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Regularity results for degenerate elliptic systems ☆

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Abstract

We prove regularity results for certain degenerate quasilinear elliptic systems with coefficients which depend on two different weights. By using Sobolev- and Poincaré inequalities due to Chanillo and Wheeden [S. Chanillo, R.L. Wheeden, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions, Amer. J. Math. 107 (1985) 1191–1226; S. Chanillo, R.L. Wheeden, Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations, Comm. Partial Differential Equations 11 (1986) 1111–1134] we derive a new weak Harnack inequality and adapt an idea due to L. Caffarelli [L.A. Caffarelli, Regularity theorems for weak solutions of some nonlinear systems, Comm. Pure Appl. Math. 35 (1982) 833–838] to prove a priori estimates for bounded weak solutions. For example we show that every bounded weak solution of the system $-D_\alpha(a^{\alpha\beta}(x,u,\nabla u)D_\beta u^i)=0$ with $|x|^2|\xi|^2 \leqslant a^{\alpha\beta}\xi_\alpha\xi_\beta \leqslant |x|^\tau|\xi|^2$, $|x|<1$, $\tau\in(1,2)$ is Hölder continuous. Furthermore we derive a Liouville theorem for entire solutions of the above systems.

Résumé

Nous prouvons des résultats de régularité pour certains systèmes elliptiques quasi linéaires dégénérés avec des coefficients dépendant de deux poids différents. En employant des inégalités de Sobolev- et Poincaré dues à Chanillo et Wheeden [S. Chanillo, R.L. Wheeden, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions, Amer. J. Math. 107 (1985) 1191–1226 ; S. Chanillo, R.L. Wheeden, Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations, Comm. Partial Differential Equations 11 (1986) 1111–1134] nous déduisons une nouvelle inégalité de Harnack et adaptons une idée due à L. Caffarelli [L.A. Caffarelli, Regularity theorems for weak solutions of some nonlinear systems, Comm. Pure Appl. Math. 35 (1982) 833–838] pour prouver des évaluations a priori pour des solutions limitées et faibles. Par exemple, chaque solution limitée et faible du système $-D_\alpha(a^{\alpha\beta}(x,u,\nabla u)D_\beta u^i)=0$ avec $|x|^2|\xi|^2 \leqslant a^{\alpha\beta}\xi_\alpha\xi_\beta \leqslant |x|^\tau|\xi|^2$, $|x|<1$, $\tau\in(1,2)$ est continue selon Hölder. De plus, nous déduisons un théorème de Liouville pour les solutions entières des systèmes ci-dessus.

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Keywords: Regularity of weak solutions; Degenerate elliptic systems; Harnack inequality; Liouville theorems

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1. Introduction

We consider weak solutions of degenerate elliptic systems of the form

$$-D_\alpha\big(A^{\alpha\beta}(x,u,\nabla u)D_\beta u^i\big)=f^i(x,u,\nabla u)\quad (i=1,\dots,m) \tag{1}$$

in a domain $\Omega\subset\mathbb{R}^n$, where $a^{\alpha\beta}(x):=A^{\alpha\beta}(x,u(x),\nabla u(x))$ are measurable and symmetric coefficients and $f(x,u,\nabla u)$ is a measurable function. Here and in the sequel, we use the summation convention: repeated Greek indices are to be summed from 1 to n, repeated Latin indices from 1 to m. We assume there exist measurable weights $v(x), w(x)>0$ a.e. in Ω with the property

$$w(x)|\xi|^2\leqslant a^{\alpha\beta}(x)\xi_\alpha\xi_\beta\leqslant v(x)|\xi|^2\quad \forall\xi\in\mathbb{R}^n. \tag{2}$$

Furthermore we require the following structure conditions:

1. $\sup_\Omega |u|\leqslant M<\infty$.
2. $|f(x,u,p)|\leqslant aQ(x,p)$ and $u(x)\cdot f(x,u,p)\leqslant a^*Q(x,p)$ for a.e. $x\in\Omega$ and for all $p\in\mathbb{R}^{n\times m}$ with some $a\geqslant 0$, $a^*\in\mathbb{R}$, where $Q(x,p):=a^{\alpha\beta}(x)p^i_\alpha p^i_\beta$.

The notion of a weak solution of (1) will be defined in Section 4; to prove regularity for weak solutions of (1) the weights v and w have to satisfy three further conditions, which we will state exactly in Section 2. Roughly speaking w and $z:=\frac{v^2}{w}$ have to be doubling weights and have to fulfill a weighted Poincaré- and a weighted Sobolev inequality. We will show that the weights $v(x)=|x|$ and $w(x)=|x|^\tau$ with $\tau\in[1,2)$ in $B_1(0)\subset\mathbb{R}^n$, $n\geqslant 3$ satisfy these conditions.

Optimal regularity results for weak solutions of uniformly elliptic systems of type (1) are well known and due to Hildebrandt and Widman [9], Wiegner [17,18] and Caffarelli [3]. For the case of equal weights, i.e. $v=w$, which belong to the Muckenhoupt class A_2 (see Section 2 for explicit definitions), Fabes, Kenig and Serapioni [7] have proven Hölder continuity for weak solutions of an elliptic equation. For certain different weights, Chanillo and Wheeden [5] proved regularity for weak solutions of elliptic equations, while for degenerate elliptic systems only very little is known. Baldes [1] and Baoyao [2] proved some results for equal weights, e.g. weak solutions of systems with bounded weights $v=w\in A_2$ are Hölder continuous provided the smallness condition $a^*+aM<1$ holds. The results in this paper are of much more general nature than in [1] or [2], and, in fact, are the first regularity results for singular systems with different weights.

