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An asymmetric Neumann problem with weights

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Abstract

We prove the existence of a first nonprincipal eigenvalue for an asymmetric Neumann problem with weights involving the p-Laplacian (cf. (1.2) below). As an application we obtain a first nontrivial curve in the corresponding Fučik spectrum (cf. (1.4) below). The case where one of the weights has meanvalue zero requires some special attention in connexion with the (PS) condition and with the mountain pass geometry.

Résumé

Nous démontrons l'existence d'une première valeur propre non principale pour un problème de Neumann asymétrique avec poids faisant intervenir le p-laplacien (cf. (1.2) ci-dessous). Comme application nous obtenons une première courbe non triviale dans le spectre de Fučik correspondant (cf. (1.4) ci-dessous). Le cas où l'un des poids est de moyenne nulle demande une attention particulière en liaison avec la condition de Palais–Smale et avec la géométrie du col.

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1. Introduction

In a previous work [2], we investigated the eigenvalues of the following asymmetric Dirichlet problem with weights:

$$-\Delta_p u = \lambda\big[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}\big] \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Δ_p is the p-Laplacian, Ω is a bounded domain in $\mathbb{R}^N$ and m, n satisfy some summability conditions together with $m^+ \not\equiv 0$, $n^+ \not\equiv 0$. We proved the existence of a first nonprincipal positive eigenvalue for (1.1). Various applications were given to the study of the Fučik spectrum and to the study of nonresonance. The construction of this

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distinguished eigenvalue was obtained by applying a version of the mountain pass theorem to the functional $\int_\Omega |\nabla u|^p$ restricted to the manifold $\{u \in W_0^{1,p}(\Omega)\colon \int_\Omega [m(u^+)^p + n(u^-)^p] = 1\}$. In this process the (PS) condition was shown to hold at all levels and the geometry of the mountain pass was derived from the observation that $\varphi_1(m)$ and $-\varphi_1(n)$ were strict local minima (where $\varphi_1(m)$ denotes the normalized positive first eigenfunction of the Dirichlet p-Laplacian with weight m).

Our purpose in the present paper is to investigate the corresponding Neumann problem:

$$-\Delta_p u = \lambda\big[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}\big] \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where ν denotes the unit exterior normal. When trying to adapt the preceding approach to the present situation, the relevant functional is still $\int_\Omega |\nabla u|^p$ but now restricted to the manifold

$$M_{m,n} := \left\{ u \in W^{1,p}(\Omega)\colon\ B_{m,n}(u) := \int_\Omega \big[m(u^+)^p + n(u^-)^p\big] = 1 \right\}. \tag{1.3}$$

A first difficulty arises in connexion with the (PS) condition. It turns out that the (PS) condition remains satisfied at all levels when $\int_\Omega m \neq 0$ and $\int_\Omega n \neq 0$, but it is not satisfied anymore at level 0 when $\int_\Omega m = 0$ or $\int_\Omega n = 0$. In this latter case, which we will call the singular case, we do not know whether the (PS) condition still holds at all positive levels (see Remark 3.4). However one can show that the Palais–Smale condition of Cerami (abbreviated into (PSC)) holds at all positive levels. Another difficulty arises when dealing with problem (1.2), which is now connected with the geometry of the functional. It turns out that in the singular case, at least one of the two natural candidates for local minimum fails to belong to the manifold $M_{m,n}$. To bypass this difficulty we will consider a minimax procedure defined from a family of paths having free endpoints (cf. (3.1)).

The existence of a first nonprincipal positive eigenvalue for (1.2) is derived in Section 3. The argument uses a version of the mountain pass theorem for a C^1 functional restricted to a C^1 manifold and which satisfies the (PSC) condition at certain levels. Section 4 is devoted to such a theorem. In Section 5 we briefly indicate some properties of the eigenvalue constructed in Section 3 as a function of the weights m, n and in Section 6 we apply our results to the study of the Fučik spectrum. Recall that the latter is defined as the set Σ of those $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$-\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

has a nontrivial solution. As in the Dirichlet case we obtain for (1.4) the existence in Σ of hyperbolic-like first curves. Note however that contrary to what was happening in the Dirichlet case, the asymptotic behaviour of these first curves does not depend on the supports of the weights (at least when the weights are bounded, cf. Proposition 6.4 and Remark 6.5).

In the preliminary Section 2 we collect some results relative to the usual eigenvalue problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

We also recall there some general definitions relative to (PS) and (PSC) conditions.

2. Preliminaries

Throughout this paper Ω will be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and the weights m, n will be assumed to belong to $L^r(\Omega)$ with $r > \frac{N}{p}$ if $p \leqslant N$ and $r = 1$ if $p > N$. We also assume unless otherwise stated

$$m^+ \text{ and } n^+ \not\equiv 0 \quad \text{in } \Omega. \tag{2.1}$$

Solutions of (1.2) or of related equations are always understood in the weak sense, i.e. $u \in W^{1,p}(\Omega)$ with

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \lambda \int_\Omega \big[m(u^+)^{p-1} - n(u^-)^{p-1}\big]\varphi, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Regularity results from [13] on general quasilinear equations imply that such a solution u is locally Hölder continuous in Ω; moreover the derivation of the L^∞ estimates in [1] can be adapted to the present situation to show that $u \in$

$L^\infty(\Omega)$. Note that if in addition $m, n \in L^\infty(\Omega)$ and Ω is of class $C^{1,1}$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ (cf. [12]).

Our main purpose in this preliminary section is to collect some results relative to the eigenvalue problem (1.5).

Clearly 0 is a principal eigenvalue of (1.5), with the constants as eigenfunctions. The search for another principal eigenvalue involves the following quantity:

$$\lambda^*(m) = \inf\left\{ \int_\Omega |\nabla u|^p\text{: } u \in W^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p = 1 \right\}. \tag{2.2}$$

By (2.1), $\lambda^*(m) < \infty$.

Proposition 2.1.

(i) *Suppose $\int_\Omega m < 0$. Then $\lambda^*(m) > 0$ and $\lambda^*(m)$ is the unique nonzero principal eigenvalue; this eigenvalue is simple and admits an eigenfunction which can be chosen > 0 in Ω; moreover the interval $]0, \lambda^*(m)[$ does not contain any other eigenvalue.*

(ii) *Suppose $\int_\Omega m > 0$. Then $\lambda^*(m) = 0$ and 0 is the unique nonnegative principal eigenvalue.*

(iii) *Suppose $\int_\Omega m = 0$. Then $\lambda^*(m) = 0$ and 0 is the unique principal eigenvalue.*

Proposition 2.1 is proved in [10] (see also [6,11]) when $m \in L^\infty(\Omega)$, but the arguments can easily be adapted to the present situation. We observe in this respect that in the case of an unbounded weight, Harnack's inequality as given in [13,9] should be used instead of Vazquez maximum principle [14] to derive in case (i) that the eigenfunction can be chosen > 0 in Ω. See [4] for similar considerations in the Dirichlet case. In case (i) or (ii) of Proposition 2.1, the positive eigenfunction associated to $\lambda^*(m)$ and normalized so as to satisfy the constraint in (2.2) will be denoted by φ_m. The infimum (2.2) is then achieved at φ_m. In case (iii) the fact that $\lambda^*(m) = 0$ is easily verified by considering the sequence

$$v_k = \frac{(1 + \psi/k)^{1/p}}{[\int_\Omega m(1 + \psi/k)]^{1/p}}, \tag{2.3}$$

where ψ is any fixed smooth function with $\psi \geqslant 0$ and $\int_\Omega m\psi > 0$. Note that in that case (iii), the infimum (2.2) is not achieved (since no constant satisfies the constraint in that case).

