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Hardy inequalities and dynamic instability of singular Yamabe metrics

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Abstract

We study the Cauchy problem

$$\text{(P)} \quad \begin{cases} u_t = \Delta u + |u|^{\sigma-1}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

for nonnegative functions $u : \mathbb{R}^N \times (0,+\infty) \to \mathbb{R}^+$. Here $N \geqslant 3$, $\sigma + 1 = 2^* = \frac{2N}{N-2}$ is the Sobolev exponent of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $u_0 = U$ is a time independent positive solution with nonempty singular set $\Sigma = \mathrm{Sing}(U)$, e.g. a distributional solution associated to a singular Yamabe metric on S^N. We show that, if Σ is a finite set, then problem (P) has a weak solution which is smooth for positive time. Hence, time independent singular solutions may be unstable and the Cauchy problem (P) may have infinitely many weak solutions. A similar weaker result is proved for any nonnegative distributional solution U when Σ is a compact set.

Résumé

Nous étudions le problème de Cauchy

$$\text{(P)} \quad \begin{cases} u_t = \Delta u + |u|^{\sigma-1}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

pour fonctions nonnégatives $u : \mathbb{R}^N \times (0,+\infty) \to \mathbb{R}^+$. Ici $N \geqslant 3$, $\sigma + 1 = 2^* = \frac{2N}{N-2}$ est la puissance critique pour l'injection de Sobolev $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ et $u_0 = U$ est une solution stationnaire singulière, par exemple une solution distributionelle associée à une métrique de Yamabe singulière sur S^N. Nous montrons que, si $\Sigma = \mathrm{Sing}(U)$ est un ensemble fini, alors le problème (P) a une solution faible qui est régulière pour temps positives. Par conséquent, solutions stationnaires singulières peuvent être instables et le problème de Cauchy (P) peut avoir un nombre infini de solutions faibles. De plus, nous montrons un rèsultat similaire pour chaque solution distributionelle U avec ensemble singulier compact.

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1. Introduction

In this paper we study global nonnegative weak solutions for the Cauchy problem

$$\text{(P)}\quad \begin{cases} u_t = \Delta u + |u|^{\sigma-1}u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

when $\sigma = \frac{N+2}{N-2}$ and the initial condition u_0 is neither bounded nor satisfies the usual integrability assumption $u_0 \in L^q(\mathbb{R}^N)$ for some $q \geqslant \frac{N}{2}(\sigma-1) = 2^*$, the Sobolev exponent associated to the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. It is well known that under this integrability assumption we can use the heat semigroup $S(t)$ to recast problem (P) in the integral form

$$\text{(E)}\quad u(t) = S(t)u_0 + \int_0^t S(t-s)\big|u(s)\big|^{\sigma-1}u(s)\,\mathrm{d}s, \tag{1.1}$$

and we can find a weak solution u of (E) (a so-called *mild* solution) in the space $C([0,T_0]; L^q(\mathbb{R}^N))$ for $T_0 = T_0(u_0) > 0$ sufficiently small. Such a solution exists by a contraction mapping argument, as shown e.g. in [54, 55,18]. See [50] and [24] for related results in more sophisticated function spaces of Lorentz and Besov–Morrey type. See also [43] for a proof based on energy method also for nonlinear Leray–Lions operators but in bounded domains. Moreover, each solution can be extended up to a maximal time $\overline{T} > T_0$ so that $u \in C([0,\overline{T}); L^q(\mathbb{R}^N))$, $u \in L^\infty_{\mathrm{loc}}((0,\overline{T}); L^\infty(\mathbb{R}^{\mathbb{N}}))$ (actually $u \in C^\infty(\mathbb{R}^N \times (0,\overline{T}))$ for nonnegative initial data) and $\|u(t)\|_\infty \to \infty$ as $t \nearrow \bar{T}$ by the classical blow-up alternative. Existence proof relies on the classical Kato trick of introducing an artificial seminorm $|||v||| = \operatorname{ess\,sup}_{0<t<T_0} t^\gamma \|v(t)\|_r$, $\gamma = \frac{N}{2}(\frac{1}{q} - \frac{1}{r})$, for $q < r \leqslant \sigma q$ to choose properly, whose finiteness, by standard parabolic theory, is responsible for the extra smoothness. Existence follows from the contraction mapping theorem for the operator

$$T(v)(t) = S(t)u_0 + \int_0^t S(t-s)\big|v(s)\big|^{\sigma-1}v(s)\,\mathrm{d}s, \tag{1.2}$$

on a suitable subset K of the space $X = \{v \in L^\infty(0,T_0; L^q(\mathbb{R}^N)),\ |||v||| < \infty\}$. Uniqueness holds in the subset K of this space and the same statement in the whole $C([0,T_0]; L^q(\mathbb{R}^N))$ holds but it has to be proved separately (see [54,55] and [4] for the case of bounded domains). However, we stress that uniqueness is a delicate issue and in the limiting case $q = \frac{N}{2}(\sigma-1)$, it depends on the assumption $q > \sigma$ which holds by our choice of σ. Otherwise, if $q = \frac{N}{2}(\sigma-1) = \sigma$, the so-called doubly critical case, nonuniqueness occurs (see [37,50]).

If the integrability assumption $u_0 \in L^q(\mathbb{R}^N)$ for some $q \geqslant \frac{N}{2}(\sigma-1)$ is not satisfied then the contraction argument breaks down and if $q < \frac{N}{2}(\sigma-1)$ there is some evidence that for suitable u_0 there is no solution in any reasonable weak sense (see [4,54]). On the other hand in this case nonuniqueness is well known (see [20]). An example of data of particular relevance is the family of singular initial conditions $u_0(x) = \lambda U(x)$, $\lambda > 0$, where

$$U(x) = \left(\frac{N-2}{2}\right)^{\frac{N-2}{2}} |x|^{\frac{2-N}{2}}, \quad U \in L^{2^*,\infty}(\mathbb{R}^N),\ U \notin L^{2^*}(\mathbb{R}^{\mathbb{N}}), \tag{1.3}$$

and the Lorentz space $L^{2^*,\infty}$ can be identified with the usual weak-L^{2^*} space of measurable functions satisfying $\sup_{s>0} s|\{|f|>s\}|^{1/2^*} < \infty$. Indeed, in this situation both the initial data and the equation are invariant under the transformation $U \to U_\delta$, $u \to u_\delta$, given by $U_\delta(x) = \delta^{\frac{2}{\sigma-1}}U(\delta x)$ and $u_\delta(x,t) = \delta^{\frac{2}{\sigma-1}}u(\delta x, \delta^2 t)$ for any $\delta > 0$. Furthermore, the L^{2^*}, the $L^{2^*,\infty}$ and even the artificial seminorm $\operatorname{ess\,sup}_{0<t<T_0} t^\gamma \|v(t)\|_r$, $\gamma = \frac{N}{2}(\frac{1}{q} - \frac{1}{r})$ for $r > q$ are invariant under the same scaling. As a consequence, in our example this invariance rules out the contraction argument unless a smallness assumption on λ is made (see e.g. [8,24,34]). The same kind of smallness assumption is required for a number of evolution equations in critical scale-invariant spaces, e.g. the nonlinear Schrödinger equation, the nonlinear wave equation, the Navier–Stokes system (see e.g. [8,44,26] Chapters 22 and 23 [24,34]). On the other hand, if we drop the smallness assumption and we take $\lambda = 1$ in the family above, then $u_0(x) = U(x)$ is a singular steady state but it is well known (see [15]) that problem (P) admits a weak solution (according to the definition below) which is smooth for positive time (quite surprisingly this regularisation phenomenon occurs even for $\lambda > 1$, $\lambda - 1 \ll 1$ as shown in the recent paper [47]). Thus, once smallness is dropped nonuniqueness may occur. This phenomenon happens also for some geometric flows when the initial data is a cone-like (homogeneous) time independent singular solution, like the mean curvature flow (see [22]), the wave map system in $\mathbb{R}^{2+1}$ with values into S^2 (see [10]) and it can be also proved for the gradient flow for harmonic maps from $\mathbb{R}^2$ to S^2 even for quasi-homogeneous data (see [42]). Here we stress that, except for the last paper cited, both the nonuniqueness results just mentioned and other existence results for similar problems with cone-like initial condition (see [16,13]) are obtained by reduction to ODE.

The aim of this paper is to shed some light in problem (P) for some initial condition $u_0 \in L^{2^*,\infty}(\mathbb{R}^N)$ including the one in (1.3) (actually, for even much more rough data) without any smallness assumption on the scale invariant norms of u_0 and without any reduction to ODE analysis.

For suitable positive functions u_0 we construct by the monotone iteration method weak solutions $\underline{u}$ as the pointwise limit of the suite $\{T^n(0)\}$ constructed inductively from (1.2). Due to the positivity of the initial data these solutions turn out to be the minimal positive solutions of (P). To be more precise, we assume $0 \leqslant u_0 \leqslant \bar{\Psi}$, for some (possibly discontinuous) $\bar{\Psi} \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ with suitable decay at infinity and satisfying $\Delta\bar{\Psi} + \bar{\Psi}^{\frac{N+2}{N-2}} \leqslant 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Under these assumptions the sequence $v_n = T^n(0)$ is increasing and pointwise convergent to a function $\underline{u} \leqslant \bar{\Psi}$ which is an a.e. solution of the integral equation (1.1). Actually this function is also a globally defined weak solution of (P) according to the following definition.

Definition 1.1. Let $u_0 \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $u_0 \geqslant 0$ a.e., and $u_0(x) = \mathcal{O}(\mathrm{e}^{C|x|^2})$ as $|x| \to \infty$ for some $C > 0$. Let $u : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ be a measurable function such that for some $C' > 0$ we have $|u(x,t)| = \mathcal{O}(\mathrm{e}^{C'|x|^2})$ as $|x| \to \infty$ uniformly on t. We say that u is a weak solution of problem (P) if $u \geqslant 0$ a.e., $u \in C^0(\mathbb{R}^+; L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N))$, $u(0) = u_0$ and for any $\psi \in C_0^\infty(\mathbb{R}^N \times \mathbb{R})$ we have

$$\int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x + \int_{\mathbb{R}^N\times\mathbb{R}^+} u(x,t)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N\times\mathbb{R}^+} u(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}^N\times\mathbb{R}^+} u(x,t)^{\frac{N+2}{N-2}}\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{1.4}$$

Weak supersolution can be obtained choosing $\bar{\Psi} = \lambda U$, where $\lambda \in (0,1]$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ is any positive distributional solution of $\Delta U + U^{\frac{N+2}{N-2}} = 0$. A plethora of such solutions with nonempty singular set is well known to exist (see e.g., [45,40,29,30,32,12]) and to be of relevance in the singular Yamabe problem (see [46], see [35] for a survey and Section 3 for a quick introduction). Here and throughout the paper $\Sigma = \operatorname{Sing} U$ is the complement of the largest open set where U is C^∞.

The first existence result we have is the following.

Theorem 1. *Let $N \geqslant 3$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $\Sigma = \operatorname{Sing} U$ is a nonempty compact set. Let u_0 a measurable function such that $0 \leqslant u_0 \leqslant \lambda U$ a.e. for some $\lambda \in (0,1]$. Then there exist a unique minimal weak solution $\underline{u}$ of* (P)*, i.e. a weak solution such that $\underline{u} \leqslant v$ a.e. in $\mathbb{R}^N \times \mathbb{R}^+$ for any weak solution v of* (P)*. This solution satisfies $0 \leqslant \underline{u} \leqslant \lambda U$ a.e. in $\mathbb{R}^N \times \mathbb{R}^+$. If $\lambda \in (0,1)$ then $\underline{u} \in C^\infty(\mathbb{R}^N \times (0,\infty))$ and for each $t > 0$*

$$\|\underline{u}(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant \left(\frac{4}{N-2}\left(\lambda^{\frac{2-N}{4}} - 1\right)\right)^{\frac{2-N}{4}} t^{\frac{2-N}{4}}. \tag{1.5}$$

If in addition $u_0 = \lambda U$, $\lambda \in (0,1]$, then $\underline{u}$ has the following additional properties.

(1) (*monotonicity*) *$\underline{u}$ is nonincreasing in time.*
(2) (*regularity vs minimality*) *If v is a weak solution of* (P) *such that $0 \leqslant v \leqslant \lambda U$ a.e., $\lambda \in (0,1]$, and $v \in C^\infty(\mathbb{R}^N \times (0,\infty))$ then $v = \underline{u}$.*
(3) (*uniqueness*) *If v is a weak solution such that $0 \leqslant v \leqslant \lambda U$ a.e., $\lambda \in (0,1)$, and Σ is a finite set then $v = \underline{u}$.*

The convergence of the monotone iteration method is classical topic, at least if we assume the continuity of the weak supersolution $\bar{\Psi}$. Under this hypothesis the universal bound (1.5) has already appeared in [53], giving the L^∞ decay rate for large time. Here we extend the monotone iteration to singular data u_0 and singular weak supersolution $\bar{\Psi}$. The universal bound (1.5) still holds and, quite surprisingly, the L^∞ blow-up rate as $t \to 0^+$ turns out to be independent of the integrability of u_0. About claim (2) we remark that the assumption $v \leqslant \lambda U$, $\lambda \leqslant 1$ cannot be removed. Indeed, as proved in [47], if $u_0 = \lambda U$, $0 \leqslant \lambda - 1 \ll 1$ and U is given by (1.3) then there are at least two weak solutions u_λ which are positive and smooth for $t > 0$ (the same multiplicity result seems to be true even for $0 < \lambda \ll 1$, see [36]). By the way, it is not hard to see that for $\lambda > 1$ these solutions do not satisfy the pointwise bound $u_\lambda \geqslant u_0$ despite u_0 is a weak subsolution. Indeed we would get $u_\lambda \not\equiv 0$ and

$$\partial_t u_\lambda \geqslant \Delta u_\lambda + \frac{c_\lambda}{|x|^2} u_\lambda \quad \text{in } \mathbb{R}^N \times (0,\infty), \quad c_\lambda = \lambda^{\frac{4}{N-2}} \left(\frac{N-2}{2}\right)^2 > \left(\frac{N-2}{2}\right)^2 = c_1,$$

the optimal constant of the Hardy inequality (5.8) below. Using u_λ as a supersolution away from the origin it is not difficult to contradict the result of [3] (see also [6]) about complete blow-up for the linear heat equation with inverse square potential with constant $c_\lambda > c_1$. On the other hand, for $\lambda = 1$, there is at least one solution which does not satisfy $u_\lambda \leqslant u_0$, despite $u_0 = U$ is a weak (super)solution. Thus, the parabolic comparison principle fails both for singular subsolutions and for singular supersolutions.

In proving claim (3) we use suitable extensions of the classical Hardy inequality

$$\int_{\mathbb{R}^N} V\varphi^2 \, dx \leqslant \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx \quad \text{for any } \varphi \in \mathcal{D}(\mathbb{R}^N), \quad V(x) = \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}, \tag{1.6}$$

which gives the (form) positivity of the Schrödinger operator $Lu = -\Delta u - V(x)u$. Here the idea is to derive smoothness from the pointwise bound $v \leqslant \lambda U$ and from an Hardy inequality, and to infer uniqueness from claim (2). For any weak supersolution $\bar{\Psi}$ as above we are able to show that if we set $V(x) = \bar{\Psi}(x)^{\frac{4}{N-2}}$, then (1.6) still holds, the choice $\bar{\Psi} = U$ and U as in (1.3) giving the classical Hardy inequality with best constant. More generally, if $U > 0$ is a distributional solution with finite singular set Σ and $V(x) = U(x)^{\frac{4}{N-2}}$ then inequality (1.6) holds and it is sharp, i.e.

$$\inf\left\{ \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \ \int_{\mathbb{R}^N} V\varphi^2 \, dx = 1 \right\} = 1. \tag{1.7}$$

Combining (1.6) for $V(x) = U(x)^{\frac{4}{N-2}}$ with the pointwise bound $v \leqslant \lambda U$, $\lambda < 1$, we are able to control the nonlinear term with the linear part, at least when Σ is a finite set (see Section 5), and obtain smoothness for t positive. At the beginning of our research we introduced these generalised Hardy inequalities in proving smoothness of the minimal positive solution, under suitable assumption on Σ. Actually for such purpose a much simpler argument, originally introduced in [53] for continuous data, can be used, assuming Σ to be just a compact set. However, this argument does not extend to nonminimal weak solutions, and this is exactly where the generalised Hardy inequalities come into play.

As final remark we observe that inequality (1.5), which holds for any $\lambda < 1$, can be regarded as an instability result for the singular steady state U (e.g. in the $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ topology) in the sense that the difference $U - u_\lambda(t)$ cannot be made arbitrarily small uniformly for $t \geqslant 0$, no matter how small $(1-\lambda)U = U - u_\lambda(0)$ is.

So far, the main question we want to address is, in view of the explicit dependence on λ in (1.5), what happens to $u_\lambda(t) = \underline{u}(t, \lambda U)$ as $\lambda \nearrow 1$. In particular, do we have nonuniqueness or the increasing sequence u_λ verifies $u_\lambda \to U$ as $\lambda \nearrow 1$? In other terms we can ask the following question. If $u_0 = U$ do we have $\underline{u}(t, U) \equiv U$? In both cases the answer is not obvious and it depends in a critical way on the smoothness of U. Indeed it is well known that if $U \in L^{2^*}_{\mathrm{loc}}(\mathbb{R}^N)$ then $U \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and in turn $U \in C^\infty(\mathbb{R}^N)$ (see [51]). By the classification of [7] (see also [27] for a much simpler proof), $U \in L^{2^*}(\mathbb{R}^N)$ and it is given by a well known formula (see Section 3). Due to the aforementioned uniqueness theorem for problem (P) when $u_0 \in L^{2^*}(\mathbb{R}^N)$, it is not hard to see that in this case $\underline{u}(t, \lambda U) \to \underline{u}(t, U) \equiv U$. On the other hand, as already mentioned, if U is given by (1.3) then $\underline{u}(t, \lambda U) \to \underline{u}(t, U) \not\equiv U$ and $\underline{u} \in C^\infty(\mathbb{R}^N \times (0,\infty))$. Since the regularity of U plays a crucial role in the nonuniqueness phenomena for problem (P), we find this topic worth of deeper investigation. Therefore, we sharpen the $L^{2^*}_{\mathrm{loc}}$-regularity condition which is implicit in [51] into an

ε-regularity theorem in the Lorentz space $L^{2^*,\infty}$ in the spirit of the analogous results for generalised harmonic maps (see [1,17]). We have the following result.

Theorem 2. *Let $N \geqslant 3$, $\Omega \subset \mathbb{R}^N$ an open set and let $u \in L_{\mathrm{loc}}^{\frac{N+2}{N-2}}(\Omega)$ be such that $u > 0$ a.e. in Ω and $\Delta u + u^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\Omega)$.*

(1) *There exists $\varepsilon_0 > 0$ depending only on N such that if $B_R(x_0) \subset \Omega$ and $\|u\|_{L^{2^*,\infty}(B_R(x_0))} = \varepsilon < \varepsilon_0$ then $u \in C^\infty(B_R(x_0))$.*
(2) *If $B_R(x_0) \subset \Omega$ and $u \in L^{2^*,q}(B_R(x_0))$ for some $q \in [2^*, \infty)$ then $u \in C^\infty(B_R(x_0))$.*

The smallness assumption in the previous theorem cannot be removed in view of the explicit example (1.3). Going back to the dynamic instability of U and the behaviour of u_λ as $\lambda \nearrow 1$ we remark the following. Due to the monotonicity of $\underline{u}$ with respect to the initial data, the two questions are both related to the validity of an a-priori estimate for the suite $\{T^n(0)\}$, $u_0 = U$, in the scale invariant norm $|||\cdot|||$ introduced above. A direct derivation of this estimate seems difficult and instead we are forced to use monotonicity methods and a suitable blow-up argument. We confine ourselves to the case when Σ is a finite set. Such distributional solutions with finitely many point singularities arbitrarily prescribed do exist and were first constructed in the important paper [45]. For such initial condition the main result of the paper answers the previous question negatively.

Theorem 3. *Let $N \geqslant 3$ and $U \in L_{\mathrm{loc}}^{\frac{N+2}{N-2}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Assume that $\Sigma = \operatorname{Sing} U = \{P_1, \ldots, P_k\}$ is a finite set. Let $u_0(x) = U(x)$. Then there exists a unique weak solution u of* (P) *such that $0 < u \leqslant U$ a.e. and $u \in C^\infty(\mathbb{R}^N \times (0,\infty))$. Moreover u is decreasing in time, and if $\Sigma = \{0\}$, i.e. if U is radial, then u is radial and radially decreasing for all $t > 0$. We have $u = \underline{u}$, the corresponding minimal weak solution given by Theorem* 1. *Moreover $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$.*

In proving the previous result the key point it to derive a suitable a-priori estimate on the solutions u_λ in the "subcritical" case $\lambda < 1$, using a blow up argument originally introduced in [19] to obtain the L^∞ decay rate of classical global solutions of (P) as t tends to infinity. Combining a suitable variant of it with the precise asymptotic analysis of U at isolated singularities developed in [7] and [23] we are able to prove that $u_\lambda \nearrow u \in C^\infty(\mathbb{R}^N \times (0,\infty))$. On the other hand we are able to show that there is no regular steady-states for (P) lying below U. In turn this forces the solution u to converge to zero uniformly as $t \to \infty$. Moreover, when U is radial the solution u is radial and it has the same (either discrete or continuous) scale invariance of the initial data. Due to this possibly discrete invariance, we call these solutions (quasi-)selfsimilar.

Using the global solution of Theorem 3 for radial initial data and taking into account the asymptotic analysis of U at isolated singularities developed in [7] and [23], we are able to give a much more precise description of the asymptotic behaviour of u, both near Σ as $t \to 0^+$ and as $t \to \infty$. We have the following result.

Theorem 4. *Let $N \geqslant 3$, U as in Theorem* 3 *and $u_0(x) = U(x)$. Let u be the unique weak solution of* (P) *constructed in Theorem* 3, *so that $0 < u \leqslant U$ a.e. and $u \in C^\infty(\mathbb{R}^N \times (0,\infty))$. Then*

(1) *for each $2^* < p \leqslant \infty$ there exists $C(p) > 0$ such that for each $t > 0$*
$$\|u(t)\|_{L^p(\mathbb{R}^N)} \leqslant C t^{-\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})}. \tag{1.8}$$
(2) *For each $P_j \in \Sigma$ let $U^j(x)$ be the unique radial singular solution such that for some $\alpha_j > 0$ we have $U(x) - U^j(x) = \mathcal{O}(|x-P_j|^{\frac{2-N}{2}+\alpha_j})$ as $|x-P_j| \to 0$. Let $u^j(x,t)$ the corresponding radial solution as given by Theorem* 3. *There exists $r_i \searrow 0$ such that as $i \to \infty$*
$$r_i^{\frac{N-2}{2}} u\big(r_i(x-P_j), r_i^2 t\big) \to u^j(x,t) \qquad \text{in} \quad C^{2,1}_{\mathrm{loc}}\big(\mathbb{R}^N \times (0,\infty)\big). \tag{1.9}$$
Moreover, for each $\eta > 0$ such that $\Sigma \cap B_\eta(P_j) = \{P_j\}$ and for each $2^* < p \leqslant \infty$ we have
$$t^{\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} \|u(t) - u^j(t)\|_{L^p(B_\eta(P_j))} \to 0 \quad \text{as } t \to 0^+. \tag{1.10}$$

(3) *Assume that U has a nonremovable singularity at infinity, i.e. $U(x) \geqslant C|x|^{\frac{2-N}{2}}$ for large x, so that there exist a unique radial singular solution U^∞ such that for some $\alpha_\infty > 0$ we have $U(x) - U^\infty(x) = \mathcal{O}(|x|^{\frac{2-N}{2}-\alpha_\infty})$ as $|x| \to \infty$. Let $u^\infty(x,t)$ the corresponding radial solution as given by Theorem* 3. *There exists $r_i \nearrow \infty$ such that as $i \to \infty$*

$$r_i^{\frac{N-2}{2}} u\big(r_i x, r_i^2 t\big) \to u^\infty(x,t) \quad \text{in } C^{2,1}_{\mathrm{loc}}\big(\mathbb{R}^N \times (0,\infty)\big). \tag{1.11}$$

Moreover, for each $2^* < p \leqslant \infty$ we have

$$t^{\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} \big\| u(t) - u^\infty(t) \big\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty. \tag{1.12}$$

Thus, the solution u turns out to be asymptotically (quasi-)selfsimilar both as $(x,t) \to (P,0)$, $P \in \Sigma$, and as t tends to infinity in the sense that the "tangent flows" obtained by scaling u both at the singular points and at infinity turns out to be the (quasi-)selfsimilar radial flows given by Theorem 3, associated to the radial "tangent maps" of U at the corresponding points. It is easy to prove that the same statement holds for the solutions corresponding to each $\lambda \in (0,1)$. We observe also that for λ small enough we can improve (1.10) and (1.12) to a power-like decay. Indeed, for example, the asymptotic property $U(x) - U^\infty(x) = \mathcal{O}(|x|^{\frac{2-N}{2}-\alpha_\infty})$ as $|x| \to \infty$ easily yields $t^{\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} \| S(t)U - S(t)U^\infty \|_{L^p(\mathbb{R}^N)} = \mathcal{O}(t^{-\delta})$ as $t \to \infty$ for some $\delta > 0$. Hence, using semigroup techniques the claim follows arguing as in [8], Theorem 6.1. We conjecture that the same conclusion holds for $\lambda = 1$, i.e. for the solutions considered in Theorems 3 and 4.

An immediate consequence of Theorem 3 is the following result.

Corollary 1. *Let $N \geqslant 3$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $\Sigma = \operatorname{Sing} U$ is a nonempty finite set. Let $u_0(x) = U(x)$. Then problem* (P) *has infinitely many weak solutions.*

The plan of the paper is as follows. In Section 2 we present some preliminary results concerning the monotone iteration method. In Section 3 we review the basic properties of singular solutions corresponding to singular Yamabe metrics on S^N which will be used in the sequel. In Section 4 we prove an ε-regularity theorem (Theorem 2) for these singular solutions using Lorentz spaces. In Section 5 we obtain the extended Hardy inequalities and we prove Theorem 1. In Section 6 we present a simpler direct proof of Theorem 3 for radial singular steady states U and we construct the corresponding (quasi-)selfsimilar solutions. In Section 7 we prove Theorem 3 in the general case and we derive Corollary 1 as a straightforward consequence. In Section 8 we use some asymptotic analysis and prove Theorem 4. A very weak form of the maximum principle for the heat equation is confined in an appendix.