Our proof uses an idea of L. Caffarelli [3] who proved a priori estimates for weak solutions of certain uniformly elliptic systems. His main tool was a weak Harnack inequality for supersolutions of a uniformly elliptic linear equation; we will prove such a Harnack inequality for solutions of degenerate (in the above sense) elliptic equations in Section 3. The proof of this Harnack inequality is based upon a method of Trudinger [16] in which a Harnack inequality for solutions of some mildly degenerate elliptic equations was shown. Our regularity result reads as follows:

Theorem 1.1. *Let u be a bounded, weak solution of* (1) *in $\Omega\subset\mathbb{R}^n$. The coefficients $a^{\alpha\beta}$ are required to fulfill* (2) *with admissible weights w and v* (*see Section* 2)*. Under the assumption $a^*+aM<2$ u is Hölder continuous and for every $\Omega'\Subset\Omega$ there exist constants $C=C(n,a,a^*,M,\Omega,\Omega')>0$ and $\alpha=\alpha(n,a,a^*,M)>0$, such that*

$$[u]_{\alpha,\Omega'}\leqslant C. \tag{3}$$

In the last section we also show a Liouville theorem for entire solutions of elliptic systems, whose coefficients are degenerate in an arbitrary large compact subset of $\mathbb{R}^n$ and uniformly elliptic outside this compact set, more precisely:

Theorem 1.2. *Let u be a bounded, weak solution of* (1) *in $\mathbb{R}^n$. The coefficients $a^{\alpha\beta}$ are assumed to be of type* (2) *in a ball $B_R(0)\subset\mathbb{R}^n$ with admissible weights w and v and to be uniformly elliptic outside this ball. If $a^*+aM<2$, then $u=const.$ a.e. in $\mathbb{R}^n$.*

This result extends a Liouville theorem for uniformly elliptic systems due to Hildebrandt and Widman [10] and Meier [11].

2. The Muckenhoupt classes A_p and conditions for the weights

The Muckenhoupt classes are defined in the paper [12] by Muckenhoupt in connection with Hardy functions. Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a nonnegative function.

Definition 2.1. Let $1 < p < \infty$. The weight w is an element of A_p, if

$$\sup_{B_R \subset \mathbb{R}^n} \left(\frac{1}{|B_R|} \int\limits_{B_R} w(x)\,dx\right)\left(\frac{1}{|B_R|} \int\limits_{B_R} w(x)^{\frac{-1}{p-1}}\,dx\right)^{p-1} =: C_p < \infty, \tag{4}$$

w is to be said of class A_∞, if for every $\epsilon > 0$ there exists a $\delta > 0$ with the property that for every measurable $E \subset B_R$ with $|E| < \delta |B_R|$ the inequality $w(E) \leqslant \epsilon w(B_R)$ holds, where $w(E) = \int_E w(x)\,dx$.

From [13] and [6] we infer $A_\infty = \bigcup_{p>1} A_p$. A result due to Muckenhoupt and Wheeden [14], p. 223 implies the *doubling property* for any $w \in A_\infty$:

$$w(B_{2R}) \leqslant K w(B_R) \quad \text{with some } K > 0. \tag{5}$$

We require the following conditions for the weights w and $z = \frac{v^2}{w}$ (cf. [5]):

(1) $w, z \in D_\infty$, i.e. the doubling property holds: $w(B_{2R}) \leqslant C w(B_R)$ and $z(B_{2R}) \leqslant C z(B_R)$ with a constant $C > 0$ independent of R.
(2) The following Poincaré inequality holds: There exists a $k > 1$ such that for all $B_R \subset \Omega$ and all $f \in C^1(\overline{B_R})$ the inequality

$$\left(\frac{1}{z(B_R)} \int\limits_{B_R} \left| f - \frac{1}{z(B_R)} \int\limits_{B_R} f z\,dx \right|^{2k} z\,dx\right)^{\frac{1}{2k}} \leqslant C R \left(\frac{1}{w(B_R)} \int\limits_{B_R} |\nabla f|^2 w\,dx\right)^{\frac{1}{2}} \tag{6}$$

holds with a constant C independent of f.
(3) The following Sobolev inequality holds: There exists a $k > 1$ such that for all $B_R \subset \Omega$ and all $f \in C^1_0(B_R)$ the inequality

$$\left(\frac{1}{z(B_R)} \int\limits_{B_R} |f|^{2k} z\,dx\right)^{\frac{1}{2k}} \leqslant C R \left(\frac{1}{w(B_R)} \int\limits_{B_R} |\nabla f|^2 w\,dx\right)^{\frac{1}{2}} \tag{7}$$

holds with a constant C independent of f.

Fabes, Kenig and Serapioni [7] showed that in the case $v = w \in A_2$ conditions (2) and (3) are satisfied. In the case of different weights, Chanillo and Wheeden [4] proved that condition (2) and (3) hold, if $w \in A_2$, $z \in D_\infty$ and if there is a $q > 2$ such that for all balls B_R, whose centers are in B_{2R}, the *balance condition*

$$s\left[\frac{z(B_{sR})}{z(B_R)}\right]^{\frac{1}{q}} \leqslant C\left[\frac{w(B_{sR})}{w(B_R)}\right]^{\frac{1}{2}} \tag{8}$$

holds for all $s \in (0, 1)$.

3. A weak Harnack inequality

To give a definition of a weak solution of a degenerate elliptic equation

$$D_\alpha\big(a^{\alpha\beta}(x) D_\beta u\big) = 0 \tag{9}$$

with coefficients $a^{\alpha\beta}(x)$ which satisfy (2) we first need to define the space $H^1_2(\Omega, v, w)$, where v and w are weights with the properties (1)–(3) of Section 2.

