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A Liouville-type theorem for the p-Laplacian with potential term

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Abstract

In this paper we prove a sufficient condition, in terms of the behavior of a ground state of a singular p-Laplacian problem with a potential term, such that a nonzero subsolution of another such problem is also a ground state. Unlike in the linear case ($p = 2$), this condition involves comparison of both the functions and of their gradients.

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1. Introduction

Positivity properties of quasilinear elliptic equations, in particular those with the p-Laplacian term in the principal part, have been extensively studied over the recent decades (see for example [2,3,6,10] and the references therein). Fix $p \in (1, \infty)$, and a domain $\Omega \subseteq \mathbb{R}^d$. In this paper we use positivity properties of such equations to prove a general Liouville comparison principle for equations of the form

$$-\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega,$$

where $\Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, and $V \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R})$ is a given potential. Throughout this paper we assume that

$$Q(u) := \int_\Omega \left(|\nabla u|^p + V|u|^p\right) \mathrm{d}x \geqslant 0 \tag{1.1}$$

for all $u \in C_0^\infty(\Omega)$.

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Definition 1.1. We say that a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (weak) solution of the equation

$$\frac{1}{p} Q'(v) := -\Delta_p(v) + V|v|^{p-2}v = 0 \quad \text{in } \Omega, \tag{1.2}$$

if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi\right) \mathrm{d}x = 0. \tag{1.3}$$

We say that a real function $v \in C^1_{\mathrm{loc}}(\Omega)$ is a *supersolution* (resp. *subsolution*) of Eq. (1.2) if for every nonnegative $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi + V|v|^{p-2}v\varphi\right) \mathrm{d}x \geqslant 0 \quad (\text{resp. } \leqslant 0). \tag{1.4}$$

Remark 1.2. It is well known that any weak solution of (1.2) admits Hölder continuous first derivatives, and that any nonnegative solution of (1.2) satisfies the Harnack inequality [12,13,15].

Definition 1.3. We say that the functional Q has a *weighted spectral gap in Ω* if there is a positive continuous function W in Ω such that

$$Q(u) \geqslant \int_\Omega W|u|^p \,\mathrm{d}x \quad \forall u \in C_0^\infty(\Omega). \tag{1.5}$$

Definition 1.4. Let Q be a nonnegative functional on $C_0^\infty(\Omega)$. We say that a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions is a *null sequence* of the functional Q in Ω, if there exists an open set $B \Subset \Omega$ (i.e., $\overline{B}$ is compact in Ω) such that $\int_B |u_k|^p \,\mathrm{d}x = 1$, and

$$\lim_{k\to\infty} Q(u_k) = \lim_{k\to\infty} \int_\Omega \left(|\nabla u_k|^p + V|u_k|^p\right) \mathrm{d}x = 0. \tag{1.6}$$

We say that a positive function $v \in C^1_{\mathrm{loc}}(\Omega)$ is a *ground state* of the functional Q in Ω if v is an $L^p_{\mathrm{loc}}(\Omega)$ limit of a null sequence of Q.

Remark 1.5. The requirement that $\{u_k\} \subset C_0^\infty(\Omega)$, can clearly be weakened by assuming only that $\{u_k\} \subset W_0^{1,p}(\Omega)$. Also the requirement that $\int_B |u_k|^p \,\mathrm{d}x = 1$ can be replaced by $\int_B |u_k|^p \,\mathrm{d}x \asymp 1$, where $f_k \asymp g_k$ means that there exists a positive constant C such that $C^{-1} g_k \leqslant f_k \leqslant C g_k$ for all $k \in \mathbb{N}$.

The following theorem was proved in [10].

Theorem 1.6. *Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $V \in L^\infty_{\mathrm{loc}}(\Omega)$, and $p \in (1,\infty)$. Suppose that the functional Q is nonnegative on $C_0^\infty(\Omega)$. Then*

(a) *Q has either a weighted spectral gap or a ground state.*
(b) *If Q admits a ground state v, then $v > 0$ and v satisfies* (1.2).
(c) *The functional Q admits a ground state if and only if* (1.2) *admits a unique positive supersolution.*

Example 1.7. Consider the functional $Q(u) := \int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{d}x$. It follows from [7, Theorem 2] that if $d \leqslant p$, then Q admits a ground state $\varphi =$ constant in $\mathbb{R}^d$. On the other hand, if $d > p$, then

$$u(x) := \left[1 + |x|^{p/(p-1)}\right]^{(p-d)/p}, \qquad v(x) = \text{constant}$$

are two positive supersolutions of the equation $-\Delta_p(u) = 0$ in $\mathbb{R}^d$. Therefore, Theorem 1.6(c) implies that if $d > p$, then Q has a weighted spectral gap in $\mathbb{R}^d$. See also Example 3.2.

In a recent paper [9], Theorem 1.6 was used in order to prove, for $p = 2$, the following Liouville-type statement.

Theorem 1.8. ([9]) *Let Ω be a domain in $\mathbb{R}^d$, $d \geqslant 1$. Consider two strictly elliptic Schrödinger operators defined on Ω of the form*

$$P_j := -\nabla \cdot (A_j \nabla) + V_j, \quad j = 0, 1, \tag{1.7}$$

where $V_j \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R})$ for some $p > d/2$, and $A_j : \Omega \to \mathbb{R}^{d^2}$ are measurable symmetric matrices such that for any $K \Subset \Omega$ there exists $\mu_K > 1$ such that

$$\mu_K^{-1} I_d \leqslant A_j(x) \leqslant \mu_K I_d \quad \forall x \in K. \tag{1.8}$$

(Here I_d is the d-dimensional identity matrix, and the matrix inequality $A \leqslant B$ means that $B - A$ is a nonnegative matrix on $\mathbb{R}^d$.)

