Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 271–284
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Exponential convergence for a periodically driven semilinear heat equation

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Received 21 January 2007; accepted 3 January 2008
Available online 31 January 2008

Abstract

We consider a semilinear heat equation in one space dimension, with a periodic source at the origin. We study the solution, which describes the equilibrium of this system and we prove that, as the space variable tends to infinity, the solution becomes, exponentially fast, asymptotic to a steady state. The key to the proof of this result is a Harnack type inequality, which we obtain using probabilistic ideas.

Résumé

On considère une équation de chaleur semilinéaire dans l'espace unidimensionnel, avec une source périodique à l'origine. On étudie la solution qui décrit l'équilibre de ce système, et on montre que, si la variable spatiale tend vers l'infini, la solution devient asymptotiquement équivalente à une solution stationnaire à vitesse exponentielle. On utilise une inégalité de type Harnack, qu'on obtient par des méthodes probabilistiques.

MSC: 35K60; 60H15

Keywords: Semilinear heat equation; Boundary homogenization; Harnack inequality

1. Introduction

In the recent years there has been an increasing interest in the study of the effect of a boundary perturbation on the long time behavior of dynamical systems described by nonlinear partial differential equations. In examples like the Navier–Stokes [5] and Ginzburg–Landau [4] equations the boundary perturbation appears as a degenerate random forcing in the frequency space. In these examples it is shown that there is a unique stochastic stationary state, where there is balance between the force term and the dissipation. On the other hand very little is known about these states. For instance, one would be interested to know how the stationary state behaves at locations "far away" from the perturbation. To answer such a question one needs to obtain a thorough understanding of the way the nonlinear dynamics mixes the boundary perturbation and for many models this question seems for the moment to be out of reach.

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2008.01.003

We will give here an example, where a satisfactory description of the behavior of the stationary state far away from the perturbation can be given. The model we are considering is described by the one-dimensional, periodically driven semilinear equation

$$u_t = u_{xx} - u^2 + \lambda(t)\delta_0(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1}$$

where $\delta_0(x)$ is the Dirac function at the origin and $\lambda(t)$ is a periodic function, such that

$$0 < \lambda_1 \leqslant \lambda(\cdot) \leqslant \lambda_2 < \infty, \tag{2}$$

for some λ_1, λ_2. Condition (2) will guarantee the existence of a positive, bounded solution of Eq. (1).

The reason we study Eq. (1), in particular, is that it describes the equilibrium density of an annihilating particle system model. Particles are born at the origin, periodically, at a rate $\lambda(t)$. Subsequently, they perform independent random walks on the $\mathbb{Z}^1$, described in the continuous limit by u_{xx}, and when two particles meet, they kill each other, described in the continuous limit by $-u^2$. Scaling limits for such types of models have been treated in [7].

The balance between the input term $\lambda(t)\delta_0(x)$ and the dissipation $-u^2$ will drive the system to a periodic, equilibrium state, which is proved to be unique. This state is given as the solution of (1). In fact, one can obtain the solution of (1) in the following way.

First, prescribe some initial data, say $u(T,\cdot) = 0$, at time $T < 0$ and denote by $u^T(t,x)$ the solution of the initial value problem

$$u_t = u_{xx} - u^2 + \lambda(t)\delta_0(x), \quad x \in \mathbb{R},\ t > T,$$
$$u(T,x) = 0, \quad x \in \mathbb{R}.$$

We will see that as $T \to -\infty$, $u^T(t,x)$ converges pointwise to a function $u(t,x)$, which solves Eq. (1). This justifies that the solution of (1) describes the equilibrium state of the system. Moreover, we will see, that (1) has a unique bounded solution, thus the equilibrium state is unique and periodic in time.

The steady states of this system are given as solutions to Eq. (1), corresponding to constant forcing $\lambda(t) = \lambda_0$. We will see that these solutions will have the form

$$u(x) = \frac{6}{(|x| + \alpha_0)^2}, \tag{3}$$

where the constant $\alpha_0 = (24/\lambda_0)^{1/3}$. We will prove that, as the spatial variable tends to infinity, the solution to (1) becomes asymptotic to a steady state solution. In other words, as we go far away from the boundary, the effect of the boundary oscillations is averaged out and the system feels a constant boundary perturbation. Moreover, we will prove that the speed of this convergence is exponential. More precisely, we have

Theorem 1.1. *Let $\lambda(t)$ be a continuous, periodic function with – without loss of generality – period equal to* 1. *Let λ_1, λ_2, be constants such that $0 < \lambda_1 \leqslant \lambda(\cdot) \leqslant \lambda_2 < \infty$. Then there exists a number $0 < \delta < 1$, a positive constant C and a positive constant α_* – all of them independent of time – such that if u is the unique positive, $C^{1,2}(\mathbb{R}, \mathbb{R} \setminus \{0\}) \cap C(\mathbb{R} \times \mathbb{R})$ solution of*

$$u_t = u_{xx} - u^2 + \lambda(t)\,\delta_0(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{4}$$

then

$$\sup_{t \in \mathbb{R}} \left| u(t,x) - \frac{6}{(|x| + \alpha_*)^2} \right| \leqslant C\delta^{|x|},$$

for any $x \in \mathbb{R}$.

The method, that we develop can also be used to prove similar results for the case that the nonlinearity $-u^2$ is replaced by any nonlinearity of the form $-u^p$, for $p \geqslant 1$, and the proof carries out with only minor modifications. Moreover, the method works also in the case that instead of a periodic input of Dirac type, we consider other type of periodic boundary perturbations, such as Dirichlet condition at the origin. Moreover, we stated the result in terms of a continuous input $\lambda(\cdot)$, but the reader will realize, that the extension to a larger class of inputs, e.g. of bounded variation, is straightforward.

Essentially the result of Theorem 1.1 is a nonstandard boundary homogenization result, of a nonlinear operator in an unbounded domain. The method we follow is a combination of parabolic comparison principles and probabilistic ideas. In particular, the key of the proof is a Harnack type inequality, which we obtain using arguments from the ergodic theory of Markov processes, and in particular what is known as Doeblin's type argument.

2. Some auxiliary results

We gather in this section some basic PDE and stochastic processes tools on which our analysis is based. On the PDE side, the tools are related to the parabolic maximum or comparison principle.

We will need the following standard definition and comparison principle (see for example [6, Chapters 4 and 7]).

