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On the geometry of null cones in Einstein-vacuum spacetimes

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Abstract

In this paper we study the geometry of null cones in smooth Einstein vacuum spacetimes. We provide the L^∞ estimate for the trace of the null second fundamental form, as well as estimates for other geometric quantities. This paper is based on the work of Klainerman and Rodnianski [S. Klainerman, I. Rodnianski, Causal geometry of Einstein-vacuum spacetimes with finite curvature flux, Invent. Math. 159 (3) (2005) 437–529; S. Klainerman, I. Rodnianski, Sharp trace theorems for null hypersurfaces on Einstein metrics with finite curvature flux, Geom. Funct. Anal. 16 (1) (2006) 164–229; S. Klainerman, I. Rodnianski, A geometric Littlewood–Paley theory, Geom. Funct. Anal. 16 (1) (2006) 126–163].

1. Introduction

This paper is concerned with the geometry of null cones in $3+1$ smooth Einstein vacuum spacetimes, i.e. $3+1$ Lorentzian manifolds $(M, \mathbf{g})$ with Ricci flat metrics,

$$R_{\alpha\beta}(\mathbf{g}) = 0.$$

Let $p \in M$ be a fixed point and let T be a fixed timelike vector at p satisfying $\langle T, T\rangle = -1$. We choose all future null vectors L_ω, $\omega \in \mathbb{S}^2$, at p such that $\langle L_\omega, T\rangle = -1$ and $\langle L_\omega, L_\omega\rangle = 0$. For each $\omega \in \mathbb{S}^2$ let $\Gamma(s,\omega)$ denote the outgoing null geodesic parametrized by the affine parameter s with the initial data $\Gamma(0,\omega) = p$ and $\frac{d}{ds}\Gamma(0,\omega) = L_\omega$. The union of all these outgoing null geodesics forms a 3-D null cone starting from p which is denoted by $\mathcal{H}$.

We define the vector field L by $L := \frac{d}{ds}\Gamma$. Obviously $L(0,\omega) = L_\omega$ and L satisfies

$$\mathbf{g}(L, L) = 0 \quad \text{and} \quad D_L L = 0.$$

The parameter s can be regarded as a function on $\mathcal{H}$ verifying $L(s) = 1$ and $s(p) = 0$. We introduce the one parameter flow $\Gamma_s(\omega) := \Gamma(s,\omega)$. It generates a family of 2-D closed surfaces $\{S_s\}$ by $S_s := \Gamma_s(\mathbb{S}^2)$, which form the geodesic foliation of $\mathcal{H}$. It is clear that each S_s is diffeomorphic to $\mathbb{S}^2$ for $s > 0$ sufficiently small. By rescaling the metric $\mathbf{g}$ we may assume without loss of generality that for $0 < s \leqslant 1$ each slice S_s is diffeomorphic to $\mathbb{S}^2$. Let $\mathcal{H}_t$ be the portion of $\mathcal{H}$ when s varies in $(0, t]$. For simplicity, we still denote by $\mathcal{H}$ the portion $\mathcal{H}_1$. Every point q in $\mathcal{H}$ can be parametrized by the coordinates (s,ω) for which $q = \Gamma_s(\omega)$. We then call (s,ω) the transport local coordinates.

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Let D denote the Levi-Civita connection of Einstein vacuum metric $\mathbf{g}$. Let γ be the induced metric on S_s, and ∇ its induced covariant derivative. At any point $q \in S_s \subset \mathcal{H}$ we denote by $\underline{L}$ the null vector conjugate to L relative to the S_s foliation, i.e. $\langle L, \underline{L}\rangle = -2$ and $\langle \underline{L}, X\rangle = 0$ for all $X \in T_q(S_s)$. A smooth choice of an orthonormal frame $\{e_a\}_{a=1,2}$ in $T_q(S_s)$ combined with $L, \underline{L}$ forms a null frame associated to the foliation.

We introduce the null components of the curvature tensor R of the spacetime metric $\mathbf{g}$ relative to L and $\underline{L}$ as follows (see [2, Section 7.3] and [4, Section 3.1.2]):

$$\begin{aligned} &\alpha_{ab} = R(L, e_a, L, e_b), && \beta_a = \frac{1}{2}R(e_a, L, \underline{L}, L),\\ &\rho = \frac{1}{4}R(\underline{L}, L, \underline{L}, L), && \sigma = \frac{1}{4}{}^\star R(\underline{L}, L, \underline{L}, L),\\ &\underline{\beta}_a = \frac{1}{2}R(e_a, \underline{L}, \underline{L}, L), && \underline{\alpha}_{ab} = R(\underline{L}, e_a, \underline{L}, e_b), \end{aligned} \tag{1.1}$$

where ${}^\star R_{\mu\nu\gamma\delta} = \frac{1}{2}\epsilon_{\mu\nu\lambda\tau}R^{\lambda\tau}{}_{\gamma\delta}$ and $\epsilon_{\mu\nu\lambda\tau}$ are components of the volume element in $(M, \mathbf{g})$. The total curvature flux $\mathcal{R}_0$ is then defined by

$$\mathcal{R}_0 = \big(\|\alpha\|^2_{L^2(\mathcal{H})} + \|\beta\|^2_{L^2(\mathcal{H})} + \|\rho\|^2_{L^2(\mathcal{H})} + \|\sigma\|^2_{L^2(\mathcal{H})} + \|\underline{\beta}\|^2_{L^2(\mathcal{H})}\big)^{\frac{1}{2}}.$$

The geometry of $\mathcal{H}$ in particular depends on the null second fundamental form

$$\chi(X, Y) = \langle D_X L, Y\rangle$$

with X and Y being arbitrary vector fields tangent to S_s. We will denote $\operatorname{tr}\chi$ and $\hat{\chi}$ the trace and traceless part of χ respectively. Other important geometric quantities are the dual null second fundament form and the torsion

$$\underline{\chi}(X, Y) = \langle D_X \underline{L}, Y\rangle \quad \text{and} \quad \zeta(X) = \frac{1}{2}\langle D_X L, \underline{L}\rangle.$$

We will also use $\operatorname{tr}\underline{\chi}$ and $\hat{\underline{\chi}}$ to denote the trace and traceless part of $\underline{\chi}$.

The mass aspect function μ is defined by

$$\mu = -\operatorname{div}\zeta + \frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}} - \rho + |\zeta|^2. \tag{1.2}$$

We are now ready to state the main theorem in this paper.

Theorem 1.1. *Consider an outgoing null hypersurface $\mathcal{H}$ in a smooth $3+1$ Einstein vacuum spacetime $(M, \mathbf{g})$, initiating from a point p and foliated by the geodesic foliation associated to the affine parameter s with $s|_p = 0$. Assume that the total curvature flux $\mathcal{R}_0$ is sufficiently small. Then we have*

$$\left\|\operatorname{tr}\chi - \frac{2}{t}\right\|_{L_t^\infty L_\omega^\infty} \lesssim \mathcal{R}_0$$

and

$$\left\|\int_0^1 |\hat{\chi}|^2\, dt\right\|_{L_\omega^\infty} + \left\|\int_0^1 |\zeta|^2\, dt\right\|_{L_\omega^\infty} \lesssim \mathcal{R}_0,$$

$$\left\|\sup_{t\leqslant 1}|t\nabla\operatorname{tr}\chi|\right\|_{L_\omega^2} + \left\|\sup_{t\leqslant 1}t^{\frac{3}{2}}|\mu|\right\|_{L_\omega^2} + \|\mu\|_{L^2(\mathcal{H})} \lesssim \mathcal{R}_0,$$

$$\mathcal{N}_1(\hat{\chi}) + \mathcal{N}_1(\zeta) + \mathcal{N}_1\left(\operatorname{tr}\chi - \frac{2}{t}\right) \lesssim \mathcal{R}_0,$$

$$\|\nabla\operatorname{tr}\chi\|_{\mathcal{B}^0} + \|t^{1/2}\mu\|_{\mathcal{B}^0} + \|\mu\|_{\mathcal{P}^0} \lesssim \mathcal{R}_0,$$

$$\left\|\sup_{t\leqslant 1}t^{\frac{1}{2}}\left|\operatorname{tr}\underline{\chi} + \frac{2}{t}\right|\right\|_{L_\omega^2} + \left\|\sup_{t\leqslant 1}t^{\frac{1}{2}}|\hat{\underline{\chi}}|\right\|_{L_\omega^2} \lesssim \mathcal{R}_0,$$

$$\left\|\operatorname{tr}\underline{\chi}+\frac{2}{t}\right\|_{L_t^2L_\omega^2}+\|\hat{\underline{\chi}}\|_{L_t^2L_\omega^2}\lesssim\mathcal{R}_0,$$

$$\left\|\nabla_L\left(\operatorname{tr}\underline{\chi}+\frac{2}{t}\right)\right\|_{L^2(\mathcal{H})}\lesssim\mathcal{R}_0,$$

$$\left\|t^{-1/2}\left(\operatorname{tr}\underline{\chi}+\frac{2}{t}\right)\right\|_{\mathcal{B}^0}+\|t^{-1/2}\hat{\underline{\chi}}\|_{\mathcal{B}^0}\lesssim\mathcal{R}_0,$$

$$\left\|t^{-1}\left(\operatorname{tr}\underline{\chi}+\frac{2}{t}\right)\right\|_{\mathcal{P}^0}+\|t^{-1}\hat{\underline{\chi}}\|_{\mathcal{P}^0}\lesssim\mathcal{R}_0.$$

The various norms appearing in the statement will be defined in Section 3 (see (3.3)–(3.6) and (3.20), (3.21)). Throughout this paper we will use the notation $A\lesssim B$ to mean $A\leqslant C\cdot B$ for some appropriate universal constants C.

In [9,11,12] Klainerman and Rodnianski developed systematic methods to prove that, on truncated null hypersurfaces initiating from a 2-D surface diffeomorphic to $\mathbb{S}^2$, within the radius of injectivity, $\operatorname{tr}\chi$ can be controlled by appropriate norms of the small initial data and small total curvature flux $\mathcal{R}_0$, which is one of their steps toward the answer of the minimal local regularity of the initial data that guarantees the existence and uniqueness of local developments for Einstein vacuum equation. See [7] and [8] for the best known regularity result. For the background of the initial data problem of Einstein vacuum equations and related results, please refer to [1,3,21,5,10]. In this work, we extend their result to null cones in smooth Einstein vacuum spacetimes. Our result shows that $\operatorname{tr}\chi-\frac{2}{s}$ can be bounded only by small total curvature flux before the formation of caustics or geodesic loops. This result is used in [13] to provide the uniform lower bound on the radius of injectivity of null boundaries in Einstein vacuum spacetimes. Such lower bound is essential in understanding the causal structure and propagation properties of solutions to the Einstein equations, and is important in construction of a Kirchoff–Sobolev type parametric for wave equations on M (see [14]), which is used in [15] to prove a large data break-down criterion for solutions of the Einstein vacuum equations.

We will follow the framework of [9] to prove the main theorem by the bootstrap principle. Since our null hypersurface $\mathcal{H}$ initiates from a point, many quantities behave like s^{-a} for some number $a>0$ as $s\to 0$, we have to keep track the weight s^a in each step. Note that the Besov norm estimates (see [9, Proposition 5.11])

$$\|\nabla\cdot\mathcal{D}^{-1}F\|_{\mathcal{P}^0}\lesssim\|F\|_{\mathcal{P}^0}$$

of the 0-order Hodge operator $\nabla\cdot\mathcal{D}^{-1}$ were used in [9, Section 6] to control the terms such as the commutator $[\nabla_L,\nabla\mathcal{D}^{-2}]\check{R}$, where $\mathcal{P}^0$ is a certain Besov norm. However, these estimates hold true only when some additional terms involving the $L_s^{4+}L_\omega^2$ norm of $\mathcal{D}^{-1}F$ is added,[1] due to the limited regularity of Gauss curvature K of each slice S_s. The corrected versions we will present on the $\mathcal{P}^0$ estimates of the 0-order Hodge operators and on some product estimates add much complexity to the commutator estimates.

This paper is organized as follows. In Section 2, we recall the structure equations on various geometric quantities on $\mathcal{H}$ and provide the results on the initial data. In Section 3 we present the complete set of bootstrap assumptions, introduce some important norms and establish some preliminary estimates. In Section 4, we provide the L^2 type estimates and the $\mathcal{P}^0$ type estimates of 0-order Hodge operators $\nabla\cdot\mathcal{D}^{-1}$. The result on $\mathcal{P}^0$ estimate, which has special significance to Section 6, is a correction of [9, Proposition 5.11]. The proof is based on the unpublished notes of Klainerman and Rodnianski [6]. In Section 5, we prove some important product estimates. In Section 6, we use the results in Sections 4 and 5 to fulfill the decomposition of the commutators. Finally, in Section 7 we use the results in previous sections to complete the proof of the main theorem.

2. Structure equations and initial data

As the starting point we state the results on the behaviors of the main geometric quantities near the vertex of the null cone which can be obtained by local analysis, see [16] or [22, Appendix] for the proofs.

Proposition 2.1. *Near the vertex of the null cone $\mathcal{H}$ there hold*

[1] In this paper we will use $a+$ to represent a number $q>a$, and $a-$ to represent a number $q<a$.

$$s\,\mathrm{tr}\,\chi = 2 + \mathrm{O}(s^3) \quad \textit{and} \quad s\,\mathrm{tr}\,\underline{\chi} \to -2 \quad \textit{as } s \to 0, \tag{2.1}$$

$$\hat{\chi} = -\frac{1}{3} sU + \mathrm{O}(s^2) \quad \textit{and} \quad \hat{\underline{\chi}} \to 0 \quad \textit{as } s \to 0, \tag{2.2}$$

$$\zeta = -\frac{1}{6} s\eta + \mathrm{O}(s^2) \quad \textit{as } s \to \infty \tag{2.3}$$

and

$$s\nabla\,\mathrm{tr}\,\chi,\ s\nabla\hat{\chi},\ s\,\mathrm{div}\,\zeta,\ s\nabla\zeta,\ s\mu \to 0 \quad \textit{as } s \to 0 \tag{2.4}$$

where U is a symmetric traceless 2-tensor and η is a 1-form, both of which are finite at the vertex, depending on the curvature tensor in $(M, \mathbf{g})$.

Let $\gamma^{(0)}$ be the canonical metric on the standard 2-sphere $\mathbb{S}^2$ and let $\mathring{\gamma} = s^{-2}\gamma_s$. Set $a_s = \sqrt{|\gamma_s|}/\sqrt{|\gamma^{(0)}|}$ and $r = r(s) = \sqrt{(4\pi)^{-1}|S_s|}$ with $|S_s|$ being the area of S_s. Then

$$\mathring{\gamma} = \gamma^{(0)} + \mathrm{O}(s^2), \quad s^{-2}a_s \to 1 \quad \textit{and} \quad \frac{r}{s} \to 1 \qquad \textit{as } s \to 0. \tag{2.5}$$

We call $r := r(s)$ the radius of each leaf S_s.

We also state the structure equations of the geodesic foliation (see [2] or [9, Section 2.12] for the derivations)

$$\frac{d}{ds}\,\mathrm{tr}\,\chi = -\frac{1}{2}(\mathrm{tr}\,\chi)^2 - |\hat{\chi}|^2, \tag{2.6}$$

$$\nabla_L \hat{\chi} = -\,\mathrm{tr}\,\chi \cdot \hat{\chi} - \alpha, \tag{2.7}$$

$$\nabla_L(\nabla\,\mathrm{tr}\,\chi) = -\frac{3}{2}\,\mathrm{tr}\,\chi \cdot \nabla\,\mathrm{tr}\,\chi - \hat{\chi} \cdot \nabla\,\mathrm{tr}\,\chi - 2\hat{\chi} \cdot \nabla\hat{\chi}, \tag{2.8}$$

$$\nabla_L \zeta = -\,\mathrm{tr}\,\chi\zeta - 2\hat{\chi} \cdot \zeta - \beta, \tag{2.9}$$

$$\frac{d}{ds}\,\mathrm{tr}\,\underline{\chi} = -\frac{1}{2}\,\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} - 2\,\mathrm{div}\,\zeta - \hat{\chi} \cdot \hat{\underline{\chi}} + 2|\zeta|^2 + 2\rho, \tag{2.10}$$

$$\nabla_L \hat{\underline{\chi}} = -\nabla \widehat{\otimes} \zeta - \frac{1}{2}(\mathrm{tr}\,\chi \cdot \hat{\underline{\chi}} + \mathrm{tr}\,\underline{\chi} \cdot \hat{\chi}) + \zeta \widehat{\otimes} \zeta, \tag{2.11}$$

$$\mathrm{div}\,\hat{\chi} = \frac{1}{2}\nabla\,\mathrm{tr}\,\chi - \frac{1}{2}\,\mathrm{tr}\,\chi \cdot \zeta - \hat{\chi} \cdot \zeta - \beta, \tag{2.12}$$

$$\mathrm{curl}\,\zeta = -\frac{1}{2}\hat{\chi} \wedge \hat{\underline{\chi}} + \sigma, \tag{2.13}$$

$$\mathrm{div}\,\hat{\underline{\chi}} = \frac{1}{2}\nabla\,\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,\mathrm{tr}\,\underline{\chi}\zeta + \zeta \cdot \hat{\underline{\chi}} + \underline{\beta}, \tag{2.14}$$

$$K = -\frac{1}{4}\,\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}} - \rho \tag{2.15}$$

and the renormalized null Bianchi identities

$$\nabla_L \beta = \mathrm{div}\,\alpha + \zeta \cdot \alpha, \tag{2.16}$$

$$L(\check{\rho}) + \frac{3}{2}\,\mathrm{tr}\,\chi \cdot \check{\rho} = \mathrm{div}\,\beta - \zeta \cdot \beta + \frac{1}{2}\hat{\chi} \cdot \left(\nabla \widehat{\otimes} \zeta + \frac{1}{2}\,\mathrm{tr}\,\underline{\chi} \cdot \hat{\chi} - \zeta \widehat{\otimes} \zeta\right), \tag{2.17}$$

$$L(\check{\sigma}) + \frac{3}{2}\,\mathrm{tr}\,\chi \cdot \check{\sigma} = -\,\mathrm{curl}\,\beta + \zeta \wedge \beta + \frac{1}{2}\hat{\chi} \wedge (\nabla \widehat{\otimes} \zeta - \zeta \widehat{\otimes} \zeta), \tag{2.18}$$

$$\begin{aligned} \nabla_L \check{\underline{\beta}} = {} & -\nabla\rho + (\nabla\sigma)^\star - 2(\nabla \widehat{\otimes} \zeta) \cdot \zeta + 3\big(\zeta \cdot \rho - \zeta^\star \sigma\big) - \mathrm{tr}\,\chi\underline{\beta}, \\ & + 2\zeta \cdot \left(-\frac{1}{2}\,\mathrm{tr}\,\chi \cdot \hat{\underline{\chi}} - \frac{1}{2}\,\mathrm{tr}\,\underline{\chi} \cdot \hat{\chi} + \zeta \widehat{\otimes} \zeta\right) - 4\chi \cdot \hat{\underline{\chi}} \cdot \zeta \end{aligned} \tag{2.19}$$

where K denotes the Gauss curvature of each leaf $S := S_s$, and

$$\check{\rho} = \rho - \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}}, \quad \check{\sigma} = \sigma - \frac{1}{2}\hat{\chi} \wedge \hat{\underline{\chi}}, \quad \check{\underline{\beta}} = \underline{\beta} + 2\hat{\underline{\chi}} \cdot \zeta,$$

moreover, $\nabla_L \pi$, for any S-tangent tensor field π, is defined as in [9, Definition 2.9], i.e. the projection of $D_L \pi$ on each leaf S_s.

The transport equation for the mass aspect function μ defined by (1.2) takes the form

$$\frac{d}{ds}\mu + \frac{3}{2}\operatorname{tr}\chi\mu = \hat{\chi}\cdot(\nabla\widehat{\otimes}\zeta) + \frac{1}{2}\operatorname{tr}\chi\check{\rho} + 2\zeta\cdot\nabla\operatorname{tr}\chi - 4\hat{\chi}\cdot\zeta\cdot\zeta + \operatorname{tr}\chi|\zeta|^2 - \frac{1}{4}\operatorname{tr}\underline{\chi}|\hat{\chi}|^2. \tag{2.20}$$

3. The bootstrap assumptions

3.1. A preliminary bootstrap assumption

As a preliminary bootstrap assumption, we require that there exists a sufficiently small positive constant $0 < \Delta_0 < 1/2$ such that

$$\|V\|_{L^\infty(\mathcal{H})} \leqslant \Delta_0, \tag{3.1}$$

where $V(s,\omega) = \operatorname{tr}\chi - \frac{2}{s}$. We also set $\underline{V}(s,\omega) = \operatorname{tr}\underline{\chi} + \frac{2}{s}$, which will be used later.

In the following we will provide some preliminary estimates under (3.1). Recall that for the induced metric $\gamma := \gamma_s$ on $S := S_s$, we have $\frac{d}{ds}\gamma = 2\chi$. Thus $\frac{d}{ds}a_s = \operatorname{tr}\chi a_s$. In view of (2.5), we then get $t^{-2}a_t = \exp(\int_0^t V_s\,ds)$. Therefore for $0 < t \leqslant 1$ there holds $|t^{-2}a(t,\omega) - 1| \lesssim t\Delta_0$. Thus for small Δ_0 we have

$$\frac{1}{2}t^2 \leqslant a(t,\omega) \leqslant 2t^2. \tag{3.2}$$

In view of [9, Lemma 2.26] which says $\frac{dr}{ds} = \frac{r}{2}\overline{\operatorname{tr}\chi}$, it is easy to check

$$\frac{d}{ds}\log\frac{r}{s} = \frac{1}{2}\overline{V_s}.$$

Using (2.5) and integrating the above equation along any null geodesic yields $\frac{r}{s} = \exp(s\cdot O(\Delta_0))$. Therefore we get

Proposition 3.1. *Under the bootstrap assumption* (3.1)*, the radius $r(s)$ of each leaf S_s and the affine parameter s are always comparable in the sense that $|\frac{1}{r} - \frac{1}{s}| \lesssim \Delta_0$ for $0 < s \leqslant 1$.*

In view of (3.2), we have for any S-tangent tensor F on $\mathcal{H}$

$$\frac{1}{2}\int_{|\omega|=1}\left|F(s,\omega)\right|^p s^2\,d\omega \leqslant \|F\|^p_{L^p(S_s)} \leqslant 2\int_{|\omega|=1}\left|F(s,\omega)\right|^p s^2\,d\omega,$$

where $|F|$ denotes the norm of F under the induced metric γ on each leaf S_s.

For S-tangent tensor fields F on $\mathcal{H}$, we introduce the following norms. For $1 \leqslant p, q \leqslant \infty$ we define the $L_t^q L_x^p$ norm

$$\|F\|_{L_t^q L_x^p} := \left(\int_0^1\left(\int_{|\omega|=1}|F(s,\omega)|^p s^2 d\omega\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}} \tag{3.3}$$

and the $L_x^p L_t^\infty$ norm

$$\|F\|_{L_x^p L_t^\infty} := \left(\int_{|\omega|=1}\left(\sup_s s^{\frac{2}{p}}\left|F(s,\omega)\right|\right)^p\right)^{\frac{1}{p}}. \tag{3.4}$$

We also define the norms

$$\mathcal{N}_1(F) := \|t^{-1}F\|_{L_t^2 L_x^2} + \|\nabla_L F\|_{L_t^2 L_x^2} + \|\nabla F\|_{L_t^2 L_x^2} \tag{3.5}$$

and

$$\mathcal{N}_2(F) := \|t^{-2}F\|_{L_t^2 L_x^2} + \left\|t^{-1}\nabla_L F\right\|_{L_t^2 L_x^2} + \|t^{-1}\nabla F\|_{L_t^2 L_x^2} + \|\nabla\nabla_L F\|_{L_t^2 L_x^2} + \|\nabla^2 F\|_{L_t^2 L_x^2}. \tag{3.6}$$

On each slice $S = S_s$, we have the following Sobolev inequalities for scalar functions f and tensor fields F (see [12] for the proofs):

$$\|f\|_{L^2(S)} \lesssim \|\nabla f\|_{L^1(S)} + \|s^{-1} f\|_{L^1(S)}, \tag{3.7}$$

$$\|f\|_{L^\infty(S)} \lesssim \|\nabla^2 f\|_{L^1(S)} + \|\nabla f\|_{L^2(S)} + \|s^{-2} f\|_{L^1(S)}, \tag{3.8}$$

$$\|F\|_{L^p(S)} \lesssim \|\nabla F\|_{L^2(S)}^{1-\frac{2}{p}} \cdot \|F\|_{L^2(S)}^{\frac{2}{p}} + \big\|s^{\frac{2}{p}-1} F\big\|_{L^2(S)}, \quad 2 \leqslant p < \infty, \tag{3.9}$$

$$\|F\|_{L^\infty(S)} \lesssim \|\nabla^2 F\|_{L^2(S)}^{\frac{1}{p}} \big(\|\nabla F\|_{L^2(S)}^{\frac{p-2}{p}} \|F\|_{L^2(S)}^{\frac{1}{p}} + s^{\frac{2}{p}-1} \|F\|_{L^2(S)}^{\frac{p-1}{p}}\big) + \|\nabla F\|_{L^2(S)}. \tag{3.10}$$

The following preliminary estimates will be used routinely.