Assume that the following assumptions hold true.

(i) *The operator P_1 admits a ground state φ in Ω.*
(ii) *$P_0 \geqslant 0$ on $C_0^\infty(\Omega)$, and there exists a real function $\psi \in H^1_{\mathrm{loc}}(\Omega)$ such that $\psi_+ \neq 0$, and $P_0 \psi \leqslant 0$ in Ω, where $u_+(x) := \max\{0, u(x)\}$.*
(iii) *The following matrix inequality holds*

$$(\psi_+)^2(x) A_0(x) \leqslant C \varphi^2(x) A_1(x) \quad \text{a.e. in } \Omega, \tag{1.9}$$

where $C > 0$ is a positive constant.

Then the operator P_0 admits a ground state in Ω, and ψ is the corresponding ground state. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution of the equation $P_0 u = 0$ in Ω.

The purpose of this paper is to find an analog of Theorem 1.8 when $p \neq 2$. The main statement is as follows.

Theorem 1.9. *Let Ω be a domain in $\mathbb{R}^d$, $d \geqslant 1$, and let $p \in (1, \infty)$. For $j = 0, 1$, let $V_j \in L^\infty_{\mathrm{loc}}(\Omega)$, and let*

$$Q_j(u) := \int_\Omega \left(\left|\nabla u(x)\right|^p + V_j(x)\left|u(x)\right|^p\right) \mathrm{d}x \quad u \in C_0^\infty(\Omega).$$

Assume that the following assumptions hold true.

(i) *The functional Q_1 admits a ground state φ in Ω.*
(ii) *$Q_0 \geqslant 0$ on $C_0^\infty(\Omega)$, and the equation $Q_0'(u) = 0$ in Ω admits a subsolution $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying $\psi_+ \neq 0$.*
(iii) *The following inequality holds almost everywhere in Ω*

$$\psi_+ \leqslant C \varphi, \tag{1.10}$$

where $C > 0$ is a positive constant.
(iv) *The following inequality holds almost everywhere in $\Omega \cap \{\psi > 0\}$*

$$|\nabla \psi|^{p-2} \leqslant C |\nabla \varphi|^{p-2}, \tag{1.11}$$

where $C > 0$ is a positive constant.

Then the functional Q_0 admits a ground state in Ω, and ψ is the ground state. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution of the equation $Q_0'(u) = 0$ in Ω.

Remark 1.10. Condition (1.11) is redundant for $p = 2$. For $p \neq 2$ it is equivalent to the assumption that the following inequality holds in Ω:

$$\begin{cases} |\nabla \psi_+| \leqslant C |\nabla \varphi| & \text{if } p > 2, \\ |\nabla \psi_+| \geqslant C |\nabla \varphi| & \text{if } p < 2, \end{cases} \tag{1.12}$$

where $C > 0$ is a positive constant.

Remark 1.11. This theorem holds if, in addition to (1.10), one assumes instead of $|\nabla\psi|^{p-2} \leqslant C|\nabla\varphi|^{p-2}$ in $\Omega \cap \{\psi > 0\}$ (see (1.11)), that the following inequality holds true almost everywhere in $\Omega \cap \{\psi > 0\}$

$$\psi^2|\nabla\psi|^{p-2} \leqslant C\varphi^2|\nabla\varphi|^{p-2}, \tag{1.13}$$

where $C > 0$ is a positive constant. This can be easily observed by repeating the proof of Theorem 1.9 with the equivalent energy functional represented in the form (2.14) instead of (2.13).

Remark 1.12. Suppose that $1 < p < 2$, and assume that the ground state $\varphi > 0$ of the functional Q_1 is such that $w = \mathbf{1}$ is a ground state of the functional

$$E_1^{\varphi}(w) = \int_{\Omega} \varphi^p|\nabla w|^p \,\mathrm{d}x, \tag{1.14}$$

that is, there is a sequence $\{w_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions satisfying $E_1^\varphi(w_k) \to 0$, and $\int_B |w_k|^p = 1$ for a fixed $B \Subset \Omega$ (this implies that $w_k \to \mathbf{1}$ in $L^p_{\mathrm{loc}}(\Omega)$). In this case, the conclusion of Theorem 1.9 holds if there is a nonnegative subsolution ψ_+ of $Q_0'(u) = 0$ satisfying (1.10) alone, without any assumption on the gradients (like (1.11) or (1.13)). This statement follows from the proof of Theorem 1.9 together with the trivial inequality

$$\int_{\Omega} v^2|\nabla w|^2\big(w|\nabla v| + v|\nabla w|\big)^{p-2} \,\mathrm{d}x \leqslant \int_{\Omega} v^p|\nabla w|^p \,\mathrm{d}x$$

which actually holds pointwise. We use this observation in Example 3.2.

Remark 1.13. By Picone identity, a nonnegative functional Q can be represented as the integral of a nonnegative Lagrangian L. Although the expression for L contains an indefinite term (see (2.2)), it admits a two-sided estimate by a simplified Lagrangian with nonnegative terms (see Lemma 2.2). We call the functional associated with this simplified Lagrangian the *simplified energy*. It plays a crucial role in the proof of Theorem 1.9.