Definition 3.1. $H_2^1(\Omega, v, w)$ is defined as completion of $C^1(\Omega)$ with respect to the norm

$$\|u\|_{1,2,\Omega} = \sqrt{\int_\Omega a^{\alpha\beta}(x) D_\alpha u^i D_\beta u^i \, dx + \int_\Omega u^2 v \, dx}.$$

$\mathring{H}_2^1(v, w, \Omega)$ denotes the completion of $C_c^1(\Omega)$ with respect to the norm

$$\|u\|_{1,2,0,\Omega} = \sqrt{\int_\Omega a^{\alpha\beta}(x) D_\alpha u^i D_\beta u^i \, dx}.$$

Remark. It is possible to estimate $\|\cdot\|_{1,2,\Omega}$ as follows:

$$\int_\Omega |\nabla u|^2 w \, dx + \int_\Omega u^2 v \, dx \leqslant \|u\|_{1,2,\Omega}^2 \leqslant \int_\Omega |\nabla u|^2 v \, dx + \int_\Omega u^2 v \, dx < \infty.$$

If $u_k \in C^1(\Omega)$ is a sequence with $u_k \to u$ in $H_2^1(\Omega, v, w)$, then u_k and ∇u_k converge in $L_2(\Omega, v)$ and $L_2(\Omega, w)$ resp. If $\lim_{k\to\infty} \nabla u_k = v$, define $\nabla u := v$; ∇u is well defined (cf. [5], §2).

Definition 3.2. $u \in H_2^1(\Omega, v, w)$ is a weak subsolution of (9), if

$$\int_\Omega a^{\alpha\beta}(x) D_\beta u D_\alpha \phi \, dx \leqslant 0 \tag{10}$$

holds for every $\phi \in \mathring{H}_2^1(\Omega, v, w)$, $\phi \geqslant 0$. u is called a weak supersolution, if $-u$ is a weak subsolution and u is called a weak solution, if u is a weak subsolution and a weak supersolution.

The main result of this section is

Theorem 3.3. *Let u be a nonnegative weak supersolution of* (9) *in $\Omega \subset \mathbb{R}^n$. Then for any ball $B_R \subset \Omega$ with $\frac{z(B_R)}{w(B_R)} \leqslant C_1$ and any α, β, γ satisfying $0 < \alpha < \beta < 1$, $0 < \gamma < k$ the estimate*

$$\left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} |u|^\gamma z \, dx\right)^{\frac{1}{\gamma}} \leqslant C(n, \alpha, \beta, \gamma, C_1) \inf_{B_{\alpha R}} u \tag{11}$$

holds, where $k > 1$ is the constant from the Sobolev- and Poincaré inequalities.

The proof of Theorem 3.3 is divided into three lemmatas, extended proofs of these lemmatas can be found in [15]. All these lemmatas are based on a method developed by Trudinger [16].

Lemma 3.4. *Let u be a weak subsolution of* (9) *in $\Omega \subset \mathbb{R}^n$. Then for every $B_R \subset \Omega$ with $\frac{z(B_R)}{w(B_R)} \leqslant C_1$ we have for any $0 < \alpha < \beta < 1$ the estimate*

$$\sup_{B_{\alpha R}} u \leqslant C(n, \alpha, \beta, C_1) \left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} |u^+|^2 z \, dx\right)^{\frac{1}{2}}. \tag{12}$$

Proof. For $\delta \geqslant 1$ and $0 < N < \infty$ we define

$$F(u) = F_\delta^N(u) = \begin{cases} (u^+)^\delta, & u \leqslant N, \\ \delta N^{\delta-1} u - (\delta - 1) N^\delta, & u > N. \end{cases}$$

Use $\phi(x) = \eta^2(x) F(u)$, $\eta \geqslant 0$, $\eta \in C_c^1(B_R)$ as test function in (10). We arrive at

$$\int_\Omega \eta^2 F'(u) |\nabla u|^2 w \, dx \leqslant 2 \int_\Omega \eta |\eta_x| F |\nabla u| v \, dx. \tag{13}$$

The inequality $F(u) \leqslant u^+ F'(u)$ is easily derived; by using this relation, the Hölder inequality yields

$$\int_\Omega \eta^2(x) F'(u) |\nabla u|^2 w \, dx \leqslant C \int_\Omega \eta_x^2 (u^+)^2 F' z \, dx. \tag{14}$$

Define

$$G(u) := \int_0^u \left| F'(t) \right|^{\frac{1}{2}} dt = \begin{cases} \sqrt{\delta} \frac{2}{\delta+1} |u^+|^{\frac{\delta+1}{2}}, & u \leqslant N, \\ \sqrt{\delta} N^{\frac{\delta-1}{2}} |u|, & u > N. \end{cases}$$

With (14) we infer

$$\int_\Omega \eta^2 |\nabla G|^2 w \, dx \leqslant C \int_\Omega \eta_x^2 (u^+ G')^2 z \, dx.$$

In connection with the Sobolev inequality and $G \leqslant u^+ G'$ this estimate implies

$$\left(\frac{1}{z(B_R)} \int_{B_R} |\eta G|^{2k} z \, dx \right)^{\frac{1}{2k}} \leqslant C R \underbrace{\sqrt{\frac{z(B_R)}{w(B_R)}}}_{\leqslant \sqrt{C_1}} \left(\frac{1}{z(B_R)} \int_{B_R} \eta_x^2 (u^+ G')^2 z \, dx \right)^{\frac{1}{2}}. \tag{15}$$

Set $q := \frac{\delta+1}{2}$ and take the qth root of (15). Then choose ϱ and σ in a way that $\alpha \leqslant \varrho < \sigma \leqslant \beta$ and η in a way that $\operatorname{supp} \eta \subset B_{\sigma R}$, $\eta \equiv 1$ in $B_{\varrho R}$, $|\eta_x| \leqslant \frac{2}{(\sigma - \varrho) R}$. If $N = \infty$ we see $G(u) = \frac{\sqrt{\delta}}{q} (u^+)^q$; by using the doubling property for z we obtain

$$\left(\frac{1}{z(B_{\varrho R})} \int_{B_{\varrho R}} (u^+)^{2kq} z \, dx \right)^{\frac{1}{2kq}} \leqslant \left(\frac{Cq}{\sigma - \varrho} \right)^{\frac{1}{q}} \left(\frac{1}{z(B_{\beta R})} \int_{B_{\sigma R}} (u^+)^{2q} z \, dx \right)^{\frac{1}{2q}}. \tag{16}$$

Iteration of (16):

