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Stability results for obstacle problems with measure data

Résultats de stabilité pour des problèmes d'obstacle avec données mesure

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Abstract

We study the convergence properties of the solutions of some elliptic obstacle problems with measure data, under the simultaneous perturbation of the operator, the forcing term and the obstacle.

Résumé

On étudie les propriétés de convergence des solutions de problémes d'obstacles elliptiques avec données mesures, lorsque l'opérateur differentiel, les données ou les obstacles changent.

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1. Introduction

Obstacle problems when the data do not belong to the dual of the energy space have been considered in [6,5,13, 19,20], where the authors studied the notion of solution of a unilateral problem for a monotone operator $A(u) = -\operatorname{div}(a(x, \nabla u))$ acting on $W_0^{1,p}(\Omega)$, $p > 1$, when the forcing term is a bounded Radon measure μ vanishing on all sets of p-capacity zero (see Section 1 for the definition of p-capacity).

The problem we deal in this paper regards the behaviour of the obstacle problem with measure data under perturbation of the operator, of the forcing term, and of the obstacle.

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We begin with some remarks on the obstacle problem in the variational framework. For any datum $F \in W^{-1,p'}(\Omega)$ and for any function $\psi : \Omega \to \widetilde{\mathbb{R}}$, the unilateral problem relative to A, F, and the obstacle ψ (denoted by $VI(A, F, \psi)$) is the problem of finding a function u such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad u \geqslant \psi, \\ \langle A(u), v - u\rangle \geqslant \langle F, v - u\rangle, \\ \forall v \in W_0^{1,p}(\Omega), \quad v \geqslant \psi. \end{cases} \tag{1.1}$$

This problem has a unique solution whenever the convex set

$$K_\psi := \{ z \in W_0^{1,p}(\Omega) \colon z \geqslant \psi \ C_p\text{-q.e. in } \Omega \}$$

is nonempty.

In [16] (see also [8]) the authors proved some results on the convergence of variational inequalities for monotone operators, when both the operator and the obstacle are perturbed. They considered a sequence of variational inequalities $VI(A_h, F_h, \psi_h)$ and the corresponding convex sets

$$K_{\psi_h} := \{ z \in W_0^{1,p}(\Omega) \colon z \geqslant \psi_h \ C_p\text{-q.e. in } \Omega \},$$

assuming that

$$F_h \text{ converges to } F \text{ strongly in } W_0^{1,p}(\Omega),$$
$$a_h \ G\text{-converges to } a,$$
$$K_{\psi_h} \text{ converges to } K_\psi \text{ in the sense of Mosco.}$$

Denoting the solutions of $VI(A_h, F_h, \psi_h)$ and $VI(A, F, \psi)$ by u_h and u, respectively, Theorem 3.1 of [16] shows that

$$u_h \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),$$
$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \quad \text{weakly in } L^{p'}(\Omega)^N,$$
$$\int_\Omega a_h(x, \nabla u_h)\nabla u_h \, \mathrm{d}x \to \int_\Omega a(x, \nabla u)\nabla u \, \mathrm{d}x.$$

In this paper we extend the stability result stated above to the case when the forcing term μ is a bounded Radon measure which vanishes on all sets of p-capacity zero, that is to say $\mu \in \mathcal{M}_{b,0}^p(\Omega)$.

We point out that, if the forcing term $\mu \in \mathcal{M}_{b,0}^p(\Omega)$, the classical definition (1.1) given above fails. In this paper we will adopt the notion of solution considered in [22] to solve uniquely the obstacle problem (denoted by $OP(A, \mu, \psi)$), when the forcing term μ is a measure in $\mathcal{M}_{b,0}^p(\Omega)$.

We consider a sequence of obstacle problems $OP(A_h, \mu_h, \psi_h)$, when the measures μ_h vanish on sets of p-capacity zero, and we assume that

$$\mu_h(B) \to \mu(B), \quad \text{for every Borel set } B \subseteq \Omega,$$
$$a_h \ G\text{-converges to } a,$$
$$K_{\psi_h} \text{ converge to } K_\psi \text{ in the sense of Mosco.}$$

Denoting the solutions of $OP(A_h, \mu_h, \psi_h)$ and $OP(A, \mu, \psi)$ by u_h and u, respectively, we will prove in Theorem 6.1 that

$$T_j(u_h) \rightharpoonup T_j(u) \quad \text{weakly in } W_0^{1,p}(\Omega), \quad \text{for every } j > 0,$$
$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \quad \text{weakly in } L^q(\Omega)^N, \text{ for every } q < \frac{N}{N-1},$$
$$\int_\Omega a_h(x, \nabla u_h) \nabla T_j(u_h)\,\mathrm{d}x \to \int_\Omega a(x, \nabla u) \nabla T_j(u)\,\mathrm{d}x, \quad \text{for every } j > 0,$$

where $T_j(\cdot)$ is the truncation function at level j (see Section 1 for the definition).

In the special case where $a_h = a$, for every h, we obtain also that $T_j(u_h)$ converges to $T_j(u)$ strongly in $W_0^{1,p}(\Omega)$, for every $j > 0$.

Other results in the case $a_h = a$, under different hypotheses on μ_h and ψ_h, can be found in [12].

2. Assumptions and notations

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geqslant 2$. Let p be a real constant, $1 < p \leqslant N$, and let p' its dual exponent, $1/p + 1/p' = 1$.

Given two constants $c_0, c_1 > 0$ and two constants α and β, with $0 \leqslant \alpha \leqslant 1 \wedge (p-1)$ and $p \vee 2 \leqslant \beta < +\infty$, we consider the family $\mathcal{L}(c_0, c_1, \alpha, \beta)$ of Carathéodory functions $a(x, \xi): \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that:

$$\big|a(x,\xi) - a(x,\eta)\big| \leqslant c_0\big(1 + |\xi| + |\eta|\big)^{p-1-\alpha} |\xi - \eta|^\alpha, \tag{2.1}$$
$$\big(a(x,\xi) - a(x,\eta)\big)(\xi - \eta) \geqslant c_1\big(1 + |\xi| + |\eta|\big)^{p-\beta} |\xi - \eta|^\beta, \tag{2.2}$$
$$a(x, 0) = 0, \tag{2.3}$$

for almost every $x \in \Omega$, for every $\xi, \eta \in \mathbb{R}^N$.

Under the assumptions (2.1)–(2.3), the operator $A : u \mapsto -\operatorname{div}(a(x, \nabla u))$ maps $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, and for any F in $W^{-1,p'}(\Omega)$ there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of the equation

$$\begin{cases} A(u) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

since, in particular, A is coercive, continuous, bounded and strictly monotone (see [22]).

Remark 2.1. For a particular choice of the constants α and β, i.e. if $1 < p \leqslant 2$, $\alpha = p - 1$, and $\beta = 2$, the inequalities (2.1) and (2.2) become

$$\big|a(x,\xi) - a(x,\eta)\big| \leqslant c_0 |\xi - \eta|^{p-1},$$
$$\big(a(x,\xi) - a(x,\eta)\big)(\xi - \eta) \geqslant c_1\big(1 + |\xi| + |\eta|\big)^{p-2} |\xi - \eta|^2.$$

Moreover, if $2 \leqslant p < +\infty$, $\alpha = 1$, and $\beta = p$, the continuity and monotonicity assumptions (2.1) and (2.2) for the function a take the form

$$\big|a(x,\xi) - a(x,\eta)\big| \leqslant c_0\big(1 + |\xi| + |\eta|\big)^{p-2} |\xi - \eta|,$$
$$\big(a(x,\xi) - a(x,\eta)\big)(\xi - \eta) \geqslant c_1 |\xi - \eta|^p.$$

We recall that, given a compact set $K \subseteq \Omega$, its p-capacity with respect to Ω is given by

$$C_p(K) = \inf\left\{ \int_\Omega |\nabla z|^2\,\mathrm{d}x : \ z \in C_0^\infty(\Omega), \ z \geqslant \chi_K \right\},$$

where χ_K is the characteristic function of K. This definition can be extended to any open subset B of Ω in the following way:

$$C_p(B) = \sup\{C_p(K):\ K \text{ compact}, K \subseteq B\}.$$

Finally, it is possible to define the p-capacity of any set $A \subseteq \Omega$ as:

$$C_p(A) = \inf\{C_p(B):\ B \text{ open}, A \subseteq B\}.$$

A property holds C_p-quasi everywhere (abbreviated as C_p-q.e.) when it holds up to sets of p-capacity zero.

A function $v:\Omega \to \overline{\mathbb{R}}$ is C_p-quasi Borel if there exists a Borel function $u:\Omega \to \overline{\mathbb{R}}$ such that $v = u$ C_p-q.e. in Ω. A function $v:\Omega \to \overline{\mathbb{R}}$ is C_p-quasi continuous (resp. C_p-quasi upper semicontinuous) if, for every $\varepsilon > 0$ there exists a set E such that $C_p(E) < \varepsilon$ and $v_{|\Omega\setminus E}$ is continuous (resp. upper semicontinuous) in $\Omega \setminus E$. Thus, every C_p-quasi continuous (resp. C_p-quasi upper semicontinuous) v is a C_p-quasi Borel function.

A function $u \in W_0^{1,p}(\Omega)$ always has a C_p-quasi continuous representative, which is uniquely defined (and finite) up to a set of p-capacity zero. In the sequel we shall always identify u with its C_p-quasi continuous representative, so that the pointwise values of u are defined C_p-quasi everywhere.

A set $E \subseteq \Omega$ is said to be C_p-quasi open if for every $\varepsilon > 0$ there exists an open set U such that $E \subseteq U \subseteq \Omega$ and $C_p(U \setminus E) \leqslant \varepsilon$.

Let $\mathcal{M}_b(\Omega)$ the space of Radon measures μ on Ω whose total variation $|\mu|$ is bounded on Ω, while $\mathcal{M}_{b,0}^p(\Omega)$ is the special subspace of $\mathcal{M}_b(\Omega)$ of all measures, which are absolutely continuous with respect to the p-capacity, that is a measure $\mu \in \mathcal{M}_b(\Omega)$ belongs to $\mathcal{M}_{b,0}^p(\Omega)$ if and only if $\mu(A) = 0$ for every Borel set $A \subseteq \Omega$ such that $C_p(A) = 0$. As usual, we identify $\mathcal{M}_b(\Omega)$ with the dual of the Banach space $C_0(\Omega)$ of continuous functions that are zero on the boundary; so that the duality is $\langle \mu, u\rangle = \int_\Omega u\,\mathrm{d}\mu$, for every u in $C_0(\Omega)$ and the norm is $\|\mu\|_{\mathcal{M}_b(\Omega)} = |\mu|(\Omega)$. Moreover, we denote the positive cones of $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_{b,0}^p(\Omega)$ by $\mathcal{M}_b^+(\Omega)$ and $\mathcal{M}_{b,0}^{p,+}(\Omega)$, respectively.

It is well known that, if μ belongs to $W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$, then μ is in $\mathcal{M}_{b,0}^p(\Omega)$, every u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is summable with respect to μ and

$$\langle \mu, u\rangle = \int_\Omega u\,\mathrm{d}\mu,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, while in the right-hand side u denotes the C_p-quasi continuous representative and, consequently, the pointwise values of u are defined μ-almost everywhere.