Remark 1.14. Condition (1.11) is essential when $p > 2$, and presumably also when $p < 2$. When $p > 2$, $\Omega = \mathbb{R}^d$ and V is radially symmetric, Proposition 4.2 shows that the simplified energy functional is not equivalent to either of its two terms that lead to conditions (1.10) and (1.11), respectively (see also Remark 4.1).

The outline of the paper is as follows. In Section 2 we study the representation of Q as a functional with a positive Lagrangian, and derive the equivalent simplified energy. Theorem 1.9 is proved in Section 3, and Section 4 is devoted to the irreducibility of the simplified energy to either of its terms. In Section 5, we study a connection between the ground states of the functional Q and of its linearization.

2. Picone identity

Let $v > 0$, $v \in C^1_{\mathrm{loc}}(\Omega)$, and $u \geqslant 0$, $u \in C_0^\infty(\Omega)$. Denote

$$R(u, v) := |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right) \cdot |\nabla v|^{p-2}\nabla v, \tag{2.1}$$

and

$$L(u, v) := |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u \cdot |\nabla v|^{p-2}\nabla v. \tag{2.2}$$

Then the following *(generalized) Picone identity* holds [5,2,3]

$$R(u, v) = L(u, v). \tag{2.3}$$

Write $L(u, v) = L_1(u, v) + L_2(u, v)$, where

$$L_1(u, v) := |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla u||\nabla v|^{p-1}, \tag{2.4}$$

and

$$L_2(u,v) := p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\big(|\nabla u||\nabla v| - \nabla u\cdot\nabla v\big) \geqslant 0. \tag{2.5}$$

From the obvious inequality $t^p + (p-1) - pt \geqslant 0$, we also have that $L_1(u,v) \geqslant 0$. Therefore, $L(u,v) \geqslant 0$ in Ω. Let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a positive solution (resp. subsolution) of (1.2). Using (2.3) and (1.3) (resp. (1.4)), we infer that for every $u \in C_0^\infty(\Omega)$, $u \geqslant 0$,

$$Q(u) = \int_\Omega L(u,v)\,\mathrm{d}x, \quad \text{resp. } Q(u) \leqslant \int_\Omega L(u,v)\,\mathrm{d}x. \tag{2.6}$$

Let now $w := u/v$, where v is a positive solution of (1.2) and $u \in C_0^\infty(\Omega)$, $u \geqslant 0$. Then 2.6 implies

$$Q(vw) = \int_\Omega \big[|v\nabla w + w\nabla v|^p - w^p|\nabla v|^p - pw^{p-1}v|\nabla v|^{p-2}\nabla v\cdot\nabla w\big]\,\mathrm{d}x. \tag{2.7}$$

Similarly, if v is a nonnegative subsolution of (1.2), then

$$Q(vw) \leqslant \int_\Omega \big[|v\nabla w + w\nabla v|^p - w^p|\nabla v|^p - pw^{p-1}v|\nabla v|^{p-2}\nabla v\cdot\nabla w\big]\,\mathrm{d}x. \tag{2.8}$$

A need to study the linearized operator arises at a certain step in this paper. This linearized operator is a Schrödinger operator of the form

$$Pu := \big(-\nabla\cdot(A\nabla) + V\big)u \quad \text{in } \Omega. \tag{2.9}$$

We assume that $V \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R})$, and $A : \Omega \to \mathbb{R}^{d^2}$ is a measurable (symmetric) matrix valued function satisfying (1.8). We consider the quadratic form

$$\mathbf{a}[u] := \int_\Omega \big(A\nabla u\cdot\nabla u + V|u|^2\big)\,\mathrm{d}x \tag{2.10}$$

on $C_0^\infty(\Omega)$ associated with the operator P. We have the following version of Picone identity (see [10]).

Lemma 2.1. *Let ψ be a (real valued) solution of the equation $P\psi = 0$ in Ω. Then for any $v \in C_0^\infty(\Omega)$ we have*

$$\mathbf{a}[\psi v] = \int_\Omega \psi^2 A\nabla v\cdot\nabla v\,\mathrm{d}x. \tag{2.11}$$

Moreover, if $\psi \in H^1_{\mathrm{loc}}(\Omega)$ is a nonnegative subsolution of the equation $P\psi = 0$ in Ω, then for any nonnegative $v \in C_0^\infty(\Omega)$ we have

$$\mathbf{a}[\psi v] \leqslant \int_\Omega \psi^2 A\nabla v\cdot\nabla v\,\mathrm{d}x. \tag{2.12}$$

So, in the linear case, the quadratic form induces a convenient weighted (Dirichlet-type) norm

$$\|v\|^2 := \int_\Omega \psi^2 A\nabla v\cdot\nabla v\,\mathrm{d}x \quad \text{on } C_0^\infty(\Omega).$$

Recall that in the quasilinear case ($p \neq 2$), the Lagrangian L in Picone's identity and (2.7) contain indefinite terms. Therefore, it is more convenient to replace identity (2.7) by two-sided inequalities with a simpler expression which we call the *simplified energy*.