Lemma 3.1. *Let F be an arbitrary S-tangent tensor field on $\mathcal{H}$. Then*

$$\big\|t^{-\frac{1}{2}} F\big\|_{L_t^\infty L_\omega^\infty} + \|t^{-1} F\|_{L_t^2 L_\omega^\infty} \lesssim \mathcal{N}_2(F),$$

$$\|F\|_{L_x^4 L_t^\infty} + \|F\|_{L_t^6 L_x^6} + \big\|t^{-1/2} F\big\|_{L_x^2 L_t^\infty} \lesssim \mathcal{N}_1(F), \tag{3.11}$$

$$\big\|t^{-\frac{1}{q}-\frac{2}{p}+\frac{1}{2}} F\big\|_{L_t^q L_x^p} \lesssim \mathcal{N}_1(F) \quad \textit{with } 2 \leqslant p < \infty, 2 < q \leqslant \frac{2p}{p-4}. \tag{3.12}$$

Proof. The proof of the first inequality can be found in [2]. In the following we show the second inequality in (3.11). Note that it suffices to prove it for scalar functions f. First

$$t^2 \big|f(t,\omega)\big|^4 \lesssim \int_0^t s\big|f(s,\omega)\big|^4 ds + \int_0^t |f|^3 \cdot \big|\nabla_L f(s,\omega)\big| s^2 ds$$
$$\lesssim \left(\int_0^t \big|f(s,\omega)\big|^6 s^2 ds\right)^{\frac{1}{2}} \left(\int_0^t s^{-2}\big|f(s,\omega)\big|^2 s^2 ds + \int_0^t \big|\nabla_L f(s,\omega)\big|^2 s^2 ds\right)^{\frac{1}{2}}.$$

Hence, integrating on $|\omega| = 1$, we obtain

$$\|f\|_{L_x^4 L_t^\infty} \lesssim \|f\|_{L^6(\mathcal{H})}^{\frac{3}{4}} \big(\|s^{-1} f\|_{L^2(\mathcal{H})} + \|\nabla_L f\|_{L^2(\mathcal{H})}\big)^{\frac{1}{4}}. \tag{3.13}$$

On the other hand, by (3.7) we have on each S_s that

$$\int_{S_s} |f|^6 dA_s \lesssim \|f^3\|_{L_x^2}^2 \lesssim \left(\int_{S_s} \big(|\nabla f| + s^{-1}|f|\big)|f|^2 dA_s\right)^2$$
$$\lesssim \int_{S_s} \big(|\nabla f|^2 + s^{-2}|f|^2\big) dA_s \cdot \int_{S_s} |f|^4 dA_s.$$

Then integrating in s yields

$$\|f\|_{L^6(\mathcal{H})} \lesssim \|f\|_{L_t^\infty L_x^4}^{\frac{2}{3}} \big(\|s^{-1} f\|_{L^2(\mathcal{H})} + \|\nabla f\|_{L^2(\mathcal{H})}\big)^{\frac{1}{3}}. \tag{3.14}$$

Finally we note that

$$\|t^{-1/2} f\|_{L_x^2 L_t^\infty} \leqslant \left\|\int_t^1 \frac{d}{ds}\big(s|f|^2\big) ds\right\|_{L_\omega^2}^{\frac{1}{2}} + \|f(1)\|_{L_\omega^2}.$$

The first term is bounded by $\mathcal{N}_1(f)$ by using Hölder inequality. The second term can be estimated by taking a test function $0 \leqslant \theta \leqslant 1$ supported in $[1/2, 1]$ with $\theta(1) = 1$ and vanishing identically at $0 < t \leqslant 1/2$, and $\sup_{t \leqslant s \leqslant 1} |\theta'(s)| < \frac{1}{1-t}$. Since

$$\|f(1)\|_{L^2_\omega}^2 = \int_{|\omega|=1} \left| \int_{1/2}^{1} \frac{d}{ds}\big(s\theta|f|^2\big)\, ds \right| d\omega \lesssim \|\nabla_L f\|_{L^2(\mathcal{H})}^2 + \|s^{-1} f\|_{L^2(\mathcal{H})}^2 \lesssim \mathcal{N}_1(f)^2$$

we conclude $\|t^{-1/2} f\|_{L_x^2 L_t^\infty} \lesssim \mathcal{N}_1(f)$. Combining (3.13) with (3.14), we conclude that the estimates in the second line of (3.11) hold true. We can prove (3.12) similarly. □

3.2. *The full set of bootstrap assumptions*

In order to provide the full set of bootstrap assumptions, we introduce the following conventions.

- R denotes the full collection of null curvature components $\alpha, \beta, \rho, \sigma, \underline{\beta}$.
- R_0 denotes the collection of the null curvature components $\beta, \rho, \sigma, \underline{\beta}$.
- $\check{R}$ denotes the collection of the renormalized null curvature components $(\check{\rho}, -\check{\sigma})$, $\check{\underline{\beta}}$.
- $\underline{A}$ denotes the collection $V, \hat{\chi}, \zeta, \underline{V}, \underline{\hat{\chi}}$.
- A denotes the collection of $V, \hat{\chi}, \zeta$.
- M denotes the collection $\nabla \operatorname{tr}\chi$, μ.
- ∇A denotes the collection $\nabla \operatorname{tr}\chi$, $\nabla\hat{\chi}$, $\nabla\zeta$.

The bootstrap assumptions we will rely on in this paper are

$$\|\operatorname{tr}\chi - 2/t\|_{L_t^\infty L_\omega^\infty}, \|A\|_{L_\omega^\infty L_t^2}, \mathcal{N}_1(A), \|\nabla \operatorname{tr}\chi\|_{L_x^2 L_t^\infty} \leqslant \Delta_0, \tag{BA1}$$

and

$$\|t^{-1/2}\underline{A}\|_{L_x^2 L_t^\infty} \leqslant \Delta_0, \qquad \|\nabla_L \underline{A}\|_{L^2(\mathcal{H})} \leqslant \Delta_0 \tag{BA2}$$

where $0 < \mathcal{R}_0 \leqslant \Delta_0 < \frac{1}{2}$ is a sufficiently small constant. Note that the preliminary bootstrap assumption (3.1) is a part of (BA1).

In order to complete the proof of Theorem 1.1, by the bootstrap principle it suffices to show, under (BA1) and (BA2), that all the inequalities in them still hold true with Δ_0 replaced by $\Delta_0/2$ when $0 < \mathcal{R}_0 \leqslant \Delta_0$ is sufficiently small. This will be done in Section 7 after the preparations given in the next three sections.

Lemma 3.2. *Under the bootstrap assumption* (BA1)*, the metric $\mathring{\gamma}_{ij}(s)$ on each S_s verifies weakly spherical conditions, i.e. relative to the transport local coordinates (s, ω_1, ω_2) the metric components $\mathring{\gamma}_{ij}(s)$ satisfy*

$$\big\|\mathring{\gamma}_{ij}(s) - \gamma_{ij}^{(0)}\big\|_{L_\omega^\infty} \lesssim \Delta_0, \tag{3.15}$$

$$\big\|\partial_k\mathring{\gamma}_{ij}(s) - \partial_k\gamma_{ij}^{(0)}\big\|_{L_\omega^2} \lesssim \Delta_0, \tag{3.16}$$

where Δ_0 is a small constant.

Proof. Recall that relative to the transport local coordinates (s, ω_1, ω_2) on $\mathcal{H}$, Proposition 2.1 says

$$\lim_{s\to 0} \mathring{\gamma}_{ij} = \gamma_{ij}^{(0)} \quad \text{and} \quad \lim_{s\to 0} \partial_k\mathring{\gamma}_{ij} = \partial_k\gamma_{ij}^{(0)} \tag{3.17}$$

where $i, j, k = 1, 2$. Recall also that the metric γ verifies $\frac{d}{ds}\gamma_{ij} = 2\chi_{ij}$ with $i, j = 1, 2$. Consequently,

$$\frac{d}{ds}\mathring{\gamma}_{ij} = \mathring{\gamma}_{ij} V + 2s^{-2}\hat{\chi}_{ij}.$$

Integrating this equation along any null geodesic on $\mathcal{H}$ and using (3.17) we derive

$$\sum_{i,j}\|\mathring{\gamma}_{ij}-\gamma_{ij}^{(0)}\|_{L_t^\infty L_\omega^\infty}\lesssim\sum_{i,j}\sup_t\left\|\int_0^t \mathring{\gamma}_{ij}V+s^{-2}\hat{\chi}_{ij}\,ds\right\|_{L_\omega^\infty}$$
$$\lesssim\left(\|V\|_{L_t^\infty L_\omega^\infty}+\|\hat{\chi}\|_{L_\omega^\infty L_t^2}\right)\left(\sum_{ij}\|\mathring{\gamma}_{ij}-\gamma_{ij}^{(0)}\|_{L_t^\infty L_\omega^\infty}+1\right).$$

This gives (3.15), by using (BA1) and the smallness of Δ_0.

The proof of (3.16) is similar by noting that

$$\frac{d}{ds}\partial_k\mathring{\gamma}_{ij}=\partial_k\mathring{\gamma}_{ij}V+\mathring{\gamma}_{ij}\partial_k V+2s^{-2}\partial_k\hat{\chi}_{ij},$$

where $i, j, k=1,2$. □

On each $S:=S_s$ we will use the geometric Littlewood–Paley (GLP) projections P_k introduced in [12] which take the form

$$P_kF:=\int_0^\infty m_k(\tau)U(\tau)F\,d\tau$$

for any tensor field F, where $m_k(\tau):=2^{2k}m(2^{2k}\tau)$ for some smooth function m on $[0,\infty)$ vanishing sufficiently fast and verifying the vanishing moment property

$$\int_0^\infty \tau^{k_1}\partial^{k_2}m(\tau)\,d\tau=0,\quad k_1+k_2\leqslant N,$$

and $U(\tau)F$ is defined by the heat flow on $(S_s,\mathring{\gamma})$

$$\frac{\partial}{\partial\tau}U(\tau)F-\Delta_{\mathring{\gamma}}U(\tau)F=0,\quad U(0)F=F. \tag{3.18}$$

One may refer to [12] for various properties of GLP projections, such as the finite band property and the Bernstein inequalities, etc, which will be frequently used in this paper.

We will also use the notations

$$F_n:=P_nF,\quad F_{\leqslant 0}:=\sum_{k\leqslant 0}P_kF\quad\text{and}\quad F_{>0}:=\sum_{k>0}P_kF$$

for any S-tangent tensor field F.

Let $0\leqslant\theta<1$, we define the Besov norm $B_{2,1}^\theta$ for tensor fields F on 2-D surface S by

$$\|F\|_{B_{2,1}^\theta}=\sum_{k>0}\left\|\left(2^kt^{-1}\right)^\theta P_kF\right\|_{L_x^2}+\|t^{-\theta}F\|_{L_x^2}. \tag{3.19}$$

We also define the Besov $\mathcal{B}^\theta$ and $\mathcal{P}^\theta$ norms for S-tangent tensor fields F on $\mathcal{H}$ as follows:

$$\|F\|_{\mathcal{B}^\theta}=\sum_{k>0}\left\|\left(2^kt^{-1}\right)^\theta P_kF\right\|_{L_t^\infty L_x^2}+\|t^{-\theta}F\|_{L_t^\infty L_x^2}, \tag{3.20}$$

$$\|F\|_{\mathcal{P}^\theta}=\sum_{k>0}\left\|\left(2^kt^{-1}\right)^\theta P_kF\right\|_{L_t^2 L_x^2}+\|t^{-\theta}F\|_{L_t^2 L_x^2}. \tag{3.21}$$

By using the heat flow (3.18), we can define the operator Λ^a with $a\leqslant 0$ such that for any S-tangent tensor fields F

$$\Lambda^aF:=\frac{s^{-a}}{\Gamma(-a/2)}\int_0^\infty \tau^{-\frac{a}{2}-1}e^{-\tau}U(\tau)F\,d\tau.$$

The definition of Λ^a extends to the range $a > 0$ by defining for $0 < a \leqslant 2m$ that

$$\Lambda^a F = \Lambda^{a-2m} \cdot \left(s^{-2}\,\mathrm{Id} - \Delta_\gamma\right)^m F.$$

We record the basic properties of Λ^a in the following result (see [12,20]).

Proposition 3.2.

(i) $\Lambda^0 = \mathrm{Id}$ *and* $\Lambda^a \cdot \Lambda^b = \Lambda^{a+b}$ *for any* $a, b \in \mathbb{R}$.

(ii) *For any* S*-tangent tensor field* F *and any* $a \leqslant 0$

$$s^a \|\Lambda^a F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(iii) *For any* S*-tangent tensor field* F *and any* $b \geqslant a \geqslant 0$

$$s^a \|\Lambda^a F\|_{L^2(S)} \lesssim s^b \|\Lambda^b F\|_{L^2(S)} \quad \text{and} \quad \|\Lambda^a F\|_{L^2(S)} \lesssim \|\Lambda^b F\|_{L^2(S)}^{\frac{a}{b}} \|F\|_{L^2(S)}^{1-\frac{a}{b}}.$$

(iv) *For any* S*-tangent tensor fields* F *and* G *and any* $0 \leqslant a < 1$

$$\left\|\Lambda^a (F \cdot G)\right\|_{L^2(S)} \lesssim \|\Lambda F\|_{L^2(S)} \|\Lambda^a G\|_{L^2(S)} + \|\Lambda^a F\|_{L^2(S)} \|\Lambda G\|_{L^2(S)}.$$

(v) *For any* S*-tangent tensor field* F *there holds with* $2 < p < \infty$ *and* $a > 1 - \frac{2}{p}$

$$\|F\|_{L^p(S)} \lesssim \|\Lambda^a F\|_{L^2(S)}.$$

(vi) *For any* $a \in \mathbb{R}$ *and any* S*-tangent tensor field* F

$$\|F\|_{H^a(S)}^2 := \|\Lambda^a F\|_{L^2(S)}^2 \approx \sum_{k \geqslant 0} 2^{2ka} s^{-2a} \|P_k F\|_{L^2(S)}^2 + s^{-2a} \|P_{\leqslant 0} F\|_{L^2(S)}^2.$$

Under (BA1) and (BA2) we can also derive

Proposition 3.3. *Under* (BA1) *and* (BA2)*, if* $0 < \mathcal{R}_0 < \Delta_0$ *are sufficiently small, then for all* $\frac{1}{2} < a < 1$ *there holds*

$$\underline{K}_a := \left\|\Lambda^{-a}\left(K - s^{-2}\right)\right\|_{L_t^\infty L_x^2} \lesssim \Delta_0.$$

The proof of Proposition 3.3 is a little involved. Noting that our definition of Λ^{-a} involves s^{-2}, by keeping track the powers of s, the argument in the proof of Proposition 4.13 in [9] still goes through. For details please refer to [22, Chapter 4.3].

Sometimes it is convenient to work with the Besov norms defined by the classical Littlewood–Paley (LP) projections E_k. Recall that (see [17–19]) for any scalar function f on $\mathbb{R}^2$ we can define

$$E_k f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \eta\left(\xi/2^k\right) \hat{f}(\xi) e^{ix\xi}\, d\xi,$$

where η is a smooth function with support in the dyadic shell $\{\frac{1}{2} \leqslant |\xi| \leqslant 2\}$ and satisfying $\sum_{k\in\mathbb{Z}} \eta(2^{-k}\xi) = 1$ when $\xi \neq 0$.

Now for any scalar function f on $\mathcal{H}$, we define for any $0 \leqslant a < 1$ its $\tilde{\mathcal{B}}^a$ and $\tilde{\mathcal{P}}^a$ norms by

$$\|f\|_{\tilde{\mathcal{B}}^a} := \sum_{k>0} \left\|\left(2^k t^{-1}\right)^a E_k f\right\|_{L_t^\infty L_x^2} + \|t^{-a} f\|_{L_t^\infty L_x^2}, \tag{3.22}$$

$$\|f\|_{\tilde{\mathcal{P}}^a} = \sum_{k>0} \left\|\left(2^k t^{-1}\right)^a E_k f\right\|_{L_t^2 L_x^2} + \|t^{-a} f\|_{L_t^2 L_x^2}. \tag{3.23}$$

It is worthy to say a few words about this definition. Recall that the geodesic flow $\Gamma_s : \mathbb{S}^2 \to S_s$ for each $s > 0$ is a diffeomorphism. Let $(U_i, \eta^{(i)})$ be a finite atlas on $\mathbb{S}^2$ with charts $\eta^{(i)}$ mapping U_i into the unit disc in $\mathbb{R}^2$, and let $\{\phi\}$ be a subordinated partition of unity on $\mathbb{S}^2$. Then $\{\phi \circ \Gamma_s^{-1}\}$ is a partition of unity on the slice S_s for $0 < s \leqslant 1$ which will be denoted as ϕ_s. Let $\eta_s^{(i)} := \eta^{(i)} \circ \Gamma_s^{-1}$. The $E_k f$ in the above definition is defined as $E_k((\phi_s f) \circ {\eta_s^{(i)}}^{-1})$ on each S_s and the L_x^2 norms are understood to be the L^2 norm on $\mathbb{R}^2$.

Using Lemma 3.2, (BA1) and (BA2), we can adapt [11, Proposition 3.28] to obtain the following lemma.

Lemma 3.3. *Under the bootstrap assumptions* (BA1) *and* (BA2)*, there exists a finite number of vector fields* $\{X_i\}_{i=1}^l$ *verifying the conditions*

$$\begin{cases} \|X, t\nabla_0 X\|_{L_t^\infty L_\omega^\infty} \lesssim 1, \quad \|t\nabla(\nabla_0 X)\|_{L_x^2 L_t^\infty} \lesssim 1, \\ \|(\nabla - \nabla_0)X\|_{L_x^2 L_t^\infty} \lesssim \Delta_0, \quad \nabla_L X = 0, \end{cases}$$

where ∇_0 *represents the covariant derivative induced by the metric* $s^2\gamma^{(0)}$*. For appropriate* S*-tangent tensor* $F \in L_t^\infty L_x^2$, $F \in \mathcal{B}^a$ *if and only if* $F \cdot X_i \in \mathcal{B}^a$*, and*

$$C^{-1} \sum_i \|F \cdot X_i\|_{\mathcal{B}^a} \leqslant \|F\|_{\mathcal{B}^a} \leqslant C \sum_i \|F \cdot X_i\|_{\mathcal{B}^a}, \quad \text{with } 0 \leqslant a < 1,$$

where C *is a universal constant. The same results hold for the spaces* $\mathcal{P}^\alpha$*. Moreover*

$$\mathcal{N}_1(F \widehat{\otimes} X) + \|F \widehat{\otimes} X\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2},$$

where $\widehat{\otimes}$ *stands for either a tensor product or a contraction.*

Lemma 3.3 allows us to define Besov norms for arbitrary S-tangent tensor fields F on $\mathcal{H}$ by the classical LP projections.

Definition 3.1. Let F be an (m,n) S-tangent tensor field on $\mathcal{H}$ and let $F_{i_1 i_2 \dots i_n}^{j_1 j_2 \cdots j_m}$ be the local components of F relative to $\{X_i\}_{i=1}^l$. We define the $\tilde{\mathcal{B}}^a$ and $\tilde{\mathcal{P}}^a$ norms of F by

$$\|F\|_{\tilde{\mathcal{B}}^a} = \sum \|F_{i_1 i_2 \dots i_n}^{j_1 j_2 \cdots j_m}\|_{\tilde{\mathcal{B}}^a} \quad \text{and} \quad \|F\|_{\tilde{\mathcal{P}}^a} = \sum \|F_{i_1 i_2 \dots i_n}^{j_1 j_2 \cdots j_m}\|_{\tilde{\mathcal{P}}^a},$$

where the summation is taken over all possible $(i_1 \dots i_n; j_1 \dots j_m)$.

Finally we state the following equivalence results between $\mathcal{B}^a$, $\mathcal{P}^a$ norms and $\tilde{\mathcal{B}}^a$, $\tilde{\mathcal{P}}^a$ norms, whose proof can be found in [22].

Proposition 3.4. *Under the bootstrap assumptions* (BA1) *and* (BA2)*, for arbitrary* S*-tangent tensor fields* F *on* $\mathcal{H}$ *there hold*

$$\|F\|_{\tilde{\mathcal{B}}^a} \approx \|F\|_{\mathcal{B}^a} \quad \text{and} \quad \|F\|_{\tilde{\mathcal{P}}^a} \approx \|F\|_{\mathcal{P}^a}$$

with $0 \leqslant a < 1$.

4. Elliptic estimates of Hodge operators on $\mathcal{H}$

In view of the structure equations given in Section 2, it is important to consider the following Hodge operators on 2-surface S diffeomorphic to the standard sphere $\mathbb{S}^2$:

- The operator $\mathcal{D}_1$ takes any 1-form F into the pair of functions $(\operatorname{div} F, \operatorname{curl} F)$.
- The operator $\mathcal{D}_2$ takes any symmetric traceless 2-tensor F on S into the 1-form $\operatorname{div} F$.
- The operator ${}^\star\mathcal{D}_1$ takes the pairs of scalar functions (ρ, σ) into the 1-forms $-\nabla\rho + (\nabla\sigma)^\star$ on S.
- The operator ${}^\star\mathcal{D}_2$ takes 1-forms F on $\mathcal{S}$ into the 2-covariant, symmetric, traceless tensors $-\frac{1}{2}\widehat{\mathcal{L}_F\gamma}$, where

$$\big(\widehat{\mathcal{L}_F\gamma}\big)_{ab} = \nabla_b F_a + \nabla_a F_b - (\operatorname{div} F)\gamma_{ab}.$$

For various properties of these operators please refer to [2,9].

4.1. L^2 estimates for Hodge operators

In this subsection we will give the L^2 estimates for the Hodge operators on $\mathcal{H}$ under the bootstrap assumptions (BA1), (BA2) and the smallness conditions on $\mathcal{R}_0$ and Δ_0.

Proposition 4.1. *The following estimates hold on each leaf $S = S_s \subset \mathcal{H}$:*

(i) *The operator $\mathcal{D}_1$ is invertible on its range and its inverse $\mathcal{D}_1^{-1}$ takes pair of function $f = (\rho, \sigma)$ in the range of $\mathcal{D}_1$ into S-tangent 1-forms F with $\operatorname{div} F = \rho$, $\operatorname{curl} F = \sigma$. Moreover*
$$\|\nabla \mathcal{D}_1^{-1} f\|_{L^2(S)} + \|s^{-1}\mathcal{D}_1^{-1} f\|_{L^2(S)} \lesssim \|f\|_{L^2(S)}.$$

(ii) *The operator $\mathcal{D}_2$ is invertible on its range and its inverse $\mathcal{D}_2^{-1}$ takes S-tangent 1-forms F (in the range of $\mathcal{D}_2$) into S-tangent symmetric, traceless, 2-tenorfields Z with $\operatorname{div} Z = F$. Moreover*
$$\|\nabla \mathcal{D}_2^{-1} F\|_{L^2(S)} + \|s^{-1}\mathcal{D}_2^{-1} F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(iii) *The operator $(-\Delta)$ is invertible on its range and its inverse $(-\Delta)^{-1}$ verifies the estimate*
$$\|\nabla^2(-\Delta)^{-1} f\|_{L^2(S)} + \|s^{-1}\nabla(-\Delta)^{-1} f\|_{L^2(S)} \lesssim \|f\|_{L^2(S)}.$$

(iv) *The operator ${}^\star\mathcal{D}_1$ is invertible as an operator defined for pairs of H^1 functions with mean zero (i.e. the quotient of H^1 by the kernel of ${}^\star\mathcal{D}_1$) and its inverse ${}^\star\mathcal{D}_1^{-1}$ takes S-tangent L^2 1-forms F(i.e. the full range of ${}^\star\mathcal{D}_1$) into pair of functions (ρ, σ) with mean zero, such that $-\nabla\rho + (\nabla\sigma)^\star = F$, verifies the estimate*
$$\|\nabla {}^\star\mathcal{D}_1^{-1} F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(v) *The operator ${}^\star\mathcal{D}_2$ is invertible as an operator defined on the quotient of H^1-vector fields by the kernel of ${}^\star\mathcal{D}_2$. Its inverse ${}^\star\mathcal{D}_2^{-1}$ takes S-tangent 2-forms Z which is in L^2 space into S tangent 1-forms F (orthogonal to the kernel of $\mathcal{D}_2$), such that ${}^\star\mathcal{D}_2 F = Z$, verifies the estimate*
$$\|\nabla \cdot {}^\star \mathcal{D}_2^{-1} Z\|_{L^2(S)} \lesssim \|Z\|_{L^2(S)}.$$

As a consequence of (i)–(v)*, let $\mathcal{D}^{-1}$ be one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^\star\mathcal{D}_1^{-1}$ or ${}^\star\mathcal{D}_2^{-1}$. By dual argument, we have the following estimate for appropriate[2] tensor fields F,*
$$\|\mathcal{D}^{-1} \operatorname{div} F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

The proof of this result is essentially the same as the proof of [9, Proposition 4.22]. Note that in our situation, $0 < s \leqslant 1$, which is different from [9] where $s \approx 1$. Therefore we must keep the weight s^{-1} in some of the estimates, which will be crucial for later applications.

Using the formula (2.15) for the Gauss curvature K of S_s and the bootstrap assumptions (BA1) and (BA2) we can easily obtain

Proposition 4.2. *For $\underline{K} := K - s^{-2}$ there holds $\|\underline{K}\|_{L^2(\mathcal{H})} \lesssim \Delta_0$.*

For later applications, we will use the renormalized Gauss curvature
$$\check{K} = K - r^{-2} \tag{4.1}$$
which, in view of Proposition 3.1, Proposition 3.3 and Proposition 4.2, verifies
$$\|\check{K}\|_{L^2(\mathcal{H})} \lesssim \Delta_0 \quad \text{and} \quad \check{K}_\alpha := \|\Lambda^{-a}\check{K}\|_{L_x^2} \lesssim \Delta_0 \tag{4.2}$$
for any $\frac{1}{2} < a < 1$.

Using Proposition 3.3 and Proposition 4.2 and following the similar argument in [9] we can obtain (see [22])

[2] By "appropriate", we mean the tensor F such that $\operatorname{div} F$ is in the space where $\mathcal{D}^{-1}$ is well-defined.

Lemma 4.1. *For appropriate S-tangent tensor field F there hold*

$$\mathcal{N}_2(\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F) \quad \text{and} \quad \mathcal{N}_1(\nabla\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F).$$

4.2. Elliptic $\mathcal{P}^\sigma$ estimates of Hodge operators on $\mathcal{H}$

In this subsection we provide $\mathcal{P}^\sigma$ estimates for 0-order Hodge operators. We begin with a few preliminary estimates which are frequently used in Section 6.

Proposition 4.3. *Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ and ${}^\star\mathcal{D}_1$. Then for $1 < p \leqslant 2$ and any S-tangent tensor F on $\mathcal{H}$ there holds*

$$\|\mathcal{D}^{-1}F\|_{L^2(S)} \lesssim \big\|s^{2-\frac{2}{p}}F\big\|_{L^p(S)}.$$

Proof. From (3.9) and Proposition 4.1 we infer for $p' \geqslant 2$ satisfying $\frac{1}{p}+\frac{1}{p'}=1$ that

$$\big\|s^{\frac{2}{p}-2}{}^\star\mathcal{D}^{-1}F\big\|_{L^{p'}(S)} \lesssim \|\nabla{}^\star\mathcal{D}^{-1}F\|_{L^2(S)}^{1-\frac{2}{p'}}\|s^{-1}{}^\star\mathcal{D}^{-1}F\|_{L^2(S)}^{\frac{2}{p'}} + \|s^{-1}\mathcal{D}^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L_x^2}.$$

We then complete the proof by duality. □

Lemma 4.2. *Let $\mathcal{D}$ denote one of the Hodge operators $\mathcal{D}_1$, $\mathcal{D}_2$, ${}^\star\mathcal{D}_1$ and ${}^\star\mathcal{D}_2$, let $\mathcal{D}^{-1}$ denote the inverse of $\mathcal{D}$. For P_kF with P_k being the GLP projections associated to the heat equation* (3.18) *there hold for $k > 0$*

$$\big\|\mathcal{D}^{-1}P_kF\big\|_{L_x^2} \lesssim 2^{-k}\|sF\|_{L_x^2} \quad \text{and} \quad \big\|P_k\mathcal{D}^{-1}F\big\|_{L_x^2} \lesssim 2^{-(2-\frac{2}{p})k}\big\|s^{2-\frac{2}{p}}F\big\|_{L_x^p}.$$

Proof. The first inequality can be proved by using the finite band property and Proposition 4.1. The second can be proved by a dual argument with the help of the first one and the Sobolev inequality. □

The following result follows from the second estimate in Lemma 4.2 immediately.