2. Preliminary results

Let us denote by $K_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}$, $t > 0$, the standard heat kernel in $\mathbb{R}^N$ and by $S(t)$ the associated heat semigroup, $S(t)v_0 = K_t * v_0$. The following lemma expresses the well-known smoothing effect of the heat semigroup. The proof is an easy application of Young inequality in L^p spaces and it will be omitted.

Lemma 2.1. *Let $1 \leqslant \beta \leqslant \gamma \leqslant \infty$. For all $t > 0$ and all $v_0 \in L^\beta(\mathbb{R}^N)$*

$$\big\| S(t)v_0 \big\|_{L^\gamma(\mathbb{R}^N)} \leqslant \frac{1}{t^{\frac{N}{2}(\frac{1}{\beta}-\frac{1}{\gamma})}} \|v_0\|_{L^\beta(\mathbb{R}^N)}, \tag{2.1}$$

and for $1 \leqslant \beta < \infty$

$$\big\| S(t)v_0 - v_0 \big\|_{L^\beta(\mathbb{R}^N)} \to 0 \quad \text{as } t \to 0^+. \tag{2.2}$$

Using the previous lemma we can prove the following existence result of the minimal and the maximal weak solutions $\underline{u}$ and $\bar{u}$ of problem (P). The assumption on the behaviour of $\bar{\Psi}$ at infinity is far from being optimal but it is modelled on the applications we have in mind.

Proposition 2.1. *Let $N \geqslant 3$ and $\bar{\Psi} \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $\bar{\Psi} > 0$ a.e., such that*

$$\Delta\bar{\Psi} + \bar{\Psi}^{\frac{N+2}{N-2}} \leqslant 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{2.3}$$

and $\bar{\Psi}(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$. Let u_0 a measurable function such that $0 \leqslant u_0 \leqslant \bar{\Psi}$ a.e.. Then there exist two weak solutions $\underline{u}$, $\bar{u}$ of (P) *such that $0 \leqslant \underline{u} \leqslant \bar{u} \leqslant \bar{\Psi}$ in $\mathbb{R}^N \times \mathbb{R}^+$. Moreover $\underline{u} \leqslant v \leqslant \bar{u}$ a.e. in $\mathbb{R}^N \times \mathbb{R}^+$ for any other weak solution v of* (P) *such that $v \leqslant \bar{\Psi}$ a.e. in $\mathbb{R}^N \times \mathbb{R}^+$. If in addition $u_0 = \bar{\Psi}$ then for all $0 \leqslant t_1 < t_2$ we have $\underline{u}(t_1) \geqslant \underline{u}(t_2)$ and $\bar{u}(t_1) \geqslant \bar{u}(t_2)$ a.e. in $\mathbb{R}^N$, i.e. $\underline{u}$ and $\bar{u}$ are decreasing in t.*

Proof. Define $\mathcal{M}_{\bar{\Psi}} := \{v : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R},\ 0 \leqslant v \leqslant \bar{\Psi}, \text{a.e.}\}$. For any $v \in \mathcal{M}_{\bar{\Psi}}$ we set

$$\begin{aligned} T(v) &= S(t)u_0 + \int_0^t S(t-s)v(s)^{\frac{N+2}{N-2}}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^N} K_t(x-y)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} K_{t-s}(x-y)\big(v(y,s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

We have the following

Lemma 2.2. *For any $v \in \mathcal{M}_{\bar{\Psi}}$ the function $T(v)$ is well defined and $T(v) \in \mathcal{M}_{\bar{\Psi}}$.*

Proof. Since both u_0 and v are positive functions, $T(v)$ is always well defined, possibly infinite. More precisely, by (2.4) for each $t > 0$ the function $T_t(\bar{\Psi})(\cdot) = T(\bar{\Psi})(\cdot, t)$ is defined a.e. in $\mathbb{R}^N$. As T is monotonically increasing both in u_0 and in v, it is enough to prove that $T(v) \in \mathcal{M}_{\bar{\Psi}}$ when $u_0 = \bar{\Psi}$ and $v = \bar{\Psi}$.

Now we are going to prove that $T(\bar{\Psi}) \in \mathcal{M}_{\bar{\Psi}}$, i.e. that $T(\bar{\Psi}) \leqslant \bar{\Psi}$ a.e. in $\mathbb{R}^N \times \mathbb{R}^+$. Let $t > 0$ be fixed. For each $0 < \varepsilon < t$ we set

$$T_t^{\varepsilon}(\bar{\Psi}) = S(t)\bar{\Psi} + \int_0^{t-\varepsilon} S(t-s)\bar{\Psi}^{\frac{N+2}{N-2}}\,\mathrm{d}s. \tag{2.5}$$

Thus $T_t^{\varepsilon}(\bar{\Psi}) \leqslant T_t(\bar{\Psi})$ and, by the monotone convergence theorem, $T_t^{\varepsilon}(\bar{\Psi}) \to T_t(\bar{\Psi})$ a.e. as $\varepsilon \searrow 0$.

Observe that since $\Delta\bar{\Psi} \leqslant 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $\bar{\Psi}$ decays to zero at infinity, by approximation we can test this inequality with $K_t(x - \cdot)$ because for $t > 0$, K_t belongs to the Schwartz class. Since $S(t)\bar{\Psi}$ is smooth for $t > 0$, differentiating under integral sign and using the identity $\partial_t K_t = \Delta K_t$ we easily conclude that $S(t)\bar{\Psi}$ is decreasing in t. Moreover $S(t)\bar{\Psi} \nearrow \bar{\Psi}$ a.e. as $t \searrow 0$ because $S(t)\bar{\Psi}$ satisfies the heat equation with initial condition $\bar{\Psi}$.

Since $\Delta\bar{\Psi} + \bar{\Psi}^{\frac{N+2}{N-2}} \leqslant 0$ in $\mathcal{D}'(\mathbb{R}^N)$ we can test this inequality with the heat kernel $K_{t-s}(x - \cdot)$, $0 < s < t - \varepsilon$ because K_t, ΔK_t and $\partial_t K_t$ are bounded and decay exponentially as $|x| \to \infty$ locally uniformly for $t > 0$ (a rigorous justification can be done as in the proof of Proposition A.1). Thus we have

$$\begin{aligned} T_t^{\varepsilon}(\bar{\Psi}) &= S(t)\bar{\Psi} + \int_0^{t-\varepsilon}\int_{\mathbb{R}^N} K_{t-s}(x-y)\bar{\Psi}(y)^{\frac{N+2}{N-2}}\,\mathrm{d}y\,\mathrm{d}s \leqslant S(t)\bar{\Psi} - \int_0^{t-\varepsilon}\int_{\mathbb{R}^N} \Delta_y K_{t-s}(x-y)\bar{\Psi}(y)\,\mathrm{d}y\,\mathrm{d}s \\ &= S(t)\bar{\Psi} + \int_0^{t-\varepsilon}\int_{\mathbb{R}^N} \partial_s K_{t-s}(x-y)\bar{\Psi}(y)\,\mathrm{d}y\,\mathrm{d}s = S(t)\bar{\Psi} + S(\varepsilon)\bar{\Psi} - S(t)\bar{\Psi} = S(\varepsilon)\bar{\Psi}. \end{aligned}$$

Since we have already shown that $S(\varepsilon)\bar{\Psi} \leqslant \bar{\Psi}$ a.e. we obtain $T_t^{\varepsilon}(\bar{\Psi}) \leqslant \bar{\Psi}$ a.e. in $\mathbb{R}^N$. As $\varepsilon \to 0$ we have $T_t(\bar{\Psi}) \leqslant \bar{\Psi}$ a.e. and the conclusion follows since $t > 0$ can be chosen arbitrarily. □

The function $u = T(v)$ inherits from the heat kernel some regularity in time.

Lemma 2.3. *Let $v \in \mathcal{M}_{\bar\Psi}$ and let $u = T(v)$. Then $u(t) \to u_0$ in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0$ and $u \in C^0(\mathbb{R}^+; L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N))$.*

Proof. Since $u \leqslant \bar\Psi$ it is enough to prove that $u(t) \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0$ and $u \in C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ and the conclusion follows easily applying the dominated convergence theorem. By assumption there exists $R_0 > 0$ such that $\bar\Psi$ is bounded for $|x| \geqslant R_0$. We write $\bar\Psi = \bar\Psi_1 + \bar\Psi_2$, where $\bar\Psi_1 = \bar\Psi \chi_{\{|x|<R_0\}} \geqslant 0$, $\bar\Psi_2 = \bar\Psi - \bar\Psi_1 \geqslant 0$. Clearly by the assumptions on $\bar\Psi$ we have $\bar\Psi_1 \in L^{\frac{N+2}{N-2}}(\mathbb{R}^N)$ and also $\bar\Psi_1 \in L^1(\mathbb{R}^N)$ because $\bar\Psi_1$ has compact support. On the other hand $\bar\Psi_2 \in L^p(\mathbb{R}^N)$ for any $p > 2^*$ by the decay assumption on $\bar\Psi$ at infinity. Similarly, we split $u_0 = u_0^1 + u_0^2$, $u_0^1 = u_0 \chi_{\{|x|<R_0\}}$, so that $u_0^1 \in L^{\frac{N+2}{N-2}}(\mathbb{R}^N)$ and $u_0^2 \in L^p(\mathbb{R}^N)$ for any $p > 2^*$. For each $t > 0$ and $p > 2^*$, using (2.4), $u \leqslant \bar\Psi$, Holder inequality and (2.1) we have

$$\begin{aligned}
\|u(t) - u_0\|_{L^1(B_{R_0})} &\leqslant \|S(t)u_0 - u_0\|_{L^1(B_{R_0})} + \int_0^t \|S(t-s)\bar\Psi^{\frac{N+2}{N-2}}\|_{L^1(B_{R_0})}\,\mathrm{d}s \\
&\leqslant \|S(t)u_0^1 - u_0^1\|_{L^1(\mathbb{R}^N)} + C(R_0,p)\|S(t)u_0^2 - u_0^2\|_{L^p(\mathbb{R}^N)} + \int_0^t \|S(t-s)\bar\Psi_1^{\frac{N+2}{N-2}}\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}s \\
&\quad + C(R_0,p)\int_0^t \|S(t-s)\bar\Psi_2^{\frac{N+2}{N-2}}\|_{L^p(\mathbb{R}^N)}\,\mathrm{d}s \\
&\leqslant \|S(t)u_0^1 - u_0^1\|_{L^1(\mathbb{R}^N)} + C(R_0,p)\|S(t)u_0^2 - u_0^2\|_{L^p(\mathbb{R}^N)} + t\|\bar\Psi_1\|^{\frac{N+2}{N-2}}_{L^{\frac{N+2}{N-2}}(\mathbb{R}^N)} \\
&\quad + tC(R_0,p)\|\bar\Psi_2\|^{\frac{N+2}{N-2}}_{L^{p\frac{N+2}{N-2}}(\mathbb{R}^N)},
\end{aligned} \tag{2.6}$$

whence the r.h.s. goes to 0 as $t \to 0$ by the assumptions on u_0^1, u_0^2, $\bar\Psi_1$, $\bar\Psi_2$, and (2.2).

Now let $T > 0$ be fixed and choose $0 < t_1 < t_2 < T$. We will consider one of them fixed and we will prove only one-side continuity as $t_2 - t_1 \to 0$. First let us argue as above and split $v = v_1 + v_2$ where $v_1 = v\chi_{\{|x|<R\}} \geqslant 0$ and $v_2 = v - v_1 \geqslant 0$, so that $v_i \leqslant \bar\Psi_i$ for $i = 1, 2$.

Write

$$\begin{aligned}
\|u(t_2) - u(t_1)\|_{L^1(B_{R_0})} &\leqslant \|S(t_2)u_0 - S(t_1)u_0\|_{L^1(B_{R_0})} + \Bigg\| \int_0^{t_2} S(t_2 - s)\big(v(s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}s \\
&\qquad - \int_0^{t_1} S(t_1 - s)\big(v(s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}s \Bigg\|_{L^1(B_{R_0})} = I_1 + I_2.
\end{aligned}$$

Clearly we can argue as in (2.6) to prove that $I_1 \to 0$ as $t_2 - t_1 \to 0$. Thus it suffices to prove that $I_2 \to 0$ as $t_2 - t_1 \to 0$. Splitting v as above and using the pointwise inequalities $v_i \leqslant \bar\Psi_i$, the semigroup property and (2.1), we get

$$\begin{aligned}
I_2 &\leqslant \Bigg\| \int_0^{t_1} S(t_1 - s)\Big\{S(t_2 - t_1)\big(v_1(s)\big)^{\frac{N+2}{N-2}} - \big(v_1(s)\big)^{\frac{N+2}{N-2}}\Big\}\,\mathrm{d}s \Bigg\|_{L^1(B_{R_0})} + \Bigg\| \int_{t_1}^{t_2} S(t_2 - s)\big(v_1(s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}s \Bigg\|_{L^1(B_{R_0})} \\
&\quad + \Bigg\| \int_0^{t_1} S(t_1 - s)\Big\{S(t_2 - t_1)\big(v_2(s)\big)^{\frac{N+2}{N-2}} - \big(v_2(s)\big)^{\frac{N+2}{N-2}}\Big\}\,\mathrm{d}s \Bigg\|_{L^1(B_{R_0})} + \Bigg\| \int_{t_1}^{t_2} S(t_2 - s)\big(v_2(s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}s \Bigg\|_{L^1(B_{R_0})} \\
&\leqslant \int_0^T \Big\| S(t_2 - t_1)\big(v_1(s)\big)^{\frac{N+2}{N-2}} - \big(v_1(s)\big)^{\frac{N+2}{N-2}} \Big\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}s
\end{aligned}$$

$$+ C(R_0, p) \int_0^T \left\| S(t_2 - t_1)\big(v_2(s)\big)^{\frac{N+2}{N-2}} - \big(v_2(s)\big)^{\frac{N+2}{N-2}} \right\|_{L^p(\mathbb{R}^N)} \mathrm{d}s$$

$$+ (t_2 - t_1) \|\bar{\Psi}_1\|^{\frac{N+2}{N-2}}_{L^{\frac{N+2}{N-2}}(\mathbb{R}^N)} + (t_2 - t_1) C(R_0, p) \|\bar{\Psi}_2\|^{\frac{N+2}{N-2}}_{L^{p\frac{N+2}{N-2}}(\mathbb{R}^N)}. \tag{2.7}$$

Since

$$G_1(t_1, t_2)(s) := \left\| S(t_2 - t_1)\big(v_1(s)\big)^{\frac{N+2}{N-2}} - \big(v_1(s)\big)^{\frac{N+2}{N-2}} \right\|_{L^1(\mathbb{R}^N)} \leqslant 2 \|\bar{\Psi}_1\|^{\frac{N+2}{N-2}}_{L^{\frac{N+2}{N-2}}(\mathbb{R}^N)}$$

and similarly

$$G_2(t_1, t_2)(s) := \left\| S(t_2 - t_1)\big(v_2(s)\big)^{\frac{N+2}{N-2}} - \big(v_2(s)\big)^{\frac{N+2}{N-2}} \right\|_{L^p(\mathbb{R}^N)} \leqslant 2 \|\bar{\Psi}_2\|^{\frac{N+2}{N-2}}_{L^{p\frac{N+2}{N-2}}(\mathbb{R}^N)},$$

and $G_1, G_2 \to 0$ a.e. as $t_2 - t_1 \to 0$ by (2.2), the conclusion follows from (2.7) and the dominated convergence theorem. $\square$

Remark 1. Given $v \in \mathcal{M}_{\bar{\Psi}}$, $u = T(v)$ is formally a mild solution of the Cauchy problem

$$(\mathrm{P}_v) \quad \begin{cases} u_t = \Delta u + v^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{2.8}$$

Since $\bar{\Psi} \notin L^{\frac{N+2}{N-2}}(\mathbb{R}^N)$ a direct application of the semigroup method is not possible. Indeed local integrability and integrability at infinity for the initial data do not match and it would be necessary to introduce weighted spaces. It is more convenient for us to interpret (2.8) in the sense of distributions according to the lemma below. This way $u = T(v)$ is a classical solution of (2.8) for $t > 0$ and $u \in C^\infty(\mathbb{R}^N \times (0, \infty))$ whenever $v \in C^\infty(\mathbb{R}^N \times (0, \infty))$ according to the standard regularity theory for the heat equation.

The function $u = T(v)$ solves (2.8) in the sense of distributions.

Lemma 2.4. *Let $v \in \mathcal{M}_{\bar{\Psi}}$ and let $u = T(v)$ as in (2.4). Then $u \in C^0(\mathbb{R}_+; L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N))$, $u(0) = u_0$ and for any $\psi \in C_0^\infty(\mathbb{R}^N \times \mathbb{R})$ we have*

$$\int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x + \int_{\mathbb{R}^N \times \mathbb{R}^+} u(x,t)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N \times \mathbb{R}^+} u(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_{\mathbb{R}^N \times \mathbb{R}^+} v(x,t)^{\frac{N+2}{N-2}} \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.9}$$

Proof. The first two claims hold by Lemma 2.3. In order to check the third we choose $\{u_0^n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant u_0^n \leqslant \bar{\Psi}$ a.e., $u_0^n \to u_0$ in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e.. Similarly, since $\bar{\Psi} \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ and $v \in \mathcal{M}_{\bar{\Psi}}$, we may extend $v \geqslant 0$ to 0 for $t < 0$ and find $\{v^n\} \subset C_0^\infty(\mathbb{R}^N \times \mathbb{R})$ such that $0 \leqslant v^n \leqslant \bar{\Psi}$, $v^n \to v$ in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ and a.e.. For example, if $S(t)$ (resp. $\overline{S}(t)$) is the heat semigroup in $\mathbb{R}^N$ (resp. in $\mathbb{R}^{N+1}$) then we can take $u_0^n = \varphi_n S(1/n)u_0$ (resp. $v^n = \varphi_n \overline{S}(1/n)v$) and $\varphi_n \in C_0^\infty(\mathbb{R}^N)$ (resp. $\varphi_n \in C_0^\infty(\mathbb{R}^{N+1})$) satisfying $0 \leqslant \varphi_n \leqslant 1$, $\varphi_n \to 1$ in $C^\infty(\mathbb{R}^N)$ (resp. in $C^\infty(\mathbb{R}^{N+1})$).

Thus, if we set

$$u^n(t) = S(t)u_0^n + \int_0^t S(t-s)\big(v^n(s)\big)^{\frac{N+2}{N-2}}\,\mathrm{d}s,$$

then $u^n \in C^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ because u^n is the unique, global, classical solution of

$$(\mathrm{P}^n_v) \quad \begin{cases} u^n_t = \Delta u^n + (v^n)^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u^n(0) = u^n_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{2.10}$$

and smoothness follows from the standard regularity theory for linear heat equation (see [25]).

Arguing as in Lemma 2.2, by dominated convergence we conclude $u^n \to u$ a.e., hence $u^n \to u$ in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R}^+)$ because $0 \leqslant u^n \leqslant \bar{\Psi}$ for each n. Multiplying (2.10) by ψ and integrating by parts we obtain (2.9) for each (u^n, v^n) and the conclusion follows as $n \to \infty$. □

Finally we are in the position to finish the proof of Proposition 2.1. Set $v_0 = 0$, $v_1 = T(v_0) = S(t)u_0$ and for each $n \geqslant 1$ let us set $v_{n+1} = T(v_n)$. By Lemma 2.2 the sequence $\{v_n\}$ is well defined and $\{v_n\} \subset \mathcal{M}_{\bar{\Psi}}$. Since $v_1 \geqslant 0 = v_0$ a simple induction argument based on Proposition A.1 and similar to the one below shows that $\{v_n\}$ is pointwise increasing, hence $v_n \to \underline{u}$ for some $\underline{u} \in \mathcal{M}_{\bar{\Psi}}$. Using the monotone convergence theorem in (2.4) we immediately obtain $\underline{u} = T(\underline{u})$ a.e. whence, applying Lemma 2.4, $\underline{u}$ is a weak solution of (P) in the sense of Definition 1.1. Similarly we can take $v^0 = \bar{\Psi}$, $v^1 = T(v^0)$ and for each $n \geqslant 1$ we can set $v^{n+1} = T(v^n)$. By Lemma 2.2 the sequence $\{v^n\}$ is well defined and $\{v^n\} \subset \mathcal{M}_{\bar{\Psi}}$. An induction argument based on Proposition A.1 and similar to the one below shows that $\{v^n\}$ is pointwise decreasing, hence $v^n \to \bar{u}$ for some $\bar{u} \in \mathcal{M}_{\bar{\Psi}}$ such that $\bar{u} = T(\bar{u})$ and $\bar{u}$ is a weak solution of (P) in the sense of Definition 1.1.

In order to prove the minimality of $\underline{u}$, let v be any other weak solution in the sense of Definition 1.1. Let v_n as above, $v_0 \equiv 0$ and $v_n \nearrow \underline{u}$ a.e.. Let us set $W_n := v - v_n$. By definition of weak solution and Lemma 2.2, $\{W_n\} \subset C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^{\mathbb{N}}))$ and $W_n(0) \equiv 0$ for each $n \geqslant 0$. On the other hand, since $v(x,t) = \mathcal{O}(\mathrm{e}^{C|x|^2})$ for $|x| \to \infty$ uniformly on t and $|v_n(x,t)| \leqslant \bar{\Psi}(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$, we clearly have $W_n(x,t) = \mathcal{O}(\mathrm{e}^{C|x|^2})$ for $|x| \to \infty$ uniformly on $t \geqslant 0$ and on $n \geqslant 0$.

We claim that for each $n \geqslant 0$ we have $W_n \geqslant 0$ a.e., i.e. $v \geqslant v_n$ a.e., whence the minimality follows as $n \to \infty$. We prove the claim by an induction argument. By definition of weak solution $W_0 = v \geqslant 0$, hence the claim holds for $n = 0$. For $n \geqslant 1$ by (1.4) and (2.9) we have $\partial_t W_n - \Delta W_n = v^{\frac{N+2}{N-2}} - v_{n-1}^{\frac{N+2}{N-2}}$ in $\mathcal{D}'(\mathbb{R}^N \times (0,\infty))$. Since the r.h.s. of this equation is positive by the inductive assumption $W_{n-1} \geqslant 0$ and $W_n(0) = u_0 - u_0 = 0$, we can apply Proposition A.1 to conclude $W_n \geqslant 0$ a.e. and the claim is proved.

The same argument applied to $\{v^n\}$ and v shows that $\bar{u}$ is the maximal weak solution in $\mathcal{M}_{\bar{\Psi}}$.

Now let us assume that $u_0 = \bar{\Psi}$ and let us prove that $\underline{u}$ is nonincreasing in t. For a given nonnegative initial function $0 \leqslant f \leqslant \bar{\Psi}$, let us denote with $\underline{u}(t,f)$ the corresponding minimal solution at time $t > 0$, whose existence is guaranteed by the first part of the proposition. Clearly by (2.4) for each $t > 0$ we have

$$f_1 \leqslant f_2 \quad \Longrightarrow \quad \underline{u}(t, f_1) \leqslant \underline{u}(t, f_2) \quad \text{a.e.} \tag{2.11}$$

We claim that $\underline{u}$ satisfies the semigroup property, i.e. for any $s, t \geqslant 0$

$$\underline{u}(s+t, f) = \underline{u}\big(s, \underline{u}(t, f)\big). \tag{2.12}$$

Assuming (2.12) for a moment, let us prove that $\underline{u}(t, \bar{\Psi})$ is decreasing in time. Since $\underline{u}(t, \bar{\Psi}) \leqslant \bar{\Psi}$ for each $t \geqslant 0$, then for any $0 \leqslant t_1 < t_2$ we have $\underline{u}(t_1, \bar{\Psi}) \geqslant \underline{u}(t_1, \underline{u}(t_2 - t_1, \bar{\Psi})) = \underline{u}(t_2, \bar{\Psi})$ by (2.11), (2.12), and the proof is completed.

In order to check (2.12) it suffice to prove that for each $t_0 > 0$, if we set

$$v(t,f) = \begin{cases} \underline{u}(t,f) & t \in [0, t_0], \\ \underline{u}\big(t - t_0, \underline{u}(t_0, f)\big) & t \geqslant t_0, \end{cases} \tag{2.13}$$

then $\underline{u}(t,f) = v(t,f)$ for every $t \geqslant 0$. To verify this property, first we observe that v is also a weak solution in the sense of Definition 1.1. Indeed, given $\underline{u}(t,f)$ we use a standard approximation argument for the test function ψ in (1.4) in the time interval $[0, t_0]$ by $\psi\theta_n$, $\theta_n = \theta_n(t) \in C^\infty_0(\mathbb{R})$ such that $0 \leqslant \theta_n \leqslant 1$ and $\theta_n \to \chi_{[0,t_0]}$. As $n \to \infty$ by dominated convergence we obtain

$$\int_{\mathbb{R}^N} f(x)\psi(x,0)\,\mathrm{d}x - \int_{\mathbb{R}^N} \underline{u}(t_0, f)(x)\psi(x,t_0)\,\mathrm{d}x + \int_{\mathbb{R}^N \times (0,t_0)} \underline{u}(t,f)(x)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{\mathbb{R}^N\times(0,t_0)} \underline{u}(t,f)(x)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^N\times(0,t_0)} \underline{u}(t,f)(x)^{\frac{N+2}{N-2}}\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t=0.$$

Testing Eq. (1.4) for $\underline{u}(s,\underline{u}(t_0,f))$ with $\psi(x,s+t_0)$, changing variables as $s=t-t_0$ and adding to the previous relation we conclude that v is a weak solution. To prove that $\underline{u}(t,f)=v(t,f)$ we observe that the "$\leqslant$" is obvious because of the minimality of $\underline{u}(t,f)$. The "$\geqslant$" is trivial for $t\in[0,t_0]$ and it follows from the minimality of $\underline{u}(t-t_0,\underline{u}(t_0,f))$ for any $t-t_0\geqslant 0$. Thus, (2.12) holds for any initial condition $0\leqslant f\leqslant\bar{\Psi}$. The proof of the analogous statements for $\bar{u}$ are entirely similar, therefore it will be omitted. □

As a consequence of the previous result we obtain the following sufficient condition for minimality.