For every $j > 0$ we define the truncation function $T_j:\mathbb{R} \mapsto \mathbb{R}$ by

$$T_j(t) = \begin{cases} t & \text{if } |t| \leqslant j, \\ j\,\mathrm{sign}(t) & \text{if } |t| > j. \end{cases}$$

Let us consider the space $\mathcal{T}_0^{1,p}(\Omega)$ of all functions $u:\Omega \mapsto \overline{\mathbb{R}}$ which are almost everywhere finite and such that $T_j(u) \in W_0^{1,p}(\Omega)$ for every $j > 0$. It is easy to see that every function $u \in \mathcal{T}_0^{1,p}(\Omega)$ has a C_p-quasi continuous representative with values in $\overline{\mathbb{R}}$, that will always be identified with the function u. Moreover, for every $u \in \mathcal{T}_0^{1,p}(\Omega)$ there exists a measurable function $\Phi:\Omega \mapsto \mathbb{R}^N$ such that $\nabla T_j(u) = \Phi\chi_{\{|u|\leqslant j\}}$ a.e. in Ω (see Lemma 2.1 in [3]). This function Φ, which is unique up to almost everywhere equivalence, will be denoted by ∇u. Note that ∇u coincides with the distributional gradient of u whenever

$$u \in \mathcal{T}_0^{1,p}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \nabla u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N).$$

3. Entropy solutions and obstacle problems

We are now in position to recall the notion of entropy solution introduced in [3] for L^1 data and extended to measures in $\mathcal{M}^p_{b,0}(\Omega)$ in [7], which ensures us that, when $\mu \in \mathcal{M}^p_{b,0}(\Omega)$, the equation

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

has a unique entropy solution.

We point out that the theory of entropy solutions works for general Carathéodory functions $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that, for almost every $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$,

$$\left|a(x,\xi)\right| \leqslant c_2\left[k(x) + |\xi|^{p-1}\right], \tag{3.2}$$

$$a(x,\xi)\cdot\xi \geqslant c_3|\xi|^p - g(x), \tag{3.3}$$

$$\big(a(x,\xi) - a(x,\eta)\big)(\xi - \eta) > 0, \tag{3.4}$$

$$a(x,0) = 0, \tag{3.5}$$

where c_2 and c_3 are two positive real constants, g is a nonnegative function in $L^1(\Omega)$ and k is a nonnegative function in $L^{p'}(\Omega)$ (see Remark 2.4 of [19]).

We note that, if $a \in \mathcal{L}(c_0, c_1, \alpha, \beta)$, these conditions are satisfied, with g and k replaced by positive real constants depending on c_0, c_1, α, and β.

Definition 3.1. Let $\mu \in \mathcal{M}^p_{b,0}(\Omega)$. A function u is an entropy solution of problem (3.1) if u belongs to $\mathcal{T}^{1,p}_0(\Omega)$, and

$$\int_\Omega a(x,\nabla u)\nabla T_j(u-\varphi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u-\varphi)\,\mathrm{d}\mu, \tag{3.6}$$

for every φ in $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and every $j > 0$.

Remark 3.2. If $F \in W^{-1,p'}(\Omega)$ we can consider as data also $\mu + F$, the definition of entropy solution being

$$\int_\Omega a(x,\nabla u)\nabla T_j(u-\varphi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u-\varphi)\,\mathrm{d}\mu + \big\langle F, T_j(u-\varphi)\big\rangle. \tag{3.7}$$

Remark 3.3. Actually, it is possible to prove that equality holds in (3.6) and (3.7) (see [21]).

Remark 3.4. Using $\varphi = 0$ in (3.7), by (3.3) and by Young's inequality, we easily get

$$\int_\Omega \left|\nabla T_j(u)\right|^p \leqslant c(j+1), \quad \forall j > 0 \tag{3.8}$$

where the constant c depends on $\|\mu\|_{\mathcal{M}_b(\Omega)}$, $\|F\|_{W^{-1,p'}(\Omega)}$, p, c_3, and $\|g\|_{L^1(\Omega)}$.

By standard arguments of capacity theory, (3.8) implies

$$C_p\big(\{|u| > j\}\big) \leqslant \frac{c(j+1)}{j^p}; \tag{3.9}$$

that is, if u is the entropy solution of (3.1) relative to $\mu + F$, then (the C_p-quasi continuous representative of) u is finite up to a set of capacity zero.

Now, using $\varphi = T_i(u)$ in (3.7), by (3.9) we get

$$\lim_{i\to+\infty} \int\limits_{\{i<|u|\leqslant i+j\}} |\nabla u|^p \,\mathrm{d}x = 0. \tag{3.10}$$

Remark 3.5. By means of (3.8), we can apply Lemma 4.2 of [3] which implies that, for every $1 < q < \frac{N}{N-1}$, $|\nabla u|^{p-1}$ is bounded in $L^q(\Omega)$ by some constant depending only by N and c. Moreover, the procedure used in [7] to obtain the entropy formulation (3.7), combined with the uniqueness of u, allows to prove that, for every $\Phi \in W_0^{1,q'}(\Omega)$, with $1 < q < \frac{N}{N-1}$,

$$\int\limits_\Omega a(x,\nabla u)\nabla\Phi \,\mathrm{d}x = \int\limits_\Omega \Phi \,\mathrm{d}\mu + \langle F, \Phi\rangle, \tag{3.11}$$

as well as

$$\int\limits_\Omega a(x,\nabla u)\nabla\big(T_j(u-\varphi)\phi\big)\,\mathrm{d}x = \int\limits_\Omega T_j(u-\varphi)\phi \,\mathrm{d}\mu + \big\langle F, T_j(u-\varphi)\phi\big\rangle, \tag{3.12}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and for every $\phi \in C^1(\overline{\Omega})$. Let us observe that if $q < \frac{N}{N-1}$, then $q' > N$, so that, by Sobolev embedding theorems, $W_0^{1,q'}(\Omega) \subseteq C(\overline{\Omega})$.

We recall also the following stability result (see Theorem 1.2 in [21] and Remark 3.3 in [19]):

Theorem 3.6. *Let $\mu_h \in \mathcal{M}^p_{b,0}(\Omega)$ and $F_h \in W^{-1,p'}(\Omega)$ be such that*

$$\mu_h \to \mu \quad \textit{strongly in } \mathcal{M}_b(\Omega), \tag{3.13}$$

$$F_h \to F \quad \textit{strongly in } W^{-1,p'}(\Omega); \tag{3.14}$$

let u_h be the entropy solutions of (3.1) *relative to $\mu_h + F_h$, and let u be the entropy solution of* (3.1) *relative to $\mu + F$. Then*

$$\lim_{h\to\infty} T_j(u_h) = T_j(u) \quad \textit{strongly in } W_0^{1,p}(\Omega),$$

for every $j > 0$.

Before specifying the notion of solution we will adopt in this paper in order to study obstacle problems when the forcing term is a measure, we want to mention here these two facts, concerning the solution u of $VI(A,F,\psi)$, when $F \in W^{-1,p'}(\Omega)$.

Characterization 1. The solution u can be characterized (see, e.g., Chapters II and III in [18]) as the smallest function in $W_0^{1,p}(\Omega)$, greater than or equal to ψ, such that

$$\begin{cases} A(u) - F = \lambda & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

for some nonnegative element λ of $W^{-1,p'}(\Omega)$.

Characterization 2. Finally, when the obstacle ψ is C_p-quasi upper semicontinuous u, is also characterized (see, e.g., Theorem 3.2 in [1]) by the complementarity system

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad u \geqslant \psi,\\ A(u) = F + \lambda,\\ \lambda \in W^{-1,p'}(\Omega), \quad \lambda \geqslant 0,\\ \lambda(\{u-\psi > 0\}) = 0, \end{cases} \tag{3.16}$$

where the pointwise values of u are defined C_p-quasi everywhere. Since λ is a nonnegative element of $W^{-1,p'}(\Omega)$, by the Riesz Representation Theorem, it is a nonnegative Radon measure; this explains the meaning of the last line of (3.16), which can be written also as $u=\psi$ λ-almost everywhere in Ω.

Let us observe that without loss of generality we may suppose that ψ is C_p-quasi upper semicontinuous thanks to the following proposition (see Proposition 1.5 in [15]).

Proposition 3.7. *Let $\psi:\Omega\mapsto\overline{\mathbb{R}}$, with K_ψ nonempty. Then there exists a C_p-quasi upper semicontinuous function $\hat{\psi}:\Omega\mapsto\overline{\mathbb{R}}$ such that*:

(1) *$\hat{\psi}\geqslant\psi$ C_p-q.e. in Ω;*
(2) *if $\varphi:\Omega\mapsto\overline{\mathbb{R}}$ is C_p-quasi upper semicontinuous and $\varphi\geqslant\psi$ C_p-q.e. in Ω, then $\varphi\geqslant\hat{\psi}$ C_p-q.e. in Ω.*

Thus, in particular, $K_\psi=K_{\hat{\psi}}$.

Besides, let us observe that if p is greater than the dimension N of the ambient space, then it is easily seen, by Sobolev embedding and duality arguments, that the space $\mathcal{M}_b(\Omega)$ is a subset of $W^{-1,p'}(\Omega)$, so that existence, uniqueness, and continuous dependence of solutions in $W_0^{1,p}(\Omega)$ to the obstacle problem was studied as part of the theory of the variational inequality (1.1).

In [19] the following definition for unilateral problems with measure data was introduced.

Definition 3.8. We say that u is the solution of the Obstacle Problem with datum $\mu\in\mathcal{M}_{b,0}^p(\Omega)$ and obstacle ψ if

(1) there exists a measure $\lambda\in\mathcal{M}_{b,0}^{p,+}(\Omega)$ such that u is the entropy solution of

$$\begin{cases} A(u)=\mu+\lambda & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

and $u\geqslant\psi$ C_p-quasi everywhere in Ω.

(2) for any $\nu\in\mathcal{M}_{b,0}^{p,+}(\Omega)$ such that the entropy solution v of (3.17) relative to $\mu+\nu$ satisfies $v\geqslant\psi$ C_p-quasi everywhere in Ω, we have $u\leqslant v$ C_p-q.e. in Ω.

By definition, it is clear that, if such a solution exists, it is unique.

The nonnegative measure λ, which is uniquely defined, will be called the obstacle reaction relative to u, or the measure associated with it.

The only restriction required on the choice of the obstacle is that there exists a measure $\rho\in\mathcal{M}_b(\Omega)\cap W^{-1,p'}(\Omega)$ such that the solution u_ρ of

$$\begin{cases} A(u_\rho)=\rho & \text{in } \Omega,\\ u_\rho=0 & \text{on } \partial\Omega \end{cases}$$

is such that

$$\psi\leqslant u_\rho \quad C_p\text{-q.e. in } \Omega. \tag{3.18}$$

The following theorem was proved in [19].

Theorem 3.9. *Let ψ satisfy* (3.18) *and let $\mu\in\mathcal{M}_{b,0}^p(\Omega)$. Then there exists a unique solution of $OP(A,\mu,\psi)$. Moreover the corresponding obstacle reaction λ satisfies*

$$\|\lambda\|_{\mathcal{M}_b(\Omega)}\leqslant\left\|(\mu-\rho)^-\right\|_{\mathcal{M}_b(\Omega)}. \tag{3.19}$$

The solution found can be characterized by the complementarity system.