Proposition 4.4. *Let $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$, ${}^\star\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, then for appropriate S-tangent tensor fields F on $\mathcal{H}$ and any $1 < p \leqslant 2$,*

$$\|\mathcal{D}^{-1}F\|_{\mathcal{B}^0} \lesssim \big\|t^{2-\frac{2}{p}}F\big\|_{L_t^\infty L_x^p}. \tag{4.3}$$

Moreover for $0 \leqslant \theta < \frac{1}{2}$ and $\frac{2}{2-\theta} < p \leqslant 2$,

$$\|\mathcal{D}^{-1}F\|_{\mathcal{P}^\theta} \lesssim \big\|t^{2-\frac{2}{p}-\theta}F\big\|_{L_t^2 L_x^p}. \tag{4.4}$$

In order to state the next result succinctly, we introduce the notation

$$K(1+\gamma,k) := (2^k s^{-1})^{(1+\gamma)} + 2^k s^{-1}\|\underline{K}\|_{L_x^2}^\gamma + (2^k s^{-1})^{1+\gamma(1-\theta)}\|\underline{K}\|_{L_x^2}^{\theta\gamma},$$

where $\gamma > 1/2$, $k \in \mathbb{N}$ and θ is a number slightly greater than 1. For simplicity, the last terms in $K(1+\gamma,k)$ can be ignored in applications.

Lemma 4.3. *For any smooth S-tangent tensor field F and $0 \leqslant \gamma \leqslant 1$*

$$\big\|P_k\nabla\Lambda^\gamma F\big\|_{L_x^2} \lesssim K(1+\gamma,k)\|F\|_{L_x^2} \quad \text{and} \quad \big\|\Lambda^\gamma\nabla P_kF\big\|_{L_x^2} \lesssim K(1+\gamma,k)\|F\|_{L_x^2}.$$

Proof. Recall (see [12, Section 10]) the Böchner identity combined with finite band property gives for $2 \leqslant p < \infty$ that

$$\big\|\nabla^2P_kG\big\|_{L_x^2} \lesssim 2^{2k}s^{-2}\|P_kG\|_{L_x^2} + \|\underline{K}\|_{L_x^2}^{\frac{p}{p-1}}(2^ks^{-1})^{\frac{p-2}{p-1}}\|P_kG\|_{L_x^2} + 2^ks^{-1}\|\underline{K}\|_{L_x^2}\|P_kG\|_{L_x^2}. \tag{4.5}$$

By Proposition 3.2(iii), combining (4.5) with the finite band inequality, $\|\nabla P_k G\|_{L^2_x} \lesssim 2^k s^{-1} \|P_k G\|_{L^2_x}$, we conclude the second inequality holds true for any $0 \leqslant \gamma \leqslant 1$. The other one follows by duality. □

Now we are ready to state the main result on the elliptic $\mathcal{P}^\sigma$ estimates on Hodge operators.

Theorem 4.1. *Let $\mathcal{D}$ denote either $\mathcal{D}_1$, $\mathcal{D}_2$ or their adjoint operators ${}^\star\mathcal{D}_1$ and ${}^\star\mathcal{D}_2$. Then for any S-tangent tensor fields ξ and F satisfying $\mathcal{D}\xi = F$ and any $\frac{1}{2} > \sigma \geqslant 0$,*

$$\|\nabla\xi\|_{\mathcal{P}^\sigma} \lesssim \|F\|_{\mathcal{P}^\sigma} + \Delta_0 \|\mathcal{D}^{-1}F\|^q_{L^a_t L^2_x} \|F\|^{1-q}_{L^2(\mathcal{H})}, \tag{4.6}$$

where $1 - \sigma > q > \gamma_0$ and $a > 4$.

The estimate (4.6) was stated in [9] where the second term on the right-hand side is not included. The proof of the corrected version is based on the unpublished notes of Klainerman and Rodnianski [6]. To prove Theorem 4.1, we rely on the following two propositions.

Proposition 4.5. *Let F be an S-tangent tensor field. Then for any $1 > \gamma > \gamma_0 > \frac{1}{2}$, where γ_0 is determined by the condition that $\check{K}_{\gamma_0} < \infty$, we have the following estimate for $k, m > 0$*

$$\begin{aligned}\|P_k \nabla P_m F\|_{L^2(\mathcal{H})} \lesssim{}& 2^{\min(m,k)} 2^{-2|m-k|} \big\|t^{-1} P_m F\big\|_{L^2_t L^2_x} + 2^{\min(m,k)} 2^{-(1-\gamma)\max(m,k)} \check{K}_{\gamma_0} \big\|t^{-\gamma} P_m F\big\|_{L^2_t L^2_x} \\ &+ 2^{-|m-k|} \|\underline{K}\|^\gamma_{L^2(\mathcal{H})} \check{K}_{\gamma_0} \|P_m F\|_{L_t^{\frac{2}{1-\gamma}} L^2_x}.\end{aligned} \tag{4.7}$$

For low frequency terms, the following estimate holds

$$\begin{aligned}\big\|P_k \nabla P_{\leqslant 0} F\big\|_{L^2(\mathcal{H})} \lesssim{}& 2^{-k} \|t^{-1} P_{\leqslant 0} F\|_{L^2(\mathcal{H})} + 2^{-(1-\gamma)k} \check{K}_{\gamma_0} \|t^{-\gamma} P_{\leqslant 0} F\|_{L^2(\mathcal{H})} \\ &+ 2^{-k} \|\check{K}\|^\gamma_{L^2(\mathcal{H})} \check{K}_{\gamma_0} \|P_{\leqslant 0} F\|_{L_t^{\frac{2}{1-\gamma}} L^2_x}.\end{aligned} \tag{4.8}$$

Proposition 4.6. *Let F be an S-tangent tensor field and $\mathcal{D}$ be the corresponding elliptic Hodge operator. Then for $k, l > 0$,*

$$\big\|P_k \mathcal{D}^{-1} P_l F\big\|_{L^2(S)} \lesssim 2^{-\max(k,l)} 2^{-|k-l|} s \|P_l F\|_{L^2(S)} + 2^{-\max(k,l)} 2^{-(1-\gamma)\max(k,l)} s^{2-\gamma} \check{K}_{\gamma_0} \|P_l F\|_{L^2(S)}, \tag{4.9}$$

where $1 > \gamma > \gamma_0 > \frac{1}{2}$ and γ_0 is determined by the condition that $\check{K}_{\gamma_0} < \infty$.

We first give the proof of Theorem 4.1 by assuming Propositions 4.5 and 4.6.

Proof of Theorem 4.1. It suffices to consider the case $\sigma = 0$. For the case $1/2 > \sigma > 0$, by taking $q < 1 - \sigma$ we can follow the same argument.

According to the definition of $\|\nabla\mathcal{D}^{-1}F\|_{\mathcal{P}^0}$ and Proposition 4.1, we only need to estimate $\sum_{k>0} \|P_k \nabla\mathcal{D}^{-1}F\|_{L^2(\mathcal{H})}$. Using Proposition 4.5 we have

$$\big\|P_k \nabla\mathcal{D}^{-1}F\big\|_{L^2(\mathcal{H})} \lesssim \sum_m \big\|P_k \nabla P_m \mathcal{D}^{-1}F\big\|_{L^2(\mathcal{H})} + \big\|P_k \nabla\big(P_{\leqslant 0}\mathcal{D}^{-1}F\big)\big\|_{L^2(\mathcal{H})} = H_k + L_k.$$

We first estimate the high frequency terms

$$\begin{aligned}H_k \lesssim{}& \sum_m 2^{\min(m,k)} 2^{-2|m-k|} \big\|t^{-1} P_m \mathcal{D}^{-1}F\big\|_{L^2(\mathcal{H})} + \sum_m 2^{\min(m,k)} 2^{-(1-\gamma)\max(m,k)} \check{K}_{\gamma_0} \big\|t^{-\gamma} P_m \mathcal{D}^{-1}F\big\|_{L^2(\mathcal{H})} \\ &+ \check{K}_{\gamma_0} \|\underline{K}\|^\gamma_{L^2(\mathcal{H})} \sum_m 2^{-|m-k|} \big\|P_m \mathcal{D}^{-1}F\big\|_{L_t^{\frac{2}{1-\gamma}} L^2_x} \\ ={}& A^{(1)}_k + \check{K}_{\gamma_0} A^{(2)}_k + \check{K}_{\gamma_0} \|\underline{K}\|^\gamma_{L^2(\mathcal{H})} A^{(3)}_k.\end{aligned}$$

We first estimate the term $A_k^{(3)}$. Note by interpolation we can easily obtain with $\frac{q}{a}+\frac{1-q}{2}=\frac{1-\gamma}{2}$,

$$\|P_m\mathcal{D}^{-1}F\|_{L_t^{\frac{2}{1-\gamma}}L_x^2}\lesssim\|P_m\mathcal{D}^{-1}F\|_{L_t^aL_x^2}^q\|P_m\mathcal{D}^{-1}F\|_{L^2(\mathcal{H})}^{1-q},$$

then by Lemma 4.2,

$$\sum_k A_k^{(3)}\lesssim\sum_m 2^{-(1-q)m}\|\mathcal{D}^{-1}F\|_{L_t^aL_x^2}^q\|tF\|_{L^2(\mathcal{H})}^{1-q}\lesssim\|\mathcal{D}^{-1}F\|_{L_t^aL_x^2}^q\|tF\|_{L^2(\mathcal{H})}^{1-q}.$$

Now we define

$$A_{kl}^{(1)}=\sum_m 2^{\min(m,k)}2^{-2|m-k|}\|t^{-1}P_m\mathcal{D}^{-1}P_lF\|_{L^2(\mathcal{H})},$$
$$A_{kl}^{(2)}=\sum_m 2^{\min(m,k)}2^{-(1-\gamma)\max(m,k)}\|t^{-\gamma}P_m\mathcal{D}^{-1}P_lF\|_{L^2(\mathcal{H})}.$$

We use the GLP projections to decompose F and ignore the low frequency terms,[3] then we infer

$$\sum_k A_k^{(1)}\leqslant\sum_{k,l}A_{kl}^{(1)}\quad\text{and}\quad\sum_k A_k^{(2)}\leqslant\sum_{k,l}A_{kl}^{(2)}.$$

Therefore, it suffices to establish the following estimates

$$\sum_k A_{kl}^{(1)}\lesssim\|P_lF\|_{L^2(\mathcal{H})}\quad\text{and}\quad\sum_k A_{kl}^{(2)}\lesssim\|P_lF\|_{L^2(\mathcal{H})}.\tag{4.10}$$

We estimate the term $A_{kl}^{(1)}$ with the help of Proposition 4.6.

$$\begin{aligned}A_{kl}^{(1)}&\lesssim\sum_m 2^{\min(m,k)}2^{-2|m-k|}2^{-\max(m,l)}2^{-|m-l|}\|P_lF\|_{L^2(\mathcal{H})}\\&\quad+\sum_m 2^{\min(m,k)}2^{-2|m-k|}2^{-\max(m,l)}2^{-(1-\gamma)\max(m,l)}\|t^{1-\gamma}P_lF\|_{L^2(\mathcal{H})}\\&\lesssim 2^{-|l-k|}\sum_m 2^{-2|m-k|}\|P_lF\|_{L^2(\mathcal{H})}\\&\lesssim 2^{-|l-k|}\|P_lF\|_{L^2(\mathcal{H})}.\end{aligned}$$

The first estimate in (4.10) now follows after summing over k. The second estimate in (4.10) can be proved similarly.

As to low frequency terms L_k, taking L_t^2 norm with the help of (4.8), we infer

$$\begin{aligned}\|P_k\nabla(P_{\leqslant 0}\mathcal{D}^{-1}F)\|_{L_t^2L_x^2}&\lesssim 2^{-k}\|t^{-1}P_{\leqslant 0}\mathcal{D}^{-1}F\|_{L^2(\mathcal{H})}+2^{-(1-\gamma)k}\check{K}_{\gamma_0}\|t^{-\gamma}P_{\leqslant 0}\mathcal{D}^{-1}F\|_{L^2(\mathcal{H})}\\&\quad+2^{-k}\|\underline{K}\|_{L^2(\mathcal{H})}^{\gamma}\check{K}_{\gamma_0}\|P_{\leqslant 0}\mathcal{D}^{-1}F\|_{L_t^{\frac{2}{1-\gamma}}L_x^2}.\end{aligned}$$

By elliptic estimates and interpolation again, we can get the desired result. □

In order to give the proofs of Propositions 4.5 and 4.6, the following lemma is crucial.

Lemma 4.4. *Let S be a weakly regular surface with Gauss curvature K satisfying the condition $\check{K}_{\gamma_0}<\infty$ for some $1>\gamma_0>\frac{1}{2}$. Then for any smooth S-tangent tensor F, $1>\gamma>\gamma_0$ and $k>0$, there hold*

$$\|\Lambda^{-\gamma}(\underline{K}F)\|_{L^2(S)}\lesssim\check{K}_{\gamma_0}(\|\nabla F\|_{L^2(S)}+\|t^{-1}F\|_{L^2(S)}).\tag{4.11}$$

For $m,l>0$ there hold

$$\|\Lambda^{-\gamma}(\underline{K}P_mF)\|_{L^2(S)}\lesssim\check{K}_{\gamma_0}2^ms^{-1}\|P_mF\|_{L^2(S)}\tag{4.12}$$

[3] The low frequency are the terms similar to $A_{kl}^{(1)}$ and $A_{kl}^{(2)}$ with l in the expressions replaced by $\leqslant 0$. These terms are actually much easier to estimate. We omit the detail.

and

$$\big\| P_m\big(\underline{K}\mathcal{D}^{-1}P_l F\big)\big\|_{L^2(S)} \lesssim \check{K}_{\gamma_0}2^{\gamma m}s^{-\gamma_0}\|P_l F\|_{L^2(S)}, \tag{4.13}$$

where $\mathcal{D}$ is an elliptic Hodge operator.

We remark that the $\check{K}_{\gamma_0}$ in Lemma 4.4 and Proposition 4.5 should be replaced by $\check{K}_{\gamma_0} + \Delta_0$, where the presence of Δ_0 is due to the difference $r^{-2} - s^{-2}$ which is relatively trivial in the calculation. We can simply ignore Δ_0, however, without hurting the proof of Theorem 4.1.

Proof. We first show (4.11). It is clear that

$$\big\|\Lambda^{-\gamma}(\underline{K}\cdot F)\big\|_{L_x^2} \lesssim \big\|\Lambda^{-\gamma}(\check{K}\cdot F)\big\|_{L_x^2} + |r^{-2}-s^{-2}|\|\Lambda^{-\gamma}F\|_{L_x^2}. \tag{4.14}$$

Due to Propositions 3.1 and 3.2(ii), for any $0 < s \leqslant 1$

$$|r^{-2}-s^{-2}|\|\Lambda^{-\gamma}F\|_{L_x^2} \lesssim \Delta_0\|s^{-1+\gamma}F\|_{L_x^2},$$

which obviously is a lower order term. Hence, it only remains to estimate the first term on the right-hand side of (4.14). By Proposition 3.2

$$\big\|\Lambda^{-\gamma}(\check{K}\cdot F)\big\|_{L_x^2}^2 \approx \sum_{m>0} 2^{-2m\gamma}\big\|s^\gamma P_m(\check{K}\cdot F)\big\|_{L_x^2}^2 + \big\|s^\gamma P_{\leqslant 0}(\check{K}\cdot F)\big\|_{L_x^2}^2. \tag{4.15}$$

In order to estimate the low frequency term $s^{\gamma_0}\|P_{\leqslant 0}(\check{K}\cdot F)\|_{L_x^2}$, we use Proposition 3.2 to obtain for any appropriate S tangent tensor field G,

$$\begin{aligned}\big\langle s^{\gamma_0}P_{\leqslant 0}(\check{K}\cdot F), G\big\rangle &= \langle \check{K}\cdot F, P_{\leqslant 0}G\rangle \leqslant \|\Lambda^{-\gamma_0}\check{K}\|_{L_x^2}\cdot \big\|s^{\gamma_0}\Lambda^{\gamma_0}(F\cdot P_{\leqslant 0}G)\big\|_{L_x^2}\\ &\lesssim \check{K}_{\gamma_0}\big(s^{\gamma_0}\|\Lambda^{\gamma_0}F\|_{L_x^2}\|\Lambda P_{\leqslant 0}G\|_{L_x^2} + s^{\gamma_0}\|\Lambda F\|_{L_x^2}\|\Lambda^{\gamma_0}P_{\leqslant 0}G\|_{L_x^2}\big)\\ &\lesssim \check{K}_{\gamma_0}\|\Lambda F\|_{L_x^2}\|G\|_{L_x^2}.\end{aligned}$$

Hence

$$\big\|s^{\gamma_0}P_{\leqslant 0}(\check{K}\cdot F)\big\|_{L_x^2} \lesssim \check{K}_{\gamma_0}\big(\|\nabla F\|_{L_x^2} + s^{-1}\|F\|_{L_x^2}\big).$$

In order to estimate the first term on the right-hand side of (4.15), by the GLP decomposition we write $\check{K} = \sum_n P_n^2\check{K} := \sum_{n\in\mathbb{N}}\check{K}_n + \check{K}_{\leqslant 0}$, then

$$\big\|P_m(\check{K}\cdot F)\big\|_{L_x^2} \leqslant \sum_{n\in\mathbb{N}}\big\|P_m(\check{K}_n\cdot F)\big\|_{L_x^2} + \big\|P_m(\check{K}_{\leqslant 0}\cdot F)\big\|_{L_x^2}. \tag{4.16}$$

For any $2 < p < \infty$ let p^* satisfy $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{2}$. We will employ the finite band property, the weak and sharp Bernstein inequalities for the GLP projections, the Sobolev inequalities and the Young's inequality. For the low frequency in (4.16), we choose p such that $\frac{2}{p} < \gamma_0$, then for any appropriate tensor field G

$$\begin{aligned}\big\langle P_m(\check{K}_{\leqslant 0}\cdot F), G\big\rangle &= \langle \check{K}_{\leqslant 0}\cdot F, P_m G\rangle \lesssim \|\check{K}_{\leqslant 0}\|_{L_x^2}\|F\|_{L_x^p}\|P_m G\|_{L_x^{p^*}}\\ &\lesssim \|\check{K}_{\leqslant 0}\|_{L_x^2}2^{\frac{2m}{p}}s^{-\frac{2m}{p}}\big(\|\nabla F\|_{L_x^2}^{1-\frac{2}{p}}\|s^{-1}F\|_{L_x^2}^{\frac{2}{p}} + \|s^{-1}F\|_{L_x^2}\big)\|G\|_{L_x^2}.\end{aligned}$$

Therefore

$$\sum_{m>0}2^{-\gamma m}\big\|s^\gamma P_m(\check{K}_{\leqslant 0}\cdot F)\big\|_{L_x^2} \lesssim \Delta_0\big(\|\nabla F\|_{L_x^2} + \|s^{-1}F\|_{L_x^2}\big). \tag{4.17}$$

Now it remains to estimate $\|P_m(\check{K}_n\cdot F)\|_{L_x^2}$. When $m > n$, we choose p_1 such that $\frac{2}{p_1} < \gamma - \gamma_0$. Then

$$\begin{aligned}\|P_m(\check K_n\cdot F)\|_{L_x^2}&\lesssim 2^{-m}s\|\nabla(\check K_n\cdot F)\|_{L_x^2}\lesssim 2^{-m}s\big(\|\nabla\check K_n\|_{L_x^{p_1^*}}\|F\|_{L_x^{p_1}}+\|\check K_n\|_{L_x^\infty}\|\nabla F\|_{L_x^2}\big)\\&\lesssim 2^{-m+n+\frac{2n}{p_1}}\|\check K_n\|_{L_x^2}\big(\|\nabla F\|_{L_x^2}+\|s^{-1}F\|_{L_x^2}\big)+2^{-m+n}\|\check K_n\|_{L_x^2}\|\nabla F\|_{L_x^2}\\&\lesssim 2^{-m+n+\frac{2n}{p_1}}\|\check K_n\|_{L_x^2}\big(\|\nabla F\|_{L_x^2}+\|s^{-1}F\|_{L_x^2}\big).\end{aligned}$$

When $m<n$, choose p_2 such that $1-\frac{2}{p_2}<\gamma-\gamma_0$. Then

$$\begin{aligned}\|P_m(\check K_n\cdot F)\|_{L_x^2}&=2^{-2n}s^2\|P_m(\Delta\check K_n\cdot F)\|_{L_x^2}\\&\lesssim 2^{-2n}s^2\|P_m\operatorname{div}(\nabla\check K_n\cdot F)\|_{L_x^2}+2^{-2n}s^2\|P_m(\nabla\check K_n\cdot\nabla F)\|_{L_x^2}\\&\lesssim 2^{-2n+m}s\|\nabla\check K_n\cdot F\|_{L_x^2}+2^{-2n+\frac{3m}{4}}s^{5/4}\|\nabla\check K_n\cdot\nabla F\|_{L_x^{\frac{8}{7}}}\\&\lesssim 2^{-2n+m}s\|\nabla\check K_n\|_{L_x^{p_2}}\|F\|_{L_x^{p_2^*}}+2^{-2n+\frac{3m}{4}}s^{5/4}\|\nabla F\|_{L_x^2}\|\nabla\check K_n\|_{L_x^{\frac{8}{3}}}\\&\lesssim 2^{m-\frac{2n}{p_2}}\|\check K_n\|_{L_x^2}\|\nabla F\|_{L_x^2}^{1-\frac{2}{p_2^*}}\|s^{-1}F\|_{L_x^2}^{\frac{2}{p_2^*}}+2^{\frac{3}{4}(m-n)}\|\nabla F\|_{L_x^2}\|\check K_n\|_{L_x^2}\\&\lesssim 2^{m-\frac{2n}{p_2}}\|\check K_n\|_{L_x^2}\big(\|\nabla F\|_{L_x^2}+\|s^{-1}F\|_{L_x^2}\big).\end{aligned}$$

Therefore, by using Proposition 3.2(vi), we obtain

$$\begin{aligned}&\sum_{m,n>0}2^{-m\gamma}s^\gamma\|P_m(\check K\cdot F)\|_{L_x^2}\\&\quad\lesssim\sum_{m>n>0}2^{-(\gamma-\gamma_0-\frac{2}{p_1})n}2^{(1+\gamma)(n-m)}\|s^{\gamma_0}2^{-\gamma_0 n}\check K_n\|_{L_x^2}\big(\|\nabla F\|_{L_x^2}+\|t^{-1}F\|_{L_x^2}\big)\\&\qquad+\sum_{n>m>0}2^{(1-\frac{2}{p_2}-\gamma+\gamma_0)n}2^{(1-\gamma)(m-n)}s^{\gamma-\gamma_0}\|2^{-\gamma_0 n}s^{\gamma_0}\check K_n\|_{L_x^2}\big(\|\nabla F\|_{L_x^2}+\|s^{-1}F\|_{L_x^2}\big)\\&\quad\lesssim\check K_{\gamma_0}\big(\|\nabla F\|_{L^2(S)}+\|s^{-1}F\|_{L^2(S)}\big).\end{aligned}$$

The proof of (4.11) is thus complete.

Now (4.12) follows by applying (4.11) to P_mF and using the finite band property. In order to show (4.13), we write

$$P_m\big(\underline{K}\mathcal{D}^{-1}P_lF\big)=P_m(\check K\mathcal{D}^{-1}P_lF)+(s^{-2}-r^{-2})P_m\mathcal{D}^{-1}P_lF.$$

Then (4.13) follows by estimating the first term as we did for $\|P_m(\check K\cdot F)\|_{L_x^2}$ in the above and applying Propositions 3.1 and 4.1 to the second term. □

Now we are ready to prove Propositions 4.5 and 4.6.

Proof of Proposition 4.5. We may assume F is a 1-form without loss of generality. By duality we can assume that $k\geqslant m$. Then from the finite band property we have

$$\|P_k\nabla P_mF\|_{L_t^2L_x^2}\lesssim 2^{-2k}\|t^2P_k\Delta\nabla P_mF\|_{L_t^2L_x^2}.$$

We denote by $\underline{R}$ the Riemann curvature tensors on 2-surface (S_s,γ). Then

$$\underline{R}_{abcd}=(\gamma_{ac}\gamma_{bd}-\gamma_{ad}\gamma_{bc})K\quad\text{and}\quad\underline{R}_{ab}=\gamma_{ab}K.$$

Note that the commutation formula

$$\nabla_a(\Delta F_i)=\Delta(\nabla_aF_i)+\nabla_c(\underline{R}_{idac}F_d)-\underline{R}_{da}\nabla_dF_i+\underline{R}_{idac}\nabla_cF_d.$$

Then we have $\|P_k\nabla P_mF\|_{L_t^2L_x^2}\lesssim\mathcal{A}_1+\mathcal{A}_2$, where

$$\mathcal{A}_1 := 2^{-2k}\big(\|t^2 P_k\nabla\Delta P_m F\|_{L_t^2L_x^2} + \|P_k\nabla P_m F\|_{L_t^2L_x^2}\big),$$
$$\mathcal{A}_2 := 2^{-2k}\|t^2 P_k\nabla(\underline{K}P_m F)\|_{L_t^2L_x^2} + 2^{-2k}\|t^2 P_k(\underline{K}\nabla P_m F)\|_{L_t^2L_x^2}.$$

It is easy to see

$$\mathcal{A}_1 \lesssim 2^{-2(k-m)}\|P_k\nabla P_m F\|_{L_t^2L_x^2} + 2^{-2k+m}\|t^{-1}P_m F\|_{L_t^2L_x^2}$$
$$\lesssim 2^{-2(k-m)}2^m\|t^{-1}P_m F\|_{L_t^2L_x^2}.$$

In order to estimate $\mathcal{A}_2$, it follows from Lemmas 4.3 and 4.4 that

$$\|P_k\nabla(\underline{K}P_m F)\|_{L_x^2} \lesssim K(1+\gamma,k)\|\Lambda^{-\gamma}(\underline{K}P_m F)\|_{L_x^2} \lesssim 2^m s^{-1}K(1+\gamma,k)\|P_m F\|_{L_x^2}\check{K}_{\gamma_0}.$$

Thus, by a standard dual argument, we obtain for any appropriate S-tangent tensor field G

$$\langle P_k(\underline{K}\cdot\nabla P_m F), G\rangle = \langle P_m\,\mathrm{div}(\underline{K}\cdot P_k G), P_m F\rangle \lesssim \|P_m\,\mathrm{div}(\underline{K}\cdot P_k G)\|_{L_x^2}\|P_m F\|_{L_x^2}$$
$$\lesssim 2^k s^{-1}K(1+\gamma,m)\check{K}_{\gamma_0}\|P_k G\|_{L_x^2}\|P_m F\|_{L_x^2}$$
$$\lesssim 2^m s^{-1}K(1+\gamma,k)\check{K}_{\gamma_0}\|P_m F\|_{L_x^2}\|G\|_{L_x^2}.$$

The last inequality is derived by using $k \geqslant m$. Hence,

$$\mathcal{A}_2 \lesssim 2^{-(1-\gamma)k}2^m\check{K}_{\gamma_0}\|t^{-\gamma}P_m F\|_{L_t^2L_x^2} + 2^{m-k}\check{K}_{\gamma_0}\|\underline{K}\|^{\gamma}_{L_t^2L_x^2}\|P_m F\|_{L_t^{\frac{2}{1-\gamma}}L_x^2}.$$

The proof of (4.7) is complete by combining the estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$. The inequality (4.8) can be proved similarly. □

Proof of Proposition 4.6. For the case $k \geqslant l$, we use the finite band property for GLP projections, the representation formula $\Delta = {}^\star\mathcal{D}\mathcal{D} \pm \underline{K} \pm t^{-2}\mathrm{Id}$ and (4.13) in Lemma 4.4 to obtain

$$\|P_k\mathcal{D}^{-1}P_l F\|_{L^2(S)} \lesssim 2^{-2k}t^2\|P_k\Delta\mathcal{D}^{-1}P_l F\|_{L^2(S)}$$
$$\lesssim 2^{-2k}t^2\|P_k{}^\star\mathcal{D}P_l F\|_{L^2(S)} + 2^{-2k}t^2\|P_k(\underline{K}\mathcal{D}^{-1}P_l F)\|_{L^2(S)} + 2^{-2k}\|P_k\mathcal{D}^{-1}P_l F\|_{L^2(S)}$$
$$\lesssim 2^{-2k}2^l t\|P_l F\|_{L^2(S)} + 2^{-2k+\gamma k}t^{2-\gamma_0}\check{K}_{\gamma_0}\|P_l F\|_{L^2(S)} + 2^{-3k}t\|P_l F\|_{L^2(S)}.$$

Then the desired estimate follows for this case.