Lemma 2.2. *Let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a positive solution of* (1.2) *and let $w \in C^1_0(\Omega)$ be a nonnegative function. Then*

$$Q(vw) \asymp \int_\Omega v^2|\nabla w|^2\big(w|\nabla v| + v|\nabla w|\big)^{p-2}\,\mathrm{d}x. \tag{2.13}$$

Moreover, for all $p \neq 2$

$$Q(vw) \asymp C \int_{\Omega} |\nabla w|^2 \big(w|\nabla v| v^{\frac{2}{p-2}} + v^{\frac{p}{p-2}} |\nabla w|\big)^{p-2} \, \mathrm{d}x \tag{2.14}$$

In particular, for $p > 2$ *we have*

$$Q(vw) \asymp \int_{\Omega} \big(v^p |\nabla w|^p + v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2\big) \, \mathrm{d}x. \tag{2.15}$$

If v *is only a nonnegative subsolution of* (1.2)*, then*

$$Q(vw) \leqslant C \int_{\Omega \cap \{v>0\}} v^2 |\nabla w|^2 \big(w|\nabla v| + v|\nabla w|\big)^{p-2} \, \mathrm{d}x. \tag{2.16}$$

If $p \neq 2$*, then*

$$Q(vw) \leqslant C \int_{\Omega \cap \{v>0\}} |\nabla w|^2 \big(w|\nabla v| v^{\frac{2}{p-2}} + v^{\frac{p}{p-2}} |\nabla w|\big)^{p-2} \, \mathrm{d}x. \tag{2.17}$$

Moreover, for $p > 2$ *we have*

$$Q(vw) \leqslant C \int_{\Omega} \big(v^p |\nabla w|^p + v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2\big) \, \mathrm{d}x. \tag{2.18}$$

Proof. Let $1 < p < \infty$. We need the following elementary algebraic vector inequality (cf. [4,14])

$$|a+b|^p - |a|^p - p|a|^{p-2} a \cdot b \asymp |b|^2 \big(|a| + |b|\big)^{p-2} \tag{2.19}$$

for all $a, b \in \mathbb{R}^d$.

Indeed, let $t = |b|/|a|$ and $\theta = (a \cdot b)/(|a||b|)$. Note that for $-1 \leqslant \theta \leqslant 1$

$$\lim_{t \to \infty} \frac{|t^2 + 2\theta t + 1|^{p/2} - 1 - p\theta t}{t^2 (1+t)^{p-2}} = 1 \quad \text{uniformly}, \tag{2.20}$$

and

$$\lim_{t \to 0_+} \frac{|t^2 + 2\theta t + 1|^{p/2} - 1 - p\theta t}{t^2 (1+t)^{p-2}} = \frac{p}{2} \big(1 + (p-2)\theta^2\big) > C_p > 0. \tag{2.21}$$

Finally, we claim that for $t > 0$ and $-1 \leqslant \theta \leqslant 1$ we have

$$f(t, \theta) := \big|t^2 + 2\theta t + 1\big|^{p/2} - 1 - p\theta t > 0. \tag{2.22}$$

Indeed, set $s := (t^2 + 2\theta t + 1)^{1/2} \geqslant 0$, then

$$f(t, \theta) = \big[s^p + (p-1) - ps\big] + p\big[\big(t^2 + 2\theta t + 1\big)^{1/2} - 1 - \theta t\big].$$

Clearly, for $s \geqslant 0$ we have, $g(s) := [s^p + (p-1) - ps] \geqslant 0$, and $g(s) = 0$ if and only if $s = 1$, which holds if and only if $t = -2\theta$.

On the other hand, let

$$h(t, \theta) := p\big[\big(t^2 + 2\theta t + 1\big)^{1/2} - 1 - \theta t\big].$$

Then $h(t, \theta) \geqslant 0$, and $h(t, \theta) = 0$ if and only if $\theta = \pm 1$. Note that if $\theta = -1$ and $t = -2\theta$, then we have $f(2, -1) = 2p > 0$. Thus, $f(t, \theta) > 0$ for all $t > 0$ and $-1 \leqslant \theta \leqslant 1$.

Therefore, for $1 < p < \infty$, relations (2.20)–(2.22) imply

$$\big|t^2 + 2\theta t + 1\big|^{p/2} - 1 - p\theta t \asymp t^2 (1+t)^{p-2}.$$

Thus, (2.19) holds true for all $a, b \in \mathbb{R}^d$.

Set now $a := w|\nabla v|$, $b := v|\nabla w|$. Then we obtain (2.13) and (2.16) by applying (2.19) to (2.7) and (2.8), respectively. □

The following Allegretto–Piepenbrink-type theorem was proved in [10].

Theorem 2.3. ([10, Theorem 2.3]) *Let Q be a functional of the form* (1.1). *Then the following assertions are equivalent*

(i) *The functional Q is nonnegative on $C_0^\infty(\Omega)$.*
(ii) *Eq.* (1.2) *admits a global positive solution.*
(iii) *Eq.* (1.2) *admits a global positive supersolution.*

The next lemma is well known for $p = 2$ (see for example [1, Lemma 2.9]).

Lemma 2.4. *Let $v \in C^1_{\mathrm{loc}}(\Omega)$ be a subsolution of Eq.* (1.2). *Then v_+ is also a subsolution of* (1.2).

