

Essential Dynamics for Lorenz maps on the real line and the Lexicographical World[☆]

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Dedicated to the memory of Professor Jorge Billeke G.

Abstract

In this paper we describe some topological and geometric properties of the set of sequences $LW = \{(a, b) \in \Sigma_0 \times \Sigma_1; a \leq \sigma^n(a) \leq b, a \leq \sigma^n(b) \leq b, \forall n \in \mathbb{N}\}$, which essentially represents all the allowed dynamics for piecewise continuous increasing maps with one discontinuity. In particular, we describe the first main bifurcations in LW which generate non-trivial dynamics, and we study (fractal) geometric properties of LW and of the phase spaces $\Sigma_{a,b}$ associated to it.

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Résumé

Dans ce travail nous décrivons quelques propriétés topologiques et géométriques de l'ensemble de suites $LW = \{(a, b) \in \Sigma_0 \times \Sigma_1; a \leq \sigma^n(a) \leq b, a \leq \sigma^n(b) \leq b, \forall n \in \mathbb{N}\}$, que représentent essentiellement toutes les dynamiques permises pour des fonctions continues et croissantes par morceaux avec un point de discontinuité. En particulier, on décrit les premières bifurcations dans LW qui produisent des dynamiques non-triviales et nous étudions des propriétés géométriques (fractales) de LW et des espaces de phase $\Sigma_{a,b}$ associés.

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1. Introduction

In the remarkable work [13], a meteorologist, E.N. Lorenz, showed numerical evidence of the existence of a strange attractor for a quadratic system of ordinary differential equations in three variables. Some time later J. Guckenheimer, [7], produced a work where he introduced symbolic dynamics in order to understand the topological equivalence classes for nearly similar attractors. At that time R.F. Williams, [26], introduced a geometrical model in order to understand the dynamics of these Lorenz attractors. In Fig. 1 we give a sketch of the geometric attractor. Moreover,

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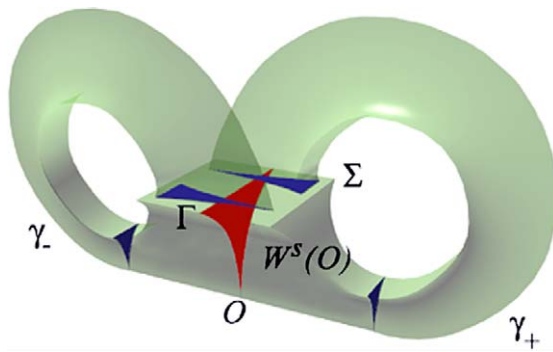


Fig. 1. Geometric Lorenz attractor.

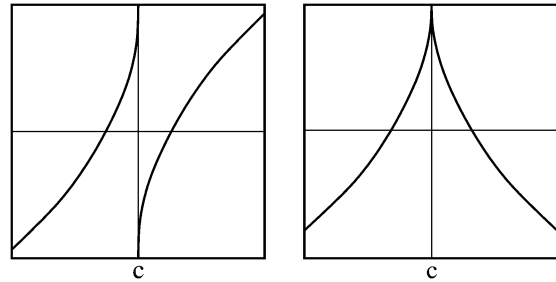


Fig. 2. One-dimensional return map.

in Fig. 2 we represent the one-dimensional models associated to the attractor. The right-hand side of this picture corresponds to the numerical experiments of Lorenz, who used a cross-section different from that of Guckenheimer and Williams, who obtained a map as in the left-hand side of the figure. The combinatorial dynamics of both one-dimensional maps sketched in this figure are equivalent. In this work we will concentrate our attention on piecewise increasing one-dimensional maps.

Using this geometrical model the dynamical behavior of the three-dimensional vector field can be reduced to the dynamical behavior of a one-dimensional map with one discontinuity and Guckenheimer and Williams, [8], used this fact to show uncountable many classes of non-equivalent geometric Lorenz attractors. The evidence of the non-equivalence were the kneading sequences associated to these one-dimensional maps. The class of one-dimensional maps defined in this way is included in the class of one-dimensional maps which we work here (see the definition of the set DM_0 given in Section 2.1).

Associated to any $f \in DM_0$ we have two kneading sequences $(a_f, b_f) = I(f)$ (see Section 2.3 for the definition of these sequences) that satisfy $a_f = \inf\{\sigma^k(a_f), k \in \mathbb{N}\}$, $b_f = \sup\{\sigma^k(b_f), k \in \mathbb{N}\}$ and $\{a_f, b_f\} \subset \Sigma_{a_f, b_f}$ (here $\Sigma_{a,b}$ denotes the set $\bigcap_{i=0}^{\infty} \sigma^i([a, b])$; see Section 2.3 for details). These properties inspired (see [22]) the following definitions. A sequence of two symbols $a = (0, \dots) \in \Sigma_2$ (resp. $b = (1, \dots) \in \Sigma_2$) is called *minimal* (resp. *maximal*) if $a = \inf\{\sigma^k(a), k \in \mathbb{N}\}$ (resp. $b = \sup\{\sigma^k(b), k \in \mathbb{N}\}$). Here $\sigma : \Sigma_2 \rightarrow \Sigma_2$ denotes the usual shift map. We will denote by Min_2 (resp. Max_2) the set of minimal (resp. maximal) sequences in Σ_2 . These two properties allow us to define the *Lexicographical World* as $LW = \{(a, b) \in \text{Min}_2 \times \text{Max}_2, \{a, b\} \subset \Sigma_{a,b}\}$. The itinerary

$$I : DM_0 \rightarrow LW, \quad f \rightarrow (a_f, b_f)$$

defines a continuous and surjective map (see Section 2.5). We will say that the map $f \in DM_0$ has *essentially* the same dynamics as $g \in DM_0$ if $I(f) = I(g)$. It is clear that two topologically equivalent maps are essentially equivalent.

Notice that one of the main topological obstructions for two maps in DM_0 with essentially the same dynamics being conjugated is the presence of wandering intervals for some of these maps. We prove in Proposition 1 that generic C^2 maps in DM_0 do not have non-trivial wandering intervals.

Therefore, the lexicographical world provide a *universal model* for (essentially) equivalent dynamics in this context. This means the following: given $(a, b) \in LW$, there is $f \in DM_0$ such that $\Sigma_{a,b} = \Sigma_{a_f, b_f}$ and a surjective map $I_f : \Gamma_f \rightarrow \Sigma_{a,b}$ such that $I_f \circ f = \sigma \circ I_f$ and reciprocally (see Section 2.3 for the definition of the set Γ_f , the definition of the map I_f and the proof of the realization lemma). Clearly, if we are able to describe the different dynamics present in this universal model then we are able to prove some dynamical properties of the elements in DM_0 . Also our set, DM_0 , contains the topologically expanding Lorenz maps as defined in [9]. The kneading sequences, $(a, b) \in LW$ associated to expanding maps satisfies the condition $a \leq \sigma^n(a) < b$ and $a < \sigma^n(b) \leq b, \forall n \geq 0$. Let denote by $TE \subset LW$ this set. One of the problems posed in [9] is to describe the set TE . In Section 4 we characterize the local fractal properties of TE (and LW).