Define $q_0 := 1$, $q_i := k q_{i-1} = k^i$, furthermore set $\varrho_i = \alpha + (\beta - \alpha)^{1+i}$, $\sigma_i = \varrho_{i-1}$. With this choice of q_i and ϱ_i we infer

$$\sup_{B_{\alpha R}} u \leqslant \prod_{l=0}^{\infty} \left(\frac{C q_l}{\varrho_l - \varrho_{l+1}} \right)^{\frac{1}{q_l}} \left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} (u^+)^2 z \, dx \right)^{\frac{1}{2}}. \tag{17}$$

We can estimate the infinite product in (17) by using the geometric sum. Thus, we have

$$\sup_{B_{\alpha R}} u \leqslant C(n, \alpha, \beta, C_1) \left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} |u^+|^2 z \, dx \right)^{\frac{1}{2}}.$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *Under the hypotheses of Theorem* 3.3 *and $\alpha < \beta$, we have*

$$\frac{1}{\inf_{B_{\alpha R}} u} \leqslant \exp\left(C - \frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} \log u \, z \, dx \right). \tag{18}$$

Proof. W.l.o.g. we assume $u \geqslant \epsilon > 0$ (in case $u \geqslant 0$ we use Levi's Theorem to derive the assertion). Testing (10) with the function $\phi(x) = \eta(x) u^{-1}(x)$, $\eta \in C_c^1(\Omega)$, $\eta \geqslant 0$ yields the estimate

$$\int_\Omega a^{\alpha\beta}(x) D_\beta u D_\alpha \eta u^{-1} \, dx - \int_\Omega a^{\alpha\beta}(x) D_\beta u D_\alpha u \eta u^{-2} \, dx \geqslant 0.$$

Set $v := \log(\frac{t}{u})$, where t denotes a positive constant which will be specified later. We see that v is a weak subsolution of (9) and with Lemma 3.4 we infer

$$\sup_{B_{\alpha R}} v \leqslant C\left(\frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} |v^+|^2 z\,dx\right)^{\frac{1}{2}}. \tag{19}$$

To estimate the right-hand side of (19) we test (10) with $\phi(x) = \eta^2(x)u^{-1}(x)$, $\eta \in C_c^1(\Omega)$. With (2) and the Hölder inequality we arrive at

$$\int_\Omega \eta^2 u^{-2}|\nabla u|^2 w\,dx \leqslant C\int_\Omega \eta|\eta_x\|\nabla u|u^{-1}v\,dx \leqslant C\left(\int_\Omega \eta_x^2 z\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \eta^2|\nabla u|^2u^{-2}w\,dx\right)^{\frac{1}{2}}.$$

It follows $\int_\Omega \eta^2 u^{-2}|\nabla u|^2 w\,dx \leqslant C\int_\Omega \eta_x^2 z\,dx$.

Choose η in a way that $\eta \equiv 1$ in $B_{\beta R}$, supp $\eta \subset B_R$, $|\eta_x| \leqslant \frac{2}{(1-\beta)R}$. From the last inequality we conclude together with the doubling property and the fact $|\nabla v|^2 = u^{-2}|\nabla u|^2$ the estimate

$$\int_{B_{\beta R}} |\nabla v|^2 w\,dx \leqslant C\left(\frac{1}{R^2}\int_{B_{\beta R}} z\,dx\right).$$

We define t by means of $\log t = \frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} \log u z\,dx$, then the weighted mean value of v is zero and the Poincaré inequality in connection with the above inequality yields

$$\left(\frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} |v|^{2k} z\,dx\right)^{\frac{1}{2k}} \leqslant C\sqrt{\frac{z(B_{\beta R})}{w(B_{\beta R})}}\left(\frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} z\,dx\right)^{\frac{1}{2}} \leqslant C(n,\beta,C_1).$$

Combining this estimate with (19) we infer

$$\sup_{B_{\alpha R}} v = \log t + \log\left(\frac{1}{\inf_{B_{\alpha R}} u}\right) \leqslant C.$$

By considering the definition of t we finally arrive at

$$\left(\inf_{B_{\alpha R}} u\right)^{-1} \leqslant \exp\left(C - \frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} \log u z\,dx\right). \qquad \Box$$

Lemma 3.6. *Under the hypotheses of Theorem* 3.3 *and* $\alpha < \beta$*, we have*

$$\left(\frac{1}{z(B_{\alpha R})}\int_{B_{\alpha R}} |u|^\gamma z\,dx\right)^{\frac{1}{\gamma}} \leqslant \exp\left(C + \frac{1}{z(B_{\beta R})}\int_{B_{\beta R}} \log u z\,dx\right). \tag{20}$$

Proof. We may again assume $u \geqslant \epsilon > 0$. Set $f = v^- = \log(\frac{u}{t})^+$ (for the definition of t see the proof of Lemma 3.5) and test the weak formulation with $\phi(x) = \eta^2(x)u^{-1}(x)(f^\delta(x) + (2\delta)^\delta)$, where $\delta \geqslant 1$, $\eta \in C_c^1(B_R)$, $\eta \geqslant 0$. By using the ellipticity condition we conclude

$$\int_\Omega \eta^2 u^{-2}\bigl(f^\delta + (2\delta)^\delta - \delta f^{\delta-1}\bigr)|\nabla u|^2 w\,dx \leqslant C\int_\Omega \eta|\eta_x|u^{-1}\bigl(f^\delta + (2\delta)^\delta\bigr)|\nabla u|v\,dx.$$

Now we use the inequality $\delta f^{\delta-1} \leqslant \frac{1}{2}(f^\delta + (2\delta)^\delta)$ in connection with $|\nabla f|^2 = u^{-2}|\nabla u|^2$ and the Hölder inequality to infer

$$\int_\Omega \eta^2\bigl(f^\delta + (2\delta)^\delta\bigr)|\nabla f|^2 w\,dx \leqslant C\int_\Omega \eta_x^2\bigl(f^\delta + (2\delta)^\delta\bigr)z\,dx.$$