Definition 2.1. Let $x_0, T \in \mathbb{R}$ arbitrary and consider the boundary value problem

$$\begin{aligned} &V_t = V_{xx} + f(t,x,V), \quad x > x_0,\ t > T, \\ &-aV_x(t,x_0) + bV(t,x_0) = p(t), \quad t > T, \\ &V(T,x) = q(x), \quad x > x_0, \end{aligned} \tag{5}$$

where $a, b \geqslant 0$ are general constants, with $a + b > 0$, $f \in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ and p, q continuous functions. The function $\hat{V} \in C^{1,2}((T,\infty),(x_0,\infty)) \cap C([T,\infty),[x_0,\infty))$, such that $\hat{V}_x(t,x_0)$ exists, for every $t > T$, is called a super-solution of problem (5) if it satisfies

$$\begin{aligned} &V_t \geqslant V_{xx} + f(t,x,V), \quad x > x_0,\ t > T, \\ &-aV_x(t,x_0) + bV(t,x_0) \geqslant p(t), \quad t > T, \\ &V(T,x) \geqslant q(x), \quad x > x_0. \end{aligned} \tag{6}$$

It is called a sub-solution if the above inequalities are reversed.

The next proposition summarizes the comparison principle.

Proposition 2.2. *Assume that $V_1(t,x), V_2(t,x) \in C^{1,2}((T,\infty),(x_0,\infty)) \cap C([T,\infty),[x_0,\infty))$ are bounded, sub- and super-solutions of the boundary value problem* (5). *Assume, also, that $f(t,x,v)$ is a smooth function, such that*

$$\left| f\big(t,x,v_1(t,x)\big) - f\big(t,x,v_2(t,x)\big)\right| \leqslant c(t,x)\left|v_1(t,x) - v_2(t,x)\right|,$$

for some bounded, nonnegative function $c(t,x)$, and for all functions v_1, v_2, such that $V_1(t,x) \leqslant v_i(t,x) \leqslant V_2(t,x)$, for $i = 1,2$ and $(t,x) \in (T,\infty)\times(x_0,\infty)$. Then $V_2(t,x) \geqslant V_1(t,x)$, for $(t,x) \in (T,\infty)\times(x_0,\infty)$.

The typical situation that we will be using this proposition is when $f(t,x,v(t,x)) = -v^2(t,x)$. Notice, also, that in this case $f(t,x,v_1(t,x)) - f(t,x,v_2(t,x)) = -(v_1(t,x)+v_2(t,x))(v_1(t,x)-v_2(t,x))$, and the sub- and super-solution of the problems, we will be dealing with, will be positive and bounded.

We prove in Appendix A that for each continuous $\lambda(\cdot)$, satisfying condition (2), Eq. (1) has a unique, bounded and smooth solution. More precisely,

Proposition 2.3. *Consider Eq.* (1)*, corresponding to a continuous source term $\lambda(\cdot)$, that satisfies the bounds $0 < \lambda_1 \leqslant \lambda(\cdot) \leqslant \lambda_2 < \infty$, for some λ_1, λ_2. Then there exists a unique positive and bounded, $C^{1,2}(\mathbb{R}\times\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R}\times\mathbb{R})$ solution $u(t,x)$ of this equation.*

The uniqueness part of this proposition is based on the following proposition, the proof of which will also be given in Appendix A.

Proposition 2.4. *Let u_λ, $u_\mu \in C^{1,2}(\mathbb{R}\times\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R}\times\mathbb{R})$ be bounded, positive solutions of* (1)*, corresponding to continuous inputs $\lambda(t)$ and $\mu(t)$, respectively. If $\lambda_1 \leqslant \lambda(t) \leqslant \mu(t) \leqslant \lambda_2$ for every $t \in \mathbb{R}$, then $u_\lambda(t,x) \leqslant u_\mu(t,x)$ for every $t, x \in \mathbb{R}$.*

This proposition is essentially Proposition 2.2, with the only difference, that the time horizon is considered to be doubly infinite, that is $T=-\infty$. We include, for completeness, the proofs of these two propositions in Appendix A since we do not want to distract the reader from the proof of the main result of Theorem 1.1. Notice that the uniqueness of positive, bounded solutions to (1), along with the periodicity of the input $\lambda(\cdot)$, implies, that the solution to (1), obtained by Proposition 2.3, is itself periodic, with the same period as the one of $\lambda(\cdot)$.

We now want to obtain a first representation of the solution to (1). To this end, let us first discuss the steady state solutions of Eq. (1). These are the solutions to (1), when the input $\lambda(t)\delta_0(x)$ is time independent. That is, when $\lambda(t)=\lambda_0$, for every $t\in(-\infty,\infty)$, and λ_0 a constant. As one can expect, these solutions are time independent. This follows by the uniqueness part of Proposition 2.3. Therefore a steady state solution satisfies the equation

$$u_{xx}-u^2+\lambda_0\delta_0(x)=0, \quad x\in\mathbb{R}. \tag{7}$$

Eq. (7) is an ordinary differential equation, which can be solved explicitly. The only physically relevant solution turns out to be $u(x)=\frac{6}{(|x|+\alpha_0)^2}$, where the constant α_0 is equal to $(24/\lambda_0)^{1/3}$. This is the explicit form of a steady state solution.

In the time dependent case, the solution to Eq. (1) cannot be written explicitly. Nevertheless, we can write it in a form, which is reminiscent to the form of the steady state solutions. This can be done as follows.

Let $\alpha_1=(24/\lambda_1)^{1/3}$ and $\alpha_2=(24/\lambda_2)^{1/3}$, where λ_1,λ_2 are the lower and upper bounds of $\lambda(t)$. The steady state solutions that correspond to λ_1,λ_2 are $6/(|x|+\alpha_1)^2$ and $6/(|x|+\alpha_2)^2$, respectively. Proposition 2.4 now implies that the solution $u(t,x)$ of (1) satisfies the bound

$$\frac{6}{(|x|+\alpha_1)^2}\leqslant u(t,x)\leqslant\frac{6}{(|x|+\alpha_2)^2}, \quad t,x\in\mathbb{R}. \tag{8}$$

This allows us to write the solution to (1) implicitly in the form

$$u(t,x)=\frac{6}{(|x|+\alpha(t,x))^2}.$$

In fact, one can define the function $\alpha(t,x)$ in terms of $u(t,x)$ as

$$\alpha(t,x)=\sqrt{\frac{6}{u(t,x)}}-|x|. \tag{9}$$

This representation is very useful, since it reduces the proof of Theorem 1.1 to proving that there exists a constant α_*, such that $\alpha(t,x)$ converges to α_* exponentially fast and uniformly in time.