In order to show the result for the case $0 < k < l$, we note that

$$\|P_k\mathcal{D}^{-1}(\underline{K}P_l F)\|_{L_x^2} \lesssim 2^{\gamma l}t^{-\gamma_0}\check{K}_{\gamma_0}\|P_l F\|_{L_x^2} \tag{4.18}$$

which follows from (4.13) in Lemma 4.4 by a dual argument. Similar as above, we may use the representation formula $\Delta = \mathcal{D}^\star\mathcal{D} \pm K$ to complete the proof. □

5. Product estimates in Besov norms

We will provide a series of product estimates in Besov norms which are of fundamental importance for the later applications.

5.1. Non-sharp product estimates

The following non-sharp product estimates will be used in Section 6.

Proposition 5.1. *For any S-tangent tensor fields F and G,*

$$\|F\cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F)\big(\|t^{-\frac{1}{a}}G\|_{L_t^aL_x^2} + \|t^{\frac12}\nabla G\|_{L_t^2L_x^2}\big) \text{ with } a > 4, \tag{5.1}$$
$$\|F\cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(t^{1/2}F)\|G\|_{\mathcal{P}^0}, \tag{5.2}$$
$$\|F\cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(t^{\frac12}F)\big(\|\nabla G\|_{L_t^2L_x^2} + \|G\|_{L_\omega^\infty L_t^2}\big). \tag{5.3}$$

Before giving the proof, we recall the notion of $\mathcal{N}_1$-envelopes of tensor fields introduced in [11].

Definition 5.1. For a given S-tangent tensor field F and a sufficiently small $\epsilon > 0$ we define its $\mathcal{N}_1$-envelope (of order ϵ) to be any sequence of positive real numbers $\mathcal{N}_1[F_k]$ satisfying

$$\begin{aligned}&\mathcal{N}_1[F_k] \lesssim 2^{\epsilon|k-k'|}\mathcal{N}_1[F_{k'}], \quad \text{for any } k, k',\\ &\sum_k \mathcal{N}_1[F_k]^2 \approx \mathcal{N}_1(F)^2.\end{aligned} \tag{5.4}$$

By the same way as in [11, page 31–33], we can obtain the following result whose proof can be found in [22].

Lemma 5.1. *For any smooth S-tangent tensor field F there always exists an envelope $\mathcal{N}_1[F_k]$ such that*

(i) *For $2 \leqslant q \leqslant \infty$, there hold the dyadic Gagliardo–Nirenberg inequalities*

$$\big\|t^{-\frac{1}{2}-\frac{1}{q}} F_k\big\|_{L_t^q L_x^2} \lesssim 2^{-\frac{1}{2}k-\frac{1}{q}k}\mathcal{N}_1[F_k], \qquad \big\|t^{-1/2} F_k\big\|_{L_x^2 L_t^\infty} \lesssim 2^{-k/2}\mathcal{N}_1[F_k]. \tag{5.5}$$

(ii) *For all q with $1 \leqslant q < 2$ sufficiently close to 2,*

$$\big\|t^{\frac{1}{2}-\frac{1}{q}+}\nabla_L F_k\big\|_{L_t^q L_x^2} \lesssim \mathcal{N}_1[F_k]. \tag{5.6}$$

(iii) *For any $2 \leqslant q < 4$,*

$$\big\|t^{1-\frac{1}{q}}\nabla F_k\big\|_{L_t^q L_x^4} \lesssim 2^{k(1-\frac{1}{q})}\mathcal{N}_1[F_k]. \tag{5.7}$$

(iv) *For any $2 \leqslant q < \infty$,*

$$\big\|t^{\frac{1}{2}-\frac{1}{q}} F_k\big\|_{L_t^q L_x^\infty} \lesssim 2^{k(\frac{1}{2}-\frac{1}{q})}\mathcal{N}_1[F_k]. \tag{5.8}$$

Remark 5.1. The above dyadic inequalities can be adapted to low frequency terms $F_{\leqslant 0} = P_{\leqslant 0}F$, for instance

$$\begin{aligned}&\big|t^{-\frac{1}{2}-\frac{1}{q}} F_{\leqslant 0}\big\|_{L_t^q L_x^2} \lesssim \mathcal{N}_1(F), \qquad \big\|t^{-1/2}F_{\leqslant 0}\big\|_{L_x^2 L_t^\infty} \lesssim \mathcal{N}_1(F),\\ &\big\|t^{1-\frac{1}{q}}\nabla F_{\leqslant 0}\big\|_{L_t^q L_x^4} \lesssim \mathcal{N}_1(F), \quad 2 \leqslant q < 4,\\ &\big\|t^{\frac{1}{2}-\frac{1}{q}} F_{\leqslant 0}\big\|_{L_t^q L_x^\infty} \lesssim \mathcal{N}_1(F), \quad 2 \leqslant q < \infty.\end{aligned}$$

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. In what follows we will prove (5.1) only. (5.2) and (5.3), which have been stated in [9] without proof, can be proved in a similar but easier way (see [22, Section 5.2]). We will frequently use Lemma 4.1, (5.7), (5.5), (5.4) and (3.12).

We choose λ and b such that

$$2 \leqslant b < 4, \quad \frac{2}{a+2} < \lambda \leqslant 1, \quad \text{and} \quad \frac{1-\lambda}{a} + \frac{\lambda}{2} = \frac{1}{b^*},$$

where $\frac{1}{b} + \frac{1}{b^*} = \frac{1}{2}$. This is always possible for $a > 4$. Moreover

$$\frac{1}{2} - \lambda - \frac{1}{b} < 0. \tag{5.9}$$

We will use the notations

$$\Pi_n^\lambda := \big\|t^{\frac{1}{2}}\nabla G_n\big\|_{L^2(\mathcal{H})}^\lambda \big\|t^{-\frac{1}{a}} G\big\|_{L_t^a L_x^2}^{1-\lambda}, \qquad \Pi^\lambda := \big\|t^{\frac{1}{2}}\nabla G\big\|_{L^2(\mathcal{H})}^\lambda \big\|t^{-\frac{1}{a}} G\big\|_{L_t^a L_x^2}^{1-\lambda}.$$

Clearly, by interpolation and Bernstein inequality

$$\big\|t^{\frac{1}{b}+\frac{1}{2}}\nabla G_n\big\|_{L_t^{b*} L_x^2} \lesssim \big\|t^{\frac{1}{2}}\nabla G_n\big\|_{L^2(\mathcal{H})}^\lambda \big\|t^{1-\frac{1}{a}}\nabla G_n\big\|_{L_t^a L_x^2}^{1-\lambda} \lesssim 2^{(1-\lambda)n}\Pi_n^\lambda \tag{5.10}$$

and

$$\|t^{\frac{1}{b}-\frac{1}{2}}G_n\|_{L_t^{b*}L_x^2} \lesssim 2^{-\lambda n}\Pi_n^\lambda, \qquad \|t^{\frac{1}{2}+\frac{1}{b}-\frac{2}{p}}G_n\|_{L_t^{b*}L_x^p} \lesssim 2^{(1-\frac{2}{p}-\lambda)n}\Pi_n^\lambda. \tag{5.11}$$

Similarly, we have

$$\|t^{\frac{1}{b}+\frac{1}{2}}\nabla G_{\leqslant 0}\|_{L_t^{b*}L_x^2} \lesssim \Pi^\lambda, \qquad \|t^{\frac{1}{b}-\frac{1}{2}}G_{\leqslant 0}\|_{L_t^{b*}L_x^2} \lesssim \Pi^\lambda. \tag{5.12}$$

Now we prove (5.1). We begin with expanding $F\cdot G$ by GLP decomposition as

$$F\cdot G = F_{>0}\cdot G_{>0} + D(F,G),$$

where

$$D(F,G) = F_{\leqslant 0}\cdot G_{\leqslant 0} + F_{\leqslant 0}\cdot G_{>0} + F_{>0}\cdot G_{\leqslant 0}.$$

We first estimate $\|F_{>0}\cdot G_{>0}\|_{\mathcal{P}^0}$. Set $\frac{1}{q}+\frac{1}{q*}=\frac{1}{2}$ with $q<4$, by the Hölder inequality and Lemma 5.1 we obtain

$$\|F_m\cdot G_n\|_{L^2(\mathcal{H})} \lesssim \|F_m\|_{L_t^\infty L_x^q}\|G_n\|_{L_t^2 L_x^{q*}} \lesssim 2^{(\frac{1}{2}-\frac{2}{q})m}2^{-\frac{2n}{q*}}\mathcal{N}_1[F_m]\|t^{1/2}\nabla G\|_{L_t^2L_x^2}.$$

Summing over $m,n>0$ gives

$$\|F_{>0}\cdot G_{>0}\|_{L^2(\mathcal{H})} \lesssim \mathcal{N}_1(F)\|t^{1/2}\nabla G\|_{L^2(\mathcal{H})}.$$

It remains to consider the following high frequency terms

$$\begin{aligned}P_k(F_{>0}\cdot G_{>0}) &= \sum_{k>n,m>0} P_k(F_m\cdot G_n) + \sum_{k<n,m} P_k(F_m\cdot G_n) + \Big(\sum_{n<k<m} + \sum_{m<k<n}\Big)P_k(F_m\cdot G_n)\\ &= \mathcal{A}_1^k + \mathcal{A}_2^k + \mathcal{A}_3^k.\end{aligned}$$

Step 1. Estimate of $\mathcal{A}_1^k$. By finite band property and the weak Bernstein inequality of GLP projections, we have

$$\|\mathcal{A}_1^k\|_{L_t^2L_x^2} \leqslant \sum_{k>n,m>0} 2^{-2k}\|t^2 P_k\Delta(F_m\cdot G_n)\|_{L_t^2L_x^2} \lesssim \sum_{k>n,m>0}\big(A_{mn}^k + 2^{\frac{-3k}{2}}B_{mn}\big) \tag{5.13}$$

where for $m,n,k>0$,

$$A_{mn}^k = 2^{-2k+2\max(m,n)}\|P_k(F_m\cdot G_n)\|_{L_t^2L_x^2} \quad\text{and}\quad B_{mn} = \|t^{\frac{3}{2}}\nabla F_m\cdot\nabla G_n\|_{L_t^2 L_x^{\frac{4}{3}}}.$$

Using (5.10) and (5.7),

$$B_{mn} \lesssim \|t^{1-\frac{1}{b}}\nabla F_m\|_{L_t^bL_x^4}\|t^{\frac{1}{2}+\frac{1}{b}}\nabla G_n\|_{L_t^{b*}L_x^2} \lesssim 2^{(1-\frac{1}{b})m+(1-\lambda)n}\mathcal{N}_1[F_m]\Pi_n^\lambda. \tag{5.14}$$

As to A_{mn}^k, by weak Bernstein inequality, (3.12), (5.10) and (5.11)

$$\begin{aligned}A_{mn}^k &\lesssim 2^{-2k+2\max(m,n)}\|t^{\frac{1}{2}-\frac{1}{b}-\frac{2}{q}}F_m\|_{L_t^bL_x^q}\|t^{\frac{1}{b}+\frac{2}{q}-\frac{1}{2}}G_n\|_{L_t^{b*}L_x^{q*}}\\ &\lesssim 2^{-2k+2\max(m,n)+(\frac{1}{2}-\frac{1}{b}-\frac{2}{q})m}2^{(1-\lambda-\frac{2}{q*})n}\mathcal{N}_1(F)\Pi_n^\lambda.\end{aligned} \tag{5.15}$$

Due to (5.9), choosing $q>2$ such that $\frac{1}{2}-\frac{1}{b}<\frac{2}{q}<\lambda$, combining (5.13) with (5.14) and (5.15), we conclude

$$\sum_{k>0}\|\mathcal{A}_1^k\|_{L_t^2L_x^2} \lesssim \mathcal{N}_1(F)\big(\|t^{\frac{1}{2}}\nabla G\|_{L_t^2L_x^2} + \|t^{-\frac{1}{a}}G\|_{L_t^aL_x^2}\big).$$

Step 2. Estimate of $\mathcal{A}_2^k$. Using the weak Bernstein inequality and (5.5), we have

$$\|P_k(F_m\cdot G_n)\|_{L_t^2L_x^2} \lesssim 2^{\frac{k}{2}}\|t^{-1/2}F_m\|_{L_t^\infty L_x^2}\|G_n\|_{L_t^2L_x^4} \lesssim 2^{\frac{k-m-n}{2}}\mathcal{N}_1[F_m]\|t^{1/2}\nabla G\|_{L_t^2L_x^2}.$$

Summing over $k<m,n$ for the respective case, we conclude

$$\sum_{k>0}\|\mathcal{A}_2^k\|_{L_t^2L_x^2} \lesssim \mathcal{N}_1(F)\|t^{\frac{1}{2}}\nabla G\|_{L_t^2L_x^2}.$$

Step 3. Estimate of $\mathcal{A}_3^k$. We first consider the case $n < k < m$ by using finite band property

$$\begin{aligned}\|P_k(F_m \cdot G_n)\|_{L_t^2 L_x^2} &\lesssim 2^{-2m} \|t^2 P_k(\Delta F_m \cdot G_n)\|_{L_t^2 L_x^2} \\ &\lesssim 2^{-2m}\big(\|t^2 P_k \nabla(\nabla F_m \cdot G_n)\|_{L_t^2 L_x^2} + \|t^2 P_k(\nabla F_m \cdot \nabla G_n)\|_{L_t^2 L_x^2}\big) \\ &\lesssim 2^{-2m}\big(2^k \|t \nabla F_m \cdot G_n\|_{L_t^2 L_x^2} + 2^{\frac{k}{2}} B_{mn}\big).\end{aligned}$$

For the first term, using Hölder inequality, (3.12), (5.10) and (5.11), it can be bounded by

$$2^{-2m+k} \|t^{1-\frac{1}{b}} \nabla F_m\|_{L_t^b L_x^4} \|t^{\frac{1}{b}} G_n\|_{L_t^{b*} L_x^4} \lesssim 2^{-2m+k+(1-\frac{1}{b})m+\frac{n}{2}-\lambda n} \mathcal{N}_1[F_m] \Pi_n^\lambda.$$

Combined with (5.14), it implies

$$\|P_k(F_m \cdot G_n)\|_{L_t^2 L_x^2} \lesssim \big(2^{-2m+k+(1-\frac{1}{b})m+\frac{n}{2}-\lambda n} + 2^{-2m+\frac{k}{2}+(1-\frac{1}{b})m+n-\lambda n}\big) \mathcal{N}_1[F_m] \Pi_n^\lambda.$$

Summing over $n < k < m$ gives

$$\sum_{n<k<m} \|P_k(F_m \cdot G_n)\|_{L_t^2 L_x^2} \lesssim \mathcal{N}_1(F)\big(\|t^{\frac{1}{2}} \nabla G\|_{L_t^2 L_x^2} + \|t^{-\frac{1}{a}} G\|_{L_t^a L_x^2}\big).$$

It remains to consider the last case $m < k < n$. With $1 < p < 2$ and $\frac{1}{p*} + \frac{1}{2} = \frac{1}{p}$, we have from the finite band property and the weak Bernstein inequality that

$$\begin{aligned}\|P_k(F_m \cdot G_n)\|_{L_x^2} &\lesssim 2^{k(\frac{2}{p}-1)} t^{1-\frac{2}{p}} \|F_m \cdot G_n\|_{L_x^p} \lesssim 2^{k(\frac{2}{p}-1)} t^{1-\frac{2}{p}} \|F_m\|_{L_x^{p*}} \|G_n\|_{L_x^2} \\ &\lesssim 2^{k(\frac{2}{p}-1)} 2^{m(1-\frac{2}{p*})-n} \|F_m\|_{L_x^2} \|\nabla G\|_{L_x^2}.\end{aligned}$$

By using (5.5) and (5.4), we infer

$$\begin{aligned}\sum_{n>k>m} \|P_k(F_m \cdot G_n)\|_{L_t^2 L_x^2} &\lesssim \sum_{n>k>m} 2^{k(\frac{2}{p}-1)} 2^{m(\frac{3}{2}-\frac{2}{p})-n} \mathcal{N}_1[F_m] \|t^{1/2} \nabla G\|_{L^2(\mathcal{H})} \\ &\lesssim \mathcal{N}_1(F) \|t^{\frac{1}{2}} \nabla G\|_{L_t^2 L_x^2}.\end{aligned}$$

Therefore

$$\sum_{k>0} \|\mathcal{A}_3^k\|_{L_t^2 L_x^2} \lesssim \mathcal{N}_1(F)\big(\|t^{\frac{1}{2}} \nabla G\|_{L_t^2 L_x^2} + \|t^{-\frac{1}{a}} G\|_{L_t^a L_x^2}\big).$$

Step 4. Finally we need to show for the low frequency term $D(F, G)$ that

$$\|D(F, G)\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F)\big(\|t^{\frac{1}{2}} \nabla G\|_{L^2(\mathcal{H})} + \|t^{-\frac{1}{a}} G\|_{L_t^a L_x^2}\big). \tag{5.16}$$

The estimates for $F_{\leqslant 0} \cdot G_{>0}$ and $F_{>0} \cdot G_{\leqslant 0}$ can be derived by adapting the arguments in step 1 and step 3. While the estimate for $F_{\leqslant 0} \cdot G_{\leqslant 0}$ can be obtained by using Bernstein inequality and Hölder inequality. We omit the details. □

5.2. Transport product estimates

The main purpose of this subsection is to provide the transport-product estimates. We will always assume the bootstrap assumptions (BA1) and (BA2). In the following theorems, $k > 0$ is a given number.

Theorem 5.1. *Assume that the S-tangent tensor fields W, F and G satisfy the transport equation* $\nabla_L W + \frac{k}{s} W = \nabla_L F \cdot G$ *along* $\mathcal{H}$ *and* $\lim_{s\to 0} s|W| = 0$. *Then*

$$\|W\|_{\mathcal{B}^0} \lesssim \mathcal{N}_1(F) \cdot \big(\mathcal{N}_1(G) + \|G\|_{L_\omega^\infty L_t^2}\big).$$

Theorem 5.2. *Assume that the S-tangent tensor fields W, F and G satisfy the transport equation* $\nabla_L W + \frac{k}{s} W = F \cdot G$ *along* $\mathcal{H}$ *and* $\lim_{s\to 0} s|W| = 0$. *Then*

$$\|W\|_{\mathcal{B}^0} \lesssim \|F\|_{\mathcal{P}^0}\big(\mathcal{N}_1(G) + \|G\|_{L_\omega^\infty L_t^2}\big).$$

Theorem 5.3. *For any pair of S-tangent tensor fields G and W such that W satisfies the transport equation of the form* $\nabla_L W + \frac{k}{s} W = F$ *along* $\mathcal{H}$, *there holds*

$$\|G \cdot W\|_{\mathcal{P}^0} \lesssim \Big(\lim_{s\to 0} \|W\|_{B^0_{2,1}(S)} + \|F\|_{\mathcal{P}^0}\Big)\big(\mathcal{N}_1(G) + \|G\|_{L^\infty_\omega L^2_t}\big).$$

The proof of these results can be carried out by using the reduction argument given in [11, Lemma 4.13]), that is, it is enough to prove these results for the scalar transport equations, with W being a scalar function. After this reduction, the results follow immediately from the sharp trace inequalities which in our situation take the following forms.

Proposition 5.2. *For any S-tangent tensor fields F and G of the same type, there holds*

$$\left\| t^{-1} \int_0^t s \nabla_L F \cdot G\, ds \right\|_{\mathcal{B}^0} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G), \tag{5.17}$$

and for any scalar functions F and G there hold

$$\left\| t^{-1} \int_0^t s F \cdot G\, ds \right\|_{\mathcal{B}^0} \lesssim \big(\mathcal{N}_1(G) + \|G\|_{L^\infty_\omega L^2_t}\big)\|F\|_{\mathcal{P}^0}, \tag{5.18}$$

$$\left\| t^{-1} G \cdot \int_0^t s F\, ds \right\|_{\mathcal{P}^0} \lesssim \big(\mathcal{N}_1(G) + \|G\|_{L^\infty_\omega L^2_t}\big)\|F\|_{\mathcal{P}^0}. \tag{5.19}$$

One can follow essentially the method in [11] to complete the proof. One can also find a proof in [22] by combining the GLP theory and classical LP theory through an equivalence argument. We thus omit the proof.

Theorem 5.4. *Let F be an S-tangent tensor field which admits a decomposition of the form* $\nabla F = \nabla_L P + E$ *with tensor fields P and E of the same type. If* $|s^{0-} F|$ *and* $|s\nabla F|$ *are uniformly bounded when* $s \to 0$, *then*

$$\|F\|_{L^\infty_\omega L^2_t} \lesssim \mathcal{N}_1(F) + \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0}.$$

Proof. We set $\varphi(t) = \int_0^t |F|^2\, ds$, then $\nabla_L \varphi = |F|^2$. Due to [9, Proposition 5.1] we have

$$\|\varphi\|_{L^\infty(\mathcal{H})} \lesssim \|\nabla\varphi\|_{\mathcal{B}^0} + \|t^{-1}\varphi\|_{L^\infty_t L^2_x}. \tag{5.20}$$

It is easy to see

$$\|t^{-1}\varphi\|_{L^\infty_t L^2_x} \lesssim \|F\|_{L^\infty_\omega L^2_t} \cdot \|t^{-1}F\|_{L^2_t L^2_x}. \tag{5.21}$$

We now estimate $\|\nabla\varphi\|_{\mathcal{B}^0}$. In view of the commutation formula $[\nabla_L, \nabla]\varphi = -\chi \cdot \nabla\varphi$, we know $\nabla\varphi$ satisfies the transport equation

$$\nabla_L \nabla\varphi + \frac{1}{s}\nabla\varphi = 2F \cdot \nabla F - \frac{1}{2} V \cdot \nabla\varphi - \hat{\chi} \cdot \nabla\varphi. \tag{5.22}$$

In order to apply Theorem 5.1 and Theorem 5.2, we need to show $\lim_{s\to 0} s|\nabla\varphi| = 0$. By the weakly spherical regularity of $(S, \mathring{\gamma})$ proved in Lemma 3.2, we only need to show $\lim_{s\to 0} \frac{\partial\varphi}{\partial\omega_i} = 0$ under the transport coordinate (s, ω_1, ω_2), where $i = 1, 2$. Note that

$$\frac{\partial\varphi}{\partial\omega_i} = 2\int_0^t \big\langle \nabla_{\frac{\partial}{\partial\omega_i}} F, F\big\rangle\, ds,$$

we infer

$$\left|\frac{\partial\varphi}{\partial\omega_i}\right| \lesssim \left|\int_0^t \big\langle \nabla_{\frac{\partial}{\partial\omega_i}} F, F\big\rangle\, ds\right| \lesssim \sup_{0<s\leqslant t} |s^{0-}F| \cdot |s\nabla_{\frac{\partial}{\partial\omega_i}} F| \left|\int_0^t s^{-1+}\, ds\right|. \tag{5.23}$$

From the conditions on F, we conclude $|\frac{\partial\varphi}{\partial\omega_i}| \to 0$ as $s \to 0$.

Now we substitute the decomposition $\nabla F = \nabla_L P + E$ into (5.22) and use Theorem 5.1 and Theorem 5.2 to conclude

$$\|\nabla\varphi\|_{\mathcal{B}^0} \lesssim (\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0})(\mathcal{N}_1(F) + \|F\|_{L^\infty_\omega L^2_t}) + (\mathcal{N}_1(A) + \|A\|_{L^\infty_\omega L^2_t})\|\nabla\varphi\|_{\mathcal{P}^0}.$$

This inequality, together with (BA1) and the fact $\|\nabla\varphi\|_{\mathcal{P}^0} \lesssim \|\nabla\varphi\|_{\mathcal{B}^0}$ yield

$$\|\nabla\varphi\|_{\mathcal{B}^0} \lesssim (\mathcal{N}_1(F) + \|F\|_{L^\infty_\omega L^2_t})(\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0}).$$

Combining the above inequality with (5.20) and (5.21), we get

$$\|F\|^2_{L^\infty_\omega L^2_t} \lesssim (\mathcal{N}_1(F) + \|F\|_{L^\infty_\omega L^2_t})(\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0}) + \|F\|_{L^\infty_\omega L^2_t} \cdot \|t^{-1}F\|_{L^2_t L^2_x}$$

which implies the desired result. □

In Theorem 5.4, we require F to satisfy certain initial conditions. Note that we will only apply this theorem to $\hat{\chi}$ and ζ to derive their $L^\infty_\omega L^2_t$ estimates. For $F = \hat{\chi}$ and ζ, in view of Proposition 2.1, there hold $|s^{0-}F| \to 0$ and $|s\nabla F| \to 0$ when $s \to 0$. Thus the initial conditions in Theorem 5.4 are satisfied. In Section 7 where the calculations for $\|\hat{\chi}\|_{L^\infty_\omega L^2_t}$ and $\|\zeta\|_{L^\infty_\omega L^2_t}$ are carried out, we will not mention the initial condition any more.