By using once again $\delta f^{\delta-1} \leqslant \frac{1}{2}(f^\delta + (2\delta)^\delta)$ and taking the elementary inequality $f^\delta + (2\delta)^\delta \leqslant 2(f^{\delta+1} + (2\delta)^\delta)$ into account we obtain

$$\delta \int_\Omega \eta^2 f^{\delta-1} |\nabla f|^2 w \, dx \leqslant C \int_\Omega \eta_x^2 \big(f^{\delta+1} + (2\delta)^\delta\big) z \, dx. \tag{21}$$

Let $q := \frac{\delta+1}{2} > 1$, by applying the Sobolev inequality to $\eta f^q \in \mathring{H}_2^1(B_R, v, w)$ we find under consideration of (21)

$$\begin{aligned}\left(\frac{1}{z(B_R)} \int_{B_R} |\eta f^q|^{2k} z \, dx\right)^{\frac{1}{2k}} &\leqslant CR\left(\frac{1}{w(B_R)} \int_{B_R} \big(\eta_x^2 f^{\delta+1} + \eta^2 (\delta+1)^2 f^{\delta-1} |\nabla f|^2\big) w \, dx\right)^{\frac{1}{2}} \\ &\leqslant C\sqrt{q} R \underbrace{\sqrt{\frac{z(B_R)}{w(B_R)}}}_{\leqslant \sqrt{C_1}} \left(\frac{1}{z(B_R)} \int_{B_R} (\eta_x f^q)^2 z \, dx + (2\delta)^\delta \sup_{B_R} |\eta_x|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Choose ϱ and σ in a way that $\alpha \leqslant \varrho < \sigma \leqslant \beta$ and η in a way that supp $\eta \subset B_{\sigma R}$, $\eta \equiv 1$ in $B_{\varrho R}$ and $|\eta_x| \leqslant \frac{2}{(\sigma - \varrho)R}$. With this choice of ϱ, σ and η, taking the qth root in the last estimate yields

$$\left(\frac{1}{z(B_{\varrho R})} \int_{B_{\varrho R}} f^{2qk} z \, dx\right)^{\frac{1}{2kq}} \leqslant C q^{\frac{1}{q}} (\sigma - \varrho)^{-\frac{1}{q}} \left[Cq + \left(\frac{1}{z(B_{\sigma R})} \int_{B_{\sigma R}} f^{2q} z \, dx\right)^{\frac{1}{2q}}\right]. \tag{22}$$

Now set $q_i = k^i \geqslant 1$, $\varrho_i = \alpha + 2^{-i}(\beta - \alpha)$, $\sigma_i = \varrho_i + 2^{-i}(\beta - \alpha)$, we obtain

$$\left(\frac{1}{z(B_{\varrho_i R})} \int_{B_{\varrho_i R}} f^{2k^{i+1}} z \, dx\right)^{\frac{1}{2k^{i+1}}} \leqslant (C 2^i k^i)^{\frac{1}{k^i}} \left[Ck^i + \left(\frac{1}{z(B_{\sigma_i R})} \int_{B_{\sigma_i R}} f^{2k^i} z \, dx\right)^{\frac{1}{2k^i}}\right].$$

In the next step we iterate this inequality; after $i - 1$ iteration steps we arrive at

$$\left(\frac{1}{z(B_{\varrho_i R})} \int_{B_{\varrho_i R}} f^{2k^{i+1}} z \, dx\right)^{\frac{1}{2k^{i+1}}} \leqslant \sum_{j=1}^{i} Ck^j \prod_{l=j}^{i} (Ck^l 2^l)^{\frac{1}{k^l}} + \prod_{j=1}^{i} (Ck^j 2^j)^{\frac{1}{k^j}} \left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} f^{2k} z \, dx\right)^{\frac{1}{2k}}. \tag{23}$$

We estimate the series and products in (23) and then we find with the doubling property and the Hölder inequality the following estimate for all $p > 2k$:

$$\left(\frac{1}{z(B_{\alpha R})} \int_{B_{\alpha R}} f^p z \, dx\right)^{\frac{1}{p}} \leqslant C\left[p + \left(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} f^{2k} z \, dx\right)^{\frac{1}{2k}}\right]. \tag{24}$$

By considering the power series expansion of $\mathrm{e}^{p_0 f}$ for $p_0 \in (0, \mathrm{e}^{-1})$ we infer by using (24) and the Stirling approximation $n! \approx \sqrt{2\pi n}(\frac{n}{\mathrm{e}})^n$ for large n the estimate

$$\frac{1}{z(B_{\alpha R})} \int_{B_{\alpha R}} \mathrm{e}^{p_0 f} z \, dx \leqslant C \mathrm{e}^{(\frac{1}{z(B_{\beta R})} \int_{B_{\beta R}} f^{2k} z \, dx)^{\frac{1}{2k}}}. \tag{25}$$

Since $f = v^-$ the right-hand side of (25) is bounded by the proof of Lemma 3.5. Thus

$$\left(\frac{1}{z(B_{\alpha R})} \int_{B_{\alpha R}} \mathrm{e}^{p_0 f} z \, dx\right)^{\frac{1}{p_0}} \leqslant C. \tag{26}$$

In the remainder of the proof we have to estimate $\|\frac{u}{t}\|_{L_\gamma(z, B_{\alpha R})}$ by $\|\frac{u}{t}\|_{L_{p_0}(z, B_{\alpha' R})}$ $(\alpha < \alpha' < \beta)$. For this, we remark that $-u$ is a subsolution of (9); by modifying the function $F(u)$ appearing in the proof of Lemma 3.4 in the sense that $\delta \in (-1, 0)$ we see that the estimate (16) holds also for $q \in (0, \frac{1}{2})$. For the iteration process we set $q_0 := \frac{\gamma}{2k}$,