Let us close with a final remark. Because of the uniqueness of the solution of (1) and the symmetry of this equation with respect to the origin, we have that $u(t,x)=u(t,-x)$ for every $t,x\in\mathbb{R}$. This symmetry implies that the input term $\lambda(t)\delta(x)$ corresponds to the Neumann boundary condition $u_x(t,0)=-\frac{1}{2}\lambda(t)$, for $t\in\mathbb{R}$. One can easily obtain this fact, by integrating Eq. (1) with respect to x around 0. In other words, (1) is equivalent to the Neumann boundary value problem

$$\begin{aligned} &u_t=u_{xx}-u^2, \quad x>0,\ t\in\mathbb{R},\\ &u_x(t,0)=-\frac{1}{2}\lambda(t), \quad t\in\mathbb{R}. \end{aligned} \tag{10}$$

From now on, we will freely interplay between these two forms of the problem.

The probabilistic framework. We now develop the probabilistic framework in which we are going to work. A good reference for the facts stated in this paragraph, as well as the interplay between stochastic processes (in particular Brownian motion) and elliptic and parabolic PDEs is [3] and in particular Chapter 4 of it.

Let W_x denote the Wiener measure on continuous paths $\{\beta(t)\colon t\geqslant 0\}$ starting from position $x\in\mathbb{R}$, and speeded by a factor of 2, and E^{W_x} the expectation with respect to this measure. The infinitesimal generator corresponding to the measure W_x is $\partial^2/\partial x^2$.

Let also $\tau_y:=\inf\{t>0\colon \beta(t)=y\}$ be the hitting time of a level y. Its distribution under the measure W_x is given by

$$W_x(\tau_y \in ds) = \frac{|x-y|}{\sqrt{4\pi s^3}} \exp\left(-\frac{|x-y|^2}{4s}\right) ds.$$

Consider, now for arbitrary $x_0 \in \mathbb{R}$ and $T \in \mathbb{R}$, the boundary value problem

$$U_t = U_{xx} - V(t,x)\,U, \quad x > x_0,\ t > T,$$
$$U(t,x_0) = U_1(t), \quad t > T,$$
$$U(T,x) = U_2(x), \quad x > x_0.$$

Under mild assumptions, e.g $V(t,x)$ is $C^{1,2}((T,\infty),(x_0,\infty))$ and nonnegative in the above domain, as well as continuity and boundedness of the functions U_1, U_2, a representation of the $C^{1,2}((T,\infty),(x_0,\infty)) \cap C([T,\infty),[x_0,\infty))$ solution of this problem can be given via the Feynman–Kac formula. This representation reads as

$$U(t,x) = E^{W_x}\left[U_2\big(\beta(t-T)\big)\exp\left(-\int_0^{t-T} V\big(t-s,\beta(s)\big)\,ds\right);\ \tau_{x_0} \geqslant t-T\right]$$
$$+ E^{W_x}\left[U_1(t-\tau_{x_0})\exp\left(-\int_0^{\tau_{x_0}} V\big(t-s,\beta(s)\big)\,ds\right);\ \tau_{x_0} < t-T\right]. \tag{11}$$

A special case of the above problem, which suits better to our considerations, is the one with doubly infinite time horizon

$$U_t = U_{xx} - V(t,x)U, \quad x > x_0,\ t \in \mathbb{R},$$
$$U(t,x_0) = U_1(t), \quad t \in \mathbb{R}.$$

In this case, the Feynman–Kac formula writes as

$$U(t,x) = E^{W_x}\left[U_1(t-\tau_{x_0})\exp\left(-\int_0^{\tau_{x_0}} V\big(t-s,\beta(s)\big)\,ds\right)\right]. \tag{12}$$

The above formulae are dealing with the case of linear PDEs, while the actual problem (1), that we are dealing with, is a nonlinear one. Nevertheless, the Feynman–Kac formula can be used to linearize it by providing a useful implicit formula. More specifically, the solution $u(t,x)$ of Eq. (1) satisfies, for $x > 0$, the equation

$$u_t = u_{xx} - u^2.$$

Let x_0 an arbitrary positive number. Then $u(t,x)$ can be regarded as the solution of the boundary value problem

$$U_t = U_{xx} - u(t,x)\,U, \quad x > x_0,\ t \in \mathbb{R},$$
$$U(t,x_0) = u(t,x_0), \quad t \in \mathbb{R}.$$

Since, by Proposition 2.3 $u(t,x)$ is nonnegative and smooth, the Feynman–Kac formula (12) can be used, with $V = u$ and $U_1(t) = u(t,x_0)$, to provide the implicit representation of the solution $u(t,x)$ of the above problem, as

$$u(t,x) = E^{W_x}\left[u(t-\tau_{x_0},x_0)\exp\left(-\int_0^{\tau_{x_0}} u\big(t-s,\beta(s)\big)\,ds\right)\right]. \tag{13}$$

This formula should be thought of as a variation of constants formula, and although implicit, it will turn to be very useful for the proof of Theorem 1.1.

3. Proof of Theorem 1.1

Idea of the proof. To begin with, let us denote by $\bar{u}(x) = \sup_{-\infty<t<+\infty} u(t,x)$, and by $\underline{u}(x) = \inf_{-\infty<t<+\infty} u(x,t)$. Notice that the above supremum and infimum need only to be taken over one period interval. Recall, that the solution $u(t,x)$ will be periodic in time, with the same period as the one of $\lambda(\cdot)$. Recall also that the function $\alpha(t,x)$ is given in terms of $u(t,x)$ by formula (9). Denote, also, for $x>0$, by

$$\bar{\alpha}(x) := \sqrt{\frac{6}{\bar{u}(x)}} - x \quad \text{and} \quad \underline{\alpha}(x) := \sqrt{\frac{6}{\underline{u}(x)}} - x. \tag{14}$$

It is clear that

$$u(t,x) \leqslant \bar{u}(x) = 6/\big(x+\bar{\alpha}(x)\big)^2 \tag{15}$$

for any x, t and this bound is optimal, in the sense that, since the supremum is taken over a finite interval, there will be a t, such that equality holds.