Corollary 2. *Let $u_0=\bar{\Psi}$, $\underline{u}$ as in Proposition* 2.1 *and let u be a weak solution of* (P). *If $u\leqslant\bar{\Psi}$ a.e. and $u\in C^\infty(\mathbb{R}^N\times(0,\infty))$, then $u=\underline{u}$, i.e. u is the minimal solution.*

Proof. By Proposition 2.1 we have a well defined minimal solution $\underline{u}(t,\bar{\Psi})\leqslant u(t)$ for all $t\geqslant 0$. In order to prove the reverse inequality let us fix $t_0>0$. Clearly for $t\geqslant 0$ the function $u(t+t_0)$ is a classical (hence a weak) solution of (P) with $u(t_0)\leqslant\bar{\Psi}(x)$ as initial condition. Observe that $u(t_0)\in L^\infty(\mathbb{R}^N)$ because it is positive, smooth and $u(t_0)\leqslant\bar{\Psi}(x)\to 0$ as $|x|\to\infty$. Similarly $\|u(t+t_0)\|_\infty$ is locally bounded for $t\geqslant 0$. By Proposition 2.1 the minimal solution $\underline{u}(t,u(t_0))$ is well defined for all $t\geqslant 0$ and, by minimality, $\underline{u}(t,u(t_0))\leqslant u(t+t_0)$ for all $t\geqslant 0$. Thus, $\|\underline{u}(t,u(t_0))\|_\infty$ is locally bounded for $t\geqslant 0$ and $\underline{u}(t,u(t_0))$ has smooth initial condition, therefore it is smooth by the standard regularity theory for the heat equation (see [25]). By the short time uniqueness of the bounded classical solution of (P) with bounded smooth initial data $u(t_0)$ we obtain $u(t+t_0)\equiv\underline{u}(t,u(t_0))$ for t small enough. Repeating this argument we actually obtain $u(t+t_0)\equiv\underline{u}(t,u(t_0))$ for all $t\geqslant 0$. Since $u(t_0)\leqslant\bar{\Psi}$, using (2.11) we obtain $u(t+t_0)\equiv\underline{u}(t,u(t_0))\leqslant\underline{u}(t,\bar{\Psi})$ a.e. for all $t\geqslant 0$. Since $u\in C^0(\mathbb{R}^+;L^1_{\mathrm{loc}}(\mathbb{R}^N))$, as $t_0\to 0$ we have $u(t)\leqslant\underline{u}(t,\bar{\Psi})$ a.e. for all $t\geqslant 0$ and the conclusion follows. □

Another consequence of Proposition 2.1 is the following corollary which guarantees the smoothness of the minimal solution and an explicit L^∞-bound of $\underline{u}(t)$ for each $t>0$. This bound has been already established in [53], Theorem 4.1, when $\bar{\Psi}$ is a continuous weak supersolution.

Corollary 3. *Let $\bar{\Psi}$ as in Proposition* 2.1, *$\lambda\in(0,1)$ and $u_0=\lambda\bar{\Psi}$. Let $\underline{u}$ the minimal weak solution of problem* (P) *as in Proposition* 2.1. *Then $0<\underline{u}\leqslant\lambda\bar{\Psi}$ a.e., $\underline{u}\in C^\infty(\mathbb{R}^N\times(0,\infty))$ and for each $t>0$*

$$\|\underline{u}(t)\|_{L^\infty(\mathbb{R}^N)}\leqslant\left(\frac{4}{N-2}\left(\lambda^{\frac{2-N}{4}}-1\right)\right)^{\frac{2-N}{4}}t^{\frac{2-N}{4}}. \tag{2.14}$$

Proof. First we regularise suitably the initial data. Given $u_0(x)=\lambda\bar{\Psi}(x)$, $\lambda\in(0,1)$, for each $\tau>0$ we set $u_{0\tau}(x)=u_0*K_\tau(x)$, where K_τ is the standard heat kernel in $\mathbb{R}^N\times\mathbb{R}_+$ at time $t=\tau$. Clearly $u_{0\tau}$ is smooth for each $\tau>0$, $0<u_{0\tau}\leqslant u_0$ by Lemma 2.2 and $u_{0\tau}\to u_0$ in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ as $\tau\to 0$ by Lemma 2.3. On the other hand, for each $\lambda\in(0,1)$, arguing as in the proof of Proposition 5.1, we can easily prove that $u_{0\tau}$ satisfy (2.3) in a classical sense (regularisation with the heat kernel involves probability measures). Thus, $\partial_\tau u_{0\tau}=\Delta u_{0\tau}\leqslant 0$, i.e. $u_{0\tau}$ is decreasing in τ. Hence $u_{0\tau}\nearrow u_0$ a.e. as $\tau\to 0$.

For each $\tau>0$ let us consider the Cauchy problem

$$(\mathrm{P}_\tau)\quad\begin{cases}u_t=\Delta u+|u|^{\frac{4}{N-2}}u & \text{in } \mathbb{R}^n\times(0,+\infty),\\ u(x,0)=u_{0\tau}(x) & \text{in } \mathbb{R}^n.\end{cases} \tag{2.15}$$

Since the initial function is bounded, smooth and decays fast enough at infinity together with its derivatives of any order, by classical theory (see [25]) problem (P_τ) has a unique bounded and infinitely smooth solution u_τ defined in some strip $\mathbb{R}^N\times[0,T]$. As already observed $u_{0,\tau}\geqslant 0$ is a classical solution of inequality (2.3), i.e. $u_{0,\tau}$ is a smooth time-independent supersolution for problem (P_τ). Thus, by standard comparison principle on (P_τ) we conclude

$$0<S(t)u_{0\tau}(x)\leqslant u_\tau(x,t)\leqslant u_{0\tau}(x),\quad \partial_t u_\tau\leqslant 0\quad\text{for any } (x,t)\in\mathbb{R}^N\times(0,T], \tag{2.16}$$

By the well known blow-up alternative, the a-priori L^∞ bound on u_τ guarantees that the solution can be extended globally in time. Thus for each $\tau > 0$ problem (P_τ) has a unique bounded global-in-time solution $u_\tau \in C^\infty(\mathbb{R}^N \times \mathbb{R}_+)$. Moreover, combining the smoothness up to $t = 0$ with the translation invariance of the equation and the decay properties of $u_{0\tau}$ and its derivatives at infinity, it is easy to check that $\partial_t u_\tau$ and $u_\tau^{\frac{N+2}{N-2}}$ belong to $L^\infty(0, T; L^\infty(\mathbb{R}^N))$ for any $T > 0$ and they are smooth.

Let us set $\bar{\Psi}_\tau = K_\tau * \bar{\Psi}$, so that $\bar{\Psi}_\tau$ are classical solutions of (2.3) by the same argument used for $u_{0\tau}$. Following [53], Theorem 4.1, let us set $v = \partial_t u_\tau + \eta u_\tau^{\frac{N+2}{N-2}}$, where $\eta > 0$ is to be chosen later. Standard calculation yields

$$\partial_t v - \Delta v \leqslant \frac{N+2}{N-2} u_\tau^{\frac{4}{N-2}} v \quad \text{on } \mathbb{R}^N \times [0,\infty).$$

Choosing $\eta = \lambda^{\frac{4}{2-N}} - 1 > 0$ we have

$$\begin{aligned} v(0) &= \big(\partial_t u_\tau + \eta u_\tau^{\frac{N+2}{N-2}}\big)(0) = \big(\Delta u_\tau + (1+\eta) u_\tau^{\frac{N+2}{N-2}}\big)(0) = \Delta u_{0\tau} + (1+\eta) u_{0\tau}^{\frac{N+2}{N-2}} \\ &= \lambda\big(\Delta \bar{\Psi}_\tau + (1+\eta)\lambda^{\frac{4}{N-2}} \bar{\Psi}_\tau^{\frac{N+2}{N-2}}\big) = \lambda\big(\Delta \bar{\Psi}_\tau + \bar{\Psi}_\tau^{\frac{N+2}{N-2}}\big) \leqslant 0, \end{aligned}$$

because $\bar{\Psi}_\tau$ are classical solutions of (2.3). Since v belong to $L^\infty(0, T; L^\infty(\mathbb{R}^N))$ for any $T > 0$ and is smooth we can apply the standard comparison principle to conclude $v \leqslant 0$ everywhere, because $T > 0$ can be chosen arbitrarily. Thus, $\partial_t u_\tau + \eta u_\tau^{\frac{N+2}{N-2}} \leqslant 0$ everywhere, whence a direct integration yields

$$\big\|u_\tau(t)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \left(\frac{4}{N-2}\big(\lambda^{\frac{2-N}{4}} - 1\big)\right)^{\frac{2-N}{4}} t^{\frac{2-N}{4}}. \tag{2.17}$$

As already observed $u_{0\tau}$ is pointwise increasing to u_0 as $\tau \to 0$, hence by standard comparison principle the same holds for u_τ. By (2.17) and (2.16) there exists a locally bounded pointwise limit $u : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ such that $0 \leqslant u \leqslant u_0$, u is nonincreasing in time and u satisfies (2.14). On the other hand, combining (2.17) with the standard L^p and Schauder theory for the heat equation, we easily infer that $\{u_\tau\}$ is a compact sequence in $C^{2,1}_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$. Therefore u is in $C^{2,1}_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$ and it is a classical positive solution of the equation. By (2.16) we also have $S(t)u_0 \leqslant u \leqslant u_0$, hence u is C^∞-smooth for $t > 0$ by standard theory (see [25]). Moreover $u(t) \to u_0$ in $L^{\frac{N+2}{N-2}}_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0^+$ and $u \in C^0(\mathbb{R}^+; L^{\frac{N+2}{N-2}}_{\text{loc}}(\mathbb{R}^N))$ by dominated convergence. Thus, u is a weak solution of problem (P) and using Corollary 2 we conclude $u = \underline{u}$. □

Remark 2. The proof of Corollary 3 clearly shows that, at least for $\lambda < 1$, the minimal positive solution $\underline{u}(t, \lambda\bar{\Psi})$ can be constructed by solving (P_τ) and passing to the limit as $\tau \to 0$. Actually it is not hard to see that the same holds for $\lambda = 1$. Indeed $u_{0\tau} \nearrow u_0$ as $\tau \to 0^+$ and $u_\tau \to u$ for some weak solution $u \in \mathcal{M}_{\bar{\Psi}}$ of problem (P). On the other hand, by Corollary 2 and (2.11) we have $u_\tau(t) = \underline{u}(t, u_{0\tau}) \leqslant \underline{u}(t, \bar{\Psi})$, whence $u(t) \leqslant \underline{u}(t, \bar{\Psi})$ and the conclusion follows from the minimality of $\underline{u}$.

3. A review of singular steady states in $\mathbb{R}^N$

In this section we recall well known results about singular stationary solutions of problem (P) which we need to prove our main theorems. Let $N \geqslant 3$ and let (S^N, g_0) the N-dimensional sphere with the standard metric, so that S^N has constant positive scalar curvature R_{g_0}. If g and g_0 are conformally related metrics, i.e. $g = u^{\frac{4}{N-2}} g_0$ for some positive function $u \in C^\infty(S^N)$, then the scalar curvatures R_g and R_{g_0} are related by the equation

$$\Delta_{g_0} u - \frac{N-2}{4(N-1)} R_{g_0} u + \frac{N-2}{4(N-1)} R_g u^{\frac{N+2}{N-2}} = 0. \tag{3.1}$$

For any Riemaniann manifold the classical Yamabe problem (resp. the prescribing scalar curvature problem) consists in finding a positive solution u of (3.1) for prescribed constant scalar curvature R_g (resp. for prescribed smooth function R_g). After the important paper [46] there has been a lot of interest in solving equation (3.1), especially in the positive case $R_g = const > 0$, on open subdomains $\Omega_0 \subsetneq S^N$ in connection with the embedding problem for locally conformally flat manifolds. In particular one requires the metric g to be of constant positive scalar curvature in Ω_0, i.e. u has to solve (3.1) in Ω_0, and to be complete in Ω_0, hence the factor u has to blow-up suitably as $x \to \Sigma_0 = S^N \setminus \Omega_0$. This is the so-called singular Yamabe problem on S^N (see [35] for an introduction and a detailed review on the subject).

Assuming $R_g = const > 0$ and using the stereographic projection $\Pi_{P_0} : S^N \to \mathbb{R}^N$ from a point $P_0 \in \Omega_0$, Eq. (3.1) is equivalent to

$$\Delta U + U^{\frac{N+2}{N-2}} = 0, \quad x \in \Omega \subsetneq \mathbb{R}^N, \tag{3.2}$$

where $\Omega = \Pi(\Omega_0)$, coupled with the "boundary condition" $U \to \infty$ suitably as $x \to \Sigma = \Pi(\Sigma_0)$ which guarantees the completeness of the corresponding metric g on Ω_0.

Existence results for complete metrics heavily depends on the "size" of the singular set Σ and in order to guarantee solvability a bound on the Hausdorff dimension of Σ, namely $0 \leqslant d_H(\Sigma) \leqslant \frac{N-2}{2}$, is necessary (see [46]). Under this assumption solutions of (3.2) such that g is complete on Ω_0 do exists whenever Σ is a finite union of compact submanifolds without boundary Σ_i of dimension $0 \leqslant d_i \leqslant \frac{N-2}{2}$ (see e.g. [45,32,29,31,40,30]). On the other hand singular solutions do exist even when Σ is a submanifold with boundary [12] or certain purely unrectifiable sets (see [46]). Under a further mild geometric assumption (see [23]) which always holds for finite union of smooth compact submanifolds satisfying the previous dimensional bound, any solution U belongs to $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$ and extends to a distributional solution in the whole $\mathbb{R}^N$.

A related major task in this subject is the understanding of the blow-up rate of positive solutions $U \in C^\infty(\Omega)$ as $x \to \Sigma$. The first result in this direction has been obtained in [7], Theorem 1.2, when Σ is a finite set. Under this assumption,

$$C_1|x - P|^{\frac{2-N}{2}} \leqslant U(x) \leqslant C_2|x - P|^{\frac{2-N}{2}} \quad \text{as } x \to P \in \Sigma, \tag{3.3}$$

for some $C_1(U), C_2(U) > 0$. Such blow-up rate is sharp as it is shown by the explicit example (1.3) and the well-known classification of radial singular solutions which we recall below. Inequalities (3.3) also show that if Σ is a finite set then $U \in L^{2^*,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and has no higher integrability. Here the Lorentz space $L^{2^*,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ can be viewed as the weak-L^p space, $p = 2^*$, of measurable functions satisfying $\sup_{s>0} s|\{|f| > s\} \cap B|^{1/p} < \infty$ for any ball $B \subset \mathbb{R}^N$ (for further information about regularity results in the setting of Lorentz spaces see the next section). Using conformal invariance of (3.2) and the Kelvin transform it is possible to deduce some information on the behaviour of U at infinity from (3.3) whenever Σ is a compact set (which is always the case if $P_0 \in \Omega_0$), namely

$$\text{either} \quad U(x) \sim |x|^{\frac{2-N}{2}} \quad \text{or} \quad U(x) \sim |x|^{2-N} \quad \text{as } |x| \to \infty. \tag{3.4}$$

In the first case we say that U is singular at infinity while in the second we say that U is regular. The second case actually occur for positive smooth solutions U. According to [51], if $U \in L^{2^*}_{\mathrm{loc}}(\mathbb{R}^N)$ and $U > 0$ a.e. is a weak solution of (3.2) then $U \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ by standard linear regularity theory, hence $U \in C^\infty(\mathbb{R}^N)$ by [51], Theorem 3. By the classification result of [7], Corollary 8.2, we have $U \in L^{2^*}(\mathbb{R}^N)$ and

$$U(x) = \overline{U}_\delta(x - x_0) = \delta^{\frac{N-2}{2}}\, \overline{U}\big(\delta(x - x_0)\big), \quad \overline{U}(x) = \left(\frac{\sqrt{N(N-2)}}{1 + |x|^2}\right)^{\frac{N-2}{2}}, \tag{3.5}$$

for some $x_0 \in \mathbb{R}^N$, $\delta > 0$.

Going back to singular solutions the upper bound in (3.3) has been generalised to any distributional solution with compact singular set Σ of zero 2-capacity. According to [9] if $\mathrm{cap}_2(\Sigma) = 0$ then $U(x) = \mathcal{O}(\mathrm{dist}(x, \Sigma))^{\frac{2-N}{2}}$ as $x \to \Sigma$. Assuming some smoothness property on Σ and some upper bound k on its dimension as above (or just k-rectifiability), the k-dimensional upper Minkowski contents of Σ is finite, hence it is not hard to derive for integer $0 \leqslant k < \frac{N-2}{2}$ the weak-L^p bound $U \in L^{\frac{2(N-k)}{N-2},\infty}_{\mathrm{loc}}$. This bound roughly shows that "the thicker the singular set is the lower is the local integrability of U". Even though this argument do not apply for the limiting case $k = \frac{N-2}{2}$, the solution constructed in [40] for $N \geqslant 4$ even and Σ a finite union of $k = \frac{N-2}{2}$-dimensional compact submanifolds turn out to be exactly in $L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, i.e. they have the lowest possible integrability to give a meaning to the equation in the sense of distributions.

A further step in the understanding of singular solutions of (3.2) is the study of the asymptotic behaviour of solutions U as $x \to \Sigma$. Here we focus on singular solutions with finitely many isolated singularities. In this case a complete picture has been obtained in the works [7,23]. A major role in this study is played by entire radial solutions of (3.2) which we are going to review in some detail below. For higher dimensional singular sets there is no complete picture of the asymptotic behaviour, due to the lack of tangential regularity (see [28]). As a consequence we are not able to extend Theorem 3 to more general singular steady states $u_0 = U$. In the presentation below we essentially follow [23], even if with different normalisations.

Let U a positive singular solution of (3.2) such that $\Sigma = \operatorname{Sing} U = \{O\}$. Then U is radial (see [7], Theorem 8.1) and

$$U(x) = |x|^{\frac{2-N}{2}} g\big(-\log|x|\big), \quad t = -\log|x|, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} g = \left(\frac{N-2}{2}\right)^2 g - g^{\frac{N+2}{N-2}}. \tag{3.6}$$

Converting the previous equation into a system in the phase space $(g, h) = (g, g')$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g = h, \qquad \frac{\mathrm{d}}{\mathrm{d}t} h = \left(\frac{N-2}{2}\right)^2 g - g^{\frac{N+2}{N-2}},$$

which is an Hamiltonian system with energy function

$$H(g, h) = \frac{1}{2} h^2 - \frac{1}{2}\left(\frac{N-2}{2}\right)^2 g^2 + \frac{N-2}{2N} g^{\frac{2N}{N-2}}. \tag{3.7}$$

In particular, if $g(t)$ is a positive solution of (3.6) then the path parametrised by $(g(t), h(t))$ is contained in a level set of H and H is constant along the trajectory. All the admissible solutions (i.e. the ones such that g is a positive solution of (3.6) and U is a singular solution of (3.2)) correspond to a one parameter family of closed level sets of H in the region $\{H < 0\} \cap \{g > 0\}$. The level set $\{H = 0\} \cap \{g > 0\}$ is an homoclinic trajectory, which, up to a translation, corresponds to the unique solution $g_0 > 0$ of (3.6) such that $g(\pm\infty) = 0$ and $g(0) = \max g$. If $g(t) = g_0(t - \log\delta)$ then (3.6) gives the family of smooth solutions $\overline{U}_\delta(x)$ as in (3.5). Therefore, an elementary calculation gives the universal bound $g \leqslant L_0$, $L_0 = \max g_0 = (\frac{N-2}{4})^{\frac{N-2}{4}}$, for any positive solution g of (3.6). On the other hand, if $\varepsilon_0 = (\frac{N-2}{2})^{\frac{N-2}{2}}$ then $H \geqslant H(\varepsilon_0, 0) \equiv H_{\varepsilon_0}$ in $\{H < 0\} \cap \{g > 0\}$ and the level set $\{H = H_{\varepsilon_0}\}$ consists of the constant solution $g_{\varepsilon_0} \equiv (\frac{N-2}{2})^{\frac{N-2}{2}}$ which corresponds to the singular solution (1.3). In order to parametrise the other level sets we introduce the necksize ε of any positive solution g such that $H(g, g') < 0$, as $\varepsilon = \min g$. This way $0 < \varepsilon \leqslant \varepsilon_0 = (\frac{N-2}{2})^{\frac{N-2}{2}}$, $H_\varepsilon \equiv H(\varepsilon, 0)$ and g is contained in the level set $\{H = H_\varepsilon\}$. Since the system is autonomous for each $0 < \varepsilon < \varepsilon_0$ there exists a unique solution g_ε such that $H(g_\varepsilon, g'_\varepsilon) = H_\varepsilon$ and $g_\varepsilon(0) = \varepsilon$. Such a solution is periodic of period $T_\varepsilon > 0$ and for each $s \in [0, T_\varepsilon)$ (actually for each $s \in \mathbb{R}$ by periodicity) the function $g(t) = g_\varepsilon(t + s)$ is still a solution.

This way the radial singular solutions are given by (1.3) and

$$U(x) = U_{\varepsilon,\delta}(x) = \delta^{\frac{N-2}{2}} U_\varepsilon(\delta x), \quad \delta > 0, \qquad U_\varepsilon(x) = |x|^{\frac{2-N}{2}} g_\varepsilon\big(-\log|x|\big), \quad 0 < \varepsilon < \varepsilon_0, \tag{3.8}$$

the former being formally included in (3.8) for $\varepsilon = \varepsilon_0$ and $\delta > 0$ chosen arbitrarily. For $0 < \varepsilon < \varepsilon_0$ and T_ε as above, if $U = U_{\varepsilon,\delta}$ and we set $\delta_\varepsilon = \mathrm{e}^{-T_\varepsilon}$, by periodicity of g_ε we have $U(x) = \delta_\varepsilon^{\frac{N-2}{2}} U(\delta_\varepsilon x)$. The following two facts are the crucial ingredients in the proof of Theorems 3 and 5. The first one is that any radial singular solution of (3.2) is radially decreasing. Indeed, for each $0 < \varepsilon \leqslant \varepsilon_0$ we have $H(g_\varepsilon(t), g'_\varepsilon(t)) \equiv H_\varepsilon < 0$, hence $g'_\varepsilon(t) > -\frac{N-2}{2} g_\varepsilon(t)$ for each $t \in \mathbb{R}$ and the conclusion follows easily by differentiating (3.8). The other key fact is related to the intersection property of classical and singular solutions of (3.2). For each $0 < \varepsilon \leqslant \varepsilon_0$ we set $L_\varepsilon = \max g_\varepsilon$. Clearly $H(L_\varepsilon, 0) = H_\varepsilon < 0$ and the definitions above yield easily $\varepsilon \leqslant g_\varepsilon \leqslant L_\varepsilon < L_0$ for each $t \in \mathbb{R}$. Thus formula (3.8) gives

$$0 < \inf_{x \neq 0} |x|^{\frac{N-2}{2}} U(x) \leqslant \sup_{x \neq 0} |x|^{\frac{N-2}{2}} U(x) < \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}} \tag{3.9}$$

for any radial singular solution $U_{\varepsilon,\delta}$ of (3.2). On the other hand, since $g_0 > 0$, $g_0(\pm\infty) = 0$ and $\max g_0 = L_0 = (\frac{N(N-2)}{4})^{\frac{N-2}{4}}$, by (3.5), (3.6) we conclude

$$\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} U(x) = 0, \quad \lim_{|x|\to\infty} |x|^{\frac{N-2}{2}} U(x) = 0 \quad \text{and} \quad \max_{x\neq 0} |x|^{\frac{N-2}{2}} U(x) = \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}} \tag{3.10}$$

for any radial regular solution $\overline{U}_\delta$ of (3.2).

Conditions (3.9), (3.10) clearly show that each pair of a regular and a singular solutions do intersect (transversally by ODE uniqueness) for at least two radii (actually at most finitely many times, due to the different behaviours at $|x| \to 0$, at $|x| \to \infty$, and ODE uniqueness).

Using the radial singular solutions discussed above it is possible to describe the asymptotic behaviour of U near its singular set when Σ consists of finitely many points. Such solutions were first constructed in [45] (see also [30]) and were discussed in more detail in [33]. Let $\Sigma = \operatorname{Sing} U$ be a finite set and let $P \in \Sigma$. Then (see [7], Theorem 1.2, [23], Proposition 5, respectively) there exist unique $\delta > 0$, $\varepsilon \in (0, \varepsilon_0]$ and $\alpha > 0$ such that

$$\begin{aligned} &U(x) = |x-P|^{\frac{2-N}{2}}\bigl(g_\varepsilon\bigl(-\log\bigl(\delta|x-P|\bigr)\bigr) + \mathrm{o}(1)\bigr), \\ &U(x) = U_{\varepsilon,\delta}(x-P) + \mathcal{O}\bigl(|x-P|^{\frac{2-N}{2}+\alpha}\bigr) \quad \text{as } x \to P. \end{aligned} \tag{3.11}$$

Similarly when $\Sigma = \operatorname{Sing} U$ is a compact set and $U(x) \sim |x|^{\frac{2-N}{2}}$ as $|x| \to \infty$ (compare with (3.4)), the asymptotic behaviour of U at infinity is given by a radial singular solution. Indeed, by conformal invariance of (3.2), Kelvin transformation and (3.11), there exist unique $\delta > 0$, $\varepsilon \in (0, \varepsilon_0]$ and $\alpha > 0$ such that

$$U(x) = |x|^{\frac{2-N}{2}}\bigl(g_\varepsilon\bigl(-\log\bigl(\delta|x|\bigr)\bigr) + \mathrm{o}(1)\bigr), \quad U(x) = U_{\varepsilon,\delta}(x) + \mathcal{O}\bigl(|x|^{\frac{2-N}{2}-\alpha}\bigr) \quad \text{as} \quad |x| \to \infty. \tag{3.12}$$

4. Lorentz spaces and ε-regularity

In this section we prove Theorem 2 using Holder, Young and Sobolev inequalities in Lorentz spaces. Since we do not assume the reader to be familiar with these spaces first we recall their definition and the results we need. For more details we refer to papers [1,21,38,49] and the books [48,56].