Let us conclude this section with some general definitions relative to the (PS) condition. Let E be a real Banach space and let $M := \{u \in E\text{: } g(u) = 1\}$ where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M. The differential $\tilde{f}'$ at $u \in M$, has a norm which will be denoted by $\| \tilde{f}'(u) \|_*$ and which is given by the norm of the restriction of $f'(u) \in E^*$ to the tangent space of M at u

$$T_u(M) := \{ v \in E\text{: } \langle g'(u), v \rangle = 0 \},$$

where $\langle , \rangle$ denotes the pairing between E^* and E. A critical point of $\tilde{f}$ is a point $u \in M$ such that $\| \tilde{f}'(u) \|_* = 0$; $\tilde{f}(u)$ is then called a critical value of $\tilde{f}$.

We recall that $\tilde{f}$ is said to satisfy the $(\mathrm{PS})_c$ condition (resp. $(\mathrm{PSC})_c$ condition) at level $c \in \mathbb{R}$ if for any sequence $u_k \in M$ such that $\tilde{f}(u_k) \to c$ and $\| \tilde{f}'(u_k) \|_* \to 0$ (resp. $\tilde{f}(u_k) \to c$ and $(1 + \|u_k\|_E) \| \tilde{f}'(u_k)\|_* \to 0$), one has that u_k admits a convergent subsequence. We will also say that $\tilde{f}$ satisfies the $(\mathrm{PS})_c$ condition along bounded sequences if for any bounded sequence $u_k \in M$ such that $\tilde{f}(u_k) \to c$ and $\| \tilde{f}'(u_k) \|_* \to 0$, one has that u_k admits a convergent subsequence. Condition $(\mathrm{PSC})_c$ was introduced in [3] as a weakening of the classical $(\mathrm{PS})_c$ condition.

Going back to case (iii) of Proposition 2.1, one can see that the functional $\int_\Omega |\nabla u|^p$ restricted to the manifold $M_{m,n}$ (cf. (1.3)) does not satisfy the $(\mathrm{PS})_0$ condition. Indeed the sequence v_k from (2.3) provides an unbounded $(\mathrm{PS})_0$ sequence. That the $(\mathrm{PSC})_0$ condition does not hold neither will follow from Proposition 4.3.

3. A first nontrivial eigenvalue

The assumptions on m, n in this section are those indicated at the beginning of Section 2. We look for nonnegative eigenvalues λ of (1.2).

Clearly the only nonnegative principal eigenvalue of (1.2) are $0, \lambda^*(m)$ and $\lambda^*(n)$. Moreover multiplying by u^+ or u^-, one easily sees that if (1.2) with $\lambda \geqslant 0$ has a solution which changes sign, then $\lambda > \max\{\lambda^*(m), \lambda^*(n)\}$. Proving the existence of such a solution which changes sign, and which in addition corresponds to a minimum value of λ, is our purpose in this section.

As indicated in the introduction we will use a variational approach and consider the functional $A(u) := \int_\Omega |\nabla u|^p$ on $W^{1,p}(\Omega)$, the manifold $M_{m,n}$ defined in (1.3) and the restriction $\tilde{A}$ of A to $M_{m,n}$. In this context one easily verifies that $\lambda > 0$ is an eigenvalue of (1.2) if and only if λ is a critical value of $\tilde{A}$. The case of the eigenvalue $\lambda = 0$ is particular: it is a critical value of $\tilde{A}$ iff $M_{m,n}$ contains a constant function, i.e. iff $\int_\Omega m > 0$ or $\int_\Omega n > 0$. It follows in particular from these considerations that if $\int_\Omega m \neq 0$, then $\lambda^*(m)$ is a critical value of $\tilde{A}$ corresponding to the critical point φ_m, and similarly for $\lambda^*(n)$ and $-\varphi_n$ if $\int_\Omega n \neq 0$.

To state our main result let us introduce the following family of paths in $M_{m,n}$:

$$\Gamma := \{\gamma \in C([0,1], M_{m,n}):\ \gamma(0) \leqslant 0 \text{ and } \gamma(1) \geqslant 0\}. \tag{3.1}$$

Lemma 3.1. *Γ is nonempty.*

Proof. Choose $u \in W^{1,p}(\Omega)$ such that $\int_\Omega m(u^+)^p > 0$ and $\int_\Omega n(u^-)^p > 0$, which is possible by (2.1), and define $\gamma_1(t) := t^{1/p}u^+ - (1-t)^{1/p}u^-$ for $t \in [0,1]$. Using the fact that u^+ and u^- have disjoint supports, one obtains

$$B_{m,n}(\gamma_1(t)) = t\int_\Omega m(u^+)^p + (1-t)\int_\Omega n(u^-)^p \geqslant \min\left\{\int_\Omega m(u^+)^p, \int_\Omega n(u^-)^p\right\} > 0.$$

The path $\gamma_2(t) := \gamma_1(t)/(B_{m,n}(\gamma_1(t)))^{1/p}$ is thus well defined and clearly belongs to Γ. □

Define now the minimax value

$$c(m,n) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma[0,1]} \tilde{A}(u), \tag{3.2}$$

which is finite by Lemma 3.1.

Theorem 3.2. *$c(m,n)$ is an eigenvalue of* (1.2) *which satisfies*

$$\max\{\lambda^*(m), \lambda^*(n)\} < c(m,n). \tag{3.3}$$

Moreover there is no eigenvalue of (1.2) *between* $\max\{\lambda^*(m), \lambda^*(n)\}$ *and* $c(m,n)$.

The rest of this section is devoted to the proof of Theorem 3.2. As indicated in the introduction, some difficulty arises in connexion with the (PS) condition.

Proposition 3.3.

(i) *$\tilde{A}$ satisfies* $(\mathrm{PS})_c$ *along bounded sequences for all* $c \geqslant 0$.
(ii) *$\tilde{A}$ satisfies* $(\mathrm{PSC})_c$ *for all* $c > 0$.
(iii) *If $\int_\Omega m \neq 0$ and $\int_\Omega n \neq 0$, then $\tilde{A}$ satisfies* $(\mathrm{PS})_c$ *for all* $c \geqslant 0$.

Remark 3.4. One can show that if $p = 2$, then $\tilde{A}$ satisfies $(\mathrm{PS})_c$ for all $c > 0$, but the case $p \neq 2$ remains undecided. On the other hand, if $\int_\Omega m = 0$ or $\int_\Omega n = 0$, then $\tilde{A}$ does not satisfy $(\mathrm{PSC})_0$. This latter fact can be seen as in Section 2: assuming $\int_\Omega m = 0$, one first observes that v_k from (2.3) provides an unbounded $(\mathrm{PS})_0$ sequence for $\tilde{A}$, and then one applies Proposition 4.3 below; similar argument when $\int_\Omega n = 0$.