On the other hand, suppose that we fix an arbitrary $x_0>0$, and let w solve the boundary problem

$$\begin{aligned} &w_t = w_{xx} - w^2, \quad x \geqslant x_0,\ t\in\mathbb{R},\\ &w(t,x_0) = \bar{u}(x_0), \quad t\in\mathbb{R}. \end{aligned} \tag{16}$$

It is clear that $u(t,x)$ is a subsolution of the above Dirichlet problem, since it satisfies the PDE, but also, on the boundary, $u(t,x_0) \leqslant \bar{u}(x_0)$, for every $t\in\mathbb{R}$. Therefore, by the comparison principle, Proposition 2.2, where we consider $a=0, b=1$ and $T=-\infty$, it follows that $u(t,x)\leqslant w(t,x)$, for any $x\geqslant x_0$, $t\in\mathbb{R}$. Since problem (16) is time homogeneous, it can be solved explicitly and yields that $w(t,x) = 6/(x+\bar{\alpha}(x_0))^2$, $x\geqslant x_0, t\in\mathbb{R}$. This implies the bound

$$u(t,x) \leqslant 6/\big(x+\bar{\alpha}(x_0)\big)^2, \quad x\geqslant x_0,\ t\in\mathbb{R} \tag{17}$$

Consider, now, the bounds (15) and (17), when u is evaluated at $x=x_0+1$, and recall that the first is optimal. We then get that $6/(x_0+1+\bar{\alpha}(x_0+1))^2 \leqslant 6/(x_0+1+\bar{\alpha}(x_0))^2$, and consequently $\bar{\alpha}(x_0)\leqslant\bar{\alpha}(x_0+1)$. In the same way, by considering the corresponding to (15) and (17) lower bounds, $u(t,x)\geqslant \underline{u}(x) = 6/(x+\underline{\alpha}(x))^2$ for any x, t and $u(t,x)\geqslant 6/(x+\underline{\alpha}(x_0))^2$, for $x\geqslant x_0$ and $t\in\mathbb{R}$, we can get that $\underline{\alpha}(x_0+1)\leqslant\underline{\alpha}(x_0)$. Moreover, by the definition, $\bar{\alpha}(x_0)\leqslant\alpha(t,x_0)\leqslant\underline{\alpha}(x_0)$ and, thus, we have proved the following monotonicity property:

$$\bar{\alpha}(x_0)\leqslant\bar{\alpha}(x_0+1)\leqslant\alpha(t,x_0+1)\leqslant\underline{\alpha}(x_0+1)\leqslant\underline{\alpha}(x_0). \tag{18}$$

By (8), and since x_0 is arbitrary, Theorem 1.1 boils down to proving that $\underline{\alpha}(x)-\bar{\alpha}(x)$ decays exponentially fast. Finally, by definition (14) we have that

$$\underline{\alpha}(x)-\bar{\alpha}(x) = \frac{\sqrt{6}}{\sqrt{\bar{u}(x)\underline{u}(x)}(\sqrt{\underline{u}(x)}+\sqrt{\bar{u}(x)})}\big(\bar{u}(x)-\underline{u}(x)\big). \tag{19}$$

Since $\bar{u}(x)$, $\underline{u}(x)$ decay like $1/x^2$, Theorem 1.1 will be established once we prove that the difference $\bar{u}(x)-\underline{u}(x)$ decays exponentially fast.

Let us denote by $\rho(x) := \bar{u}(x)-\underline{u}(x)$. In order to get the exponential decay we are going to show that there exists a constant $0<\delta<1$, such that,

$$\rho(x+1) < \delta\rho(x), \tag{20}$$

for every x large enough. This is the Harnack inequality we are after, since it implies that

$$\frac{\sup_t u(t,x)}{\inf_t u(t,x)} = 1 + \mathrm{O}\big(x^2\delta^x\big).$$

for x large enough.

In order to obtain the contraction estimate (20), we will use ideas from the theory of Markov processes. In particular, we will use a similar idea to what is known as Doeblin's argument. This argument is used to describe the ergodic properties of certain Markov processes, and in a linear PDE setting it can be used to prove homogenization results [2, Chapter 3]. Here, we develop a Doeblin's type argument, suitable to our nonlinear setting.

Proof of (20). The proof of (20) is given through a series of lemmas. The first one shows that $\rho(x)$ is nonincreasing in x.

Lemma 3.1. *If ρ is defined as above, and $x_0 > 0$ is arbitrary, then for any $x \geqslant x_0$, $\rho(x) \leqslant \rho(x_0)$.*

Proof. For arbitrary $t_1, t_2 \in \mathbb{R}$ define the function

$$v(t,x) := u(t_1 + t, x) - u(t_2 + t, x).$$

Then, using the fact that $u(t_i + t, x)$, $i = 1, 2$ satisfy (4), we have that $v(t,x)$ will solve the boundary value problem

$$\frac{\partial V(t,x)}{\partial t} = \frac{\partial^2 V(t,x)}{\partial x^2} - \big(u(t_1+t,x) + u(t_2+t,x)\big)V(t,x), \quad x > x_0,\ t \in \mathbb{R}$$
$$V(t,x_0) = v(t,x_0), \quad t \in \mathbb{R}.$$

Since $v(x_0,t) \leqslant \bar{u}(x_0) - \underline{u}(x_0) = \rho(x_0)$ and

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \big(u(x,t+t_1) + u(x,t+t_2)\big)\right)\rho(x_0) \geqslant 0,$$

then, $\rho(x_0)$ is a supersolution for the above Dirichlet problem and so, $v(t,x) \leqslant \rho(x_0)$, or $u(t_1+t,x) - u(t_2+t,x) \leqslant \rho(x_0)$. Since t_1, t_2 are arbitrary, it follows that $\rho(x) := \sup_{t_1} u(t_1+t,x) - \inf_{t_2} u(t_2+t,x) \leqslant \rho(x_0)$ for $x \geqslant x_0$. □

The next step is to use the Feynman–Kac formula to provide a representation of the solution $u(t,x)$. Let us recall the definition of the hitting time $\tau_y := \inf\{t > 0\colon\ \beta(t) = y\}$. Also, its distribution is given by

$$W_x(\tau_y \in ds) = \frac{|x-y|}{\sqrt{4\pi s^3}} \exp\left(-\frac{|x-y|^2}{4s}\right) ds.$$

Let us also recall Eq. (13), which provides the formula

$$u(t,x) = E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u\big(t-r,\beta(r)\big)\,dr\right)\right]. \tag{21}$$

This formula is valid for any $y > 0$, but we will be choosing y to be less than x.