We consider real valued measurable functions f defined on the measure space $(\Omega, |\cdot|)$, where $\Omega \subset \mathbb{R}^N$ is an open set and $|\cdot|$ is the Lebesgue measure. We assume all the functions to be finite a.e. and such that $|\{|f| > s\}| < \infty$ for any $s > 0$. Given $f : \Omega \to \mathbb{R}$ the distribution function λ_f is defined as

$$\lambda_f(s) = \bigl|\bigl\{x \in \Omega\colon \bigl|f(x)\bigr| > s\bigr\}\bigr|, \quad s > 0,$$

and the nonincreasing rearrangement f^* is defined as

$$f^*(t) = \inf\bigl\{s > 0\colon \lambda_f(s) \leqslant t\bigr\},$$

and it is a.e. finite. The averaged nondecreasing rearrangement f^{**} is defined as $f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)\,\mathrm{d}\tau$, $t > 0$. For $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, we set

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left[\dfrac{q}{p}\displaystyle\int_0^{|\Omega|} \bigl(t^{1/p} f^{**}(t)\bigr)^q \,\dfrac{\mathrm{d}t}{t}\right]^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 < p < \infty,\ q = \infty. \end{cases} \tag{4.1}$$

By definition $f \in L^{p,q}(\Omega)$ iff $\|f\|_{L^{p,q}(\Omega)} < \infty$ (see e.g. [38]; see also e.g. [49] and [56] for equivalent definitions). Finally, we say that $f \in L^{p,q}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, if $f \in L^{p,q}(\Omega')$ for each open set $\Omega' \subset \mathbb{R}^N$ with compact closure such that $\overline{\Omega}' \subset \Omega$.

When $1 < q = p < \infty$ we have $L^{p,p}(\Omega) = L^p(\Omega)$ and when $1 < p < \infty$, $q = \infty$ this definition is equivalent to the one of the weak-L^p (see e.g. [56]). For $1 < p < \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ we have the inclusion $L^{p,q_1}(\Omega) \subset L^{p,q_2}(\Omega)$ with continuous embedding and $\|f\|_{L^{p,q_2}(\Omega)} \leqslant \frac{p}{p-1}\|f\|_{L^{p,q_1}(\Omega)}$. When $|\Omega| < \infty$, for any $1 < p_1 < p_2 < \infty$ and

$q_1, q_2 \in [1, \infty]$ we have $L^{p_2,q_2}(\Omega) \subset L^{p_1,q_1}(\Omega)$ with continuous embedding. If $|\Omega| < \infty$ and $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, we also have $L^\infty(\Omega) \subset L^{p,q}(\Omega)$ with continuous embedding.

Continuity of multiplication and convolution (when $\Omega = \mathbb{R}^N$) hold under certain restriction on the exponents. Here confine to the cases we are interested in. For the general case see [38], Theorems 3.5 and 2.6 respectively.

If $f_1 \in L^{p_1,q_1}(\Omega)$, $f_2 \in L^{p_2,q_2}(\Omega)$, $1 < p_1, p_2 < \infty$ and $1 \leqslant q_1, q_2 \leqslant \infty$, then $f_1 f_2 \in L^1(\Omega)$ and

$$\|f_1 f_2\|_{L^1(\Omega)} \leqslant C(p_1, p_2, q_1, q_2)\|f_1\|_{L^{p_1,q_1}(\Omega)}\|f_2\|_{L^{p_2,q_2}(\Omega)}, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1,\ \frac{1}{q_1} + \frac{1}{q_2} = 1. \tag{4.2}$$

If $f_1 \in L^{p,q}(\mathbb{R}^N)$, $f_2 \in L^1(\mathbb{R}^N)$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then $f_1 * f_2 \in L^{p,q}(\mathbb{R}^N)$ and

$$\|f_1 * f_2\|_{L^{p,q}} \leqslant C'(p,q)\|f_1\|_{L^{p,q}}\|f_2\|_{L^1}. \tag{4.3}$$

The following preliminary result concerning interior elliptic regularity in Lorentz spaces will be used in the sequel.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$. Let $u, f \in L^1_{\mathrm{loc}}(\Omega)$ such that $-\Delta u = f$ in $\mathcal{D}'(\Omega)$. Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. If $f \in L^{p,q}_{\mathrm{loc}}(\Omega)$, then $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ for all $1 < r < p$ and $\nabla^2 u \in L^{p,q}_{\mathrm{loc}}(\Omega)$.*

Proof. It is enough to prove that for any ball $B_{3R} \Subset \Omega$ and any $1 < r < p$ we have

$$\|u\|_{W^{2,r}(B_R)} + \|\nabla^2 u\|_{L^{p,q}(B_R)} \leqslant C(p,q,r,R)\big(\|f\|_{L^{p,q}(B_{3R})} + \|u\|_{L^1(B_{3R})}\big). \tag{4.4}$$

Let $f_0 = f\chi_{B_{3R}}$, so that $f_0 \in L^{p,q}(B_{3R})$, $f_0 \in L^r(B_{3R})$ for each $1 < r < p$ and f_0 vanishes identically outside B_{3R}. Let us denote with $H(x)$ the fundamental solution of the Laplace equation in $\mathbb{R}^N$, i.e. the Newtonian potential $H(x) = C(N)|x|^{2-N}$, $C(N) > 0$, satisfying $-\Delta H = \delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$. For any $x \in \mathbb{R}^N$ we set

$$v(x) = H * f_0(x) = C(N)\int_{\mathbb{R}^N} |x-y|^{2-N} f_0(y)\,\mathrm{d}y.$$

Using the Calderon–Zygmund theorem we have $v \in W^{2,r}(B_{3R})$ and

$$\|v\|_{W^{2,r}(B_{3R})} \leqslant C(r,R)\|f_0\|_{L^r(B_{3R})}. \tag{4.5}$$

On the other hand $-\Delta v = f_0$ in $\mathcal{D}'(B_{3R})$, hence $w = u - v$ satisfies $\Delta w = 0$ in $\mathcal{D}'(B_{3R})$. Thus, $w \in C^\infty(B_{3R})$ by Weyl's lemma and by standard regularity results for harmonic functions

$$\|w\|_{W^{2,r}(B_R)} \leqslant C(r,R)\|w\|_{C^2(B_R)} \leqslant C(r,R)\|w\|_{L^1(B_{2R})} \leqslant C(r,R)\big(\|v\|_{L^r(B_{3R})} + \|u\|_{L^1(B_{3R})}\big).$$

Since $u = v + w$, combining the previous estimate with (4.5) we easily infer

$$\|u\|_{W^{2,r}(B_R)} \leqslant C(r,R)\big(\|f_0\|_{L^r(B_{3R})} + \|u\|_{L^1(B_{3R})}\big),$$

i.e. the first part of (4.4) is proved. To prove the second part we choose $1 < p_1 < p < p_2$ and we argue by interpolation. Indeed, if we set $Tf_0 = \nabla^2 v$, arguing as in (4.5) we see that $T : L^{p_i}(B_{3R}) \to (L^{p_i}(B_{3R}))^{\frac{N(N-1)}{2}}$, for $i = 1, 2$. Using the general Marcinkiewicz interpolation theorem (see [48], Theorem V.3.15) with $\theta \in (0,1)$ such that $1/p = \theta/p_1 + (1-\theta)/p_2$ we have $T : L^{p,q}(B_{3R}) \to (L^{p,q}(B_{3R}))^{\frac{N(N-1)}{2}}$ boundedly, i.e. there exists $C = C(p,q,R) > 0$ such that

$$\|\nabla^2 v\|_{L^{p,q}(B_{3R})} \leqslant C(p,q,R)\|f_0\|_{L^{p,q}(B_{3R})}. \tag{4.6}$$

On the other hand $w = u - v \in C^\infty(B_{3R})$ satisfies $\Delta w = 0$ in $\mathcal{D}'(B_{3R})$ and the previous argument gives also

$$\|\nabla^2 w\|_{L^{p,q}(B_R)} \leqslant C(p,q,R)\|w\|_{C^2(B_R)} \leqslant C(p,q,R)\|w\|_{L^1(B_{2R})} \leqslant C(p,q,R)\big(\|v\|_{L^{p,q}(B_{3R})} + \|u\|_{L^1(B_{3R})}\big),$$

for any $1 < r < p$. Since $u = v + w$, combining the previous estimate with (4.5), (4.6) and the embedding $L^{p,q}(B_{3R}) \subset L^r(B_{3R})$ we easily infer

$$\|\nabla^2 u\|_{L^{p,q}(B_R)} \leqslant C(p,q,r,R)\big(\|f_0\|_{L^{p,q}(B_{3R})} + \|u\|_{L^1(B_{3R})}\big),$$

and the proof is complete. $\square$

The crucial tool in the proof our regularity result is given by the following well-known improved Sobolev inequality in Lorentz spaces (see e.g. [49], Theorem 8). Namely, for any $1 < p < N$, $1 \leqslant q \leqslant \infty$, there exists $C' = C'(N, p, q) > 0$ such that

$$\|f\|_{L^{p^*,q}} \leqslant C''(N, p, q)\|\nabla f\|_{L^{p,q}}, \quad \frac{1}{p^*} = \frac{1}{N} - \frac{1}{p}, \tag{4.7}$$

for any measurable function f such that ∇f exists and satisfies $\|\nabla f\|_{L^{p,q}} < \infty$. Here we confine ourselves to the case $1 < p < N$, the case $p = 1$ being slightly different.

Using these tools we are ready to prove the ε-regularity theorem in $L^{2^*,\infty}$.

Proof of Theorem 2. We are going to show that $u \in L^{2^*}_{\mathrm{loc}}(B_R(x_0))$, whence $u \in H^1_{\mathrm{loc}}(B_R(x_0))$ by Lemma 4.1 and the conclusion follows from [51], Theorem 3. To this end let us set $\varepsilon_0 = (\frac{N^2}{2(N-2)})^{\frac{2-N}{4}}(C'(N)C''(N)^{\frac{N-2}{2}})^{-1}$, where $C'(N)$, $C''(N)$ are the ones defined in (4.3), (4.7) with $p = q = 2$. We fix two radii R_1, R_2 such that $0 < R_1 < R_2 < R$ and for simplicity (up to a translation) we may assume $x_0 = 0$. Let $\varphi \in C_0^\infty(B_{R_2})$ be such that $\varphi \equiv 1$ for $|x| \leqslant R_1$.

Clearly $f = u^{\frac{N+2}{N-2}} \in L^s(B_R)$ for any $1 < s < \frac{2N}{N+2}$ and $f \in L^{\frac{2N}{N+2},\infty}(B_R)$ by the definition of the Lorentz norm (see Lemma 13 in [1]). Using Lemma 4.1 with $p = \frac{2N}{N+2}$ and $q = \infty$ and taking into account the improved Sobolev embedding (4.7) we infer $u \in W^{1,s}(B_{R_2})$ for any $s < 2$ and $\nabla u \in L^{2,\infty}(B_{R_2})$. Unfortunately this is not enough to test the equation with $\varphi^2 u$, therefore we need to approximate the equation suitably. Let $0 < \delta < R - R_2$ and let ζ_δ be a standard mollifier supported in the ball B_δ. Set $u_\delta = u * \zeta_\delta$ and $f_\delta = f * \zeta_\delta$, so that $u_\delta, f_\delta \in C^\infty(B_{R_2})$ and

$$\begin{aligned} &-\Delta u_\delta = f_\delta \quad \text{in } C^2(B_{R_2}), \quad u_\delta \to u \quad \text{in } L^r(B_{R_2}),\ 1 \leqslant r < 2^*, \\ &f_\delta \to f \quad \text{in } L^s(B_{R_2}),\ 1 \leqslant s < \frac{2N}{N+2}. \end{aligned} \tag{4.8}$$

Let $0 < \beta_0 < 1/2$ such that $4\beta_0^2/(1 - 4\beta_0^2) \leqslant \frac{1}{2}(1 - (\varepsilon/\varepsilon_0)^{4/N-2})$. For each $0 < \beta \leqslant \beta_0$ we set

$$v_\delta = \varphi^2 \frac{u_\delta}{(1 + u_\delta^2)^\beta}, \quad w_\delta = \frac{u_\delta \varphi}{(1 + u_\delta^2)^{\beta/2}}, \quad g_\delta = \frac{\varphi}{(1 + u_\delta^2)^{\beta/2}}. \tag{4.9}$$

It is easy to check that $v_\delta, w_\delta, g_\delta \in C_0^\infty(B_{R_2})$. Moreover, we have the following pointwise inequalities

$$\left|\nabla \frac{1}{(1 + u_\delta^2)^{\beta/2}}\right| \leqslant \frac{\beta|\nabla u_\delta|}{(1 + u_\delta^2)^{(\beta+1)/2}}, \quad |\nabla g_\delta|^2 \leqslant 2|\nabla\varphi|^2 + 2\varphi^2\left|\nabla \frac{1}{(1 + u_\delta^2)^{\beta/2}}\right|^2, \tag{4.10}$$

and

$$u_\delta^2|\nabla g_\delta|^2 \leqslant 2|\nabla\varphi|^2 u_\delta^2 + \frac{2\beta^2\varphi^2|\nabla u_\delta|^2}{(1 + u_\delta^2)^\beta}, \qquad |\nabla w_\delta|^2 \leqslant \frac{2\varphi^2|\nabla u_\delta|^2}{(1 + u_\delta^2)^\beta} + 2u_\delta^2|\nabla g_\delta|^2. \tag{4.11}$$

As in (4.11) we also have $\frac{\varphi^2|\nabla u_\delta|^2}{(1+u_\delta^2)^\beta} \leqslant 2u_\delta^2|\nabla g_\delta|^2 + 2|\nabla w_\delta|^2$. Thus, using (4.11), the monotonicity of $\frac{t}{1-t}$ and the choice of β_0, we obtain

$$u_\delta^2|\nabla g_\delta|^2 \leqslant \frac{2}{1 - 4\beta^2}|\nabla\varphi|^2 u_\delta^2 + \frac{1}{2}\left(1 - \left(\frac{\varepsilon}{\varepsilon_0}\right)^{4/N-2}\right)|\nabla w_\delta|^2. \tag{4.12}$$

Since $|\nabla w_\delta|^2 = \nabla u_\delta \nabla v_\delta + u_\delta^2|\nabla g_\delta|^2$, combining this identity with (4.12) and integrating, we obtain

$$\frac{1}{2}\left(1 + \left(\frac{\varepsilon}{\varepsilon_0}\right)^{4/N-2}\right)\int_{B_{R_2}} |\nabla w_\delta|^2\,\mathrm{d}x \leqslant \frac{2}{1 - 4\beta_0^2}\int_{B_{R_2}} |\nabla\varphi|^2 u_\delta^2\,\mathrm{d}x + \int_{B_{R_2}} \nabla u_\delta \nabla v_\delta\,\mathrm{d}x. \tag{4.13}$$

On the other hand, multiplying (4.8) by $v_\delta \in C_0^\infty(B_{R_2})$ and integrating by parts we obtain $\int_{B_{R_2}} \nabla u_\delta \nabla v_\delta\,\mathrm{d}x = \int_{B_{R_2}} f_\delta v_\delta\,\mathrm{d}x$, whence

$$\frac{1}{2}\left(1 + \left(\frac{\varepsilon}{\varepsilon_0}\right)^{4/N-2}\right)\int_{B_{R_2}} |\nabla w_\delta|^2\,\mathrm{d}x \leqslant \frac{2}{1 - 4\beta_0^2}\int_{B_{R_2}} |\nabla\varphi|^2 u_\delta^2\,\mathrm{d}x + \int_{B_{R_2}} f_\delta v_\delta\,\mathrm{d}x. \tag{4.14}$$

We rewrite the last integral as follows

$$\int\limits_{B_{R_2}} f_\delta v_\delta \,\mathrm{d}x = \int\limits_{B_{R_2}} \left(f_\delta v_\delta - u_\delta^{\frac{4}{N-2}} w_\delta^2\right) \mathrm{d}x + \int\limits_{B_{R_2}} u_\delta^{\frac{4}{N-2}} w_\delta^2 \,\mathrm{d}x = I_\delta + II_\delta. \tag{4.15}$$

Setting $\overline{C}(N) = \frac{N}{N-2} C'(N)^{\frac{4}{N-2}}$, by Young inequality (4.3) and the definition of the Lorentz norm (see [1], Lemma 13) we have

$$\left\| u_\delta^{\frac{4}{N-2}} \right\|_{L^{N/2,\infty}(B_{R_2})} \leqslant \frac{N}{N-2} \|u_\delta\|^{\frac{4}{N-2}}_{L^{2^*,\infty}(B_{R_2})} \leqslant \overline{C}(N) \|u\|^{\frac{4}{N-2}}_{L^{2^*,\infty}(B_R)} = \overline{C}(N) \varepsilon^{\frac{4}{N-2}}.$$

Using Holder inequality (4.2) with parameters $(N/2, \infty)$ and $(2^*/2, 1)$ respectively, since $\|w_\delta^2\|_{L^{2^*/2,1}} \leqslant \frac{N}{2} \|w_\delta\|^2_{L^{2^*,2}}$, by the Sobolev embedding (4.7) and the previous inequality we have

$$\begin{aligned} II_\delta = \int\limits_{B_{R_2}} u_\delta^{\frac{4}{N-2}} w_\delta^2 \,\mathrm{d}x &\leqslant \left\| u_\delta^{\frac{4}{N-2}} \right\|_{L^{N/2,\infty}(B_{R_2})} \frac{N}{2} \|w_\delta\|^2_{L^{2^*,2}(B_{R_2})} \\ &\leqslant \frac{N^2}{2(N-2)} C'(N)^{\frac{4}{N-2}} C''(N)^2 \varepsilon^{\frac{4}{N-2}} \int\limits_{B_{R_2}} |\nabla w_\delta|^2 \,\mathrm{d}x, \end{aligned}$$

i.e.

$$II_\delta \leqslant \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{4}{N-2}} \int\limits_{B_{R_2}} |\nabla w_\delta|^2 \,\mathrm{d}x. \tag{4.16}$$

Combining (4.14)–(4.16) and using Sobolev inequality we deduce

$$\frac{1}{2}\left(1 - \left(\frac{\varepsilon}{\varepsilon_0}\right)^{4/N-2}\right) \left(\int\limits_{B_{R_2}} |w_\delta|^{2^*} \,\mathrm{d}x\right)^{2/2^*} \leqslant \frac{2C(N)}{1-4\beta_0^2} \int\limits_{B_{R_2}} |\nabla\varphi|^2 u_\delta^2 \,\mathrm{d}x + C(N) I_\delta. \tag{4.17}$$

Now we claim that $I_\delta \to 0$ as $\delta \to 0$. Assuming the claim for a moment, we have $u_\delta \to u$ in $L^2(B_{R_2})$ by (4.8). Up to subsequences we may assume $w_\delta \to \varphi u(1+u^2)^{-\beta/2}$ a.e. in B_{R_2}, and applying Fatou's lemma in (4.17) we conclude

$$\frac{1}{2}\left(1 - \left(\frac{\varepsilon}{\varepsilon_0}\right)^{4/N-2}\right) \left(\int\limits_{B_{R_2}} \left|\frac{\varphi u}{(1+u^2)^{\beta/2}}\right|^{2^*} \mathrm{d}x\right)^{2/2^*} \leqslant \frac{2C(N)}{1-4\beta_0^2} \int\limits_{B_{R_2}} |\nabla\varphi|^2 u^2 \,\mathrm{d}x. \tag{4.18}$$

Since the r.h.s. of (4.18) is finite and independent of β, by Fatou lemma and the choice of φ, as $\beta \to 0$ we conclude

$$\int\limits_{B_{R_1}} u^{2^*} \,\mathrm{d}x < \infty.$$

As $0 < R_1 < R_2 < R$ have been chosen arbitrarily we have $u \in L^{2^*}_{\mathrm{loc}}(B_R)$ as claimed and the proof is complete.

In order to prove that $I_\delta \to 0$ as $\delta \to 0$, first we define v and w from u as in (4.9). Clearly $fv - u^{\frac{4}{N-2}} w^2 \equiv 0$ and both the terms are absolutely integrable because of (4.9) and the assumption $u \in L^s(B_R)$ for each $1 \leqslant s < 2^*$. Recall that

$$I_\delta = \int\limits_{B_{R_2}} \left(f_\delta v_\delta - u_\delta^{\frac{4}{N-2}} w_\delta^2\right) \mathrm{d}x,$$

so the conclusion follows if we can pass to the limit under integral sign. Since $0 < \beta < 1/2$, if we set $s_1 = \frac{2^*(1-\beta)}{1-\beta/2}$, $s_2 = \frac{2^*}{2-\beta}$, then $1 < s_1 < 2^*$ and $2^*/2 < s_2 < \infty$. Thus $u_\delta \to w_\delta^2$ is a bounded (hence continuous) superposition operator from $L^{s_1}(B_{R_2})$ to $L^{s_2}(B_{R_2})$. If s_2' is the conjugate exponent to s_2 and w set $s_3 = \frac{4}{N-2} s_2'$, then $1 < s_2' < N/2$

and $4/N-2 < s_3 < 2^*$. Therefore $u_\delta \to u_\delta^{\frac{4}{N-2}}$ is a bounded (hence continuous) superposition operator from $L^{s_3}(B_{R_2})$ to $L^{s_2'}(B_{R_2})$. Combining these remarks, by (4.8) we conclude

$$\int_{B_{R_2}} u_\delta^{\frac{4}{N-2}} w_\delta^2 \,\mathrm{d}x \to \int_{B_{R_2}} u^{\frac{4}{N-2}} w^2 \,\mathrm{d}x \quad \text{as } \delta \to 0.$$

Similarly if we set $s_4 = \frac{2^*}{1-\beta}$, $s_5 = 2^* \frac{1-2\beta}{1-\beta}$, then $1 < s_5 < 2^*$, $2^* < s_4 < \infty$ and $1 < s_4' < \frac{2N}{N+2}$. Thus $u_\delta \to v_\delta$ is a bounded (hence continuous) superposition operator from $L^{s_5}(B_{R_2})$ to $L^{s_4}(B_{R_2})$ and $f_\delta \to f$ in $L^{s_4'}(B_{R_2})$ because of (4.8). Combining these remarks we conclude that

$$\int_{B_{R_2}} f_\delta v_\delta \,\mathrm{d}x \to \int_{B_{R_2}} f v \,\mathrm{d}x \quad \text{as } \delta \to 0,$$

and the claim follows.

(2) We give here a proof which is independent of the previous part and is based just on Lemma 4.1 and the improved Sobolev embedding (4.7). For alternative proofs see Remark 4 below.

Consider the equation $-\Delta u = f$ in $\mathcal{D}'(\Omega)$, where $f = |u|^{\frac{4}{N-2}} u$. Let $B_R(x_0) \subset \Omega$ and $0 < R_0 < R$. Let $R_0 < R_1 < R_2 \leqslant R$. If $u \in L^{2^*,q}(B_{R_2}(x_0))$, $2^* \leqslant q < \infty$ then $f \in L^{\frac{2N}{N+2}, \frac{N-2}{N+2}q}(B_{R_2}(x_0))$ (see [1], Lemma 13). By Lemma 4.1 we have $u \in W^{2,r}(B_{R_1})$ for any $1 < r < \frac{2N}{N+2}$ and $\nabla^2 u \in L^{\frac{2N}{N+2}, \frac{N-2}{N+2}q}(B_{R_1}(x_0))$. Applying (4.7) twice we conclude $u \in L^{2^*, \frac{N-2}{N+2}q}(B_{R_1}(x_0))$. With finitely many choices of radii R_1, R_2 we can easily infer $u \in L^{2^*, (\frac{N-2}{N+2})^m q}(B_{R_0}(x_0))$, for any positive integer m such that $(\frac{N-2}{N+2})^m q > 1$. Clearly we can get $1 < (\frac{N-2}{N+2})^m q \leqslant 2^*$ for m large enough, whence $u \in L^{2^*, (\frac{N-2}{N+2})^m q}(B_{R_0}(x_0)) \subset L^{2^*}(B_{R_0}(x_0))$ and $L^{2^*}_{\mathrm{loc}}(B_{R_0}(x_0))$ because R_0 can be chosen arbitrarily. By Lemma 4.1 we have $u \in H^1_{\mathrm{loc}}(B_{R_0}(x_0))$ and smoothness follows from [51], Theorem 3. □

Remark 3. As already observed in the Introduction about (1.3), the smallness assumption in the previous theorem cannot be removed. Indeed, if u is any distributional solution with finite nonempty singular set as constructed in [45], then $u \in L^{2^*,\infty}$ (see (3.3)) but the singularities are not removable.

Remark 4. Alternative proofs of claim (2) can be obtained arguing as in [1] or [17] respectively. Indeed, arguing as in [1], p. 233–234, one can use claim (1) to prove that $\Sigma = \operatorname{Sing} u \cap B_R(x_0)$ is a finite set. Then the asymptotic results of [7] actually show that isolated singularities which are $L^{2^*,q}$-integrable for some $q < \infty$ are removable by (3.3). Alternatively, following a remark contained in [17], one can prove that the smallness assumption of claim (1) holds at suitably smaller scales. Namely, since $u \in L^{2^*,q}(B_R(x_0))$ then $\sup_{B_\rho \subset B_R(x_0)} \|u\|_{L^{2^*,\infty}(B_\rho)} \to 0$ as $\rho \to 0$. Thus, smoothness follows from claim (1).