Theorem 3.10. *Let μ be in $\mathcal{M}^p_{b,0}(\Omega)$ and ψ satisfy* (3.18); *then the following statements are equivalent*:

(1) *u is the solution of $OP(A,\mu,\psi)$ and λ is the associated obstacle reaction*;
(2) *$u \geqslant \psi$ C_p-q.e. in Ω, $\lambda \in \mathcal{M}^{p,+}_{b,0}(\Omega)$, u is the entropy solution of* (3.17) *relative to $\mu+\lambda$, and*

$$\begin{cases} \int_\Omega T_k(u-\varphi)\,\mathrm{d}\lambda \leqslant \int_\Omega T_k(v-\varphi)\,\mathrm{d}\lambda, \\ \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \\ \forall v \in \mathcal{T}^{1,p}_0(\Omega),\ v \geqslant \psi \quad C_p\text{-q.e. in } \Omega; \end{cases} \tag{3.20}$$

(3) *$u \geqslant \psi$ C_p-q.e. in Ω, $\lambda \in \mathcal{M}^{p,+}_{b,0}(\Omega)$, u is the entropy solution of* (3.17) *relative to $\mu+\lambda$, and*

$$u = \psi \quad \lambda\text{-a.e. in } \Omega. \tag{3.21}$$

Remark 3.11. Observe that if ψ is C_p-q.e. upper bounded, we can consider in (3.20) $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, $\varphi \geqslant \psi$ C_p-q.e. in Ω and $v = \varphi$, so that, taking into account that u is the entropy solution of (3.17) relative to $\mu+\lambda$, u satisfies

$$\int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi)\,\mathrm{d}x \leqslant \int_\Omega T_k(u-\varphi)\,\mathrm{d}\mu, \tag{3.22}$$

which is quite similar to the usual variational formulation. Formula (3.22) was just obtained in [5] when the datum μ is a function in $L^1(\Omega)$. In that paper L. Boccardo and G.R. Cirmi proved also that formulation (3.22) characterizes uniquely the function u. In the same way this can be done also when $\mu \in \mathcal{M}^p_{b,0}(\Omega)$.

4. *G*-convergence, Mosco-convergence, and weak convergence in $\mathcal{M}_b(\Omega)$

The study of the properties of the solutions to the obstacle problems under perturbations of the operator a is based on a notion of convergence in $\mathcal{L}(c_0,c_1,\alpha,\beta)$, called G-convergence.

Definition 4.1. We say that a sequence of functions $a_h(x,\xi)$ belonging to $\mathcal{L}(c_0,c_1,\alpha,\beta)$ G-converges to a function $a(x,\xi)$ satisfying the same hypotheses (possibly with different constants $\tilde{c}_0,\tilde{c}_1,\tilde{\alpha},\tilde{\beta}$) if for any $F \in W^{-1,p'}(\Omega)$, the solution u_h of

$$\begin{cases} A_h(u_h) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

satisfies

$$u_h \rightharpoonup u \quad \text{weakly in } W^{1,p}_0(\Omega) \tag{4.2}$$

and

$$a_h(x,\nabla u_h) \rightharpoonup a(x,\nabla u) \quad \text{weakly in } L^{p'}(\Omega)^N, \tag{4.3}$$

where u is the unique solution of (2.4).

The following theorem justifies the definition of G-convergence.

Theorem 4.2. *Any sequence $a_h(x,\xi)$ of functions belonging to $\mathcal{L}(c_0,c_1,\alpha,\beta)$ admits a subsequence which G-converges to a function $a(x,\xi) \in \mathcal{L}(\tilde{c}_0,\tilde{c}_1,\frac{\alpha}{\beta-\alpha},\beta)$, where $\tilde{c}_0,\tilde{c}_1$ depend only on N,p,α,β,c_0,c_1.*

This compactness theorem was obtained by L. Tartar (see [24] and Theorem 1.1 of [17]) in the case of nonlinear monotone operators defined from $\mathrm{H}_0^1(\Omega)$ into $\mathrm{H}^{-1}(\Omega)$, when $p=2$ and the functions $a_h \in \mathcal{L}(c_0, c_1, 1, 2)$, and then extended in the version of Theorem 4.2 in [11] (see Theorem 4.1).

The investigations of the properties of obstacle problems when the obstacle varies relies on a notion of convergence for sequences of convex sets introduced by U. Mosco in [23].

Definition 4.3. Let K_h be a sequence of subsets of a Banach space X. The strong lower limit

$$s\text{-}\liminf_{h\to+\infty} K_h$$

of the sequence K_h is the set of all $v \in X$ such that there exists a sequence $v_h \in K_h$, for h large, converging to v strongly in X.

The weak upper limit

$$w\text{-}\limsup_{h\to+\infty} K_h$$

of the sequence K_h is the set of all $v \in X$ such that there exists a sequence v_k converging to v weakly in X and a sequence of integers h_k converging to $+\infty$, such that $v_k \in K_{h_k}$.

The sequence K_h converges to the set K in the sense of Mosco, shortly $K_h \xrightarrow{M} K$, if

$$s\text{-}\liminf_{h\to+\infty} K_h = w\text{-}\limsup_{h\to+\infty} K_h = K.$$

Mosco proved that this type of convergence is the right one for the stability of variational inequalities with respect to obstacles. This is the main theorem of his theory.

Theorem 4.4. *Let K_{ψ_h} and K_ψ be nonempty. Then*

$$K_{\psi_h} \xrightarrow{M} K_\psi$$

if and only if, for any $F \in W^{-1,p'}(\Omega)$,

$$u_h \to u \quad \text{strongly in } W_0^{1,p}(\Omega),$$

where u_h and u are the solutions of $VI(A, F, \psi_h)$ and $VI(A, F, \psi)$, respectively.

Several stability results can be proved as corollaries of this theorem by Mosco. In particular, the strong convergence

$$\psi_h \to \psi \quad \text{strongly in } W_0^{1,p}(\Omega)$$

easily implies the convergence of K_{ψ_h} to K_ψ in the sense of Mosco, but the weak convergence

$$\psi_h \rightharpoonup \psi \quad \text{weakly in } W^{1,r}(\Omega),\ r > p,$$

also implies the same result (see [9,1]). Moreover, if

$$\psi_h \leqslant \psi \quad C_p\text{-q.e. in } \Omega, \qquad \psi_h \to \psi \quad C_p\text{-q.e. in } \Omega,$$

then K_{ψ_h} converges to K_ψ in the sense of Mosco.

A necessary and sufficient condition for the convergence of K_{ψ_h}, expressed in terms of the convergence of the C_p-capacity of the level sets $\{x \in \Omega\colon\ \psi_h(x) > t\}$ has been given in [15].

Remark 4.5. It has been proved in [15] that if K_{ψ_h} converges to K_ψ in the sense of Mosco, then also $K_{T_i(\psi_h)}$ converges to $K_{T_i(\psi)}$ in the sense of Mosco, for every $i > 0$.

We recall now some properties of the $*$-weak and the weak convergence of measures in $\mathcal{M}_b(\Omega)$.

Definition 4.6. If $\mu_h, \mu \in \mathcal{M}_b(\Omega)$, we say that μ_h converges to μ $*$-weakly in $\mathcal{M}_b(\Omega)$ if

$$\lim_{h\to+\infty} \int_\Omega u\,\mathrm{d}\mu_h = \int_\Omega u\,\mathrm{d}\mu,$$

for every $u \in C_0(\Omega)$.

For nonnegative measures we have a characterization of the $*$-weak convergence in terms of convergence of sets.

Proposition 4.7. *Given $\mu_h, \mu \in \mathcal{M}_b^+(\Omega)$, the following conditions are equivalent*:

(1) *μ_h converges to μ $*$-weakly in $\mathcal{M}_b(\Omega)$;*
(2) *$\mu(A) \leqslant \liminf_{h\to+\infty} \mu_h(A)$, for every A open subset of Ω, $\mu(K) \geqslant \limsup_{h\to+\infty} \mu_h(K)$, for every K compact subset of Ω.*

Concerning the weak convergence in $\mathcal{M}_b(\Omega)$, the following result shows that it is stronger than the $*$-weak one.

Proposition 4.8. *Given $\mu_h, \mu \in \mathcal{M}_b(\Omega)$, the following conditions are equivalent*:

(1) *μ_h converges to μ weakly in $\mathcal{M}_b(\Omega)$;*
(2) *$\lim_{h\to+\infty}\mu_h(B) = \mu(B)$, for every Borel set B contained in Ω.*

The proof of this result (see, e.g., Theorem 6.6 in [2]) relies on the Vitali–Hahn–Sacks Theorem (see, e.g., Theorem 6.4 in [2]), which is similar to the Banach–Steinhaus uniform boundedness theorem and gives a useful condition for the equiintegrability of a sequence of summable functions.

Theorem 4.9. *Let ν be a measure in $\mathcal{M}_b(\Omega)$, let g_h be a sequence in $L^1(\Omega, \nu)$ and set $\mu_h = g_h\nu$. Assume that, for every Borel set $B \subseteq \Omega$, the $\lim_{h\to+\infty} \mu_h(B)$ exists and is finite*; *then g_h is equiintegrable.*

In the last part of this section we give a weak notion of convergence in capacity, similar to that one considered in [10], and some properties related to it.

Definition 4.10. Let $u_j, u : \Omega \to \mathbb{R}$ be C_p-quasi Borel functions. We say that u_j converges to u weakly in capacity if, for every measure $\mu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$, u_j converges to u in μ-measure, i.e.,

$$\lim_{j\to+\infty} \mu\big(\big\{x \in \Omega\colon \big|u_j(x) - u(x)\big| > \varepsilon\big\}\big) = 0, \tag{4.4}$$

for every $\varepsilon > 0$.

The following proposition (see Proposition 3.5 in [10]) shows the relationship between weak convergence in $W_0^{1,p}(\Omega)$ and weak convergence in capacity.

Proposition 4.11. *Let $u_j, u \in W_0^{1,p}(\Omega)$ be such that u_j converges weakly to u in $W_0^{1,p}(\Omega)$. Then u_j converges to u weakly in capacity.*

Remark 4.12. Actually, Definition 4.10 is not equivalent to Definition 3.1 of [10], where the measures μ are positive elements of $W^{-1,p'}(\Omega)$, hence positive Radon measures (not bounded), and the convergence in μ-measure is only local. However, it is easy to check that (4.4) turns out to be equivalent to the condition considered in Definition 3.1 of [10], when $\mu \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$. On the other hand, for every measure $\mu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$ there exists a nonnegative measure $\gamma \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$ and a nonnegative Borel measurable function $g \in L^1(\Omega, \gamma)$ such that $\mu(A) = (g\gamma)(A)$ for every C_p-quasi open subset A of Ω (see Theorem 2.2 in [14]). Hence, nothing essential changes in the proof of Proposition 3.5 in [10], when μ belongs to $\mathcal{M}_{b,0}^{p,+}(\Omega)$.

5. Preliminary results

Actually, Theorem 3.6 can be improved in the following way.

Theorem 5.1. *Let $\mu_h \in \mathcal{M}_{b,0}^p(\Omega)$ and $F_h \in W^{-1,p'}(\Omega)$ be such that*

$$\mu_h \rightharpoonup \mu \quad \text{weakly in } \mathcal{M}_b(\Omega), \tag{5.1}$$

$$F_h \to F \quad \text{strongly in } W^{-1,p'}(\Omega); \tag{5.2}$$

let u_h be the entropy solutions of (3.1) *relative to $\mu_h + F_h$, and let u be the entropy solution of* (3.1) *relative to $\mu + F$. Then*

$$\lim_{h\to\infty} T_j(u_h) = T_j(u) \quad \text{strongly in } W_0^{1,p}(\Omega),$$

for every $j > 0$.