In these directions are the main results of the present paper: we describe some metric and geometrical properties of the lexicographical world which “essentially” represents all the allowed dynamics for piecewise continuous increasing maps and we use these properties to establish some results for the elements in DM_0 . For instance, among other results, we prove

Theorem 1. *The maps $\varphi, \psi, \chi : \text{Min}_2 \rightarrow \text{Max}_2$ given by*

$$\begin{aligned} \varphi(a) &= \inf\{b \in \Sigma_1; \Sigma_{a,b} \neq \emptyset\}, \quad \psi(a) = \inf\{b \in \Sigma_1; \Sigma_{a,b} \text{ is infinite}\} \quad \text{and} \\ \chi(a) &= \inf\{b \in \Sigma_1; \Sigma_{a,b} \text{ is uncountable}\} \end{aligned}$$

satisfy the following recursive formulas (see Section 3 for the definition of the maps $T_{a,b}$ and $T_{a,b}^$)*

- (i) (1) *for $a \leq 00\underline{1}$ we have $0\varphi(a) = T_{0,01} \circ \varphi \circ T_{0,01}^*(a)$,*
 (2) *for $0\underline{1} < a < 0\underline{1}$ we have $\varphi(a) = T_{10,1} \circ \varphi \circ T_{10,1}^*(1a)$,*
 (3) *for $00\underline{1} \leq a \leq 0\underline{1}$ we have $\varphi(a) = \underline{10}$.*
- (ii) (1) *for $a < 00\underline{1}$, $\chi(a) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(a)$,*
 (2) *for $00\underline{1} \leq a \leq 001\underline{10}$, $\chi(a) = 1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(a))$,*
 (3) *for $001\underline{10} \leq a \leq 0\underline{1}$, $\chi(a) = \underline{110}$,*
 (4) *for $0\underline{1} \leq a \leq 0\underline{1}$, $\chi(a) = T_{10,1} \circ \chi \circ T_{10,1}^*(1a)$.*
- (iii) (1) *for $a \leq 00\underline{1}$, $\psi(a) = \sigma \circ T_{0,01} \circ \psi \circ T_{0,01}^*(a)$,*
 (2) *for $00\underline{1} < a \leq 001\underline{10}$, $\psi(a) = 1T_{01,10} \circ \psi \circ T_{01,10}^*(\sigma(a))$,*
 (3) *for $001\underline{10} \leq a \leq 0\underline{1}$, $\psi(a) = \underline{110}$ and*
 (4) *for $0\underline{1} < a \leq 0\underline{1}$ we have $\psi(a) = T_{10,1} \circ \psi \circ T_{10,1}^*(1a)$.*

We notice that $\{(a, b) \in \Sigma_0 \times \Sigma_1; \Sigma_{a,b} \neq \emptyset\} = \{(a, b) \in \Sigma_0 \times \Sigma_1; b \geq \varphi(a)\}$.

Another characterization of the map χ is given by the following

Theorem 2. *The set $\{(a, b) \in \Sigma_0 \times \Sigma_1; \text{the topological entropy of the map } (\sigma|_{\Sigma_{a,b}}) : \Sigma_{a,b} \rightarrow \Sigma_{a,b} \text{ is zero}\}$ is equal to the set $\{(a, b) \in \Sigma_0 \times \Sigma_1; b \leq \chi(a)\}$.*

We observe that a consequence of Theorem 2 is the following: The set $EZ_0 = \{\text{maps in } DM_0 \text{ with zero topological entropy}\}$ is equal to the set $\{f \in DM_0; b_f \leq \chi(a_f)\}$ (we observe that related problems of characterizing the boundary of the set of maps of zero entropy were focused by several authors in this and other contexts; see for instance [2,23,19,15,24,5,20,21]).

Also we prove the following result about (fractal) geometric properties of LW :

Theorem 3. *Let $D : \Sigma_0 \times \Sigma_1 \rightarrow \mathbb{R}$ be the map defined as $D(a, b) = HD(\Sigma_{a,b})$, where $HD(\Sigma_{a,b})$ denotes the Hausdorff dimension of the set $\Sigma_{a,b}$ (here we consider the set $\Sigma_0 \times \Sigma_1$ equipped with the usual diadic metric; see Section 2.2). Then D is a continuous map.*

For $(a, b) \in \Sigma_0 \times \Sigma_1$ define $\Omega(a, b) = \{(\alpha, \beta) \in LW; a \leq \alpha \leq \beta \leq b\}$ and $\bar{\Omega}(a, b) = \{(\alpha, \beta) \in \Omega(a, b); (\alpha, \beta) \in TE\}$ then

$$HD(\bar{\Omega}(a, b)) = HD(\Omega(a, b)) = 2D(a, b) = \frac{2}{\log(2)} h_{\text{top}}(\Sigma_{a,b})$$

(here $h_{\text{top}}(\Sigma_{a,b})$ means the topological entropy of the restriction of the shift map to the set $\Sigma_{a,b}$).

Clearly, the described structure of the set LW reflects in the bifurcation theory associated to any parameterized family of maps in DM_0 (also an extremely interesting problem focused for several authors in this and other contexts, see for instance [14,2,1,4]). In [10,11] and [12] we applied these results for contracting, expanding and linear families of Lorenz maps.

This paper is organized as follows: In Section 2 we introduce the lexicographical world, we describe the set DM_0 (that we recognize as the set of *Lorenz Maps*) and we prove the realization lemma. In Section 3 we prove Theorem 1 and part of Theorem 2. In Section 4 we prove Theorem 3 and we complete the proof of Theorem 2.

There are several works related to the symbolic dynamics associated to one dimensional Lorenz maps. For instance, in the papers by Hubbard and Sparrow [9] and Glendinnig and Sparrow [6] a study of the symbolic dynamics associated to topologically expansive one-dimensional Lorenz maps is performed. We note that the equality $D(a, b) = \frac{1}{\log(2)} h_{\text{top}}(\Sigma_{a,b})$ follows from results by H. Furstenberg [3] and also by Urbański [25].

2. Lorenz maps and symbolic dynamics

2.1. The set DM_0

In the sequel DM_0 will denote the set of maps $f : (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}$ such that:

- (1) The restriction maps $f|_{(-\infty, 0)} : (-\infty, 0) \mapsto \mathbb{R}$ and $f|_{(0, \infty)} : (0, \infty) \rightarrow \mathbb{R}$ are continuous and non-decreasing maps.
- (2) $f(0^+) = \lim_{x \downarrow 0} f(x) \in (-\infty, 0]$ and $f(0^-) = \lim_{x \uparrow 0} f(x) \in [0, \infty[$.

2.2. The lexicographical order

Let Σ_2 denote the set of sequences $\theta : \mathbb{N} \rightarrow \{0, 1\}$ endowed with the topology given by the metric

$$d(\alpha, \beta) = \frac{1}{2^n},$$

where

$$n = \min\{k \in \mathbb{N}; \alpha_k \neq \beta_k\}.$$

Let $\sigma : \Sigma_2 \rightarrow \Sigma_2$ be the shift map $\sigma(\theta_0, \theta_1, \theta_2, \dots) = (\theta_1, \theta_2, \dots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2; \theta_0 = 0\}$ and $\{\theta \in \Sigma_2; \theta_0 = 1\}$ respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$.

In Σ_2 we consider the *lexicographical order*: $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in \mathbb{N}$ such that $\theta_i = \alpha_i$ for $i = 0, 1, 2, \dots, n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $a \leq b$ in Σ_2 let $[a, b]$ denote the interval $\{\theta \in \Sigma_2 \mid a \leq \theta \leq b\}$. $\Sigma_{a,b}$ will denote the set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a, b])$.

Let $a = a_0, a_1, \dots, a_n$ be a finite word. We will denote by \underline{a} the infinite sequence $(a_0, a_1, \dots, a_n; a_0, a_1, \dots, a_n; a_0, a_1, \dots, a_n; \dots) \in \Sigma_2$.

2.3. The set Σ_{a_f, b_f}

For $f \in DM_0$ let $\Gamma_f = (\mathbb{R} \setminus \bigcup_{j=0}^{\infty} f^{-j}(0))$ denote the set of “continuity” of the map f .