5. Hardy inequalities and instability in the subcritical case

In this section we derive some extensions of the classical Hardy inequality (1.6) and we prove Theorem 1. We start with an Hardy-type inequality for supersolutions of semilinear equations of quite general form. The connection between Hardy-type inequalities like (1.6) and linear elliptic equations is well known (see e.g. [39], Chapter 2). On the other hand, at least in the author's knowledge, the connection with semilinear differential inequalities is new.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $N \geqslant 1$, and let $u \in L^1_{\mathrm{loc}}(\Omega)$, $u \geqslant 0$, $u \not\equiv 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function. Assume that $f(u) \in L^1_{\mathrm{loc}}(\Omega)$ and u satisfies*

$$\Delta u + f(u) \leqslant 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

Then, $u > 0$ a.e. in Ω and

$$\int_\Omega \frac{f(u)}{u} \varphi^2 \,\mathrm{d}x \leqslant \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x \quad \text{for any } \varphi \in \mathcal{D}(\Omega). \tag{5.2}$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and let ζ_ε be a family of standard mollifiers, i.e. $\zeta_\varepsilon \geqslant 0$, $\zeta_\varepsilon \in C_0^\infty(B_\varepsilon)$ and $\int \zeta_\varepsilon = 1$ for each $\varepsilon > 0$. As it is well known, if $\Omega' \Subset \Omega$ is such that $\operatorname{spt}\varphi \subset \Omega'$ and ε is small enough, then $u_\varepsilon = u * \zeta_\varepsilon$ are nonnegative and smooth in Ω', $u_\varepsilon \to u$ in $L^1(\Omega')$ and (up to a subsequence) $u_\varepsilon \to u$ a.e. in Ω'.

Since f is positive and convex and φ has compact support in Ω, for $x \in \Omega'$ and $\varepsilon < \operatorname{dist}(\partial\Omega, \overline{\Omega}')$, using Jensen inequality in (5.1) we have

$$\Delta u_\varepsilon(x) + f\big(u_\varepsilon(x)\big) = \int_\Omega u(y)\Delta_y \zeta_\varepsilon(x-y)\,\mathrm{d}y + f\bigg(\int_\Omega u(y)\zeta_\varepsilon(x-y)\,\mathrm{d}y\bigg)$$
$$\leqslant \big\langle \Delta u(y) + f\big(u(y)\big), \zeta_\varepsilon(x-y)\big\rangle \leqslant 0.$$

From the previous relation we deduce that u_ε are (classical) local solutions of the differential inequality (5.1). In particular they are nonnegative superharmonic functions. Since $u \not\equiv 0$ we have $u_\varepsilon \not\equiv 0$ and, by the classical mean-value inequality $u_\varepsilon > 0$ in Ω'. Using a standard covering argument, as $\varepsilon \to 0$ the mean value inequality gives $u > 0$ a.e. in Ω, since Ω' can be chosen arbitrarily. Let $\psi_\varepsilon \in C_0^\infty(\Omega')$ to be specified later. Writing (5.1) for u_ε, multiplying by $u_\varepsilon\psi_\varepsilon^2$ and integrating by parts we have

$$\int_{\Omega'} f(u_\varepsilon)u_\varepsilon\psi_\varepsilon^2\,\mathrm{d}x \leqslant \int_{\Omega'} -u_\varepsilon\psi_\varepsilon^2\Delta u_\varepsilon\,\mathrm{d}x = \int_{\Omega'} \nabla u_\varepsilon \nabla\big(u_\varepsilon\psi_\varepsilon^2\big)\,\mathrm{d}x = \int_{\Omega'} \big(\big|\nabla(u_\varepsilon\psi_\varepsilon)\big|^2 - u_\varepsilon^2|\nabla\psi_\varepsilon|^2\big)\,\mathrm{d}x$$
$$\leqslant \int_{\Omega'} \big|\nabla(u_\varepsilon\psi_\varepsilon)\big|^2\,\mathrm{d}x.$$

Let $\varphi \in C_0^\infty(\Omega')$ be fixed. Choosing $\psi_\varepsilon = \varphi/u_\varepsilon \in C_0^\infty(\Omega')$ in the previous inequality we obtain

$$\int_\Omega \frac{f(u_\varepsilon)}{u_\varepsilon}\varphi^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x,$$

and the conclusion follows from Fatou's lemma and the continuity of f as $\varepsilon \to 0$. $\square$

As a consequence of the previous proposition we obtain the following result.

Proposition 5.2. *Let $U \in L_{\mathrm{loc}}^{\frac{N+2}{N-2}}(\mathbb{R}^N)$, $N \geqslant 3$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} \leqslant 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and let $V(x) = U(x)^{\frac{4}{N-2}}$. Then*

$$\int_{\mathbb{R}^N} V\varphi^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla\varphi|^2\,\mathrm{d}x \quad \text{for any } \varphi \in \mathcal{D}\big(\mathbb{R}^N\big), \tag{5.3}$$

and for any $\varphi \in H^1(\mathbb{R}^N)$.

If in addition $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $\Sigma = \operatorname{Sing} U$ is a nonempty finite set then

$$\inf\bigg\{\int_{\mathbb{R}^N} |\nabla\varphi|^2\,\mathrm{d}x,\ \varphi \in C_0^\infty\big(\mathbb{R}^N\big),\ \int_{\mathbb{R}^N} V\varphi^2\,\mathrm{d}x = 1\bigg\} = 1. \tag{5.4}$$

Proof. Clearly (5.3) follows readily from Proposition 5.1 and the case $\varphi \in H^1(\mathbb{R}^N)$ can be obtained by approximation. In order to prove (5.4) we set

$$m_U = \inf\bigg\{\int_{\mathbb{R}^N} |\nabla\varphi|^2\,\mathrm{d}x,\ \varphi \in C_0^\infty\big(\mathbb{R}^N\big),\ \int_{\mathbb{R}^N} V\varphi^2\,\mathrm{d}x = 1\bigg\}, \tag{5.5}$$

therefore $m_U \geqslant 1$ by (5.3) and it suffice to prove that $m_U = 1$. First we assume that $U = U_0$ is a radial singular solution and we prove (5.4) with an argument similar to the one in [41] for the homogeneous case. Let us denote by $V_0 = U_0^{\frac{4}{N-2}}$ the corresponding potential. Since U_0 is radial, by (3.8) we may assume $U_0(x) = |x|^{\frac{2-N}{2}} g(-\log|x|)$ for some bounded smooth function $g:\mathbb{R}\to\mathbb{R}$ which is a positive solution of $g''=(\frac{N-2}{2})^2 g - g^{\frac{N+2}{N-2}}$ bounded away from zero. Moreover either g is periodic of period $T>0$ or $g\equiv(\frac{N-2}{2})^{\frac{N-2}{2}}$ (see the discussion in Section 3). In both cases there exists $T>0$ such that $g(t+T)=g(t)$ for each $t\in\mathbb{R}$. Arguing by contradiction, we assume that $m_{U_0}>1$. Let $r_0>0$, $t_0=-\log r_0$ and for each integer $n\geqslant 1$ let us set $r_n=r_0\,\mathrm{e}^{nT}$, so that $r_n\to\infty$ as $n\to\infty$. Multiplying the equation $\Delta U_0+U_0^{\frac{N+2}{N-2}}=0$ by U_0, integrating over $\Omega_n=\{r_0<|x|<r_n\}$, taking polar coordinates and integrating by parts we have

$$\begin{aligned}\int_{\Omega_n} V_0U_0^2\,\mathrm{d}x &= -\int_{\Omega_n} U_0\Delta U_0\,\mathrm{d}x = -\omega_N\int_{r_0}^{r_n} r^{\frac{2-N}{2}}g(-\log r)\partial_r\big(r^{N-1}\partial_r\big(r^{\frac{2-N}{2}}g(-\log r)\big)\big)\,\mathrm{d}r\\ &= \int_{\Omega_n}|\nabla U_0|^2\,\mathrm{d}x-\omega_N\big[r^{N-1}\big(r^{\frac{2-N}{2}}g(-\log r)\big)\partial_r\big(r^{\frac{2-N}{2}}g(-\log r)\big)\big]_{r_0}^{r_n}.\end{aligned}$$

Clearly

$$\big[r^{N-1}\big(r^{\frac{2-N}{2}}g(-\log r)\big)\partial_r\big(r^{\frac{2-N}{2}}g(-\log r)\big)\big]_{r_0}^{r_n}=\left[g(-\log r)\left(g(-\log r)\frac{2-N}{2}-g'(-\log r)\right)\right]_{r_0}^{r_n}=0,$$

because $r_n=r_0\,\mathrm{e}^{nT}$ and g is a T-periodic function. Since $\int_{|x|>r_0}|\nabla U_0|^2\,\mathrm{d}x=\infty$ we conclude

$$\int_{\Omega_n} V_0U_0^2\,\mathrm{d}x=\int_{\Omega_n}|\nabla U_0|^2\,\mathrm{d}x\to\infty\quad\text{as } n\to\infty. \tag{5.6}$$

Let $\varepsilon>0$ any fixed positive number and for each $n\geqslant 1$ let us set

$$\varphi_n(x)=\begin{cases} U_0\Big(r_0\dfrac{x}{|x|}\Big) & \text{if } |x|\leqslant r_0,\\ U_0(x) & \text{if } r_0<|x|<r_n,\\ U_0\Big(r_n\dfrac{x}{|x|}\Big)\Big(\dfrac{|x|}{r_n}\Big)^{\frac{2-N}{2}-\varepsilon} & \text{if } |x|\geqslant r_n.\end{cases} \tag{5.7}$$

Clearly φ_n is constant in B_{r_0}, $\varphi_n\in L^{2^*}(\mathbb{R}^N)$, $\nabla\varphi_n\in L^2(\mathbb{R}^N)$ and combining the definition of m_{U_0} given by (5.5) with a standard approximation argument we deduce

$$\int_{\mathbb{R}^N} V_0\varphi_n^2\,\mathrm{d}x\leqslant\frac{1}{m_{U_0}}\int_{\mathbb{R}^N}|\nabla\varphi_n|^2\,\mathrm{d}x,$$

for each $n\geqslant 1$. On the other hand, since $U_0(x)=|x|^{\frac{2-N}{2}}g(-\log|x|)\leqslant C|x|^{\frac{2-N}{2}}$ for some $C=C(N)>0$ independent of g (see the discussion after (3.7)), a straightforward computation yields $\int_{|x|>r_n}|\nabla\varphi_n|^2\,\mathrm{d}x+\int_{|x|>r_n}V\varphi_n^2\leqslant C$. Combining the previous two inequalities with (5.6) we have

$$\begin{aligned}0&\geqslant\int_{\mathbb{R}^N}V_0\varphi_n^2\,\mathrm{d}x-\frac{1}{m_{U_0}}\int_{\mathbb{R}^N}|\nabla\varphi_n|^2\,\mathrm{d}x\\ &=\int_{|x|<r_0}V_0\varphi_n^2\,\mathrm{d}x+\left(1-\frac{1}{m_{U_0}}\right)\int_{\Omega_n}|\nabla U_0|^2\,\mathrm{d}x+\int_{|x|>r_n}V_0\varphi_n^2\,\mathrm{d}x-\int_{|x|>r_n}|\nabla\varphi_n|^2\,\mathrm{d}x\\ &\geqslant\left(1-\frac{1}{m_{U_0}}\right)\int_{\Omega_n}|\nabla U_0|^2\,\mathrm{d}x-C.\end{aligned}$$

If $m_{U_0}>1$, letting $n\to\infty$ and taking (5.6) into account we get a contradiction. Thus $m_{U_0}=1$ and the claim holds when U is radial.

In the general case we use a blow-up argument. Let U be any positive weak solution such that $\Sigma=\operatorname{Sing}U$ is a nonempty finite set, so that, up to a translation, we may assume $0\in\Sigma$. Following [7,23], there exists a radial singular solution U_0 and $\alpha > 0$ such that $U(x) - U_0(x) = \mathcal{O}(|x|^{\frac{2-N}{2}+\alpha})$ as $x \to 0$ (see (3.11)). Hence, if $U_0(x) = |x|^{\frac{2-N}{2}} g(-\log|x|)$ and g is T-periodic as above, and if we set $\lambda_n = \mathrm{e}^{-nT} \to 0$ as $n \to \infty$, then $\lambda_n^{\frac{N-2}{2}} U(\lambda_n x) \to U_0(x)$ a.e. as $n \to \infty$. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and for each $n \geqslant 1$ let $\varphi_n(x) = \lambda_n^{\frac{2-N}{2}} \varphi(\lambda_n^{-1} x)$. Taking into account the definition of m_U and changing variables in both the integrals below we infer

$$\frac{1}{m_U} \int\limits_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x = \frac{1}{m_U} \int\limits_{\mathbb{R}^N} |\nabla\varphi_n|^2 \,\mathrm{d}x \geqslant \int\limits_{\mathbb{R}^N} V\varphi_n^2 \,\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(\lambda_n^{\frac{N-2}{2}} U(\lambda_n x)\right)^{\frac{4}{N-2}} \varphi^2 \,\mathrm{d}x.$$

Using Fatou's lemma we obtain

$$\frac{1}{m_U} \int\limits_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \int\limits_{\mathbb{R}^N} V_0\varphi^2 \,\mathrm{d}x,$$

for $V_0 = U_0^{\frac{4}{N-2}}$. Optimising with respect to $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} V_0\varphi^2 \,\mathrm{d}x = 1$ we conclude $m_U \leqslant m_{U_0}$. Combining this inequality with (5.3) and the previous part we have $1 \leqslant m_U \leqslant m_{U_0} = 1$ and the conclusion follows. □

Remark 5. If U is the homogeneous solution given by (1.3), $U(x) = (\frac{N-2}{2})^{\frac{N-2}{2}} |x|^{\frac{2-N}{2}}$, then $V(x) = (\frac{N-2}{2})^2 \frac{1}{|x|^2}$ and

$$\left(\frac{N-2}{2}\right)^2 \int\limits_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \,\mathrm{d}x \leqslant \int\limits_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x \quad \text{for any } \varphi \in \mathcal{D}\big(\mathbb{R}^N\big), \tag{5.8}$$

i.e., we recover the classical Hardy inequality with best constant.

Remark 6. We conjecture that (5.4) holds for distributional solutions U with more general singular set Σ. A necessary and sufficient condition for the validity of (5.4) involving capacity is known (see [11]), but we are not able to check it in the present situation, even under the assumptions of Proposition 5.2. On the other hand, in view of the recent results contained in [5,52,2] and [14] it would be of interest to know if the Hardy inequalities (5.3) can be improved on bounded domains by adding various lower order terms.

Using the generalised Hardy inequalities (5.3), (5.4) we are able to prove regularity properties of weak solutions for the Cauchy problem (P).

Proposition 5.3. *Let $N \geqslant 3$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Assume that $\Sigma = \operatorname{Sing} U$ is a finite set, so that $U(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$. Let $\lambda \in (0,1)$, $0 \leqslant u_0(x) \leqslant \lambda U(x)$, and let $\bar{u}$ be the maximal solution of problem* (P) *such that $0 < \bar{u}(x,t) \leqslant \lambda U(x)$ for a.e. $(x,t) \in \mathbb{R}^N \times \mathbb{R}_+$, as constructed in Proposition* 2.1. *Then $\bar{u} \in C^\infty(\mathbb{R}^N \times (0,\infty))$.*

Proof. First we observe that $U \in L^{2^*,\infty}(\mathbb{R}^N)$, because Σ is a finite set and (3.3) holds near each singular point. Thus, $U \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p < 2^*$, by the standard embedding properties of weak-L^p spaces. Let $v(t) = \bar{u}(t, u_0)$ be the maximal solution of (P) satisfying $0 \leqslant v(t) \leqslant \lambda U$ a.e. in $\mathbb{R}^N$ for each $t \geqslant 0$, as constructed in Proposition 2.1. Clearly by monotonicity with respect to u_0 we may assume $u_0 = \lambda U$. Indeed by (2.11) we would infer that v is locally bounded for $t > 0$ whenever $\bar{u}(t, \lambda U)$ is, whence the conclusion follows from the standard smoothing effect for the heat equation. Thus, by Proposition 2.1 we may also assume that v is decreasing in t.

Let $\Sigma = \operatorname{Sing} U$, $\eta_0 > 0$ such that $d(P,Q) \geqslant 4\eta_0$ for each $P, Q \in \Sigma$, $P \neq Q$. For each $0 < \eta < \eta_0$ let us set $\Sigma_\eta = \{x\colon \operatorname{dist}(x,\Sigma) \leqslant \eta\}$, so that Σ_η are compact sets, and let us set $\Omega = \mathbb{R}^N \setminus \Sigma$. If we define $\Omega_\eta = \mathbb{R}^N \setminus \Sigma_\eta$, then, by assumption, v is bounded in each open set $\Omega_\eta \times (0,\infty)$, therefore it is smooth by standard smoothing effect of the heat equation. Thus, $v \in C^\infty(\Omega \times (0,\infty))$ because $\eta > 0$ can be chosen arbitrarily small. Let $\overline{R} > 0$ such that $\Sigma_{2\eta_0} \subset B_{\overline{R}}(0)$ and let $R > 2\overline{R}$. Let $\varphi \in C_0^\infty(B_{2R}(0))$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ for $|x| \leqslant R$.

We need the following integrability result.

Lemma 5.1. *Let U, u_0 and v as above. Then for any $0 < t_1 < t_2$ and $R > 2\overline{R}$, we have $\nabla v \in L^2(B_R \times (t_1, t_2))$ and $vv_t \in L^1(B_R \times (t_1, t_2))$.*

Proof. Let $\psi \in C_0^\infty(B_1)$ such that $0 \leqslant \psi \leqslant 1$ and $\psi \equiv 1$ for $|x| \leqslant 1/2$ and let φ as above, i.e. $\varphi \in C_0^\infty(B_{2R}(0))$, $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ for $|x| \leqslant R$. Let us define $\{\zeta_\eta\} \subset C_0^\infty(B_{2R})$ as

$$\zeta_\eta(x) = \varphi(x)\psi_\eta(x) = \varphi(x)\left(1 - \sum_{P \in \Sigma} \psi\left(\frac{x-P}{2\eta}\right)\right), \quad \text{so that} \quad 0 \leqslant \zeta_\eta \leqslant \varphi, \quad \zeta_\eta \equiv 0 \quad \text{on } \Sigma_\eta,$$
$$|\nabla \zeta_\eta| \leqslant \frac{C}{\eta}, \tag{5.9}$$

for some $C > 0$ depending on φ but independent of η. Let $\theta(t) \in C_0^\infty((t_1/2, 2t_2))$ such that $0 \leqslant \theta \leqslant 1$ and $\theta \equiv 1$ for $t_1 \leqslant t \leqslant t_2$. Then $v\zeta_\eta^2\theta \in C_0^\infty(\mathbb{R}^N \times (0,\infty))$. Testing (1.4) with $v\zeta_\eta^2\theta$ and integrating by parts in the first two terms we have

$$\int_{\mathbb{R}^N \times (0,\infty)} v_t v \zeta_\eta^2 \theta \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N \times (0,\infty)} \nabla v \cdot \nabla\big(v\zeta_\eta^2\big)\theta \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^N \times (0,\infty)} v^{\frac{4}{N-2}} (v\zeta_\eta)^2 \theta \,\mathrm{d}x\,\mathrm{d}t.$$

If we choose $\theta = \theta_n \to \chi_{\{t_1 < t < t_2\}}$ a.e. in $(t_1/2, 2t_2)$, by dominated convergence (both v and ∇v are bounded on $(B_{2R} \setminus \Sigma_\eta) \times (t_1/2, 2t_2)$) we infer

$$\int_{\mathbb{R}^N \times (t_1,t_2)} \nabla v \cdot \nabla\big(v\zeta_\eta^2\big) \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^N \times (t_1,t_2)} v^{\frac{4}{N-2}} (v\zeta_\eta)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N \times (t_1,t_2)} \big(-v_t v \zeta_\eta^2\big) \,\mathrm{d}x\,\mathrm{d}t. \tag{5.10}$$

Since $v \leqslant \lambda U$ is decreasing in time, then $v_t \leqslant 0$ in $\Omega \times (0,\infty)$ and from (5.9) we easily obtain

$$\int_{\mathbb{R}^N \times (t_1,t_2)} -v v_t \zeta_\eta^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^N \times \{t=t_1\}} \frac{1}{2} v^2 \zeta_\eta^2 \,\mathrm{d}x - \int_{\mathbb{R}^N \times \{t=t_2\}} \frac{1}{2} v^2 \zeta_\eta^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} U^2 \varphi^2 \,\mathrm{d}x < \infty. \tag{5.11}$$

Since the r.h.s. of (5.11) is independent of η, $\zeta_\eta \to \varphi$ a.e. as $\eta \to 0$ and $\varphi \equiv 1$ on B_R, by Fatou's lemma we immediately infer $vv_t \in L^1(B_R \times (t_1, t_2))$.

Combining (5.11), (5.3) with (5.10) we deduce

$$\int_{\mathbb{R}^N \times (t_1,t_2)} \nabla v \cdot \nabla\big(v\zeta_\eta^2\big) \,\mathrm{d}x\,\mathrm{d}t \leqslant \lambda^{\frac{4}{N-2}} \int_{\mathbb{R}^N \times (t_1,t_2)} \big|\nabla(v\zeta_\eta)\big|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} U^2 \varphi^2 \,\mathrm{d}x. \tag{5.12}$$

Since $\nabla v \cdot \nabla(v\zeta_\eta^2) = |\nabla(v\zeta_\eta)|^2 - v^2|\nabla\zeta_\eta|^2$ and $v \leqslant U$, we have

$$\big(1 - \lambda^{\frac{4}{N-2}}\big) \int_{\mathbb{R}^N \times (t_1,t_2)} \big|\nabla(v\zeta_\eta)\big|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant (t_2 - t_1) \int_{\mathbb{R}^N} U^2 |\nabla\zeta_\eta|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} U^2 \varphi^2 \,\mathrm{d}x. \tag{5.13}$$

Since Σ is a finite set, using the asymptotic results of [7] (see (3.3)) we know that there exists $C = C(U) > 0$ such that $U(x) \leqslant C \sum_{P \in \Sigma} |x - P|^{\frac{2-N}{2}} \chi_{B_{2\eta_0}(P)}$ for every $x \in \Sigma_{2\eta_0}$. Thus, taking (5.9) into account, we conclude

$$\begin{aligned}
\int_{\mathbb{R}^N} U^2 |\nabla\zeta_\eta|^2 \,\mathrm{d}x &\leqslant 2 \int_{\mathbb{R}^N} U^2 |\nabla\varphi|^2 \,\mathrm{d}x + 2 \int_{\mathbb{R}^N} \varphi^2 U^2 |\nabla\psi_\eta|^2 \,\mathrm{d}x \\
&\leqslant 2 \int_{\mathbb{R}^N} U^2 |\nabla\varphi|^2 \,\mathrm{d}x + \sum_{P \in \Sigma} \int_{\eta < |x-P| < 2\eta} C/\eta^2 U^2 \,\mathrm{d}x \\
&\leqslant 2 \int_{\mathbb{R}^N} U^2 |\nabla\varphi|^2 \,\mathrm{d}x + C\#\Sigma,
\end{aligned} \tag{5.14}$$

where $C > 0$ is an absolute constant. Combining (5.13) and (5.14) we conclude that there exists $C = C(\lambda, U, \varphi) > 0$ such that

$$\int_{\mathbb{R}^N \times (t_1,t_2)} |\nabla v \zeta_\eta|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant C.$$

Clearly $v\zeta_\eta \to v\varphi$ in $L^2(\mathbb{R}^N \times (t_1, t_2))$, and using the previous inequality we deduce that $\nabla(v\varphi)$ exists and $\nabla(v\varphi) \in L^2(\mathbb{R}^N \times (t_1, t_2))$. Since $\varphi \equiv 1$ in B_R the conclusion follows. $\square$

Once we know the L^2-integrability of the gradient we can use another perturbation argument to obtain higher integrability of v.

Lemma 5.2. *Let U, u_0 and v as above and let $\lambda \in (0, 1)$. There exists $p = p(\lambda) > 2^*$ and for each $t_0 > 0$ a constant $C = C(p, t_0, U) > 0$ such that*

$$\sup_{t \geqslant 2t_0} \|v(t)\|_{L^p(\mathbb{R}^N)} \leqslant C. \tag{5.15}$$

Proof. Due to our choice of $\overline{R}$ we have $\Sigma \subset B_{\overline{R}}$, hence $U(x) \leqslant C|x|^{\frac{2-N}{2}}$ for $|x| \geqslant \overline{R}$ by the asymptotic decay rate (3.4). Since $v \leqslant U$ it suffices to show that

$$\sup_{t \geqslant 2t_0} \|v(t)\|_{L^p(B_R)} \leqslant C, \tag{5.16}$$

for some $R > \overline{R}$.