Since the proof of Theorem 5.1 can be obtained by following the same scheme of the proof of Theorem 3.6, we have to enter into details only when assumption (5.1), instead of (3.13), requires some modifications. As a matter of fact, it is enough to prove here the following lemma, by which we deduce the strong convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h)$ to $T_j(u)$, exactly as in the proof of Theorem 3.6.

Lemma 5.2. *Let $\mu_h, \mu \in \mathcal{M}_{b,0}^p(\Omega)$ be such that μ_h converges to μ weakly in $\mathcal{M}_b(\Omega)$. Let $\Phi_h, \Phi \in W_0^{1,p}(\Omega) \cap$ $L^\infty(\Omega)$ be such that $\sup_h \|\Phi_h\|_{L^\infty(\Omega)}$ is bounded and Φ_h converges to Φ weakly in $W_0^{1,p}(\Omega)$. Then*

$$\lim_{h\to+\infty} \int_\Omega \Phi_h \, \mathrm{d}\mu_h = \int_\Omega \Phi \, \mathrm{d}\mu.$$

Proof. We define the measure $\nu \in \mathcal{M}_{b,0}^{p,+}(\Omega)$ as

$$\nu := \sum_{h=1}^\infty \frac{1}{2^h} \frac{|\mu_h|}{|\mu_h|(\Omega)},$$

so that $|\mu_h| \ll \nu$. This implies that $\mu_h = g_h \nu$, with $g_h \in L^1(\Omega, \nu)$; on the other hand, thanks to Proposition 4.8, we have that $\mu_h(B)$ tends to $\mu(B)$, for every Borel set $B \subseteq \Omega$. Applying Theorem 4.9 we deduce that the sequence g_h is equiintegrable, and, in conclusion, it converges to a function g weakly in $L^1(\Omega, \nu)$, with $\mu = g\nu$.

Now, we can prove that $\int_\Omega \Phi_h \,d\mu_h$ tends to $\int_\Omega \Phi \,d\mu$, when Φ_h, Φ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $\sup_h \|\Phi_h\|_{L^\infty(\Omega)} < +\infty$, and Φ_h converges to Φ weakly in $W_0^{1,p}(\Omega)$. By Proposition 4.11, indeed, the convergence of Φ_h to Φ is, in particular, in ν-measure. At this point, it is easy to obtain that

$$\lim_{h\to+\infty} \int_\Omega \Phi_h \,d\mu_h = \lim_{h\to\infty} \int_\Omega \Phi_h g_h \,d\nu = \int_\Omega \Phi g \,d\nu = \int_\Omega \Phi \,d\mu. \qquad \square$$

Proposition 5.3. *Assume* (3.3), (3.2), (3.4), *and* (3.5). *Let $\mu \in \mathcal{M}_{b,0}^p(\Omega)$, and let u be the entropy solution of* (3.1). *Then, for every $z \in W_0^{1,p}(\Omega)$ the function $u - z$ belongs to $\mathcal{T}_0^{1,p}(\Omega)$; more precisely, for every $j > 0$, we have*:

$$\left\|T_j(u-z)\right\|^p_{W_0^{1,p}(\Omega)} \leqslant c(j+1), \tag{5.3}$$

where the constant c depends only on $\|\mu\|_{\mathcal{M}_b(\Omega)}$, $\|z\|_{W_0^{1,p}(\Omega)}$, p, c_3, and $\|g\|_{L^1(\Omega)}$.

Proof. Let us consider a sequence $\mu_n \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega)$ such that μ_n converges to μ strongly in $\mathcal{M}_b(\Omega)$. Denoting the variational solution of the problem (3.1) relative to μ_n by u_n, we know that u_n tends to u in the sense of Theorem 3.6. If $z \in W_0^{1,p}(\Omega)$, define the operator $B(v) = -\operatorname{div}(a(x, \nabla v + \nabla z) - a(x, \nabla z))$, which satisfies (3.3), (3.2), (3.4), and (3.5) with different coercitivity and growth parameters depending by c_2, c_3, k, g, p, and z. Let v_n be the solution of

$$\begin{cases} B(v) = \mu_n - A(z) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega; \end{cases} \tag{5.4}$$

that is

$$\langle B(v_n), w\rangle = \langle \mu_n - A(z), w\rangle,$$

or, equivalently

$$\int_\Omega a(x, \nabla v_n + \nabla z)\nabla w \,dx = \langle \mu_n, w\rangle,$$

for every $w \in W_0^{1,p}(\Omega)$. By the uniqueness of the solution of (5.4), it follows that $u_n = v_n + z$, and, since v_n tends to the entropy solution v of the problem (5.4) relative to $\mu - Az$ (see Theorem 3.6) we obtain that $u = v + z$.

At this point, the result follows by (3.8). $\square$

Remark 5.4. By the previous proposition we deduce also that if $z_n, z \in W_0^{1,p}(\Omega)$, with z_n converging to z weakly in $W_0^{1,p}(\Omega)$, then, for every $j > 0$, $T_j(u - z_n)$ converges to $T_j(u - z)$ weakly in $W_0^{1,p}(\Omega)$, where u is the entropy solution of (3.1) relative to $\mu \in \mathcal{M}_{b,0}^p(\Omega)$.

6. Convergence results

The problem we deal with in this section regards the behaviour of the obstacle problems in the sense of Definition 3.8 under perturbations of the operator A, of the right hand side μ, and of the obstacle ψ.

We consider a sequence a_h of functions in $\mathcal{L}(c_0, c_1, \alpha, \beta)$, a sequence of measures $\rho_h \in \mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$, and the variational solution $u_{\rho_h}^{A_h}$ of

$$\begin{cases} A_h(u_{\rho_h}^{A_h}) = \rho_h & \text{in } \Omega, \\ u_{\rho_h}^{A_h} \in W_0^{1,p}(\Omega). \end{cases}$$

We assume that

$$\sup_h \|\rho_h\|_{\mathcal{M}_b(\Omega)} < +\infty \tag{6.1}$$

and that the function ψ_h satisfies:

$$\psi_h \leqslant u_{\rho_h}^{A_h} \quad C_p\text{-q.e. in } \Omega. \tag{6.2}$$

Moreover we suppose that

$$\psi \leqslant 0 \quad C_p\text{-q.e. in } \Omega. \tag{6.3}$$

We can now state the main result of this section.

Theorem 6.1. *Let a_h be a sequence in $\mathcal{L}(c_0, c_1, \alpha, \beta)$, which G-converges to a function a, and let A_h and A be the operators associated to a_h and a, respectively. Let us assume* (6.1), (6.2), *and* (6.3), *with K_{ψ_h} converging to K_ψ in the sense of Mosco. Finally, consider $\mu_h, \mu \in \mathcal{M}_{b,0}^p(\Omega)$, with μ_h converging to μ weakly in $\mathcal{M}_b(\Omega)$. Then the solutions u_h and u of the obstacle problems $OP(A_h, \mu_h, \psi_h)$ and $OP(A, \mu, \psi)$, respectively, satisfy*

$$T_j(u_h) \rightharpoonup T_j(u) \quad \textit{weakly in } W_0^{1,p}(\Omega), \quad \textit{for every } j > 0, \tag{6.4}$$

$$a_h(x, \nabla u_h) \rightharpoonup a(x, \nabla u) \quad \textit{weakly in } L^q(\Omega)^N, \quad \textit{for every } q < \frac{N}{N-1}, \tag{6.5}$$

$$\int_\Omega a_h(x, \nabla u_h)\nabla T_j(u_h)\,\mathrm{d}x \to \int_\Omega a(x, \nabla u)\nabla T_j(u)\,\mathrm{d}x, \quad \textit{for every } j > 0. \tag{6.6}$$

Remark 6.2. By formal modifications we can prove Theorem 6.1 replacing (6.3) with (3.18) and

$$\psi \leqslant M \quad C_p\text{-q.e. in } \Omega,$$

where M is a positive constant.

Proof of Theorem 6.1. To simplify the exposition, it is convenient to divide the proof into various steps.

Step 1. We will prove (6.4).

Proof of Step 1. Let us recall that the solution u_h of the obstacle problem $OP(A_h, \mu_h, \psi_h)$ is the entropy solution of Eq. (4.1) relative to $\mu + \lambda_h$, i.e., for every $\varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, u_h satisfies:

$$\int_\Omega a_h(x, \nabla u_h)\nabla T_j(u_h - \varphi)\,\mathrm{d}x = \int_\Omega T_j(u_h - \varphi)\,\mathrm{d}\mu_h + \int_\Omega T_j(u_h - \varphi)\,\mathrm{d}\lambda_h, \tag{6.7}$$

where the obstacle reaction $\lambda_h \in \mathcal{M}_{b,0}^{p,+}(\Omega)$ satisfies (3.19), i.e.

$$\|\lambda_h\|_{\mathcal{M}_b(\Omega)} \leqslant \left\|(\mu_h - \rho_h)^-\right\|_{\mathcal{M}_b(\Omega)}.$$

Combining the previous estimate with (6.1) and (3.8), we obtain that, for every $j > 0$,

$$\int_\Omega \left|\nabla T_j(u_h)\right|^p \mathrm{d}x \leqslant cj, \tag{6.8}$$

where the constant c does not depend on j and h. Working as in the proof of Theorem 6.1 of [3] we have that there exists a subsequence of u_h (still denoted by u_h) and a function $u^* \in \mathcal{T}_0^{1,p}(\Omega)$ such that u_h converges to u^* a.e. in Ω and, for every $j > 0$, $T_j(u_h)$ converges to $T_j(u^*)$ weakly in $W_0^{1,p}(\Omega)$. Since also $K_{T_j(\psi_h)}$ converges to $K_{T_j(\psi)}$ in the sense of Mosco (see Remark 4.5), by the weakly convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h)$ to $T_j(u^*)$ we deduce that $T_j(u^*) \geqslant T_j(\psi)$ C_p-q.e. in Ω, for every $j > 0$, so that $u^* \geqslant \psi$ C_p-q.e. in Ω. Furthermore, by the weak convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h)$ to $T_j(u)$, we obtain that also $T_j(u^*)$ satisfies (6.8), which implies (3.9) for u^*.

Let us consider a function $\Phi \in W_0^{1,p}(\Omega)$, with $\Phi \geqslant \psi$, and the solution w_h of the variational inequality $VI(A_h, A(\Phi), \psi_h)$. By Theorem 3.1 of [16], w_h satisfies:

$$w_h \rightharpoonup w \quad \text{weakly in } W_0^{1,p}(\Omega),$$
$$a_h(x, \nabla w_h) \rightharpoonup a(x, \nabla w) \quad \text{weakly in } L^{p'}(\Omega)^N,$$
$$\langle a_h(x, \nabla w_h), w_h \rangle \to \langle a(x, \nabla w), w \rangle,$$

where w is the solution of $VI(A, A(\Phi), \psi)$; so that $w = \Phi$ (see Characterization 1).