Proof. Fix $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. As in [1, Lemma 2.9], define for $\varepsilon > 0$

$$v_\varepsilon := \left(v^2 + \varepsilon^2\right)^{1/2}, \quad \text{and} \quad \varphi_\varepsilon := \frac{v_\varepsilon + v}{2v_\varepsilon}\varphi.$$

Then $v_\varepsilon \to |v|$, $\nabla v_\varepsilon \to \nabla|v|$, and $\varphi_\varepsilon \to (\operatorname{sgn} v_+)\varphi$ as $\varepsilon \to 0$. An elementary computation shows that

$$\nabla v_\varepsilon \cdot \nabla\varphi \leqslant \nabla v \cdot \nabla\left(\frac{v\varphi}{v_\varepsilon}\right),$$

and therefore,

$$\nabla\left(\frac{v_\varepsilon + v}{2}\right) \cdot \nabla\varphi \leqslant \nabla v \cdot \nabla\varphi_\varepsilon.$$

Since

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla v \cdot \nabla\varphi_\varepsilon + V|v|^{p-2}v\varphi_\varepsilon\right) \mathrm{d}x \leqslant 0, \tag{2.23}$$

it follows that

$$\int_\Omega \left(|\nabla v|^{p-2}\nabla\left(\frac{v_\varepsilon + v}{2}\right) \cdot \nabla\varphi + V|v|^{p-2}v\varphi_\varepsilon\right) \mathrm{d}x \leqslant 0. \tag{2.24}$$

Letting $\varepsilon \to 0$ we obtain

$$\int_\Omega \left(|\nabla v_+|^{p-2}\nabla v_+ \cdot \nabla\varphi + V|v_+|^{p-2}v_+\varphi\right) \mathrm{d}x \leqslant 0. \qquad \square \tag{2.25}$$

3. Proof of the main result

Proof of Theorem 1.9. By Lemma 2.4, we may assume that $\psi \geqslant 0$.

Let $\{u_k\}$ be a null sequence for Q_1, that is $Q_1(u_k) \to 0$ and, for some nonempty open set $B \Subset \Omega$, $\int_B u_k^p \, \mathrm{d}x = 1$. Without loss of generality, we may assume that $B \subset \operatorname{supp}\psi$. Let $w_k := u_k/\varphi$. From (2.13) it follows that with some $C > 0$

$$\int_\Omega \varphi^2|\nabla w_k|^2\left(w_k|\nabla\varphi| + \varphi|\nabla w_k|\right)^{p-2} \mathrm{d}x \leqslant C Q_1(u_k) \to 0.$$

Fix $\alpha, \beta \in \mathbb{R}_+$, then the function $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ defined by

$$f(s,t) := \alpha^2 t^2\left(\beta s^{1/(p-2)} + \alpha t\right)^{p-2}$$

is nondecreasing monotone function in each variable separately. Hence, assumptions (1.10) and (1.11) imply that

$$\int_\Omega \psi^2|\nabla w_k|^2\big(w_k|\nabla\psi|+\psi|\nabla w_k|\big)^{p-2}\,\mathrm{d}x\to 0.$$

Together with (2.13) this implies that $Q_0(\psi w_k)\to 0$. On the other hand, since $w_k\to 1$ in $L^p_{\mathrm{loc}}(\Omega)$, it follows that $\psi w_k\to\psi$ in $L^p_{\mathrm{loc}}(\Omega)$. Consequently, $\int_B \varphi^p w_k^p\,\mathrm{d}x=1$ implies that $\int_B \psi^p w_k^p\,\mathrm{d}x\asymp 1$. In light of Remark 1.5, we conclude that ψ is a ground state of Q_0. $\square$

Example 3.1. Assume that $1\leqslant d\leqslant p\leqslant 2$, $p>1$, $\Omega=\mathbb{R}^d$, and consider the functional $Q_1(u):=\int_{\mathbb{R}^d}|\nabla u|^p\,\mathrm{d}x$. By Example 1.7, the functional Q_1 admits a ground state $\varphi=\text{constant}$ in $\mathbb{R}^d$.

Let Q_0 be a functional of the form (1.1) satisfying $Q_0\geqslant 0$ on $C_0^\infty(\mathbb{R}^d)$. Let $\psi\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $\psi_+\neq 0$ be a subsolution of the equation $Q_0'(u)=0$ in $\mathbb{R}^d$, such that $\psi_+\in L^\infty(\mathbb{R}^d)$. It follows from Theorem 1.9 that ψ is the ground state of Q_0 in $\mathbb{R}^d$. In particular, ψ is (up to a multiplicative constant) the unique positive supersolution and unique bounded solution of the equation $Q_0'(u)=0$ in $\mathbb{R}^d$. Note that there is no assumption on the behavior of the potential V_0 at infinity. This result generalizes some striking Liouville theorems for Schrödinger operators on $\mathbb{R}^d$ that hold for $d=1,2$ and $p=2$ (see [9, Theorems 1.4–1.6]).

Example 3.2. Let $d>1$, $d\neq p$, and $\Omega:=\mathbb{R}^d\setminus\{0\}$ be the punctured space. Let $c^*_{p,d}:=|(p-d)/p|^p$ be the Hardy constant, and consider the functional

$$Q(u):=\int_\Omega\left(|\nabla u|^p-c^*_{p,d}\frac{|u|^p}{|x|^p}\right)\mathrm{d}x,\quad u\in C_0^\infty(\Omega).\tag{3.1}$$

By Hardy's inequality, Q is nonnegative on $C_0^\infty(\Omega)$. The proof of Theorem 1.3 in [11] shows that Q admits a null sequence. It can be easily checked that the function $v(r):=|r|^{(p-d)/p}$ is a positive solution of the corresponding radial equation:

$$-|v'|^{p-2}\left[(p-1)v''+\frac{d-1}{r}v'\right]-c^*_{p,d}\frac{|v|^{p-2}v}{r^p}=0,\quad r\in(0,\infty).$$

Therefore, $\varphi(x):=|x|^{(p-d)/p}$ is the ground state of the equation

$$-\Delta_p(u)-c^*_{p,d}\frac{|u|^{p-2}u}{|x|^p}=0\quad\text{in }\Omega.\tag{3.2}$$

Note that $\varphi\notin W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for $p\neq d$. In particular, φ is not a positive supersolution of the equation $\Delta_p(u)=0$ in $\mathbb{R}^d$.

Let Q_0 be a functional of the form (1.1) satisfying $Q_0\geqslant 0$ on $C_0^\infty(\Omega)$. Let $\psi\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1<p<\infty$, $p\neq d$, $\psi_+\neq 0$, be a subsolution of the equation $Q_0'(u)=0$ in Ω satisfying

$$\psi_+(x)\leqslant C|x|^{(p-d)/p},\quad x\in\Omega.\tag{3.3}$$

When $p>2$, we require in addition that the following inequality is satisfied

$$\psi_+(x)^2\big|\nabla\psi_+(x)\big|^{p-2}\leqslant C|x|^{2-d},\quad x\in\Omega.\tag{3.4}$$

It follows from Theorem 1.9, Remark 1.11 and Remark 1.12 that ψ is the ground state of Q_0 in Ω. The reason why (3.4) is stated only for $p>2$ hinges on the fact that for $p\leqslant 2$,

$$C^{-1}Q_0(\varphi w)\leqslant E_1^\varphi(w)=\int_\Omega |x|^{p-d}|\nabla w|^p\,\mathrm{d}x,\tag{3.5}$$

and for all $p>1$ the functional E_1^φ has a ground state $\mathbf{1}$. The null sequence convergent to this ground state is given by [11], relation (2.2) with $R\to\infty$. Therefore, Remark 1.12 applies.

Next, we present a family of functionals Q_0 for which the conditions of Example 3.2 are satisfied.

Example 3.3. Let $d \geqslant 2$, $1 < p < d$, $\alpha \geqslant 0$, and $\Omega := \mathbb{R}^d \setminus \{0\}$. Let

$$W_\alpha(x) := -\left(\frac{d-p}{p}\right)^p \frac{\alpha\, dp/(d-p) + |x|^{\frac{p}{p-1}}}{(\alpha + |x|^{\frac{p}{p-1}})^p}.$$

Note that if $\alpha = 0$ this is the Hardy potential as in the Example 3.2. If Q_0 is the functional (1.1) with the potential $V := W_\alpha$, then

$$\psi_\alpha(x) := \left(\alpha + |x|^{\frac{p}{p-1}}\right)^{-\frac{(d-p)(p-1)}{p^2}}$$

is a solution of $Q_0'(u) = 0$ in Ω, and therefore $Q_0 \geqslant 0$ on $C_0^\infty(\Omega)$. Moreover, one can use the calculations of Example 3.2 to show that ψ_α is a ground state of Q_0. Indeed, we note first that $\psi = \psi_\alpha$ satisfies (3.3). If $\frac{d}{d-1} < p < d$, then ψ_α satisfies also (3.4) and therefore, it is a ground state in this case. In the remaining case $p \leqslant \frac{d}{d-1} \leqslant 2$, Example 3.2 concludes that ψ_α is a ground state from the property of the functional (3.5).

4. The simplified energy

In this section we give examples showing that none of the terms in the simplified energy (2.15) for $p > 2$ is dominated by the other, so that (2.15) cannot be further simplified. In particular, neither condition (1.10) nor condition (1.11) in Theorem 1.9 can be omitted.

Let $p > 2$, and fix $v > 0$, $v \in C^1_{\mathrm{loc}}(\Omega)$. For $w \in C_0^\infty(\Omega)$ we denote

$$E_1^v(w) := \int_\Omega v^p |\nabla w|^p \,\mathrm{d}x, \tag{4.1}$$

and

$$E_2^v(w) := \int_\Omega v^2 |\nabla v|^{p-2} w^{p-2} |\nabla w|^2 \,\mathrm{d}x. \tag{4.2}$$

Remark 4.1. Suppose that $V = 0$ in Ω, and assume that Q has a weighted spectral gap in Ω. In this case, the constant function $v = \mathbf{1}$ is a positive solution of (1.2) which is not a ground state in Ω. Clearly, $E_2^v = 0$ on $C_0^\infty(\Omega)$. Therefore, the inequality $Q(vu) \leqslant C E_2^v(u)$ for $u \in C_0^\infty(\Omega)$ is generally false, or in other words, the first term in the simplified energy (2.15) is necessary. The above assumptions are satisfied if Q' is the p-Laplacian operator, and either $\operatorname{int}(\mathbb{R}^d \setminus \Omega) \neq \emptyset$, or $\Omega = \mathbb{R}^d$ and $p < d$ (see Example 1.7).

In the following proposition we restrict our consideration to the case $\Omega = \mathbb{R}^d$ and a radial positive solution v.

Proposition 4.2. *Let $\Omega = \mathbb{R}^d$ and $p > 2$.*

- *There exists a positive continuous radial function φ on $\mathbb{R}^d$ such that the simplified energy $E_1^\varphi(w) + E_2^\varphi(w)$ has a weighted spectral gap, but there exist a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^d)$ with $u_k \geqslant 0$, and an open set $B \Subset \mathbb{R}^d$ such that $E_1^\varphi(u_k) \to 0$ and $\int_B |u_k|^p \,\mathrm{d}x = 1$.*
- *Moreover, there exists a positive radial continuous function ψ on $\mathbb{R}^d$ such that the simplified energy $E_1^\psi(w) + E_2^\psi(w)$ has a weighted spectral gap, but there exist a sequence $\{v_k\} \subset C_0^\infty(\mathbb{R}^d)$ with $v_k \geqslant 0$, and an open set $B \Subset \mathbb{R}^d$ such that $E_2^\psi(v_k) \to 0$ and $\int_B |v_k|^p \,\mathrm{d}x = 1$.*

Proof. *Step 1.* Let Q be a functional of the form (1.1) on $\mathbb{R}^d$ with a radial potential V. By the proof of Theorem 2.3 (see [10, Theorem 2.3]), it follows that a *radial* functional Q is nonnegative on $C_0^\infty(\mathbb{R}^d)$ if and only if the equation $Q'(u) = 0$ admits a positive *radial* solution in $\mathbb{R}^d$. On the other hand, from the standard rearrangement argument it is evident that Q has a weighted spectral gap if and only if it has a radial weighted spectral gap with a radial, decreasing and fast decaying weight W. Therefore, Q has a weighted spectral gap if and only if there exists a positive continuous radial potential W such that the Euler–Lagrange equation for the functional $Q(u) - \int_{\mathbb{R}^d} W|u|^p \,\mathrm{d}x$ has a positive *radial* solution. But such a solution is a (radial) supersolution of the equation $Q'(u)=0$ in $\mathbb{R}^d$ which is not a solution. Therefore, it is sufficient to consider the restrictions of Q and Q' to radial functions. (So, in fact, we are dealing with a one-dimensional problem.)

Step 2. We establish for a radial function φ a criterion for the existence of null sequences for E_1^φ and E_2^φ using a change of variable. First, for a positive continuous function φ on $[0,\infty)$ define $\rho_1(r)$ by

$$\rho_1(r) := \int_1^r \left[\varphi^p(s)s^{d-1}\right]^{-1/(p-1)}\,\mathrm{d}s. \tag{4.3}$$

Assume further that $p \leqslant d$, then ρ_1 is well-defined on $(0,\infty)$ and $\rho_1(0)=-\infty$. Since

$$\frac{\mathrm{d}r}{\mathrm{d}\rho_1} = \left(\varphi^p r^{d-1}\right)^{1/(p-1)},$$

it follows that for a radially symmetric $w \in C_0^\infty(\mathbb{R}^d)$ we have

$$E_1^\varphi(w) = \int_{\mathbb{R}^d} \varphi^p |\nabla w|^p\,\mathrm{d}x = C_d \int_{-\infty}^{M_1} \left|w'(\rho_1)\right|^p\,\mathrm{d}\rho_1, \tag{4.4}$$

where

$$M_1 = M_1^\varphi := \int_1^\infty \left(\varphi^p r^{d-1}\right)^{-1/(p-1)}\,\mathrm{d}r. \tag{4.5}$$

Recall that from Example 1.7 it follows that for $p>1$ the p-Laplacian on (a,b) admits a ground state if and only if $(a,b)=\mathbb{R}$. Therefore, E_1^φ has a null sequence if and only if $M_1^\varphi=\infty$ and $p \leqslant d$ (cf. [8, Theorem 3.1]).

Consider now the functional E_2^φ. The substitution $u := w^{p/2}$ implies that

$$E_2^\varphi(w) = \int_{\mathbb{R}^d} \varphi^2\,|\nabla\varphi|^{p-2} w^{p-2} |\nabla w|^2\,\mathrm{d}x = (2/p)^2 \int_{\mathbb{R}^d} \varphi^2 |\nabla\varphi|^{p-2} |\nabla u|^2\,\mathrm{d}x. \tag{4.6}$$

Let

$$\rho_2(r) := \int_1^r \varphi(s)^{-2}\left|\varphi'(s)\right|^{2-p} s^{1-d}\,\mathrm{d}s, \tag{4.7}$$

and assume further that $d \geqslant 2$, then ρ_2 is well-defined function on $(0,\infty)$ and $\rho_2(0)=-\infty$.

Using spherical coordinates for radial w, and then the substitution ρ_2, we obtain

$$\int_{\mathbb{R}^d} \varphi^2\,|\nabla\varphi|^{p-2} w^{p-2} |\nabla w|^2\,\mathrm{d}x = C_d \int_{-\infty}^{M_2} |u'(\rho_2)|^2\,\mathrm{d}\rho_2,$$

where

$$M_2 = M_2^\varphi = \int_1^\infty \varphi^{-2} |\varphi'|^{2-p} r^{1-d}\,\mathrm{d}r. \tag{4.8}$$

Therefore, E_2^φ has a null sequence (for $p>2$ and $d \geqslant 2$) if and only if $M_2^\varphi=\infty$ (cf. [8, Theorem 3.1]).

Therefore, in order to prove the proposition it is sufficient to find two positive radial functions φ and ψ satisfying $M_1^\varphi=\infty$ and $M_2^\varphi<\infty$, while $M_1^\psi<\infty$ and $M_2^\psi=\infty$.