Proof of Proposition 3.3. (i) Let $u_k \in M_{m,n}$ be a bounded $(\mathrm{PS})_c$ sequence for $\tilde{A}$. So $\int_\Omega |\nabla u_k|^p \to c$ and

$$\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla\xi\right| \leqslant \varepsilon_k\|\xi\| \quad \forall\xi \in T_{u_k}M_{m,n}, \tag{3.4}$$

where $\varepsilon_k \to 0$ and $\|\cdot\|$ denotes the $W^{1,p}(\Omega)$ norm. For a subsequence and some $u_0 \in W^{1,p}(\Omega)$, one has that $u_k \rightharpoonup u_0$ in $W^{1,p}(\Omega)$. Let us write for $w \in W^{1,p}(\Omega)$

$$a_k(w) := w - \left[\int_\Omega \left(m(u_k^+)^{p-1} - n(u_k^-)^{p-1}\right) w\right] u_k \in T_{u_k} M_{m,n}.$$

Taking $\xi = a_k(w)$ in (3.4), one deduces

$$\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla w - \left[\int_\Omega \left(m(u_k^+)^{p-1} - n(u_k^-)^{p-1}\right) w\right]\int_\Omega |\nabla u_k|^p\right| \leqslant \varepsilon_k \left\|a_k(w)\right\| \leqslant D\varepsilon_k\left(\|u_k\|^p + 1\right)\|w\|$$

for some constant D; taking now $w = u_k - u_0$ in the above, one obtains

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k - u_0) \to 0.$$

It then follows from the (S^+) property that $u_k \to u_0$ in $W^{1,p}(\Omega)$, which yields the conclusion of part (i).

(ii) Let now $u_k \in M_{m,n}$ be a $(\mathrm{PSC})_c$ sequence for $\tilde{A}$, with $c > 0$. So $\int_\Omega |\nabla u_k|^p \to c$ and (3.4) is replaced by

$$\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla \xi\right| \leqslant \frac{\varepsilon_k}{1 + \|u_k\|}\|\xi\| \quad \forall \xi \in T_{u_k} M_{m,n} \tag{3.5}$$

where $\varepsilon_k \to 0$. We will show that u_k remains bounded so that part (i) applies and yields the conclusion of part (ii). Let us assume by contradiction that, for a subsequence, $\|u_k\| \to \infty$. Write $v_k = u_k/\|u_k\|$. For a further subsequence and some $v_0 \in W^{1,p}(\Omega)$, one has that $v_k \rightharpoonup v_0$ in $W^{1,p}(\Omega)$. Since $\int_\Omega |\nabla u_k|^p$ remains bounded, one has $\int_\Omega |\nabla v_k|^p \to 0$ and it follows easily that $v_0 \equiv cst \neq 0$ and that $v_k \to v_0$ in $W^{1,p}(\Omega)$. On the other hand, taking $\xi = a_k(w)$ in (3.5) and dividing by $\|u_k\|^{p-1}$, one gets

$$\left|\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla w - \left[\int_\Omega \left(m(v_k^+)^{p-1} - n(v_k^-)^{p-1}\right) w\right]\int_\Omega |\nabla u_k|^p\right| \\ \leqslant \varepsilon_k \frac{\|u_k\|}{1 + \|u_k\|}\left\|\frac{w}{\|u_k\|^p} - \left[\int_\Omega \left(m(v_k^+)^{p-1} - n(v_k^-)^{p-1}\right) w\right] v_k\right\|.$$

This implies that v_0 is a solution of

$$-\Delta_p v_0 = c\left[m(v_0^+)^{p-1} - n(v_0^-)^{p-1}\right] \quad \text{in } \Omega, \qquad \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{3.6}$$

where c is the level appearing in the $(\mathrm{PSC})_c$ sequence. Since $v_0 \equiv cst$, the right-hand side of (3.6) is $\equiv 0$, and since $c > 0$, one gets $m(v_0^+)^{p-1} - n(v_0^-)^{p-1} \equiv 0$. This relation with a nonzero constant v_0 implies $m \equiv 0$ or $n \equiv 0$, which contradicts (2.1).

(iii) Let us finally consider the case where $\int_\Omega m \neq 0$, $\int_\Omega n \neq 0$, and let $u_k \in M_{m,n}$ be a $(\mathrm{PS})_c$ sequence for $\tilde{A}$ with $c \geqslant 0$. We will show that u_k remains bounded so that part (i) applies and yields the conclusion. Assume that for a subsequence $\|u_k\| \to +\infty$. For a further subsequence one obtains as above that $v_k \to v_0$ in $W^{1,p}(\Omega)$ with v_0 a nonzero constant. But $B_{m,n}(u_k) = 1$ and so, dividing by $\|u_k\|^p$ and going to the limit, one obtains

$$\int_\Omega \left[m(v_0^+)^p + n(v_0^-)^p\right] = 0.$$

This is a contradiction since v_0 is a nonzero constant and $\int_\Omega m \neq 0$, $\int_\Omega n \neq 0$. $\square$

We now turn to the geometry of $\tilde{A}$. The situation here is again simpler in the nonsingular case where the following proposition applies.

Proposition 3.5. *If $\int_\Omega m \neq 0$, then $\varphi_m \in M_{m,n}$ is a strict local minimum of $\tilde{A}$, with in addition for some $\varepsilon_0 > 0$ and all $0 < \varepsilon < \varepsilon_0$,*

$$\tilde{A}(\varphi_m) = \lambda^*(m) < \inf\{\tilde{A}(u): u \in M_{m,n} \cap \partial B(\varphi_m, \varepsilon)\}, \tag{3.7}$$

where $B(\varphi_m, \varepsilon)$ denotes the ball in $W^{1,p}(\Omega)$ of center φ_m and radius ε. Similar conclusion for $-\varphi_n$ if $\int_\Omega n \neq 0$.

Proof. We only sketch it since it is adapted from [2]. One first shows that for some $\varepsilon_0 > 0$,

$$\tilde{A}(\varphi_m) < \tilde{A}(u) \quad \forall u \in M_{m,n} \cap B(\varphi_m, \varepsilon_0), u \neq \varphi_m. \tag{3.8}$$

To prove (3.8) one distinguishes two cases: (i) $\lambda^*(m) = 0$ or (ii) $\lambda^*(m) > 0$. In case (i) one chooses ε_0 such that $M_{m,n} \cap B(\varphi_m, \varepsilon_0)$ only contains φ_m as constant function. This clearly implies (3.8). In case (ii) one assumes by contradiction the existence of a sequence $u_k \in M_{m,n}$ with $u_k \neq \varphi_m$, $u_k \to \varphi_m$ in $W^{1,p}(\Omega)$ and $\tilde{A}(u_k) \leqslant \lambda^*(m)$. One then deduces, as on p. 585 of [2], that u_k changes sign for k sufficiently large. One also has

$$\lambda^*(m) \int_\Omega \left[m(u_k^+)^p + n(u_k^-)^p\right] = \lambda^*(m) \geqslant \tilde{A}(u_k) \geqslant \lambda^*(m) \int_\Omega m(u_k^+)^p + \int_\Omega |\nabla u_k^-|^p$$

and consequently

$$\lambda^*(m) \int_\Omega n^+(u_k^-)^p \geqslant \lambda^*(m) \int_\Omega n(u_k^-)^p \geqslant \int_\Omega |\nabla u_k^-|^p.$$

Since $u_k \to \varphi_m$, $|u_k^- > 0| \to 0$ where $|u_k^- > 0|$ denotes the measure of the set where u_k^- is > 0. The desired contradiction then follows from Lemma 3.6 below. Thus (3.8) is proved.