Moreover, w_h satisfies

$$\int_\Omega a_h(x, \nabla w_h)\nabla(w_h - v)\,\mathrm{d}x \leqslant \langle A(\Phi), w_h - v \rangle = \int_\Omega a(x, \nabla\Phi)\nabla(w_h - v), \tag{6.9}$$

for every $v \in W_0^{1,p}(\Omega)$, with $v \geqslant \psi_h$. Now, using the monotonicity of the operator A_h, we can rewrite (6.7) as

$$\int_\Omega a_h(x, \nabla\varphi)\nabla T_j(u_h - \varphi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u_h - \varphi)\,\mathrm{d}\mu_h + \int_\Omega T_j(u_h - \varphi)\,\mathrm{d}\lambda_h, \tag{6.10}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. We would like to use w_h in (6.10), but, a priori, we do not know that w_h is a bounded function. Let us note, nevertheless, that if a function φ is in $W_0^{1,p}(\Omega)$, for every $i > 0$, we can use $T_i(\varphi)$ as function test in (6.10). Observe now that, letting i tend to infinity, $T_i(\varphi)$ converges to φ strongly in $W_0^{1,p}(\Omega)$, so that, on one hand $a_h(x, \nabla T_i(\varphi))$ tends to $a_h(x, \nabla\varphi)$ strongly in $L^{p'}(\Omega)^N$, on the other $T_j(u_h - T_i(\varphi))$ converges to $T_j(u_h - \varphi)$ weakly in $W_0^{1,p}(\Omega)$, as observed in Remark 5.4. Now we can rewrite (6.10) for every $\varphi \in W_0^{1,p}(\Omega)$, and, in particular, choosing w_h as function test we obtain

$$\begin{aligned}\int_\Omega a_h(x, \nabla w_h)\nabla T_j(u_h - w_h)\,\mathrm{d}x &\leqslant \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\mu_h + \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\lambda_h \\ &\leqslant \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\mu_h,\end{aligned} \tag{6.11}$$

where the last inequality follows by the complementarity system (3.21) and by the fact that $w_h \geqslant \psi_h$ C_p-q.e. in Ω.

The choice of the function $v = v_h := w_h - T_j(w_h - u_h)$ as test in (6.9) is admissible and gives:

$$\int_\Omega a_h(x, \nabla w_h)\nabla T_j(w_h - u_h)\,\mathrm{d}x \leqslant \int_\Omega a(x, \nabla\Phi)\nabla T_j(w_h - u_h)\,\mathrm{d}x, \tag{6.12}$$

which, with (6.11), implies

$$\int_\Omega a(x, \nabla\Phi)\nabla T_j(u_h - w_h)\,\mathrm{d}x \leqslant \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\mu_h. \tag{6.13}$$

By the estimate (5.3), it is easy to prove that $T_j(u_h - w_h)$ converges to $T_j(u - \Phi)$ weakly in $W_0^{1,p}(\Omega)$, and, thanks to Lemma 5.2 we easily pass to the limit in (6.13). In conclusion, we obtain

$$\int_\Omega a(x, \nabla\Phi)\nabla T_j(u^* - \Phi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u^* - \Phi)\,\mathrm{d}\mu, \tag{6.14}$$

for every $\Phi \in W_0^{1,p}(\Omega)$, $\Phi \geqslant \psi$ C_p-q.e. in Ω.

Thanks to the next lemma, we have the following fact:

$$\int_\Omega a(x, \nabla u^*)\nabla T_j(u^* - \varphi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u^* - \varphi)\,\mathrm{d}\mu,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\varphi \geqslant \psi$ C_p-q.e. in Ω. As observed in Remark 3.11, the previous formulation characterizes uniquely the function u^*. Thus, having denoted the solution of $OP(A, \mu, \psi)$ by u, we have $u^* = u$; this implies that the whole sequence $T_j(u_h)$ (and not only a subsequence) converge to $T_j(u)$.

Hence, to conclude, we have to prove the following lemma, which is inspired by Lemma 1.2 of [4]. We give here the proof for the sake of completeness.

Lemma 6.3. *Assume μ be in $\mathcal{M}_{b,0}^p(\Omega)$ and ψ satisfy* (6.3)*. Under hypotheses* (3.2), (3.3), (3.4)*, and* (3.5) *a solution u of*

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega), \quad u \geqslant \psi, \\ \int_\Omega a(x, \nabla \Phi)\nabla T_j(u - \Phi)\,\mathrm{d}x \leqslant \int_\Omega T_j(u - \Phi)\,\mathrm{d}\mu, \\ \forall j > 0,\ \forall \Phi \in W_0^{1,p}(\Omega), \quad \Phi \geqslant \psi, \end{cases} \tag{6.15}$$

satisfying (3.8)*, is also a solution of*

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega), \quad u \geqslant \psi, \\ \int_\Omega a(x, \nabla u)\nabla T_k(u - \varphi)\,\mathrm{d}x \leqslant \int_\Omega T_k(u - \varphi)\,\mathrm{d}\mu, \\ \forall k > 0,\ \forall \varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega), \quad \varphi \geqslant \psi. \end{cases} \tag{6.16}$$

The converse is also true.

Proof. Let u be a solution of (6.15) and $\varphi \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\varphi \geqslant \psi$. The choice of $\Phi = tT_i(u) + (1-t)\varphi$, with $i > 0$ and $t \in (0,1)$, in (6.15) is admissible and gives

$$\begin{cases} I_i \leqslant J_i, \\ I_i = \int_\Omega a(x, t\nabla T_i(u) + (1-t)\nabla\varphi)\nabla T_j(u - tT_i(u) - (1-t)\varphi)\,\mathrm{d}x, \\ J_i = \int_\Omega T_j(u - tT_i(u) - (1-t)\varphi)\,\mathrm{d}\mu. \end{cases} \tag{6.17}$$

Now,

$$I_i = \int_{\{|u| \leqslant i\}} a\big(x, t\nabla u + (1-t)\nabla\varphi\big)\nabla T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}x$$
$$+ \int_{\{|u|>i\}\cap\{|u - tT_i(u) - (1-t)\varphi| \leqslant j\}} a\big(x, (1-t)\nabla\varphi\big)\nabla\big(u - (1-t)\varphi\big)\,\mathrm{d}x,$$

since $\nabla T_j(u - tT_i(u) - (1-t)\varphi) = 0$ where $|u - tT_i(u) - (1-t)\varphi| > j$. The set $\{|u| > i\} \cap \{|u - tT_i(u) - (1-t)\varphi| \leqslant j\}$ is empty if we choose $i > \|\varphi\|_{\mathrm{L}^\infty(\Omega)}$ and $0 < j \leqslant (1-t)(i - \|\varphi\|_{\mathrm{L}^\infty(\Omega)})$; hence

$$I_i = \int_{\{|u| \leqslant i\}} a\big(x, t\nabla u + (1-t)\nabla\varphi\big)\nabla T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}x.$$

Let us consider J_i:

$$J_i = \int\limits_{\{|u| \leqslant i\}} T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}\mu + \int\limits_{\{|u|>i\}} T_j\big(u - tT_i(u) - (1-t)\varphi\big)\,\mathrm{d}\mu$$
$$\leqslant \int\limits_{\{|u| \leqslant i\}} T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}\mu + j|\mu|\big(\{|u|>i\}\big).$$

Now we pass to the limit as i tends to $+\infty$ in (6.17); taking into account (3.9), we obtain, by the previous remarks about I_i and J_i that

$$I := \lim_{i\to+\infty} I_i = \int\limits_{\Omega} a\big(x, t\nabla u + (1-t)\nabla\varphi\big)\nabla T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}x$$
$$\leqslant \lim_{i\to\infty} J_i = \int\limits_{\Omega} T_j\big((1-t)(u-\varphi)\big)\,\mathrm{d}\mu =: J,$$

for every $j > 0$. Let us write I as

$$I = (1-t) \int\limits_{\{(1-t)|u-\varphi| \leqslant j\}} a\big(x, t\nabla u + (1-t)\nabla\varphi\big)\nabla(u-\varphi)\,\mathrm{d}x,$$

while

$$J = (1-t) \int\limits_{\{(1-t)|u-\varphi| \leqslant j\}} (u-\varphi)\,\mathrm{d}\mu + \int\limits_{\{(1-t)(u-\varphi)>j\}} j\,\mathrm{d}\mu + \int\limits_{\{(1-t)(u-\varphi)<-j\}} (-j)\,\mathrm{d}\mu.$$

Let $k > 0$ and j such that $j = k(1-t)$, so that $I \leqslant J$ implies

$$(1-t)\int\limits_{\Omega} a\big(x, t\nabla u + (1-t)\nabla\varphi\big)\nabla T_k(u-\varphi)\,\mathrm{d}x \leqslant (1-t)\int\limits_{\Omega} T_k(u-\varphi)\,\mathrm{d}\mu.$$

Dividing by $(1-t)$ and passing to the limit with respect to $t \to 1^-$, we obtain (6.16).

The converse is just the monotonicity of the operator A. Let us note that, if u solves (6.16), then u satisfies (3.8), since we can use $\varphi = 0$ as test in (6.16) and use (3.3). □

Step 2. Denoting the obstacle reactions of u_h and u by λ_h and λ, respectively, we will prove that

$$\lim_{h\to+\infty} \int\limits_{\Omega} \Phi_h\,\mathrm{d}\lambda_h = \int\limits_{\Omega} \Phi\,\mathrm{d}\lambda, \tag{6.18}$$

for every $\Phi \in W_0^{1,q'}(\Omega)$, with $q < \frac{N}{N-1}$, and for every $\Phi_h \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, with $\sup_h \|\Phi_h\|_{\mathrm{L}^\infty(\Omega)} < +\infty$, converging to Φ strongly in $W_0^{1,p}(\Omega)$.

Proof of Step 2. For every $i > 0$ and for every $t \in \mathbb{R}$, $t \neq 0$, we consider the solution v_h of the variational inequality $VI(A_h, A(T_i(u) + t\Phi_h), \psi_h + t\Phi_h)$ and the obstacle reaction η_h associated with it. Observing that $A(T_i(u) + t\Phi_h)$ converges to $A(T_i(u) + t\Phi)$ strongly in $W^{-1,p'}(\Omega)$, and that $K_{\psi_h + t\Phi_h}$ converges to $K_{\psi + t\Phi}$ in the sense of Mosco, we can apply Theorem 3.1 of [16] to deduce:

$$v_h \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega),$$
$$\eta_h \rightharpoonup \eta \quad \text{weakly in } W^{-1,p'}(\Omega),$$

where v is the solution of $VI(A, A(T_i(u) + t\Phi), \psi + t\Phi)$ and η is the obstacle reaction associated with it. On the other hand, thanks to (6.3), for every $i > 0$, we have that $T_i(u) \geqslant \psi$ C_p-q.e. in Ω, so that $v = T_i(u) + t\Phi$ and $\eta = 0$ (see Characterization 1).