Step 3. Let us simplify now (4.5) and (4.8) in order to investigate when M_j^φ are finite or infinite for a specific φ and $j = 1, 2$. Without loss of generality, we assume that the integration in (4.5) and (4.8) is from r_0 to ∞, where $r_0 \gg 1$. We set first $\varphi(r) := r^{1-d/p}\eta(r)$, where $2 < p < d$. Then (4.5) becomes, up to a constant multiple,

$$M_1 = \int_{r_0}^{\infty} \left(\varphi^p r^{d-1}\right)^{-1/(p-1)} \mathrm{d}r = \int_{r_0}^{\infty} \eta^{-p/(p-1)} r^{-1} \,\mathrm{d}r. \tag{4.9}$$

Set now

$$\eta(r) := \left[t^{(p-1)/(p-2)} (\log t)^\gamma\right]^{(p-2)/p}, \quad \text{where } t := \log r, \text{ and } \gamma > 0.$$

Then we have

$$M_1 = \int_{r_0}^{\infty} \eta^{-p/(p-1)} r^{-1} \,\mathrm{d}r = \int_{t_0}^{\infty} \left[t^{(p-1)/(p-2)} (\log t)^\gamma\right]^{(2-p)/(p-1)} \mathrm{d}t = \int_{t_0}^{\infty} \frac{(\log t)^{\gamma(2-p)/(p-1)}}{t} \,\mathrm{d}t.$$

On the other hand,

$$M_2 = \int_{r_0}^{\infty} \varphi^{-2} |\varphi'|^{2-p} r^{1-d} \,\mathrm{d}r = \int_{r_0}^{\infty} \left|\frac{p-d}{p} + \frac{r\eta'}{\eta}\right|^{2-p} \eta^{-p} r^{-1} \,\mathrm{d}r. \tag{4.10}$$

Denote

$$\tilde{M}_2 := \int_{r_0}^{\infty} \eta^{-p} r^{-1} \,\mathrm{d}r = \int_{t_0}^{\infty} t^{1-p} (\log t)^{\gamma(2-p)} \,\mathrm{d}t. \tag{4.11}$$

Since $r|\eta'|/\eta \ll 1$, it follows from (4.10) that there exist $C > 0$ such that

$$C^{-1} \tilde{M}_2 \leqslant M_2 \leqslant C \tilde{M}_2. \tag{4.12}$$

Consequently, for $0 < \gamma \leqslant (p-1)/(p-2)$ and $2 < p < d$, we have $M_1^\varphi = \infty$ and $M_2^\varphi < \infty$.

On the other hand, fix $\beta \in \mathbb{R}$, $\beta \neq 0$, and let $\psi : (1, \infty) \to [1, \infty)$, be a smooth monotone function such that $\psi(r) \asymp r^\beta$, and such that ψ' satisfies for any $n = 1, 2, \ldots$,

$$\left|\psi'(r)\right| = \begin{cases} \mathrm{e}^{-r}, & r \in [2n + 1/4, 2n + 3/4], \\ |\beta| r^{\beta-1}, & r \in [2n + 1, 2n + 2]. \end{cases}$$

Therefore, if $\beta > (p-d)/p$, then $M_1^\psi < \infty$.

Consider now M_2^ψ, and recall that $2 < p \leqslant d$. Consequently, there exist $\varepsilon > 0$ and $C > 0$ such that the integrand of M_2^ψ satisfies

$$\psi^{-2} |\psi'|^{2-p} r^{1-d} \geqslant C \mathrm{e}^{\varepsilon r}$$

on a set of infinite measure. Hence, $M_2^\psi = \infty$. $\square$

Remark 4.3. The proof above takes into account that (4.5), (4.8) both converge or both diverge when $\varphi(r)$ is of the form $r^\alpha (\log r)^\beta$, and the differentiation occurs only with respect to γ for $\varphi(r) = r^{1-d/p} (\log r)^{(p-1)/p} (\log \log r)^\gamma$.

5. Application: ground state of the linearized functional

We consider the linearized problem associated with the functional $Q \geqslant 0$. Let φ be a positive solution of the equation $Q'(u) = 0$ in Ω, and let

$$\mathbf{a}[u] := \int_{\Omega} \left(|\nabla\varphi|^{p-2} |\nabla u|^2 + V(x) \varphi^{p-2} u^2\right) \mathrm{d}x. \tag{5.1}$$

Proposition 5.1. *Let φ be a positive solution of the equation $Q'(u) = 0$ in Ω satisfying $\nabla\varphi \neq 0$.*

1. *If $p > 2$ and φ is a ground state of Q, then φ is a ground state of* **a**.
2. *If $p < 2$ and φ is a ground state of* **a**, *then φ is a ground state of Q.*

Proof. Consider first the case $p > 2$. Assume that φ is a ground state of Q. Let $\{u_k\}$ be a null sequence of nonnegative functions, and let $w_k := u_k/\varphi$. By (2.15),

$$\int_\Omega \varphi^2 |\nabla\varphi|^{p-2} w_k^{p-2} |\nabla w_k|^2 \,\mathrm{d}x \to 0. \tag{5.2}$$

Set $v_k := w_k^{p/2}$. Then

$$\int_\Omega \varphi^2 |\nabla\varphi|^{p-2} |\nabla v_k|^2 \,\mathrm{d}x \to 0 \tag{5.3}$$

which by (2.11) yields

$$\mathbf{a}[\varphi v_k] \to 0.$$

Taking into account Remark 1.5, we conclude that $\{\varphi v_k\}$ is a null sequence for **a**.

The case $p < 2$ is similar. If $\{z_k\}$ is a null sequence of nonnegative functions for the form **a**, then (5.3) is satisfied with $v_k := z_k/\varphi$. This implies (5.2) with $w_k = v_k^{2/p}$, which by (2.13) yields $Q(u_k) \to 0$ with $u_k = \varphi w_k$. Therefore, $\{u_k\}$ is a null sequence for Q and the proposition is proved. □

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