The fact that (3.8) implies (3.7) follows from Lemma 6 in [2], after observing that it suffices in this lemma that the functional satisfies (PS) along bounded sequences, a property which holds here by Proposition 3.3. This concludes the proof of Proposition 3.5 when $\int_\Omega m \neq 0$. Similar arguments when $\int_\Omega n \neq 0$. □

Lemma 3.6. *Let $v_k \in W^{1,p}(\Omega)$ with $v_k \geqslant 0$, $v_k \not\equiv 0$ and $|v_k > 0| \to 0$. Let n_k be bounded in $L^r(\Omega)$. Then*

$$\int_\Omega n_k v_k^p \Big/ \int_\Omega |\nabla v_k|^p \to 0.$$

Proof. Without loss of generality, one can assume $\|v_k\| = 1$. So for a subsequence, $v_k \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_k \to v$ in $L^p(\Omega)$. The assumption on $|v_k > 0|$ implies $v \equiv 0$ and consequently $\int_\Omega |\nabla v_k|^p \to 1$. The conclusion then follows since, by Hölder inequality, $\int_\Omega n_k v_k^p \to 0$. □

In the singular case, one at least of the two local minima provided by Proposition 3.5 is missing. The search for suitable endpoints of paths which allow the application of a mountain pass argument will be based on the following lemmas (see in particular Lemma 3.10).

Lemma 3.7. *Inequality* (3.3) *holds.*

Proof. The inequality $\leqslant$ easily follows from the definition of $\lambda^*(m)$ and $\lambda^*(n)$. Indeed for any $\gamma \in \Gamma$, $\gamma(1)$ belongs to $M_{m,n}$, is $\geqslant 0$ and so satisfies the constraint in the definition (2.2) of $\lambda^*(m)$. Consequently $c(m,n) \geqslant \lambda^*(m)$, and a similar argument applies to $\lambda^*(n)$. To prove the strict inequality assume by contradiction that for instance $\lambda^*(m) = c(m,n)$. So, there exists a sequence $\gamma_k \in \Gamma$ such that

$$\max_{t\in[0,1]} \tilde{A}(\gamma_k(t)) \to \lambda^*(m). \tag{3.9}$$

Put $u_k := \gamma_k(1)$. Since $u_k \geqslant 0$, one has

$$\lambda^*(m) \leqslant \int_\Omega |\nabla u_k|^p \leqslant \max_{t\in[0,1]} \tilde{A}(\gamma_k(t)) \to \lambda^*(m), \tag{3.10}$$

and consequently $\int_\Omega |\nabla u_k|^p \to \lambda^*(m)$. Let us now distinguish two cases: either (i) $\|u_k\|$ remains bounded or (ii) for a subsequence $\|u_k\| \to \infty$.

In case (i), for a subsequence and for some $u_0 \in W^{1,p}(\Omega)$, one has that $u_k \rightharpoonup u_0$ in $W^{1,p}(\Omega)$. Since $u_k \geqslant 0$, one has

$$\int_\Omega m|u_0|^p = 1, \tag{3.11}$$

and so

$$\lambda^*(m) \leqslant \int_\Omega |\nabla u_0|^p \leqslant \liminf \int_\Omega |\nabla u_k|^p = \lambda^*(m),$$

which implies that $\int_\Omega |\nabla u_0|^p = \lambda^*(m)$. Consequently $u_k \to u_0$ in $W^{1,p}(\Omega)$. If $\int_\Omega m = 0$, then $\lambda^*(m) = 0$ and so $u_0 \equiv cst$, which leads to a contradiction with (3.11). So $\int_\Omega m \neq 0$ and we conclude that $u_0 = \varphi_m$. Let us now choose $\varepsilon > 0$ such that (3.7) holds and $B(\varphi_m, \varepsilon)$ does not contain any function v with $v \leqslant 0$, which is clearly possible. For k sufficiently large $u_k = \gamma_k(1) \in B(\varphi_m, \varepsilon)$, while $\gamma_k(0) \notin B(\varphi_m, \varepsilon)$ since $\gamma_k(0) \leqslant 0$. It follows that the path γ_k intersects $\partial B(\varphi_m, \varepsilon)$ and consequently

$$\max_{t\in[0,1]} \tilde{A}\big(\gamma_k(t)\big) \geqslant \inf\big\{\tilde{A}(u)\colon u \in M_{m,n} \cap \partial B(\varphi_m, \varepsilon)\big\} > \lambda^*(m).$$

This contradicts (3.9).

In case (ii) we put $v_k = u_k/\|u_k\|$. For a subsequence and some $v_0 \in W^{1,p}(\Omega)$, $v_k \rightharpoonup v_0$ in $W^{1,p}(\Omega)$. Since $\int_\Omega |\nabla u_k|^p$ remains bounded, we obtain $\int_\Omega |\nabla v_k|^p \to 0$ and so $v_0 \equiv cst$; also $v_0 \not\equiv 0$ since $\|v_k\| = 1$ implies $\|v_0\| = 1$. Moreover $\int_\Omega m|v_0|^p = 0$ since $\int_\Omega m|u_k|^p = 1$. We have reached a contradiction if $\int_\Omega m \neq 0$. So let us assume from now on that $\int_\Omega m = 0$. We first observe that for any $\gamma \in \Gamma$ there exists $t_0 = t_0(\gamma) \in [0, 1]$ such that

$$\int_\Omega m\big(\gamma(t_0)^+\big)^p = \int_\Omega n\big(\gamma(t_0)^-\big)^p = \frac{1}{2}. \tag{3.12}$$

Consider now $w_k := \gamma_k(t_0(\gamma_k))$. We have now instead of (3.10)

$$0 \leqslant \int_\Omega |\nabla w_k|^p \leqslant \max_{t\in[0,1]} \tilde{A}\big(\gamma_k(t)\big) \to \lambda^*(m) = 0. \tag{3.13}$$

We again distinguish two cases: either $\|w_k\|$ remains bounded, or for a subsequence $\|w_k\| \to \infty$. In the first case, for a subsequence and some $w_0 \in W^{1,p}(\Omega)$, $w_k \rightharpoonup w_0$ in $W^{1,p}(\Omega)$. It follows from (3.13) that $w_0 \equiv cst$ and that $w_k \to w_0$ in $W^{1,p}(\Omega)$. A contradiction then follows from

$$\int_\Omega m(w_0^+)^p = \int_\Omega n(w_0^-)^p = \frac{1}{2}.$$

In the second case we put $z_k := w_k/\|w_k\|$. For a subsequence and some $z_0 \in W^{1,p}(\Omega), z_k \rightharpoonup z_0$ in $W^{1,p}(\Omega)$. It follows from (3.13) that $z_0 \equiv cst$ and that $z_k \to z_0$ in $W^{1,p}(\Omega)$; consequently $\|z_0\| = 1$. If $z_0 > 0$ then $|w_k < 0| = |z_k < 0| \to 0$; moreover w_k changes sign and by (3.12)

$$\frac{\int_\Omega n^+|w_k^-|^p}{\int_\Omega |\nabla w_k^-|^p} \geqslant \frac{1/2}{\int_\Omega |\nabla w_k|^p} \to +\infty.$$

This yields a contradiction with Lemma 3.6. A similar argument applies if $z_0 < 0$. □

Lemma 3.8. *For any $d > 0$, the set*

$$\mathcal{O} := \big\{u \in M_{m,n}\colon u \geqslant 0 \text{ and } \tilde{A}(u) < d\big\}$$

is arcwise connected. Similar conclusion if $u \geqslant 0$ is replaced by $u \leqslant 0$.