For $x \in \Gamma_f$ we define $I_f(x) \in \Sigma_2$ by

$$I_f(x)(i) = 0 \quad \text{if } f^i(x) < 0 \quad \text{and} \quad I_f(x)(i) = 1 \quad \text{if } f^i(x) > 0.$$

For $x = 0$ we define:

$$I_f(0^+) = \lim_{x \downarrow 0, x \in \Gamma_f} I_f(x)$$

and

$$I_f(0^-) = \lim_{x \uparrow 0, x \in \Gamma_f} I_f(x).$$

In the same way to any $x \in \bigcup_{j=0}^{\infty} f^{-j}(0)$ such that $f^i(x) \neq 0$, $0 \leq i < n$; $f^n(x) = 0$ we associate the sequences:

$$I_f(x^+) = (I_f(x)(0), \dots, I_f(x)(n-1), I_f(0^+))$$

and

$$I_f(x^-) = (I_f(x)(0), \dots, I_f(x)(n-1), I_f(0^-)).$$

For $x \in \Gamma_f$ we define $I_f(x^+) = I_f(x^-) = I_f(x)$.

Let $I_f = \{I_f(x^+); x \in [f(0^+), f(0^-)]\} \cup \{I_f(x^-); x \in]f(0^+), f(0^-)]\}$.

Let us denote $a_f = I_f((f(0^+))^+)$ and $b_f = I_f((f(0^-))^-)$. The following lemma is a classical fact which associates a symbolic dynamical system to a Lorenz map on the interval, via kneading sequences. See, for instance, [22].

Lemma 1. $I_f = \bigcap_{n=0}^{\infty} \sigma^{-n}([a_f, b_f]) = \Sigma_{a_f, b_f}$.

Proof. We have $I_f(f(x)) = \sigma(I_f(x))$, by definition of $I_f(x)$, so $\sigma(I_f) \subset I_f$. This also implies that, since $I_f \subset [a_f, b_f]$, $I_f \subset \sigma^{-n}([a_f, b_f])$ for every natural number n . Let now $\theta \in \bigcap_{n=0}^{\infty} \sigma^{-n}([a_f, b_f])$, that is $a_f \leq \sigma^n(\theta) \leq b_f$ for $n = 0, 1, 2, \dots$. Clearly we must have $a_f \leq \sigma^n(\theta) \leq 0b_f$ or $1a_f \leq \sigma^n(\theta) \leq b_f$ for every $n = 0, 1, 2, \dots$. Let $I_0 = [f(0^+), 0[$ and $I_1 =]0, f(0^-)]$. The opposite inclusion follows from the following facts:

- (i) $I_f(0^+) = 1a_f, I_f(0^-) = 0b_f$;
- (ii) $I_{\theta_0} \cap f^{-1}(I_{\theta_1}) \cap f^{-2}(I_{\theta_2}) \cap \dots \cap f^{-n}(I_{\theta_n}) \neq \emptyset$ for $n = 0, 1, \dots$ and
- (iii) the continuity of the map f on $I_0 \cup I_1$. \square

We observe that associated to any $f \in DM_0$ we can define a continuous map (see [16] for details)

$$h : [f(0^+), f(0^-)] \cap \Gamma_f \rightarrow \Sigma_{a_f, b_f} \subset \Sigma_2,$$

such that $h \circ f = \sigma \circ h$. The map h is given by $h(x) = I_f(x)$ and could collapse some intervals into points. This map cannot be extended, continuously, to the set $\bigcup_{i=0}^{\infty} f^{-i}(0)$.

There are two kinds of intervals that are collapsed by the map h : The wandering intervals and the intervals that are contained in the stable manifolds of the periodic sinks. An interval $I \subset [f(0^+), f(0^-)]$ is called a *wandering interval*, for the map f , if for any $x \in I$ we have that x is a wandering point. We will call a point x a *non-wandering point* if for any neighborhood U_x of x and any positive integer N we can find $n \geq N$ such that $f^n(U_x) \cap U_x \neq \emptyset$. The set of non-wandering points of the map f is denoted by Ω_f . A point $x \notin \Omega_f$ is called a *wandering point*. Given any interval I , the *orbit* of this interval is the sequence of iterations $(f^n(I), n \in \mathbb{N})$.

We say that a wandering interval is *non-trivial* if it is not contained in a basin of attraction of a periodic orbit.

Concerning the existence of wandering intervals we have the following:

Proposition 1. Let $\{\varphi_\lambda, \lambda \in \mathbb{R}\} \subset DM_0$ be a one parameter family of C^2 piecewise increasing maps (for instance, elements of DM_0) such that for each λ there are sequences $\lambda_n \rightarrow \lambda$ and $\mu_n \rightarrow \lambda$ with $\varphi_{\lambda_n}(x) > \varphi_\lambda(x)$ and $\varphi_{\mu_n}(x) < \varphi_\lambda(x), \forall x$ then there is a residual set of parameters λ for which φ_λ has no non-trivial wandering intervals.

Proof. The orbit of any non-trivial wandering interval must accumulate in the discontinuity (it is a consequence of the Schwarz lemma or of Mañé’s Theorem on hyperbolicity for one-dimensional maps; see [16]). To get the result it is enough to prove that for any $a, b \in \mathbb{Q}$, with $a < b$ the interval $]a, b[$ is not contained in a wandering interval for an open and dense set of parameter λ . If, for some parameter λ the interval $]a, b[$ is contained in a wandering interval then its orbit accumulates the discontinuity. Assume that 0 is accumulated by the left side by the orbit of the interval $]a, b[$. Let $\lambda_n \rightarrow \lambda$ be the sequence, associated to λ , given by the hypothesis. Since $\varphi_{\lambda_n} > \varphi_\lambda$ we can find an iterate of $]a, b[$, say $\varphi_\lambda^{k_n}(]a, b[)$ such that $\varphi_\lambda^{k_n}(]a, b[) \subset]-\infty, 0[$ and $\varphi_{\lambda_n}^{k_n}(]a, b[) \subset]0, \infty[$ (since the length of $\varphi^n(]a, b[)$ must converge to 0 and all the maps φ_λ are increasing). Joining λ to λ_n by a continuous path we find a parameter value, ρ_n between λ and λ_n such that $\varphi_{\rho_n}^{k_n}(]a, b[)$ contains 0 and then $]a, b[$ cannot be contained in a wandering interval for φ_{ρ_n} . The same is true for parameters near ρ_n and also $\rho_n \rightarrow \lambda$ as $n \rightarrow \infty$. This completes the proof of the proposition. \square

Remark 1.

- (a) The previous argument also shows that the set

$$\{f \in DM_0; f \text{ has no non-trivial wandering intervals}\}$$

is residual in DM_0 .

- (b) The result is true also with an arbitrary number of parameters.

Definition 1. Let $f, g \in DM_0$. We will say that f has *essentially* the same dynamics as g if $I_f = I_g$.

We observe that in this situation, up to the existence of some intervals where the itineraries of the points are the same, the dynamics of the maps f and g are topologically equivalent (see [16]).

2.4. The Lexicographical World

Let $\text{Min}_2 = \{a \in \Sigma_0; \sigma^k(a) \geq a, \forall k \in \mathbb{N}\}$ and $\text{Max}_2 = \{b \in \Sigma_b; \sigma^k(b) \leq b, \forall k \in \mathbb{N}\}$. Elements in Min_2 (resp. Max_2) will be called *minimal* (resp. *maximal*) elements in Σ_2 .

Remark 2.

- (1) Assume that $a \in \Sigma_0$ is a periodic sequence in Min_2 . Let $a_0 a_1 \cdots a_k$ be its period. Then, necessarily, we have $a_0 = 0$ and $a_k = 1$.
- (2) Assume that $b \in \Sigma_1$ is a periodic sequence in Min_2 . Let $b_0 b_1 \cdots b_k$ be its period then, necessarily, we have $b_0 = 1$ and $b_k = 0$.
- (3) Clearly, Min_2 and Max_2 are closed sets in Σ_2 .

Definition 2. The set $LW = \{(a, b) \in \text{Min}_2 \times \text{Max}_2; \{a, b\} \subset \Sigma_{a,b}\}$ will be called the *lexicographical world*.