Consider now, for every $l > 0$ and for every $j > 0$, the inequality

$$\int_\Omega \big(a_h(x, \nabla u_h) - a_h\big(x, \nabla T_l(v_h)\big)\big)\nabla T_j\big(u_h - T_l(v_h)\big)\,\mathrm{d}x \geqslant 0,$$

which follows by the monotonicity of a_h. If we use the entropy formulation of u_h in the previous inequality we obtain

$$\int_\Omega T_j\big(u_h - T_l(v_h)\big)\,\mathrm{d}\lambda_h + \int_\Omega T_j\big(u_h - T_l(v_h)\big)\,\mathrm{d}\mu_h \geqslant \int_\Omega a_h\big(x, \nabla T_l(v_h)\big)\nabla T_j\big(u_h - T_l(v_h)\big)\,\mathrm{d}x; \tag{6.19}$$

passing to the limit as l tends to $+\infty$ thanks to Proposition 5.3 (see also Remark 5.4), and using the variational formulation (3.15) satisfied by v_h, we rewrite (6.19) as

$$\begin{cases} I_h + II_h \geqslant III_h + IV_h, \\ I_h = \int_\Omega T_j(u_h - v_h)\,\mathrm{d}\lambda_h, \\ II_h = \int_\Omega T_j(u_h - v_h)\,\mathrm{d}\mu_h, \\ III_h = \int_\Omega a(x, \nabla(T_i(u) + t\Phi_h))\nabla T_j(u_h - v_h)\,\mathrm{d}x, \\ IV_h = \langle \eta_h, T_j(u_h - v_h)\rangle. \end{cases} \tag{6.20}$$

By the complementarity system (3.16), we have that

$$IV_h = \int_\Omega T_j(u_h - \psi_h - t\Phi_h)\,\mathrm{d}\eta_h \geqslant \int_\Omega T_j(-t\Phi_h)\,\mathrm{d}\eta_h = \big\langle \eta_h, T_j(-\Phi_h)\big\rangle,$$

which tends to 0 as h goes to $+\infty$, i.e.

$$\liminf_{h\to+\infty} IV_h \geqslant 0. \tag{6.21}$$

Moreover, by (5.3), it is easy to check that $T_j(u_h - v_h)$ converges to $T_j(u - T_i(u) - t\Phi)$ weakly in $W_0^{1,p}(\Omega)$, so that we can apply Lemma 5.2 to deduce that

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j\big(u - T_i(u) - t\Phi\big)\,\mathrm{d}\mu. \tag{6.22}$$

Since, thanks to (2.1), $a(x, \nabla(T_i(u) + t\Phi_h))$ converges to $a(x, \nabla(T_i(u) + t\Phi))$ strongly in $L^{p'}(\Omega)^N$, we pass to the limit also in III_h, obtaining

$$\lim_{h\to+\infty} III_h = \int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla T_j\big(u - T_i(u) - t\Phi\big)\,\mathrm{d}x. \tag{6.23}$$

Combining (6.21), (6.22) and (6.23) we have

$$\begin{aligned} &\liminf_{h\to+\infty}\int_\Omega T_j(u_h - v_h)\,\mathrm{d}\lambda_h + \int_\Omega T_j\big(u - T_i(u) - t\Phi\big)\,\mathrm{d}\mu \\ &\quad\geqslant \int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla T_j\big(u - T_i(u) - t\Phi\big)\,\mathrm{d}x, \end{aligned} \tag{6.24}$$

which can be written also as

$$\begin{cases} \liminf_{h\to+\infty} I_h - I^i \geqslant II^i, \\ I^i = \int_\Omega T_j(u - T_i(u) - t\Phi)\,\mathrm{d}\lambda, \\ II^i = \int_\Omega (a(x, \nabla(T_i(u) + t\Phi)) - a(x, \nabla u))\nabla T_j(u - T_i(u) - t\Phi)\,\mathrm{d}x, \end{cases} \tag{6.25}$$

using the entropy formulation satisfied by u. By the complementarity system (3.21), we have that

$$\liminf_{h\to+\infty} I_h = \liminf_{h\to+\infty} \int_\Omega T_j(\psi_h - v_h)\,\mathrm{d}\lambda_h;$$

on the other hand, since $v_h \geqslant \psi_h + t\Phi_h$ we obtain by the previous equality that

$$\liminf_{h\to+\infty} I_h \leqslant \liminf_{h\to+\infty} \int_\Omega T_j(-t\Phi_h)\,\mathrm{d}\lambda_h. \tag{6.26}$$

On the other hand, thanks to (3.9) it is easy to check that

$$\lim_{i\to+\infty} I^i = \int_\Omega T_j(-t\Phi)\,\mathrm{d}\lambda. \tag{6.27}$$

Finally, in II^i we split the integral into the sets where $|u| \leqslant i$ and where $|u| > i$, getting

$$\begin{aligned} II^i = &\int_{\{|u|\leqslant i\}} \big(a\big(x, \nabla(u+t\Phi)\big) - a(x,\nabla u)\big)\nabla T_j(-t\Phi)\,\mathrm{d}x \\ &+ \int_{\{|u|>i\}\cap\{|u-T_i(u)-t\Phi|\leqslant j\}} \big(a\big(x,\nabla(t\Phi)\big) - a(x,\nabla u)\big)\nabla(u-t\Phi)\,\mathrm{d}x, \end{aligned}$$

since $\nabla T_j(u - T_i(u) - t\Phi) = 0$ where $|u - T_i(u) - t\Phi| > j$. Let us observe that $\{|u - T_i(u) - t\Phi| \leqslant j\} \subseteq \{|u| \leqslant i + j + |t|\|\Phi\|_{\mathrm{L}^\infty(\Omega)}\}$, so that, by the growth conditions assumed on a and by (3.10), it is easy to prove that

$$\lim_{i\to+\infty} \int_{\{|u|>i\}\cap\{|u-T_i(u)-t\Phi|\leqslant j\}} \big(a\big(x,\nabla(t\Phi)\big) - a(x,\nabla u)\big)\nabla(u-t\Phi)\,\mathrm{d}x = 0, \tag{6.28}$$

as well as

$$\begin{aligned} &\lim_{i\to+\infty} \int_{\{|u|\leqslant i\}} \big(a\big(x,\nabla(u+t\Phi)\big) - a(x,\nabla u)\big)\nabla T_j(-t\Phi)\,\mathrm{d}x \\ &\quad = \int_\Omega \big(a\big(x,\nabla(u+t\Phi)\big) - a(x,\nabla u)\big)\nabla T_j(-t\Phi)\,\mathrm{d}x, \end{aligned} \tag{6.29}$$

since $a(x, \nabla(u + t\Phi)) - a(x, \nabla u) \in L^q(\Omega)^N$ and, by hypothesis, $\Phi \in W_0^{1,q'}(\Omega)$. Combining (6.26)–(6.28) and (6.29) we have

$$\liminf_{h\to+\infty} \int_\Omega T_j(-t\Phi_h)\,\mathrm{d}\lambda_h - \int_\Omega T_j(-\Phi)\,\mathrm{d}\lambda \geqslant \int_\Omega \big(a\big(x,\nabla(u+t\Phi)\big) - a(x,\nabla u)\big)\nabla T_j(-t\Phi)\,\mathrm{d}x,$$

and, for $j \geqslant |t|(\|\Phi_h\|_{\mathrm{L}^\infty(\Omega)} \vee \|\Phi\|_{\mathrm{L}^\infty(\Omega)})$

$$\liminf_{h\to+\infty} \int_\Omega -t\Phi_h\,\mathrm{d}\lambda_h + t\int_\Omega \Phi\,\mathrm{d}\lambda \geqslant -t\int_\Omega \big(a\big(x,\nabla(u+t\Phi)\big) - a(x,\nabla u)\big)\nabla\Phi\,\mathrm{d}x.$$

At this point, dividing by $|t|$ and passing to the limit with respect to $t \to 0$, we obtain (6.18).

Step 3. We will prove (6.5).

Proof of Step 3. We recall that u_h satisfies (3.11), i.e.

$$\int_\Omega a_h(x,\nabla u_h)\nabla\Phi\,\mathrm{d}x=\int_\Omega \Phi\,\mathrm{d}\mu_h+\int_\Omega \Phi\,\mathrm{d}\lambda_h, \tag{6.30}$$

for every $\Phi\in W_0^{1,q'}(\Omega)$, with $1<q<\frac{N}{N-1}$. We just observed that $W_0^{1,q'}(\Omega)\subseteq C(\overline{\Omega})$, so that, thanks to Lemma 5.2 and (6.18), we can pass to the limit as h goes to $+\infty$ in the last two terms of (6.30), obtaining

$$\lim_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla\Phi\,\mathrm{d}x=\int_\Omega \Phi\,\mathrm{d}\mu+\int_\Omega \Phi\,\mathrm{d}\lambda=\int_\Omega a(x,\nabla u)\nabla\Phi\,\mathrm{d}x,$$

where the last equality follows by Eq. (3.11) satisfied by u. In other words, we proved that

$$-\operatorname{div}\big(a_h(x,\nabla u_h)\big)\rightharpoonup-\operatorname{div}\big(a\big(x,\nabla u\big)\big)\quad\text{weakly in } W^{-1,q}(\Omega),\ \text{for every } q<\frac{N}{N-1}.$$

On the other hand, $a_h(x,\nabla u_h)$ is equibounded (with respect to h) in the L^q-norm, as observed in Remark 3.5. By this fact we easily deduce that

$$a_h(x,\nabla u_h)\rightharpoonup\sigma \text{ weakly in } L^q(\Omega)^N, \tag{6.31}$$

where $\operatorname{div}(a(x,\nabla u)-\sigma)=0$. As we will see later, to prove (6.5), it is enough to show, by Minty's trick, that

$$\lim_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla\Phi\phi\,\mathrm{d}x=\int_\Omega a(x,\nabla u)\nabla\Phi\phi\,\mathrm{d}x, \tag{6.32}$$

for every $\Phi\in W_0^{1,q'}(\Omega)$ and for every $\phi\in C^1(\overline{\Omega})$.

With minor changes with respect to the proof of Step 2, we will prove (6.32). Let $\Phi\in W_0^{1,q'}(\Omega)$ and $t\in\mathbb{R}$, with $t\neq 0$; then the solution v_h of $VI(A_h,A(T_i(u)+t\Phi),\psi_h+t\Phi)$ and the obstacle reaction η_h associated with it are such that

$$\begin{aligned}
&v_h\rightharpoonup T_i(u)+t\Phi\quad\text{weakly in } W_0^{1,p}(\Omega),\\
&a_h(x,\nabla v_h)\rightharpoonup a\big(x,\nabla\big(T_i(u)+t\Phi\big)\big)\quad\text{weakly in } L^{p'}(\Omega)^N,\\
&\eta_h\rightharpoonup 0\quad\text{weakly in } W^{-1,p'}(\Omega),
\end{aligned}$$

since $T_i(u)+t\Phi$ is the solution of $VI(A,A(T_i(u)+t\Phi),\psi+t\Phi)$. By the monotonicity assumption on $a_h(x,\cdot)$ we have, for every $l,j>0$

$$\int_\Omega\big(a_h(x,\nabla u_h)-a_h\big(x,\nabla T_l(v_h)\big)\big)\nabla T_j\big(u_h-T_l(v_h)\big)\phi\,\mathrm{d}x\geqslant 0,$$

where $\phi\in C^1(\overline{\Omega})$, with $\phi\geqslant 0$. For convenience we write the previous inequality in the form

$$\begin{aligned}
&\int_\Omega\big(a_h(x,\nabla u_h)-a_h\big(x,\nabla T_l(v_h)\big)\big)\nabla\big(T_j\big(u_h-T_l(v_h)\big)\phi\big)\,\mathrm{d}x\\
&\quad\geqslant\int_\Omega\big(a_h(x,\nabla u_h)-a_h\big(x,\nabla T_l(v_h)\big)\big)\nabla\phi\,T_j\big(u_h-T_l(v_h)\big)\,\mathrm{d}x,
\end{aligned}$$

which gives, using the entropy formulation (3.12) of u_h and letting l tend to $+\infty$, as in the proof of Step 2,

$$\begin{cases} I_h + II_h + III_h \geqslant IV_h + V_h, \\ I_h = \int_\Omega T_j(u_h - v_h)\phi \,\mathrm{d}\lambda_h, \\ II_h = \int_\Omega T_j(u_h - v_h)\phi \,\mathrm{d}\mu_h, \\ III_h = -\int_\Omega a_h(x, \nabla u_h)\nabla\phi\, T_j(u_h - v_h)\,\mathrm{d}x, \\ IV_h = \int_\Omega a_h(x, \nabla v_h)\nabla(T_j(u_h - v_h)\phi)\,\mathrm{d}x, \\ V_h = -\int_\Omega a_h(x, \nabla v_h)\nabla\phi T_j(u_h - v_h)\,\mathrm{d}x. \end{cases}$$