6. Error estimates

In this section, we study various kinds of error terms which arise from commuting ∇_L with Hodge operators. Recall that we have introduced the conventions R, $\check{R}$, R_0, A, $\underline{A}$ and ∇A in Section 3. Then the null Bianchi equations (2.17), (2.18) and (2.19) can be symbolically written as

$$L(\check{\rho}, -\check{\sigma}) = \mathcal{D}_1\beta + s^{-1}\check{R} + A \cdot \bar{R}, \tag{6.1}$$

$$\nabla_L\underline{\check{\beta}} = {}^\star\mathcal{D}_1(\rho, \sigma) + s^{-1}\check{R} + A \cdot \bar{R}, \tag{6.2}$$

where

$$\bar{R} := R_0 + \nabla A + A \cdot \underline{A} + s^{-1}\underline{A}.$$

It follows easily from (BA1) and (BA2) that

$$\|\bar{R}\|_{L^2(\mathcal{H})} \leqslant \mathcal{R}_0 + \Delta_0. \tag{6.3}$$

We will also use the commutation formulas given in [9, Proposition 2.16] which symbolically can be written as

$$[\nabla_L, \nabla]F = (A + s^{-1}) \cdot \nabla F + (A + s^{-1}) \cdot A \cdot F + \beta \cdot F \tag{6.4}$$

for any S-tangent tensor field F. When F are scalar functions, the right-hand side is simply $(A + s^{-1}) \cdot \nabla F$.

In the remaining parts of this paper, we will employ the following conventions:

- $\check{R}$ denotes either the pair $(\check{\rho}, -\check{\sigma})$ or $\underline{\check{\beta}}$
- $\mathcal{D}^{-1}\check{R}$ denotes either $\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or ${}^\star\mathcal{D}_1^{-1}\underline{\check{\beta}}$
- $\mathcal{D}^{-2}\check{R}$ denotes either $\mathcal{D}_2^{-1} \cdot \mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1} \cdot {}^\star\mathcal{D}_1^{-1}\underline{\check{\beta}}$
- $\mathcal{D}^{-1}\nabla_L\check{R}$ denotes either ${}^\star\mathcal{D}_1^{-1}\nabla_L\underline{\check{\beta}}$ or $\mathcal{D}_1^{-1}L(\check{\rho}, -\check{\sigma})$
- $C_0(\check{R})$ denotes $[\nabla_L, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})$ or $[\nabla_L, {}^\star\mathcal{D}_1^{-1}]\underline{\check{\beta}}$
- $\mathcal{D}^{-2} \cdot \nabla_L\check{R}$ denotes $\mathcal{D}_2^{-1} \cdot \mathcal{D}_1^{-1}\nabla_L(\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1} \cdot {}^\star\mathcal{D}_1^{-1}\nabla_L\underline{\check{\beta}}$
- $\mathcal{D}^{-1} \cdot C_0(\check{R})$ denotes $\mathcal{D}_2^{-1} \cdot [\nabla_L, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1} \cdot [\nabla_L, {}^\star\mathcal{D}_1^{-1}]\underline{\check{\beta}}$

In this section we will consider the commutators

$$C(\check{R}) = (C_1(\check{R}), C_2(\check{R}), C_3(\check{R})) \tag{6.5}$$

given in [9, Definition 6.3] which, by the above conventions, can be written symbolically as

$$C_1(\check{R}) = \nabla \cdot \mathcal{D}^{-1} \cdot \left[\nabla_L, \mathcal{D}^{-1}\right]\check{R},$$
$$C_2(\check{R}) = \nabla \cdot \left[\nabla_L, \mathcal{D}^{-1}\right] \cdot \mathcal{D}^{-1}\check{R},$$
$$C_3(\check{R}) = [\nabla_L, \nabla] \cdot \mathcal{D}^{-2}\check{R}.$$

Corresponding to (6.1) and (6.2), we introduce the error terms

$$Err := \mathcal{D}_1^{-1}\nabla_L(\check{\rho}, -\check{\sigma}) - \beta \quad \text{and} \quad \widetilde{Err} := {}^\star\mathcal{D}_1^{-1}\nabla_L\check{\underline{\beta}} - (\rho, \sigma). \tag{6.6}$$

Let

$$F_1 := (Err, \widetilde{Err}).$$

Then symbolically F_1 has the form

$$F_1 = \mathcal{D}^{-1}\left(s^{-1}\check{R} + A \cdot \bar{R}\right).$$

Consequently we infer from (6.1) and (6.2) the symbolic expressions

$$\mathcal{D}^{-1}\nabla_L\check{R} = R_0 + F_1. \tag{6.7}$$

By using (4.4) with $\theta = 0$ and $p = 2$, Proposition 4.1 and the Hölder inequality we infer that

$$\|F_1\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.8}$$

Now we are ready to state the main results in this section. The first one is

Proposition 6.1. *For the error terms $C_0(\check{R})$, $C_1(\check{R})$, $C_2(\check{R})$ and $C_3(\check{R})$ there hold*

$$\left\|C_0(\check{R})\right\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.9}$$
$$\left\|C_1(\check{R})\right\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.10}$$
$$C_2(\check{R}) = \nabla \cdot \mathcal{D}^{-1}\left(\beta \cdot \mathcal{D}^{-2}\check{R}\right) + err, \tag{6.11}$$
$$C_3(\check{R}) = \beta \cdot \mathcal{D}^{-2}(\check{R}) + err \tag{6.12}$$

with

$$\|err\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

We remark that the terms $\nabla \cdot \mathcal{D}^{-1}(\beta \cdot \mathcal{D}^{-2}\check{R})$ and $\beta \cdot \mathcal{D}^{-2}\check{R}$ in $C_2(\check{R})$ and $C_3(\check{R})$ cannot be bounded in $\mathcal{P}^0$ norm. The next main result provides tools to deal with such terms.

Proposition 6.2. *All the commutators $C(\check{R})$ can be expressed as follows*

$$C(\check{R}) = \nabla_L P + E,$$

where P and E are tensors verifying

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

We mention that the above two results have been proved in [9]. The proofs in [9], however, rely on the following Hodge-elliptic $\mathcal{P}^0$ estimate and product estimate (see [9, (196),(190)])

$$\|\nabla \cdot \mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \|F\|_{\mathcal{P}^0}, \quad \text{and} \quad \|F \cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F) \cdot \left(\|\nabla G\|_{L_t^2 L_x^2} + \|G\|_{L_t^2 L_x^2}\right),$$

for appropriate tensor fields F and G. Unfortunately, these inequalities are not quite accurate since some terms were missed. Instead we will use the corrected versions (4.6) and (5.1). The presence of the terms in $L_t^a L_x^2$ norms in these corrected inequalities requires us to modify the proof in [9, Section 6.12] by establishing $L_t^a L_x^2$ estimates for some commutators, which add much complexity.

6.1. Proof of Proposition 6.1: Part I

We first note that by (3.11), the Hölder inequality and the Sobolev inequality, we can obtain from (BA1) and (BA2) that

$$\begin{cases} \|s^{-1}A\|_{L^2(\mathcal{H})}, \|\nabla A\|_{L^2(\mathcal{H})}, \|A\|_{L_t^\infty L_x^4}, \|A\|_{L_t^6 L_x^6}, \|\nabla \operatorname{tr}\chi\|_{L_x^2 L_t^\infty} \lesssim \Delta_0, \\ \|A\cdot \underline{A}\|_{L^2(\mathcal{H})}, \|A\cdot A\|_{L_t^{2+}L_x^2} \lesssim \Delta_0^2, \|\nabla_L \underline{A}\|_{L_t^2 L_x^2} \lesssim \Delta_0, \end{cases} \tag{6.13}$$

In this subsection, with the help of Propositions 4.3 and 4.4, we will prove (6.9), (6.11) and (6.12).

We first prove (6.9). We use (6.4) to write

$$C_0(\check{R}) = \mathcal{D}^{-1}\big(\big(A+s^{-1}\big)\big(\nabla\cdot\mathcal{D}^{-1}\check{R}\big) + \big(A+s^{-1}\big)\cdot A\cdot \mathcal{D}^{-1}\check{R} + \beta\cdot\mathcal{D}^{-1}\check{R}\big). \tag{6.14}$$

Then, by using (4.4) with $\theta = 0$ and $p = \frac{4}{3}$, Proposition 4.1 and the Hölder inequality we can estimate the various terms in the above equation to get

$$\big\|C_0(\check{R})\big\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0 + \Delta_0\cdot\mathcal{N}_1\big(\mathcal{D}^{-1}\check{R}\big). \tag{6.15}$$

By the definition of $\mathcal{N}_1(\mathcal{D}^{-1}\check{R})$ and Proposition 4.1 it follows that

$$\mathcal{N}_1(\mathcal{D}^{-1}\check{R}) \lesssim \mathcal{R}_0 + \Delta_0^2 + \big\|\mathcal{D}^{-1}\nabla_L\check{R}\big\|_{L_t^2L_x^2} + \big\|C_0(\check{R})\big\|_{L_t^2L_x^2}.$$

While it follows from (6.7) and (6.8) that

$$\big\|\mathcal{D}^{-1}\nabla_L\check{R}\big\|_{L_t^2L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Combining the above three inequalities and using the smallness of Δ_0 we obtain (6.9).

In the above proof, together with Lemma 4.1 we have actually verified the following

Proposition 6.3.

$$\big\|\mathcal{D}^{-1}\nabla_L\check{R}\big\|_{L_t^2L_x^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.16}$$

$$\big\|\big[\nabla_L,\mathcal{D}^{-1}\big]\check{R}\big\|_{L_t^2L_x^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.17}$$

$$\mathcal{N}_1\big(\mathcal{D}^{-1}\check{R}\big) \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.18}$$

$$\mathcal{N}_1\big(\nabla\cdot\mathcal{D}^{-2}\check{R}\big) \lesssim \mathcal{R}_0 + \Delta_0^2, \mathcal{N}_2\big(\mathcal{D}^{-2}\check{R}\big) \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{6.19}$$

In order to prove (6.11) and (6.12), we first use (6.4) to write

$$C_2(\check{R}) = \nabla\cdot\big[\nabla_L,\mathcal{D}^{-1}\big]_g\mathcal{D}^{-1}\check{R} + \nabla\cdot\mathcal{D}^{-1}\big(\beta\cdot\mathcal{D}^{-2}\check{R}\big), \tag{6.20}$$

$$C_3(\check{R}) = [\nabla_L,\nabla]_g\cdot\mathcal{D}^{-2}\check{R} + \beta\cdot\mathcal{D}^{-2}\check{R} \tag{6.21}$$

where

$$[\nabla_L,\nabla]_g F := (A+s^{-1})\cdot\nabla F + (A+s^{-1})\cdot A\cdot F,$$

$$\big[\nabla_L,\mathcal{D}^{-1}\big]_g F := \mathcal{D}^{-1}\big((A+s^{-1})\cdot\nabla\mathcal{D}^{-1}F + (A+s^{-1})\cdot A\cdot\mathcal{D}^{-1}F\big)$$

are the "good" parts in the corresponding commutators consisting of those terms not involving the curvature β. Then the proof can be complete by using (6.18) and the following result.

Lemma 6.1. *For appropriate S-tangent tensor field F, there hold*

$$\big\|[\nabla_L,\nabla]_g\mathcal{D}^{-1}F\big\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F) \quad \text{and} \quad \big\|\nabla\cdot\big[\nabla_L,\mathcal{D}^{-1}\big]_g F\big\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{6.22}$$

Proof. Noting that the simple inequality

$$\|F\|_{\mathcal{P}^0} \leqslant \|t\nabla F\|_{L^2(\mathcal{H})} + \|F\|_{L^2(\mathcal{H})}$$

for any S-tangent tensor field F. By Proposition 5.1 with $a > 4$ and Lemma 4.1 we then have

$$\begin{aligned}\big\|[\nabla_L, \nabla]_g \mathcal{D}^{-1} F\big\|_{\mathcal{P}^0} &\lesssim \mathcal{N}_1(\nabla\mathcal{D}^{-1}F)\big(\big\|t^{\frac{1}{2}}\nabla A\big\|_{L_t^2 L_x^2} + \big\|t^{-\frac{1}{a}} A\big\|_{L_t^a L_x^2}\big)\\ &\quad + \|t^{-1}\nabla\mathcal{D}^{-1}F\|_{\mathcal{P}^0} + \mathcal{N}_2(\mathcal{D}^{-1}F)\big(\|A\cdot A\|_{\mathcal{P}^0} + \big\|t^{-\frac{1}{2}}A\big\|_{\mathcal{P}^0}\big)\\ &\lesssim \mathcal{N}_1(\nabla\mathcal{D}^{-1}F) + \mathcal{N}_2(\mathcal{D}^{-1}F)\\ &\lesssim \mathcal{N}_1(F).\end{aligned}$$

This proves the first inequality. In order to prove the second inequality, in view of (4.6), it suffices to show for appropriate S-tangent tensor fields F there holds[4]

$$\big\|\big[\nabla_L, \mathcal{D}^{-1}\big]_g F\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(F) \quad \text{with } 4 < a < \infty, \tag{6.23}$$

which can be proved, by using Proposition 4.3 with $p = 4/3$, Proposition 4.1, (3.11), (6.13) and Lemma 4.1, as follows:

$$\begin{aligned}\big\|\big[\nabla_L, \mathcal{D}^{-1}\big]_g F\big\|_{L_t^a L_x^2} &\lesssim \big\|t^{1/2}A\cdot\nabla\mathcal{D}^{-1}F\big\|_{L_t^a L_x^{4/3}} + \big\|t^{1/2}A\cdot A\cdot\mathcal{D}^{-1}F\big\|_{L_t^a L_x^{4/3}}\\ &\quad + \big\|t^{-1}\mathcal{D}^{-1}\nabla\mathcal{D}^{-1}F\big\|_{L_t^a L_x^2} + \big\|t^{-1/2}A\cdot\mathcal{D}^{-1}F\big\|_{L_t^a L_x^{4/3}}\\ &\lesssim \|A\|_{L_t^a L_x^2}\big\|\nabla\cdot\mathcal{D}^{-1}F\big\|_{L_t^\infty L_x^4} + \|A\cdot A\|_{L_t^a L_x^2}\big\|\mathcal{D}^{-1}F\big\|_{L_t^\infty L_x^4}\\ &\quad + \big\|\nabla\mathcal{D}^{-1}F\big\|_{L_t^a L_x^2} + \|A\|_{L_t^a L_x^2}\big\|t^{-1/2}\mathcal{D}^{-1}F\big\|_{L_t^\infty L_x^4}\\ &\lesssim \mathcal{N}_2\big(\mathcal{D}^{-1}F\big) + \mathcal{N}_1\big(\nabla\mathcal{D}^{-1}F\big)\\ &\lesssim \mathcal{N}_1(F). \qquad \square\end{aligned}$$

6.2. Proof of Proposition 6.1: Part II

In order to complete the proof of Proposition 6.1, it remains only to prove (6.10). Observe that $C_1(\check{R})$ can be written symbolically in the form

$$C_1(\check{R}) = \nabla\cdot\mathcal{D}^{-1}C_0(\check{R}).$$

We then obtain from the Hodge-elliptic estimate (4.6), (6.9) and (6.17) that

$$\begin{aligned}\big\|C_1(\check{R})\big\|_{\mathcal{P}^0} &\lesssim \big\|C_0(\check{R})\big\|_{\mathcal{P}^0} + \Delta_0\big\|\mathcal{D}^{-1}C_0(\check{R})\big\|_{L_t^a L_x^2}^q\big\|C_0(\check{R})\big\|_{L^2(\mathcal{H})}^{1-q}\\ &\lesssim \Delta_0^2 + \mathcal{R}_0 + \Delta_0\big\|\mathcal{D}^{-1}C_0(\check{R})\big\|_{L_t^a L_x^2}^q\big(\Delta_0^2 + \mathcal{R}_0\big)^{1-q},\end{aligned}$$

where $\gamma_0 < q < 1$ and $4 < a < \infty$.

We will complete the proof of (6.10) by establishing the following

Proposition 6.4. *For* $4 < a < \infty$ *there holds*

$$\big\|t^{-\frac{1}{a}}\mathcal{D}^{-1}C_0(\check{R})\big\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Before proving Proposition 6.4, let us state the following two useful results.

Proposition 6.5. *Let* $\mathcal{D}^{-1}$ *denote either* $\mathcal{D}_1^{-1}$ *or* ${}^\star\mathcal{D}_1^{-1}$. *Then for any S-tangent tensor fields F and G on $\mathcal{H}$ there holds*

$$\big\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(F\cdot\nabla G)\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G), \quad \textit{with } 4 < a < \infty.$$

[4] We will improve the right-hand side of (6.23) to be $\mathcal{N}_1(\mathcal{D}^{-1}F)$ in the next section.

Proof of Proposition 6.5. By the GLP decomposition we first write

$$t^{-\frac{1}{a}}\mathcal{D}^{-1}(F\cdot\nabla G)=t^{-\frac{1}{a}}\mathcal{D}^{-1}(F_{>0}\cdot\nabla G_{>0})+\mathcal{L}(F,G)$$

where $\mathcal{L}(F,G)$ denotes low frequency terms. For the high frequency terms, we use the GLP decomposition again and ignore the low frequency part $\sum_{l\leqslant 0}P_l$, we only need to consider $\mathcal{I}=\sum_{l,m,n>0}\mathcal{I}_{lnm}$, where

$$\mathcal{I}_{lnm}:=\left\|t^{-\frac{1}{a}}P_l\mathcal{D}^{-1}(P_nF\cdot\nabla P_mG)\right\|_{L_t^aL_x^2}.$$

We will estimate such terms by considering several cases. When $l<m<n$, by using Lemma 4.2 we obtain

$$\mathcal{I}_{lnm}\lesssim 2^{l(\frac{2}{p}-2)}\left\|t^{2-\frac{2}{p}-\frac{1}{a}}P_nF\cdot\nabla P_mG\right\|_{L_t^aL_x^p} \tag{6.24}$$

where $2>p>1$ is sufficiently close to 1.

Let p^* be such that $\frac{1}{p^*}+\frac{1}{2}=\frac{1}{p}$. By using Lemma 5.1 and the Sobolev inequality it is easy to derive that

$$\left\|t^{-(\frac{2}{p^*}+\frac{1}{a}-\frac{1}{2})}P_nF\right\|_{L_t^aL_x^{p^*}}\lesssim 2^{n(\frac{1}{2}-\frac{1}{a}-\frac{2}{p^*})}\mathcal{N}_1[F_n], \tag{6.25}$$

$$\left\|t^{1/2}\nabla P_mG\right\|_{L_t^\infty L_x^2}\lesssim 2^{\frac{m}{2}}\mathcal{N}_1[G_m]. \tag{6.26}$$

Using both of them together with Hölder inequality, it follows from (6.24) that

$$\mathcal{I}_{lnm}\lesssim 2^{(-2+\frac{2}{p})l+\frac{m-n}{2}+(1-\frac{1}{a}-\frac{2}{p*})n}\mathcal{N}_1[F_n]\mathcal{N}_1[G_m].$$

Thus we can obtain

$$\sum_{0<l<m<n}\mathcal{I}_{lnm}\lesssim\mathcal{N}_1(F)\mathcal{N}_1(G)$$

as long as $1-\frac{1}{a}-\frac{2}{p*}<0$, which is possible when p is sufficiently close to 1.

When $m<l<n$, by using Lemma 4.2 with $4/3<p<2$, (6.25), (6.26), and defining p^* by $\frac{1}{p*}+\frac{1}{2}=\frac{1}{p}$ we obtain

$$\begin{aligned}\mathcal{I}_{lnm}&\lesssim 2^{-(2-\frac{2}{p})l}\left\|t^{2-\frac{2}{p}-\frac{1}{a}}P_nF\cdot\nabla P_mG\right\|_{L_t^aL_x^p}\\&\lesssim 2^{-(2-\frac{2}{p})l}\left\|t^{1/2}\nabla P_mG\right\|_{L_t^\infty L_x^2}\left\|t^{\frac{3}{2}-\frac{2}{p}-\frac{1}{a}}P_nF\right\|_{L_t^aL_x^{p*}}\\&\lesssim 2^{(\frac{m}{2}-\frac{l}{2})+(\frac{2}{p}-\frac{3}{2})l+(\frac{n}{2}-\frac{n}{a}-\frac{2n}{p*})}\mathcal{N}_1[G_m]\mathcal{N}_1[F_n].\end{aligned}$$

We can choose p sufficiently close to $4/3$ so that $\frac{3}{2}-\frac{2}{p}-\frac{1}{a}<0$. Then we have

$$\sum_{0<m<l<n}\mathcal{I}_{lnm}\lesssim\mathcal{N}_1(G)\mathcal{N}_1(F).$$

When $l<n<m$, we note that

$$P_nF\cdot\nabla P_mG=\nabla(P_nF\cdot P_mG)-\nabla P_nF\cdot P_mG,$$

thus we need to consider the two terms

$$\mathcal{I}^1_{lnm}:=\left\|t^{-\frac{1}{a}}P_l\mathcal{D}^{-1}(\nabla P_nF\cdot P_mG)\right\|_{L_t^aL_x^2}$$

$$\mathcal{I}^2_{lnm}:=\left\|t^{-\frac{1}{a}}P_l\mathcal{D}^{-1}\nabla(P_nF\cdot P_mG)\right\|_{L_t^aL_x^2}.$$

Observe that by the same method for establishing $\sum_{0<l<m<n}\mathcal{I}_{lnm}$, we can obtain

$$\sum_{0<l<n<m}\mathcal{I}^1_{lnm}\lesssim\mathcal{N}_1(F)\mathcal{N}_1(G).$$

For $\mathcal{I}^2_{lnm}$ we have from Proposition 4.1, (5.5) and (5.8) that

$$\begin{aligned}\mathcal{I}^2_{lnm} &\lesssim \big\|t^{-\frac{1}{a}}(P_nF\cdot P_mG)\big\|_{L^a_tL^2_x}\\ &\lesssim \big\|t^{-(\frac{1}{a'}-\frac{1}{2})}P_nF\big\|_{L^{a'}_tL^\infty_x}\big\|t^{(-\frac{1}{a''}-\frac{1}{2})}P_mG\big\|_{L^{a''}_tL^2_x}\\ &\lesssim 2^{n(\frac{1}{2}-\frac{1}{a'})-m(\frac{1}{a''}+\frac{1}{2})}\mathcal{N}_1[F_n]\mathcal{N}_1[G_m]\end{aligned}$$

where $\frac{1}{a'}+\frac{1}{a''}=\frac{1}{a}$. Summing over $l<n<m$ gives

$$\sum_{0<l<n<m}\mathcal{I}^2_{lnm}\lesssim \mathcal{N}_1(F)\mathcal{N}_1(G).$$

When $m>l>n$, we can follow exactly the same way as for the case $l<n<m$ to obtain

$$\sum_{0<n<l<m}\mathcal{I}_{lnm}\lesssim \mathcal{N}_1(F)\mathcal{N}_1(G).$$

Finally when $l>m$ and $l>n$, we derive by Lemma 4.2, (6.26) and (5.8) that

$$\begin{aligned}\mathcal{I}_{lnm} &\lesssim 2^{-l}\big\|t^{1-\frac{1}{a}}P_nF\cdot\nabla P_mG\big\|_{L^a_tL^2_x}\\ &\lesssim 2^{-l}\big\|t^{\frac{1}{2}-\frac{1}{a}}P_mG\big\|_{L^a_tL^\infty_x}\big\|t^{1/2}\nabla P_nF\big\|_{L^\infty_tL^2_x}\\ &\lesssim 2^{-l+\frac{n}{2}+m(\frac{1}{2}-\frac{1}{a})}\mathcal{N}_1[F_n]\mathcal{N}_1[G_m],\end{aligned}$$

which yields

$$\sum_{l>m,l>n}\mathcal{I}_{lnm}\lesssim \mathcal{N}_1(F)\mathcal{N}_1(G).$$

Thus we conclude $\mathcal{I}\lesssim \mathcal{N}_1(F)\mathcal{N}_1(G)$.

It remains to show

$$\big\|\mathcal{L}(F,G)\big\|_{L^a_tL^2_x}\lesssim \mathcal{N}_1(F)\mathcal{N}_1(G)$$

for low frequency term $\mathcal{L}(F,G)$, which can be done similarly. Since the argument is much easier, we omit the details. □

Lemma 6.2. *For S-tangent tensor fields F on $\mathcal{H}$ there hold*

$$\big\|t^{-\epsilon}\Lambda^{-\epsilon}\mathcal{D}_1^{-1}F\big\|_{L^2_x}\lesssim \|F\|_{L^1_x}, \tag{6.27}$$

$$\big\|t^{-\epsilon}\Lambda^{-\epsilon\star}\mathcal{D}_1^{-1}F\big\|_{L^b_tL^2_x}\lesssim \|F\|_{L^{b+}_tL^1_x} \tag{6.28}$$

where $1\leqslant b<\infty$ and $0<\epsilon\leqslant 1$.

Proof of Lemma 6.2. To show (6.28), by duality and ignoring lower order terms, we only need to show for appropriate tensor field G,

$$\big\|t^{-\epsilon}\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\big\|_{L^\infty_x}\lesssim 2^{-\epsilon k}\big(1+\|t\underline{K}\|^\theta_{L^2_x}\big)\|G\|_{L^2_x}, \tag{6.29}$$

where $0<\theta<1$ is close to 0, then summing over $k>0$ and integrating in t.

We decompose $\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G$ using $\sum_l P_l^2=\mathrm{Id}$. Since $P_l\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G$ are not scalar functions, to which we cannot apply sharp Bernstein inequality, we use the following curvature dependent estimates, ([12, Section 10])

$$\|P_lH\|_{L^\infty_x}\lesssim 2^lt^{-1}\big(1+2^{-\theta l}\|t\underline{K}\|^\theta_{L^2_x}\big)\|P_lH\|_{L^2_x},\ l>0 \tag{6.30}$$

and

$$\|P_{\leqslant 0}H\|_{L^\infty_x}\lesssim t^{-1}\big(1+\|t\underline{K}\|^\theta_{L^2_x}\big)\|H\|_{L^2_x}. \tag{6.31}$$

From (6.31), we infer

$$\big\|t^{-\epsilon}P_{\leqslant 0}\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\big\|_{L^\infty_x}\lesssim 2^{-\epsilon k-k}\big(1+\|t\underline{K}\|^\theta_{L^2_x}\big)\|P_kG\|_{L^2_x}. \tag{6.32}$$

When $l>0$, it follows from (6.30) that

$$\left\|P_l\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\right\|_{L_x^\infty}\lesssim 2^l t^{-1}\left(1+2^{-\theta l}\|t\underline{K}\|_{L_x^2}^{\theta}\right)\left\|P_l\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\right\|_{L_x^2}.$$

Using (4.9) we have

$$\left\|t^{-\epsilon}P_l\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\right\|_{L_x^\infty}\lesssim 2^{-\epsilon k}2^{-|l-k|}\left(1+2^{-\theta l}\|t\underline{K}\|_{L_x^2}^{\theta}\right)\|P_kG\|_{L_x^2}.$$

Summing the above inequality over $l,k>0$, combined with (6.32), gives (6.29).