$q_i := \frac{q_{i-1}}{k} \to 0, i \to \infty$. After finitely many iteration steps we achieve $2q_i < p_0 < \mathrm{e}^{-1}$. From (26) and (16) (for $q \in (0, 1/2)$) we infer with the definition of t

$$\left(\frac{1}{z(B_{\alpha R})} \int\limits_{B_{\alpha R}} u^\gamma z\,dx\right)^{\frac{1}{\gamma}} \leqslant \exp\left(C + \frac{1}{z(B_{\beta R})} \int\limits_{B_{\beta R}} \log u z\,dx\right). \qquad \square$$

Proof of Theorem 3.3. Multiply (18) and (20). □

4. Results for weak solutions of degenerate elliptic systems

Now we define what we will understand under a *weak solution* of a system of type (1):

Definition 4.1. $u \in H_2^1(\Omega, v, w, \mathbb{R}^m)$ is called a weak solution of (1), if

$$\int\limits_{\Omega} a^{\alpha\beta}(x) D_\beta u D_\alpha \phi\,dx = \int\limits_{\Omega} f(x, u, \nabla u)\phi\,dx \tag{27}$$

holds for all $\phi \in \mathring{H}_2^1(\Omega, v, w, \mathbb{R}^m)$.

For the proof of Theorem 1.1 we now can use an idea of L. Caffarelli [3]. In fact we only have to replace the weak Harnack inequality for weak supersolutions of uniformly elliptic equations by the weak Harnack inequality proven in Section 3 (Theorem 3.3).

Examples.

1) $v(x) = w(x) = |x|^\alpha$, $x \in B_R(0) \subset \mathbb{R}^n$ and $\alpha > -n$. If $\alpha \in (-n, n)$ it is easy to show that $v = w \in A_2$ and if $\alpha > -n + 2$ we can interpret $|x|^\alpha$ as a weight which arises from a quasiconformal mapping (cf. [7], pp. 105–112). This weight has also the properties which were needed in the proof of Theorem 3.3 (cf. [7]) and so it is an admissible weight for the system (1).
2) $v(x) = w(x) = (\log|x|)^k$, $x \in B_{1/2}(0) \subset \mathbb{R}^n$, $k \in 2\mathbb{N}$.
3) $v(x) = w(x) = |x|^\alpha(\log|x|)^2$, $x \in B_{1/2}(0) \subset \mathbb{R}^n$, $\alpha \in (-n, n)$.
4) $v(x) = |x|$, $w(x) = |x|^\tau$, $\tau \in (1, 2)$, $x \in B_1(0) \subset \mathbb{R}^n$, $n \geqslant 3$. It is obvious that $w \in A_2$, $z = |x|^{2-\tau} \in D_\infty$. In view of a result due to Chanillo and Wheeden [4] it is enough to show that the balance condition (8) holds. We remark that for $\alpha > 0$ and $a \in B_R(0)$ there are positive constants c_1 and c_2 with the property

$$c_1 R^n \big(R + |a|\big)^\alpha \leqslant \int\limits_{B_R(a)} |x|^\alpha\,dx \leqslant c_2 R^n \big(R + |a|\big)^\alpha. \tag{28}$$

From (28) we infer $\frac{z(B_R)}{w(B_R)} \leqslant C_1$; for $q \in (2, \frac{2n}{n+\tau-2}]$ we have for any $s \in (0, 1)$ the estimates

$$s\left[\frac{z(B_{sR}(a))}{z(B_R(a))}\right]^{\frac{1}{q}} \leqslant Css^{\frac{n}{q}} \quad \text{and} \quad \left[\frac{w(B_{sR}(a))}{w(B_R(a))}\right]^{\frac{1}{2}} \geqslant s^{\frac{n}{2}} s^{\frac{\tau}{2}}.$$

Since $ss^{\frac{n}{q}} \leqslant Cs^{\frac{n+\tau}{2}}$ the validity of (8) is shown.

Liouville theorem for entire solutions. Here, we assume the coefficients $a^{\alpha\beta}(x)$ satisfy the estimate

$$\frac{1}{C} s(x)|\xi|^2 \leqslant a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant Ct(x)|\xi|^2 \tag{29}$$

with $C \geqslant 1$ and

$$s(x) = \begin{cases} w(x), & |x| < L, \\ 1, & |x| \geqslant L, \end{cases} \qquad t(x) = \begin{cases} v(x), & |x| < L, \\ 1, & |x| \geqslant L, \end{cases}$$

where w and v are weights which satisfy the conditions of Section 2.

The proof of the Liouville Theorem uses an idea of Meier [11], who proved the corresponding Liouville theorem for uniformly elliptic systems. First we have to consider some lemmatas:

Lemma 4.2. *Let u be a bounded, weak solution of* (1) *in a domain $\Omega \subset \mathbb{R}^n$. If $a^* < 1$ and $\xi \in \mathbb{R}^m$ is a vector with $|\xi| \leqslant \frac{1-a^*}{a}$, then $-D_\alpha(a^{\alpha\beta}(x) D_\beta |u-\xi|^2) \leqslant 0$ in Ω.*

Proof. We use $\phi = \eta(u-\xi)$, $\eta \in C_c^\infty(\Omega)$, $\eta \geqslant 0$ as a test function in the weak formulation (27) and take the structure conditions of the introduction into account. □

With the notation $z_1(x) := \frac{t^2(x)}{s(x)}$ we can formulate the next lemmatas.

Lemma 4.3. *Let $B_{4L}(0) \subset \Omega$ and u be a bounded, weak, nonnegative supersolution of $D_\alpha(a^{\alpha\beta}(x) D_\beta u) = 0$ in $\Omega \subset \mathbb{R}^n$ with coefficients $a^{\alpha\beta}(x)$ of the form* (29)*, furthermore let $\frac{z_1(B_L)}{s(B_L)} \leqslant C_1$. Then we have for any $R > 0$ with $B_{4R}(0) \subset \Omega$ the estimate*

$$\frac{1}{z_1(B_{2R})} \int_{B_{2R}(0)} u z_1 \, dx \leqslant C(n, C_1) \inf_{B_R(0)} u.$$

Proof. If $B_{4R}(0) \subset B_L(0)$, the lemma is a direct consequence of Theorem 3.3 with $\gamma = 1$ and suitable α, β. If $B_L(0) \subset B_{4R}(0)$ we can prove similarly to [8], pp. 195–198 a Harnack inequality for supersolutions of $Lu = 0$ with uniformly elliptic coefficients on annular regions $B_{4S} - B_S$ $(S \geqslant L)$, i.e.

$$\frac{1}{z_1(B_{\beta_1 S} - B_{\beta_2 S})} \int_{B_{\beta_1 S} - B_{\beta_2 S}} u z_1 \, dx \leqslant C \inf_{B_{\alpha_1 S} - B_{\alpha_2 S}} u \tag{30}$$

with $1 < \beta_2 < \alpha_2 < \alpha_1 < \beta_1 < 4$.