Let us rewrite the formula, by conditioning with respect to the hitting time, as

$$\begin{aligned}
&E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u\big(t-r,\beta(r)\big)\,dr\right)\right] \\
&\quad = E^{W_x}\left[E^{W_x}\left[u(t-\tau_y, y)\exp\left(-\int_0^{\tau_y} u\big(t-r,\beta(r)\big)\,dr\right) \,\middle|\, \tau_y\right]\right] \\
&\quad = \int_0^{\infty} u(t-s,y) E^{W_x}\left[\exp\left(-\int_0^{s} u\big(t-r,\beta(r)\big)\,dr\right) \,\middle|\, \tau_y = s\right] W_x(\tau_y \in ds) \\
&\quad := \int_0^{\infty} u(t-s,y) g(s;t)\,ds \\
&\quad = \int_{-\infty}^{t} u(s,y) g(t-s;t)\,ds,
\end{aligned} \tag{22}$$

where the measure $g(s;t)\,ds$ (we have made explicit the dependence on t, but suppressed the one on u, x and y), that appears in the above formula is equal to

$$g(s;t)\,ds := E^{W_x}\left[\exp\left(-\int_0^{\tau_y} u\big(t-r,\beta(r)\big)\,dr\right)\,\middle|\,\tau_y = s\right]W_x(\tau_y \in ds). \tag{23}$$

Let us finally write

$$g_t(s) := g(t-s;t), \tag{24}$$

and then write, by (22), the representation of u as

$$u(t,x) = \int_{-\infty}^{t} u(s,y)g_t(s)\,ds.$$

Having this representation, the proof of the main estimate will follow the lines of Doeblin's argument [2, Chapter 3]. In our case there is an extra difficulty in applying this argument, because of the fact that the density $g_t(s)$ depends on u, and, moreover, the total mass,

$$\int_{-\infty}^{t} g_t(s)\,ds = E^{W_x}\left[\exp\left(-\int_0^{\tau_y} u\big(t-s,\beta(s)\big)\,ds\right)\right], \tag{25}$$

is not constant in time.

Let us denote the difference in masses between different times by

$$m(t_1,t_2) := \int_{-\infty}^{t_1} g_{t_1}(s)\,ds - \int_{-\infty}^{t_2} g_{t_2}(s)\,ds. \tag{26}$$

For the rest of the section the densities $g_t(s)$ and the mass difference $m(t_1,t_2)$ will correspond to $y = x-1$.

Lemma 3.2. *Let us define*

$$g_{t_1t_2}(s) := g_{t_1}(s) - g_{t_2}(s),$$
$$I_+ := \big\{s \in (-\infty,t_1)\colon\ g_{t_1t_2}(s) \geqslant 0\big\},$$

and

$$I_- := (-\infty,t_1) \setminus I_+.$$

Then, for arbitrary $0 \leqslant t_1, t_2 \leqslant 1$, $x > 1$, *we have that*

$$u(t_1,x) - u(t_2,x) \leqslant \big(\bar{u}(x-1) - \underline{u}(x-1)\big)\int_{I_+} g_{t_1t_2}(s)\,ds + \underline{u}(x-1)m(t_1,t_2).$$

Proof. Without loss of generality suppose that $t_1 < t_2$. By the representation (22) we have that

$$\begin{aligned} u(t_1,x) - u(t_2,x) &= \int_{-\infty}^{t_1} u(s,x-1)g_{t_1}(s)\,ds - \int_{-\infty}^{t_2} u(s,x-1)g_{t_2}(s)\,ds \\ &= \int_{-\infty}^{t_1} u(s,x-1)g_{t_1t_2}(s)\,ds - \int_{t_1}^{t_2} u(s,x-1)g_{t_2}(s)\,ds \end{aligned}$$

$$= \int_{I_+} u(s,x-1) g_{t_1 t_2}(s)\,ds + \int_{(-\infty,t_1)\setminus I_+} u(s,x-1) g_{t_1 t_2}(s)\,ds - \int_{t_1}^{t_2} u(s,x-1) g_{t_2}(s)\,ds$$

$$\leqslant \bar{u}(x-1) \int_{I_+} g_{t_1 t_2}(s)\,ds + \underline{u}(x-1) \int_{(-\infty,t_1)\setminus I_+} g_{t_1 t_2}(s)\,ds - \int_{t_1}^{t_2} u(s,x-1) g_{t_2}(s)\,ds. \tag{27}$$

Notice that

$$\int_{(-\infty,t_1)\setminus I_+} g_{t_1 t_2}(s)\,ds = \int_{-\infty}^{t_1} g_{t_1 t_2}(s)\,ds - \int_{I_+} g_{t_1 t_2}(s)\,ds$$
$$= \int_{-\infty}^{t_1} g_{t_1}(s)\,ds - \int_{-\infty}^{t_1} g_{t_2}(s)\,ds - \int_{I_+} g_{t_1 t_2}(s)\,ds$$
$$= \int_{-\infty}^{t_1} g_{t_1}(s)\,ds - \int_{-\infty}^{t_2} g_{t_2}(s)\,ds + \int_{t_1}^{t_2} g_{t_2}(s)\,ds - \int_{I_+} g_{t_1 t_2}(s)\,ds. \tag{28}$$

Substitution of (28) into (27) gives us

$$u(t_1,x) - u(t_2,x) \leqslant \big(\bar{u}(x-1) - \underline{u}(x-1)\big) \int_{I_+} g_{t_1 t_2}(s)\,ds + \underline{u}(x-1) \int_{t_1}^{t_2} g_{t_2}(s)\,ds$$
$$- \int_{t_1}^{t_2} u(s,x-1) g_{t_2}(s)\,ds + \underline{u}(x-1) m(t_1,t_2),$$

where recall that $m(t_1,t_2)$ was defined in (26). The result now follows by noticing that $u(s,x-1) \geqslant \underline{u}(x-1)$ and, therefore, the second line in the last inequality is negative. □

Lemma 3.3. *There exists a $0 < \delta_1 < 1$ such that, for x large enough, we have*

$$\sup_{0 \leqslant t_1, t_2 \leqslant 1} \int_{I_+} g_{t_1 t_2}(s)\,ds \leqslant \delta_1.$$

Proof. First, by the definition of g_t and since the solution u is positive, we see, by (25), with $y = x-1$, that

$$\int_{-\infty}^{t} g_t(s)\,ds = E^{W_x}\left[\exp\left(-\int_0^{\tau_{x-1}} u\big(t-s,\beta(s)\big)\,ds\right)\right] \leqslant 1.$$

Also recall that

$$\int_{I_+} g_{t_1 t_2}(s)\,ds = \int_{I_+} g_{t_1}(s)\,ds - \int_{I_+} g_{t_2}(s)\,ds = \int_{-\infty}^{t_1} g_{t_1}(s)\,ds - \int_{I_-} g_{t_1}(s)\,ds - \int_{I_+} g_{t_2}(s)\,ds. \tag{29}$$

Let us denote by $W_x(s; x-1)$ the density of the measure $W_x(\tau_{x-1} \in ds)$. Then (29) can be bounded above by

$$1 - \int_{I_-} g_{t_1}(s)\,ds - \int_{I_+} g_{t_2}(s)\,ds = 1 - \int_{I_-} E^{W_x}\big[e^{-\int_0^{\tau_{x-1}} u(t_1-r,\beta(r))\,dr} \mid \tau_{x-1} = t_1 - s\big] W_x(t_1 - s; x-1)\,ds$$
$$- \int_{I_+} E^{W_x}\big[e^{-\int_0^{\tau_{x-1}} u(t_2-r,\beta(r))\,dr} \mid \tau_{x-1} = t_2 - s\big] W_x(t_2 - s; x-1)\,ds. \tag{30}$$

So, in order to derive the statement of the proposition, it suffices to bound from below the sum of the two integrals in (30), by a positive number.