Note that by the definition of $c(m,n)$, $\{u \in M_{m,n}\colon\ \tilde{A}(u) < d\}$ is not arcwise connected when $\max\{\lambda^*(m), \lambda^*(n)\} < d < c(m,n)$.

Proof of Lemma 3.8. Since $\mathcal{O}$ is empty if $d \leqslant \lambda^*(m)$, we can assume from now on $d > \lambda^*(m)$. We first consider the case where $\int_\Omega m \neq 0$. Using Lemma 3.9 below, one constructs a weight $\hat{n} \in L^r(\Omega)$ such that $\hat{n}^+ \not\equiv 0$, $\hat{n} \leqslant m$, $\int_\Omega \hat{n} < 0$ and $\lambda^*(\hat{n}) > d$. When $m^- \not\equiv 0$, it suffices in this construction to take $\hat{n} = \varepsilon m^+ - m^-$ with $\varepsilon > 0$ sufficiently small; when $m^- = 0$ i.e. $m \geqslant 0$, it suffices to take $\hat{n} = \varepsilon m - k\chi_B$ with ε sufficiently small and k sufficiently large, where χ_B is the characteristic function of a ball $B \Subset \Omega$ such that $m^+ \not\equiv 0$ on $\Omega \setminus B$. We then consider the manifold $M_{m,\hat{n}}$ and the sublevel set

$$\widehat{\mathcal{O}} := \{u \in M_{m,\hat{n}}\colon\ A(u) < d\}.$$

By part (iii) of Proposition 3.3, the restriction $\hat{A}$ of A to $M_{m,\hat{n}}$ satisfies $(\mathrm{PS})_c$ for all $c \geqslant 0$. Lemma 14 from [2] then implies that any (nonempty) component of $\widehat{\mathcal{O}}$ contains a critical point of $\hat{A}$. But the first two critical levels $\lambda^*(m), \lambda^*(\hat{n})$ of $\hat{A}$ verify $\lambda^*(m) < d < \lambda^*(n)$, and consequently $\hat{A}$ admits only one critical point in $\widehat{\mathcal{O}}$. We can conclude in this way that $\widehat{\mathcal{O}}$ is arcwise connected.

Let now $u_1, u_2 \in \mathcal{O}$. Since they are $\geqslant 0$, they also belong to $\widehat{\mathcal{O}}$. Let γ be a path in $\widehat{\mathcal{O}}$ from u_1 to u_2 and consider the path

$$\gamma_1(t) := \frac{|\gamma(t)|}{(\int_\Omega m|\gamma(t)|^p)^{1/p}}.$$

By the choice of $\hat{n}$,

$$\int_\Omega m\big|\gamma(t)\big|^p \geqslant \int_\Omega \big[m\big(\gamma(t)^+\big)^p + \hat{n}\big(\gamma(t)^-\big)^p\big] = 1, \tag{3.14}$$

and consequently γ_1 is a well defined path in $M_{m,n}$, which clearly goes from u_1 to u_2 and is made of nonnegative functions. Moreover, by (3.14),

$$A\big(\gamma_1(t)\big) = \frac{A(\gamma(t))}{\int_\Omega m|\gamma(t)|^p} \leqslant A\big(\gamma(t)\big) < d$$

for all t, and we conclude that the path γ_1 lies in $\mathcal{O}$.

Consider now the case where $\int_\Omega m = 0$. Let $u_1, u_2 \in \mathcal{O}$. One starts by decreasing a little bit the weight m into a weight $\hat{m} \in L^r(\Omega)$ such that $\hat{m} \leqslant m$, $\int_\Omega \hat{m} < 0$, $\int_\Omega \hat{m}u_1^p > 0$, $\int_\Omega \hat{m}u_2^p > 0$ and

$$\frac{\int_\Omega |\nabla u_1|^p}{\int_\Omega \hat{m}u_1^p} < d, \qquad \frac{\int_\Omega |\nabla u_2|^p}{\int_\Omega \hat{m}u_2^p} < d,$$

which is clearly possible since $\lambda^*(m) < d$. Put

$$v_1 := \frac{u_1}{(\int_\Omega \hat{m}u_1^p)^{1/p}} \quad \text{and} \quad v_2 := \frac{u_2}{(\int_\Omega \hat{m}u_2^p)^{1/p}}.$$

By the first part of this proof, there exists a path γ in $M_{\hat{m},\hat{m}}$ which goes from v_1 to v_2, is made of nonnegative functions and is such that $A(\gamma(t)) < d$ for all t. Consider now the path

$$\gamma_1(t) := \frac{\gamma(t)}{(\int_\Omega m|\gamma(t)|^p)^{1/p}}.$$

By the choice of $\hat{m}$,

$$\int_\Omega m\big|\gamma(t)\big|^p \geqslant \int_\Omega \hat{m}\big|\gamma(t)\big|^p = 1, \tag{3.15}$$

and consequently γ_1 is a well defined path in $M_{m,n}$, which clearly goes from u_1 to u_2 and is made of nonnegative functions. Moreover, by (3.15),

$$A\big(\gamma_1(t)\big) = \frac{A(\gamma(t))}{\int_\Omega m|\gamma(t)|^p} \leqslant A\big(\gamma(t)\big) < d$$

for all t. This concludes the proof of Lemma 3.8 for $\mathcal{O}$ with $u \geqslant 0$. Similar argument in the case $u \leqslant 0$. □

Lemma 3.9. *Let $m_k \in L^r(\Omega)$ with $m_k^+ \not\equiv 0$ and $m_k \to m$ in $L^r(\Omega)$ where $m \leqslant 0$, $m \not\equiv 0$. Then $\lambda^*(m_k) \to +\infty$.*

Proof. Suppose by contradiction that for a subsequence, $\lambda^*(m_k) \to \lambda < +\infty$. Let φ_k be the positive eigenfunction associated to $\lambda^*(m_k)$ and normalized by $\|\varphi_k\|_{pr'} = 1$, where $\|\cdot\|_q$ denotes the $L^q(\Omega)$ norm. One has

$$\int_\Omega |\nabla\varphi_k|^p = \lambda^*(m_k)\int_\Omega m_k\varphi_k^p \leqslant \lambda^*(m_k)\|m_k^+\|_r.$$