The same tools used to deduce (6.26) give:

$$I_h \leqslant \int_\Omega T_j(-t\Phi)\phi \,\mathrm{d}\lambda_h;$$

choosing $j \geqslant |t| \|\Phi\|_{L^\infty(\Omega)}$ and using the formulation (3.11) satisfied by u_h, we have:

$$I_h \leqslant -t\int_\Omega a_h(x, \nabla u_h)\nabla(\Phi\phi)\,\mathrm{d}x + t\int_\Omega \Phi\phi \,\mathrm{d}\mu_h. \tag{6.33}$$

Thanks to the variational formulation satisfied by v_h we write IV_h as

$$IV_h = \int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla\big(T_j(u_h - v_h)\phi\big)\,\mathrm{d}x + \big\langle \eta_h, T_j(u_h - v_h)\phi\big\rangle$$

and we obtain that

$$\liminf_{h\to+\infty} IV_h \geqslant \int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla\big(T_j\big(u - T_i(u) - t\Phi\big)\phi\big)\,\mathrm{d}x, \tag{6.34}$$

since we can work as in the proof of (6.21) and (6.23). Analogously, as we prove (6.22), we have also that

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j\big(u - T_i(u) - t\Phi\big)\,\phi\,\mathrm{d}\mu. \tag{6.35}$$

On the other hand, it is easy to check that

$$\lim_{h\to+\infty} III_h = t\lim_{h\to+\infty}\int_\Omega a_h(x, \nabla u_h)\nabla\phi\,\Phi\,\mathrm{d}x + \int_\Omega \sigma\nabla\phi\big(-t\Phi - T_j\big(u - T_i(u) - t\Phi\big)\big)\,\mathrm{d}x, \tag{6.36}$$

as well as

$$\lim_{h\to+\infty} V_h = -\int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla\phi\, T_j\big(u - T_i(u) - t\Phi\big)\,\mathrm{d}x. \tag{6.37}$$

Combining (6.33)–(6.36) and (6.37) we obtain

$$\begin{aligned} &\liminf_{h\to+\infty} -t\int_\Omega a_h(x, \nabla u_h)\nabla\Phi\phi\,\mathrm{d}x + t\int_\Omega \Phi\phi\,\mathrm{d}\mu + \int_\Omega T_j\big(u - T_i(u) - t\Phi\big)\phi\,\mathrm{d}\mu \\ &\quad + \int_\Omega \sigma\nabla\phi\big(-t\Phi - T_j\big(u - T_i(u) - t\Phi\big)\big)\,\mathrm{d}x \\ &\geqslant \int_\Omega a\big(x, \nabla\big(T_i(u) + t\Phi\big)\big)\nabla T_j\big(u - T_i(u) - t\Phi\big)\,\phi\,\mathrm{d}x, \end{aligned}$$

which gives, letting $i \to +\infty$

$$\liminf_{h\to+\infty} -t \int_\Omega a_h(x, \nabla u_h)\nabla \Phi \phi \,\mathrm{d}x \geqslant -t \int_\Omega a\big(x, \nabla(u + t\Phi)\big)\nabla \Phi \phi \,\mathrm{d}x.$$

Finally, dividing by $|t|$ and passing to the limit with respect to $t \to 0$, we obtain (6.32).

Combining (6.32) and (6.31), we have

$$\int_\Omega \big(\sigma - a(x, \nabla u)\big)\nabla \Phi \phi \,\mathrm{d}x = 0, \tag{6.38}$$

for every $\Phi \in W_0^{1,q'}(\Omega)$, with $q < \frac{N}{N-1}$, and for every $\phi \in C^1(\overline{\Omega})$.

Let $\xi \in \mathbb{R}^N$, with $\xi \neq 0$, and let $\zeta \in C_c^\infty(\Omega)$; then the choice of $\Phi(x) = \xi x\, \zeta(x)$ in (6.38) is admissible, and gives

$$\int_\Omega \big(\sigma - a(x, \nabla u)\big)\xi \zeta \phi \,\mathrm{d}x = 0,$$

since $\xi x\, \phi(x) \in C^1(\overline{\Omega})$. Now we let ζ tend to 1, obtaining

$$\int_\Omega \big(\sigma - a(x, \nabla u)\big)\xi \phi \,\mathrm{d}x = 0,$$

for every $\phi \in C^1(\overline{\Omega})$, and, finally, $(\sigma(x) - a(x, \nabla u(x)))\xi = 0$, for every $\xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$, so that (6.5) is proved.

Step 4. We will prove the lower semicontinuity of the "energy", that is

$$\int_\Omega a(x, \nabla u)\nabla T_j(u)\,\mathrm{d}x \leqslant \liminf_{h\to+\infty} \int_\Omega a_h(x, \nabla u_h)\nabla T_j(u_h)\,\mathrm{d}x, \tag{6.39}$$

for every $j > 0$.

Proof of Step 4. To prove (6.39) we need an approximation result for the G-convergence (see Lemma 2.3 of [16]).

Lemma 6.4. *Let a_h be a sequence in $\mathcal{L}(c_0, c_1, \alpha, \beta)$ G-converging to a function a, and let A_h and A be the operators associated to a_h and a, respectively. Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and v_h the solution of* (4.1) *relative to $A(v)$. Then there exist a decreasing sequence ε_h converging to* 0 *and a sequence $w_h \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that*

$$w_h \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{6.40}$$

$$\big(a_h(x, \nabla w_h) - a_h(x, \nabla v_h)\big) \to 0 \quad \text{strongly in } L^{p'}(\Omega)^N, \tag{6.41}$$

$$\big|w_h(x) - v(x)\big| \leqslant \varepsilon_h \quad C_p\text{-q.e. in } \Omega. \tag{6.42}$$

Let v, $w_h \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as in Lemma 6.4. By the monotonicity assumption on $a_h(x, \cdot)$ we have, for every $j > 0$:

$$\int_\Omega \big(a_h(x, \nabla u_h) - a_h(x, \nabla w_h)\big)\nabla T_j(u_h - w_h)\,\mathrm{d}x \geqslant 0.$$

We use the entropy formulation of u_h to obtain

$$\int_\Omega T_j(u_h - w_h)\,\mathrm{d}\lambda_h + \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\mu_h - \int_\Omega a_h(x, \nabla w_h)\nabla T_j(u_h - w_h)\,\mathrm{d}x \geqslant 0.$$

We rewrite the previous inequality as

$$\begin{cases} I_h + II_h + III_h \geqslant 0, \\ I_h = \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\lambda_h, \\ II_h = \int_\Omega T_j(u_h - w_h)\,\mathrm{d}\mu_h, \\ III_h = -\int_\Omega a_h(x, \nabla w_h)\nabla T_j(u_h - w_h)\,\mathrm{d}x, \end{cases}$$

and the term III_h as

$$III_h = \int_\Omega \big(a_h(x, \nabla v_h) - a_h(x, \nabla w_h)\big)\nabla T_j(u_h - w_h)\,\mathrm{d}x - \int_\Omega a_h(x, \nabla v_h)\nabla T_j(u_h - w_h)\,\mathrm{d}x.$$

By (5.3), $T_j(u_h - w_h)$ is uniformly bounded (with respect to h) in $W_0^{1,p}(\Omega)$, so that $T_j(u_h - w_h)$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_j(u - v)$. Thanks to this fact and to (6.41), it is easy to pass to the limit in the first term of III_h. For the second one it is sufficient to use the definition of v_h and, again, the weak convergence in $W_0^{1,p}(\Omega)$ of $T_j(u_h - w_h)$, so that

$$\lim_{h\to+\infty} III_h = -\int_\Omega a(x, \nabla v)\nabla T_j(u - v)\,\mathrm{d}x. \tag{6.43}$$

Analogously we have

$$\lim_{h\to+\infty} II_h = \int_\Omega T_j(u - v)\,\mathrm{d}\mu, \tag{6.44}$$

since we can apply Lemma 5.2. Finally, thanks to (6.42) and by the lipschitzianity of the truncation function, we have:

$$\liminf_{h\to+\infty} I_h = \liminf_{h\to+\infty} \int_\Omega T_j(u_h - v)\,\mathrm{d}\lambda_h. \tag{6.45}$$

Combining (6.43), (6.44), and (6.45) we obtain

$$\liminf_{h\to+\infty} \int_\Omega T_j(u_h - v)\,\mathrm{d}\lambda_h + \int_\Omega T_j(u - v)\,\mathrm{d}\mu \geqslant \int_\Omega a(x, \nabla v)\nabla T_j(u - v)\,\mathrm{d}x, \tag{6.46}$$

for every $v \in W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$.

Let $t \in (0,1)$; for $i > 0$, we use $v = tT_i(u)$ as function test in (6.46). Since $tT_i(u) \geqslant tT_i(\psi)$ C_p-q.e. in Ω and since $K_{tT_i(\psi_h)}$ converges to $K_{tT_i(\psi)}$ in the sense of Mosco (see Remark 4.5), there exist $k \in \mathbb{N}$ and a sequence z_h converging to $tT_i(u)$ strongly in $W_0^{1,p}(\Omega)$ such that $z_h \in K_{tT_i(\psi_h)}$, for every $h \geqslant k$. We consider the function $\Phi_h = T_i(z_h) - tT_i(u)$, which belongs to $W_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and tends to 0 strongly in $W_0^{1,p}(\Omega)$; so we can use (6.18) and the lipschitzianity of the truncation function to deduce that

$$\liminf_{h\to+\infty} \int_\Omega T_j\big(u_h - tT_i(u)\big)\,\mathrm{d}\lambda_h = \liminf_{h\to+\infty} \int_\Omega T_j\big(u_h - T_i(z_h)\big)\,\mathrm{d}\lambda_h. \tag{6.47}$$

Moreover, since, for every $h \geqslant k$, $T_i(z_h) \geqslant tT_i(\psi_h)$ C_p-q.e. in Ω, we estimate the right-hand side of (6.47) as

$$\liminf_{h\to+\infty}\int_\Omega T_j\big(u_h - T_i(z_h)\big)\,\mathrm{d}\lambda_h \leqslant \liminf_{h\to+\infty}\int_\Omega T_j\big(u_h - tT_i(\psi_h)\big)\,\mathrm{d}\lambda_h = \liminf_{h\to+\infty}\int_\Omega T_j\big(u_h - tT_i(u_h)\big)\,\mathrm{d}\lambda_h,$$

where the last equality follows by the complementarity system (3.21). Finally, using the entropy formulation of u_h, we get

$$\begin{aligned}
&\liminf_{h\to+\infty}\int_\Omega T_j\big(u_h - T_i(z_h)\big)\,\mathrm{d}\lambda_h \\
&\quad\leqslant \liminf_{h\to+\infty}\bigg(\int_\Omega a_h(x,\nabla u_h)\nabla T_j\big(u_h - tT_i(u_h)\big)\,\mathrm{d}x - \int_\Omega T_j\big(u_h - tT_i(u_h)\big)\,\mathrm{d}\mu_h\bigg) \\
&\quad= \liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j\big(u_h - tT_i(u_h)\big)\,\mathrm{d}x - \int_\Omega T_j\big(u - tT_i(u)\big)\,\mathrm{d}\mu, \qquad (6.48)
\end{aligned}$$

since $T_j(u_h - tT_i(u_h))$ converges to $T_j(u - tT_i(u))$ weakly in $W_0^{1,p}(\Omega)$ and we can apply Lemma 5.2.