By (8), we have that $u(t,x) \leqslant 6/(x+\alpha_2)^2$, for any $x>0$ and $t\in\mathbb{R}$. Therefore, for $r \leqslant \tau_{x-1}$, and when the Brownian motion starts from x, we have that $\beta(r) \geqslant x-1$ and so $u(t_1-r,\beta(r)) \leqslant 6/(x-1+\alpha_2)^2$. This implies, that the first integral in (30) is bounded below by

$$\int_{I_-} e^{-\frac{6}{(x-1+\alpha_2)^2}(t_1-s)} W_x(t_1-s;x-1)\,ds \geqslant \int_{I_-\cap(-2,-1)} e^{-\frac{6}{(x-1+\alpha_2)^2}(t_1-s)} W_x(t_1-s;x-1)\,ds$$

and similarly for the second one. It can be checked that for any $c>0$, $e^{-cs}W_x(s;x-1)$ is decreasing for $s\geqslant 1$, and since $t_1,t_2\in[0,1]$, we see that the last integrand is bounded below by $e^{-18/(x-1+\alpha_2)^2}W_x(3;x-1)\geqslant\varepsilon$, for some positive ε, uniformly in x, whenever x is large enough. In the same way we can bound the second integral in (30) from below, and hence estimate their sum by

$$\varepsilon\big|I_-\cap(-2,-1)\big| + \varepsilon\big|I_+\cap(-2,-1)\big| \geqslant \varepsilon.$$

Thus, we see that we can choose δ_1 to be $1-\varepsilon$. □

Remark. The proof of this lemma shows also that the number δ_1 that appears here, as well as in Theorem 1.1, does not depend on $\lambda(\cdot)$. In other words the exponential decay rate depends only on the dynamics of the system.

Lemma 3.4. *For the mass difference $m(t_1,t_2)$ between arbitrary times $0\leqslant t_1,t_2\leqslant 1$, the following bound holds*:

$$m(t_1,t_2) \leqslant (x-1+\alpha_1)^2 \ln\left(\frac{x+\alpha_1}{x-1+\alpha_1}\right)\big(\bar{u}(x-1)-\underline{u}(x-1)\big),$$

where α_1 is the constant that appears in (8).

Proof. By (23) we have that $m(t_1,t_2)$ is equal to

$$\begin{aligned} &E^{W_x}\left[\exp\left(-\int_0^{\tau_{x-1}} u\big(t_1-r,\beta(r)\big)\,dr\right)\right] - E^{W_x}\left[\exp\left(-\int_0^{\tau_{x-1}} u\big(t_2-r,\beta(r)\big)\,dr\right)\right] \\ &\quad\leqslant E^{W_x}\left[\exp\left(-\int_0^{\tau_{x-1}} \underline{u}\big(\beta(r)\big)\,dr\right)\right] - E^{W_x}\left[\exp\left(-\int_0^{\tau_{x-1}} \bar{u}\big(\beta(r)\big)\,dr\right)\right]. \end{aligned} \tag{31}$$

Denote by $\underline{f}(x;x-1)$ and $\bar{f}(x;x-1)$, respectively, the first and the second terms of (31). We can replace $x-1$ with an arbitrary $x_0<x$ and note, by an application of the Feynman–Kac formula, that $\underline{f}(x;x_0)$ and $\bar{f}(x;x_0)$ solve, respectively, the equations

$$\underline{f}_{xx} - \underline{u}\,\underline{f} = 0, \quad x>x_0,$$

and

$$\bar{f}_{xx} - \bar{u}\bar{f} = 0, \quad x>x_0,$$

with boundary condition on x_0: $\underline{f}(x_0;x_0)=\bar{f}(x_0;x_0)=1$. Subtract the equations to get that

$$(\underline{f}-\bar{f})_{xx} + \bar{u}\bar{f} - \underline{u}\,\underline{f} = 0,$$

or

$$(\underline{f}-\bar{f})_{xx} - \underline{u}(\underline{f}-\bar{f}) + (\bar{u}-\underline{u})\bar{f} = 0.$$

Since $\underline{f}(x;x_0)-\bar{f}(x;x_0)$ equals to 0 on $x=x_0$, we get, again, by the Feynman–Kac formula for equations with a source term (see [3] in the case of a continuous source, and [1, Section 2.6], in the case of a Dirac), that

$$(\underline{f}-\bar{f})(x;x_0) = E^{W_x}\left[\int_0^{\tau_{x_0}} \big((\bar{u}-\underline{u})\bar{f}\big)\big(\beta(s)\big)\exp\left(-\int_0^s \underline{u}\big(\beta(r)\big)\,dr\right)ds\right].$$

By Lemma 1, this is

$$\leqslant \rho(x_0) E^{W_x}\left[\int_0^{\tau_{x_0}} \bar{f}\big(\beta(s); x_0\big) \exp\left(-\int_0^s \underline{u}\big(\beta(r)\big)\, dr\right) ds\right] < \rho(x_0) E^{W_x}\left[\int_0^{\tau_{x_0}} \bar{f}\big(\beta(s); x_0\big)\, ds\right]. \tag{32}$$

In order to bound the last quantity let us first get a bound for $\bar{f}$. Again we can use the bounds in (8) to bound $\bar{u}(\beta(r))$ below by $6/(\beta(r)+\alpha_1)^2$, and get that

$$\bar{f}(x; x_0) \leqslant E^{W_x}\left[\exp\left(-\int_0^{\tau_{x_0}} \frac{6}{(\beta(r)+\alpha_1)^2}\, dr\right)\right].$$