Let $R > 2\overline{R}$ be fixed and let $\varphi \in C_0^\infty(B_{2R}(0))$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ for $|x| \leqslant R$. Let $t_0 > 0$ be fixed, $t_1 = t_0$, $t_2 > t_1$ and $\theta(t) \in C_0^\infty((t_1/2, 2t_2))$ such that $0 \leqslant \theta \leqslant 1$ and $\theta \equiv 1$ for $t_1 \leqslant t \leqslant t_2$. Let $0 < \alpha < \min\{1/4, 2^*/2 - 1\}$ to be chosen later. For each $\delta > 0$ we define $\phi_\delta(s) = s(1 + s^2)^\alpha/(1 + \delta s^2)^\alpha$, so that ϕ_δ is a Lipschitz function, $\phi_\delta(0) = 0$ and $|\phi_\delta(s)| \leqslant |s|\delta^{-\alpha}$. From Lemma 5.1 and standard composition properties we infer $\phi_\delta(v) \in L^2_{\mathrm{loc}}(0, \infty; H^1_{\mathrm{loc}}(\mathbb{R}^N))$, whence $\Psi = \varphi^2\theta\phi_\delta(v) \in L^2(\mathbb{R}^+; H^1_0(B_{2R}))$ and has compact support in $\mathbb{R}^N \times (0, \infty)$, namely $\operatorname{spt}\Psi \subset \operatorname{spt}\varphi \times \operatorname{spt}\theta \Subset B_{2R} \times (t_1/2, 2t_2)$. Let

$$\begin{aligned} &\{\Psi_n\} \subset C_0^\infty\big(B_{2R} \times (t_1/2, 2t_2)\big) \quad \text{such that } \Psi_n \to \Psi \text{ in } L^2\big(t_1/2, 2t_2; H^1(B_{2R})\big), \\ &0 \leqslant \Psi_n \leqslant U\delta^{-\alpha} \quad \text{a.e.} \end{aligned} \tag{5.17}$$

Such a sequence can be easily constructed as $\Psi_n = \bar\varphi\overline{S}(1/n)\Psi$, where $\bar\varphi \in C_0^\infty(B_{2R} \times (t_1/2, 2t_2))$ satisfies $0 \leqslant \bar\varphi \leqslant 1$ and $\bar\varphi \equiv 1$ on $\operatorname{spt}\varphi \times \operatorname{spt}\theta$, and $\overline{S}$ is the standard heat semigroup in $\mathbb{R}^{N+1}$. Thus $0 \leqslant \Psi_n \leqslant \delta^{-\alpha}\overline{S}(1/n)U \leqslant \delta^{-\alpha}U$ because $v \leqslant U$ and $\Delta U \leqslant 0$ in $\mathcal{D}'(\mathbb{R}^{N+1})$. Arguing as in Lemma 5.1 we easily get $v_t U \in L^1(B_{2R} \times (t_1/2, 2t_2))$. Since $v \in C^\infty(\Omega \times (0, \infty))$ and Lemma 5.1 holds on the cylinder $B_{2R} \times \{t_1/2, 2t_2\} \supset \operatorname{spt}\bar\varphi$, we can test (1.4) with Ψ_n and integrate by parts in the first two terms to get

$$\int_{\mathbb{R}^N \times (t_1/2,2t_2)} v_t\Psi_n \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}^N \times (t_1/2,2t_2)} \nabla v \cdot \nabla\Psi_n \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{R}^N \times (t_1/2,2t_2)} v^{\frac{N+2}{N-2}}\Psi_n \,\mathrm{d}x \,\mathrm{d}t.$$

Using dominated convergence and (5.17) we can pass to the limit in the first two integrals as $n \to \infty$. As a consequence the integrals in r.h.s. are bounded with respect to n. Since, up to subsequences, $\Psi_n \to \Psi$ a.e., by Fatou's lemma we get $\int_{\mathbb{R}^N \times (t_1/2,2t_2)} v^{\frac{N+2}{N-2}}\Psi \,\mathrm{d}x \,\mathrm{d}t < \infty$. As $v \leqslant U$, combining Young inequality $ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, $\varepsilon > 0$, with Hardy inequality (5.3) we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^N \times (t_1/2,2t_2)} v^{\frac{N+2}{N-2}}\Psi_n \,\mathrm{d}x \,\mathrm{d}t - \int_{\mathbb{R}^N \times (t_1/2,2t_2)} v^{\frac{N+2}{N-2}}\Psi \,\mathrm{d}x \,\mathrm{d}t \right| \leqslant \int_{\mathbb{R}^N \times (t_1/2,2t_2)} U^{\frac{4}{N-2}}|\hat\varphi v| \big|(\Psi_n - \Psi)\big| \,\mathrm{d}x \,\mathrm{d}t \\ &\quad \leqslant \varepsilon \int_{\mathbb{R}^N \times (t_1/2,2t_2)} \big|\nabla(\hat\varphi v)\big|^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{4\varepsilon} \int_{\mathbb{R}^N \times (t_1/2,2t_2)} \big|\nabla(\Psi_n - \Psi)\big|^2 \,\mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

where $\hat{\varphi}$ is any smooth function $\hat{\varphi} \in C_0^\infty(B_{2R} \times (t_1/2, 2t_2))$ satisfying $0 \leqslant \hat{\varphi} \leqslant 1$ and $\hat{\varphi} \equiv 1$ on $\operatorname{spt} \bar{\varphi}$.

As $n \to \infty$ the second integral goes to zero, due to (5.17), while the first can be made arbitrarily small for a suitable choice of $\varepsilon > 0$. Thus

$$\int\limits_{\mathbb{R}^N \times (t_1/2, 2t_2)} v^{\frac{N+2}{N-2}} \Psi_n \,\mathrm{d}x\,\mathrm{d}t - \int\limits_{\mathbb{R}^N \times (t_1/2, 2t_2)} v^{\frac{N+2}{N-2}} \Psi \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty,$$

and

$$\int\limits_{\mathbb{R}^N \times (t_1/2, 2t_2)} v_t \Psi \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^N \times (t_1/2, 2t_2)} \nabla v \cdot \nabla \Psi \,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\mathbb{R}^N \times (t_1/2, 2t_2)} v^{\frac{N+2}{N-2}} \Psi \,\mathrm{d}x\,\mathrm{d}t.$$

Arguing as in Lemma 5.1, if $\theta = \theta_n \to \chi_{\{t_1 < t < t_2\}}$ a.e. in $(t_1/2, 2t_2)$, by dominated convergence we have

$$\int\limits_{\mathbb{R}^N \times (t_1, t_2)} v_t \Psi \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^N \times (t_1, t_2)} \nabla v \cdot \nabla \Psi \,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\mathbb{R}^N \times (t_1, t_2)} v^{\frac{N+2}{N-2}} \Psi \,\mathrm{d}x\,\mathrm{d}t, \tag{5.18}$$

where $\Psi = \varphi^2 \phi_\delta(v)$.

As in Section 4, for each $\delta > 0$ we introduce two other Lipschitz functions $\bar{\phi}_\delta$ and $\underline{\phi}_\delta$. We set

$$\bar{\phi}_\delta(s) = s\bigl(1 + s^2\bigr)^{\alpha/2} / \bigl(1 + \delta s^2\bigr)^{\alpha/2},$$

so that $\bar{\phi}_\delta$ satisfies $\bar{\phi}_\delta(0) = 0$ and $|\bar{\phi}_\delta(s)| \leqslant |s| \delta^{-\alpha/2}$, and $\underline{\phi}_\delta(s) = (1 + s^2)^{\alpha/2} / (1 + \delta s^2)^{\alpha/2}$ which is a bounded and satisfies $|\bar{\phi}_\delta(s)| \leqslant \delta^{-\alpha/2}$. Thus, if we define $w = \varphi \bar{\phi}_\delta(v)$, from Lemma 5.1 and standard composition properties we infer $w \in L^2(t_1, t_2; H_0^1(B_{2R}))$, $\bar{\phi}_\delta(v) \in L^2(t_1, t_2; H^1(B_{2R})) \cap L^\infty$ and (5.18) can be rewritten as

$$\int\limits_{\mathbb{R}^N \times (t_1, t_2)} v_t \varphi^2 \phi_\delta(v) \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^N \times (t_1, t_2)} \nabla v \cdot \nabla\bigl(\varphi^2 \underline{\phi}_\delta^2(v) v\bigr) \,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\mathbb{R}^N \times (t_1, t_2)} v^{\frac{4}{N-2}} w^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.19}$$

Combining the identity $\nabla v \cdot \nabla(\varphi^2 \underline{\phi}_\delta^2(v) v) = |\nabla w|^2 - v^2 |\nabla(\varphi \underline{\phi}_\delta(v))|^2$, the pointwise inequality $v \leqslant \lambda U$ and (5.3) we infer

$$\bigl(1 - \lambda^{\frac{4}{N-2}}\bigr) \int\limits_{\mathbb{R}^N \times (t_1, t_2)} |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \int\limits_{\mathbb{R}^N \times (t_1, t_2)} -\varphi^2 v_t \phi_\delta(v) \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^N \times (t_1, t_2)} v^2 \bigl|\nabla\bigl(\varphi \underline{\phi}_\delta(v)\bigr)\bigr|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.20}$$

Now we estimate the r.h.s. of (5.20). Since $v_t \leqslant 0$ a.e.,v is smooth in $\Omega \times (0, \infty)$ and $\Sigma = \mathbb{R}^N \setminus \Omega$ is a finite set, by Fubini's theorem we have

$$\int\limits_{\mathbb{R}^N \times (t_1, t_2)} -\varphi^2 v_t \phi_\delta(v) \,\mathrm{d}x\,\mathrm{d}t \leqslant \int\limits_{\mathbb{R}^N} \varphi^2 \int\limits_{t_1}^{t_2} -v_t v\bigl(1 + v^2\bigr)^\alpha \,\mathrm{d}t\,\mathrm{d}x \leqslant \int\limits_{\mathbb{R}^N} \varphi^2 \frac{1}{2(1+\alpha)} \bigl(1 + U^2\bigr)^{1+\alpha} < \infty,$$

because φ has compact support, $\alpha < 2^*/2 - 1$ and $U \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \leqslant p < 2^*$. Thus

$$\bigl(1 - \lambda^{\frac{4}{N-2}}\bigr) \int\limits_{\mathbb{R}^N \times (t_1, t_2)} |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \int\limits_{\mathbb{R}^N} \frac{\varphi^2}{2(1+\alpha)} \bigl(1 + U^2\bigr)^{1+\alpha} + \int\limits_{\mathbb{R}^N \times (t_1, t_2)} v^2 \bigl|\nabla\bigl(\varphi \underline{\phi}_\delta(v)\bigr)\bigr|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.21}$$

It is easy to check that $\underline{\phi}_\delta$ satisfies $|s \underline{\phi}'_\delta(s)| \leqslant 2\alpha \underline{\phi}_\delta(s)$, whence

$$v^2 \bigl|\nabla\bigl(\varphi \underline{\phi}_\delta(v)\bigr)\bigr|^2 \leqslant 2|\nabla\varphi|^2 v^2 \underline{\phi}_\delta^2(v) + 2\varphi^2 \bigl|v \bar{\phi}'_\delta(v)\bigr|^2 |\nabla v|^2 \leqslant 2|\nabla\varphi|^2 \bigl(1 + U^2\bigr)^{1+\alpha} + 8\alpha^2 \varphi^2 \underline{\phi}_\delta^2(v) |\nabla v|^2.$$

On the other hand $\nabla w = v \nabla(\varphi \underline{\phi}_\delta(v)) + \varphi \underline{\phi}_\delta(v) \nabla v$, hence

$$\varphi^2 \underline{\phi}_\delta^2(v) |\nabla v|^2 \leqslant 2|\nabla w|^2 + 2v^2 \bigl|\nabla\bigl(v \underline{\phi}_\delta(v)\bigr)\bigr|^2.$$

Since $\alpha < \frac{1}{4}$, combining the previous two inequalities we obtain

$$v^2\left|\nabla\left(\varphi\underline{\phi}_\delta(v)\right)\right|^2 \leqslant \frac{2|\nabla\varphi|^2}{1-16\alpha^2}\left(1+U^2\right)^{1+\alpha} + \frac{16\alpha^2}{1-16\alpha^2}|\nabla w|^2,$$

and (5.21) can be rewritten as

$$\begin{aligned}
&\left(1-\lambda^{\frac{4}{N-2}}\right)\int\limits_{\mathbb{R}^N\times(t_1,t_2)} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \int\limits_{\mathbb{R}^N}\left(\frac{\varphi^2}{2(1+\alpha)} + \frac{2(t_2-t_1)|\nabla\varphi|^2}{1-16\alpha^2}\right)\left(1+U^2\right)^{1+\alpha}\mathrm{d}x + \frac{16\alpha^2}{1-16\alpha^2}\int\limits_{\mathbb{R}^N\times(t_1,t_2)}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t. \qquad (5.22)
\end{aligned}$$

Choosing α possibly smaller so that $\frac{16\alpha^2}{1-16\alpha^2} \leqslant \frac{1}{2}(1-\lambda^{\frac{4}{N-2}})$ we easily get

$$\int\limits_{\mathbb{R}^N\times(t_1,t_2)} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C(\lambda,\alpha,t_1,t_2)\|\varphi\|_{C^1(\mathbb{R}^N)}\int\limits_{B_{2R}}\left(1+U^2\right)^{1+\alpha}\mathrm{d}x. \qquad (5.23)$$

Observe that w is decreasing in t because $v_t \leqslant 0$ a.e. and $(s\underline{\phi}_\delta(s))' \geqslant (1-2\alpha)\underline{\phi}_\delta(s) \geqslant 0$ by our choice of α. Choosing $t_1 = t_0$, $t_2 = 2t_0$, applying Sobolev embedding and taking into account the previous observation we obtain

$$t_0\Bigg(\int\limits_{\mathbb{R}^N\times\{t=2t_0\}}\left(\varphi v\frac{(1+v^2)^{\alpha/2}}{(1+\delta v^2)^{\alpha/2}}\right)^{2^*}\mathrm{d}x\Bigg)^{2/2^*} \leqslant C(\lambda,\alpha,t_0)\|\varphi\|_{C^1(\mathbb{R}^N)}\int\limits_{B_{2R}}\left(1+U^2\right)^{1+\alpha}\mathrm{d}x.$$

If we set $p = 2^*(1+\alpha) > 2^*$ and we use Fatou's lemma, as $\delta \to 0$ we easily conclude $\|v(2t_0)\|_{L^p(B_R)} < \infty$ because $\varphi \equiv 1$ on B_R and the r.h.s. is finite and independent of δ. Thus, (5.16) follows immediately because v is decreasing w.r.t. t. The lemma is completely proved. □

From the previous lemmas we infer $v \in L^\infty_{\mathrm{loc}}(0,\infty; L^p(\mathbb{R}^N))$ for some $p > 2^*$, $\nabla v \in L^2_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$, and the nonlinear term $V = v^{\frac{4}{N-2}}$ belongs to $L^\infty_{\mathrm{loc}}(0,\infty; L^q(\mathbb{R}^N))$ for $q = \frac{N-2}{4}p > \frac{N}{2}$. By standard L^p theory for the linear equation $v_t = \Delta v + Vv$ (see [25]) v belongs to $L^\infty_{\mathrm{loc}}((0,\infty); L^\infty_{\mathrm{loc}}(\mathbb{R}^N))$. As $\bar{u} = v$ is locally bounded, the smoothness of $\bar{u}$ follows from the classical bootstrap argument in L^p and C^α spaces (see [25]). □

We are ready to prove the first theorem of the paper.

Proof of Theorem 1. Let $\bar{\Psi} = \lambda U$. Since $\Sigma = \operatorname{Sing} U$ is a compact set we have $\bar{\Psi}(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$ by the asymptotic results of [7] (see (3.4)). Let $\underline{u}$ the solution with initial data $u_0 \leqslant \lambda U$ as constructed in Proposition 2.1, so that $0 < \underline{u} \leqslant \lambda U$ a.e. in $\mathbb{R}^N\times\mathbb{R}^+$ and $\underline{u}$ is minimal. Hence $\underline{u}$ is unique and, by Proposition 2.1, if $u_0 = \lambda U$ then $\underline{u}$ is also decreasing in time. By Corollary 3, if $\lambda \in (0,1)$ then $\underline{u}$ is smooth for $t > 0$ and (1.5) holds, i.e. claim (1) is completely proved. Claim (2) holds by Corollary 2. To prove claim (3) we observe that $\bar{u}(t,\lambda U)$ as constructed in Proposition 2.1 is maximal, i.e. $\bar{u}$ satisfies $0 \leqslant v \leqslant \bar{u} \leqslant \lambda U$ a.e. in $\mathbb{R}^N\times\mathbb{R}^+$ for any weak solution of (P) such that $v \leqslant \lambda U$ a.e. in $\mathbb{R}^N\times\mathbb{R}^+$. On the other hand from Proposition 5.3 we know that $\bar{u} \in C^\infty(\mathbb{R}^N\times(0,\infty))$ whenever Σ is a finite set and $\lambda \in (0,1)$. Under these assumptions we conclude $\underline{u} = \bar{u}$ by Corollary 2, and claim (3) follows. □

6. Instability and nonuniqueness: radial solutions

In this section we improve the result obtained in the previous section up to the critical value $\lambda = 1$ when U is a radial distributional solution with an isolated singularity at the origin. The proof evilly relies on the scale invariance of radial singular solutions U explained just after (3.8), namely $U(x) = \delta_\varepsilon^{\frac{N-2}{2}}U(\delta_\varepsilon x)$ when $U = U_{\varepsilon,\delta}$. In this case the corresponding minimal solution $\underline{u}$ is (quasi-)selfsimilar, i.e. $\underline{u}(x,t) = \delta_\varepsilon^{\frac{N-2}{2}}\underline{u}(\delta_\varepsilon x, \delta_\varepsilon^2 t)$ for the same $\delta_\varepsilon > 0$. More precisely we are going to prove the following version of Theorem 3.

Theorem 5. *Let $N \geqslant 3$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Assume that U is radial, $\Sigma = \operatorname{Sing} U = \{O\}$, and let $u_0(x) = U(x)$. Then the minimal weak solution $\underline{u}$ of* (P) *exists and for each $t > 0$ $\underline{u}(t)$ is radial and radially decreasing. If $U(x) = \delta^{\frac{N-2}{2}} U(\delta x)$ for some $\delta > 0$, then $\underline{u}(x,t) = \delta^{\frac{N-2}{2}} \underline{u}(\delta x, \delta^2 t)$. Moreover $\underline{u} \in C^\infty(\mathbb{R}^N \times (0,\infty))$, $\underline{u}_t \leqslant 0$ and for each $2^* < p \leqslant \infty$ there exists $C = C(p) > 0$ such that for each $t > 0$*

$$\left\|\underline{u}(t)\right\|_{L^p(\mathbb{R}^N)} \leqslant C t^{-\frac{N}{2}(\frac{1}{2^*} - \frac{1}{p})}. \tag{6.1}$$

Before going into the proof let us make some preliminary observations. As recalled in Section 3, $U = U_{\varepsilon,\delta}$ is given by (3.8) for some $0 < \varepsilon \leqslant \varepsilon_0$ and some $\delta > 0$. We may assume $\varepsilon < \varepsilon_0$, otherwise $U = U^s$ is given by (1.3), the theorem holds and the minimal solution is self-similar (see [15] or [47]). The function $g(s) = U(x)|x|^{\frac{N-2}{2}}$, $s = -\log|x|$, is continuous in $\mathbb{R}$ and periodic of period $T = T_\varepsilon > 0$. We have $g(s) \equiv g_\varepsilon(s - \log\delta)$ and $\min_s g(s) = \varepsilon$. Moreover some standard phase-plane analysis shows that ε is attained precisely once in the period, i.e. only on a sequence $\{s_i\}_{i\in\mathbb{Z}}$ satisfying $s_{i+1} = s_i - T$. Let us denote by $\{r_i\}_{i\in\mathbb{Z}}$ the corresponding sequence of radii $r_i = \mathrm{e}^{-s_i}$, so that $r_i \to +\infty$ as $i \to +\infty$.

Recall that the radial classical solutions of $\Delta U + U^{\frac{N+2}{N-2}} = 0$ are explicitly given in (3.5) by the one parameter family $\{\overline{U}_\delta\}_{\delta>0}$. As explained in Section 3, due to (3.9), (3.10), for each $\delta > 0$ the graphs of the radial profile of $\overline{U}_\delta$ and U intersect for finitely many values of $r = |x|$ (always transversally and at least twice); there is also a unique choice of the parameter $\delta_0 > 0$ such that $\overline{U}_{\delta_0}(x)|x|^{\frac{N-2}{2}} = \varepsilon$ for $|x| = r_0$ and $\overline{U}_{\delta_0}(x)|x|^{\frac{N-2}{2}} \leqslant \varepsilon$ for $|x| \leqslant r_0$. Thus, if we set

$$U^0(x) = \begin{cases} \overline{U}_{\delta_0}(x) & \text{if } |x| \leqslant r_0, \\ U(x) & \text{if } |x| \geqslant r_0, \end{cases} \tag{6.2}$$

then U^0 is a continuous radial and radially decreasing function, $U^0 \leqslant U$ and U^0 is weak solution of (5.1) (indeed, if $r > r_0$, $r - r_0 \ll 1$, we have $U(x) < \overline{U}_{\delta_0}(x)$ for $r_0 < |x| \leqslant r$ and the conclusion follows from [19], Proposition 2.1, choosing $R_1 \in (r_0, r)$ and $0 < R_2 < r_0$). The crucial ingredient in the proof of the theorem is the following auxiliary result.

Proposition 6.1. *Let U^0 as in (6.2) and $u_0 = U^0$. Then problem* (P) *has a global classical solution $v \in C^0(\mathbb{R}^N \times \mathbb{R}^+) \cap C^\infty(\mathbb{R}^N \times (0,\infty))$. The solution v is decreasing in t and it is radial and radially decreasing for each $t > 0$. Moreover, there exists $C > 0$ such that for each $t > 0$ we have*

$$\left\|v(t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{\frac{2-N}{4}}. \tag{6.3}$$

Proof. Let v the minimal positive solution as constructed in Proposition 2.1. Since U^0 is a bounded continuous weak supersolution, then $v \leqslant U^0$, it is smooth for $t > 0$ and it is decreasing in time, hence it is continuous up to $t = 0$. Moreover, since U^0 is radial and radially decreasing the same holds for v (compare [19], Proposition 2.2). Indeed the operator $T : \mathcal{M}_{\bar\Psi} \to \mathcal{M}_{\bar\Psi}$ used in Proposition 2.1 is a convolution operator with radial and radially decreasing kernels (with respect to the space variables). Hence, T maps functions w such that $w(t)$ is radial and radially decreasing for all $t \geqslant 0$ into functions with the same property by repeated use of [53], Lemma 1.4. Thus, the same holds for $v = \lim_{n\to\infty} T^n(0)$. It remains to prove the asymptotic decay (6.3). To this end we apply exactly the same blow-up argument used in [19], pp. 609–611, under the assumption

$$\limsup_{|x|\to\infty} U^0(x)|x|^{\frac{N-2}{2}} \leqslant \left(\frac{N-2}{2}\right)^{\frac{N-2}{2}}.$$

First observe that we must have $\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$. Indeed v is globally bounded and decreasing in t. Therefore there exists some nonnegative radial regular steady state v_∞ such that $v(t) \to v_\infty$ locally smoothly by standard Schauder estimates (see [25]). Moreover $\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$ if and only if $v_\infty \equiv 0$. As already mentioned, if $v_\infty \not\equiv 0$ then $v_\infty = \overline{U}_\delta$ for some $\delta > 0$, where $\overline{U}_\delta$ is given by (3.5). Hence $\overline{U}_\delta \leqslant v(t) \leqslant U^0 \leqslant U$ in the whole $\mathbb{R}^N$. This is a contradiction, because U and $\overline{U}_\delta$ must intersect transversally. Thus $\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as

$t \to \infty$ as claimed. Now we can apply exactly the same argument used in [19], pp. 609–611. We assume that (6.3) is false and that there exists a sequence $t_k \to \infty$ such that

$$\alpha_k := \sup_{t\in[0,t_k]} t^{\frac{N-2}{4}} \left\|v(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} = t_k^{\frac{N-2}{4}} \left\|v(\cdot,t_k)\right\|_{L^\infty(\mathbb{R}^N)} \to \infty. \tag{6.4}$$

Following their argument and taking (3.5) into account we find that there exists a sequence $\{M_k\}$, $M_k \to \infty$, such that

$$\overline{U}_{\bar\delta}(z) \leqslant \limsup_{k\to\infty} M_k^{\frac{N-2}{2}} U^0(M_k z), \quad \bar\delta = \left[N(N-2)\right]^{-1/2}. \tag{6.5}$$

As $U^0 \leqslant U$, $U = U_{\varepsilon,\delta}$, and $L_\varepsilon \equiv \sup_{x\in\mathbb{R}^N} |x|^{\frac{N-2}{2}} U(x)$, from (6.5) we obtain

$$\overline{U}_{\bar\delta}(z) \leqslant \limsup_{k\to\infty} M_k^{\frac{N-2}{2}} U(M_k z) \leqslant L_\varepsilon |z|^{\frac{2-N}{2}},$$

which contradicts (3.9) and (3.10). The proposition is completely proved. □

Proof of Theorem 5. For each $i \geqslant 1$ we set $U^i(x) = r_i^{\frac{N-2}{2}} U^0(r_i x)$, so that, by the scale invariance of U we know that U^i are continuous bounded weak supersolutions satisfying $U^i \leqslant U$. Let us set $u_0^i(x) = U^i(x)$ and let $\tilde v^i$ be the corresponding sequence of minimal weak solutions as constructed in Proposition 2.1. Since U^i are bounded weak supersolutions and $\tilde v^i \leqslant U^i$ we conclude that the $\tilde v^i$ are continuous, smooth for $t > 0$ and decreasing in time. As $v^i(x,t) := r_i^{\frac{N-2}{2}} v(r_i x, r_i^2 t)$ have the same properties, by Corollary 2 we conclude $v^i = \tilde v^i$. On the other hand by (6.2) it's easy to check that $U^i \nearrow U$ pointwise as $i \to \infty$, hence v^i is an increasing sequence by (2.11). Thus, there exists a pointwise limit $\bar v = \lim_{i\to\infty} v^i \leqslant U$ which is also decreasing in time. Using dominated convergence theorem in (1.4) it is easy to check that $\bar v$ is a weak solution of problem (P) with $u_0 = U$ as initial data. On the other hand, taking (6.3) into account we conclude

$$\left\|v^i(t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{\frac{2-N}{4}}, \quad \text{for all } t > 0, \tag{6.6}$$

for some constant $C > 0$ independent of i (actually $C > 0$ is precisely the one in (6.3)). As $i \to \infty$ we obtain

$$\left\|\bar v(t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{\frac{2-N}{4}}, \quad \text{for all } t > 0, \tag{6.7}$$

hence, $\bar v$ is smooth for $t > 0$ by standard parabolic theory because it is locally bounded. Using Corollary 2 we have $\bar v = \underline{u}$, the minimal weak solution of problem (P) with initial data $u_0 = U$, and $\underline{u}$ inherits from v all the claimed properties except scale invariance and (6.1). The scale invariance of $\underline{u}$ follows readily from the one of U, the pointwise bound $\underline{u}(x,t) \leqslant U(x)$, the smoothness of $\underline{u}$ for $t > 0$ and the uniqueness property of Corollary 2. Alternatively it follows from the covariance of the operator T, and in turn the invariance of the sequence $\{T^n(0)\}$, under the parabolic scaling when the initial condition $u_0 = U$ has the same invariance property. Estimate (6.1) has been proved just for $p = \infty$ (Eq. (6.7)). In the remaining cases we combine (6.7) with the pointwise inequality $\underline{u}(x,t) \leqslant U(x) \leqslant L|x|^{\frac{2-N}{2}}$, with $L = L_\varepsilon$ as in (3.9). For each $t > 0$ we write

$$\left\|\underline{u}(t)\right\|_{L^p(\mathbb{R}^N)} \leqslant \left\|\underline{u}(t)\right\|_{L^p(B_{\sqrt t})} + L\Bigg(\int\limits_{|x|>\sqrt t} \frac{\mathrm{d}x}{|x|^{\frac{p(N-2)}{2}}}\Bigg)^{1/p} \leqslant \left\|\underline{u}(t)\right\|_{L^p(B_{\sqrt t})} + LC(p) t^{-\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})}. \tag{6.8}$$

By (6.7) we have $\|\underline{u}(t)\|^p_{L^p(B_{\sqrt t}(0))} \leqslant C t^{\frac{N}{2}\frac{p}{2^*}} |B_{\sqrt t}(0)| = C t^{-\frac{Np}{2}(\frac{1}{2^*}-\frac{1}{p})}$. Combining it with (6.8) the conclusion follows. □

7. Instability and nonuniqueness: nonradial solutions

In this section we extend the result obtained in the previous section to arbitrary distributional solution U with finite singular set as initial data and we prove Theorem 3. The strategy here is quite different and the crucial step is deriving suitable uniform a-priori estimates on the minimal solutions $u_\lambda(t) = \underline{u}(t, \lambda U)$, $\lambda < 1$, constructed in Theorem 1.