To show (6.27), the difference is that $P_l{}^\star\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G$ are scalar functions, to which, instead of using the inequalities (6.30) and (6.31), we employ the sharp Bernstein inequality ([12, Section 10]) which does not involve $\|\underline{K}\|_{L_x^2}$. Thus we can derive that

$$\left\|t^{-\epsilon}P_l{}^\star\mathcal{D}_1^{-1}P_k\Lambda^{-\epsilon}G\right\|_{L_x^\infty}\lesssim 2^{-|k-l|-\epsilon k}\|P_kG\|_{L_x^2}. \tag{6.33}$$

Summing over l and k, we obtain

$$\left\|t^{-\epsilon\star}\mathcal{D}_1^{-1}\Lambda^{-\epsilon}G\right\|_{L_x^\infty}\lesssim\|G\|_{L_x^2}. \tag{6.34}$$

Then (6.27) follows by a standard dual argument. □

Note that by using Proposition 3.2(iii), for appropriate S-tangent tensor fields F on $\mathcal{H}$,

$$\left\|t^{\alpha-1}\Lambda^{\alpha\star}\mathcal{D}^{-1}F\right\|_{L_x^2}\lesssim\|F\|_{L_x^2}\quad\text{with } 0\leqslant\alpha\leqslant 1, \tag{6.35}$$

By duality, the above estimate implies the following inequality for any appropriate S-tangent tensor field F on $\mathcal{H}$

$$\|t^{\alpha-1}\mathcal{D}^{-1}\Lambda^{\alpha}F\|_{L_x^2}\lesssim\|F\|_{L_x^2}. \tag{6.36}$$

With its aid, we infer from Lemma 6.2 the following result.

Corollary 6.1. *Let F be an appropriate S-tangent tensors on $\mathcal{H}$. Then on each leaf S_t there holds*

$$\left\|\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}F\right\|_{L_x^2}\lesssim\|tF\|_{L_x^1}. \tag{6.37}$$

Moreover, on $\mathcal{H}$ there holds

$$\left\|\mathcal{D}_1^{-1\star}\mathcal{D}_1^{-1}F\right\|_{L_t^bL_x^2}\lesssim\|tF\|_{L_t^{b+}L_x^1} \tag{6.38}$$

where $1\leqslant b<\infty$.

Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1$, ${}^\star\mathcal{D}_1$ or $\mathcal{D}_2$. We will use Proposition 6.5, Lemma 6.2 to estimate the error type terms in the following result.

Proposition 6.6. *For S-tangent tensors G on $\mathcal{H}$ verifying $\mathcal{N}_1(G)<\infty$, set*

$$\mathcal{E}_1(G):=s^{-1}\mathcal{D}^{-1}(A\cdot G)\quad or\quad \mathcal{D}^{-1}(A\cdot A\cdot G),$$
$$\mathcal{E}_2(G):=\mathcal{D}^{-1}(A\cdot\nabla G),\quad \mathcal{E}_3:=\mathcal{D}^{-1}(A\cdot\underline{A}\cdot A),\quad \mathcal{E}_4:=s^{-1}\mathcal{D}^{-1}(A\cdot\underline{A}),$$

where for $\mathcal{E}_3$, $\mathcal{D}$ denote either $\mathcal{D}_1$ or ${}^\star\mathcal{D}_1$. The following estimates hold

$$\left\|t^{-\frac{1}{a}}\mathcal{E}_1(G)\right\|_{L_t^aL_x^2}+\left\|t^{-\frac{1}{a}}\mathcal{E}_2(G)\right\|_{L_t^aL_x^2}\lesssim\Delta_0\mathcal{N}_1(G)$$
$$\left\|t^{-\epsilon}\Lambda^{-\epsilon}\mathcal{E}_3\right\|_{L_t^aL_x^2}+\left\|t^{-\frac{1}{a}}\mathcal{E}_4\right\|_{L_t^aL_x^2}+\left\|t^{-1}\mathcal{D}^{-1}\mathcal{E}_3\right\|_{L_t^aL_x^2}\lesssim\Delta_0^2$$

where $4<a<\infty$, $0<\epsilon\leqslant 1$.

Proof. For $\mathcal{E}_1(G) = \mathcal{D}^{-1}(A \cdot A \cdot G)$, we can use Proposition 4.3 to get

$$\big\|t^{-\frac{1}{a}}\mathcal{E}_1(G)\big\|_{L_t^a L_x^2} \lesssim \big\|t^{\frac{1}{2}-\frac{1}{a}} A \cdot A \cdot G\big\|_{L_t^a L_x^{4/3}} \lesssim \|A\|^2_{L_t^\infty L_x^4}\|G\|_{L_t^a L_x^4} \lesssim \Delta_0^2 \mathcal{N}_1(G).$$

When $\mathcal{E}_1(G) = s^{-1}\mathcal{D}^{-1}(A \cdot G)$, by Proposition 4.1 and (3.12)

$$\begin{aligned}\big\|t^{-1-\frac{1}{a}}\mathcal{D}^{-1}(A \cdot G)\big\|_{L_t^a L_x^2} &\lesssim \big\|t^{-\frac{1}{a}} A \cdot G\big\|_{L_t^a L_x^2} \lesssim \big\|t^{-\frac{1}{a}} A\big\|_{L_t^a L_x^4}\|G\|_{L_t^\infty L_x^4} \\ &\lesssim \mathcal{N}_1(A)\mathcal{N}_1(G) \lesssim \Delta_0 \mathcal{N}_1(G).\end{aligned}$$

For $\mathcal{E}_2(G)$, we infer from Proposition 6.5 that

$$\big\|t^{-\frac{1}{a}}\mathcal{E}_2(G)\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(G)\mathcal{N}_1(A) \lesssim \Delta_0 \mathcal{N}_1(G).$$

In order to estimate $\mathcal{E}_3$, we use (6.28) and the simple inequality

$$\big\|t^{-\frac{1}{a}} F\big\|_{L_t^a L_x^p} \lesssim \|\nabla_L F\|_{L_t^1 L_x^p} + \|t^{-1}F\|_{L_t^1 L_x^p} \quad \text{with } p \geqslant 1 \tag{6.39}$$

combined with the Leibnitz rule, Hölder inequality and (6.13) to get

$$\begin{aligned}\big\|t^{-\epsilon}\Lambda^{-\epsilon}\mathcal{E}_3\big\|_{L_t^a L_x^2} &\lesssim \|A \cdot \underline{A} \cdot A\|_{L_t^{a+} L_x^1} \lesssim \big\|\nabla_L(A \cdot \underline{A} \cdot A)\big\|_{L_t^1 L_x^1} + \big\|t^{\frac{1}{a+}-1} A \cdot \underline{A} \cdot A\big\|_{L_t^1 L_x^1} \\ &\lesssim \|\nabla_L A\|_{L_t^2 L_x^2}\|A \cdot \underline{A}\|_{L_x^2 L_t^2} + \|\nabla_L \underline{A}\|_{L_t^2 L_x^2}\|A \cdot A\|_{L_t^2 L_x^2} + \Delta_0^3 \\ &\lesssim \Delta_0^3.\end{aligned}$$

For $\mathcal{E}_4$, we use Proposition 4.3 with $p > 1$ sufficiently close to 1 together with Hölder inequality to get

$$\big\|t^{-\frac{1}{a}}\mathcal{E}_4\big\|_{L_t^a L_x^2} \lesssim \big\|t^{1-\frac{2}{p}-\frac{1}{a}} A \cdot \underline{A}\big\|_{L_t^a L_x^p} \lesssim \big\|t^{\frac{3}{2}-\frac{2}{p}-\frac{1}{a}} A\big\|_{L_t^a L_x^{p*}}\|t^{-1/2}\underline{A}\|_{L_t^\infty L_x^2},$$

where $\frac{1}{p} = \frac{1}{2} + \frac{1}{p*}$. By (3.12), $\|t^{\frac{3}{2}-\frac{2}{p}-\frac{1}{a}} A\|_{L_t^a L_x^{p*}} \lesssim \mathcal{N}_1(A)$, we obtain $\|t^{-\frac{1}{a}}\mathcal{E}_4\|_{L_t^a L_x^2} \lesssim \Delta_0^2$. In view of the inequality (6.36), we have $\|t^{-1}\mathcal{D}^{-1}\mathcal{E}_3\|_{L_t^a L_x^2} \lesssim \Delta_0^2$. □

By analyzing the expression of β and $C_0(F) := [\nabla_L, \mathcal{D}^{-1}]F$,[5] we have

Corollary 6.2. *The following inequalities hold for any S-tangent tensor F,*

$$\big\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\beta \cdot F)\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(F)\Delta_0, \tag{6.41}$$

$$\big\|t^{-\frac{1}{a}} C_0(F)\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1\big(t^{-\frac{1}{2}}\mathcal{D}^{-1}F\big) \tag{6.42}$$

where $4 < a < \infty$.

Proof. Using Codazzi equation (2.12), i.e. $\beta = \nabla A + A \cdot A + s^{-1}A$, we infer

$$\mathcal{D}^{-1}(\beta \cdot F) = \mathcal{E}_1(F) + \mathcal{E}_2(F).$$

Whence (6.41) follows from Proposition 6.6.

Similarly, using (6.4) we can write

$$C_0(F) = \mathcal{E}_1(\mathcal{D}^{-1}F) + \mathcal{E}_2(\mathcal{D}^{-1}F) + t^{-1}\mathcal{D}^{-1}\nabla\mathcal{D}^{-1}F.$$

For the last term, using Proposition 4.1, we infer

$$\big\|t^{-1-\frac{1}{a}}\mathcal{D}^{-1}\nabla\mathcal{D}^{-1}F\big\|_{L_t^a L_x^2} \lesssim \big\|t^{-1-\frac{1}{a}}\mathcal{D}^{-1}F\big\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1\big(t^{-\frac{1}{2}}\mathcal{D}^{-1}F\big).$$

[5] Using Proposition4.4 and Hölder inequality, we can get the simple result

$$\big\|C_0(F)\big\|_{L_t^1 L_x^2} \lesssim \|F\|_{L^2(\mathcal{H})}. \tag{6.40}$$

The desired estimate then follows from Proposition 6.6. □

Proof of Proposition 6.4. Combining Proposition 4.3, (6.42) and (6.18) we derive

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}C_0(\check{R})\|_{L_t^aL_x^2} \lesssim \|t^{1-\frac{1}{a}}C_0(\check{R})\|_{L_t^aL_x^2} \lesssim \mathcal{N}_1(t^{\frac{1}{2}}\mathcal{D}^{-1}\check{R}) \lesssim \Delta_0^2 + \mathcal{R}_0$$

as desired. □

6.3. $L_t^aL_x^2$ estimates for $\mathcal{D}^{-1}E_1^G$

For arbitrary S-tangent tensor field F, we denote by E_1^G either $[\nabla_L, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma}) \cdot F$ or $Err \cdot F$. In what follows, we establish $L_t^aL_x^2$ estimates for $\mathcal{D}^{-1}E_1^G$, which are important for the Hodge-elliptic $\mathcal{P}^0$ estimates involved in the decomposition procedure in Section 6.5.

Proposition 6.7. *Denote by $\mathcal{D}$ either $\mathcal{D}_1$ or $\mathcal{D}_2$, for appropriate S-tangent tensor fields F, the following estimates hold*

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(Err \cdot F)\|_{L_t^aL_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F) \tag{6.43}$$

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}([\nabla_L, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma}) \cdot F)\|_{L_t^aL_x^2} \lesssim \Delta_0\mathcal{N}_1(F). \tag{6.44}$$

where Err is defined in (6.6) *and* $4 < a < \infty$.

In order to prove Proposition 6.7, we may use the error type terms introduced in Proposition 6.6 to rewrite (6.6) as

$$Err = \mathcal{D}_1^{-1}(s^{-1}\check{R}) + \mathcal{E}_1(A) + \mathcal{E}_2(A) + \mathcal{E}_3 + \mathcal{E}_4. \tag{6.45}$$

We first assume the following lemma which will be used to treat the term $\mathcal{E}_3$.

Lemma 6.3. *Let $\mathcal{D}$ denote one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ or ${}^\star\mathcal{D}_1$. For appropriate S-tangent tensor fields W and F there holds*

$$\|\mathcal{D}^{-1}(W \cdot F)\|_{L_t^aL_x^2} \lesssim \|t^{\frac{3}{2}-\epsilon}\Lambda^{-\epsilon}W\|_{L_t^aL_x^2}\mathcal{N}_2(F), \tag{6.46}$$

where $0 < \epsilon < 1$.

Proof of Proposition 6.7. (6.44) can be obtained by using Proposition 4.3 and (6.42) as follows,

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(C_0(\check{R}) \cdot F)\|_{L_t^aL_x^2} \lesssim \|t^{\frac{1}{2}-\frac{1}{a}}C_0(\check{R})\|_{L_t^aL_x^2}\|F\|_{L_t^\infty L_x^4} \lesssim \mathcal{N}_1(F)\Delta_0.$$

In the same manner, we can easily check

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\mathcal{E}_1(A) \cdot F)\|_{L_t^aL_x^2} + \|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\mathcal{E}_2(A) \cdot F)\|_{L_t^aL_x^2} + \|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\mathcal{E}_4 \cdot F)\|_{L_t^aL_x^2} \lesssim \Delta_0^2\mathcal{N}_1(F).$$

Thus to prove (6.43), in view of (6.45), we only need to show

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}((\mathcal{D}_1^{-1}(t^{-1}\check{R}) + \mathcal{E}_3) \cdot F)\|_{L_t^aL_x^2} \lesssim \mathcal{N}_2(F)(\Delta_0^2 + \mathcal{R}_0).$$

Using Proposition 4.3, we infer

$$\begin{aligned}\|t^{-1-\frac{1}{a}}\mathcal{D}^{-1}(\mathcal{D}_1^{-1}\check{R} \cdot F)\|_{L_t^aL_x^2} &\lesssim \|t^{-\frac{1}{2}-\frac{1}{a}}\mathcal{D}_1^{-1}\check{R} \cdot F\|_{L_t^aL_x^{4/3}} \lesssim \|\mathcal{D}_1^{-1}\check{R}\|_{L_t^\infty L_x^4}\|t^{-\frac{1}{2}-\frac{1}{a}}F\|_{L_t^aL_x^2}\\ &\lesssim \mathcal{N}_1(\mathcal{D}_1^{-1}\check{R})\mathcal{N}_1(F).\end{aligned}$$

Using Lemma 6.3 and Proposition 6.6, we derive

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\mathcal{E}_3 \cdot F)\|_{L_t^aL_x^2} \lesssim \|t^{\frac{1}{2}-\frac{1}{a}}t^{1-\epsilon}\Lambda^{-\epsilon}\mathcal{E}_3\|_{L_t^aL_x^2}\mathcal{N}_2(F) \lesssim \Delta_0^2\mathcal{N}_2(F).$$

The proof is thus complete. □

Proof of Lemma 6.3. We will show (6.46) by using GLP decomposition. For simplicity, we ignore the low frequency terms. Note that $\mathcal{D}^{-1}(W\cdot F)=\sum_{k,n\in\mathbb{Z}} P_k\mathcal{D}^{-1}(W_n\cdot F)$. After dropping the low frequency terms, we consider the following terms

$$I=\sum_{0<k<n}\big\|P_k\mathcal{D}^{-1}(W_n\cdot F)\big\|_{L_x^2},\quad II=\sum_{0<n\leqslant k}\big\|P_k\mathcal{D}^{-1}(W_n\cdot F)\big\|_{L_x^2}.$$

By using Lemma 4.2, Proposition 4.1 and the finite band property we have

$$\begin{aligned}\big\|P_k\mathcal{D}^{-1}(W_n\cdot F)\big\|_{L_x^2}&\lesssim 2^{-2n}\big\|t^2P_k\mathcal{D}^{-1}(\Delta W_n\cdot F)\big\|_{L_x^2}\\&\lesssim 2^{-2n}\big(\big\|t^2P_k\mathcal{D}^{-1}\operatorname{div}(\nabla W_n\cdot F)\big\|_{L_x^2}+\big\|t^2P_k\mathcal{D}^{-1}(\nabla W_n\cdot\nabla F)\big\|_{L_x^2}\big)\\&\lesssim 2^{-2n}\big(\big\|t^2\nabla W_n\cdot F\big\|_{L_x^2}+2^{(\frac{2}{p}-2)k}\big\|t^{4-\frac{2}{p}}\nabla W_n\cdot\nabla F\big\|_{L_x^p}\big).\end{aligned}$$

Then by the finite band property and (3.11) we obtain with $\frac{1}{p*}+\frac12=\frac1p$ and $1<p\leqslant 4/3$

$$\begin{aligned}\big\|P_k\mathcal{D}^{-1}(W_n\cdot F)\big\|_{L_x^2}&\lesssim 2^{-2n+n+\epsilon n}\big\|2^{-\epsilon n}t^{\frac32}W_n\big\|_{L_x^2}\big\|t^{-\frac12}F\big\|_{L_x^\infty}\\&\quad+2^{-2n+(\frac{2}{p}-2)k+n+\epsilon n}\big\|2^{-\epsilon n}t^{\frac32}W_n\big\|_{L_x^2}\big\|t^{\frac32-\frac2p}\nabla F\big\|_{L_x^{p*}}.\end{aligned}$$

Summing over $0<k<n$, we conclude for $0<\epsilon<1$,

$$I\lesssim\big\|t^{\frac32-\epsilon}\Lambda^{-\epsilon}W\|_{L_x^2}\big(\big\|t^{-\frac12}F\big\|_{L_x^\infty}+\big\|t^{\frac32-\frac2p}\nabla F\big\|_{L_x^{p*}}\big).$$

Taking L_t^1 norm of I, noting $\|t^{\frac32-\frac2p}\nabla F\|_{L_t^\infty L_x^{p*}}\lesssim\mathcal{N}_2(F)$ and using (3.11), we infer

$$\|I\|_{L_t^a}\lesssim\mathcal{N}_2(F)\big\|t^{\frac32-\epsilon}\Lambda^{-\epsilon}W\big\|_{L_t^aL_x^2}. \tag{6.47}$$

As to II, by the finite band property and (3.11) we infer for $k\geqslant n>0$

$$\begin{aligned}\big\|P_k\mathcal{D}^{-1}(W_n\cdot F)\big\|_{L_x^2}&\lesssim 2^{-k}\|tW_n\cdot F\|_{L_x^2}\lesssim 2^{-k}\big\|t^{\frac32}W_n\big\|_{L_x^2}\big\|t^{-\frac12}F\big\|_{L_x^\infty}\\&\lesssim 2^{-k+\epsilon n}\big\|2^{-n\epsilon}t^{\frac32}W_n\big\|_{L_x^2}\big\|t^{-\frac12}F\big\|_{L_x^\infty}.\end{aligned}$$

Summing over $k>n>0$, then taking L_t^1 norm and using (3.11) we conclude for $0<\epsilon<1$,

$$\|II\|_{L_t^a}\lesssim\big\|t^{\frac32-\epsilon}\Lambda^{-\epsilon}W\big\|_{L_t^aL_x^2}\mathcal{N}_2(F).$$

Combine the estimates of I and II, we get (6.46) as desired. □

6.4. $L_t^aL_x^2$ estimates for $\nabla_L\mathcal{D}^{-2}\check{R}$

We will establish the following result which will be used in the next subsection.

Proposition 6.8. *The following estimate holds*

$$\big\|t^{-\frac1a}\nabla_L\mathcal{D}^{-2}\check{R}\big\|_{L_t^aL_x^2}\lesssim\mathcal{R}_0+\Delta_0^2,$$

where $4<a<\infty$.

To prove Proposition 6.8, we will rely on the following estimate of $\mathcal{D}^{-1}F_1$, which will be justified at the end of this subsection.[6]

[6] By $\mathcal{D}^{-1}F_1$, we denote either $\mathcal{D}_2^{-1}Err$ or ${}^\star\mathcal{D}_1^{-1}\widetilde{Err}$.

Proposition 6.9. *For $F_1 = (Err, \widetilde{Err})$ with Err and $\widetilde{Err}$ given by* (6.6)*, there holds*

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}F_1\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.48}$$

where $4 < a < \infty$.

Using Hodge-elliptic $\mathcal{P}^\sigma$ estimate (4.6), Proposition 6.9 and (6.8), we can obtain

Corollary 6.3. *For $F_1 = (Err, \widetilde{Err})$ there holds*

$$\|\nabla \cdot \mathcal{D}^{-1}F_1\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.49}$$

Now we will show Proposition 6.8 by assuming Proposition 6.9 first.

Proof of Proposition 6.8. In view of the formula

$$\nabla_L \mathcal{D}^{-2}\check{R} = [\nabla_L, \mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R} + \mathcal{D}^{-1}[\nabla_L, \mathcal{D}^{-1}]\check{R} + \mathcal{D}^{-2}\nabla_L\check{R},$$

we only need to show

$$\|t^{-\frac{1}{a}}[\nabla_L, \mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R}\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.50}$$

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}[\nabla_L, \mathcal{D}^{-1}]\check{R}\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.51}$$

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-2}\nabla_L\check{R}\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.52}$$

By using the fact that $\mathcal{N}_1(t^{-\frac{1}{2}}\mathcal{D}^{-2}\check{R}) \lesssim \mathcal{N}_2(\mathcal{D}^{-2}\check{R}) \lesssim \Delta_0^2 + \mathcal{R}_0$, (6.50) follows from (6.42) with $F = \mathcal{D}^{-1}\check{R}$. (6.51) was proved in Proposition 6.4. Thus it only remains to prove (6.52).

We first verify (6.52) for the case $\mathcal{D}^{-2}\nabla_L\check{R} = \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\nabla_L(\check{\rho}, \check{\sigma})$. It is clear that

$$\|\nabla_L\mathcal{D}^{-1}\beta\|_{L_t^1 L_x^2} \lesssim \|[\nabla_L, \mathcal{D}^{-1}]\beta\|_{L_t^1 L_x^2} + \|\mathcal{D}^{-1}\nabla_L\beta\|_{L_t^1 L_x^2}.$$

Applying (6.40) to the commutator and applying (2.16) and Proposition 4.3 to the other term, we obtain $\|\nabla_L\mathcal{D}^{-1}\beta\|_{L_t^1 L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0$. Then by (6.48) and (6.39), we obtain

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}(Err + \beta)\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In view of the expression $Err = \mathcal{D}_1^{-1}\nabla_L(\check{\rho}, -\check{\sigma}) - \beta$, (6.52) is proved in this case.

Next, for the case $\check{R} = \check{\underline{\beta}}$, we estimate the $L_t^a L_x^2$ norm of the term $\mathcal{D}^{-2}\nabla_L\check{\underline{\beta}}$. Using

$${}^\star\mathcal{D}_1^{-1}\nabla_L\check{\underline{\beta}} = (\rho, \sigma) + \widetilde{Err},$$

we obtain

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-2}\nabla_L\check{\underline{\beta}}\|_{L_t^a L_x^2} \lesssim \|t^{-\frac{1}{a}}\mathcal{D}_1^{-1}((\check{\rho}, \check{\sigma}) + \widetilde{Err})\|_{L_t^a L_x^2} + \|t^{1-\frac{1}{a}}\mathcal{E}_4\|_{L_t^a L_x^2}.$$

By Proposition 6.6, the second term is bounded by Δ_0^2. The first term is bounded by

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}\check{R}\|_{L_t^a L_x^2} + \|t^{-\frac{1}{a}}\mathcal{D}^{-1}\widetilde{Err}\|_{L_t^a L_x^2}.$$

It is bounded by $\Delta_0^2 + \mathcal{R}_0$ by using the inequality $\|t^{-\frac{1}{a}}\mathcal{D}^{-1}\check{R}\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(t^{\frac{1}{2}}\mathcal{D}^{-1}\check{R})$, (6.18) and (6.48). We conclude $\|\mathcal{D}^{-2}\nabla_L\check{\underline{\beta}}\|_{L_t^a L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0$. □

We will rely on the following two results to prove Proposition 6.9.

Lemma 6.4. *Let $\mathcal{D}^{-1}$ denote one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$. For any appropriate S-tangent tensor field G there holds*

$$\|\mathcal{D}^{-1}(\check{K}\cdot G)\|_{L_t^a L_x^2} \lesssim \check{K}_{\gamma_0}\mathcal{N}_1(G) \quad \text{with } 4 < a < \infty, \tag{6.53}$$

where $\gamma_0 > 1/2$ is close to $\frac{1}{2}$.

Proof of Lemma 6.4. Set $\Omega_{nl} := P_l\mathcal{D}^{-1}(\check{K}\cdot P_n G)$, it suffices to prove

$$\sum_{l,n>0}\|\Omega_{nl}\|_{L_t^a L_x^2} \lesssim \check{K}_{\gamma_0}\mathcal{N}_1(G).$$

We first consider the case $n > l > 0$, by (4.18) we have

$$\|\Omega_{nl}\|_{L_x^2} \lesssim t^{-\gamma}2^{\gamma n}\|P_n G\|_{L_x^2}\check{K}_{\gamma_0} \quad \text{with } \gamma > \gamma_0.$$

Due to (5.5), we infer for $\gamma > \gamma_0$ that

$$\|\Omega_{nl}\|_{L_t^a L_x^2} \lesssim 2^{n(\gamma-\frac{1}{2}-\frac{1}{a})}\mathcal{N}_1[G_n]\check{K}_{\gamma_0}.$$

Since $\gamma < \frac{1}{2}+\frac{1}{a}$ can be achieved when $\gamma > \gamma_0 > \frac{1}{2}$ are sufficiently close to $\frac{1}{2}$, we obtain

$$\sum_{n>l}\|\Omega_{nl}\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(G)\check{K}_{\gamma_0}.$$

Next, we consider the case $0 < n < l$. Combine (4.12) with the fact that for S-tangent tensor fields F

$$\|P_l\mathcal{D}^{-1}\Lambda^\gamma F\|_{L_x^2} \lesssim 2^{(-1+\gamma)l}t^{1-\gamma}\|F\|_{L_x^2},$$

we infer

$$\|\Omega_{nl}\|_{L_x^2} \lesssim 2^{(-1+\gamma)l+n}t^{-\gamma}\|P_n G\|_{L_x^2}\check{K}_{\gamma_0}.$$

Since $\gamma < \frac{1}{2}+\frac{1}{a}$, following the same treatment as for the case $n > l$ we derive

$$\sum_{n<l}\|\Omega_{nl}\|_{L_t^a L_x^2} \lesssim \sum_{n<l}2^{(-1+\gamma)l+\frac{n}{2}-\frac{n}{a}}\mathcal{N}_1[G_n]\check{K}_{\gamma_0} \lesssim \mathcal{N}_1(G)\check{K}_{\gamma_0}. \qquad \Box$$

Lemma 6.5. *For S-tangent tensor fields G, the following estimate holds*

$$\|[\nabla_L, \mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}]G\|_{L_t^1 L_x^2} \lesssim \|{}^\star\mathcal{D}_1^{-1}G\|_{L_t^2 L_x^p} \quad \text{with } p > 2. \tag{6.54}$$

Proof of Lemma 6.5. In this proof and the next one, we denote by $\mathcal{D}^{-2}$ the operator $\mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}$. In view of

$$[\nabla_L, \mathcal{D}^{-2}]G = C_0({}^\star\mathcal{D}_1^{-1}G) + \mathcal{D}_1^{-1}C_0(G),$$

it suffices to estimate $\|C_0({}^\star\mathcal{D}_1^{-1}G)\|_{L_t^1 L_x^2}$ and $\|\mathcal{D}_1^{-1}C_0(G)\|_{L_t^1 L_x^2}$. From (6.40) it follows

$$\|C_0({}^\star\mathcal{D}_1^{-1}G)\|_{L_t^1 L_x^2} \lesssim \|{}^\star\mathcal{D}_1^{-1}G\|_{L_t^2 L_x^2}. \tag{6.55}$$

Then we can obtain (6.54) by combining (6.55) with the following estimate

$$\|\mathcal{D}_1^{-1}C_0(G)\|_{L_t^1 L_x^2} \lesssim \|{}^\star\mathcal{D}_1^{-1}G\|_{L_t^2 L_x^p}, \tag{6.56}$$

where $p > 2$.