For $a \in \text{Min}_2$ its *LW*-fiber is the set $LW_0(a) = \{b \in \text{Max}_2; (a, b) \in LW\}$. For $b \in \text{Max}_2$ its *LW*-fiber is the set $LW_1(b) = \{a \in \text{Min}_2; (a, b) \in LW\}$.

Remark 3. It is clear that if $(a, b) \in LW$ then $\Sigma_{a,b} \neq \emptyset$, since it contains $\{a, b\}$.

2.5. The realization lemma

Let us now consider $(a, b) \in LW$.

Lemma 2. There is $f \in DM_0$ such that $I_f = \Sigma_{a,b}$.

Proof. Let us consider the map $g : (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}$ given by

$$g(x) = \begin{cases} 2x - 1, & x > 0, \\ 2x + 1, & x < 0. \end{cases}$$

In this case $I_g = \Sigma_2$. Let $x_a < 0$ and $x_b > 0$ be the points such that $I_g(x_a^+) = a, I_g(x_b^-) = b$. Let $x_a < \bar{x}_b < 0 < \bar{x}_a < x_b$, be the points that satisfy $g(\bar{x}_b) = x_b$ and $g(\bar{x}_a) = x_a$.

Let $f : (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}$ be the map defined by:

$$f(x) = \begin{cases} g(x), & x \leq \bar{x}_b, \\ x_b, & \bar{x}_b \leq x < 0, \\ x_a, & 0 < x \leq \bar{x}_a, \\ g(x), & \bar{x}_a \leq x. \end{cases}$$

The map f satisfy $I_f = \Sigma_{a,b}$. In fact: $a_f = a, b_f = b$ and the itinerary, θ , send points in the set $\{x \in [x_a, \bar{x}_b] \cup [\bar{x}_a, x_b] (= J_f); f^n(x) \in J_f \forall n \in \mathbb{N}\}$ into points in I_f which is equal to $\Sigma_{a,b}$. \square

Therefore, we have a surjective map $I : DM_0 \rightarrow LW, I(f) = (a_f, b_f)$. Also, using C^0 proximity of maps in DM_0 (with respect to $f|_{]-\infty, 0]}$ and $f|_{]0, \infty[}$) on compact sets, this map is continuous.

3. Structure of the Lexicographical World

3.1. The maps φ, ψ, χ

It is clear that $\Sigma_{a,\perp} \neq \emptyset$ for any $a \in \Sigma_0$. Hence we can define maps $\varphi, \psi, \chi : \Sigma_0 \rightarrow \Sigma_1$ by:

$$\begin{aligned} \varphi(a) &= \inf\{b \in \Sigma_1 \mid \Sigma_{a,b} \neq \emptyset\}, \\ \psi(a) &= \inf\{b \in \Sigma_1; \Sigma_{a,b} \text{ contains infinitely many elements}\} \end{aligned}$$

and

$$\chi(a) = \inf\{b \in \Sigma_1; \Sigma_{a,b} \text{ is uncountable}\}.$$

Clearly, $a_1 \leq a_2$ imply $\varphi(a_1) \leq \varphi(a_2)$, $\psi(a_1) \leq \psi(a_2)$ and $\chi(a_1) \leq \chi(a_2)$ and for all $c \in \Sigma_1$ such that $c < \varphi(a)$ we have $\Sigma_{a,c} = \emptyset$. Moreover, for any $a \in \Sigma_0$ we have that $1a \leq \varphi(a)$ (in fact, any $b < 1a$ satisfy $b \notin \Sigma_{a,d}$ for any $d \in \Sigma_1$).

Examples. We have: $\varphi(\underline{0}) = \psi(\underline{0}) = \chi(\underline{0}) = 1\underline{0}$; $\varphi(0\underline{1}) = \psi(0\underline{1}) = \underline{1}$.

If $\Sigma_{a,b}$ is uncountable then we can define

$$\tilde{a} = \inf\{\theta \in \Sigma_{a,b}, \theta \text{ is a condensation point of } \Sigma_{a,b}\}$$

and

$$\tilde{b} = \sup\{\theta \in \Sigma_{a,b}, \theta \text{ is a condensation point of } \Sigma_{a,b}\}.$$

Let us recall that a *condensation* point of a set $X \subset \Sigma_2$ is a point with the property that any neighborhood of it contains an uncountable subset of X .

It is clear that the set $(\Sigma_{a,b} \setminus \Sigma_{\tilde{a},\tilde{b}})$ is countable, since it has no condensation points, and we have

$$\tilde{a} \leq \sigma^n(\tilde{a}) < \tilde{b} \quad \text{and} \quad \tilde{a} < \sigma^n(\tilde{b}) \leq \tilde{b}, \quad \forall n \in \mathbb{N},$$

so, by [9], $\Sigma_{\tilde{a},\tilde{b}}$ is a perfect set and the restriction of the shift map, $\sigma|_{\Sigma_{\tilde{a},\tilde{b}}}$, is topologically expansive.

We observe that for any $a \in \Sigma_0$ we have $\varphi(a) \leq \psi(a) \leq \chi(a)$.

3.2. The recurrence formula of the maps φ , ψ and χ

Let $m_0 < m_1$ be two finite words of 0's and 1's. Let $T_{m_0,m_1} : \Sigma_2 \rightarrow \Sigma_2$ be the map $T_{m_0,m_1}(\theta_0, \theta_1, \dots) = (m_{\theta_0}, m_{\theta_1}, \dots)$. About this map we make the following considerations:

- (1) $\Theta_1 \leq \Theta_2$ imply $T_{m_0,m_1}(\Theta_1) \leq T_{m_0,m_1}(\Theta_2)$.
- (2) If $\Sigma_{m_0,m_1} = \{\Theta : \mathbb{N} \rightarrow \{m_0, m_1\}\}$ then $T_{m_0,m_1}(\Sigma_2) = \Sigma_{m_0,m_1}$ and it is an homeomorphism onto its image. The inverse map is constructed in the following way: let $\epsilon(m_0) = 0$, $\epsilon(m_1) = 1$ for $\alpha \in \Sigma_{m_0,m_1}$ we have $T_{m_0,m_1}^{-1}(\alpha) = (\epsilon(\alpha_0), \epsilon(\alpha_1), \dots)$.
- (3) An extension of the map T_{m_0,m_1}^{-1} is given by the map T_{m_0,m_1}^* defined, for $\alpha \leq \underline{m_1}$, as $T_{m_0,m_1}^*(\alpha) = \inf\{\beta \in \Sigma_2; T_{m_0,m_1}(\beta) \geq \alpha\}$.
- (4) We define the map $\sigma_{m_0,m_1} : \Sigma_{m_0,m_1} \rightarrow \Sigma_{m_0,m_1}$, by

$$\sigma_{m_0,m_1}(T_{m_0,m_1}(a)) = T_{m_0,m_1}(\sigma(a)), \quad \forall a \in \Sigma_2.$$

Proposition 2. *The map φ satisfies:*

- (1) for $a \leq 0\underline{01}$ we have $0\varphi(a) = T_{0,01} \circ \varphi \circ T_{0,01}^*(a)$,
- (2) for $0\underline{1} < a < 0\underline{1}$ we have $\varphi(a) = T_{10,1} \circ \varphi \circ T_{10,1}^*(1a)$,
- (3) for $0\underline{01} \leq a \leq 0\underline{1}$ we have $\varphi(a) = \underline{10}$.