The main difference in the proof of (30) compared with [8] is to construct suitable test functions on the corresponding annular regions.

Choose $\alpha, \alpha_1, \beta, \beta_1$ in a way that $1 < \alpha < \alpha_1 < 2$, $\alpha_1 < \beta_1 < \beta < 4$ and $B_{\alpha_1 L} \subset B_{\beta_1 R}$. We conclude

$$\begin{aligned}
\frac{1}{z_1(B_{\beta R})} \int_{B_{\beta R}} u z_1 \, dx &= \frac{1}{z_1(B_{\beta R} - B_{\alpha L}) + z_1(B_{\alpha L})} \left[\int_{B_{\beta R} - B_{\alpha L}} u z_1 \, dx + \int_{B_{\alpha L}} u z_1 \, dx \right] \\
&\leqslant \frac{1}{z_1(B_{\beta R} - B_{\alpha L})} \int_{B_{\beta R} - B_{\alpha L}} u z_1 \, dx + C \frac{1}{z_1(B_{2L})} \int_{B_{2L}} u z_1 \, dx \\
&\leqslant C \inf_{B_{\beta_1 R} - B_{\alpha_1 L}} u + C \inf_{B_{\alpha_1 L}} u \leqslant C \inf_{B_R} u.
\end{aligned}$$

Here, we used (30) and Theorem 3.3.

If $B_L(0) \subset B_{2R}(0)$ we choose $\beta = 2$, $\beta_1 = 3/2$, $\alpha_1 = 5/4$, $\alpha = 9/8$ to arrive at the assertion. If $B_L(0) \not\subset B_{2R}(0)$ we choose some $\beta \in (2, 4)$ with $B_L(0) \subset B_{\beta R}(0)$; the doubling property of z_1 yields the desired estimate. □

Lemma 4.4. *Let u be a weak solution of $-D_\alpha(a^{\alpha\beta} D_\beta u) \leqslant 0$ in $B_{4R}(0) \subset \mathbb{R}^n$ with coefficients of the form* (29)*. If $\frac{z_1(B_R)}{s(B_R)} \leqslant C_1$, then there is a constant $\delta(n, C_1) \in (0, 1)$ with the property*

$$\sup_{B_R(0)} u \leqslant (1-\delta) \sup_{B_{4R}(0)} u + \delta \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1 \, dx.$$

Proof. From Lemma 4.3 we infer for the nonnegative supersolution $\sup_{B_{4R}(0)} u - u$ the estimate

$$\frac{1}{z_1(B_{2R})} \int_{B_{2R}} \Big(\sup_{B_{4R}} u - u \Big) z_1 \, dx \leqslant C \inf_{B_R} \Big(\sup_{B_{4R}} u - u \Big).$$

With the help of the doubling property we can estimate the left-hand side from below by

$$\tilde{C}\frac{1}{z_1(B_R)}\int_{B_R}\Big(\sup_{B_{4R}} u - u\Big) z_1\,dx$$

and we infer

$$\frac{\tilde{C}}{C}\sup_{B_{4R}} u - \frac{\tilde{C}}{C}\frac{1}{z_1(B_R)}\int_{B_R} u z_1\,dx \leqslant \sup_{B_{4R}} u - \sup_{B_R} u. \qquad \square$$

Lemma 4.5. *Let u be a bounded, weak solution of* (1) *in $\mathbb{R}^n$ with coefficients $a^{\alpha\beta}(x)$ of the form* (29). *If $\frac{z_1(B_R)}{s(B_R)} \leqslant C_1$ for some $R \leqslant \frac{L}{2}$ and $a^* < 1$, then we have*

(i) $\lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} u(x) z_1\,dx =: \bar{u}_\infty$ *exists and* $|\bar{u}_\infty| = \sup_{\mathbb{R}^n} |u| = M$.
(ii) $\lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} |u - \bar{u}_\infty|^2 z_1\,dx = 0$.
(iii) $\sup_{\mathbb{R}^n} |u - \xi| = |\bar{u}_\infty - \xi|\ \forall \xi \in \mathbb{R}^m$ *with* $|\xi| \leqslant \frac{1-a^*}{a}$.

Proof. (i) In view of Lemma 4.2 we have $-D_\alpha(a^{\alpha\beta}(x) D_\beta |u-\xi|^2) \leqslant 0\ \forall \xi \in \mathbb{R}^m$ with $|\xi| \leqslant \frac{1-a^*}{a}$. From Lemma 4.4 we infer by letting $R \to \infty$ the estimate $\sup_{\mathbb{R}^n} |u-\xi|^2 \leqslant \lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} |u-\xi|^2 z_1\,dx$. It's obvious that the reverse inequality is also true. Thus,

$$\lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} |u-\xi|^2 z_1\,dx = \sup_{\mathbb{R}^n} |u-\xi|^2. \tag{31}$$

Since

$$\frac{1}{z_1(B_R)} \int_{B_R(0)} |u-\xi|^2 z_1\,dx = \frac{1}{z_1(B_R)} \int_{B_R(0)} |u|^2 z_1\,dx - 2\xi \cdot \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx + |\xi|^2$$

we see in view of (31) that $\lim_{R\to\infty} \xi \cdot \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx$ exists and we infer

$$\sup_{\mathbb{R}^n} |u-\xi|^2 = M^2 + |\xi|^2 - 2\xi \cdot \lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx. \tag{32}$$

Set $\tau := \frac{1-a^*}{aM}$ and choose $\bar{u}_\infty \in \mathbb{R}^m$ in a way that $|\bar{u}_\infty| = M$ and $\sup_{\mathbb{R}^n} |u + \tau\bar{u}_\infty| = (1+\tau)M$. With $\xi := -\tau\bar{u}_\infty$ we observe from (32)

$$M^2 = \lim_{R\to\infty} \bar{u}_\infty \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx.$$

Since $|\bar{u}_\infty|, |\frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx| \leqslant M$ we conclude assertion (i).