The right-hand side of the above inequality, which we denote by $h(x; x_0)$, solves the equation $h_{xx} - \frac{6}{(x+\alpha_1)^2}\, h = 0$, $x > x_0$ with boundary condition $h(x_0; x_0) = 1$. This equation can be solved and gives us $h(x; x_0) = ((x_0+\alpha_1)/(x+\alpha_1))^2$. Thus,

$$E^{W_x}\left[\int_0^{\tau_{x_0}} \bar{f}\big(\beta(s); x_0\big)\, ds\right] \leqslant E^{W_x}\left[\int_0^{\tau_{x_0}} \left(\frac{x_0+\alpha_1}{\beta(s)+\alpha_1}\right)^2 ds\right] = (x_0+\alpha_1)^2 \ln\left(\frac{x+\alpha_1}{x_0+\alpha_1}\right), \tag{33}$$

where the last equality follows, since the right-hand side expectation solves the equation

$$\hbar_{xx} + \big((x_0+\alpha_1)/(x+\alpha_1)\big)^2 = 0,$$

for $x \geqslant x_0$ and $\hbar(x_0) = 0$. The result now follows if we set $x_0 = x-1$ in (33) and plug the resulting bound in (32) and subsequently in (31). □

Proposition 3.5. *For x large enough, there is a $0 < \delta < 1$ such*

$$\rho(x) \leqslant \delta\, \rho(x-1).$$

Proof. The proof follows immediately by substitution of the estimates in Lemmas 3.3 and 3.4 into the estimate in Lemma 3.2, and noticing that, for all x large enough, $\underline{u}(x-1)\cdot(x-1+\alpha_1)^2 \ln((x+\alpha_1)/(x-1+\alpha_1)) < \delta - \delta_1$ for δ, such that $0 < \delta_1 < \delta < 1$. □

Proof of Theorem 1.1. Proposition 3.5 and relations (18) and (19) imply that there exists a positive number α_* and a positive constant C_1, such that

$$\sup_t \big|\alpha(t,x) - \alpha_*\big| \leqslant C_1 x^3 \delta^x, \tag{34}$$

for any x large enough, positive. Therefore, we obtain, that

$$\begin{aligned}\sup_t \left|u(t,x) - \frac{6}{(x+\alpha_*)^2}\right| &= \sup_t \left|\frac{6}{(x+\alpha(t,x))^2} - \frac{6}{(x+\alpha_*)^2}\right| \\ &= \sup_t \frac{6(2x+\alpha(t,x)+\alpha_*)}{(x+\alpha(t,x))^2(x+\alpha_*)^2} \sup_t \big|\alpha(t,x)-\alpha_*\big| \\ &\leqslant C\delta^x,\end{aligned}$$

by (34), with a positive constant C. This completes the proof of Theorem 1.1. □

Acknowledgement

This result is part of my PhD thesis done under the supervision of S.R.S. Varadhan. I would like to thank Professor Varadhan for his constant support, and for all the invaluable discussions, that I had with him.

Appendix A. Existence and uniqueness of the dynamics

Proof of Proposition 2.4. We start by proving a preliminary estimate.

Let $u_2(t,x)=6/(|x|+\alpha_2)^2$ be the solution to (1), corresponding to the input λ_2. Subtracting the equations satisfied by u, u_2 we obtain

$$(u-u_2)_t=(u-u_2)_{xx}-(u+u_2)(u-u_2)+\big(\lambda(t)-\lambda_2(t)\big)\delta_0(x),\quad t\in\mathbb{R},\ x\in\mathbb{R}. \tag{35}$$

For any $\sigma>-\infty$, we view $u-u_2$ as the solution of the initial value problem

$$\begin{aligned}&V_t=V_{xx}-(u+u_2)V+\big(\lambda(t)-\lambda_2(t)\big)\delta_0(x), && t>\sigma,\ x\in\mathbb{R},\\ &V(\sigma,x)=u(\sigma,x)-u_2(\sigma,x), && x\in\mathbb{R}.\end{aligned}$$

We can use a slightly different version of the Feynman–Kac formula, namely when the equation has a source term, to write the solution to the above equation as

$$\begin{aligned}u(t,x)-u_2(t,x)={}&E^{W_x}\int_0^{t-\sigma}\big(\lambda(t-s)-\lambda_2(t-s)\big)\delta_0\big(\beta(s)\big)e^{-\int_0^s(u+u_2)(t-r,\beta(r))\,dr}\,ds\\&+E^{W_x}\big[(u-u_2)\big(\sigma,\beta(t-\sigma)\big)e^{-\int_0^{t-\sigma}(u+u_2)(t-r,\beta(r))\,dr}\big].\end{aligned} \tag{36}$$

A good reference for equations with source terms and their relation with the Feynman–Kac formula is [3, Chapter 4] for continuous source, or [1, Section 2.6] for Dirac source. Let us briefly clarify the appearance of the Dirac term in the first integral. One can work with such Feynman–Kac formulae in one dimension, as if the Dirac term was a mollified version of it. Formally, $\delta_0(\beta(s))$ denotes what is defined to be the *local time* of Brownian motion at zero. For details one can refer to [1, Section 2.6].

Going back to the proof, we have, that since u, u_2 are bounded and positive and $u_2(x)=6/(|x|+\alpha_2)^2$, we can bound the second term in the above relation by $CE^{W_x}[\exp\{-\int_0^{t-\sigma}\frac{6}{(|\beta(r)|+\alpha_2)^2}\,dr\}]$, and it is a routine to see that it converges to zero, as σ tends to negative infinity. To argue for this, let us, first, bound $6/(|\beta(r)|+\alpha_2)^2$ below by $C_1 1_{\{|\beta(r)|<1\}}$ and then bound $CE^{W_x}[\exp\{-\int_0^{t-\sigma}\frac{6}{(|\beta(r)|+\alpha_2)^2}\,dr\}]$ above by $CE^{W_x}[\exp\{-C_1\int_0^{t-\sigma}1_{\{|\beta(r)|<1\}}\,dr]$. Since one-dimensional Brownian motion is recurrent we have that $\int_0^{t-\sigma}1_{\{|\beta(r)|<1\}}\,dr$ tends almost surely to infinity as σ tends to negative infinity. We can, therefore, use the dominated convergence theorem to obtain the claim.