Hence, using in (6.46) $v = tT_i(u)$ and combining (6.47) and (6.48), we obtain

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j\big(u_h - tT_i(u_h)\big)\,\mathrm{d}x \geqslant \int_\Omega a\big(x, t\nabla T_i(u)\big)\nabla T_j\big(u - tT_i(u)\big)\,\mathrm{d}x. \qquad (6.49)$$

We denote $\int_\Omega a_h(x,\nabla u_h)\nabla T_j(u_h - tT_i(u_h))\,\mathrm{d}x$ by J_h, and we split Ω into the sets where $|u_h| \leqslant i$ and where $|u_h| > i$, so that

$$J_h = \int_{\{|u_h|\leqslant i\}} a_h(x,\nabla u_h)\nabla T_j\big((1-t)(u_h)\big)\,\mathrm{d}x + \int_{\{|u_h|>i\}\cap\{|u_h - tT_i(u_h)|\leqslant j\}} a_h(x,\nabla u_h)\nabla u_h\,\mathrm{d}x.$$

Observing that $\{|u_h - tT_i(u_h)| \leqslant j\} \subseteq \{|u_h| \leqslant j + ti\}$, if we choose $j < (1-t)i$, we have that $\{|u_h| > i\} \cap \{|u_h - tT_i(u_h)| \leqslant j\}$ is empty, and

$$J_h = \int_{\{|u_h|\leqslant i\}} a_h(x,\nabla u_h)\nabla T_j\big((1-t)(u_h)\big)\,\mathrm{d}x \leqslant \int_\Omega a_h(x,\nabla u_h)\nabla T_j\big((1-t)(u_h)\big)\,\mathrm{d}x,$$

since the integrand is nonnegative. Analogously

$$\int_\Omega a\big(x, t\nabla T_i(u)\big)\nabla T_j\big(u - tT_i(u)\big)\,\mathrm{d}x = \int_{\{|u|\leqslant i\}} a\big(x, t\nabla T_i(u)\big)\nabla T_j\big((1-t)u\big)\,\mathrm{d}x,$$

so that (6.49) becomes

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j\big((1-t)(u_h)\big)\,\mathrm{d}x \geqslant \int_{\{|u|\leqslant i\}} a\big(x, t\nabla T_i(u)\big)\nabla T_j\big((1-t)u\big)\,\mathrm{d}x.$$

Letting i tend to $+\infty$, we rewrite the previous inequality as

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_j\big((1-t)(u_h)\big)\,\mathrm{d}x \geqslant \int_\Omega a(x, t\nabla u)\nabla T_j\big((1-t)u\big)\,\mathrm{d}x$$

or, equivalently,

$$(1-t)\liminf_{h\to+\infty}\int_{\{(1-t)|u_h|\leqslant j\}} a_h(x,\nabla u_h)\nabla u_h\,\mathrm{d}x \geqslant (1-t)\int_{\{(1-t)|u|\leqslant j\}} a(x, t\nabla u)\nabla u\,\mathrm{d}x \qquad (6.50)$$

for every $j > 0$. Let $n > 0$ and $j = (1-t)n$; then we can rewrite (6.50) as

$$\liminf_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_n(u_h)\,dx \geqslant \int_\Omega a(x,t\nabla u)\nabla T_n(u)\,dx.$$

Finally, letting t tend to 1^-, we obtain (6.39).

Step 5. We will prove (6.6).

Proof of Step 5. The proof is quite similar to that of Step 2, so we will often refer to it.

Let $t > 0$; then, for every $k > 0$, we have that $tT_k(u) \geqslant tT_k(\psi)$ C_p-q.e. in Ω. Since $K_{tT_k(\psi_h)}$ converges to $K_{tT_k(\psi)}$ in the sense of Mosco (see Remark 4.5), there exist $n \in \mathbb{N}$ and a sequence Φ_h converging to $tT_k(u)$ strongly in $W_0^{1,p}(\Omega)$ such that $\Phi_h \in K_{tT_k(\psi_h)}$ for every $h \geqslant n$. For $i > 0$ we consider the solution v_h of $VI(A_h, A(T_i(u)+tT_k(u)), \psi_h + \Phi_h)$ and the obstacle reaction η_h associated with it; as in the proof of Step 2, we deduce by Theorem 3.1 of [16] that

$$v_h \rightharpoonup T_i(u) + tT_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega),$$
$$\eta_h \rightharpoonup 0 \quad \text{weakly in } W^{-1,p'}(\Omega),$$

since $T_i(u) + tT_k(u)$ is the solution of $VI(A, A(T_i(u)+tT_k(u)), \psi + tT_k(u))$. We have also, by (6.24), that

$$\liminf_{h\to+\infty}\int_\Omega T_j(u_h - v_h)\,d\lambda_h + \int_\Omega T_j\big(u - T_i(u) - tT_k(u)\big)\,d\mu \geqslant \int_\Omega a\big(x,\nabla\big(T_i(u)+tT_k(u)\big)\big)\nabla T_j\big(u - T_i(u) - tT_k(u)\big)\,dx. \tag{6.51}$$

On the other hand, by (6.26), we have that

$$\liminf_{h\to+\infty}\int_\Omega T_j(u_h - v_h)\,d\lambda_h \leqslant \liminf_{h\to+\infty}\int_\Omega T_j(-\Phi_h)\,d\lambda_h \leqslant \liminf_{h\to+\infty}\int_\Omega T_j\big(-tT_k(\psi_h)\big)\,d\lambda_h = \liminf_{h\to+\infty}\int_\Omega T_j\big(-tT_k(u_h)\big)\,d\lambda_h, \tag{6.52}$$

where the last inequalities follow, on one hand, by the fact that $\Phi_h \in K_{tT_k(\psi_h)}$, for h large enough, on the other, by the complementarity system (3.21). Thanks to (6.52) we rewrite (6.51) as

$$\liminf_{h\to+\infty}\int_\Omega T_j\big(-tT_k(u_h)\big)\,d\lambda_h + \int_\Omega T_j\big(u - T_i(u) - tT_k(u)\big)\,d\mu \geqslant \int_\Omega a\big(x,\nabla\big(T_i(u)+tT_k(u)\big)\big)\nabla T_j\big(u - T_i(u) - tT_k(u)\big)\,dx. \tag{6.53}$$

Let us choose $j \geqslant tk$ and $i > k$; if we split the integral in the right-hand side of (6.53) into the sets where $|u| \leqslant i$ and where $|u| > i$ we obtain by (2.3):

$$-t\int_{\{|u|\leqslant i\}} a\big(x,\nabla\big(u + tT_k(u)\big)\big)\nabla T_k(u)\,dx = -t\int_\Omega a\big(x,\nabla\big(T_k(u)(1+t)\big)\big)\nabla T_k(u)\,dx. \tag{6.54}$$

As in the proof of *Step 2,* we let i tend to $+\infty$ in (6.53), so that, using (6.54), we easily get

$$-t\limsup_{h\to+\infty}\int_\Omega T_k(u_h)\,d\lambda_h - t\int_\Omega T_k(u)\,d\mu \geqslant -t\int_\Omega a\big(x,\nabla\big(T_k(u)(1+t)\big)\big)\nabla T_k(u)\,dx,$$

or, equivalently, using the entropy formulation of u_h

$$-t\limsup_{h\to+\infty}\left(\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,\mathrm{d}x-\int_\Omega T_k(u_h)\,\mathrm{d}\mu_h\right)-t\int_\Omega T_k(u)\,\mathrm{d}\mu$$
$$\geqslant -t\int_\Omega a\big(x,\nabla\big(T_k(u)(1+t)\big)\big)\nabla T_k(u)\,\mathrm{d}x, \tag{6.55}$$

for every $k>0$. On the other hand, Lemma 5.2 implies that

$$\lim_{h\to+\infty}\int_\Omega T_k(u_h)\,\mathrm{d}\mu_h=\int_\Omega T_k(u)\,\mathrm{d}\mu,$$

so that (6.55) becomes

$$-t\limsup_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,\mathrm{d}x\geqslant -t\int_\Omega a\big(x,\nabla\big(T_k(u)(1+t)\big)\big)\nabla T_k(u)\,\mathrm{d}x.$$

Finally, dividing by t and passing to the limit as $t\to 0^+$, we have, for every $k>0$

$$\limsup_{h\to+\infty}\int_\Omega a_h(x,\nabla u_h)\nabla T_k(u_h)\,\mathrm{d}x\leqslant\int_\Omega a(x,\nabla u)\nabla T_k(u)\,\mathrm{d}x,$$

which, combined with (6.39) gives (6.6).

Remark 6.5. If we choose in Theorem 6.1 as obstacles $\psi_h=\psi=-\infty$, we recover Theorem 3.2 of [4], which concerns the continuous dependence of the entropy solutions under perturbations of the operator A.

Remark 6.6. Let us remark that we cannot prove Theorem 6.1 under the assumption that μ_h converges to μ in the $*$-weak topology of $\mathcal{M}_b(\Omega)$ (see Example 4.5 of [13]).

Corollary 6.7. *Let a be in $\mathcal{L}(c_0,c_1,\alpha,\beta)$ and A be the operator associated with it. Let us assume* (6.1), (6.2) (*with $A_h=A$, for every $h>0$*), *and* (6.3), *with K_{ψ_h} converging to K_ψ in the sense of Mosco. Finally, consider μ_h, $\mu\in\mathcal{M}^p_{b,0}(\Omega)$, with μ_h converging to μ weakly in $\mathcal{M}_b(\Omega)$. Then the solutions u_h and u of the obstacle problems $OP(A,\mu_h,\psi_h)$ and $OP(A,\mu,\psi)$, respectively, satisfy*

$$T_j(u_h)\to T_j(u)\quad \textit{strongly in } W^{1,p}_0(\Omega),\ \textit{for every } j>0. \tag{6.56}$$

Proof. By Theorem 6.1 (with $a_h=a$, for every $h>0$) we have that $T_j(u_h)$ converges to $T_j(u)$ weakly in $W^{1,p}_0(\Omega)$, and

$$\int_\Omega a(x,\nabla u_h)\nabla T_j(u_h)\,\mathrm{d}x\to\int_\Omega a(x,\nabla u)\nabla T_j(u)\,\mathrm{d}x,\quad \text{for every } j>0. \tag{6.57}$$

On the other hand, if the function a is fixed, working as in the proof of Theorem 6.1 of [3], it can be proved that ∇u_h converges to ∇u almost everywhere in Ω. Since $a(x,\cdot)$ is a Carathéodory function, also $a(x,\nabla T_j(u_h))$ tends to $a(x,\nabla T_j(u))$ almost everywhere in Ω.

Moreover, thanks to (2.1), $a(x,\nabla T_j(u_h))$ is uniformly (with respect to h) bounded in $L^{p'}(\Omega)^N$; so we deduce that

$$a\big(x,\nabla T_j(u_h)\big)\rightharpoonup a\big(x,\nabla T_j(u)\big)\quad \text{weakly in } L^{p'}(\Omega)^N. \tag{6.58}$$

Combining (6.57) and (6.58), we have:

$$\lim_{h\to+\infty}\int_{\Omega}\big(a\big(x,\nabla T_j(u_h)\big)-a\big(x,\nabla T_j(u)\big)\big)\nabla\big(T_j(u_h)-T_j(u)\big)\,\mathrm{d}x=0,$$

which implies that $T_j(u_h)$ converges to $T_j(u)$ strongly in $W_0^{1,p}(\Omega)$. □

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