$. Let b be a vector in $\mathbb{R}^5$ which is not zero and which is orthogonal to a^j for $j = 5, \ldots, 8$. If b_5, the last component of vector b, is zero, we can reason as before getting the result. Now we analyze the case $b_5 \neq 0$. We want to estimate the eigenvalue $\mu_{9,\lambda}$ with formula (11.10), where $v = \hat{\Phi}\psi + \sum a_j v_{j,\lambda}$ where $\hat{\Phi}$ is as defined in (10.2), $\psi = \sum_{l=1}^5 b_l V^2_l((x - x_\lambda)/\delta_\lambda)$ with b as chosen before. Reasoning as in the first part of the proof, using the orthogonality of a^j and b in $\mathbb{R}^5$, we get $a_j = o(1)$ for $j = 1, \ldots, 8$. We can argue as before getting

$$\mu_{9,\lambda} \leqslant 3 + \frac{\int_\Omega \psi^2 |\nabla\hat{\Phi}|^2 + o(1)}{\lambda \int_\Omega e^{u_\lambda} \hat{\Phi}^2 \psi^2 + o(1)}.$$

Recalling the proof of Theorem 2.5, using the fact that V^2_l are uniformly bounded in $\mathbb{R}^2$ we get that the denominator is strictly positive while the numerator goes to zero as $-\frac{C}{\log\delta_\lambda}$. This proves the Theorem for $i = 5, \ldots, 9$. □

Appendix A

A.1. Uniform boundedness of $\frac{\partial\tilde{u}_\lambda}{\partial x_i}$ in Ω_λ

Lemma A.1. *Let $\tilde{u}_\lambda$ be as defined before. We have*

$$\left|\frac{\partial\tilde{u}_\lambda}{\partial x_i}\right| \leqslant \frac{C}{|x|} \quad \text{in } \Omega_\lambda. \tag{A.1}$$

Proof. Let Ω be such that $B(x_0, 1) \subset \Omega$. We know by (3.2) that $u_\lambda(x) \to 8\pi G(x, x_0)$ in $C^1(\bar{\Omega} \setminus B(x_0, 1))$. Hence we have

$$\left|\frac{\partial\tilde{u}_\lambda}{\partial x_i}\right| \leqslant C \sup_{\Omega\setminus B(x_0,\frac{1}{2})} \left|\frac{\partial G(x, x_\lambda)}{\partial x_i}\right| \leqslant \frac{C}{|x - x_\lambda|} \quad \text{for } x \in \Omega \setminus B(x_0, 2).$$

Recalling that $\tilde{u}_\lambda(y) = u_\lambda(\delta_\lambda y + x_\lambda) - \|u_\lambda\|_\infty$ a simple computation shows us that

$$\frac{\partial\tilde{u}_\lambda}{\partial y_j}(y) = \delta_\lambda \frac{\partial u_\lambda}{\partial x_j}(\delta_\lambda y + x_\lambda) \leqslant \frac{C}{|y|}$$

for $y \in \Omega_\lambda \setminus B(0, \frac{2}{\delta_\lambda})$.

Now we need to estimate the first derivatives just for $x \in B(0, \frac{2}{\delta_\lambda})$. To this end, using the Green's representation formula, we get

$$\frac{\partial\tilde{u}_\lambda}{\partial x_i}(x) = \delta_\lambda \int_{\Omega_\lambda} \frac{\partial}{\partial x_i} G(\delta_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)}\, dy.$$

From the standard decomposition of $G(x, y)$, we can write

$$\frac{\partial\tilde{u}_\lambda}{\partial x_i}(x) = \delta_\lambda \frac{1}{2\pi} \int_{\Omega_\lambda} \frac{\delta_\lambda (x - y)_i}{\delta^2_\lambda |x - y|^2} e^{\tilde{u}_\lambda(y)}\, dy + \delta_\lambda \int_{\Omega_\lambda} \frac{\partial H}{\partial x_i}(\delta_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) e^{\tilde{u}_\lambda(y)}\, dy. \tag{A.2}$$

Since H is a harmonic function we have that

$$\nabla H(x,y) \leqslant C \sup_{y\in\Omega} \nabla H(x,y) = C \sup_{y\in\partial\Omega} \nabla H(x,y) = C \sup_{y\in\Omega} \frac{(x-y)}{|x-y|^2}.$$

Then,

$$\nabla H(\delta_\lambda x + x_\lambda, \delta_\lambda y + x_\lambda) \leqslant C \sup_{y\in\partial\Omega_\lambda} \frac{1}{\delta\lambda} \frac{(x-y)}{|x-y|^2}.$$

On the other hand since $y \in \partial\Omega_\lambda$ and $x \in B(0, \frac{2}{\delta\lambda})$ we get that $\sup_{y\in\partial\Omega_\lambda} \frac{x-y}{|x-y|^2} \sim \delta_\lambda \to 0$ as $\lambda \to 0$, $\forall x \in B(0, \frac{1}{\delta\lambda})$. We turn back to the (A.2), getting

$$\begin{aligned} \left|\frac{\partial \tilde{u}_\lambda(x)}{\partial x_i}\right| &\leqslant \frac{1}{2\pi} \int_{\Omega_\lambda} \frac{1}{|x-y|} e^{\tilde{u}_\lambda(y)}\,dy + \sup_{y\in\partial\Omega_\lambda} \frac{1}{|x-y|} \int_{\Omega_\lambda} e^{\tilde{u}_\lambda(y)}\,dy \\ &\leqslant \frac{1}{2\pi} \int_{\Omega_\lambda} \frac{1}{|x-y|} e^{U(y)}\,dy + \mathrm{O}(\delta_\lambda) \leqslant C \int_{\mathbb{R}^2} \frac{1}{|x-y|(8+|y|^2)^2}\,dy + \mathrm{O}(\delta_\lambda). \end{aligned} \tag{A.3}$$

Note that for $x \in \Omega_\lambda$ we have $\delta_\lambda \leqslant \frac{1}{|x|}$ and

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|(8+|x|^2)^2} \leqslant \frac{C}{|x|}.$$

Then the claim follows. □

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