It follows that for a subsequence, $\varphi_k \rightharpoonup \varphi$ in $W^{1,p}(\Omega)$, with $\|\varphi\|_{pr'} = 1$. Moreover, by the above inequality, $\int_\Omega |\nabla\varphi_k|^p \to 0$, which implies $\varphi \equiv cst \neq 0$ (call it A) and $\varphi_k \to \varphi$ in $W^{1,p}(\Omega)$. Consequently, for k sufficiently large so that $\int_\Omega m_k < 0$, one has

$$0 < \frac{1}{\lambda^*(m)}\int_\Omega |\nabla\varphi_k|^p = \int_\Omega m_k\varphi_k^p \to A^p\int_\Omega m < 0,$$

a contradiction. □

Lemma 3.10. *There exists $u_1 \geqslant 0$ and $u_2 \leqslant 0$ in $M_{m,n}$ such that $\tilde{A}(u_1) < c(m,n)$ and $\tilde{A}(u_2) < c(m,n)$. Moreover, for any such choice of u_1, u_2, one has*

$$c(m,n) = \inf_{\gamma\in\overline{\Gamma}}\max_{u\in\gamma[0,1]}\tilde{A}(u) \tag{3.16}$$

where

$$\overline{\Gamma} := \{\gamma \in C([0,1], M_{m,n}):\ \gamma(0) = u_2 \text{ and } \gamma(1) = u_1\}.$$

Proof. If $\int_\Omega m \neq 0$, one takes $u_1 = \varphi_m$ and the inequality $\tilde{A}(u_1) < c(m,n)$ follows from Lemma 3.7. Similarly with $u_2 = -\varphi_n$ in case $\int_\Omega n \neq 0$. If now $\int_\Omega m = 0$, one takes $u_1 = v_k$ for k sufficiently large, where v_k is defined in (2.3). Indeed $\tilde{A}(v_k) \to 0$ and by Lemma 3.7, $0 < c(m,n)$, so that $\tilde{A}(v_k) < c(m,n)$ for k sufficiently large. Similar argument for the choice of u_2 in case $\int_\Omega n = 0$.

It remains to prove (3.16). Call $\bar{c}$ the right-hand side of (3.16). One clearly has $c(m,n) \leqslant \bar{c}$. To prove the converse inequality, let $\varepsilon > 0$ and take $\gamma_\varepsilon \in \Gamma$ such that

$$\max_{u\in\gamma_\varepsilon[0,1]}\tilde{A}(u) < c(m,n) + \varepsilon.$$

By Lemma 3.8 there exits a path η_1 in $M_{m,n}$ joining $\gamma_\varepsilon(1)$ and u_1, made of nonnegative functions, and such that

$$\max_{u\in\eta_1[0,1]}\tilde{A}(u) < c(m,n) + \varepsilon.$$

Similarly there exists a path η_2 in $M_{m,n}$ joining $\gamma_\varepsilon(0)$ and u_2, made of nonpositive functions, and such that

$$\max_{u\in\eta_2[0,1]}\tilde{A}(u) < c(m,n) + \varepsilon.$$

Gluing together η_2, γ_ε and η_1, one gets a path in $M_{m,n}$ joining u_2 and u_1, and such that $\tilde{A}$ remains $< c(m,n) + \varepsilon$ along this path. This implies $\bar{c} < c(m,n) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

We are now ready to give the

Proof of Theorem 3.2. Inequality (3.3) was established in Lemma 3.7. To prove that $c(m,n)$ is an eigenvalue, we pick u_1, u_2 as in Lemma 3.10 and we will show that $\bar{c}$, the right-hand side of (3.16), is a critical value of $\tilde{A}$. If $\int_\Omega m \neq 0$ and $\int_\Omega n \neq 0$, then $\tilde{A}$ satisfies $(\mathrm{PS})_c$ for all $c \geqslant 0$ and the classical mountain pass theorem for a C^1 functional on a C^1 manifold (cf. e.g. Proposition 4 from [2]) yields the conclusion. If either $\int_\Omega m = 0$ or $\int_\Omega n = 0$, then we only know that $\tilde{A}$ satisfies $(\mathrm{PSC})_c$ for all $c > 0$. It is then Theorem 4.1 from the following section which yields the conclusion.

It remains to show that there is no eigenvalue between $\max\{\lambda^*(m), \lambda^*(n)\}$ and $c(m,n)$. Assume by contradiction the existence of such an eigenvalue λ and let u be the corresponding nontrivial solution of (1.2). We know that u changes sign (since $\lambda > \max\{\lambda^*(m), \lambda^*(n)\}$); moreover

$$0 < \int_\Omega |\nabla u^+|^p = \lambda \int_\Omega m(u^+)^p, \qquad 0 < \int_\Omega |\nabla u^-|^p = \lambda \int_\Omega n\big(u^{-p}\big),$$

and we can normalize u so that $u \in M_{m,n}$. The functions

$$u_1 := \frac{u^+}{(\int_\Omega m(u^+)^p)^{1/p}}, \qquad u_2 := \frac{-u^-}{(\int_\Omega n(u^-)^p)^{1/p}}$$

belongs to $M_{m,n}$, with $u_1 \geqslant 0$, $u_2 \leqslant 0$. We will construct a path γ in $M_{m,n}$ joining u_1 and u_2, and such that $\tilde{A}$ remains equal to λ along that path. This will give a contradiction with the definition of $c(m,n)$. To construct γ we first go from u_1 to u by the path

$$\gamma_1(t) := \frac{u^+ - tu^-}{(B_{m,n}(u^+ - tu^-))^{1/p}}$$

and then from u to u_2 by the path

$$\gamma_2(t) := \frac{tu^+ - u^-}{(B_{m,n}(tu^+ - u^-))^{1/p}}.$$

It is easily verified that the path constructed in this way is well defined and satisfies all the required conditions. □

Remark 3.11. Reproducing the end of the above proof with λ replaced by $c(m,n)$, we conclude that the infimum in (3.2) is achieved.

4. A mountain pass theorem

Our purpose in this section is to derive a mountain pass theorem for a C^1 functional on a C^1 manifold and which satisfies the (PSC) condition.

We put ourselves in the general setting of the end of Section 2: E is a real Banach space, $g \in C^1(E,\mathbb{R})$, $M := \{u \in E\colon\ g(u) = 1\}$ with 1 a regular value of g, $f \in C^1(E,\mathbb{R})$, $\tilde{f}$ the restriction of f to M. The space E in this section is assumed to be uniformly convex.

Theorem 4.1. *Let K be a compact metric space, $K_0 \subset K$, and $h_0 \in C(K_0, M)$. Consider the family of extensions of h_0:*

$$\mathcal{H} := \big\{h \in C(K,M)\colon\ h_{|K_0} = h_0\big\}.$$

Assume $\mathcal{H}$ nonempty as well as the following condition:

$$\max_{t\in K_0} f\big(h_0(t)\big) < \max_{t\in K} f\big(h(t)\big)$$

for any $h \in \mathcal{H}$. Define

$$c := \inf_{h\in H} \max_{t\in K} f\big(h(t)\big). \tag{4.1}$$

Assume that $\tilde{f}$ satisfies $(\mathrm{PSC})_c$ *for c given in* (4.1). *Then c is a critical value of $\tilde{f}$.*

Typically, as in the application in Section 3, $K = [0,1]$ and $K_0 = \{0,1\}$.