(ii) We have

$$\frac{1}{z_1(B_R)} \int_{B_R(0)} |u-\bar{u}_\infty|^2 z_1\,dx = |\bar{u}_\infty|^2 + \frac{1}{z_1(B_R)} \int_{B_R(0)} |u|^2 z_1\,dx - 2\bar{u}_\infty \cdot \frac{1}{z_1(B_R)} \int_{B_R(0)} u z_1\,dx.$$

By letting $R \to \infty$ we infer from the proof of (i)

$$\lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R} |u-\bar{u}_\infty|^2 z_1\,dx = M^2 + M^2 - 2M^2 = 0.$$

(iii) (32) and (i) yield for every $\xi \in \mathbb{R}^m$ with $|\xi| \leqslant \frac{1-a^*}{a}$ the equation

$$\sup_{\mathbb{R}^n} |u-\xi|^2 = |\bar{u}_\infty|^2 + |\xi|^2 - 2\xi \cdot \bar{u}_\infty = |\bar{u}_\infty - \xi|^2. \qquad \square$$

Now we can start with the proof of Theorem 1.2:

Proof of Theorem 1.2. Define for $t \in [0,1]$ the function $u_t := u - t\bar{u}_\infty$ with $\bar{u}_\infty = \lim_{R\to\infty} \frac{1}{z_1(B_R)} \int_{B_R(0)} uz_1\,dx$. Furthermore, set $M_t := \sup_{\mathbb{R}^n} |u_t|$ (note: M_t depends continuously on t) and let $I := \{t \in [0,1];\ M_t \leqslant (1-t)M_0\}$. We denote by T the biggest number in I and we assume $T < 1$.

$u_T = u - T\bar{u}_\infty$ is a weak solution of a system of type (1) with $|f| \leqslant aQ(x,\nabla u)$ and $(u - T\bar{u}_\infty)\cdot f \leqslant (a^* + aT|\bar{u}_\infty|)Q(x,\nabla u)$. Since $\bar{u}_\infty$ has been chosen in the direction of u, we infer with $a_T^* := a^* + aT|\bar{u}_\infty|$ the estimates

$$a_T^* + a\sup_{\mathbb{R}^n} |u - T\bar{u}_\infty| < 2 \quad \text{and} \quad a_T^* < 1.$$

Define $t := \min(1, T + \frac{1-a_T^*}{aM})$; with this t we have $T < t \leqslant 1$ and $|(t-T)\bar{u}_\infty| \leqslant \frac{1-a_T^*}{a}$. With $\xi := (t-T)\bar{u}_\infty$ we conclude from Lemma 4.5(iii) $\sup_{\mathbb{R}^n} |u - t\bar{u}_\infty| = \sup_{\mathbb{R}^n} |u_T - \xi| = (1-t)|\bar{u}_\infty|$ and therefore $M_t \leqslant (1-t)M_0$. This means $t \in I$, but since $T < t$ this is a contradiction to our assumption that T is the biggest number in I. We infer $T = 1$ and the proof is complete. □

Examples.

1) Let $L > 0$ and $\tau \in (-n, \infty)$. Choose
$$a^{\alpha\beta}(x) = \begin{cases} |x|^\tau \delta_{\alpha\beta}, & |x| < L, \\ \delta_{\alpha\beta}, & |x| \geqslant L. \end{cases}$$
With the same argument as above we see that these coefficients are admissible.
2) Let $k \in 2\mathbb{N}$ and
$$a^{\alpha\beta}(x) = \begin{cases} \log(|x|)^k \delta_{\alpha\beta}, & |x| < \dfrac{1}{2}, \\ \delta_{\alpha\beta}, & |x| \geqslant \dfrac{1}{2}. \end{cases}$$
3) Let $\tau \in (-n, n)$ and
$$a^{\alpha\beta}(x) = \begin{cases} |x|^\tau \log(|x|)^2 \delta_{\alpha\beta}, & |x| < \dfrac{1}{2}, \\ \delta_{\alpha\beta}, & |x| \geqslant \dfrac{1}{2}. \end{cases}$$
4) Let $\tau \in (1,2)$ and choose coefficients $a^{\alpha\beta}(x)$ with
$$s(x)|\xi|^2 \leqslant a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant t(x)|\xi|^2,$$
where
$$s(x) = \begin{cases} |x|^\tau, & |x| < 1 \\ 1, & |x| \geqslant 1 \end{cases} \quad \text{and} \quad t(x) = \begin{cases} |x|, & |x| < 1, \\ 1, & |x| \geqslant 1. \end{cases}$$

By using the same methods as above, it is easy to see that these weights satisfy (8) and $\frac{z_1(B_R)}{s(B_R)} \leqslant C_1$ for all balls $B_R(a) \subset B_1(0)$.

References

[1] A. Baldes, Degenerate elliptic operators, diagonal systems and variational integrals, Manuscripta Math. 55 (1986) 467–486.
[2] C. Baoyao, Regularity of weak solutions for a class of degenerate elliptic systems in diagonal form, in: Proc. Asian Math. Conference 1990, World Scientific, Singapore, New Jersey, London, Hongkong, 1992, pp. 51–54.
[3] L.A. Caffarelli, Regularity theorems for weak solutions of some nonlinear systems, Comm. Pure Appl. Math 35 (1982) 833–838.
[4] S. