The Neumann problem for the Hénon equation, trace inequalities and Steklov eigenvalues ☆

Le problème de Neumann pour l'équation de Hénon, inégalités de trace et valeurs propres de Steklov

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Abstract

We consider the Neumann problem for the Hénon equation. We obtain existence results and we analyze the symmetry properties of the ground state solutions. We prove that some symmetry and variational properties can be expressed in terms of eigenvalues of a Steklov problem. Applications are also given to extremals of certain trace inequalities.
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Résumé

On considère le problème de Neumann pour l'équation de Hénon. On obtient des résultats d'existence et on analyse les propriétés de symétrie des solutions à énergie minimale. Nous démontrons que certaines propriétés variationnelles et de symétrie peuvent être exprimées à l'aide des valeurs propres d'un problème de Steklov. En outre, nous appliquons ces résultats aux fonctions réalisant la meilleure constante de certaines inégalités de trace.
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1. Introduction and statement of the main results

The elliptic equation appearing in the Dirichlet problem

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is the unit ball of $\mathbf{R}^N$, $p > 2$ and $\alpha > 0$, was introduced in the paper [12] by M. Hénon and now bears his name. In [12], problem (1) was proposed as a model for spherically symmetric stellar clusters and was investigated numerically for some definite values of p and α.

In the last few years it became evident that in spite of (or thanks to) its simple appearance, the Hénon equation exhibits very interesting features concerning existence, multiplicity and, above all, symmetry properties of solutions.

Research has been directed up to now only on the Dirichlet problem (1) with the intent of classifying the range of solvability (in p) and especially of analyzing the symmetry properties of the *ground state* solutions.

In order to provide a clear motivation for the results contained in the present paper, we briefly recall some of the main achievements concerning problem (1). We denote, as usual, $2^* = \frac{2N}{N-2}$.

The first existence result is due to Ni, who in [14] proved that for every $p \in (2, 2^* + \frac{2\alpha}{N-2})$, problem (1) admits at least one *radial* solution and pointed out that it is the presence of the weight $|x|^\alpha$ that enlarges the existence range beyond the usual critical exponent.

The most important question for our results is that of symmetry of solutions. The starting point is the fact that since the function $r \mapsto r^\alpha$ is increasing, Gidas–Ni–Nirenberg type results [11] do not apply, and therefore nonradial solutions could be expected. This is the content of the paper [18] by Smets, Su and Willem, who studied the ground state solutions associated to (1). They proved in particular the following symmetry breaking result.

Theorem 1.1 (*Smets, Su, Willem*). *For every $p \in (2, 2^*)$ no ground state for problem* (1) *is radial provided α is large enough.*

Further results on the Dirichlet problem can be found in [19,6,4,5] for residual symmetry properties and asymptotic behavior of ground states (for $p \to 2^*$ or $\alpha \to \infty$) and in [16,3,15] for existence and multiplicity of nonradial solutions for critical, supercritical and slightly subcritical growth; see also [17] for symmetry breaking results for Trudinger–Moser type nonlinearities.

We stress once more that all the above results have been obtained for the homogeneous Dirichlet problem, while it seems that so far the *Neumann* problem has never been studied. The purpose of this paper is to fill this gap and to point out a series of new, interesting and unexpected phenomena that arise in passing from Dirichlet to Neumann boundary conditions.

To describe our results we let Ω be the unit ball of $\mathbf{R}^N$, with $N \geqslant 3$, and we consider the Neumann analogue to (1), namely

$$\begin{cases} -\Delta u + u = |x|^\alpha u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where again $p > 2$ and $\alpha > 0$. We have denoted by ν the outer normal to $\partial\Omega$.

Solutions of (2) arise from critical points of the functional $Q_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbf{R}$ defined by

$$Q_\alpha(u) = \frac{\int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx}{(\int_\Omega |u|^p |x|^\alpha \, dx)^{2/p}} = \frac{\|u\|^2}{(\int_\Omega |u|^p |x|^\alpha \, dx)^{2/p}}.$$

This functional of course is well defined if $p \in (2, 2^*)$, but as we shall see, its restriction to the space of $H^1_{\text{rad}}(\Omega)$ of H^1 radial functions is still well defined if $p \in (2, 2^* + \frac{2\alpha}{N-2})$.

We will call *ground states* the functions that minimize Q_α over $H^1(\Omega)$, while we reserve the term "radial minimizer" to functions that minimize Q_α over $H^1_{\text{rad}}(\Omega)$.

The purpose of this paper is to investigate the existence of solutions to (2), in the spirit of [14], and especially to carry out the analysis of the symmetry properties of the ground states of Q_α.

As far as existence of solutions is concerned, our first result matches completely the one for the Dirichlet problem obtained by Ni in [14].

Theorem 1.2. *For every $\alpha > 0$ and every $p \in (2, 2^* + \frac{2\alpha}{N-2})$, there exists $u \in H^1_{\mathrm{rad}}(\Omega)$ such that*

$$Q_\alpha(u) = \inf_{v \in H^1_{\mathrm{rad}}(\Omega) \setminus \{0\}} Q_\alpha(v).$$

A suitable multiple of u is a classical solution of (2).

The main point of interest in the previous theorem is the fact that the presence of the weight $|x|^\alpha$ allows one to obtain existence of solutions beyond the usual critical threshold.

When $p \in (2, 2^*)$, the functional Q_α can be minimized directly in $H^1(\Omega)$, without the symmetry constraint; of course in this case the infimum is attained because p is subcritical. The natural question that arises is to ascertain whether this minimizer, the ground state, is still a radial function. This is the question addressed in [18] for the Dirichlet problem and answered in the negative for all p, provided α is large enough.

In the Neumann problem the situation is much more complex, and we point out from the beginning that we cannot give a complete solution in the whole interval $(2, 2^*)$.

The main breakthrough for the study of the symmetry properties of the ground states is the possibility to describe the precise asymptotic behavior, as $\alpha \to \infty$, of radial minimizers. This is obtained by the construction of a "limit" functional in terms of a classical eigenvalue problem, the Steklov problem (see Definition 2.6). A further important point in all the symmetry questions is played by the number $2_* = \frac{2N-2}{N-2}$, the critical exponent for the embedding of $H^1(\Omega)$ into $L^p(\partial\Omega)$. Indeed, in contrast to the Dirichlet problem, as $\alpha \to \infty$, the denominator in Q_α behaves like a trace norm (in a sense to be made precise) resulting in worse growth properties of the levels of radial minimizers.

The consequence is the following result; Theorem 1.1 should be kept in mind for comparison.

Theorem 1.3. *For every $p \in (2_*, 2^*)$, no ground state of Q_α is radial provided α is large enough.*

The new limitation $p > 2_*$ does not come from a weakness of the arguments, but is a structural fact, peculiar of the Neumann problem. Indeed, we will prove in Section 6 the following symmetry result, in opposition to the Dirichlet case.

Theorem 1.4. *For every p close enough to 2, the ground state of Q_α is radial for every α large.*

The two above theorems are our main results concerning symmetry of ground states. They are completed by a further analysis of the variational properties of radial minimizers, which unveils a rather unexpected phenomenon. Indeed it is quite natural to expect that symmetry of ground states breaks down for α large because the second derivative $Q''_\alpha(u_\alpha)$ at a radial minimizer u_α becomes indefinite on $H^1(\Omega)$ when α exceeds a certain threshold. This is what is described in [18].

In the Neumann problem, the situation is completely different: radial minimizers continue to be local minima for Q_α over $H^1(\Omega)$ no matter how large α is. This means, for example for $p \in (2_*, 2^*)$, that the formation of nonsymmetric ground states does not manifest locally around radial minimizers, but can be justified only by *global* properties of the functional. We think that this fact is rather interesting, also in connection with an analogous result for minimizers of the trace inequality (see below). The precise result is the following.

Theorem 1.5. *For all $p \in (2, 2^*)$ if $N \geqslant 4$ (resp. for all $p \in (2, \bar{p})$, for some $\bar{p} \in (2_*, 2^*)$ if $N = 3$) and all α large enough, the minimizers of Q_α over $H^1_{\mathrm{rad}}(\Omega)$ are local minima of Q_α over the whole space $H^1(\Omega)$. The limitation for $N = 3$ is not removable.*

As an application of the analysis of the sign of the second derivative of Q_α we obtain two uniqueness results, for radial minimizers in essentially the whole interval $(2, 2^*)$, and for ground states when p is close to 2.

Theorem 1.6.

 (i) *In the same range of p's as in the previous theorem, Q_α has a unique positive radial minimizer* (*of unitary norm*) *for all α large enough.*
(ii) *There exists $\hat{p} \in (2, 2_*)$ such that for all $p \in (2, \hat{p})$, Q_α has a unique positive ground state* (*of unitary norm*) *for all α large enough.*

Throughout the paper we use more or less explicitly a link between the functional Q_α and the functional $S_p : H^1(\Omega) \setminus H^1_0(\Omega) \to \mathbf{R}$ defined, for $p \in [2, 2_*]$ by

$$S_p(u) = \frac{\|u\|^2}{(\int_{\partial\Omega} |u|^p \, d\sigma)^{2/p}},$$

whose infimum is the best constant in the Sobolev trace inequality. Existence of extremals for this functional and their symmetry properties have been studied in [7,10,9] and [13] (see also the references therein). As we have anticipated earlier, the functional S_p plays the role of a limiting functional for Q_α when $\alpha \to \infty$. It is therefore clear that many properties of minimizers of Q_α for α large and of S_p should coincide. In particular, we obtain the following analogue of Theorem 1.5 for S_p.

Theorem 1.7. *For all $p \in (2, 2_*)$ the minimizers of S_p over $H^1_{\mathrm{rad}}(\Omega)$ are local minima of S_p over the whole space $H^1(\Omega)$.*

We point out that it is not known whether the ground states for S_p are radial for a generic $p \in (2, 2_*)$, but only for p close to 2. The contribution of the previous theorem and of the discussion that surrounds it in Section 5 is the fact that nonradial ground states for S_p, if they exist, cannot *bifurcate* from the branch $p \mapsto u_p$ of radial minimizers, and should therefore appear as "separated" objects, whose existence springs out only after a certain $\hat{p}$ not too close to 2. In view of these considerations, and also of the arguments presented in Section 5, we propose the following

Conjecture. *For all $N \geqslant 3$ and for all $p \in (2, 2_*)$, the best constant S_p for the trace inequality on the* unit ball of $\mathbf{R}^N$ *is attained by a* radial *function.*

The paper is structured as follows. In Section 2 we establish the main existence result (Theorem 1.2), we carry out the asymptotic analysis of radial minimizers, and we prove Theorem 1.3. Section 3 is devoted to the variational properties of radial minimizers, and contains the proof of Theorem 1.5. In Section 4 we prove the first part of the uniqueness result Theorem 1.7. Section 5 is entirely devoted to the applications of our results to the trace inequality. Finally, in Section 6, we prove Theorem 1.4 and the second part of the uniqueness result.

Notation. The space $H^1(\Omega)$ is endowed with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx$. Any $L^p(\Omega)$ norm is denoted by $\| \cdot \|_p$. The expression $\langle \cdot, \cdot \rangle$ is the scalar product in $H^1(\Omega)$. If u is a radial function, we will write freely $u(x)$ as $u(|x|)$ or $u(\rho)$, for $\rho = |x|$. The symbol $d\sigma$ stands for the standard $(N-1)$-dimensional measure on the unit sphere. When we deal with functionals Q defined on $H^1(\Omega) \setminus \{0\}$, we will frequently write $\inf_{H^1(\Omega)} Q$ instead of the heavier $\inf_{H^1(\Omega) \setminus \{0\}} Q$, and likewise for similar expressions. The letter C denotes positive constants that may change from line to line.

2. Radial minimizers and their asymptotic properties

Throughout the paper Ω is the unit ball of $\mathbf{R}^N$, with $N \geqslant 3$; the numbers

$$2^* = \frac{2N}{N-2} \quad \text{and} \quad 2_* = \frac{2N-2}{N-2}$$

are the critical exponents for the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ and $L^q(\partial\Omega)$ respectively.

We begin with the study of radial minimizers of the functional $Q_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbf{R}$ defined by

$$Q_\alpha(u) = \frac{\int_\Omega |\nabla u|^2 \, dx + \int_\Omega u^2 \, dx}{(\int_\Omega |u|^p |x|^\alpha \, dx)^{2/p}} = \frac{\|u\|^2}{(\int_\Omega |u|^p |x|^\alpha \, dx)^{2/p}}.$$

This functional is well defined and of class C^2 over $H^1(\Omega)$ if $p \leqslant 2^*$. We shall see in a while that its restriction to $H^1_{\mathrm{rad}}(\Omega)$ is still well defined and C^2 for a much wider interval of p's.

To begin with we first establish some properties that have been first proved by Ni in [14] in the context of the Dirichlet problem; we now give the H^1 versions.

Lemma 2.1. *There exists a positive constant C such that for all $u \in H^1_{\mathrm{rad}}(\Omega)$ there results*

$$\bigl|u(x)\bigr| \leqslant C \frac{\|u\|}{|x|^{(N-2)/2}} \tag{3}$$

for all $x \in \Omega \setminus \{0\}$.

Proof. By the *radial lemma* of [14], we have that there exists $C > 0$ such that

$$\bigl|u(x)\bigr| \leqslant \bigl|u(1)\bigr| + C \frac{\|\nabla u\|_2}{|x|^{(N-2)/2}}$$

for all u radial. Since $|u(1)|$ can be controlled by $\|u\|$, the inequality follows. $\square$

Denote by $L^p(\Omega, |x|^\alpha \, dx)$ the space of L^p functions on Ω with respect to the measure $|x|^\alpha \, dx$. The previous lemma allows us to establish the following essential property.

Proposition 2.2. *The space $H^1_{\mathrm{rad}}(\Omega)$ embeds compactly into $L^p(\Omega, |x|^\alpha \, dx)$ for every $p \in [1, 2^* + \frac{2\alpha}{N-2})$.*

Proof. By the growth estimate (3) we see that

$$\int\limits_{\Omega} |u|^p |x|^\alpha \, dx \leqslant C^p \|u\|^p \int\limits_{\Omega} |x|^{\alpha - p(N-2)/2} \, dx.$$

The last integral is finite for every $p \in [1, 2^* + \frac{2\alpha}{N-2})$, which shows that for all these p's the embedding is continuous. With a standard interpolation argument one obtains the compactness of the embedding in the same range. $\square$

Notice that the continuity of the embedding makes Q_α a C^2 functional over $H^1_{\mathrm{rad}}(\Omega)$, for every $p \in [1, 2^* + \frac{2\alpha}{N-2})$.

We are now ready to give the main existence result. It matches completely the analogous one for the Dirichlet problem obtained in [14].

Theorem 2.3. *For every $\alpha > 0$ and every $p \in (2, 2^* + \frac{2\alpha}{N-2})$, there exists $u \in H^1_{\mathrm{rad}}(\Omega)$ such that*

$$Q_\alpha(u) = \inf_{v \in H^1_{\mathrm{rad}}(\Omega)} Q_\alpha(v).$$

Proof. The proof is standard. Notice that for every $p \in (1, 2^* + \frac{2\alpha}{N-2})$, by Proposition 2.2, we have $\inf_{H^1_{\mathrm{rad}}(\Omega)} Q_\alpha > 0$. Let u_n be a minimizing sequence for Q_α, normalized by $\|u_n\| = 1$. Then some subsequence u_n has a weak limit u in $H^1_{\mathrm{rad}}(\Omega)$. By Proposition 2.2 the limit u cannot vanish identically, since in that case we would have $\int_\Omega u_n^p |x|^\alpha \, dx \to 0$. Then $Q_\alpha(u) \leqslant \liminf Q_\alpha(u_n) = \inf_{H^1_{\mathrm{rad}}(\Omega)} Q_\alpha$. $\square$

Corollary 2.4. *For every α and p as in Theorem 2.3, (a suitable multiple of) the minimizer u is a classical solution of problem*

$$\begin{cases} -\Delta u + u = |x|^\alpha u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial \Omega. \end{cases}$$

Moreover u is strictly positive in $\overline{\Omega}$.

Proof. This follows from the symmetric criticality principle, standard elliptic regularity and the maximum principle. $\quad\Box$

For every $\alpha > 0$ and for a given $p \in (2, 2^* + \frac{2\alpha}{N-2})$, let

$$m_{\alpha,r} = \min_{H^1_{\mathrm{rad}}(\Omega)} Q_\alpha. \tag{4}$$

Then any (positive) minimizer u_α of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, when normalized by $\|u_\alpha\| = 1$, satisfies

$$\begin{cases} -\Delta u_\alpha + u_\alpha = m_{\alpha,r}^{p/2} |x|^\alpha u_\alpha^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u_\alpha}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

We are interested in the behavior of $m_{\alpha,r}$ and u_α when $\alpha \to \infty$. It turns out that it is possible to describe it in terms of a classical eigenvalue problem.

We begin with a fundamental result.

Lemma 2.5. *The asymptotic relation*

$$(\alpha + N) \int_\Omega |u|^p |x|^\alpha \, \mathrm{d}x = \int_{\partial\Omega} |u|^p \, \mathrm{d}\sigma + \mathrm{o}(1) \quad \text{as } \alpha \to \infty \tag{5}$$

holds

(i) *uniformly on bounded subsets of $H^1_{\mathrm{rad}}(\Omega)$, if $p \in (2, 2^*)$,*
(ii) *uniformly on bounded subsets of $H^1(\Omega)$, if $p \in (2, 2_*)$.*

Proof. Notice that $(\alpha + N)|x|^\alpha = \mathrm{div}(|x|^\alpha x)$. For $u \in H^1_{\mathrm{rad}}(\Omega)$ or $u \in H^1(\Omega)$, and p accordingly, we can write, applying the divergence theorem,

$$(\alpha + N) \int_\Omega |u|^p |x|^\alpha \, \mathrm{d}x = \int_\Omega |u|^p \, \mathrm{div}\big(|x|^\alpha x\big) \, \mathrm{d}x = \int_{\partial\Omega} |u|^p |x|^\alpha x \cdot \nu \, \mathrm{d}\sigma - p \int_\Omega |u|^{p-2} u \nabla u \cdot x |x|^\alpha \, \mathrm{d}x$$

$$= \int_{\partial\Omega} |u|^p \, \mathrm{d}\sigma - p \int_\Omega |u|^{p-2} u \nabla u \cdot x |x|^\alpha \, \mathrm{d}x$$

since on $\partial\Omega$ we have $\nu = x$ and $|x| = 1$. We just have to show that the last integral is $\mathrm{o}(1)$ as $\alpha \to \infty$ with the required uniformity.

To this aim, we first use the Hölder inequality to write

$$\left| \int_\Omega |u|^{p-2} u \nabla u \cdot x |x|^\alpha \, \mathrm{d}x \right| \leqslant \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |u|^{2p-2} |x|^\alpha \, \mathrm{d}x \right)^{1/2}$$

$$\leqslant \|u\| \left(\int_\Omega |u|^{2p-2} |x|^\alpha \, \mathrm{d}x \right)^{1/2}.$$

Assume now that $p \in (2, 2^*)$ and that u is radial. By Lemma 2.1 we have

$$\int_\Omega |u|^{2p-2} |x|^\alpha \, \mathrm{d}x \leqslant C \|u\|^{2p-2} \int_\Omega |x|^{\alpha - (2p-2)(N-2)/2} \, \mathrm{d}x = \|u\|^{2p-2} \mathrm{o}(1)$$

as $\alpha \to \infty$.

If, on the other hand, u is not radial, but p is strictly less than 2_*, we notice that $2p - 2 < 2^*$, so that, by the Hölder and Sobolev inequalities,

$$\int\limits_{\Omega} |u|^{2p-2}|x|^{\alpha}\,dx \leqslant \left(\int\limits_{\Omega} |u|^{2^*}\,dx\right)^{\frac{2p-2}{2^*}} \left(\int\limits_{\Omega} |x|^{\alpha \frac{2^*}{2^*-2p+2}}\right)^{\frac{2^*-2p+2}{2^*}} \leqslant \|u\|^{2p-2}\mathrm{o}(1)$$

as $\alpha \to \infty$.

Thus in both cases

$$\left|\int\limits_{\Omega} |u|^{p-2}u\nabla u \cdot x|x|^{\alpha}\,dx\right| \leqslant \|u\|^{p}\mathrm{o}(1) \quad \text{as } \alpha \to \infty,$$

which gives the required uniformity. $\quad\square$

In order to state the main result of this section we need to introduce an auxiliary problem. This is one of the classical eigenvalue problems, and we refer to [2] and [13] for more details.