Proof. Let us prove (2). By the definition of φ , $\Sigma_{T_{10,1}^*(1a), \varphi(T_{10,1}^*(1a))}$ is a non-empty set. Hence,

$$T_{10,1}(\Sigma_{T_{10,1}^*(1a), \varphi(T_{10,1}^*(1a))})$$

is an invariant, non-empty set for $\sigma_{10,1} : \Sigma_{10,1} \rightarrow \Sigma_{10,1}$. So, since $T_{10,1}$ is a conjugacy between σ and $\sigma_{10,1}$,

$$T_{10,1}(\Sigma_{T_{10,1}^*(1a), \varphi(T_{10,1}^*(1a))}) \cup \sigma(T_{10,1}(\Sigma_{T_{10,1}^*(1a), \varphi(T_{10,1}^*(1a))}))$$

is a non-empty, invariant set for σ . Therefore $\varphi(a) \leq T_{10,1} \circ \varphi \circ T_{10,1}^*(1a)$. If $\varphi(a) < T_{10,1} \circ \varphi \circ T_{10,1}^*(1a)$ then $[1a, \varphi(a)] \cap \Sigma_{a, \varphi(a)} \neq \emptyset$ and $T_{10,1}^*([1a, \varphi(a)] \cap \Sigma_{a, \varphi(a)}) \subset [T_{10,1}^*(1a), \varphi(T_{10,1}^*(1a))]$ is a non-empty, σ -invariant set (since $a > 0\underline{1}$ implies $\Sigma_{a, \underline{1}} \cap \Sigma_1 \subset \Sigma_{0\underline{1}, \underline{1}}$, and so, in particular, $\varphi(a) \in \Sigma_{10,1}$). This is a contradiction.

The proof of (1) follows in a similar way since $\emptyset \neq \Sigma_{a,01} \cap \Sigma_0 \subset \Sigma_{0,01}$.

The proof of (3) follows from $\Sigma_{01,10} \neq \emptyset$ and $\varphi(001) = 10$ which gives $10 = \varphi(001) \leq \varphi(a) \leq \varphi(01) \leq 10$, $\forall a \in [001, 01]$. \square

Note 1.

- (i) We observe that φ is not a continuous map since $\lim_{a \rightarrow 01+} \varphi(a) = 110 \neq 10 = \varphi(01)$. In fact, for $a_n = (01)_n 011$ we have that $\varphi(a_n) = 1(10)_{n+1}$. Therefore $a_n \rightarrow 01$ and $\varphi(a_n) \rightarrow 110$.
- (ii) As a consequence of Proposition 2 we note that the graph of the map φ is a kind of “devil stair” (although not continuous) in $\Sigma_0 \times \Sigma_1$: it is locally constant in an open and dense set. In fact, let A denote the map $T_{0,01}$, B denote the map $\sigma \circ T_{10,1}$ and I be the interval $[0010, 01]$. Let $A_0 = I$, $A_1 = A_0 \cup A(A_0) \cup B(A_0)$, $A_2 = A_1 \cup A(A_1) \cup B(A_1)$ and, inductively, $A_{n+1} = A_n \cup A(A_n) \cup B(A_n)$. It is not hard to see, from the recursive formulas at Proposition 2 that the set $A_\infty = \bigcup_{n=0}^{\infty} A_n$ is a dense set in Σ_0 and $(\varphi|_J)$ is constant for any interval $J \subset A_\infty$.

Proposition 3. *The maps χ and ψ satisfy:*

- (a) for $a < 001$, $\chi(a) = \sigma \circ T_{0,01} \circ \chi \circ T_{0,01}^*(a)$,
 (b) for $001 \leq a \leq 00110$, $\chi(a) = 1T_{01,10} \circ \chi \circ T_{01,10}^*(\sigma(a))$,
 (c) for $00110 \leq a \leq 01$, $\chi(a) = 110$,
 (d) for $01 \leq a \leq 01$, $\chi(a) = T_{10,1} \circ \chi \circ T_{10,1}^*(1a)$,
 (a') for $a \leq 001$, $\psi(a) = \sigma \circ T_{0,01} \circ \psi \circ T_{0,01}^*(a)$,
 (b') for $001 < a \leq 00110$, $\psi(a) = 1T_{01,10} \circ \psi \circ T_{01,10}^*(\sigma(a))$,
 (c') for $00110 \leq a \leq 01$, $\psi(a) = 110$ and
 (d') for $01 < a \leq 01$ we have $\psi(a) = T_{10,1} \circ \psi \circ T_{10,1}^*(1a)$.

Proof. Let us first remark that if $\Sigma_{a,b} \neq \emptyset$ and $a \geq 00110$ begins with 00 then $b \geq \sigma^2(a) \geq 110$. If a begins with 01 then there is no pair of consecutive 0's in any element of $\Sigma_{a,b}$, so for the interesting a we have $a \geq 01$, and if b begins with 11 then (for the interesting b) we have $b \geq 110$. If b begins with 10 then there is no pair of consecutive 1's in any element of $\Sigma_{a,b}$, so $b \leq 10$ in this case and, hence $\Sigma_{a,b} \subset \Sigma_{01,10} = \{01, 10\}$ is finite. This implies $\chi(a) \geq 110$ and $\psi(a) \geq 110$ for $a \geq 00110$.

To show that $\chi(a) \leq 110$ for $a \leq 01$ it is enough to notice that the set $\Sigma_{01,1(10)^n}$ is uncountable for each $n \in \mathbb{N}$.

The proof in the cases $a \leq 001$ and $a > 01$ is analogous to the previous proof for the map φ .

In the case $001 \leq a \leq 00110$, $11001 \leq b \leq 110$, let $C_{a,b} = \{x \in \Sigma_{a,b}, 01 \leq x \leq 10\}$ then $C_{a,b} \subset \Sigma_{01,10}$, and the first return map of σ to the set $C_{a,b}$ is σ^2 , that is $T_{01,10} \circ \sigma^2 \circ T_{01,10}^*$ restricted to $C_{a,b}$. Moreover, $\Sigma_{a,b} = [a, b] \cap \bigcap_{n \in \mathbb{N}} \sigma^{-n}(C_{a,b})$ so $\Sigma_{a,b}$ is uncountable if and only if $C_{a,b}$ is uncountable, and $\Sigma_{a,b}$ is infinite if and only if $C_{a,b}$ is non-empty. This gives the result. \square

Now, Theorem 1 follows from Propositions 2 and 3.

Note 2.

- (i) The map ψ, χ are discontinuous. In fact, for $a_n = 001(01)_n$ we have $\chi(a_n) = (10)_n 101001(01)_n$ and then we obtain $\lim_{n \rightarrow \infty} \chi(a_n) = 10 \neq 110010 = \chi(00101)$. Also, for $\alpha_n = 00(10)_n 11$ we have $\psi(\alpha_n) = 1100(10)_n$ and then we get $\lim_{n \rightarrow \infty} \psi(\alpha_n) = 110010 \neq 10 = \psi(0010)$.
- (ii) As a consequence of Proposition 3 ((a)–(c) and (d)) we observe that the graph of the map χ is a devil stair in $\Sigma_0 \times \Sigma_1$. In fact, let A denote the map $T_{0,01}$, B denote the map $\sigma \circ T_{10,1}$, C denote the map $0T_{01,10}$ and J be the interval $[001101, 01]$. Let $\bar{A}_0 = J$, $\bar{A}_1 = \bar{A}_0 \cup A(\bar{A}_0) \cup B(\bar{A}_0) \cup C(\bar{A}_0)$, $\bar{A}_2 = \bar{A}_1 \cup A(\bar{A}_1) \cup B(\bar{A}_1) \cup C(\bar{A}_1)$ and, inductively, $\bar{A}_{n+1} = \bar{A}_n \cup A(\bar{A}_n) \cup B(\bar{A}_n) \cup C(\bar{A}_n)$. Now it is not hard to see that $\bar{A}_\infty = \bigcup_{n=0}^{\infty} \bar{A}_n$ is a dense set in Σ_0 and $(\chi|_L)$ is constant for any interval $L \subset \bar{A}_\infty$.
 Similarly, as a consequence of Proposition 3 ((a')–(c') and (d')) we verify that $(\psi|_L)$ is constant for any interval $L \subset \bar{A}_\infty$.
- (iii) We observe that the map ψ is lower semicontinuous while the map χ is upper semicontinuous.