In order to prove (6.56), we use the commutation formula $[\nabla_L, \nabla]\varphi = \chi\cdot\nabla\varphi$ for scalar function to write

$$\mathcal{D}_1^{-1}C_0(G) = \mathcal{D}^{-2}\big((A+s^{-1})\nabla{}^\star\mathcal{D}_1^{-1}G\big).$$

Noting that by Proposition 4.3 and Proposition 4.1,

$$\|\mathcal{D}^{-2}(t^{-1}\nabla{}^\star\mathcal{D}_1^{-1}G)\|_{L_t^1 L_x^2} \lesssim \|{}^\star\mathcal{D}_1^{-1}G\|_{L_t^1 L_x^2},$$

it remains to estimate $\|\mathcal{D}^{-2}(A\cdot\nabla^\star\mathcal{D}_1^{-1}G)\|_{L_t^1L_x^2}$. Clearly, we have the following identity.

$$\mathcal{D}^{-2}\big(A\cdot\nabla^\star\mathcal{D}_1^{-1}G\big)=\mathcal{D}^{-2}\big(\nabla\big(A\cdot{}^\star\mathcal{D}_1^{-1}G\big)-\nabla A\cdot{}^\star\mathcal{D}_1^{-1}G\big). \tag{6.57}$$

The first term of (6.57) can be estimated by using Proposition 4.3 and Proposition 4.1 as follows

$$\begin{aligned}\big\|\mathcal{D}^{-2}\nabla\big(A\cdot{}^\star\mathcal{D}_1^{-1}G\big)\big\|_{L_t^1L_x^2}&\lesssim\big\|tA\cdot{}^\star\mathcal{D}_1^{-1}G\big\|_{L_t^1L_x^2}\lesssim\|A\|_{L_t^2L_x^{p*}}\big\|t^\star\mathcal{D}_1^{-1}G\big\|_{L_t^2L_x^p}\\&\lesssim\Delta_0\big\|{}^\star\mathcal{D}_1^{-1}G\big\|_{L_t^2L_x^p},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p*}=\frac{1}{2}$ and p is close to 2. To derive the last inequality, we employed Sobolev inequality $\|A\|_{L_t^2L_x^{p*}}\lesssim\|\nabla A\|_{L_t^2L_x^2}+\|t^{-1}A\|_{L_t^2L_x^2}\lesssim\Delta_0$.

Using Proposition 4.3, we infer

$$\big\|\mathcal{D}_1^{-1\star}\mathcal{D}_1^{-1}\big(\nabla A\cdot{}^\star\mathcal{D}_1^{-1}G\big)\big\|_{L_t^1L_x^2}\lesssim\big\|t^{2-\frac{2}{p}\star}\mathcal{D}_1^{-1}G\cdot\nabla A\big\|_{L_t^1L_x^{p*}}\lesssim\Delta_0\big\|{}^\star\mathcal{D}_1^{-1}G\big\|_{L_t^2L_x^p},$$

where $\frac{1}{p}+\frac{1}{2}=\frac{1}{p*}$ and $p*$ is close to 1. Thus (6.56) is proved. □

Now we are ready to prove Proposition 6.9.

Proof of Proposition 6.9. By letting $F=1$ in (6.43), we can obtain $\|t^{-\frac{1}{a}}\mathcal{D}_2^{-1}Err\|_{L_t^aL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0$. Thus we only need to consider $\mathcal{D}_1^{-1}\widetilde{Err}$.

Recall that

$$\widetilde{Err}={}^\star\mathcal{D}_1^{-1}\big(s^{-1}\check{\underline{\beta}}+A\cdot\big(\nabla A+A\cdot\underline{A}+s^{-1}\underline{A}\big)\big)+{}^\star\mathcal{D}_1^{-1}\big(\zeta\cdot\rho-\zeta^\star\sigma+V\cdot\underline{\beta}\big). \tag{6.58}$$

By Proposition 6.6, we have

$$\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}\big(A\cdot\big(\nabla A+t^{-1}\underline{A}+A\cdot\underline{A}\big)\big)\big\|_{L_t^aL_x^2}\lesssim\Delta_0^2\quad\text{with } 4<a<\infty, \tag{6.59}$$

which allows us to renormalize the curvature terms ρ, σ and $\underline{\beta}$. It remains to estimate the following three terms:

$$\mathcal{U}=\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}(\zeta\cdot\check{\rho})\big\|_{L_t^aL_x^2},\quad\mathcal{V}=\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}(\zeta^\star\check{\sigma})\big\|_{L_t^aL_x^2},\quad\mathcal{W}=\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}(V\check{\underline{\beta}})\big\|_{L_t^aL_x^2}.$$

By (2.13), clearly $\check{\sigma}=\operatorname{curl}\zeta$. Thus we obtain $\mathcal{V}=\|t^{-\frac{1}{a}}\mathcal{D}^{-1}\mathcal{E}_2(\zeta)\|_{L_t^aL_x^2}\lesssim\mathcal{N}_1(\zeta)^2$.

Now we consider the term $\mathcal{U}$. By (2.15), we have

$$\mathcal{U}\lesssim\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}\big(\big(\check{K}+r^{-2}-s^{-2}\big)\cdot\zeta\big)\big\|_{L_t^aL_x^2}+\big\|t^{-\frac{1}{a}}\mathcal{D}^{-2}\big(A\cdot\big(A\cdot\underline{A}+t^{-1}\underline{A}\big)\big)\big\|_{L_t^aL_x^2}. \tag{6.60}$$

By using (6.59) the second term on the right-hand side of (6.60) can be bounded by Δ_0^2. Due to (6.53), Proposition 3.1 and Proposition 4.1, the first term can be bounded by $(\check{K}_{\gamma_0}+\Delta_0)\mathcal{N}_1(\zeta)$, whence $\mathcal{U}\lesssim\Delta_0^2$ follows.

To estimate $\mathcal{W}$, using (6.39), it suffices to show

$$\big\|\nabla_L\mathcal{D}^{-2}(V\check{\underline{\beta}})\big\|_{L_t^1L_x^2}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{6.61}$$

Note that

$$\big\|\nabla_L\mathcal{D}^{-2}(V\check{\underline{\beta}})\big\|_{L_t^1L_x^2}\leqslant\big\|\mathcal{D}^{-2}\nabla_L(V\check{\underline{\beta}})\big\|_{L_t^1L_x^2}+\big\|\big[\nabla_L,\mathcal{D}^{-2}\big](V\check{\underline{\beta}})\big\|_{L_t^1L_x^2}=\mathcal{W}_1+\mathcal{W}_2.$$

First, by (6.54) and (6.39), we can estimate $\mathcal{W}_2$ as follows

$$\mathcal{W}_2\lesssim\big\|{}^\star\mathcal{D}_1^{-1}(V\check{\underline{\beta}})\big\|_{L_t^2L_x^{2+}}\lesssim\|V\check{\underline{\beta}}\|_{L_t^2L_x^2}+\big\|t^{0-}\mathcal{D}_1^{-1}(V\check{\underline{\beta}})\big\|_{L_t^2L_x^2}\lesssim\Delta_0\mathcal{R}_0,$$

where we employed Proposition 4.3 to obtain the last inequality.

It remains to prove $\mathcal{W}_1=\|\mathcal{D}^{-2}\nabla_L(V\check{\underline{\beta}})\|_{L_t^1L_x^2}\lesssim\Delta_0^2+\mathcal{R}_0$. Set

$$\mathcal{W}_1^{(1)}=\big\|\mathcal{D}^{-2}(\nabla_LV\cdot\check{\underline{\beta}})\big\|_{L_t^1L_x^2},\qquad\mathcal{W}_1^{(2)}=\big\|\mathcal{D}^{-2}(V\cdot\nabla_L\check{\underline{\beta}})\big\|_{L_t^1L_x^2},$$

clearly $\mathcal{W}_1 \leqslant \mathcal{W}_1^{(1)} + \mathcal{W}_1^{(2)}$.

By (6.38), we infer

$$\mathcal{W}_1^{(1)} \lesssim \big\| t\nabla_L V \cdot \underline{\check{\beta}} \big\|_{L_t^{1+} L_x^1} \lesssim \|\underline{\check{\beta}}\|_{L_t^2 L_x^2} \|t\nabla_L V\|_{L_t^{2+} L_x^2}.$$

According to (2.6), $t\nabla_L V \approx A + tA \cdot A$, by (6.13), $\|t\nabla_L V\|_{L_t^{2+} L_x^2} \lesssim \Delta_0$. Thus $\mathcal{W}_1^{(1)} \lesssim \Delta_0 \mathcal{R}_0$.

Finally, using (6.2) we deduce

$$\mathcal{W}_1^{(2)} \lesssim \big\| \mathcal{D}_1^{-1\star} \mathcal{D}_1^{-1} \big(V \cdot {}^\star\mathcal{D}_1(\rho, \sigma) \big) \big\|_{L_t^1 L_x^2} + \big\| t^{-1} \mathcal{D}^{-2} (V \cdot \check{R}) \big\|_{L_t^1 L_x^2} + \big\| \mathcal{D}_1^{-1\star} \mathcal{D}_1^{-1} \big(V \cdot (A \cdot \bar{R}) \big) \big\|_{L_t^1 L_x^2}. \tag{6.62}$$

The last term on the right-hand side of (6.62), in view of (6.38), can be bounded by

$$\|tV \cdot A \cdot \bar{R}\|_{L_t^{1+} L_x^1} \lesssim \|tV \cdot A\|_{L_t^{2+} L_x^2} \|\bar{R}\|_{L_t^2 L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

The second term can be treated similarly. At last, we estimate the first term on the right-hand side of (6.62) with the help of the formula

$${}^\star\mathcal{D}_1\big(V(\rho, \sigma)\big) = V\,{}^\star\mathcal{D}_1(\rho, \sigma) - \rho \nabla V + (\sigma \nabla V)^\star. \tag{6.63}$$

Using (6.38), we obtain

$$\begin{aligned} \big\| \mathcal{D}_1^{-1\star} \mathcal{D}_1^{-1} \big(-\rho \nabla V + (\sigma \nabla V)^\star \big) \big\|_{L_t^1 L_x^2} &\lesssim \|t\rho \nabla V\|_{L_t^{1+} L_x^1} + \big\| t(\sigma \nabla V)^\star \big\|_{L_t^{1+} L_x^1} \\ &\lesssim \mathcal{R}_0 \|\nabla V\|_{L_t^\infty L_x^2} \lesssim \mathcal{R}_0 \Delta_0. \end{aligned}$$

Combined with

$$\big\| \mathcal{D}_1^{-1\star} \mathcal{D}_1^{-1\star} \mathcal{D}_1 \big(V(\rho, \sigma) \big) \big\|_{L_t^1 L_x^2} \lesssim \big\| tV(\rho, \sigma) \big\|_{L_t^1 L_x^2} \lesssim \Delta_0 \mathcal{R}_0,$$

which is obtained by Proposition 4.3, we conclude

$$\big\| \mathcal{D}_1^{-1\star} \mathcal{D}_1^{-1} \big(V\,{}^\star\mathcal{D}_1(\rho, \sigma) \big) \big\|_{L_t^1 L_x^2} \lesssim \Delta_0 \mathcal{R}_0.$$

Therefore, $\mathcal{W}_1^{(2)} \lesssim \Delta_0^2 + \mathcal{R}_0$. We complete the proof of Proposition 6.9. □

6.5. Decomposition and correction estimates for $C_2(\check{R})$ and $C_3(\check{R})$.

In this section we will prove Proposition 6.2. To this end, according to Proposition 6.1, it remains to consider the "bad" terms $\beta \cdot \mathcal{D}^{-2} \check{R}$ and $\nabla \cdot \mathcal{D}^{-1} (\beta \cdot \mathcal{D}^{-2} \check{R})$. We establish the following result which, together with (6.19) and Proposition 6.8, immediately completes the proof of Proposition 6.2.

Theorem 6.1. *Assume that F is an S-tangent tensor field of appropriate order on $\mathcal{H}$ verifying $\mathcal{N}_2(F) < \infty$ and $\|t^{-\frac{1}{a}} \nabla_L F\|_{L_t^a L_x^2} < \infty$ with $4 < a < \infty$. Then we have*

(i) *There exists a 1-form E_0 such that*[7]

$$\beta = \nabla_L \cdot \mathcal{D}^{-1} \check{R} + E_0 \quad \text{with } \|E_0\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.64}$$

(ii) *There exists a decomposition $\beta \cdot F = \nabla_L P + E$, where P and E are tensor fields of the same type as $\beta \cdot F$ with the estimates*

$$\mathcal{N}_1(P) \lesssim \Delta_0 \mathcal{N}_2(F), \qquad \|E\|_{\mathcal{P}^0} \lesssim \Delta_0 \cdot \big(\mathcal{N}_2(F) + \|t^{-\frac{1}{a}} \nabla_L F\|_{L_t^a L_x^2} \big). \tag{6.65}$$

(iii) *There exist tensors $\bar{P}$ and $\bar{E}$ verifying* (6.65) *so that*

$$\nabla \cdot \mathcal{D}^{-1} (\beta \cdot F) = \nabla_L \bar{P} + \bar{E}. \tag{6.66}$$

[7] In Theorem 6.1 and the following proofs, $\check{R} = (\check{\rho}, -\check{\sigma})$ and $C_0(\check{R}) = [\nabla_L, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})$, since the other case in our convention will not come up here.

Proof. In view of (6.6), we have

$$\beta = \nabla_L \cdot \mathcal{D}^{-1}\check{R} + C_0(\check{R}) + Err. \tag{6.67}$$

This proves (i) by noting that $E_0 := Err + C_0(\check{R})$ satisfies $\|E_0\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0$ in view of (6.8) and (6.9).

Now we prove (ii). We have from (6.67) that

$$\beta \cdot F = \big(\nabla_L \mathcal{D}_1^{-1}\check{R} + Err + C_0(\check{R})\big) \cdot F = \nabla_L\big(\mathcal{D}_1^{-1}\check{R} \cdot F\big) + E_1^B + E_1^G,$$

where

$$E_1^B := -\mathcal{D}_1^{-1}\check{R} \cdot \nabla_L F \quad \text{and} \quad E_1^G := \big(Err + C_0(\check{R})\big) \cdot F.$$

By (5.2), (6.9) and (6.8) we obtain

$$\big\|E_1^G\big\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(F)\big(\|Err\|_{\mathcal{P}^0} + \|C_0(\check{R})\|_{\mathcal{P}^0}\big) \lesssim \Delta_0 \mathcal{N}_2(F).$$

By (5.1) and (6.18) we have

$$\begin{aligned}\big\|E_1^B\big\|_{\mathcal{P}^0} &\lesssim \mathcal{N}_1(\mathcal{D}^{-1}\check{R})\big(\big\|t^{-\frac{1}{a}}\nabla_L F\big\|_{L_t^a L_x^2} + \big\|t^{\frac{1}{2}}\nabla\nabla_L F\big\|_{L_t^2 L_x^2}\big) \\ &\lesssim \big(\mathcal{R}_0 + \Delta_0^2\big)\big(\mathcal{N}_2(F) + \big\|t^{-\frac{1}{a}}\nabla_L F\big\|_{L_t^a L_x^2}\big).\end{aligned} \tag{6.68}$$

Now we set

$$P_1 := \mathcal{D}_1^{-1}\check{R} \cdot F \quad \text{and} \quad E_1 := E_1^B + E_1^G, \tag{6.69}$$

from the above estimates we have

$$\|E_1\|_{\mathcal{P}^0} \lesssim \Delta_0\big(\mathcal{N}_2(F) + \big\|t^{-\frac{1}{a}}\nabla_L F\big\|_{L_t^a L_x^2}\big).$$

In order to estimate $\mathcal{N}_1(P_1)$, let us estimate $\|E_1\|_{L^2(\mathcal{H})}$ first. By using Hölder's inequality and Sobolev inequalities, we can obtain

$$\begin{aligned}\big\|E_1^B\big\|_{L^2(\mathcal{H})} &= \big\|\mathcal{D}^{-1}\check{R} \cdot \nabla_L F\big\|_{L^2(\mathcal{H})} \lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\nabla_L F\|_{L_t^2 L_x^4} \\ &\lesssim \mathcal{N}_1(\mathcal{D}^{-1}\check{R})\big(\|\nabla\nabla_L F\|_{L^2(\mathcal{H})} + \big\|t^{-\frac{1}{2}}\nabla_L F\big\|_{L^2(\mathcal{H})}\big),\end{aligned}$$

and by using $\|E_1^G\|_{L^2(\mathcal{H})} \lesssim \|E_1^G\|_{\mathcal{P}^0}$ we can obtain $\|E_1^G\|_{L^2(\mathcal{H})} \lesssim \Delta_0\mathcal{N}_2(F)$. Therefore

$$\|E_1\|_{L^2(\mathcal{H})} \lesssim \Delta_0 \mathcal{N}_2(F). \tag{6.70}$$

Now we show

$$\mathcal{N}_1(P_1) \lesssim \mathcal{N}_2(F)(\Delta_0 + \mathcal{R}_0). \tag{6.71}$$

With the help of $\nabla_L P_1 = \beta \cdot F - E_1$ and (6.70) we can estimate $\|\nabla_L P_1\|_{L_t^2 L_x^2}$ as follows

$$\|\nabla_L P_1\|_{L_t^2 L_x^2} \lesssim \|\beta \cdot F\|_{L_t^2 L_x^2} + \|E_1\|_{L_t^2 L_x^2} \lesssim (\mathcal{R}_0 + \Delta_0)\mathcal{N}_2(F).$$

Similar to [9, Section 6.12], we get $\|\nabla P_1\|_{L_t^2 L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F)$. Therefore (ii) is proved.

Finally we prove (iii) by using the iteration procedure in [9, Section 6.12]. Let $P_0 := \mathcal{D}F$, then we can apply (ii) to construct iteratively two sequences of S-tangent tensor fields $\{P_i\}$ and $\{E_i\}$ such that

$$\beta \cdot \mathcal{D}^{-1} P_{i-1} = \nabla_L P_i + E_i \tag{6.72}$$

and

$$\begin{aligned}&\mathcal{N}_1(P_i) \leqslant C\Delta_0 \mathcal{N}_2\big(\mathcal{D}^{-1}P_{i-1}\big), \\ &\|E_i\|_{\mathcal{P}^0} \leqslant C\Delta_0\big(\mathcal{N}_2\big(\mathcal{D}^{-1}P_{i-1}\big) + \big\|t^{-\frac{1}{a}}\nabla_L \mathcal{D}^{-1}P_{i-1}\big\|_{L_t^a L_x^2}\big).\end{aligned}$$

Such P_i and E_i can be constructed as in the proof of (ii), in particular, P_1 and E_1 are given by (6.69).

By using Lemma 4.1 it is easy to see that

$$\mathcal{N}_1(P_k) \leqslant (C\Delta_0)^k \mathcal{N}_2\big(\mathcal{D}^{-1}P_0\big) = (C\Delta_0)^k \mathcal{N}_2(F). \tag{6.73}$$

Moreover we have

Proposition 6.10. *For* $\{P_k\}_{k=1}^{\infty}$ *and* $\{E_k\}_{k=1}^{\infty}$ *there hold*

$$\left\|t^{-\frac{1}{a}}\nabla_L\mathcal{D}^{-1}P_k\right\|_{L_t^aL_x^2} \lesssim \Delta_0\big(\mathcal{N}_2\big(\mathcal{D}^{-1}P_{k-1}\big)+\left\|\nabla_L\mathcal{D}^{-1}P_{k-1}\right\|_{L_t^aL_x^2}\big), \tag{6.74}$$

$$\left\|\nabla\cdot\mathcal{D}^{-1}E_k\right\|_{\mathcal{P}^0} \lesssim \|E_k\|_{\mathcal{P}^0}+\Delta_0\big(\mathcal{N}_2\big(\mathcal{D}^{-1}P_{k-1}\big)+\left\|\nabla_L\mathcal{D}^{-1}P_{k-1}\right\|_{L_t^aL_x^2}\big). \tag{6.75}$$

We will prove this result at the end of this section. We observe that (6.74) and Lemma 4.1 clearly imply

$$\|E_k\|_{\mathcal{P}^0} \leqslant (C\Delta_0)^k\big(\mathcal{N}_2(F)+\left\|t^{-\frac{1}{a}}\nabla_LF\right\|_{L_t^aL_x^2}\big). \tag{6.76}$$

We note that

$$\begin{aligned}\nabla\cdot\mathcal{D}^{-1}\big(\beta\cdot\mathcal{D}^{-1}P_k\big) &= \nabla\cdot\mathcal{D}^{-1}(\nabla_LP_{k+1}+E_{k+1})\\ &= \nabla_L\big(\nabla\cdot\mathcal{D}^{-1}P_{k+1}\big)+\big[\nabla\cdot\mathcal{D}^{-1},\nabla_L\big]P_{k+1}+\nabla\cdot\mathcal{D}^{-1}E_{k+1}\\ &= \nabla_L\big(\nabla\cdot\mathcal{D}^{-1}P_{k+1}\big)+\beta\cdot\mathcal{D}^{-1}P_{k+1}+\nabla\cdot\mathcal{D}^{-1}\big(\beta\cdot\mathcal{D}^{-1}P_{k+1}\big)\\ &\quad+\big[\nabla_L,\nabla\cdot\mathcal{D}^{-1}\big]_gP_{k+1}+\nabla\cdot\mathcal{D}^{-1}E_{k+1}\end{aligned}$$

where, for any appropriate S-tangent tensor field F,

$$\big[\nabla_L,\nabla\cdot\mathcal{D}^{-1}\big]_gF=[\nabla_L,\nabla]_g\mathcal{D}^{-1}F+\nabla\cdot\big[\nabla_L,\mathcal{D}^{-1}\big]_gF.$$

This together with the definition of P_k and E_k implies

$$\nabla\mathcal{D}^{-1}(\beta\cdot F)=\nabla_L\bar{P}_k+\nabla\cdot\mathcal{D}^{-1}(\nabla_LP_k)+\bar{E}_k,$$

where

$$\begin{aligned}\bar{P}_k &= \nabla\cdot\mathcal{D}^{-1}(P_1+\cdots+P_{k-1})+P_2+\cdots+P_k.\\ \bar{E}_k &= \big[\nabla\cdot\mathcal{D}^{-1},\nabla_L\big]_g(P_1+\cdots+P_{k-1})+\nabla\cdot\mathcal{D}^{-1}(E_1+\cdots+E_k)+E_2+\cdots+E_k.\end{aligned}$$

It follows from (6.73), (6.75), (6.76) and (6.22) that

$$\mathcal{N}_1(\bar{P}_k-\bar{P}_j)\leqslant\mathcal{N}_2(F)\sum_{j+1\leqslant m\leqslant k}(C\Delta_0)^m\lesssim\Delta_0\mathcal{N}_2(F),$$

and

$$\begin{aligned}\|\bar{E}_k-\bar{E}_j\|_{\mathcal{P}^0} &\leqslant \big(\mathcal{N}_2(F)+\left\|t^{-\frac{1}{a}}\nabla_LF\right\|_{L_t^aL_x^2}\big)\sum_{j+1\leqslant m\leqslant k}(C\Delta_0)^m\\ &\lesssim \Delta_0\big(\mathcal{N}_2(F)+\left\|t^{-\frac{1}{a}}\nabla_LF\right\|_{L_t^aL_x^2}\big).\end{aligned}$$

Therefore $\{\bar{P}_k\}$ forms a Cauchy sequence relative to the norm $\mathcal{N}_1(\cdot)$, while $\{\bar{E}_k\}$ forms a Cauchy sequence relative to the $\mathcal{P}^0$ norm. Denote by $\bar{P}$ and $\bar{E}$ their corresponding limits, we have

$$\mathcal{N}_1(\bar{P})\lesssim\Delta_0\mathcal{N}_2(F)\quad\text{and}\quad\|\bar{E}\|_{\mathcal{P}^0}\lesssim\Delta_0\big(\mathcal{N}_2(F)+\left\|t^{-\frac{1}{a}}\nabla_LF\right\|_{L_t^aL_x^2}\big).$$

We also observe that for sufficiently small Δ_0,

$$\left\|\nabla\mathcal{D}^{-1}(\beta\cdot F)-\nabla_L\bar{P}_k-\bar{E}_k\right\|_{L_t^2L_x^2}=\left\|\nabla\cdot\mathcal{D}^{-1}(\nabla_LP_k)\right\|_{L_t^2L_x^2}\leqslant\mathcal{N}_1(P_k).$$

Letting $k\to+\infty$, we get

$$\left\|\nabla\mathcal{D}^{-1}(\beta\cdot F)-\nabla_L\bar{P}-\bar{E}\right\|_{L_t^2L_x^2}=0.$$

Hence $\nabla\mathcal{D}^{-1}(\beta\cdot F)=\nabla_L\bar{P}+\bar{E}$. This completes the proof of (iii).