Definition 2.6. The eigenvalue problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega \end{cases} \tag{6}$$

is called the Steklov problem.

The eigenvalues λ_k of this problem are known to be (recall that Ω is the unit ball in $\mathbf{R}^N$)

$$\lambda_k = 1 - \frac{N}{2} + \frac{I'_{k+N/2-2}(1)}{I_{k+N/2-2}(1)}, \quad k = 1, 2, \dots, \tag{7}$$

where I_ν is the modified Bessel function of the first kind of order ν. The associated eigenfunctions are also known (see [13]); the *first* eigenfunction, corresponding to λ_1, is radial and never vanishes in $\overline{\Omega}$. The first eigenvalue is *simple* and it is characterized by

$$\lambda_1 = \min_{u \in H^1(\Omega)} \frac{\|u\|^2}{\int_{\partial\Omega} u^2\,d\sigma}. \tag{8}$$

With the aid of the Steklov problem we can now describe the asymptotic behavior of the radial minimizers of Q_α. The asymptotics for the solutions of the Dirichlet problem for the Hénon equation has been obtained in [4] and [5]; in that case the situation is completely different and much more complex.

In the statement of the next result, λ_1 and φ_1, positive in $\overline{\Omega}$ and normalized by $\|\varphi_1\| = 1$, are the first eigenvalue and eigenfunction of the Steklov problem (6).

Theorem 2.7. *Let $p \in (2, 2^*)$ and let u_α, with $\|u_\alpha\| = 1$, be a minimizer of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, so that $m_{\alpha,r} = Q_\alpha(u_\alpha)$. Then, as $\alpha \to \infty$,*

$$m_{\alpha,r} \sim (\alpha + N)^{2/p}|\partial\Omega|^{1-2/p}\lambda_1, \tag{9}$$

$$u_\alpha \to \varphi_1 \quad \text{in } H^1(\Omega). \tag{10}$$

Proof. Let u be any (nonnegative) function in $H^1_{\mathrm{rad}}(\Omega)$, with $\|u\| = 1$. By Lemma 2.5, as $\alpha \to \infty$,

$$\frac{Q_\alpha(u)}{(\alpha + N)^{2/p}} = \frac{1}{((\alpha + N)\int_{\Omega} u^p|x|^{\alpha}\,dx)^{2/p}} = \frac{1}{(\int_{\partial\Omega} u^p\,d\sigma + \mathrm{o}(1))^{2/p}} = \frac{1}{(\int_{\partial\Omega} u^p\,d\sigma)^{2/p}} + \mathrm{o}(1),$$

where $\mathrm{o}(1)$ does not depend on u.

Since u is radial,

$$\left(\int\limits_{\partial\Omega} u^p\,d\sigma\right)^{2/p} = |\partial\Omega|^{2/p}u(1)^2 = |\partial\Omega|^{2/p-1}\int\limits_{\partial\Omega} u^2\,d\sigma,$$

so that for $u = u_\alpha$,

$$\frac{m_{\alpha,r}}{(\alpha+N)^{2/p}} = \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/p}} = |\partial\Omega|^{1-2/p}\frac{1}{\int_{\partial\Omega}u_\alpha^2\,d\sigma} + o(1)$$

$$\geqslant |\partial\Omega|^{1-2/p}\min_{\substack{v\in H^1_{rad}(\Omega)\\ \|v\|=1}}\frac{1}{\int_{\partial\Omega}v^2\,d\sigma} + o(1) = |\partial\Omega|^{1-2/p}\lambda_1 + o(1)$$

because λ_1 is attained by a radial function.

On the other hand, for every $u \in H^1_{rad}(\Omega)$ with $\|u\| = 1$,

$$\frac{m_{\alpha,r}}{(\alpha+N)^{2/p}} = \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/p}} \leqslant \frac{Q_\alpha(u)}{(\alpha+N)^{2/p}} = |\partial\Omega|^{1-2/p}\frac{1}{\int_{\partial\Omega}u^2\,d\sigma} + o(1).$$

Choosing $u = \varphi_1$ we obtain

$$\frac{m_{\alpha,r}}{(\alpha+N)^{2/p}} \leqslant |\partial\Omega|^{1-2/p}\lambda_1 + o(1),$$

and (9) is proved.

To prove (10) notice that since $\|u_\alpha\| = 1$, there is a subsequence, still denoted u_α, that converges to some u weakly in $H^1(\Omega)$ and strongly in $L^q(\partial\Omega)$ for $q < 2_*$. By the above arguments,

$$|\partial\Omega|^{1-2/p}\frac{1}{\int_{\partial\Omega}u_\alpha^2\,d\sigma} + o(1) = \frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/p}} \leqslant |\partial\Omega|^{1-2/p}\lambda_1 + o(1),$$

from which we see that u cannot be identically zero. Then, by previous computations and the properties of u_α,

$$\lambda_1 \leqslant \frac{\|u\|^2}{\int_{\partial\Omega}u^2\,d\sigma} \leqslant \frac{1}{\int_{\partial\Omega}u^2\,d\sigma} = \lim_{\alpha\to\infty}\frac{1}{\int_{\partial\Omega}u_\alpha^2\,d\sigma} = |\partial\Omega|^{2/p-1}\lim_{\alpha\to\infty}\left(\frac{Q_\alpha(u_\alpha)}{(\alpha+N)^{2/p}} + o(1)\right) = \lambda_1.$$

This shows that $\|u\|^2 = 1$ and that $1/\int_{\partial\Omega}u^2\,d\sigma = \lambda_1$; since λ_1 is simple, it must be $u = \varphi_1$. Convergence of the norm implies that $u_\alpha \to \varphi_1$ strongly in $H^1(\Omega)$. $\quad\square$

The precise asymptotic behavior of $m_{\alpha,r}$ will be used later. However the fact that $m_{\alpha,r}$ grows like $\alpha^{2/p}$ is enough to prove a first symmetry breaking result.

Theorem 2.8. *Assume that $p \in (2_*, 2^*)$. Then for every α large enough (depending on p) we have*

$$\min_{u\in H^1(\Omega)}\frac{\|u\|^2}{(\int_\Omega |u|^p|x|^\alpha\,dx)^{2/p}} < \min_{u\in H^1_{rad}(\Omega)}\frac{\|u\|^2}{(\int_\Omega |u|^p|x|^\alpha\,dx)^{2/p}}. \tag{11}$$

Proof. We estimate the growth of the left-hand side of (11) as in [18]. Take a nonnegative function $v \in C^1_0(\Omega)$ and extend it to zero outside Ω. Let $x_\alpha = (1 - 1/\alpha, 0, \dots, 0)$ and set $v_\alpha(x) = v(\alpha(x - x_\alpha))$. Then supp $v_\alpha \subset B_{1/\alpha}(x_\alpha)$. Therefore, by standard changes of variable,

$$\int_\Omega v_\alpha^p|x|^\alpha\,dx = \int_{B_{1/\alpha}(x_\alpha)} v_\alpha^p|x|^\alpha\,dx \geqslant \left(1 - \frac{2}{\alpha}\right)^\alpha \int_{B_{1/\alpha}(x_\alpha)} v_\alpha^p\,dx = \alpha^{-N}\left(1 - \frac{2}{\alpha}\right)^\alpha \int_\Omega v^p\,dx$$

and

$$Q_\alpha(v_\alpha) \leqslant \frac{\alpha^{2-N}\int_\Omega |\nabla v|^2\,dx + \alpha^{-N}\int_\Omega v^2\,dx}{\alpha^{-2N/p}(1 - 2/\alpha)^{2\alpha/p}(\int_\Omega v^p\,dx)^{2/p}} \leqslant C\alpha^{2-N+2N/p}.$$

Since by Theorem 2.7 the right-hand side of (11) is $m_{\alpha,r} \sim \alpha^{2/p}$, we see that (11) holds for all α large because $2 - N + 2N/p < 2/p$ for all $p \in (2_*, 2^*)$. $\quad\square$

Remark 2.9. The level of radial minimizers for the quotient associated to the Dirichlet problem grows like $\alpha^{1+2/p}$, as is shown in [18]; this gives a symmetry breaking result for *all* $p \in (2, 2^*)$. We will see in the next sections that it is the loss one power in the case of the Neumann problem that causes a more subtle behavior from the point of view of symmetry of the ground states.

Remark 2.10. Theorem 2.8 gives immediately a multiplicity result: for every $p \in (2_*, 2^*)$ and every α large enough, problem (2) admits at least two solutions. One is radial and the other is the (nonradial) ground state.

3. Variational properties of radial minimizers

We have seen that for $p \in (2_*, 2^*)$ the minimizers of Q_α over $H^1_{\mathrm{rad}}(\Omega)$ are not global minimizers over $H^1(\Omega)$, at least for α large. In the interval $(2, 2_*]$ the situation is less clear, and will be analyzed in the next sections.

Now since for $\alpha = 0$ global minimizers are radial, it is quite natural to think that the symmetry breaking phenomenon described above takes place because when α becomes very large the radial minimizer u_α ceases to be a minimizer over $H^1(\Omega)$ due to the appearance of "negative directions". In other words one expects that when α grows the second derivative $Q''_\alpha(u_\alpha)$ over $H^1(\Omega)$ becomes indefinite; this is exactly the phenomenon described in [18] for the Dirichlet problem.

In this section we show, rather surprisingly, that this is not the case for the Neumann problem: although for $p \in (2_*, 2^*)$ the functions u_α are not global minimizers over $H^1(\Omega)$ for α large, they are still *local* minimizers. This fact in turn will have some interesting consequences, that we will analyze later.

We now study the sign of $Q''_\alpha(u_\alpha)$ for α large. From now on we assume that u_α is normalized by $\|u_\alpha\| = 1$. We denote by $\mathcal{S}$ the unit sphere in $H^1(\Omega)$, and by $T_{u_\alpha}\mathcal{S}$ the tangent space to $\mathcal{S}$ at u_α, namely

$$T_{u_\alpha}\mathcal{S} = \left\{ v \in H^1(\Omega) \colon \langle v, u_\alpha \rangle = 0 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $H^1(\Omega)$.