Proof of Theorem 3. We will argue by contradiction and establish the key estimate (7.1) below uniformly on λ. This is enough to construct the solution u.

Lemma 7.1. *Let $\lambda \in (0,1)$ and let $u_\lambda(t) = \underline{u}(t, \lambda U)$ as constructed in Theorem 1. There exists $C > 0$ independent of λ such that for all $\lambda \in (0,1)$ we have*

$$\left\| u_\lambda(t) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{\frac{2-N}{4}} \quad \textit{for all } t \in (0,1]. \tag{7.1}$$

Proof. We employ a blow-up argument similar to the one used in Theorem 5. However, since we are no longer dealing with radial and radially decreasing solutions the argument used there has to be modified. Arguing by contradiction, we assume that (7.1) does not hold. Due to the monotonicity w.r.t. λ, i.e. (2.11), and (2.14), there exist two sequences $\{\lambda_n\}, \{t_n\} \subset (0,1)$ such that $\lambda_n \nearrow 1$,

$$\begin{aligned} &\sup_{t\in(0,1]} t^{\frac{N-2}{4}} \left\| u_{\lambda_n}(t) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant t_n^{\frac{N-2}{4}} \left\| u_{\lambda_n}(t_n) \right\|_{L^\infty(\mathbb{R}^N)} + \frac{1}{n} \quad \text{and} \\ &\delta_n := t_n^{\frac{N-2}{4}} \left\| u_{\lambda_n}(t_n) \right\|_{L^\infty(\mathbb{R}^N)} \to \infty \quad \text{as } n \to \infty. \end{aligned} \tag{7.2}$$

Since $u_{\lambda_n}(x,t) \leqslant U(x) \to 0$ as $|x| \to \infty$ and it is smooth for $t > 0$, we may clearly assume that there exists $\{x_n\} \subset \mathbb{R}^N$ such that

$$\delta_n := t_n^{\frac{N-2}{4}} \left\| u_{\lambda_n}(t_n) \right\|_{L^\infty(\mathbb{R}^N)} = t_n^{\frac{N-2}{4}} u_{\lambda_n}(x_n, t_n). \tag{7.3}$$

Obviously $\delta_n \leqslant u_{\lambda_n}(x_n, t_n) \leqslant U(x_n)$, hence $U(x_n) \to \infty$ as $n \to \infty$. Thus, $\mathrm{dist}(x_n, \Sigma) \to 0$ as $n \to \infty$ because U is bounded far from Σ and, up to a subsequence, we may assume $x_n \to x_\infty \in \Sigma$.

Let us set $s = \log t$ and for each $n \geqslant 1$, $s_n = \log t_n$ and $v_n(y,s) = t^{\frac{N-2}{4}} u_{\lambda_n}(x_n + y\sqrt{t}, t)$, $y \in \mathbb{R}^N$, $s \in \mathbb{R}$. Then

$$\partial_s v_n = \Delta v_n + \frac{1}{2} y \cdot \nabla v_n + \frac{N-2}{4} v_n + v_n^{\frac{N+2}{N-2}}, \quad y \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{7.4}$$

and, by definition, $0 < v_n(y,s) \leqslant \delta_n + \frac{1}{n}$ for $-\infty < s \leqslant s_n$, $v_n(0, s_n) = \delta_n$. We rescale each v_n as follows

$$w_n(z,\tau) = \beta_n^{\frac{N-2}{2}} v_n\big(\beta_n z, \beta_n^2 \tau + s_n\big), \quad \text{where } \beta_n = \left(\delta_n + \frac{1}{n}\right)^{\frac{2}{2-N}} \to 0 \text{ as } n \to \infty. \tag{7.5}$$

By (7.3), (7.4) and (7.5) we have

$$\begin{cases} \partial_\tau w_n = \Delta w_n + \beta_n^2 \Big(\dfrac{1}{2} z \cdot \nabla w_n + \dfrac{N-2}{4} w_n \Big) + w_n^{\frac{N+2}{N-2}}, \\ w_n(0,0) = \dfrac{\delta_n}{\delta_n + 1/n}, \quad 0 < w_n \leqslant 1 \text{ for } z \in \mathbb{R}^N,\ -\infty < \tau \leqslant 0. \end{cases} \tag{7.6}$$

As $\beta_n \to 0$ as $n \to \infty$, using the interior L^p-estimates and Schauder estimates for linear parabolic equations, there exists a function $w \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^N \times (-\infty, 0])$ such that (up to subsequences) $w_k \to w$ in $C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\mathbb{R}^N \times (-\infty,0])$ and w satisfies

$$\begin{cases} \partial_\tau w = \Delta w + w^{\frac{N+2}{N-2}} & \text{in } (-\infty, 0], \\ w(0,0) = 1, \quad 0 \leqslant w \leqslant 1 & \text{in } (-\infty, 0]. \end{cases} \tag{7.7}$$

Since for each $\lambda \in (0,1)$ the function u_λ is decreasing in t, we have $\Delta u_{\lambda_n}(x,t) + (u_{\lambda_n}(x,t))^{\frac{N+2}{N-2}} \leqslant 0$ for $x \in \mathbb{R}^N$ and $t \in (0, \infty)$. Thus

$$w_n(z,\tau) = \big(\beta_n \mathrm{e}^{(\beta_n^2 \tau + s_n)/2}\big)^{\frac{N-2}{2}} u_{\lambda_n}\big(x_n + \beta_n \mathrm{e}^{(\beta_n^2\tau + s_n)/2} z, \mathrm{e}^{\beta_n^2 \tau + s_n}\big)$$

satisfies

$$\Delta_z w_n(z,\tau) + \big(w_n(z,\tau)\big)^{\frac{N+2}{N-2}} \leqslant 0, \quad z \in \mathbb{R}^N,\ \tau \leqslant 0,$$

hence

$$\partial_\tau w = \Delta w + w^{\frac{N+2}{N-2}} \leqslant 0, \quad z \in \mathbb{R}^N,\ \tau \leqslant 0. \tag{7.8}$$

As $w \leqslant 1$ and $w(0,0) = 1$, from the previous inequality we obtain $\partial_\tau w(0,\tau) \equiv 0$. Differentiating (7.8) in τ and taking (7.8) into account, by the strong maximum principle for parabolic equations we easily conclude $\partial_\tau w \equiv 0$. Thus $w(z,\tau) \equiv w(z)$ is a classical solution of $\Delta u + u^{\frac{N+2}{N-2}} = 0$ with maximum at the origin equal to 1. By (3.5) we conclude $w(z) = \overline{U}_{\bar\delta}(z)$, for $\bar\delta = [N(N-2)]^{-1/2}$.

Now we claim that there exists $C > 0$ such that

$$\frac{|x_n - x_\infty|}{\beta_n\sqrt{t_n}} \leqslant C, \quad \text{for each } n \geqslant 1. \tag{7.9}$$

Indeed, $x_n \to x_\infty \in \Sigma$ and $U(x) \leqslant C|x - x_\infty|^{\frac{2-N}{2}}$ whenever $|x - x_\infty|$ is sufficiently small by (3.3). Thus, by (7.2), (7.3) we have

$$\delta_n = t_n^{\frac{N-2}{4}} u_{\lambda_n}(x_n, t_n) \leqslant t_n^{\frac{N-2}{4}} U(x_n) \leqslant C\left(\frac{|x_n - x_\infty|}{\sqrt{t_n}}\right)^{\frac{2-N}{2}},$$

whence

$$\left(\frac{|x_n - x_\infty|}{\beta_n\sqrt{t_n}}\right)^{\frac{N-2}{2}} \leqslant C\frac{\delta_n + 1/n}{\delta_n} \to C \quad \text{as } n \to \infty,$$

and (7.9) is proved.

For any fixed $(z,\tau) \in \mathbb{R}^N \times (-\infty, 0]$, $z \neq 0$, we have

$$w_n(z,\tau) = \left(\beta_n\, \mathrm{e}^{(\beta_n^2\tau + s_n)/2}\right)^{\frac{N-2}{2}} u_{\lambda_n}\left(x_n + \beta_n\, \mathrm{e}^{(\beta_n^2\tau + s_n)/2} z, \mathrm{e}^{\beta_n^2\tau + s_n}\right) \leqslant M_n^{\frac{N-2}{2}} U(x_n + M_n z), \tag{7.10}$$

where we set $M_n = \beta_n\sqrt{t_n}\, \mathrm{e}^{(\beta_n^2\tau)/2}$. Observe that $M_n \to 0$ as $n \to \infty$ because $\beta_n \to 0$ by (7.5),

$$\left\|u_{\lambda_n}(\cdot, t_n)\right\|_{L^\infty(\mathbb{R}^N)} \to \infty$$

by (7.2) and

$$\left(\beta_n\sqrt{t_n}\right)^{\frac{N-2}{2}} = \frac{\delta_n}{(\delta_n + 1/n)} \frac{1}{\|u_{\lambda_n}(\cdot, t_n)\|_{L^\infty(\mathbb{R}^N)}} \to 0,$$

by (7.5), (7.2). Hence $x_n + M_n z \to x_\infty$ as $n \to \infty$.

Since $w_n \to \overline{U}_{\bar\delta}$ locally uniformly in $\mathbb{R}^N \times (-\infty, 0]$ as $n \to \infty$, from (7.10) we infer

$$\overline{U}_{\bar\delta}(z) \leqslant \limsup_{n\to\infty} M_n^{\frac{N-2}{2}} U(x_n + M_n z). \tag{7.11}$$

As $x_n + M_n z \to x_\infty$ we can apply the asymptotic results of [23] (see also [7]) to estimate the r.h.s. of (7.11). Indeed, by (3.11) there exists a radial singular solution $U^0 = U_{\varepsilon,\delta}$ and a positive number $\alpha > 0$ such that $U(x) - U^0(x - x_\infty) = \mathcal{O}(|x - x_\infty|^{\frac{2-N}{2} + \alpha})$ as $|x - x_\infty| \to 0$. We claim that (7.11) yields

$$\overline{U}_{\bar\delta}(z) \leqslant L|z|^{\frac{2-N}{2}}, \quad L = L_\varepsilon = \max_{x \in \mathbb{R}^N} U^0(x)|x|^{\frac{N-2}{2}}. \tag{7.12}$$

Clearly, if (7.12) holds, then we have a contradiction because of (3.9) and (3.10). Thus, the contradiction proves the lemma.

It remains to prove (7.12). First we observe that, due to (7.9), $\xi_n := (x_n - x_\infty)M_n^{-1}$ is a bounded sequence. If $\xi_n \to 0$ then from (7.11) we infer

$$\begin{aligned}
\overline{U}_{\bar\delta}(z) &\leqslant \limsup_{n\to\infty} M_n^{\frac{N-2}{2}} U(x_n + M_n z) \\
&\leqslant \limsup_{n\to\infty} M_n^{\frac{N-2}{2}} \left|U(x_n + M_n z) - U^0(x_n + M_n z)\right| + \limsup_{n\to\infty} M_n^{\frac{N-2}{2}} U^0(x_n + M_n z) \\
&\leqslant \limsup_{n\to\infty} C|\xi_n + z|^{\frac{2-N}{2}} |x_n - x_\infty + M_n z|^\alpha + \limsup_{n\to\infty} L|\xi_n + z|^{\frac{2-N}{2}} = L|z|^{\frac{2-N}{2}},
\end{aligned}$$

i.e. (7.12) holds. Otherwise, up to a subsequence we may assume $\xi_n \to \xi_\infty \neq 0$. Let $\xi \neq 0$ and $z = (\xi_\infty/|\xi_\infty|)|\xi|$, so that $\xi_n + z \to \xi_\infty + z = (1 + |\xi|/|\xi_\infty|)\xi_\infty \neq 0$.

Applying (7.11) with $z = (\xi_\infty/|\xi_\infty|)|\xi|$, using radial symmetry we have

$$\begin{aligned}\overline{U}_{\bar\delta}(\xi) = \overline{U}_{\bar\delta}(z) &\leqslant \limsup_{n\to\infty} M_n^{\frac{N-2}{2}} U(x_n + M_n z)\\ &\leqslant \limsup_{n\to\infty} C|\xi_n + z|^{\frac{2-N}{2}} |x_n - x_\infty + M_n z|^\alpha + \limsup_{n\to\infty} L|\xi_n + z|^{\frac{2-N}{2}}\\ &= L\left|\left(1 + \frac{|\xi|}{|\xi_\infty|}\right)\xi_\infty\right|^{\frac{2-N}{2}} \leqslant L|\xi|^{\frac{2-N}{2}},\end{aligned}$$

i.e. (7.9) holds and the proof is completed. □

By (2.11) the family of solutions $u_\lambda = \underline{u}(\lambda U) \leqslant U$ is clearly increasing as $\lambda \nearrow 1$. Thus, there exists a pointwise limit $u = \lim_{\lambda\nearrow 1} u_\lambda \leqslant U$ which is a weak solution of problem (P), with $u_0 = U$ as initial data by the same dominated convergence argument used in the previous section. This solution is also decreasing in time because the same holds for each u_λ.

Taking Lemma 7.1 into account, as $\lambda \nearrow 1$ we conclude

$$\left\|u(t)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant C t^{\frac{2-N}{4}}, \quad \text{for all } 0 < t \leqslant 1. \tag{7.13}$$

Hence, u is smooth for $t > 0$ by standard parabolic theory because it is locally bounded for $0 < t \leqslant 1$ by (7.13) and it is globally bounded for $t \geqslant 1$ because it is decreasing in time and it satisfies (7.13). Using Corollary 2 we have $u = \underline{u}$, the minimal weak solution of problem (P) with initial data $u_0 = U$, because $u \leqslant U$ and it is smooth for $t > 0$.

It remains to show that $\lim_{t\to\infty} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = 0$. We need the following lemma.

Lemma 7.2. *Let $N \geqslant 3$ and $U \in L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N)$, $U > 0$ a.e., such that $\Delta U + U^{\frac{N+2}{N-2}} = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Assume that $\Sigma = \operatorname{Sing} U = \{P_1, \dots, P_k\}$ is a finite set. Let $V \in C^2(\mathbb{R}^N)$ a positive solution of $\Delta V + V^{\frac{N+2}{N-2}} = 0$. Assume that $V \leqslant U$ a.e. Then there are infinitely many $x_0 \in \mathbb{R}^N \setminus \Sigma$ such that $V(x_0) = U(x_0)$.*

Proof. As already recalled in (3.5), from [7] we know that $V(x) = \overline{U}_\delta(x - \bar{x})$ for some $\delta > 0$ and $\bar{x} \in \mathbb{R}^N$. On the other hand it follows from (3.4) that $U(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$ and we may assume also $U(x) \geqslant C|x|^{\frac{2-N}{2}}$ for large $|x|$, i.e. we may assume that U has a nonremovable singularity at infinity. Otherwise Σ contains at least two points (see [7], Theorem 8.1) and if we select $P \in \Sigma$ and for $z = \frac{x-P}{|x-P|^2}$ we set $\widehat{U}(z) = \frac{1}{|z|^{N-2}} U(P + \frac{z}{|z|^2})$, then $\widehat{U}$ is still singular and it has all the desired properties, while $\widehat{V}(z) = \frac{1}{|z|^{N-2}} V(P + \frac{z}{|z|^2})$ is still a regular positive solution. Once the lemma is proved for $\widehat{U}$ and $\widehat{V}$, the conclusion follows going back to the original variables.

Since V is bounded and $\underline{u}(t, V) \leqslant V$, we deduce that $\underline{u}(t, V) \equiv V$ by the uniqueness of bounded solutions of the Cauchy problem (P). Let $\{\lambda_n\} \subset (0, 1)$, $\lambda_n \nearrow 1$. For each $n \geqslant 1$ we set $K_n = \{x \in \mathbb{R}^N \setminus \Sigma\colon \lambda_n U(x) \leqslant V(x)\}$. Clearly $K_{n+1} \subseteq K_n$ for each n and each K_n is a relatively closed subset of $\mathbb{R}^N \setminus \Sigma$ because both U and V are continuous in $\mathbb{R}^N \setminus \Sigma$. Moreover K_n is a compact set in $\mathbb{R}^N$, $K_n \subset \mathbb{R}^N \setminus \Sigma$. Indeed by our assumptions $U(x)/V(x) \geqslant C|x|^{\frac{N-2}{2}} \to \infty$ as $|x| \to \infty$, and $U(x)/V(x) \to \infty$ as $\operatorname{dist}(x, \Sigma) \to 0$ by (3.5) and (3.3).

We claim that each K_n is nonempty. Otherwise $V < \lambda U$ a.e. Evolving in time through the minimal positive solution and using (2.11), (2.14) we would get

$$V(x) = \underline{u}(t, V) \leqslant \underline{u}(t, \lambda U) \leqslant C(\lambda_n, N) t^{\frac{2-N}{4}}.$$

Since the r.h.s. goes to zero as $t \to \infty$ we have a contradiction. Thus $K_n \neq \emptyset$ for each n and clearly $\partial K_n \neq \emptyset$ because K_n is a compact set. As both U and V are continuous functions in a neighbourhood of K_n we have also $\lambda U \equiv V$ on ∂K_n. Obviously $K = \bigcap_n K_n \neq \emptyset$ and if $x_n \in \partial K_n$, up to subsequences we may assume $x_n \to x_0 \in K$. On the other hand $\lambda_n U(x) \to U(x)$ uniformly on K_1, therefore

$$V(x_0) \leqslant U(x_0) = \lim_{n\to\infty} \lambda_n U(x_n) \leqslant \lim_{n\to\infty} V(x_n) = V(x_0),$$

and the set $\mathcal{A} = \{x \in \mathbb{R}^N \setminus \Sigma \colon U(x) = V(x)\}$ is not empty. In order to finish the proof it is enough to show that each point of $\mathcal{A}$ is not isolated. Assume the converse. Then there exists $x_0 \in \mathcal{A}$ and $R > 0$ such that $B_R(x_0) \subset \mathbb{R}^N \setminus \Sigma$ and $V(x) < U(x)$ in $B_R(x_0) \setminus \{x_0\}$. Then $W(x) = U(x) - V(x)$ is smooth and satisfies $\Delta W \leqslant 0$ in $B_R(x_0)$; moreover $W(x_0) = 0$ and $W(x) > 0$ for each $x \in B_R(x_0) \setminus \{x_0\}$. By the strong maximum principle we have a contradiction and the lemma is completely proved. □

Let $V(x) = \lim_{t\to\infty} u(x,t)$, which is well defined and bounded because u is decreasing in time and bounded e.g. for $t = 1$. Let $t_0 \geqslant 1$ a fixed number. Multiplying the equation by $\varphi \in C_0^\infty(\mathbb{R}^N)$ and integrating by parts on $\mathbb{R}^N \times (t_0, t_0+1)$ we get

$$\int_{\mathbb{R}^N} u(x,t_0+1)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} u(x,t_0)\varphi(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N\times(t_0,t_0+1)} u(x,t)\Delta\varphi(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N\times(t_0,t_0+1)} u(x,t)^{\frac{N+2}{N-2}}\varphi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Letting $t_0 \to \infty$ in the previous equation, by dominated convergence we get

$$0 = \int_{\mathbb{R}^N} V(x)\Delta\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)^{\frac{N+2}{N-2}}\varphi(x)\,\mathrm{d}x,$$

hence, by elliptic regularity (L^p and Schauder theory) V is a classical solution because it is bounded. Obviously $V \leqslant U$. If $V \equiv 0$ then $u(\cdot,t) \to 0$ uniformly as $t \to \infty$ because u is decreasing in time and $u(x,t) \leqslant U(x) \to 0$ as $|x| \to \infty$. Thus, the last claim of the theorem follows if $V \equiv 0$. Otherwise, assume $V \not\equiv 0$, then $V > 0$ by the strong maximum principle and, applying Lemma 7.2 there exists $x_0 \in \mathbb{R}^N \setminus \Sigma$ such that $V(x_0) = U(x_0)$. On the other hand using the strong maximum principle for parabolic equations in the strip $0 < t_1 < t \leqslant t_2$ we know that u is strictly decreasing in time and we have

$$U(x_0) \geqslant u(x_0,t_1) > u(x_0,t_2) \geqslant V(x_0),$$

which is a contradiction. □

A trivial consequence of Theorem 3 is Corollary 1. The proof is standard and it will be just outlined below.

Proof of Corollary 1. Let U as in Theorem 3 and let u be the corresponding minimal solution which is smooth for $t > 0$ by the same theorem. For each $\tau > 0$ we set $u^\tau(t) = U$ for $0 \leqslant t \leqslant \tau$ and $u^\tau(t) = u(t-\tau)$ for $t > \tau$. It is easy to check that each u^τ is a weak solution, and clearly they are all distinct because u is smooth for $t > 0$. □

8. Asymptotic behaviour

Proof of Theorem 4. (1) First we prove (1.8) for $p = \infty$, and in view of (7.13) we may restrict to the case $t \geqslant 1$. In order to prove the estimate we use another blow up argument similar to the one used in Theorem 3. However, since this time the rescaling is performed at infinity we give a full detailed proof. First we observe that $\delta(t) = t^{\frac{N-2}{4}}\|u(t)\|_{L^\infty(\mathbb{R}^N)}$ is a continuous function in $[1,\infty)$ because u is smooth for $t > 0$ and $u(x,t) \leqslant U(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x| \to \infty$. Arguing by contradiction, assume (1.8) does not hold. By (7.13), there exist a sequence $\{t_n\} \subset (1,\infty)$, $t_n \to \infty$ such that

$$\delta_n := \sup_{t\in(0,t_n]} t^{\frac{N-2}{4}}\big\|u(t)\big\|_{L^\infty(\mathbb{R}^N)} = t_n^{\frac{N-2}{4}}\big\|u(t_n)\big\|_{L^\infty(\mathbb{R}^N)} \to \infty \quad \text{as } n \to \infty. \tag{8.1}$$

Since $u(x,t) \leqslant U(x) \to 0$ as $|x| \to \infty$ and u is smooth for $t > 0$, we may assume that there exists $\{x_n\} \subset \mathbb{R}^N$ such that

$$\delta_n = t_n^{\frac{N-2}{4}}\big\|u(t_n)\big\|_{L^\infty(\mathbb{R}^N)} = t_n^{\frac{N-2}{4}} u(x_n,t_n). \tag{8.2}$$

Let us set $s=\log t$ and for each $n\geqslant 1$, $s_n=\log t_n$ and $v_n(y,s)=t^{\frac{N-2}{4}}u(x_n+y\sqrt{t},t)$, $y\in\mathbb{R}^N$, $s\in\mathbb{R}$. Then

$$\partial_s v_n=\Delta v_n+\frac{1}{2}y\cdot\nabla v_n+\frac{N-2}{4}v_n+v_n^{\frac{N+2}{N-2}},\quad y\in\mathbb{R}^N,\ s\in\mathbb{R},\tag{8.3}$$

and, by definition of t_n and x_n we have $0<v_n(y,s)\leqslant\delta_n$ for $-\infty<s\leqslant s_n$, and $v_n(0,s_n)=\delta_n$. We rescale v_n as follows

$$w_n(z,\tau)=\beta_n^{\frac{N-2}{2}}v_n\big(\beta_n z,\beta_n^2\tau+s_n\big),\quad \text{where } \beta_n=(\delta_n)^{\frac{2}{2-N}}\to 0 \text{ as } n\to\infty.\tag{8.4}$$

By (8.3) we have

$$\begin{cases}\partial_\tau w_n=\Delta w_n+\beta_n^2\Big(\frac{1}{2}z\cdot\nabla w_n+\frac{N-2}{4}w_n\Big)+w_n^{\frac{N+2}{N-2}},\\ w_n(0,0)=1,\quad 0<w_n\leqslant 1 \text{ for } z\in\mathbb{R}^N,\ -\infty<\tau\leqslant 0.\end{cases}\tag{8.5}$$

By the interior L^p-estimates and Schauder estimates for linear parabolic equations, there exists a function $w\in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N\times(-\infty,0])$ such that (up to subsequences) $w_k\to w$ in $C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(\mathbb{R}^N\times(-\infty,0])$ and w satisfies

$$\begin{cases}\partial_\tau w=\Delta w+w^{\frac{N+2}{N-2}} & \text{in } (-\infty,0],\\ w(0,0)=1,\quad 0\leqslant w\leqslant 1 & \text{in } (-\infty,0].\end{cases}\tag{8.6}$$

Arguing as in (7.7) and (7.8) we conclude that w is nonincreasing in τ, hence $w(z,\tau)=\overline{U}_{\bar\delta}(z)$, $\bar\delta=[N(N-2)]^{-1/2}$, by the parabolic maximum principle, the normalisation at the origin and formula (3.5).