Now we conclude this section by proving Proposition 6.10. We first prove (6.75). By using (4.6) we have

$$\left\|\nabla\mathcal{D}^{-1}E_k\right\|_{\mathcal{P}^0}\lesssim\|E_k\|_{\mathcal{P}^0}+\Delta_0\left\|\mathcal{D}^{-1}E_k\right\|_{L_t^aL_x^2}^q\|E_k\|_{L^2(\mathcal{H})}^{1-q},$$

where $4 < a < \infty$ and $1 > q > \gamma_0$. Thus it suffices to show for $4 < a < \infty$ that

$$\|t^{-\frac{1}{a}}\mathcal{D}^{-1}E_k\|_{L_t^a L_x^2} \lesssim \Delta_0\big(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1}) + \|\nabla_L \mathcal{D}^{-1}P_{k-1}\|_{L_t^a L_x^2}\big). \tag{6.77}$$

By the construction of P_k and E_k, it suffices to show it for $k = 1$. To this end, in view of $E_1 = E_1^G + E_1^B$, we can complete the proof by using Proposition 6.7 and the estimate

$$\begin{aligned}\|t^{-\frac{1}{a}}\mathcal{D}^{-1}E_1^B\|_{L_t^a L_x^2} &\lesssim \|t^{\frac{1}{2}-\frac{1}{a}}E_1^B\|_{L_t^a L_x^{4/3}} \lesssim \|\mathcal{D}_1^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\nabla_L F\|_{L_t^a L_x^2}\\ &\lesssim (\Delta_0^2 + \mathcal{R}_0)\|\nabla_L F\|_{L_t^a L_x^2}\end{aligned}$$

which follows from Proposition 4.3 and Hölder inequality.

In order to prove (6.74), we first note that

$$\|t^{-\frac{1}{a}}\nabla_L \mathcal{D}^{-1}P_k\|_{L_t^a L_x^2} \lesssim \|t^{-\frac{1}{a}}[\nabla_L, \mathcal{D}^{-1}]P_k\|_{L_t^a L_x^2} + \|t^{-\frac{1}{a}}\mathcal{D}^{-1}\nabla_L P_k\|_{L_t^a L_x^2}. \tag{6.78}$$

By using (6.42), the first term on the right-hand side of (6.78) can be estimated as

$$\|t^{-\frac{1}{a}}C_0(P_k)\|_{L_t^a L_x^2} \lesssim \mathcal{N}_1(t^{-\frac{1}{2}}\mathcal{D}^{-1}P_k) \lesssim \mathcal{N}_2(t^{\frac{1}{2}}\mathcal{D}^{-1}P_k) \lesssim \mathcal{N}_1(P_k),$$

while by using (6.72), (6.77) and (6.41), the second term can be estimated as

$$\begin{aligned}\|t^{-\frac{1}{a}}\mathcal{D}^{-1}\nabla_L P_k\|_{L_t^a L_x^2} &\lesssim \|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\beta\cdot\mathcal{D}^{-1}P_{k-1} - E_k)\|_{L_t^a L_x^2}\\ &\lesssim \|t^{-\frac{1}{a}}\mathcal{D}^{-1}(\beta\cdot\mathcal{D}^{-1}P_{k-1})\|_{L_t^a L_x^2} + \|t^{-\frac{1}{a}}\mathcal{D}^{-1}E_k\|_{L_t^a L_x^2}\\ &\lesssim \Delta_0\big(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1}) + \|\nabla_L\mathcal{D}^{-1}P_{k-1}\|_{L_t^a L_x^2}\big).\end{aligned}$$

Therefore (6.74) is proved. □

7. Proof of main result, Theorem 1.1

In this section, we prove the main result, Theorem 1.1, based on the bootstrap principle. In addition to (BA1) and (BA2), we also make the following auxiliary bootstrap assumption

$$\|A\cdot\underline{A}\|_{\mathcal{P}^0} \leqslant \Delta_0^2.$$

In order to complete the proof of Theorem 1.1, it suffices to show that all the inequalities in (BA1), (BA2) and (BA3) still hold with Δ_0 replaced by $\Delta_0/2$ when $0 < \mathcal{R}_0 < \Delta_0$ are sufficiently small.

7.1. *Estimates for* $\operatorname{tr}\chi$ *and* $\hat{\chi}$

Step 1. Recall (2.6) and Proposition 2.1, for $V := \operatorname{tr}\chi - \frac{2}{s}$ we have

$$\nabla_L V = -\frac{2}{s}V - \frac{1}{2}V^2 - |\hat{\chi}|^2 \quad\text{and}\quad V = O(s) \text{ as } s \to 0. \tag{7.1}$$

Integrate the equation along any null geodesic and using (BA1), we obtain

$$\left\|\operatorname{tr}\chi - \frac{2}{t}\right\|_{L_t^\infty L_\omega^\infty} \lesssim \|\hat{\chi}\|^2_{L_\omega^\infty L_t^2} + \|V\|^2_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2. \tag{7.2}$$

Moreover, by using (BA1), (3.11) and the Hölder inequality, we also get from (7.1) that

$$\|\nabla_L V\|_{L_x^2 L_t^\infty} \lesssim \|t^{-1}V\|_{L_x^2 L_t^\infty} + \|V^2\|_{L_x^2 L_t^\infty} + \||\hat{\chi}|^2\|_{L_x^2 L_t^\infty} \lesssim \Delta_0^2. \tag{7.3}$$

Step 2. Estimates for $\nabla\operatorname{tr}\chi$. First we have from (5.2) that

$$\beta = \mathcal{D}_1^{-1}L(\check{\rho}, -\check{\sigma}) + F_1 \quad\text{with}\quad F_1 = Err.$$

This together with (2.12) gives

$$\hat{\chi} = -\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\nabla_L(\check{\rho}, -\check{\sigma}) + \mathcal{D}_2^{-1}F_1 + \mathcal{D}_2^{-1}\big(\nabla \operatorname{tr}\chi + A\cdot A + s^{-1}\zeta\big).$$

Set $\mathcal{D}^{-2} = \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1} = \mathcal{D}_2^{-1}$, we obtain after taking covariant derivatives

$$\nabla\hat{\chi} = -\nabla\cdot\mathcal{D}^{-2}\nabla_L(\check{R}) + F + \nabla\cdot\mathcal{D}^{-1}M, \tag{7.4}$$

where $F = \nabla\cdot\mathcal{D}^{-1}(F_1 + A\cdot A + s^{-1}\zeta)$ and $M = \nabla\operatorname{tr}\chi$.

We claim that

$$\|F\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.5}$$

Indeed, by (6.49), Theorem 4.1, the non-sharp product estimate, Propositions 6.6, 4.1, (3.11) and the bootstrap assumptions we have

$$\|F\|_{\mathcal{P}^0} \lesssim \|s^{-1}\xi\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0.$$

It remains to estimate $\|s^{-1}\zeta\|_{\mathcal{P}^0}$. In view of (1.2) and (2.13), we have the Hodge system

$$\operatorname{div}\zeta = -\mu - \check{\rho} + |\zeta|^2, \quad \operatorname{curl}\zeta = \check{\sigma}. \tag{7.6}$$

Thus it follows from Proposition 4.1 that

$$\begin{aligned}\|s^{-1}\zeta\|_{\mathcal{P}^0} &\leqslant \|\nabla\zeta\|_{L^2(\mathcal{H})} + \|t^{-1}\zeta\|_{L^2(\mathcal{H})} \lesssim \|\mu\|_{L^2(\mathcal{H})} + \|\check{R}\|_{L^2(\mathcal{H})} + \|A\cdot A\|_{L^2(\mathcal{H})}\\ &\leqslant \|\mu\|_{L^2} + \Delta_0^2 + \mathcal{R}_0.\end{aligned} \tag{7.7}$$

In order to estimate $\|\mu\|_{L^2(\mathcal{H})}$, we use (2.20), which symbolically can be written as

$$\frac{d}{ds}\mu + \frac{3}{s}\mu = \hat{\chi}\cdot\nabla\widehat{\otimes}\zeta + \frac{1}{s}\check{R} + A\cdot\bar{R}.$$

Using $\lim_{s\to 0} s\mu = 0$ given in Proposition 2.1, and integrating the above equation in s, we derive

$$\begin{aligned}\|\mu\|_{L^2(\mathcal{H})} &\lesssim \Big\|t^{-2}\int_0^t s^3\big(A\cdot\nabla A + s^{-1}\check{R} + A\cdot\bar{R}\big)\,ds\Big\|_{L_t^2 L_\omega^2}\\ &\lesssim \|t(A\cdot\nabla A + A\cdot\bar{R})\|_{L_\omega^2 L_t^1} + \|\check{R}\|_{L_t^2 L_x^2}\\ &\lesssim \|A\|_{L_\omega^\infty L_t^2}\|\nabla A\|_{L_t^2 L_x^2} + \|\check{R}\|_{L^2(\mathcal{H})} + \|A\|_{L_\omega^\infty L_t^2}\|\bar{R}\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Therefore $\|s^{-1}\xi\|_{\mathcal{P}^0} \leqslant \Delta_0^2 + \mathcal{R}_0$, and the claim (7.5) is proved.

Now we come back to (7.4). By using the notations in (6.5), we can write

$$\nabla\cdot\mathcal{D}^{-2}\nabla_L(\check{R}) = \nabla_L\big(\nabla\cdot\mathcal{D}^{-2}\check{R}\big) + C(\check{R})$$

where, by Proposition 6.2, there exist tensors P' and E' so that $C(\check{R}) = \nabla_L P' + E'$ and $\mathcal{N}_1(P') + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2$. Thus (7.4) becomes

$$\nabla\hat{\chi} = \nabla_L P + \nabla\cdot\mathcal{D}^{-1}M + E \tag{7.8}$$

where $P = \nabla\cdot\mathcal{D}^{-2}\check{R} + P'$ and $E = F + E'$, both of which satisfy, by using Corollary 6.19 and (7.5),

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

By combining (7.8) with (2.8) we obtain

$$\nabla_L M + \frac{3}{s}M = A\cdot\big(\nabla_L P + \nabla\cdot\mathcal{D}^{-1}M + E\big) - \frac{3}{2}V\cdot M.$$

Since Proposition 2.1 implies $\lim_{s\to 0} sM = 0$, we can apply Theorem 5.1, Theorem 5.2, (BA1) and (7.10) to obtain

$$\begin{aligned}\|M\|_{\mathcal{B}^0} &\lesssim \big(\mathcal{N}_1(P) + \|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0} + \|E\|_{\mathcal{P}^0} + \Delta_0^2\big)\big(\mathcal{N}_1(A) + \|A\|_{L_\omega^\infty L_t^2}\big)\\ &\leqslant \Delta_0\big(\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0\big).\end{aligned} \tag{7.9}$$

Thus we need to estimate $\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0}$. To this end, using $\lim_{s\to 0} sM=0$ we derive from (2.8) that

$$\|M\|_{L_x^2L_t^\infty}\lesssim\left\|t^{-3}\int_0^t s^3A\cdot\nabla A\,ds\right\|_{L_x^2L_t^\infty}\lesssim\|\nabla A\cdot A\|_{L_x^2L_t^1}\lesssim\Delta_0^2,$$

while, by Proposition 4.3 we have

$$\|\mathcal{D}^{-1}M\|_{L_t^aL_x^2}\lesssim\|tM\|_{L_t^aL_x^2}\lesssim\|M\|_{L_x^2L_t^\infty}\lesssim\Delta_0^2.$$

Therefore we infer from Theorem 4.1 that

$$\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0}\lesssim\|M\|_{\mathcal{P}^0}+\Delta_0^2. \tag{7.10}$$

Since $\|M\|_{\mathcal{P}^0}\lesssim\|M\|_{\mathcal{B}^0}$, we obtain from (7.9) and (7.10) that

$$\|M\|_{\mathcal{B}^0}\lesssim\Delta_0\big(\|M\|_{\mathcal{B}^0}+\Delta_0^2+\mathcal{R}_0\big). \tag{7.11}$$

Using the smallness of Δ_0 we get

$$\|M\|_{\mathcal{B}^0}+\|M\|_{\mathcal{P}^0}+\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.12}$$

From the above argument, by setting $\tilde{E}=E+\nabla\cdot\mathcal{D}^{-1}M$ we obtain from (7.8) the decomposition

$$\nabla\hat{\chi}=\nabla_L P+\tilde{E}\quad\text{and}\quad\mathcal{N}_1(P)+\|\tilde{E}\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.13}$$

Step 3. Estimates for $\mathcal{N}_1(\hat{\chi})$ and $\|\hat{\chi}\|_{L_\omega^\infty L_t^2}$. In order to estimate $\mathcal{N}_1(\hat{\chi})$, we first use Proposition 4.1 and (2.12) to get

$$\begin{aligned}\|t^{-1}\hat{\chi}\|_{L_t^2L_x^2}+\|\nabla\hat{\chi}\|_{L_t^2L_x^2}&\lesssim\|\beta\|_{L_t^2L_x^2}+\|\nabla\operatorname{tr}\chi\|_{L_t^2L_x^2}+\|A\cdot A\|_{L_t^2L_x^2}+\|t^{-1}\zeta\|_{L_t^2L_x^2}\\&\lesssim\mathcal{R}_0+\Delta_0^2.\end{aligned}$$

We then use (2.7) to obtain

$$\|\nabla_L\hat{\chi}\|_{L_t^2L_x^2}\lesssim\|V\cdot\hat{\chi}\|_{L_t^2L_x^2}+\|t^{-1}\hat{\chi}\|_{L_t^2L_x^2}+\mathcal{R}_0\lesssim\Delta_0^2+\mathcal{R}_0.$$

Therefore

$$\mathcal{N}_1(\hat{\chi})\lesssim\mathcal{R}_0+\Delta_0^2. \tag{7.14}$$

Using Theorem 5.4, (7.13) and (7.14) we have

$$\|\hat{\chi}\|_{L_\omega^\infty L_t^2}\lesssim\mathcal{N}_1(\hat{\chi})+\mathcal{N}_1(P)+\|\tilde{E}\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.15}$$

In view of (7.2), (7.3), (7.12), (7.14) and (7.15), we verify for small $\mathcal{R}_0$ that the bootstrap assumptions (BA1) for $\operatorname{tr}\chi$ and $\hat{\chi}$ hold true with Δ_0 replaced by $\Delta_0/2$.

7.2. *Estimates for μ and $\nabla\zeta$*

We first decompose $\nabla\zeta$ as we did for $\nabla\hat{\chi}$. By using (7.6) and (2.19) we derive symbolically that

$$\zeta=\mathcal{D}_1^{-1}\cdot J\cdot{}^\star\mathcal{D}_1^{-1}\left(\nabla_L\check{\underline{\beta}}+\frac{1}{s}R_0\right)+\mathcal{D}_1^{-1}\cdot J\cdot F_1-\mathcal{D}_1^{-1}(\mu,0)+\mathcal{D}_1^{-1}(A\cdot A).$$

where J is the involution $(\rho,\sigma)\to(-\rho,\sigma)$, $R_0=\underline{\beta}$ and $F_1=\widetilde{Err}$ is given by (6.6). Set $\mathcal{D}^{-2}=\mathcal{D}_1^{-1}\cdot J\cdot{}^\star\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1}=\mathcal{D}_1^{-1}$, by using (6.5) we get

$$\nabla\zeta=\nabla_L\big(\nabla\cdot\mathcal{D}^{-2}\check{\underline{\beta}}\big)+C(\check{R})+\nabla\cdot\mathcal{D}^{-1}M+F+\frac{1}{s}\nabla\mathcal{D}^{-2}R_0$$

where $M=(\mu,0)$ and $F=\nabla\cdot\mathcal{D}^{-1}(F_1+A\cdot A)$. By (7.5) we have $\|F\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0$. In view of Proposition 6.2, we can write $\nabla\zeta$, for some tensors P and E, in the form

$$\nabla\zeta=\nabla_L P+\nabla\cdot\mathcal{D}^{-1}M+E\quad\text{with}\quad\mathcal{N}_1(P)+\|E\|_{\mathcal{P}_0}\leqslant\Delta_0^2+\mathcal{R}_0. \tag{7.16}$$

Using (2.20) we see that M satisfies

$$\frac{d}{ds}M+\frac{3}{s}M=A\cdot\big(\nabla_L P+\nabla\cdot\mathcal{D}^{-1}M+E\big)+s^{-1}\check{\rho}+V\cdot\check{\rho}+A\cdot(A\cdot\underline{A}+M).$$

While by using (6.2), (BA3), and noting that $\|[\nabla_L,{}^\star\mathcal{D}_1^{-1}]\check{R}\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0$, we can find two tensors $p':={}^\star\mathcal{D}_1^{-1}\check{\underline{\beta}}$ and e' such that

$$(\check{\rho},\check{\sigma})=\nabla_L p'+e'\quad\text{with}\quad\mathcal{N}_1(p')+\|e'\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.17}$$

Thus there exist two tensors $\tilde{P}$ and $\tilde{E}$ such that

$$\nabla_L M+\frac{3}{s}M=A\cdot\nabla_L\tilde{P}+\frac{1}{s}(\nabla_L p'+e')+A\cdot\big(\nabla\cdot\mathcal{D}^{-1}M+\tilde{E}\big). \tag{7.18}$$

with

$$\mathcal{N}_1(\tilde{P})\lesssim\mathcal{R}_0+\Delta_0^2\quad\text{and}\quad\|\tilde{E}\|_{\mathcal{P}^0}\lesssim\mathcal{R}_0+\Delta_0^2+\|M\|_{\mathcal{P}^0}.$$

We first claim that

$$\left\|\frac{1}{t^3}\int_0^t s^2\nabla_L p'\,ds\right\|_{\mathcal{P}^0},\ \left\|\frac{1}{t^3}\int_0^t s^2 e'\,ds\right\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.19}$$

Indeed, for the first term in 7.19, we recall that $p'={}^\star\mathcal{D}_1^{-1}\check{\underline{\beta}}$, then

$$\lim_{s\to0}\|s^{-1}p'\|_{B^0_{2,1}}\lesssim\mathcal{N}_1(p')\lesssim\Delta_0^2+\mathcal{R}_0,\ \|s^{-1}p'\|_{\mathcal{P}^0}\lesssim\mathcal{N}_1(p')\lesssim\mathcal{R}_0+\Delta_0^2. \tag{7.20}$$

Applying Proposition 3.4 to the first term in (7.19), with the help of (7.20) we derive

$$\begin{aligned}
\left\|\frac{1}{t^2}\int_0^t s^2\nabla_L p'\,ds\right\|_{\mathcal{P}^0}&\lesssim\sum_{k>0}\left\|E_k\frac{1}{t^2}\int_0^t s^2\nabla_L p'\,ds\right\|_{L_t^2L_\omega^2}+\left\|\frac{1}{t^2}\int_0^t s^2\nabla_L p'\,ds\right\|_{L_t^2L_\omega^2}\\
&\lesssim\sum_{k>0}\left\{\left\|E_k\Big(p'-\lim_{s\to0}p'\Big)\right\|_{L_t^2L_\omega^2}+\left\|E_k t^{-2}\int_o^t sp'\,ds\right\|_{L_t^2L_\omega^2}\right\}+\|\nabla_L p'\|_{L_t^2L_x^2}\\
&\lesssim\sum_{k>0}\|E_k p'\|_{L_t^2L_\omega^2}+\sum_{k>0}\left\|\frac{1}{t^2}\int_0^t sE_k p'\,ds\right\|_{L_t^2L_\omega^2}+\lim_{s\to0}\|s^{-1}p'\|_{\tilde{B}^0_{2,1}}+\|\nabla_L p'\|_{L_t^2L_x^2}\\
&\lesssim\|t^{-1}p'\|_{\mathcal{P}^0}+\|\nabla_L p'\|_{L_t^2L_x^2}+\lim_{s\to0}\|s^{-1}p'\|_{B^0_{2,1}}\leqslant\mathcal{R}_0+\Delta_0^2.
\end{aligned}$$

The second term in (7.19) can be estimated similarly.

By the definition of μ, Proposition 4.3 and Proposition 6.6, we have for $4<a<\infty$,

$$\begin{aligned}
\|\mathcal{D}^{-1}\mu\|_{L_t^aL_x^2}&\lesssim\big\|\mathcal{D}^{-1}(\check{\rho}+A\cdot\underline{A}+\operatorname{div}\zeta)\big\|_{L_t^aL_x^2}\\
&\lesssim\mathcal{N}_1(\mathcal{D}^{-1}\check{R})+\mathcal{N}_1(\zeta)+\|t\mathcal{E}_4\|_{L_t^aL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0.
\end{aligned}$$

Thus, in view of (4.6), $\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0}\lesssim\|M\|_{\mathcal{P}^0}+\Delta_0^2+\mathcal{R}_0$. Now we can apply Proposition 5.2 to (7.18) to get

$$\|M\|_{\mathcal{P}^0}\lesssim\big(\|M\|_{\mathcal{P}^0}+\Delta_0^2+\mathcal{R}_0\big)\Delta_0+\Delta_0^2+\mathcal{R}_0.$$

Since $\mathcal{R}_0\leqslant\Delta_0<1/2$, we conclude that

$$\|M\|_{\mathcal{P}^0},\|\nabla\cdot\mathcal{D}^{-1}M\|_{\mathcal{P}^0},\|M\|_{L^2(\mathcal{H})}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.21}$$

Following the same manner as above, we can get

$$\|t^{1/2}\mu\|_{\mathcal{B}^0}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{7.22}$$

Using (2.20) and noting $s\mu \to 0$ as $s \to 0$, we can easily get

$$\|t^{1/2} M\|_{L_x^2 L_t^\infty} \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{7.23}$$

Similar to the estimates for $\mathcal{N}_1(\hat{\chi})$ and $\|\hat{\chi}\|_{L_x^\infty L_t^2}$, we may use (2.9), (7.7) and (7.16) to derive that

$$\mathcal{N}_1(\zeta) + \|\zeta\|_{L_x^\infty L_t^2} \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{7.24}$$

In view of (7.21)–(7.24) we verify for small $\mathcal{R}_0$ that the bootstrap assumptions (BA1) and (BA2) for ζ and μ hold true with Δ_0 replaced by $\Delta_0/2$.

7.3. *Estimates for* $\mathrm{tr}\underline{\chi}$ *and* $\hat{\underline{\chi}}$

It follows from (2.10) that

$$\frac{d}{ds}\underline{V} + \frac{1}{s}\underline{V} = -\frac{1}{2} V \cdot \underline{V} + 2\mu + \frac{1}{s} V + 4\check{\rho}. \tag{7.25}$$

Recall that $\lim_{s\to 0} s\underline{V}(s,\omega) = 0$ given in Proposition 2.1, we infer

$$\begin{aligned}\|\underline{V}\|_{L_\omega^2 L_t^2} &\lesssim \left\| \frac{1}{t}\int_0^t s\left\{V \cdot \underline{V} + \check{\rho} + \mu + \frac{1}{s} V\right\} ds \right\|_{L_t^2 L_\omega^2} \\ &\lesssim \|tV \cdot \underline{V}\|_{L_t^2 L_\omega^2} + \|t\check{\rho}\|_{L_t^2 L_\omega^2} + \|V\|_{L_t^2 L_\omega^2} + \|\mu\|_{L_t^2 L_x^2} \leqslant \Delta_0^2 + \mathcal{R}_0\end{aligned}$$

and

$$\begin{aligned}\|t^{-1/2}\underline{V}\|_{L_x^2 L_t^\infty} &\lesssim \left\| \frac{1}{t^{1/2}}\int_0^t s\left\{V \cdot \underline{V} + \check{\rho} + \mu + \frac{1}{s} V\right\} ds \right\|_{L_\omega^2 L_t^\infty} \\ &\lesssim \|tV \cdot \underline{V}\|_{L_t^2 L_\omega^2} + \|\check{\rho}\|_{L_x^2 L_t^2} + \|V\|_{L_t^2 L_\omega^2} + \|\mu\|_{L_t^2 L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Recall also the decomposition (7.17), we may use Proposition 3.4 and (BA3), ignoring the low frequency terms, to get

$$\begin{aligned}\|t^{-1}\underline{V}\|_{\mathcal{P}^0} &\lesssim \sum_{k>0} \left\| E_k \frac{1}{t}\int_0^t V + s\mu + sV \cdot \underline{V} + s(\nabla_L p' + e')\, ds \right\|_{L_t^2 L_\omega^2} \\ &\lesssim \|t^{-1} p'\|_{\mathcal{P}^0} + \|e'\|_{\mathcal{P}^0} + \|t^{-1} V\|_{\mathcal{P}^0} + \|V \cdot \underline{V}\|_{\mathcal{P}^0} + \|\mu\|_{\mathcal{P}^0} \\ &\lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Similarly we can obtain

$$\|t^{-1/2}\underline{V}\|_{\mathcal{B}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

From (7.25) it is easy to see

$$\|\nabla_L \underline{V}\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Using (2.11) we can derive

$$\begin{aligned}&\|t^{-1/2}\hat{\underline{\chi}}\|_{L_x^2 L_t^\infty} \lesssim \|\nabla \zeta\|_{L^2(\mathcal{H})} + \|\zeta\|_{L_\omega^\infty L_t^2} + \|A \cdot \underline{A}\|_{L^2(\mathcal{H})} + \|t^{-1} A\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0, \\ &\|t^{-1}\hat{\underline{\chi}}\|_{L^2(\mathcal{H})} \lesssim \|\nabla \zeta\|_{L^2(\mathcal{H})} + \|\zeta\|_{L_\omega^\infty L_t^2} + \|A \cdot \underline{A}\|_{L^2(\mathcal{H})} + \|t^{-1} A\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0\end{aligned}$$

and

$$\|\nabla_L \hat{\underline{\chi}}\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In order to derive the Besov norms of $\hat{\underline{\chi}}$, we employ (7.16) to (2.11) to get

$$\nabla_L \hat{\underline{\chi}} + \frac{1}{s}\hat{\underline{\chi}} = -(\nabla_L P + E) - \frac{1}{2}\left(\underline{V}\cdot\hat{\chi} + V\cdot\hat{\underline{\chi}} - \frac{2}{s}\hat{\chi}\right) + \zeta\,\widehat{\otimes}\,\zeta. \tag{7.26}$$

With the help of $\lim_{s\to 0}|\hat{\underline{\chi}}| = 0$ given by Proposition 2.1, integrating the above equation in s, and using Proposition 3.4 and Lemma 3.3, we obtain

$$\begin{aligned}\big\|t^{-\frac{1}{2}}\hat{\underline{\chi}}\big\|_{\mathcal{B}^0} &\lesssim \|A\cdot\underline{A}\|_{\tilde{\mathcal{P}}^0} + \|s^{-1}\hat{\chi}\|_{\tilde{\mathcal{P}}^0} + \|E\|_{\tilde{\mathcal{P}}^0} + \|t^{-1}P\|_{\tilde{\mathcal{P}}^0} + \|t^{-1/2}P\|_{\tilde{\mathcal{B}}^0}\\ &\lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Similarly, we can obtain

$$\|t^{-1}\hat{\underline{\chi}}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

The above argument shows that for small $\mathcal{R}_0$ the bootstrap assumptions (BA2) for $\operatorname{tr}\underline{\chi}$ and $\hat{\underline{\chi}}$ hold true with Δ_0 replaced by $\Delta_0/2$.

It remains to show the justification of (BA3). By following the argument in [9, p. 524] we can obtain

$$\|A\cdot\underline{A}\|_{\mathcal{P}^0} \lesssim \Delta_0^3 + \Delta_0\mathcal{R}_0 \leqslant \frac{1}{2}\Delta_0^2,$$

provided that $\mathcal{R}_0$ is sufficiently small relative to Δ_0. We omit the details.

The proof of Theorem 1.1 is therefore complete.

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