Thus, passing to the limit $\sigma\to-\infty$ in (36) – using the monotone convergence theorem to deal with the first term – we obtain that

$$u(t,x)-u_2(t,x)=E^{W_x}\int_0^{\infty}\big(\lambda(t-s)-\lambda_2(t-s)\big)\delta_0\big(\beta(s)\big)e^{-\int_0^s(u+u_2)(t-r,\beta(r))\,dr}\,ds. \tag{37}$$

Since $\lambda(\cdot)\leqslant\lambda_2$ we obtain that $u\leqslant u_2$. In the same way we can obtain a similar lower bound on u, and so we get that

$$\frac{6}{(|x|+\alpha_1)^2}\leqslant u(t,x)\leqslant\frac{6}{(|x|+\alpha_2)^2},\quad t\in\mathbb{R},\ x\in\mathbb{R}. \tag{38}$$

We can now use the preliminary bound (38), to compare two solutions u_λ, u_μ corresponding to sources $\lambda(\cdot), \mu(\cdot)$, such that $\lambda_1\leqslant\lambda(\cdot)\leqslant\mu(\cdot)\leqslant\lambda_2$. In the same way as (36) we obtain that

$$\begin{aligned}u_\lambda(t,x)-u_\mu(t,x)={}&E^{W_x}\int_0^{t-\sigma}\big(\lambda(t-s)-\mu(t-s)\big)\delta_0\big(\beta(s)\big)e^{-\int_0^s(u_\lambda+u_\mu)(t-r,\beta(r))\,dr}\,ds\\&+E^{W_x}\big[(u_\lambda-u_\mu)\big(\sigma,\beta(t-\sigma)\big)e^{-\int_0^{t-\sigma}(u_\lambda+u_\mu)(t-r,\beta(r))\,dr}\big].\end{aligned} \tag{39}$$

Using, now, the same argument as the one employed to obtain (37), along with (38), we can pass to the limit $\sigma\to\infty$, to obtain

$$u_\lambda(t,x)-u_\mu(t,x)=E^{W_x}\int_0^\infty \big(\lambda(t-s)-\mu(t-s)\big)\,\delta_0\big(\beta(s)\big)e^{-\int_0^s (u_\lambda+u_\mu)(t-r,\beta(r))\,dr}\,ds.$$

So the fact that $\lambda(\cdot)\leqslant\mu(\cdot)$ immediately implies that $u_\lambda\leqslant u_\mu$, which establishes the desired comparison. □

Proof of Proposition 2.3. Clearly, the uniqueness of positive, bounded solutions to Eq. (1) is implied by Proposition 2.4. Let us now prove the existence of the solution.

As we have already mentioned in the introduction, we want to think of the solution as the equilibrium state of the system governed by the same dynamics, but starting from some initial state, which for simplicity we will consider it to be zero. In other words we will construct the solution of problem (1) as the limit, as $\tau\to-\infty$, of the solution of the problem

$$\begin{aligned}&u_t=u_{xx}-u^2+\lambda(t)\delta_0(x),\quad x\in\mathbb{R},\ t>\tau,\\&u(\tau,x)=0,\quad x\in\mathbb{R}.\end{aligned}\tag{40}$$

It will be convenient to think of problem (1) as a Neumann boundary problem, as in (10). Similarly, the analogue Neumann problem for (40) is

$$\begin{aligned}&u_t=u_{xx}-u^2,\quad x>0,\ t>\tau,\\&u_x(t,0)=-\frac{1}{2}\lambda(t),\quad t>\tau,\\&u(\tau,x)=0,\quad x>0.\end{aligned}\tag{41}$$

Using the standard Perron iteration method (e.g. [6, Chapters 2 and 7]), one can show that problem (41) has a unique, $C^{1,2}((\tau,\infty)\times\mathbb{R}_+)\cap C([\tau,\infty),[0,\infty))$ solution, bounded below and above, respectively, by the sub- and supersolutions 0 and $6/(x+\alpha_2)^2$, to problem (41) – recall that $\alpha_2=(24/\lambda_2)^{1/3}$. Let us denote by $u^{(\tau)}$, the symmetric extension around 0, of the solution to (41), obtained by the Perron's method. Clearly, $u^{(\tau)}$ is a solution to (40).

Let us denote by $g(t,x;s,y)$ with $t>s$ and $x,y>0$, the Green's function for the generator $-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}$ with Neumann boundary conditions, i.e.

$$g(t,x;s,y)=\frac{1}{\sqrt{4\pi(t-s)}}\left(\exp\left(-\frac{(x-y)^2}{4(t-s)}\right)+\exp\left(-\frac{(x+y)^2}{4(t-s)}\right)\right),$$

and by $g_c(t,x;s,y)=e^{-c(t-s)}g(t,x;s,y)$, the Green's function for the generator $-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}-c$, also with Neumann boundary conditions. Consider c to be an arbitrary, positive number.

In order to guarantee, that the integrals below are convergent, we add and subtract from the right-hand side of the first equation in (41) the quantity cu. We can now use the variation of constants formula to write for $t>\tau$ and $x>0$,

$$u^{(\tau)}(t,x)=\frac{1}{2}\int_\tau^t \lambda(s)g_c(t,x;s,0)\,ds+\int_\tau^t\int_{\mathbb{R}_+}\big(-\big(u^{(\tau)}\big)^2+cu^{(\tau)}\big)(s,y)\,g_c(t,x;s,y)\,dy\,ds.\tag{42}$$

Let us note that the mapping $\tau\to u^{(\tau)}(t,x)$ is nonincreasing, for any arbitrary t,x. Indeed, consider the solutions $u^{(\tau_1)},u^{(\tau_2)}$ for $\tau_1>\tau_2$. In the domain $x>0$, $t>\tau_1$, $u^{(\tau_1)}$ satisfies (41) with initial condition 0, while in the same domain $u^{(\tau_2)}$ satisfies the same problem, but with initial condition $u^{(\tau_2)}(\tau_1,x)\geqslant0$. Hence, by the comparison principle, Proposition 2.2, it follows that for any $t>\tau_1>\tau_2$, and $x>0$, $u^{(\tau_2)}(t,x)\geqslant u^{(\tau_1)}(t,x)$. This monotonicity implies that, as $\tau\to-\infty$, $u^{(\tau)}(t,x)$ converges to a bounded and positive function $u(t,x)$. Passing now to the limit, $\tau\to-\infty$, in (42) we see, by dominated convergence, that $u(t,x)$ also satisfies (42), where the lower bound τ in the time integrals is replaced by negative infinity. Standard arguments now (see [6, Chapter 2]) imply that $u(t,x)$ is a $C^{1,2}(\mathbb{R}\times\mathbb{R}_+)\cap C(\mathbb{R}\times[0,\infty))$ function and it satisfies the equation $u_t=u_{xx}-u^2$, for any $t\in\mathbb{R}$, $x\in\mathbb{R}_+$, as well as the boundary condition $u_x(t,0)=-1/2\lambda(t)$, for $t\in\mathbb{R}$. Extending now this function symmetrically around the origin we obtain a solution to (1), with the desired regularity properties. □

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