Since $u(x,t)\leqslant U(x)$, for any fixed $(z,\tau)\in\mathbb{R}^N\times(-\infty,0]$, $z\neq 0$ we have

$$w_n(z,\tau)=\big(\beta_n\,\mathrm{e}^{(\beta_n^2\tau+s_n)/2}\big)^{\frac{N-2}{2}}u\big(x_n+\beta_n\,\mathrm{e}^{(\beta_n^2\tau+s_n)/2}z,\mathrm{e}^{\beta_n^2\tau+s_n}\big)\leqslant M_n^{\frac{N-2}{2}}U(x_n+M_n z),\tag{8.7}$$

where we set $M_n=\beta_n\sqrt{t_n}\,\mathrm{e}^{(\beta_n^2\tau)/2}$. As $w_n\to\overline{U}_{\bar\delta}$ locally uniformly in $\mathbb{R}^N\times(-\infty,0]$ as $n\to\infty$, we conclude

$$\overline{U}_{\bar\delta}(z)\leqslant\limsup_{n\to\infty}M_n^{\frac{N-2}{2}}U(x_n+M_n z).\tag{8.8}$$

Observe that $M_n\to\infty$ as $n\to\infty$ because $\beta_n\to 0$ by (8.4), $\|u(\cdot,t_n)\|_{L^\infty(\mathbb{R}^N)}\to 0$ by Theorem 3 and

$$\big(\beta_n\sqrt{t_n}\big)^{\frac{N-2}{2}}=\frac{1}{\|u(\cdot,t_n)\|_{L^\infty(\mathbb{R}^N)}}\to\infty.$$

From the asymptotic analysis of [7] (see (3.4)) we know that $U(x)=\mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x|\to\infty$. We claim that $\xi_n:=x_nM_n^{-1}$ is a bounded sequence. If this is not he case, then, up to subsequences, $|x_n|M_n^{-1}\to\infty$ and in particular $|x_n|\to\infty$, because $M_n\to\infty$. Since $u\leqslant U$, by definition of δ_n, β_n and M_n we get

$$1=\big(\beta_n\sqrt{t_n}\big)^{\frac{N-2}{2}}u(x_n,t_n)\leqslant C\big(\beta_n\sqrt{t_n}\big)^{\frac{N-2}{2}}|x_n|^{\frac{2-N}{2}}=\frac{C}{(|x_n|/(\beta_n\sqrt{t_n}))^{\frac{N-2}{2}}}\to 0\quad\text{as } n\to\infty,$$

because $|x_n|(\beta_n\sqrt{t_n})^{-1}\sim|x_n|M_n^{-1}\to\infty$ as $n\to\infty$. The contradiction shows that ξ_n is a bounded sequence.

To finish the proof we distinguish two cases according to (3.4). If U is nonsingular at infinity, then $U(x)\leqslant C|x|^{2-N}$ for large $|x|$. Up to subsequences we may assume $y_n:=\frac{x_n}{|x_n|}\to y_\infty$, and if we fix $\xi\neq 0$ and $z=y_\infty|\xi|$, then $|x_n+M_nz|=||x_n|y_n+M_n|\xi|y_\infty|\geqslant C(|x_n|+|\xi|M_n)$ for $C>0$ independent of n and n large enough, hence $|x_n+M_nz|\to\infty$ as $n\to\infty$. By (8.8) we have

$$0<\overline{U}_{\bar\delta}(\xi)=\overline{U}_{\bar\delta}(z)\leqslant\limsup_{n\to\infty}M_n^{\frac{N-2}{2}}U(x_n+M_nz)\leqslant\limsup_{n\to\infty}\frac{CM_n^{\frac{N-2}{2}}}{(|x_n|+|\xi|M_n)^{N-2}}=0,$$

which is a contradiction.

In the second case, i.e. if U has a nonremovable singularity at infinity, then there exists a radial singular solution $U^\infty=U_{\varepsilon,\delta}$ and a positive number $\alpha>0$ such that $U(x)-U^\infty(x)=\mathcal{O}(|x|^{\frac{2-N}{2}-\alpha})$ as $|x|\to\infty$ (see (3.12)). We claim that from (8.8) we obtain

$$\overline{U}_{\bar\delta}(z)\leqslant L|z|^{\frac{2-N}{2}},\qquad L=L_\varepsilon=\sup_{x\in\mathbb{R}^N}U^\infty(x)|x|^{\frac{N-2}{2}}.\tag{8.9}$$

Clearly, if (8.9) holds, then we have a contradiction because of (3.9) and (3.10). Since the proof of (8.9) is entirely similar to the one of (7.12) it will be omitted.

Thus, (1.8) holds for $p=\infty$. The other cases are obtained "by interpolation" as in Theorem 5. Indeed $u(x,t)\leqslant U(x)$ in the whole $\mathbb{R}^N\times(0,\infty)$. On the other hand by (3.4) $U(x)\leqslant C|x|^{\frac{2-N}{2}}$ for $|x|\geqslant\overline{R}$, $\overline{R}>1$ sufficiently large. Combining this pointwise inequality with (1.8) in the case $p=\infty$ on paraboloids of the form $\{|x|>\sqrt{t}\}$ as in Theorem 5, Eq. (6.8), the estimate (1.8) holds for arbitrary $p>2^*$ on each interval $[t_1,\infty)$, $t_1>0$. For time intervals $(0,t_1]$ the argument is similar and is based on the asymptotic bounds $U(x)\leqslant C|x-P|^{\frac{2-N}{2}}$ for any $P\in\Sigma$ and $|x-P|$ sufficiently small given by (3.3) and $U(x)=\mathcal{O}(|x|^{\frac{2-N}{2}})$ as $|x|\to\infty$ given by (3.4) combined with inequality (1.8) for $p=\infty$ on paraboloids of the form $\{|x-P|>\sqrt{t}\}$. Thus, for t small we can control the decay in time of the l.h.s. of (1.8) both near the singular points $(P,0)$, $P\in\Sigma$ and at the spatial infinity. The details are left to the reader.

(2) Up to a translation we may assume $P=0$. Then $U^0=U_{\varepsilon,\delta}$ is radial and radially decreasing and the same holds for the corresponding minimal solution given by Theorem 5. We know that $g(-\log|x|)=|x|^{\frac{N-2}{2}}U^0(x)$ is a bounded continuous function and it is either constant or a periodic function with period $T=T_\varepsilon>0$. We confine ourselves to the second case, the first one being even simpler to deal with because the argument is exactly the same and $T>0$ can be chosen arbitrarily. Thus, $U^0(x)=R^{\frac{N-2}{2}}U^0(Rx)$ for $R=\mathrm{e}^{-T}$ and if we set $r_i=\mathrm{e}^{-iT}$, $i\in\mathbb{Z}$, then $r_i^{\frac{N-2}{2}}U^0(r_ix)=U^0(x)$ for each i. By Theorem 5 the corresponding minimal solution $u^0(x,t)$ has the scale invariance $u^0(x,t)=r_i^{\frac{N-2}{2}}u^0(r_ix,r_i^2t)$.

Now we rescale $\underline{u}$ at 0 by setting $v_i(x,t)=r_i^{\frac{N-2}{2}}\underline{u}(r_ix,r_i^2t)$. Let $w_k=v_{i_k}$ any subsequence. Clearly w_k are still weak solutions and they satisfy (1.8) for $p=\infty$, with the same positive constant $C>0$, due to claim (1) of the theorem and the scale invariance of (1.8). Thus, by standard L^p-estimates and Schauder estimates for linear parabolic equations w_k are compact in $C^{2,1}_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ and there exists a subsequence w_{k_j} and a limit function $w\in C^{2,1}_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ such that $w_{k_j}\to w$ in $C^{2,1}_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ and w is smooth for $t>0$ because is a $C^{2,1}_{\mathrm{loc}}$ solution of the semilinear heat equation. Moreover w is decreasing in t because each w_k is decreasing in t.

Since $\underline{u}$ is a weak solution, testing (1.4) with $\psi_k(x,t)=\psi(xr_{i_k}^{-1},tr_{i_k}^{-2})$, $\psi\in C_0^\infty(\mathbb{R}^N\times\mathbb{R})$, and scaling back the variables we obtain

$$\int\limits_{\mathbb{R}^N} r_{i_k}^{\frac{N-2}{2}}U(r_{i_k}x)\psi(x,0)\,\mathrm{d}x+\int\limits_{\mathbb{R}^N\times\mathbb{R}^+} w_k(x,t)\psi_t(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\mathbb{R}^N\times\mathbb{R}^+} w_k(x,t)\Delta\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int\limits_{\mathbb{R}^N\times\mathbb{R}^+} w_k(x,t)^{\frac{N+2}{N-2}}\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{8.10}$$

As $i\to+\infty$ we have $\bar r_k:=r_{i_k}\to0$ and $\bar r_k^{\frac{N-2}{2}}U(\bar r_kx)\to U^0(x)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Indeed, $\bar r_k^{\frac{N-2}{2}}U(\bar r_kx)\to U^0(x)$ a.e. by (3.11) and, due to (3.3), we have the bound $|U(x)|\leqslant C|x|^{\frac{2-N}{2}}$ for some $C>0$ and $|x|\leqslant\eta$, $\eta>0$ small enough, and the claim follows from the dominated convergence theorem. Clearly $S(t)U\leqslant\underline{u}(t)\leqslant U$, whence $S(t)\bar r_k^{\frac{N-2}{2}}U(\bar r_kx)\leqslant w_k(x,t)\leqslant\bar r_k^{\frac{N-2}{2}}U(\bar r_kx)$ and $S(t)U^0(x)\leqslant w(x,t)\leqslant U^0(x)$ by pointwise convergence. Thus, by monotone and dominated convergence we infer $w\in C^0(\mathbb{R}^+;L^{\frac{N+2}{N-2}}_{\mathrm{loc}}(\mathbb{R}^N))$, $w(0)=U^0$, because w is also decreasing in time and smooth for $t>0$. On the other hand for each $\eta_0>0$ there exist $C=C_{\eta_0}>0$ and an integer $k_0>0$ such that for each $k\geqslant k_0$ we have $w_k(x,t)\leqslant\bar r_k^{\frac{N-2}{2}}U(\bar r_kx)\leqslant C|x|^{\frac{2-N}{2}}$ for each $x\in B_{\eta_0}$. As $w_{k_j}\to w$ in $C^{2,1}_{\mathrm{loc}}$, applying dominated convergence in (8.10) we conclude that w is a weak solution. Since w is smooth for $t>0$, $w(0)=U^0$ and $w\leqslant U^0$, by Corollary 2 we deduce $w=u^0$. As the limit is independent of the chosen subsequence w_k we conclude $v_i\to u^0$ in $C^{2,1}_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ and (1.9) is proved.

In order to check (1.10) we use a scaling argument. As already observed, by (3.3) $U(x)\leqslant C|x|^{\frac{2-N}{2}}$ for $|x|<\eta$, hence $v_i(x,t)\leqslant r_i^{\frac{N-2}{2}}U(r_ix)\leqslant C|x|^{\frac{2-N}{2}}$ for any $|x|\leqslant\eta r_i^{-1}$ and $t>0$ and the same holds for u^0 as $i\to+\infty$ in the whole $\mathbb{R}^N \times (0,\infty)$. Here we stress that the constant C depends only on U and η. Thus if we confine ourselves to the strip $r_1^2 \leqslant t \leqslant 1$, $r_1 = \mathrm{e}^{-T} < 1$ and if we choose i_0 such that $\eta r_{i_0}^{-1} > k$ and $i \geqslant i_0$, then

$$I_1 := \sup_{r_1^2 \leqslant t \leqslant 1} \left(\left\| v_i(\cdot,t) \right\|_{L^p(B_{\eta r_i^{-1}} \setminus B_k)} + \left\| \underline{u}^0(\cdot,t) \right\|_{L^p(B_{\eta r_i^{-1}} \setminus B_k)} \right) \leqslant 2C \left(\int_{k<|x|} \frac{\mathrm{d}x}{|x|^{p\frac{N-2}{2}}} \right)^{1/p}, \tag{8.11}$$

if $p < \infty$, or $I_1 \leqslant 2Ck^{\frac{2-N}{2}}$ if $p = \infty$. On the other hand, by (1.9) $v_i \to \underline{u}^0$ in $C^0(B_k \times [r_1^2, 1])$ and $r_1 = \mathrm{e}^{-T} < 1$, whence for $k > 0$ fixed we have

$$I_2 := \sup_{r_1^2 \leqslant t \leqslant 1} \left\| v_i(\cdot,t) - u^0(\cdot,t) \right\|_{L^p(B_k)} \leqslant C(k) \left\| v_i - u^0 \right\|_{L^\infty(B_k \times [r_1^2,1])} \to 0 \quad \text{as } i \to +\infty. \tag{8.12}$$

Given $0 < t \leqslant r_1^2$, let $i(t) = [\frac{-\log t}{2T}]$, so that $i(t) = k$ whenever $r_{k+1}^2 < t \leqslant r_k^2$ and $i(t) \to \infty$ as $t \to 0^+$. Let $k > 0$ be a fixed number. For i_0 such that $\eta r_{i_0}^{-1} > k$ and t so small that $i = i(t) \geqslant i_0$, taking the scaling invariance of the norms into account and using (8.11), (8.12) we have

$$\begin{aligned} t^{\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} \left\| \underline{u}(\cdot,t) - \underline{u}^0(\cdot,t) \right\|_{L^p(B_\eta)} &\leqslant \sup_{r_1^2 \leqslant t \leqslant 1} \left\| v_i(\cdot,t) - \underline{u}^0(\cdot,t) \right\|_{L^p(B_{\eta r_i^{-1}})} \\ &\leqslant \sup_{r_1^2 \leqslant t \leqslant 1} \left\| v_i(\cdot,t) - \underline{u}^0(\cdot,t) \right\|_{L^p(B_{\eta r_i^{-1}} \setminus B_k)} + \sup_{r_1^2 \leqslant t \leqslant 1} \left\| v_i(\cdot,t) - \underline{u}^0(\cdot,t) \right\|_{L^p(B_k)} \\ &\leqslant C(p) k^{-\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} + C(k) \left\| v_i - u^0 \right\|_{L^\infty(B_k \times [r_1^2,1])}. \end{aligned}$$

As $t \to 0^+$ we have $i(t) \to +\infty$ and

$$\begin{aligned} &\limsup_{t \to 0^+} t^{\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} \left\| \underline{u}(\cdot,t) - \underline{u}^0(\cdot,t) \right\|_{L^p(B_\eta)} \\ &\quad \leqslant C(p) k^{-\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})} + \limsup_{i \to \infty} C(k) \left\| v_i - u^0 \right\|_{L^\infty(B_k \times [r_1^2,1])} = C(p) k^{-\frac{N}{2}(\frac{1}{2^*}-\frac{1}{p})}, \end{aligned}$$

and the conclusion follows as $k \to \infty$.

(3) Since U has a nonremovable singularity at infinity there exists a singular solution $U^\infty = U_{\varepsilon,\delta}$, radial and radially decreasing such that $U(x) - U^\infty(x) = \mathcal{O}(|x|^{\frac{2-N}{2}-\alpha})$ as $|x| \to \infty$ for some $\alpha > 0$ (see (3.12). By (3.8) we know that $g(-\log|x|) = |x|^{\frac{N-2}{2}} U^\infty(x)$ is a bounded continuous function and it is either constant or a periodic function with period $T = T_\varepsilon > 0$. Again we confine ourselves to the second case, the first one being even simpler to deal with. Thus, $U^\infty(x) = R^{\frac{N-2}{2}} U^\infty(Rx)$ for $R = \mathrm{e}^{-T}$ and if we set $r_i = \mathrm{e}^{iT}$, $i \in \mathbb{Z}$, then $r_i \to \infty$ as $i \to +\infty$ and $r_i^{\frac{N-2}{2}} U^\infty(r_i x) = U^\infty(x)$ for each i (observe that now a sign has been changed in the definition of r_i). By Theorem 5 the corresponding minimal solution $u^\infty(x,t)$ has the scale invariance $u^\infty(x,t) = r_i^{\frac{N-2}{2}} u^\infty(r_i x, r_i^2 t)$ for each $i \in \mathbb{Z}$. We claim that if we set $v_i(x,t) = r_i^{\frac{N-2}{2}} \underline{u}(r_i x, r_i^2 t)$, then $v_i(x,t) \to u^\infty(x,t)$ in $C^{2,1}_{\mathrm{loc}}$ because $r_i^{\frac{N-2}{2}} U(r_i x) \to U^\infty(x)$ by (3.12). The proof of this claim is still based on (1.8) and it is completely analogous to the one in (2), therefore it will be omitted. Finally, a scaling argument similar to the one used in proving (1.10) and based on the asymptotic estimate $U(x) = \mathcal{O}(|x|^{\frac{2-N}{2}})$ shows that (1.12) holds. The details are left to the reader. □

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Appendix A. A weak maximum principle for the heat equation

We start with the following slight extension of (2.2) which will be useful later.

Lemma A.1. *Let $I=[0,a]$ and $w\in C^0(I;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ such that for some C we have $|w(x,t)|=\mathcal{O}(\mathrm{e}^{C|x|^2})$ as $|x|\to\infty$ uniformly on $t\in I$. Let $a\leqslant\frac{1}{8C}$ and let $v_0=v(0)$. Then $S(t)v(t)\to v_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t\to 0^+$.*

Proof. Let $R>0$ be any positive number and let $R_0\geqslant 2R$ such that for positive constants C_0, C we have $|w(x,t)|\leqslant C_0(\mathrm{e}^{C|x|^2})$ as $|x|\geqslant R_0$ uniformly on $t\in I$. Set $v^1=v\chi_{\{|x|<R_0\}}$, $v^2=v-v^1$, and similarly $v_0^1=v_0\chi_{\{|x|<R_0\}}$, $v_0^2=v_0-v_0^1$. Clearly we have

$$\big\|S(t)v(t)-v_0\big\|_{L^1(B_R)}\leqslant\big\|S(t)v^1(t)-v_0^1\big\|_{L^1(B_R)}+\big\|S(t)v^2(t)-v_0^2\big\|_{L^1(B_R)}=I_1(t)+I_2(t).$$

By assumption $v^1\in C^0(I;L^1(\mathbb{R}^N))$, hence (2.1), (2.2) yield

$$\begin{aligned}I_1(t)&=\big\|S(t)v^1(t)-v_0^1\big\|_{L^1(B_R)}\leqslant\big\|S(t)\big(v^1(t)-v_0^1\big)\big\|_{L^1(B_R)}+\big\|S(t)v_0^1-v_0^1\big\|_{L^1(B_R)}\\&\leqslant\big\|v^1(t)-v_0^1\big\|_{L^1(\mathbb{R}^N)}+\big\|S(t)v_0^1-v_0^1\big\|_{L^1(\mathbb{R}^N)}\to 0\quad\text{as } t\to 0.\end{aligned}$$

To control the second term first we observe that $K_t(x)=C(N)\,\mathrm{e}^{-\frac{|x|^2}{8t}}K_{t/2}(x)$. Thus, taking the pointwise bound of v^2 into account, we have

$$I_2(t)=\big\|S(t)v^2(t)\big\|_{L^1(B_R)}\leqslant\big\|S(t)C_0\,\mathrm{e}^{C|x|^2}\chi_{\{|x|\geqslant R_0\}}\big\|_{L^1(B_R)}=\int\limits_{|x|\leqslant R}\mathrm{d}x\int\limits_{|y|\geqslant R_0}C'\,\mathrm{e}^{-\frac{|x-y|^2}{8t}}K_{t/2}(x-y)\,\mathrm{e}^{C|y|^2}\mathrm{d}y.$$

If $t\leqslant\frac{1}{64C}$ then $C'\,\mathrm{e}^{-\frac{|x-y|^2}{8t}}\,\mathrm{e}^{C|y|^2}\leqslant C'\,\mathrm{e}^{-\frac{R_0^2}{64t}}$ because $|x|<R$ and $|y|\geqslant R_0\geqslant 2R$. Thus

$$I_2(t)\leqslant C''R^N\,\mathrm{e}^{-\frac{R_0^2}{64t}}\to 0$$

as $t\to 0$ and the lemma is completely proved.

Using the previous result we are can prove the following very weak form of the maximum principle for the heat equation.

Proposition A.1. *Let $w\in C^0(\mathbb{R}^+;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ such that for some $C>0$ we have $|w(x,t)|=\mathcal{O}(\mathrm{e}^{C|x|^2})$ as $|x|\to\infty$ uniformly on t. Assume that $\partial_t w-\Delta w\geqslant 0$ in $\mathcal{D}'(\mathbb{R}^N\times(0,\infty))$. If $u_0(x):=w(x,0)\geqslant 0$ a.e. in $\mathbb{R}^N$, then $w\geqslant 0$ a.e. in $\mathbb{R}^N\times\mathbb{R}^+$, $u(x,t)=S(t)u_0$ is defined for all $t>0$ and $w\geqslant u$ a.e. in $\mathbb{R}^N\times\mathbb{R}^+$.*

Proof. Let $T=\frac{1}{8C}$ so that $u=S(t)u_0$ is well defined and smooth in $\mathbb{R}^N\times(0,T]$ and let us fix two numbers $0<t_0<t_1\leqslant T$ and let $x_1\in\mathbb{R}^N$. We fix $R_0>0$ such that $|w(x,t)|\leqslant C'\mathrm{e}^{C|x|^2}$ for a.e. $x\in\mathbb{R}^N\setminus B_{R_0}$ and for each $t>0$. Let $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi\geqslant 0$, such that $\varphi\equiv 1$ for $|x|\leqslant R_0$. For each $m\in\mathbb{N}$ let us define $\varphi_m\in C_0^\infty(\mathbb{R}^N)$, $\varphi_m\geqslant 0$, as $\varphi_m(x)=\varphi(x/m)$ so that $\{\varphi_m\}$ is bounded in $C^2(\mathbb{R}^N)$, $\varphi_m\equiv 1$ for $|x|\geqslant m$ and $\varphi_m\to 1$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $m\to\infty$.

For each $n\in\mathbb{N}$ let $\psi_n\in C_0^\infty((0,t_0))$, $\psi_n\geqslant 0$ to be specified later. For $s<0$ let us denote by $\overline{K}_s(x)$ the backward heat kernel $\overline{K}_s(x)=(4\pi|s|)^{-N/2}\,\mathrm{e}^{-\frac{|x|^2}{4|s|}}$, so that $\partial_s\overline{K}_s+\Delta\overline{K}_s\equiv 0$ in $\mathbb{R}^N\times(-\infty,0)$.

Testing the inequality with $\overline{K}_{t-t_1}(x_1-x)\varphi_m(x)\psi_n(t)$ we have

$$\begin{aligned}&-\int\limits_{\mathbb{R}^N\times\mathbb{R}^+}w(x,t)\overline{K}_{t-t_1}(x_1-x)\varphi_m(x)\psi_n'(t)\,\mathrm{d}x\,\mathrm{d}t-\int\limits_{\mathbb{R}^N\times\mathbb{R}^+}w(x,t)\overline{K}_{t-t_1}(x_1-x)\Delta_x\varphi_m(x)\psi_n(t)\,\mathrm{d}x\,\mathrm{d}t+\\&\quad-2\int\limits_{\mathbb{R}^N\times\mathbb{R}^+}w(x,t)\nabla_x\overline{K}_{t-t_1}(x_1-x)\nabla_x\varphi_m(x)\psi_n(t)\,\mathrm{d}x\,\mathrm{d}t\geqslant 0.\end{aligned}\tag{A.1}$$

Let us use (A.1) with $\psi_n\in C_0^\infty((0,t_0))$, $0\leqslant\psi_n\leqslant 1$, $\psi_n\to\chi_{(0,t_0)}$ a.e. and $\psi_n'\to-\delta_{t_0}+\delta_0$ in $(C^0([0,t_0]))'$. Since $t_0<t_1\leqslant T=\frac{1}{8C}$, using the growth assumptions on w and the properties of the (backward) heat kernel, it is easy to check that for each $x_1\in\mathbb{R}^N$ there exists $g=g_{x_1}\in L^1(\mathbb{R}^N)$ such that for a.e. $(x,t)\in\{|x|\geqslant R_0\}\times(0,t_0)$ it holds

$$\big|w(x,t)\big|\big(\overline{K}_{t-t_1}(x_1-x)+\big|\nabla_x\overline{K}_{t-t_1}(x_1-x)\big|\big)\leqslant g_{x_1}(x).\tag{A.2}$$

Since $\nabla_x \varphi_m \equiv 0$ and $\Delta_x \varphi_m \equiv 0$ for $|x| > R_0$, using (A.2) we can pass to the limit as $n \to \infty$ in the second and the third term of (A.1) by dominated convergence.

On the other hand, for each x_1 fixed the function $t \to \int_{\mathbb{R}^N} w(x,t)\overline{K}_{t-t_1}(x_1 - x)\varphi_m(x)\,\mathrm{d}x$ belongs to $C^0([0,t_0])$ because of (A.2) and the assumption $w \in C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. Thus, by Fubini theorem, as $n \to \infty$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} w(x,t_0)\overline{K}_{t_0-t_1}(x_1 - x)\varphi_m(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} u_0(x)K_{t_1}(x_1 - x)\varphi_m(x)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N\times(0,t_0)} w(x,t)\overline{K}_{t-t_1}(x_1 - x)\Delta_x\varphi_m(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 2\int_{\mathbb{R}^N\times(0,t_0)} w(x,t)\nabla_x\overline{K}_{t-t_1}(x_1 - x)\nabla_x\varphi_m(x)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.
\end{aligned} \tag{A.3}$$

Combining (A.3), (A.2) and the support and convergence properties of φ_m, $\nabla\varphi_m$ and $\Delta\varphi_m$, as $m \to \infty$ by dominated convergence we obtain

$$\int_{\mathbb{R}^N} w(x,t_0)\overline{K}_{t_0-t_1}(x_1 - x)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} u_0(x)K_{t_1}(x_1 - x)\,\mathrm{d}x = u(x_1,t_1), \tag{A.4}$$

for each $x_1 \in \mathbb{R}^N$ and for each $0 < t_0 < t_1 \leqslant T$. Taking $v(s) = w(t_1 - s)$, $s = t_1 - t_0$, $0 \leqslant s \leqslant t_1$, and applying Lemma A.1 we easily conclude that there exists $t_0^n \nearrow t_1$ such that for a.e. $x_1 \in \mathbb{R}^N$ we have

$$\lim_{t_0^n \nearrow t_1} \int_{\mathbb{R}^N} w\big(x,t_0^n\big)\overline{K}_{t_0^n - t_1}(x_1 - x)\,\mathrm{d}x = w(x_1,t_1). \tag{A.5}$$

Since $u_0 \geqslant 0$ then $u \geqslant 0$ in $\mathbb{R}^N \times [0,T]$. Combining (A.5) and (A.4) we easily infer $w(t_1) \geqslant u(t_1)$ a.e. in $\mathbb{R}^N$ for each $t_1 \in [0,T]$. The conclusion follows from an induction argument using the same proof on each time interval $I_l = [(l-1)T, lT]$, $l \in \mathbb{N}$, $l \geqslant 1$ and the semigroup property for the standard heat kernel.

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