Notice that since u_α solves

$$\begin{cases} -\Delta u_\alpha + u_\alpha = m_{\alpha,r}^{p/2} |x|^\alpha u_\alpha^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u_\alpha}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

the condition $v \in T_{u_\alpha}\mathcal{S}$ is equivalent to $\int_\Omega u_\alpha^{p-1} v |x|^\alpha \, dx = 0$.

Lemma 3.1. *Let $p \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, normalized by $u_\alpha \in \mathcal{S}$. Then for every $v \in T_{u_\alpha}\mathcal{S}$,*

$$Q''_\alpha(u_\alpha) \cdot v^2 = 2m_{\alpha,r} \left(\|v\|^2 - (p-1)m_{\alpha,r}^{p/2} \int_\Omega u_\alpha^{p-2} v^2 |x|^\alpha \, dx \right). \tag{13}$$

Proof. Set $N(u) = \|u\|^2$ and $D(u) = (\int_\Omega u^p |x|^\alpha \, dx)^{2/p}$, so that $Q_\alpha(u) = N(u)/D(u)$. For every critical point $u \in H^1(\Omega)$ of Q_α and every $v \in H^1(\Omega)$, we have

$$Q''_\alpha(u) \cdot v^2 = \frac{D(u)N''(u) \cdot v^2 - N(u)D''(u) \cdot v^2}{D(u)^2}.$$

Now $N''(u) \cdot v^2 = 2\|v\|^2$ and

$$D''(u) \cdot v^2 = 2(2-p)\left(\int_\Omega u^p |x|^\alpha \, dx \right)^{2/p-2} \left(\int_\Omega u^{p-1} v |x|^\alpha \, dx \right)^2$$

$$+ 2(p-1)\left(\int_\Omega u^p |x|^\alpha \, dx \right)^{2/p-1} \int_\Omega u^{p-2} v^2 |x|^\alpha \, dx,$$

so that

$$Q''_\alpha(u) \cdot v^2 = \left\{ \|v\|^2 - \|u\|^2 \left((2 - p) \left(\frac{\int_\Omega u^{p-1} v |x|^\alpha \, dx}{\int_\Omega u^p |x|^\alpha \, dx} \right)^2 \right. \right.$$
$$\left. \left. + (p - 1) \frac{\int_\Omega u^{p-2} v^2 |x|^\alpha \, dx}{\int_\Omega u^p |x|^\alpha \, dx} \right) \right\} \left(\int_\Omega u^p |x|^\alpha \, dx \right)^{-2/p}. \tag{14}$$

This holds for every critical point u of Q_α and every $v \in H^1(\Omega)$. If $u = u_\alpha \in \mathcal{S}$ we have $(\int_\Omega u^p |x|^\alpha \, dx)^{-2/p} = m_{\alpha,r}$ and if $v \in T_{u_\alpha} \mathcal{S}$, then $\int_\Omega u^{p-1} v |x|^\alpha \, dx = 0$. Therefore in this case (14) reduces to

$$Q''_\alpha(u_\alpha) \cdot v^2 = 2m_{\alpha,r} \left(\|v\|^2 - (p - 1) m_{\alpha,r}^{p/2} \int_\Omega u_\alpha^{p-2} v^2 |x|^\alpha \, dx \right). \quad \square \tag{15}$$

In order to study the sign of $Q''_\alpha(u_\alpha)$ we need some more precise estimates on u_α.

In what follows u_α is a minimizer of Q_α over $H^1_{\text{rad}}(\Omega)$, normalized by $\|u_\alpha\| = 1$.

Lemma 3.2. *The functions u_α are uniformly bounded in $C^1(\overline{\Omega})$ as $\alpha \to \infty$.*

Proof. We first prove a uniform bound in $L^\infty(\Omega)$. Since $\|u_\alpha\| = 1$ and u_α is radial, there is $C > 0$ such that

$$\|u_\alpha\|_{L^\infty(\Omega \setminus B_{1/2}(0))} \leqslant C \|u_\alpha\| = C. \tag{16}$$

By Lemma 2.1, there is a positive constant C such that

$$|u_\alpha(x)| \leqslant C \frac{\|u_\alpha\|}{|x|^{(N-2)/2}} = \frac{C}{|x|^{(N-2)/2}}$$

for all $x \in \Omega \setminus \{0\}$ and all α.

Set $f_\alpha(x) = m_{\alpha,r}^{p/2} |x|^\alpha u_\alpha(x)^{p-1}$; then, recalling that $m_{\alpha,r} \leqslant C\alpha^{2/p}$,

$$\|f_\alpha\|_{L^\infty(B_{1/2}(0))} \leqslant C\alpha |x|^\alpha \frac{1}{|x|^{(p-1)(N-2)/2}} \leqslant \frac{C\alpha}{2^{\alpha - (p-1)(2-N)/2}} = o(1) \tag{17}$$

as $\alpha \to \infty$.

Therefore, by (16) and (17) we see that u_α solves

$$\begin{cases} -\Delta u_\alpha + u_\alpha = f_\alpha & \text{in } B_{1/2}(0), \\ u_\alpha \leqslant C & \text{on } \partial B_{1/2}(0) \end{cases}$$

with $\|f_\alpha\|_{L^\infty(B_{1/2}(0))} \to 0$ as $\alpha \to \infty$. By standard elliptic estimates, we obtain that u_α is uniformly bounded in $C^{1,\beta}(\overline{B_{1/2}(0)})$, for all $\beta \in (0, 1)$. In view of (16) we obtain that u_α is uniformly bounded in $L^\infty(\Omega)$ as $\alpha \to \infty$.

To complete the proof we only have to show that there is a C^1 bound also on $\overline{\Omega} \setminus B_{1/2}(0)$.

Since u_α is radial we see that it solves

$$-u''_\alpha - \frac{N-1}{\rho} u'_\alpha + u_\alpha = m_{\alpha,r}^{p/2} \rho^\alpha u_\alpha^{p-1}$$

and $u'_\alpha(1) = 0$. Integrating this equation over $[t, 1]$, with $t \geqslant \frac{1}{2}$ we obtain

$$u'_\alpha(t) = (N - 1) \int_t^1 \frac{1}{\rho} u'_\alpha \, d\rho - \int_t^1 u_\alpha + m_{\alpha,r}^{p/2} \int_t^1 u_\alpha^{p-1} \rho^\alpha \, d\rho.$$

Therefore, using the fact that u_α is bounded in $L^\infty(\Omega)$ and the growth of $m_{\alpha,r}$,

$$|u'_\alpha(t)| \leqslant (N - 1) \left(\frac{u_\alpha(\rho)}{\rho} \Big|_t^1 + \int_t^1 \frac{u_\alpha(\rho)}{\rho^2} \, d\rho \right) + C + C m_{\alpha,r}^{p/2} \frac{\rho^{\alpha+1}}{\alpha + 1} \Big|_t^1 \leqslant C + C \frac{\alpha}{\alpha + 1} \leqslant C$$

for all $t \in [\frac{1}{2}, 1]$. This, together with the estimate in $C^{1,\beta}(B_{1/2}(0))$, gives the required bound in $C^1(\overline{\Omega})$. $\quad \square$

Remark 3.3. Notice that one cannot hope to obtain uniform $C^2(\overline{\Omega})$ estimates, even though each u_α does lie in $C^2(\overline{\Omega})$. This is due to the fact that the right-hand side of the equation behaves like $\alpha|x|^\alpha$. Now, while this term goes to zero locally uniformly in Ω, on the boundary it blows up like α. Therefore Δu_α cannot be bounded in C^0 up to the boundary of Ω.

At first sight, a rather confusing consequence of the lack of C^2 bounds is that if one tries to pass naïvely to the limit in (12) as $\alpha \to \infty$, then one can do it in the equation (in the weak form, for instance), but *not in the boundary conditions*. Thus it may (and does) happen that limits of solutions of homogeneous Neumann problems do not satisfy a homogeneous Neumann condition. We have already observed this fact when we have shown that $u_\alpha \to \varphi_1$, a solution of the *Steklov* problem.

Remark 3.4. In view of the preceding lemma, we can assure that the convergence of u_α to φ_1 takes place also in $C^0(\overline{\Omega})$. Since φ_1 is strictly positive in $\overline{\Omega}$, then for some $C > 0$ and all α large,

$$\min_{x \in \overline{\Omega}} u_\alpha(x) \geqslant C.$$

We can now continue the study of the second derivative of Q_α.

Lemma 3.5. *Let $p \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, normalized by $u_\alpha \in \mathcal{S}$. Then, as $\alpha \to \infty$,*

$$(\alpha + N) \int_\Omega u_\alpha^{p-2} v^2 |x|^\alpha \, \mathrm{d}x = \int_{\partial\Omega} u_\alpha^{p-2} v^2 \, \mathrm{d}\sigma + \mathrm{o}(1), \tag{18}$$

uniformly for v in bounded subsets of $H^1(\Omega)$.

Proof. We apply the divergence theorem exactly like in Lemma 2.5. We obtain

$$(\alpha + N) \int_\Omega u_\alpha^{p-2} v^2 |x|^\alpha \, \mathrm{d}x = \int_{\partial\Omega} u_\alpha^{p-2} v^2 \, \mathrm{d}\sigma - (p-2) \int_\Omega v^2 u_\alpha^{p-3} \nabla u_\alpha \cdot x |x|^\alpha \, \mathrm{d}x - 2 \int_\Omega u_\alpha^{p-2} v \nabla v \cdot x |x|^\alpha \, \mathrm{d}x$$

and we just have to show that the two last integral vanish as $\alpha \to \infty$.

Now by Lemma 3.2 and Remark 3.4 we have

$$\left| \int_\Omega v^2 u_\alpha^{p-3} \nabla u_\alpha \cdot x |x|^\alpha \, \mathrm{d}x \right| \leqslant C \int_\Omega v^2 |x|^\alpha \, \mathrm{d}x \leqslant C \|v\|_{2_*}^2 \left(\int_\Omega |x|^{\alpha 2^*/(2^*-2)} \, \mathrm{d}x \right)^{(2^*-2)/2^*}$$

$$\leqslant C \|v\|^2 \mathrm{o}(1)$$

as $\alpha \to \infty$, while

$$\left| \int_\Omega u_\alpha^{p-2} v \nabla v \cdot x |x|^\alpha \, \mathrm{d}x \right| \leqslant C \int_\Omega |v| |\nabla v| |x|^\alpha \, \mathrm{d}x \leqslant C \|v\| \left(\int_\Omega v^2 |x|^\alpha \, \mathrm{d}x \right)^{1/2}$$

$$\leqslant C \|v\|^2 \mathrm{o}(1),$$

as in the previous computation. Thus (18) is proved. $\square$

We can now prove the main result on the sign of $Q''_\alpha(u_\alpha)$.