To complete the proof of Theorem 2 let us prove the following

Proposition 4. *Let $\chi(a) = \inf\{b \in \Sigma_1; \Sigma_{a,b} \text{ is uncountable}\}$. Then, for any $b > \chi(a)$ the topological entropy $h_{\text{top}}(\sigma|_{\Sigma_{a,b}})$ of the restriction of the shift map, σ , to $\Sigma_{a,b}$ is positive.*

Proof. Let $\tilde{b} \in \Sigma_1$ be such that $\chi(a) < \tilde{b} < b$. By the definition of $\chi(a)$, $\Sigma_{a,\tilde{b}}$ is uncountable. Let $n \in \mathbb{N}$ be such that $d(\tilde{b}, b) = 1/2^n$, and let $A = \{\alpha \in \{0, 1\}^{n+1}; \alpha \text{ appears as a subsequence of } (n+1) \text{ consecutive terms of some element } \theta \in \Sigma_{a,\tilde{b}}\}$.

Let M_A be the matrix $(a_{\alpha\beta})_{\alpha,\beta \in A}$ given by $a_{\alpha\beta} = 1$ if every subsequence of $(n+1)$ consecutive terms of $\alpha\beta$ belongs to A and $a_{\alpha\beta} = 0$ otherwise.

Let $\Sigma_A = \{\alpha_1\alpha_2\alpha_3 \cdots \mid \alpha_i \in A, a_{\alpha_i\alpha_{i+1}} = 1 \forall i \geq 1\}$ be the subshift of finite type induced by M_A . Since $\Sigma_{a,\tilde{b}}$ is invariant by σ , Σ_A is also invariant by σ , and since $\Sigma_{a,\tilde{b}}$ is uncountable, and $\Sigma_{a,\tilde{b}} \subset \Sigma_A$ then Σ_A is also uncountable and we have $h_{\text{top}}(\Sigma_A) > 0$, because Σ_A is a shift of finite type.

Notice now that $\Sigma_A \subset \Sigma_{a,b}$. In fact, for any $\theta \in \Sigma_A$ there is $\tilde{\theta} \in \Sigma_{a,\tilde{b}}$ whose first $(n+1)$ terms are the same as those of θ , so $d(\theta, \tilde{\theta}) > \frac{1}{2^{n+1}}$. Since $d(b, \tilde{b}) = \frac{1}{2^n}$, $b > \tilde{b}$ and $\tilde{b} \geq \tilde{\theta}$, we have $b > \theta$. So, we conclude that $h_{\text{top}}(\Sigma_{a,b}) \geq h_{\text{top}}(\Sigma_A) > 0$. \square

4. Hausdorff dimensions and the Lexicographical World

We will discuss in this section some results on geometrical properties of invariant sets for shifts as the sets $\Sigma_{a,b}$ and of the natural parameter space associated to them. We will study the Hausdorff dimension of such sets, which equipped with natural diadic metrics, and prove a general result of continuity. In the case of $\Sigma_{a,b}$, the Hausdorff dimension is related to the topological entropy. The continuity of the topological entropy in related cases was studied by Urbański [25] and also by Misiurewicz and Szlenk [17], among other authors.

In this section $N_n(a, b)$ will denote the number of different sequences of size n that appears as a subsequence of some element in $\Sigma_{a,b}$. Clearly, $N_{kn}(a, b) \leq (N_n(a, b))^k, \forall n, k \in \mathbb{N}$. In this situation, since $N_n(a, b)$ is an increasing function of n , the number $\lim_{n \rightarrow \infty} (\log(N_n(a, b)))/(n \cdot \log(2))$ exists and is equal to

$$\inf_{n \in \mathbb{N}^*} \frac{\log(N_n(a, b))}{n \cdot \log(2)}.$$

Indeed, given a natural number k ,

$$\limsup_{n \rightarrow \infty} \frac{\log(N_n(a, b))}{n \cdot \log(2)} \leq \limsup_{n \rightarrow \infty} \frac{\log(N_{k \cdot \lceil n/k \rceil}(a, b))}{n \cdot \log(2)} \leq \lim_{m \rightarrow \infty} \frac{m \cdot \log(N_k(a, b))}{k \cdot (m-1) \log(2)} = \frac{\log(N_k(a, b))}{k \cdot \log(2)}.$$

We will denote by $D(a, b)$ this number. Notice that $D(\underline{0}, \underline{1}) = 1$.

Definition 3. A complete shift is a subset, $\Sigma(B)$, of Σ_2 obtained by arbitrary concatenations of elements of a fixed finite set of finite words B .

Now we have the following lemma, assuming $D(a, b) > 0$ analogous to the main lemma of [18].

Lemma 3. *For any $a, b \in \Sigma_2$ with $a < \underline{0} \leq \underline{1} < b$ and $\epsilon > 0$ there are sequences $a < c < \underline{0} \leq \underline{1} < d < b$ and a complete shift contained in $\Sigma_{c,d}$ with Hausdorff dimension at least $D(a, b) - \epsilon$.*

Proof. Fix a large $n_0 \in \mathbb{N}$ and let B_{n_0} be the set of sequences of size n_0 that appear as a subsequence of some element of $\Sigma_{a,b}$. Let $N = N_{n_0}(a, b) = \#(B_{n_0})$. Without loss of generality we may assume that the number $\tilde{d} = (\log(N_{n_0}(a, b)))/(n_0 \log(2))$ is very close to $D(a, b)$, so $\lambda := 1 - \epsilon/D(a, b) < D(a, b)/\tilde{d}$. Let $k = 2N^2$. We note that the set B_{kn_0} has (at least) $2^{kn_0 D(a,b)}$ elements. An element of B_{kn_0} can be written as $\beta_1\beta_2 \cdots \beta_k$ where $\beta_i \in B_{n_0}$ for $i = 1, 2, \dots, k$.

Let $\gamma = \beta_1\beta_2 \cdots \beta_k$ be an element in B_{kn_0} . We say that $\beta_i \in B_{n_0}, 2 \leq i \leq k-1$, is *good* if there are words $\beta^{(s)} \in B_{kn_0}, s \in \{1, 2\}$, such that $\beta^{(s)} = \beta_1\beta_2 \cdots \beta_i\tilde{\beta}_{i+1}^{(s)} \cdots \tilde{\beta}_k^{(s)}$, and $\beta^{(s)} = \tilde{\beta}_1^{(s)}\tilde{\beta}_2^{(s)} \cdots \tilde{\beta}_{i-1}^{(s)}\beta_i\beta_{i+1} \cdots \beta_k$ such that $\tilde{\beta}_{i+1}^{(1)} < \beta_{i+1} < \tilde{\beta}_{i+1}^{(2)}$.

We can prove, as in [18], that at least $\frac{3}{4}k$ elements β_i of most words $\gamma \in B_{kn_0}$ are good. Indeed, we can estimate the number of sequences in B_{kn_0} for which there are (at least) $\frac{k}{4}$ positions $2 \leq i_1 < i_2 < \dots < i_{k/4} \leq (k-1)$ such that β_{i_j} is not good for $1 \leq j \leq \frac{k}{4}$ as follows: there are at most $\binom{k}{k/4} < 2^k$ choices of the i_j , $1 \leq j \leq \frac{k}{4}$, and, given a choice of i_j , $1 \leq j \leq \frac{k}{4}$, the number of sequences for which β_{i_j} is not good for $1 \leq j \leq k/4$ is bounded by $2^{k/4} N^{3k/4} \ll N^{4k/5} \ll 2^{kn_0 D(a,b)} \leq \#(B_{kn_0})$, since at the positions i_j we have at most 2 choices of β_{i_j} (the extreme options) to continue the sequence.