Proof of Theorem 4.1. Arguing as in the proof of Theorem 2.1 in [5] but using the strong form of Ekeland variational principle (cf. [8,7]) instead of the usual one, one obtains that if $h \in \mathcal{H}$ and $\varepsilon > 0$ are such that

$$\max_{t\in K} f\big(h(t)\big) < c + \frac{\varepsilon}{2}, \tag{4.2}$$

then, for each $\mu > 0$, there exists $u_\mu \in M$ with

$$c \leqslant f(u_\mu) \leqslant c + \frac{\varepsilon}{2},$$
$$\operatorname{dist}\big(u_\mu, h(K)\big) \leqslant \mu,$$
$$\big\|\tilde{f}'(u_\mu)\big\|_* \leqslant \frac{\varepsilon}{\mu}.$$

We let $\varepsilon = \frac{1}{k}$ and pick $h = h_k$ such that (4.2) holds, which is possible by the definition (4.1) of c. We also take $\mu = \mu_k = 1 + \|h_k\|_\infty$, where $\|\cdot\|_\infty$ denotes the $C(K, E)$ norm. So there exists $u_k \in M$ such that

$$c \leqslant f(u_k) \leqslant c + \frac{1}{2k},$$
$$\operatorname{dist}\big(u_k, h_k(K)\big) \leqslant 1 + \|h_k\|_\infty, \tag{4.3}$$
$$\big\|\tilde{f}'(u_k)\big\|_* \leqslant \big[k\big(1 + \|h_k\|_\infty\big)\big]^{-1}. \tag{4.4}$$

It follows from (4.3) that

$$\|u_k\|_E \leqslant \operatorname{dist}\big(u_k, h_k(K)\big) + \|h_k\|_\infty \leqslant 1 + 2\|h_k\|_\infty$$

and so $1 + \|u_k\|_E \leqslant 2(1 + \|h_k\|_\infty)$. Replacing in (4.4) gives

$$\big\|\tilde{f}'(u_k)\big\|_* \leqslant \big[2k\big(1 + \|u_k\|_E\big)\big]^{-1}.$$

Thus u_k is a $(\mathrm{PSC})_c$ sequence, and the conclusion follows. □

The following additional information will be used later (cf. Proposition 2.3 in [5]).

Proposition 4.2. *Let $K, K_0, h_0, \mathcal{H}$ and c be as in Theorem* 4.1 *and let $h \in \mathcal{H}$ satisfy*

$$\max_{t \in K} f\big(h(t)\big) = c.$$

Then $h(K)$ contains a critical point of $\tilde{f}$ at level c.

The strong form of Ekeland variational principle can also be used in a way rather similar to the above to derive

Proposition 4.3. *Assume $\tilde{f}$ bounded from below and let $c := \inf\{f(u)\colon u \in M\}$. Then $\tilde{f}$ satisfies* $(\mathrm{PSC})_c$ *if and only if $\tilde{f}$ satisfies* $(\mathrm{PS})_c$.

Proof. It clearly suffices to prove that $(\mathrm{PSC})_c$ implies $(\mathrm{PS})_c$. Let $u_k \in M$ satisfy $c \leqslant f(u_k) \leqslant c + 1/k$ with $\|\tilde{f}'(u_k)\|_* \to 0$ and $\|u_k\| \to \infty$ (if $\|u_k\|$ remains bounded then u_k is a $(\mathrm{PSC})_c$ sequence and the conclusion follows immediately). Using the strong form of Ekeland variational principle, we obtain for any k and any $\mu > 0$ the existence of $v_\mu \in M$ with

$$c \leqslant f(v_\mu) \leqslant c + \frac{1}{2k},$$
$$\|v_\mu - u_k\| \leqslant \mu,$$
$$\big\|\tilde{f}'(v_\mu)\big\|_* \leqslant \frac{1}{k\mu}.$$

We take $\mu = \|u_k\|/2$ and we write v_k instead of v_μ. We have

$$\frac{1}{2}\|u_k\| \leqslant \|v_k\| \leqslant \frac{3}{2}\|u_k\|,$$
$$\big(1 + \|v_k\|\big)\big\|\tilde{f}'(v_k)\big\|_* \leqslant \frac{1}{k}\frac{2}{\|u_k\|}\left(1 + \frac{3}{2}\|u_k\|\right) \leqslant \frac{cst}{k}, \tag{4.5}$$

and so, by $(\mathrm{PSC})_c$, v_k has a subsequence v_{n_k} which converges. Combining with (4.5) leads to a contradiction with the fact that $\|u_k\| \to \infty$. □

5. Some properties of $c(m,n)$

We briefly study here the dependence of $c(m,n)$ with respect to m,n. All the weights in this section are assume to belong to $L^r(\Omega)$ and to satisfy (2.1). The results and proofs below are similar to those of Section 4 in [2].

Proposition 5.1. *If $(m_k,n_k)\to(m,n)$ in $L^r(\Omega)\times L^r(\Omega)$, then $c(m_k,n_k)\to c(m,n)$.*

Proof. It is easily adapted from that of Proposition 22 in [2]. One successively proves upper and lower semicontinuity. In the latter part it is convenient here to normalize u_k so that $\|u_k\|_{pr'}=1$. □

Proposition 5.2. *If $m\leqslant\tilde m$ and $n\leqslant\tilde n$ a.e., then*

$$c(\tilde m,\tilde n)\leqslant c(m,n).$$

If in addition

$$\int_\Omega(\tilde m-m)(u^+)^p+\int_\Omega(\tilde n-n)(u^-)^p>0$$

for at least one eigenfunction u associated to $c(m,n)$, then $c(\tilde m,\tilde n)<c(m,n)$.

Proof. It is easily adapted from that of Propositions 23 and 25 of [2]. Proposition 4.2 is used to derive the strict monotonicity. □

Finally let us observe that $c(m,n)$ is homogeneous of degree -1. Some sort of separate sub-homogeneity also holds, which will be used later:

Proposition 5.3. *If $0<s<\hat s$, then*

$$c(\hat s m,n)<c(sm,n)\quad\text{and}\quad c(m,\hat s n)<c(m,sn).$$

Proof. It is easily adapted from that of Proposition 31 of [2]. Again Proposition 4.2 is used here. □

6. Fučik spectrum

Let $m,n\in L^r(\Omega)$ with r as before. Unless otherwise stated, we also assume (2.1). The Fučik spectrum $\Sigma=\Sigma(m,n)$ clearly contains the lines $\{0\}\times\mathbb{R}$, $\mathbb{R}\times\{0\}$, $\mathbb{R}\times\{\lambda^*(n)\}$, $\{\lambda^*(m)\}\times\mathbb{R}$ and also possibly the lines $\mathbb{R}\times\{-\lambda^*(-n)\}$ and $\{-\lambda^*(-m)\}\times\mathbb{R}$. It will be convenient to denote by $\Sigma^*=\Sigma^*(m,n)$ the set $\Sigma(m,n)$ without these 2, 3 or 4 lines.

We start by looking at the part of Σ^* which lies in $\mathbb{R}^+\times\mathbb{R}^+$. From the properties of $\lambda^*(m)$, $\lambda^*(n)$ follows that if $(\alpha,\beta)\in\Sigma^*\cap(\mathbb{R}^+\times\mathbb{R}^+)$, then $\alpha>\lambda^*(m)$ and $\beta>\lambda^*(n)$.

Theorem 6.1. *For any $s>0$, the line $\beta=s\alpha$ in the (α,β) plane intersects $\Sigma^*\cap(\mathbb{R}^+\times\mathbb{R}^+)$. Moreover the first point in this intersection is given by $\alpha(s)=c(m,sn)$, $\beta(s)=s\alpha(s)$, where $c(\cdot,\cdot)$ is defined in* (3.2).

Proof. An easy consequence of Theorem 3.2. □

Letting $s>0$ varying, we thus get a first curve $\mathcal{C}:=\{(\alpha(s),\beta(s))\colon s>0\}$ in $\Sigma^*\cap(\mathbb{R}^+\times\mathbb{R}^+)$.

Proposition 6.2. *The functions $\alpha(s)$ and $\beta(s)$ in Theorem* 6.1 *are continuous. Moreover $\alpha(s)$ is strictly decreasing and $\beta(s)$ is strictly increasing. One also has that $\alpha(s)\to+\infty$ as $s\to0$ and $\beta(s)\to+\infty$ as $s\to+\infty$.*

Proof. The first two statements are direct consequences of the results of Section 5. The last one easily follows from Lemma 6.3 below. □

Lemma 6.3. *The lines* $\mathbb{R} \times \{\lambda^*(n)\}$ *and* $\{\lambda^*(m)\} \times \mathbb{R}$ *are isolated in* $\Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$.

Proof. Assume by contradiction the existence of a sequence $(\alpha_k, \beta_k) \in \Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ such that $\alpha_k \to \alpha_0$, $\beta_k \to \beta_0$ with $\alpha_0 \in \mathbb{R}$ and say $\beta_0 = \lambda^*(n)$. Let u_k be an eigenfunction corresponding to (α_k, β_k), normalized by $\|u_k\|_{pr'} = 1$. Note that u_k changes sign. By standard arguments one gets that for a subsequence, $u_k \to u$ in $W^{1,p}(\Omega)$ with $u \not\equiv 0$ satisfying

$$-\Delta_p u = \alpha_0 m(x)(u^+)^{p-1} - \lambda^*(n)n(x)(u^-)^{p-1} \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{6.1}$$

It follows from (6.1) that

$$\int_\Omega |\nabla u^-|^p = \lambda^*(n) \int_\Omega n(x)(u^-)^p$$

and consequently either (i) $u^- \equiv 0$ or (ii) u^- is an eigenfunction associated to $\lambda^*(n)$. In case (i), $u \geqslant 0, u \not\equiv 0$ and so $u > 0$ in Ω, which implies $|u_k^- > 0| \to 0$. It then follows from Lemma 3.6 that

$$\frac{\int_\Omega \beta_k n(x)(u_k^-)^p}{\int_\Omega |\nabla u_k^-|^p} \to 0, \tag{6.2}$$

which is impossible since by the equation satisfied by u_k, the expression in (6.2) is equal to 1. In case (ii), $u < 0$ in Ω, which implies $|u_k^+ > 0| \to 0$. An argument as above applied to u_k^+ then leads to a contradiction. □

We now investigate the asymptotics values $\alpha_\infty := \lim_{s\to\infty} \alpha(s)$ and $\beta_\infty := \lim_{s\to\infty} \beta(s)$ of the first curve $\mathcal{C}$. We will limit ourselves to the study of α_∞. Similar results on β_∞ can be proved interchanging the roles of m and n.

Proposition 6.4. *If* $p \leqslant N$, *then* $\alpha_\infty = \lambda^*(m)$. *If* $p > N$ *and one of the following conditions holds*: (i) $\int_\Omega m \geqslant 0$, *or* (ii) $m \in L^\infty(\Omega)$, *or* (iii) $\operatorname{supp}(n^+) \Subset \Omega$, *then* $\alpha_\infty > \lambda^*(m)$.

Proof. The arguments are rather similar to those in the proof of Proposition 35 in [2] and we will only indicate below the main steps as well as the differences.

One starts by introducing

$$\bar{\alpha} := \inf\left\{\int_\Omega |\nabla u^+|^p \colon u \in W^{1,p}(\Omega), \int_\Omega m(u^+)^p = 1 \text{ and } \int_\Omega n(u^-)^p > 0\right\} \tag{6.3}$$

and shows that $\alpha_\infty = \bar{\alpha}$. The proof here is a direct adaption of our argument on p. 599 in [2]. In fact it is simpler since the required path has just to belong to Γ and so can be constructed directly through a normalized convex combination.

One then considers the case $p \leqslant N$. If $\int_\Omega m \neq 0$ then the argument on p. 600 in [2] adapts immediately to obtain $\bar{\alpha} = \lambda^*(m)$. If $\int_\Omega m = 0$ one considers the sequence v_k defined in (2.3) and applies to each v_k the construction on p. 600 in [2].

One now considers the case $p > N$ Assume by contradiction that $\bar{\alpha} = \lambda^*(m)$ and let (u_k) be a minimizing sequence for $\bar{\alpha}$. We claim that u_k^+ remains bounded in $W^{1,p}(\Omega)$. Indeed otherwise, one easily sees that for a subsequence $v_k := u_k^+/\|u_k^+\|$ converges uniformly on $\overline{\Omega}$ to a positive constant, which implies $v_k > 0$ (and so $u_k > 0$) in $\overline{\Omega}$ for k sufficiently large, contradicting the admissibility of u_k in definition (6.3). So u_k^+ remains bounded in $W^{1,p}(\Omega)$ and it follows by standard arguments that for a subsequence, u_k^+ converges uniformly on $\overline{\Omega}$ to φ_m when $\int_\Omega m \neq 0$, to a positive constant when $\int_\Omega m = 0$. In case (i) φ_m is also a positive constant, and we conclude as above that u_k is not admissible in definition (6.3) for k sufficiently large, a contradiction. In case (ii) $\varphi_m \in C^1(\overline{\Omega})$ by [12] and the strong maximum principle of [14] applies to guarantee that φ_m is positive on $\overline{\Omega}$; a contradiction can then be derived as above. Finally in case (iii) one has $\varphi_m \geqslant$ some $\epsilon > 0$ on $\operatorname{supp}(n^+)$, and one deduces again that u_k is not admissible in definition (2.2) for k sufficiently large, a contradiction. □

Remark 6.5. When the weight m is unbounded and $\int_\Omega m < 0$, it is unclear whether φ_m is positive on $\overline{\Omega}$. This is why we impose condition (iii) in Proposition 6.4.

We finally observe that the distribution of Σ^* in the other quadrants of $\mathbb{R} \times \mathbb{R}$ could be studied here in a manner similar to that in [2].

One could also adapt to the present setting the results of [2] relative to the study of nonresonance.

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