Proposition 3.6. *Let $p \in (2, 2^*)$ and let u_α be a minimizer of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, normalized by $u_\alpha \in \mathcal{S}$. Then the inequality*

$$\min_{\substack{v \in T_{u_\alpha}\mathcal{S} \\ \|v\|=1}} Q''_\alpha(u_\alpha) \cdot v^2 > 0 \tag{19}$$

holds

(i) *for all $p \in (2, 2^*)$ if $N \geqslant 4$,*
(ii) *for all $p \in (2, \bar{p})$, for some $\bar{p} \in (2_*, 2^*)$ if $N = 3$*

provided α is large enough (*depending on p*).

Proof. It is standard to see that the minimum in (19) is attained; we supply some details for completeness. Set

$$F(v) = 2m_{\alpha,r}\left(\|v\|^2 - (p-1)m_{\alpha,r}^{p/2}\int_\Omega u_\alpha^{p-2}v^2|x|^\alpha \, dx\right) \quad \text{and} \quad \mu = \inf_{\substack{v \in T_{u_\alpha}\mathcal{S} \\ \|v\|=1}} F(v).$$

We have to show that μ is attained. It is obvious that μ is finite and that $\mu < 2m_{\alpha,r}$. Let $v_n \in T_{u_\alpha}\mathcal{S}$, $\|v_n\| = 1$, be a minimizing sequence for F. Up to subsequences, $v_n \to v$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Notice that $v \not\equiv 0$, since otherwise

$$\mu + o(1) = F(v_n) = 2m_{\alpha,r}\left(1 - (p-1)m_{\alpha,r}^{p/2}\int_\Omega u_\alpha^{p-2}v_n^2|x|^\alpha \, dx\right) = 2m_{\alpha,r} + o(1),$$

namely $\mu = 2m_{\alpha,r}$, which is false. Also, $v \in T_{u_\alpha}\mathcal{S}$. Write now $v_n = v + w_n$, with $w_n \to 0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. A simple computation shows that

$$\mu + o(1) = F(v_n) = 2m_{\alpha,r}\left(\|v\|^2 + \|w_n\|^2 - (p-1)m_{\alpha,r}^{p/2}\int_\Omega u_\alpha^{p-2}v^2|x|^\alpha \, dx + o(1)\right)$$

$$= \|v\|^2 F\left(\frac{v}{\|v\|}\right) + 2m_{\alpha,r}\|w_n\|^2 + o(1) \geqslant \mu\|v\|^2 + 2m_{\alpha,r}\|w_n\|^2 + o(1)$$

$$= \mu\left(1 - \|w_n\|^2 + o(1)\right) + 2m_{\alpha,r}\|w_n\|^2 + o(1),$$

so that $(2m_{\alpha,r} - \mu)\|w_n\|^2 \leqslant o(1)$. Since $\mu < 2m_{\alpha,r}$, this shows that $v_n \to v$ strongly in $H^1(\Omega)$; thus $\|v\| = 1$ and $F(v) = \mu$.

We now turn to the main part of the proof. We assume that (19) is false for an unbounded sequence of α's (which we denote by A), so that

$$\min_{\substack{v \in T_{u_\alpha}\mathcal{S} \\ \|v\|=1}} Q_\alpha''(u_\alpha) \cdot v^2 \leqslant 0 \quad \text{for all } \alpha \in A. \tag{20}$$

This means that for all $\alpha \in A$ there exists $v_\alpha \in T_{u_\alpha}\mathcal{S}$, with $\|v_\alpha\| = 1$ such that $Q_\alpha''(u_\alpha) \cdot v_\alpha \leqslant 0$, namely, by (13),

$$1 - (p-1)m_{\alpha,r}^{p/2}\int_\Omega u_\alpha^{p-2}v_\alpha^2|x|^\alpha \, dx \leqslant 0. \tag{21}$$

Recalling from Theorem 2.7 the asymptotic behavior of $m_{\alpha,r}$, we can write the preceding inequality for $\alpha \to \infty$ in A as

$$1 \leqslant (p-1)\left(|\partial\Omega|^{p/2-1}\lambda_1^{p/2} + o(1)\right)(\alpha + N)\int_\Omega u_\alpha^{p-2}v_\alpha^2|x|^\alpha \, dx$$

$$= (p-1)\left(|\partial\Omega|^{p/2-1}\lambda_1^{p/2} + o(1)\right)\left(\int_{\partial\Omega} u_\alpha^{p-2}v_\alpha^2 \, d\sigma + o(1)\right),$$

where for the last equality we have used Lemma 3.5.

Now $u_\alpha \to \varphi_1$ in $C^0(\overline{\Omega})$ and v_α, being bounded in $H^1(\Omega)$, admits a subsequence (still denoted v_α) converging to some v weakly in $H^1(\Omega)$ and strongly in $L^2(\partial\Omega)$.

Passing to the limit as $\alpha \to \infty$ in A in the preceding inequality we find

$$1 \leqslant (p-1)|\partial\Omega|^{p/2-1}\lambda_1^{p/2}\int_{\partial\Omega} \varphi_1^{p-2}v^2 \, d\sigma. \tag{22}$$

If v is identically zero we have reached a contradiction and the proof is complete. Assume therefore that $v \not\equiv 0$.

Since φ_1 is radial and normalized by $\|\varphi_1\| = 1$, we have

$$\lambda_1 = \frac{1}{\int_{\partial \Omega} \varphi_1^2 \, d\sigma} = \frac{1}{\varphi_1(1)^2 |\partial \Omega|},$$

so that $\varphi_1(1)^{p-2} = |\partial \Omega|^{1-p/2} \lambda_1^{1-p/2}$. Inserting this in (22) we see that

$$1 \leqslant (p-1)|\partial \Omega|^{p/2-1} \lambda_1^{p/2} |\partial \Omega|^{1-p/2} \lambda_1^{1-p/2} \int_{\partial \Omega} v^2 \, d\sigma = (p-1)\lambda_1 \int_{\partial \Omega} v^2 \, d\sigma,$$

that is,

$$\frac{1}{\int_{\partial \Omega} v^2 \, d\sigma} \leqslant (p-1)\lambda_1.$$

Notice now that $\langle v, \varphi_1 \rangle = \lim_\alpha \langle v_\alpha, u_\alpha \rangle = 0$ by strong convergence of u_α and weak convergence of v_α. Thus v is orthogonal to φ_1 in $H^1(\Omega)$; this, together with the fact that λ_1 is simple yields

$$\lambda_2 = \min_{\substack{w \in H^1(\Omega) \\ \langle w, \varphi_1 \rangle = 0}} \frac{\|w\|^2}{\int_{\partial \Omega} w^2 \, d\sigma} \leqslant \frac{\|v\|^2}{\int_{\partial \Omega} v^2 \, d\sigma} \leqslant \frac{1}{\int_{\partial \Omega} v^2 \, d\sigma} \leqslant (p-1)\lambda_1.$$

Therefore assuming that (19) is false for an unbounded sequence of α's implies the inequality

$$\frac{\lambda_2}{\lambda_1} \leqslant p - 1 \tag{23}$$

on the eigenvalues of the Steklov problem (6). We now complete the proof by showing that this inequality cannot hold. Recall from (7) that

$$\lambda_k = 1 - \frac{N}{2} + \frac{I'_{k+N/2-2}(1)}{I_{k+N/2-2}(1)}, \quad k = 1, 2, \ldots,$$

where I_ν is the modified Bessel function of the first kind of order ν.

Since (see [1]) $I'_\nu(x) = I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x)$ holds for all x and all ν, we see that

$$\lambda_k = k - 1 + \frac{I_{k+N/2-1}(1)}{I_{k+N/2-2}(1)}. \tag{24}$$

We also recall from [1] that $I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x)$, so that $I_{\nu-1}(1)/I_\nu(1) \geqslant 2\nu$. Therefore

$$\frac{\lambda_2}{\lambda_1} = \frac{1 + I_{N/2+1}(1)/I_{N/2}(1)}{I_{N/2}(1)/I_{N/2-1}(1)} > \frac{1}{I_{N/2}(1)/I_{N/2-1}(1)} = \frac{I_{N/2-1}(1)}{I_{N/2}(1)} \geqslant 2\frac{N}{2} = N.$$

Thus, if (23) holds, then by our choice of p,

$$N < \frac{\lambda_2}{\lambda_1} \leqslant p - 1 < \frac{N+2}{N-2},$$

which is false for every $N \geqslant 4$. If $N = 3$, the inequality is false not in the whole interval $(2, 2^*) = (2, 6)$, but only in a subinterval $(2, \bar{p})$, with $\bar{p} \in (2_*, 2^*) = (4, 6)$ (an approximate value of λ_2/λ_1 is 3.8, which would locate $\bar{p}$ around 4.8). In both cases this is the required contradiction, and the proof is complete. $\quad \square$

Remark 3.7. In the previous proposition, one cannot hope to get, when $N = 3$, the whole interval $(2, 2^*)$ as for $N = 4$. Indeed testing $Q''_\alpha(u_\alpha)$ with φ_2, the second eigenfunction of the Steklov problem, one sees easily that $Q''_\alpha(u_\alpha) \cdot \varphi_2^2 < 0$ for p close to 2^* and α large. The fact that for α large $Q''_\alpha(u_\alpha)$ becomes indefinite for some $p \in (2_*, 2^*)$ is thus a peculiarity of dimension three.

Proposition 3.6, together with the fact that Q_α is homogeneous of degree zero constitutes the proof of the following result.

Theorem 3.8. *For all $p \in (2, 2^*)$ if $N \geqslant 4$ (resp. $p \in (2, \bar{p})$ if $N = 3$) and all α large enough, the minimizers u_α of Q_α over $H^1_{\mathrm{rad}}(\Omega)$ are local minima of Q_α over the whole space $H^1(\Omega)$.*

4. Uniqueness of radial minimizers

As an application of the discussion carried out in the previous section, we now give a uniqueness result for radial minimizers of Q_α.