For each of these words, in which at least $\frac{3}{4}k = \frac{3}{2}N^2$ elements are good we can choose good elements $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{3N/2}}$ with $i_{r-1} - i_r \geq N$ for $r = 1, 2, \dots, \frac{3N}{2} - 1$. The number of such words is $N^{\rho k}$ with ρ close to 1 (and bigger than λ). We note, as in [18], that there are: a fixed set of indexes $\{i_1, i_2, \dots, i_{3N/2}\} \subset \{1, 2, \dots, k\}$ with $i_{r+1} - i_r \geq N$ for $1 \leq r < \frac{3N}{2}$ and a subset $\{\overline{\beta_{i_1}}, \overline{\beta_{i_2}}, \dots, \overline{\beta_{i_{3N/2}}}\} \subset B_{n_0}$ such that for at least

$$\frac{N^{\rho k}}{C_k^{3N/2} N^{3N/2}} \geq \frac{N^{\rho k}}{(kN)^{3N/2}} \geq N^{\rho' k}$$

(with ρ' still close to 1 and bigger than λ) elements β_1, \dots, β_k of B_{kn_0} we have $\beta_{i_r} = \overline{\beta_{i_r}}$ and it is a good element for $1 \leq r \leq \frac{3N}{2}$ (here we have used $C_r^p = \frac{r!}{p!(r-p)!} \leq r^p$).

Let us call the set of these sequences B^* . For $1 \leq r < s \leq \frac{3N}{2}$ let $\pi_{r,s} : B^* \rightarrow B_{n_0}^{s-i_r}$ be defined by

$$\pi_{r,s}(\alpha_1, \dots, \alpha_k) = (\alpha_{i_r+1}, \dots, \alpha_{i_s}).$$

If $\#\pi_{r,s}(B^*) < N^{\lambda(i_s-i_r)}$, we exclude from $\{1, 2, \dots, \frac{3N}{2}\}$ the set of indexes $\{r, r+1, \dots, s-1\}$. The total number of indexes excluded is less than $\frac{N}{2}$, provided that ρ' is close enough to 1 (otherwise B^* would not have so many elements), so there is a set of indexes $\{j_0, j_1, \dots, j_N\} \subset \{0, 1, \dots, \frac{3N}{2}\}$ which are not excluded, and there are $r < s$ such that $\overline{\beta_{i_{j_r}}} = \overline{\beta_{i_{j_s}}}$.

The promised complete shift is $\Sigma(A)$, where $A = \pi_{j_r, j_s}(B^*)$ is a set of at least $N^{\lambda(i_{j_s}-i_{j_r})}$ words of length $n_0(i_{j_s} - i_{j_r})$, so the Hausdorff dimension of $\Sigma(A)$ is at least

$$\log(N^{\lambda(i_{j_s}-i_{j_r})})/n_0(i_{j_s} - i_{j_r}) \log 2 = \lambda \cdot \log N/n_0 \log 2 > D(a, b) - \epsilon.$$

Hence, in this way, we conclude the proof of the lemma. \square

We will use this lemma in order to prove

Theorem 4. *With the diadic metric, defined in Section 2.2, for every $(a, b) \in \Sigma_0 \times \Sigma_1$,*

$$HD(\Sigma_{a,b}) = D(a, b) = \lim_{n \rightarrow \infty} \frac{\log(N_n(a, b))}{n \log(2)}$$

is a continuous function of (a, b) . We have $D(a, b) = h_{\text{top}}(\sigma|_{\Sigma_{a,b}})/\log(2)$. Moreover, $\Omega(a, b) = \{(\alpha, \beta) \in LW; a \leq \alpha < \beta \leq b\}$ and $\tilde{\Omega}(a, b) = \{(\alpha, \beta) \in \Omega(a, b); \sigma^n(\alpha) < \beta \text{ and } \alpha < \sigma^n(\beta) \forall n \in \mathbb{N}\}$ have Hausdorff dimension equal to $2D(a, b)$.

Proof. Let us notice, that a finite sequence that appears as a subsequence of some element of $\Sigma_{c,d}$ for $c < a < b < d$ for c and d arbitrarily near a and b does appear as a subsequence of some element of $\Sigma_{a,b}$ by compactness. Hence, for each $n \in \mathbb{N}$ we have $\lim_{c \uparrow a, d \downarrow b} N_n(c, d) = N_n(a, b)$. Also, for each $n \in \mathbb{N}$, the number $(\log(N_n(a, b)))/(n \log(2))$ is an upper bound for $HD(\Sigma_{a,b})$. All of this imply that $HD(\Sigma_{a,b}) = D(a, b)$; that the Hausdorff dimension is a continuous function of (a, b) and that $\Sigma_{a,b}$ can be approximated, from inside, by complete shifts with almost the same dimension.

Let us now show that $\Omega(a, b) = \{(\alpha, \beta) \in LW; a \leq \alpha < \beta \leq b\}$ and $\tilde{\Omega}(a, b) = \{(\alpha, \beta) \in \Omega(a, b); \sigma^n(\alpha) < \beta \text{ and } \alpha < \sigma^n(\beta) \forall n \in \mathbb{N}\}$ have Hausdorff dimension equal to $2D(a, b)$.

We can suppose, without loss of generality, that all the finite sequences $(\beta_1, \beta_2, \dots, \beta_k)$ that generate the referred complete shift (see previous lemma) contained in $\Sigma_{c,d}$ have a large number of elements. Let us denote by $\tilde{\Sigma}$ this complete shift. Consider the σ -invariant subshift $\bar{\Sigma} = \bigcup_{n \in \mathbb{N}} \sigma^n(\tilde{\Sigma})$. Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be the smallest (resp. the largest) element in $\bar{\Sigma}$. Take a large initial finite sequence γ (resp. $\tilde{\gamma}$) of $\tilde{\alpha}$ (resp. $\tilde{\beta}$) ending with some of the β_i . We have

$\tilde{\alpha} = \gamma\beta'\beta'\beta'\dots$ (resp. $\tilde{\beta} = \bar{\gamma}\beta''\beta''\beta''\dots$), where β' and β'' are the smallest and the largest elements in $\{\beta_1, \beta_2, \dots, \beta_k\}$ respectively. Now, let $B = \{\beta_1, \beta_2, \dots, \beta_k\} \setminus \{\beta', \beta''\}$. Hence, provided that the sequences β_i are large enough, we have that $HD(\Sigma(B)) \geq D(a, b) - 2\epsilon$, and $\Sigma_\gamma \times \Sigma_{\bar{\gamma}} \subset \tilde{\Omega}(a, b) \subset \Omega(a, b)$ where the sets $\Sigma_\gamma = \{\gamma\beta'\theta; \theta \in \Sigma(B)\}$, $\Sigma_{\bar{\gamma}} = \{\bar{\gamma}\beta''\theta; \theta \in \Sigma(B)\}$ satisfy $HD(\Sigma_\gamma) = HD(\Sigma_{\bar{\gamma}}) = HD(\Sigma(B)) \geq D(a, b) - 2\epsilon$.

In order to see that $\Sigma_\gamma \times \Sigma_{\bar{\gamma}} \subset \tilde{\Omega}(a, b)$, take any $(\alpha, \beta) \in \Sigma_\gamma \times \Sigma_{\bar{\gamma}}$. Given $n \in \mathbb{N}$, $\sigma^n(\beta) = \tau\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3\dots$ where the size of τ is smaller than the size of the β_i and $\tilde{\beta}_i \in B \cup \{\beta''\}$ for each i , so $\tilde{\beta}_i > \beta'$. In particular, $\sigma^n(\beta) > \tau\beta'\beta''\beta''\beta''\dots \geq \gamma\beta'\beta''\beta''\beta''\dots > \alpha$ (since $\gamma\beta'$ is the smallest possible initial segment of its size, which is larger than the size of $\tau\beta'$, of an element of $\bar{\Sigma}$). Analogously, $\sigma^n(\alpha) < \beta$ for each $n \in \mathbb{N}$. Similar arguments show that $\sigma^n(\alpha) \geq \alpha$ and $\sigma^n(\beta) \leq \beta$, $\forall n \in \mathbb{N}$.

Therefore, $HD(\Omega(a, b)) \geq HD(\tilde{\Omega}(a, b)) \geq 2D(a, b) - 4\epsilon$ for all $\epsilon > 0$.

On the other hand it is easy to see that $\tilde{\Omega}(a, b) \subset \Omega(a, b) \subset \Sigma_{a,b} \times \Sigma_{a,b}$ and thus $HD(\Omega(a, b)) \leq 2HD(\Sigma_{a,b}) = 2D(a, b)$, since $(\log(N_n(a, b)))/(n \log(2))$ is also an upper bound for the limit capacity of $\Sigma_{a,b}$ and converges to $D(a, b)$ as $n \rightarrow \infty$.

We may also notice that, by the arguments which precede the Lemma 3 of [25], we have $D(a, b) = HD(\Sigma_{a,b}) = h_{\text{top}}(\sigma|_{\Sigma_{a,b}})/\log(2)$, where $h_{\text{top}}(\sigma|_{\Sigma_{a,b}})$ denotes the topological entropy of the restriction of the shift to its invariant subset $\Sigma_{a,b}$. \square

Remark 4. In general the local dimension of the set $\Omega(a, b)$ in a point $(\alpha, \beta) \in LW$ is not necessarily equal to $2D(\alpha, \beta)$ but it is at most this value. For instance, when $(0\beta, 1\alpha)$ is properly renormalizable (see [6]) the local dimension at (α, β) can be smaller than $2D(\alpha, \beta)$.

We also note that the set $\tilde{\Omega}(\underline{0}, \underline{1}) = TE$ is the set of all the sequences (α, β) that satisfies the Hubbard–Sparrow conditions (see [9]); we have $HD(\tilde{\Omega}(\underline{0}, \underline{1})) = 2HD(\Sigma_{\underline{0}, \underline{1}}) = 2$. The preceding theorem implies that the set, S , sketched in Fig. 3 of [9], has the following property: given $s \in [0, 2]$, there is $P = (\alpha, \beta) \in S$ such that $HD(S \cap ([\alpha, \underline{0}]) \times [\underline{0}, \beta]) = \lim_{\epsilon \rightarrow 0} HD(S \cap B(P, \epsilon)) = s$, so the local dimension of S at a point P can be any number between 0 and 2.

Proof of the Theorem 2. Let $(a, b) \in LW$. If $b < \chi(a)$, $\Sigma_{a,b}$ is countable, so $h_{\text{top}}(\sigma|_{\Sigma_{a,b}}) = 0$. The continuity of $D(a, b) = \frac{1}{\log(2)}h_{\text{top}}(\sigma|_{\Sigma_{a,b}})$ implies that $h_{\text{top}}(\sigma|_{\Sigma_{a, \chi(a)}}) = 0$. Finally, Proposition 4 gives $h_{\text{top}}(\sigma|_{\Sigma_{a,b}}) > 0$ for each $b > \chi(a)$. \square

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References

[1] K.M. Brucks, M. Misiurewicz, Ch. Tresser, Monotonicity properties of the family of trapezoidals maps, *Comm. Math. Phys.* 137 (1991) 1–12.
 [2] D.K. Campbell, R. Galeeva, Ch. Tresser, D.J. Uherka, Piecewise linear models for the quasiperiodic transition to chaos, *Chaos* 6 (2) (1996) 121–154.
 [3] H. Furstenberg, Disjointness in ergodic theory, *Math. Systems Theory* 1 (1969) 1–49.
 [4] R. Galeeva, M. Martens, Ch. Tresser, Inducing, slopes and conjugacy classes, *Israel J. Math.* 99 (1997) 123–147.
 [5] J.M. Gambaudo, Ch. Tresser, Dynamique régulière ou chaotique. Applications du cercle ou l’intervalle ayant une discontinuité, *C. R. Acad. Soc. Paris, Sér. I* 300 (10) (1985) 311–313.
 [6] P. Glendinning, C. Sparrow, Prime and renormalisable kneading invariants and the dynamics of expanding Lorenz maps, *Physica D* 62 (1993) 22–50.
 [7] J. Guckenheimer, A strange, strange attractor, in: J.E. Marsden, M. McCracken (Eds.), *Hopf Bifurcations and its Applications*, Springer-Verlag, Berlin, 1976, pp. 368–381.
 [8] J. Guckenheimer, R.F. Williams, Structural stability of Lorenz attractors, *Publ. Math. IHES* 50 (1979) 59–72.
 [9] J.H. Hubbard, C.T. Sparrow, The classification of topologically expansive Lorenz maps, *Comm. Pure Appl. Math.* XLIII (1990) 431–443.
 [10] R. Labarca, C.G. Moreira, Bifurcations of the essential dynamics of Lorenz maps and the application to Lorenz like flows: contributions to the study of the contracting case, Preprint, 2003.
 [11] R. Labarca, C.G. Moreira, Bifurcations of the essential dynamics of Lorenz maps and the application to Lorenz like flows: contributions to the study of the expanding case, *Bol. Soc. Bras. Mat. (N.S.)* 32 (2) (2001) 107–144.

- [12] R. Labarca, C.G. Moreira, Bifurcation of the essential dynamics of Lorenz maps on the real line and the bifurcation scenario for the lineal family, *Sci. Ser. A Math. Sci. (N.S.)* 7 (2001) 13–29.
- [13] E.N. Lorenz, Deterministic non-periodic flow, *J. Atmos. Sci.* 20 (1963) 130–141.
- [14] R.S. MacKay, Ch. Tresser, Boundary of topological chaos for bimodal maps of the interval, *J. London Math. Soc. (2)* 37 (1988) 164–181.
- [15] R.S. MacKay, Ch. Tresser, Some flesh on the skeleton: the bifurcation structure of bimodal maps, *Physica D* 27 (1987) 412–422.
- [16] W. de Melo, S. Van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1993.
- [17] M. Misiurewicz, W. Szlenk, Entropy of piecewise monotone mappings, *Studia Math.* 67 (1) (1980) 45–63.
- [18] C. Moreira, Geometric properties of the Markov and Lagrange spectra, Preprint IMPA, 2004.
- [19] M.V. Otero-Espinar, Ch. Tresser, Global complexity and essential simplicity: a conjectural picture of the boundary of chaos for smooth endomorphisms of the interval, *Physica D* 39 (1989) 163–168.
- [20] D. Rand, The topological classification of Lorenz attractors, *Math. Proc. Cambridge Philos. Soc.* 83 (3) (1978) 451–460.
- [21] F. Rhodes, Kneading of Lorenz type for intervals and product spaces, *Math. Proc. Cambridge Philos. Soc.* 89 (1) (1981) 167–179.
- [22] J. Ringland, Ch. Tresser, A genealogy for finite kneading sequences of bimodal maps on the interval, *TAMS* 347 (12) (1995) 4599–4624.
- [23] Ch. Tresser, Nouveaux types de transitions vers une entropie topologique positive, *C. R. Acad. Paris, Sér. I* 296 (1983) 729–732.
- [24] Ch. Tresser, R.F. Williams, Splitting words and Lorenz braids, *Physica D* 62 (1993) 15–21.
- [25] M. Urbański, On Hausdorff dimension of invariant sets for expanding maps of a circle, *Ergodic Theory Dynam. Systems* 6 (1986) 295–309.
- [26] R.F. Williams, The structure of Lorenz attractors, in: P. Bernard, T. Ratiu (Eds.), *Turbulence Seminar Berkeley 1976/1977*, Springer-Verlag, New York, 1977, pp. 94–112.