Theorem 4.1. *For every $p \in (2, 2^*)$ if $N \geqslant 4$, or every $p \in (2, \bar{p})$ if $N = 3$, there exists $\alpha(p)$ such that for all $\alpha \geqslant \alpha(p)$, the problem*

$$\min_{u \in H^1_{\mathrm{rad}}(\Omega)} \frac{\|u\|^2}{(\int_\Omega |u|^p |x|^\alpha \, \mathrm{d}x)^{2/p}}$$

has a unique positive solution (normalized by $\|u\| = 1$).

Proof. Fix p in the appropriate range, according to the value of N, and assume on the contrary that for an unbounded sequence of α's, denoted by A, there exist two minimizers u_α and v_α of Q_α over $H^1_{\mathrm{rad}}(\Omega)$, normalized by $\|u_\alpha\| = \|v_\alpha\| = 1$.

By Theorem 2.7 and Lemma 3.2 we have that, as $\alpha \to \infty$ in A,

$$u_\alpha \to \varphi_1 \quad \text{and} \quad v_\alpha \to \varphi_1 \quad \text{in } H^1(\Omega) \text{ and in } C^0(\overline{\Omega}).$$

Moreover, both u_α and v_α solve

$$\begin{cases} -\Delta u + u = m_{\alpha,r}^{p/2} |x|^\alpha u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{25}$$

with $m_{\alpha,r} \sim (\alpha + N)^{2/p} |\partial\Omega|^{1-2/p} \lambda_1$.

Subtracting (25) for v_α from (25) for u_α and setting $w_\alpha = u_\alpha - v_\alpha$, we see that w_α solves

$$\begin{cases} -\Delta w_\alpha + w_\alpha = (p-1) m_{\alpha,r}^{p/2} |x|^\alpha c_\alpha w_\alpha & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

where

$$c_\alpha = \int\limits_0^1 \left(v_\alpha + t(u_\alpha - v_\alpha)\right)^{p-2} \mathrm{d}t.$$

By assumption $u_\alpha \not\equiv v_\alpha$ for all $\alpha \in A$, so that we can divide the equations in (26) by $\|w_\alpha\|$ and set $\psi_\alpha = w_\alpha / \|w_\alpha\|$. Then we obtain that ψ_α satisfies

$$\begin{cases} -\Delta \psi_\alpha + \psi_\alpha = (p-1) m_{\alpha,r}^{p/2} |x|^\alpha c_\alpha \psi_\alpha & \text{in } \Omega, \\ \dfrac{\partial \psi_\alpha}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \|\psi_\alpha\| = 1 \end{cases} \tag{27}$$

for all $\alpha \in A$.

Since $\|\psi_\alpha\| = 1$, we can assume that (up to a subsequence), $\psi_\alpha \to \psi$ weakly in $H^1(\Omega)$ and strongly in $L^q(\partial\Omega)$ for all $q < 2_*$.

We now show that it cannot be $\psi \equiv 0$. Indeed, noticing that by Lemma 3.2 we have $\|c_\alpha\|_\infty \leqslant C$ uniformly in α, and multiplying (27) by ψ_α, we obtain (using $(\alpha + N)|x|^\alpha = \mathrm{div}\, |x|^\alpha x$)

$$1 = \|\psi_\alpha\|^2 = (p-1)m_{\alpha,r}^{p/2} \int_\Omega c_\alpha \psi_\alpha^2 |x|^\alpha \, dx \leqslant Cm_{\alpha,r}^{p/2} \int_\Omega \psi_\alpha^2 |x|^\alpha \, dx \leqslant C(\alpha+N) \int_\Omega \psi_\alpha^2 |x|^\alpha \, dx$$

$$= C\left(\int_{\partial\Omega} \psi_\alpha^2 \, d\sigma - 2\int_\Omega \psi_\alpha \nabla\psi_\alpha \cdot x |x|^\alpha \, dx \right) \leqslant C\left(\int_{\partial\Omega} \psi_\alpha^2 \, d\sigma + 2\|\psi_\alpha\| \left(\int_\Omega \psi_\alpha^2 \, dx \right)^{1/2} \right).$$

If ψ were identically zero, we would have $\psi_\alpha \to 0$ strongly in $L^2(\Omega)$ and in $L^2(\partial\Omega)$, so that the preceding inequality would yield $1 = \|\psi_\alpha\|^2 \leqslant o(1)$, as $\alpha \to \infty$ in A, a contradiction. Therefore $\psi \not\equiv 0$.

To proceed we notice that, still by Lemma 3.2, we have

$$c_\alpha \to \varphi_1^{p-2} \quad \text{in } C^0(\overline{\Omega}). \tag{28}$$

Multiplying (27) by $\phi \in H^1(\Omega)$ and integrating we obtain

$$\langle \psi_\alpha, \phi \rangle = (p-1)m_{\alpha,r}^{p/2} \int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx = (p-1)\left(|\partial\Omega|^{p/2-1}\lambda_1^{p/2} + o(1) \right)(\alpha+N) \int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx. \tag{29}$$

Now

$$\left| (\alpha+N)\int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx - (\alpha+N)\int_\Omega \varphi_1^{p-2} \psi_\alpha \phi |x|^\alpha \, dx \right| \leqslant (\alpha+N)\int_\Omega |c_\alpha - \varphi_1^{p-2}||\psi_\alpha \phi||x|^\alpha \, dx$$

$$\leqslant \|c_\alpha - \varphi_1^{p-2}\|_\infty (\alpha+N)\int_\Omega |\psi_\alpha||\phi||x|^\alpha \, dx$$

and

$$(\alpha+N)\int_\Omega |\psi_\alpha||\phi||x|^\alpha \, dx \leqslant \left((\alpha+N)\int_\Omega \psi_\alpha^2 |x|^\alpha \, dx \right)^{1/2} \left((\alpha+N)\int_\Omega \phi^2 |x|^\alpha \, dx \right)^{1/2}$$

$$\leqslant \left(\int_{\partial\Omega} \psi_\alpha^2 \, d\sigma - 2\int_\Omega \psi_\alpha \nabla\psi_\alpha \cdot x |x|^\alpha \, dx \right)^{1/2} \left(\int_{\partial\Omega} \phi^2 \, d\sigma - 2\int_\Omega \phi\nabla\phi \cdot x |x|^\alpha \, dx \right)^{1/2}$$

$$\leqslant C$$

as $\alpha \to \infty$, since all the integrals are uniformly bounded.

This and the preceding inequality show that

$$(\alpha+N)\int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx = (\alpha+N)\int_\Omega \varphi_1^{p-2} \psi_\alpha \phi |x|^\alpha \, dx + o(1)$$

as $\alpha \to \infty$ in A. Finally we notice that

$$(\alpha+N)\int_\Omega \varphi_1^{p-2} \psi_\alpha \phi |x|^\alpha \, dx = \int_{\partial\Omega} \varphi_1^{p-2} \psi_\alpha \phi \, d\sigma - \int_\Omega \nabla\left(\varphi_1^{p-2} \psi_\alpha \phi \right) \cdot x |x|^\alpha \, dx$$

$$= \int_{\partial\Omega} \varphi_1^{p-2} \psi_\alpha \phi \, d\sigma + o(1),$$

due to by now familiar computations. Inserting this in the left-hand side of (29) yields, as $\alpha \to \infty$,

$$\langle \psi_\alpha, \phi \rangle = (p-1)\left(|\partial\Omega|^{p/2-1}\lambda_1^{p/2} + o(1) \right)\left(\int_{\partial\Omega} \varphi_1^{p-2} \psi_\alpha \phi \, d\sigma + o(1) \right).$$

Letting $\alpha \to \infty$ in A (and recalling that $\varphi_1^{p-2} \equiv |\partial\Omega|^{1-p/2}\lambda_1^{1-p/2}$ on $\partial\Omega$), we obtain

$$\langle \psi, \phi \rangle = (p-1)\lambda_1 \int_{\partial\Omega} \psi\phi \, d\sigma,$$

for all $\phi \in H^1(\Omega)$. In other words, ψ is a (nontrivial) solution of the problem

$$\begin{cases} -\Delta\psi + \psi = 0 & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = (p-1)\lambda_1\psi & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Thus the number $(p-1)\lambda_1$ must be one of the eigenvalues λ_k of the Steklov problem. However $(p-1)\lambda_1 > \lambda_1$ because $p > 2$, and $(p-1)\lambda_1 < \lambda_2$ for all $p \in (2, 2^*)$ if $N \geqslant 4$ and all $p \in (2, \bar{p})$ if $N = 3$, as have already proved in Proposition 3.6. This is a contradiction, and the proof is complete. $\square$

5. A detour on the trace inequalities

In this section we analyze a little more closely the relations between the minimization of Q_α and some Sobolev trace inequalities. Although we have already used some more or less evident link between the two, we have not yet formalized the question.

In our context the trace inequalities state that the embedding of $H^1(\Omega)$ into $L^p(\partial\Omega)$ is continuous for $p \in [1, 2_*]$; that is, for all $p \in [1, 2_*]$ there exists $C > 0$ such that

$$\left(\int\limits_{\partial\Omega} |u|^p \, d\sigma \right)^{2/p} \leqslant C \|u\|^2$$

for every $u \in H^1(\Omega)$. We set

$$S_p = \inf_{u \in H^1(\Omega)} \frac{\|u\|^2}{(\int_{\partial\Omega} |u|^p \, d\sigma)^{2/p}} \tag{31}$$

and we recall that S_p is attained for $p \in [1, 2_*)$ because the corresponding embedding is compact. If $p = 2_*$ the embedding is no longer compact and the situation is more complex, see [10] and references therein, and only partial results are known. However, combining the condition of Theorem 1 of [10] with the results of [8], one can say that *for the unit ball* Ω the constant S_{2_*} is attained.

The question of the symmetry of minimizer of S_p has been treated in [13,9], and [7] (see also the references in these papers).

Roughly speaking it turns out that radial symmetry of minimizers depends on the size of the domain. Confining ourselves to the context where Ω is the unit ball, the main results about symmetry (deduced from [7,9] and [13]) take the following form: denoting by $\mu\Omega$ the ball of radius μ centered at zero, then the functions that attain S_p in (31) are radial for all μ small enough, and nonradial for all μ large enough.

The same kind of phenomenon takes place for a fixed domain, say Ω, but when p varies: it has been proved in [13] (for more general problems) that minimizers of (31) are radial for all p close enough to 2, and nonradial for p large.

For further reference we quote a part of Theorem 2 of [13], specialized to our context. In its statement λ_1 denotes, as usual, the first eigenvalue of the Steklov problem (6).

Theorem 5.1 (*Lami Dozo, Torné*). *If*

$$p - 1 > \frac{1}{\lambda_1^2}\big(1 - (N-1)\lambda_1\big), \tag{32}$$

then no minimizer of (31) *is radial.*

Below we will give an interpretation of the number appearing in the right-hand side of (32).

Notice that minimizers of S_p, normalized by $\|u\| = 1$, are solutions of

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = S_p^{p/2} u^{p-1} & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Let us now return to the Hénon problem. In the rest of this section we always assume that $p \in (2, 2_*)$. The link with the trace inequalities is given by (ii) of Lemma 2.5; indeed, denoting by $S_p : H^1(\Omega) \setminus H_0^1(\Omega) \to \mathbf{R}$ the functional

$$S_p(u) = \frac{\|u\|^2}{(\int_{\partial \Omega} |u|^p \, d\sigma)^{2/p}},$$

Lemma 2.5 shows that as $\alpha \to \infty$,

$$\frac{Q_\alpha(u)}{(\alpha + N)^{2/p}} = S_p(u) + o(1)$$

uniformly on bounded subsets of $H^1(\Omega)$. This fact, that we have already used, has further consequences that we now examine, especially in connection with the results on $Q_\alpha''(u_\alpha)$ of Section 3.

Theorem 5.2. *For all $p \in (2, 2_*)$ the minimizers of S_p over $H_{\mathrm{rad}}^1(\Omega)$ are local minima of S_p over the whole space $H^1(\Omega)$.*

Proof. It is a simplified version of the proof of Proposition 3.6. Indeed we will show that if u is a minimizer of S_p over $H_{\mathrm{rad}}^1(\Omega)$, then

$$\min_{\substack{v \in T_u \mathcal{S} \\ \|v\|=1}} S_p''(u) \cdot v^2 > 0, \tag{34}$$

where $T_u \mathcal{S} = \{v \in H^1(\Omega): \langle u, v \rangle = 0\}$.

Notice that, by (33), $\langle u, v \rangle = 0$ is equivalent to $\int_{\partial \Omega} u^{p-1} v \, d\sigma = 0$.

Since u minimizes S_p among radial functions, we have that $u = \varphi_1$ and $S_p(u) = S_p(\varphi_1) = |\partial \Omega|^{1-2/p} \lambda_1$; therefore, with the same arguments as in Lemma 3.1 we see that

$$S_p''(u) \cdot v^2 = S_p''(\varphi_1) \cdot v^2 = 2|\partial \Omega|^{1-2/p} \lambda_1 \left(\|v\|^2 - (p-1)|\partial \Omega|^{p/2-1} \lambda_1^{p/2} \int_{\partial \Omega} \varphi_1^{p-2} v^2 \, d\sigma \right) \tag{35}$$

for all $v \in T_u \mathcal{S}$.

Recalling that $\varphi_1(1)^{p-2} = |\partial \Omega|^{1-p/2} \lambda_1^{1-p/2}$, and taking $\|v\| = 1$, we arrive at

$$S_p''(u) \cdot v^2 = 2|\partial \Omega|^{1-2/p} \lambda_1 \left(1 - (p-1)\lambda_1 \int_{\partial \Omega} v^2 \, d\sigma \right).$$

Since $v \in T_u \mathcal{S}$, we have $\int_{\partial \Omega} v^2 \, d\sigma \leqslant 1/\lambda_2$, so that

$$S_p''(u) \cdot v^2 \geqslant 2|\partial \Omega|^{1-2/p} \lambda_1 0 \left(1 - (p-1)\frac{\lambda_1}{\lambda_2} \right)$$

for all $v \in T_u \mathcal{S}$, with $\|v\| = 1$. By the results at the end of the proof of Proposition 3.6 the number in the right-hand side of the preceding inequality is uniformly positive, for all $p \in (2, 2_*)$, which shows that $S_p''(u)$ is positive definite on $T_u \mathcal{S}$. Hence $u = \varphi_1$ is a local minimum for S_p over $H^1(\Omega)$. $\quad\square$

Some comments are in order. It is not known whether the best constant S_p is attained by a radial function for all $p \in (2, 2_*)$. If the ground states are not radial for some p, it is quite natural to expect that they bifurcate from the branch of radial minimizers. Theorem 5.2 shows however that this is definitely not the case: nondegeneracy of radial minimizers over the whole space $H^1(\Omega)$ rules out any bifurcation phenomenon. Nonradial ground states, if any exist, are rather "separated objects", located far away from the radial minimizers and whose existence begins only after a certain value of p.

It is also interesting to compare our results with Theorem 5.1, by Lami Dozo and Torné (the actual result from [13] is much more general and applies to a wider class of problems). Theorem 5.1 states that if

$$p - 1 > \frac{1}{\lambda_1^2} \big(1 - (N-1)\lambda_1 \big), \tag{36}$$

then no minimizer of (31) is radial; the argument consists in showing that a suitable (small nonradial) variation of a radial minimizer makes the functional $S_p(u)$ decrease. Therefore we are in the presence of a local phenomenon, around radial minimizers.

On the other hand, Theorem 5.2 shows that radial minimizers are local minima over the whole space $H^1(\Omega)$, for all $p \in (2, 2_*)$; the key argument is the fact already proved that for all these p,

$$p - 1 < \frac{\lambda_2}{\lambda_1}. \tag{37}$$

A natural question is to compare the conditions (36) and (37); our intent is to give a natural interpretation of (36). The next result shows that the two conditions are in some sense dual.

Proposition 5.3. *There results*

$$\frac{1}{\lambda_1^2}\bigl(1 - (N - 1)\lambda_1\bigr) = \frac{\lambda_2}{\lambda_1}. \tag{38}$$

Proof. We recall from (24) that

$$\lambda_k = k - 1 + \frac{I_{k+N/2-1}(1)}{I_{k+N/2-2}(1)}$$

for all $k = 1, 2, \ldots$. In particular,

$$\lambda_1 = \frac{I_{N/2}(1)}{I_{N/2-1}(1)} \quad \text{and} \quad \lambda_2 = \frac{I_{N/2+1}(1)}{I_{N/2}(1)} + 1.$$

To prove (38) we have to show that $\frac{1}{\lambda_1} - N + 1 = \lambda_2$.

The already used recursive relation $I_{\nu-1}(x) = I_{\nu+1}(x) + \frac{2\nu}{x}I_\nu(x)$, for $\nu = N/2$ and $x = 1$ reads

$$I_{N/2-1}(1) = I_{N/2+1}(1) + NI_{N/2}(1).$$

Therefore

$$\frac{1}{\lambda_1} - N + 1 = \frac{I_{N/2-1}(1)}{I_{N/2}(1)} - N + 1 = \frac{I_{N/2+1}(1) + NI_{N/2}(1)}{I_{N/2}(1)} - N + 1 = \frac{I_{N/2+1}(1)}{I_{N/2}(1)} + 1 = \lambda_2. \quad \square$$

Although (36) or (37) are only sufficient conditions for the existence of nonradial minimizers, the fact that $\lambda_2/\lambda_1 > N/(N-2) = 2_* - 1$ for all $N \geq 3$ (proved in Section 3) and the variational properties of the radial minimizers described in this section seem to provide some evidence towards the validity of the following

Conjecture. *For all $N \geq 3$ and for all $p \in (2, 2_*)$, the best constant S_p for the trace inequality on the* unit *ball of* $\mathbf{R}^N$ *is attained by a* radial *function.*

6. Symmetry for slow growth

In this final section we return to the Neumann problem for the Hénon equation. We have seen that for every $p \in (2_*, 2^*)$ the minimizers of Q_α are not radial provided α is sufficiently large. In the interval $(2, 2_*)$ the situation is less clear, since it depends on the symmetry properties of the minimizers of the trace inequality, which are not precisely known for the unit ball. We point out that even if one knows that the minimizers of S_p are radial, it is not clear a priori that also the minimizers of Q_α should be radial.

In this section we investigate the symmetry of minimizers when p is close to 2. It is interesting to keep in mind the behavior of minimizers for the Dirichlet problem described in [18]: in that case the authors showed that for p close to 2 minimizers are nonradial only if α is very large (the threshold α^* between radial and nonradial minimizers tends to infinity as $p \to 2$). However the symmetry breaking phenomenon persists, as for a fixed p close to 2 one has nonradial solutions for very large α.

We show in Theorem 6.1 below that this is not the case for minimization in $H^1(\Omega)$: for p close to 2 minimizers are radial for *all* α large enough.

Of course we take advantage of some result for the "limit" problem given by the minimization of S_p. The precise result we need is contained in Theorem 4 of [13]. There it is proved that there exists $\hat{p} \in (2, 2_*]$ such that for every $p \in (2, \hat{p}]$ the problem $\min_{u \in H^1(\Omega)} S_p(u)$ has a unique solution, which is radial. Of course this solution (with norm equal to one) is φ_1, and $S_p(\varphi_1) = |\partial\Omega|^{1-2/p}\lambda_1$.

Theorem 6.1. *Let $p \in (2, \hat{p})$. For every α large enough the problem*

$$\min_{u \in H^1(\Omega)} \frac{\|u\|^2}{(\int_\Omega |u|^p |x|^\alpha \, dx)^{2/p}} \tag{39}$$

has a unique positive solution (normalized by $\|u\| = 1$), and it is a radial function.

Proof. The proof of this theorem follows very closely that of Theorem 4.1; the main difference comes from the fact that in the present case we are not dealing with radial functions, which tends to complicate things. On the other hand we will profit of the fact that we are now working with $p < 2_*$.

By Lemma 2.5 we have that as $\alpha \to \infty$,

$$\frac{Q_\alpha(u)}{(\alpha + N)^{2/p}} = S_p(u) + o(1)$$

uniformly on bounded subsets of $H^1(\Omega)$; thus, setting $m_\alpha = \min_{H^1(\Omega) \cap \mathcal{S}} Q_\alpha$ we see that

$$m_\alpha \sim (\alpha + N)^{2/p} |\partial\Omega|^{1-2/p}\lambda_1.$$

Let $u_\alpha \in H^1(\Omega) \cap \mathcal{S}$ be (positive and) such that $Q_\alpha(u_\alpha) = m_\alpha$. Then, up to subsequences, $u_\alpha \to u$ weakly in $H^1(\Omega)$, and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$, since $p < 2_*$. Notice that $u \not\equiv 0$, because otherwise

$$|\partial\Omega|^{1-2/p}\lambda_1 + o(1) = \frac{m_\alpha}{(\alpha + N)^{2/p}} = \frac{1}{(\int_{\partial\Omega} u_\alpha^p \, d\sigma)^{2/p}} + o(1) \to \infty,$$

which is absurd.

Furthermore, since $p < \hat{p}$,

$$|\partial\Omega|^{1-2/p}\lambda_1 = \min_{v \in H^1(\Omega)} \frac{\|v\|^2}{(\int_{\partial\Omega} v^p \, d\sigma)^{2/p}} \leqslant \frac{\|u\|^2}{(\int_{\partial\Omega} u^p \, d\sigma)^{2/p}} \leqslant \frac{1}{(\int_{\partial\Omega} u^p \, d\sigma)^{2/p}}$$

$$= \lim_\alpha \frac{1}{(\int_{\partial\Omega} u_\alpha^p \, d\sigma)^{2/p}} = \lim_\alpha \left(\frac{Q_\alpha(u_\alpha)}{(\alpha + N)^{2/p}} + o(1) \right) = |\partial\Omega|^{1-2/p}\lambda_1.$$

Therefore we have that $\|u\| = 1$ and that u is a minimizer for S_p. By the above quoted results and the assumption $p < \hat{p}$, it must be $u = \varphi_1$. We conclude that every sequence of minimizers for Q_α converges to φ_1 strongly in $H^1(\Omega)$.

We now show that Q_α has a unique minimizer for α large. Suppose this is not true; then for every α in an unbounded set A, there exist two distinct minimizers u_α and v_α. As in the proof of Theorem 4.1, if we set

$$\psi_\alpha = \frac{u_\alpha - v_\alpha}{\|u_\alpha - v_\alpha\|},$$

we see that it solves

$$\begin{cases} -\Delta\psi_\alpha + \psi_\alpha = (p-1)m_\alpha^{p/2}|x|^\alpha c_\alpha \psi_\alpha & \text{in } \Omega, \\ \dfrac{\partial\psi_\alpha}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{40}$$

with

$$c_\alpha = \int_0^1 \left(v_\alpha + t(u_\alpha - v_\alpha) \right)^{p-2} dt.$$

Notice that $|c_\alpha| \leqslant (u_\alpha + v_\alpha)^{p-2}$.

Since $\psi_\alpha \in \mathcal{S}$, up to subsequences we can assume that $\psi_\alpha \to \psi$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. We claim that $\psi \not\equiv 0$. To see this we multiply the equation in (40) by ψ_α and we use Lemma 2.5 to obtain

$$
\begin{aligned}
1 = \|\psi_\alpha\|^2 &= (p-1)m_\alpha^{2/p} \int_\Omega c_\alpha \psi_\alpha^2 |x|^\alpha \, dx = (p-1)\big(|\partial\Omega|^{2/p-1}\lambda_1^{p/2} + o(1)\big)(\alpha+N)\int_\Omega c_\alpha \psi_\alpha^2 |x|^\alpha \, dx \\
&\leqslant C\left((\alpha+N)\int_\Omega c_\alpha^{p/(p-2)}|x|^\alpha \, dx\right)^{1-2/p}\left((\alpha+N)\int_\Omega \psi_\alpha^p |x|^\alpha \, dx\right)^{2/p} \\
&\leqslant C\left((\alpha+N)\int_\Omega (u_\alpha + v_\alpha)^p |x|^\alpha \, dx\right)^{1-2/p}\left(\int_{\partial\Omega} \psi_\alpha^p \, d\sigma + o(1)\right)^{2/p} \\
&\leqslant C\left(\int_{\partial\Omega}(u_\alpha+v_\alpha)^p \, d\sigma + o(1)\right)^{1-2/p}\left(\int_{\partial\Omega}\psi_\alpha^p \, d\sigma + o(1)\right)^{2/p} \leqslant C\left(\int_{\partial\Omega}\psi_\alpha^p \, d\sigma + o(1)\right)^{2/p}.
\end{aligned}
$$

If ψ is zero, then the strong convergence of ψ_α in $L^p(\partial\Omega)$ gives a contradiction.

We now pass to the limit in the weak form of (40), which is

$$
\langle \psi_\alpha, \phi \rangle = (p-1)m_\alpha^{2/p} \int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx
$$

for all $\phi \in H^1(\Omega)$.

We write

$$
\begin{aligned}
(\alpha+N)\int_\Omega c_\alpha \psi_\alpha \phi |x|^\alpha \, dx &= (\alpha+N)\left[\int_\Omega \varphi_1^{p-2}\psi_\alpha \phi |x|^\alpha \, dx + \int_\Omega \big(c_\alpha - \varphi_1^{p-2}\big)\psi_\alpha \phi |x|^\alpha \, dx\right] \\
&= (\alpha+N)\left[\int_\Omega \varphi_1^{p-2}\psi \phi |x|^\alpha \, dx + \int_\Omega \varphi_1^{p-2}(\psi_\alpha - \psi)\phi |x|^\alpha \, dx \right. \\
&\qquad \left. + \int_\Omega \big(c_\alpha - \varphi_1^{p-2}\big)\psi_\alpha \phi |x|^\alpha \, dx\right]
\end{aligned}
$$

and we evaluate the three terms in the right-hand side separately. For the first one we have

$$
(\alpha+N)\int_\Omega \varphi_1^{p-2}\psi \phi |x|^\alpha \, dx = \int_{\partial\Omega} \varphi_1^{p-2}\psi \phi \, d\sigma - \int_\Omega \nabla\big(\varphi_1^{p-2}\psi\phi\big) \cdot |x|^\alpha x \, dx,
$$

and, by Hölder inequality,

$$
\begin{aligned}
&\left|\int_\Omega \nabla\big(\varphi_1^{p-2}\psi\phi\big)\cdot |x|^\alpha x \, dx\right| \\
&\leqslant C\int_\Omega |\psi||\phi||x|^\alpha \, dx + C\int_\Omega \big(|\phi||\nabla\psi| + |\psi||\nabla\phi|\big)|x|^\alpha \, dx \\
&\leqslant C\|\psi\|_2 \|\phi\|_{2^*}\left(\int_\Omega |x|^{\alpha N} \, dx\right)^{1/N} + C\big(\|\nabla\psi\|_2\|\phi\|_{2^*} + \|\nabla\phi\|_2\|\psi\|_{2^*}\big)\left(\int_\Omega |x|^{\alpha N} \, dx\right)^{1/N} = o(1)
\end{aligned}
$$

as $\alpha \to \infty$. Therefore

$$
(\alpha+N)\int_\Omega \varphi_1^{p-2}\psi \phi |x|^\alpha \, dx = \int_{\partial\Omega} \varphi_1^{p-2}\psi \phi \, d\sigma + o(1).
$$

For the second term we apply Hölder inequality to obtain, by Lemma 2.5,

$$(\alpha + N) \int_{\Omega} \varphi_1^{p-2} (\psi_\alpha - \psi) \phi |x|^\alpha \, dx$$

$$\leqslant \left((\alpha + N) \int_{\Omega} \varphi_1^p |x|^\alpha \, dx \right)^{1-2/p} \left((\alpha + N) \int_{\Omega} |\psi_\alpha - \psi|^p |x|^\alpha \, dx \right)^{1/p} \left((\alpha + N) \int_{\Omega} |\phi|^p |x|^\alpha \, dx \right)^{1/p}$$

$$= \left(\int_{\partial\Omega} \varphi_1^p \, d\sigma + o(1) \right)^{1-2/p} \left(\int_{\Omega} |\psi_\alpha - \psi|^p \, d\sigma + o(1) \right)^{1/p} \left(\int_{\Omega} |\phi|^p \, d\sigma + o(1) \right)^{1/p} = o(1)$$

because $\psi_\alpha \to \psi$ strongly in $L^p(\partial\Omega)$.

Finally, for the third term we write

$$(\alpha + N) \int_{\Omega} \left(c_\alpha - \varphi_1^{p-2} \right) \psi_\alpha \phi |x|^\alpha \, dx \leqslant \left((\alpha + N) \int_{\Omega} \left| c_\alpha - \varphi_1^{p-2} \right|^{p/(p-2)} |x|^\alpha \, dx \right)^{(p-2)/p}$$

$$\times \left((\alpha + N) \int_{\Omega} |\psi_\alpha|^p |x|^\alpha \, dx \right)^{1/p} \left((\alpha + N) \int_{\Omega} |\phi|^p |x|^\alpha \, dx \right)^{1/p}$$

and we readily recognize, as above, that the last two integrals are uniformly bounded as $\alpha \to \infty$. Recalling the definition of c_α it is easy to see that the first integral goes to zero as $\alpha \to \infty$.

Putting together the above estimates we can say that

$$\langle \psi_\alpha, \phi \rangle = (p-1) \left(|\partial\Omega|^{p/2-1} \lambda_1^{p/2} + o(1) \right) \left(\int_{\partial\Omega} \varphi^{p-2} \psi \phi \, d\sigma + o(1) \right),$$

so that, when $\alpha \to \infty$,

$$\langle \psi, \phi \rangle = (p-1)\lambda_1 \int_{\partial\Omega} \psi \phi \, d\sigma, \tag{41}$$

for all $\phi \in H^1(\Omega)$ (we have used the fact that $\varphi_1^{p-2} \equiv |\partial\Omega|^{1-p/2} \lambda_1^{1-p/2}$ on $\partial\Omega$). Eq. (41) says that ψ is a (nontrivial) weak solution of

$$\begin{cases} -\Delta\psi + \psi = 0 & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = (p-1)\lambda_1 \psi & \text{on } \partial\Omega. \end{cases}$$

Since we know that $(p-1)\lambda_1$ is not an eigenvalue of the Steklov problem (6), we conclude that $\psi \equiv 0$, a contradiction.

Therefore $u_\alpha \equiv v_\alpha$ for all α large. In other words, problem (39) has a unique solution for $p \in (2, \hat{p})$ and α large. Since (39) is invariant under rotations, this solution must be radial. $\quad\square$

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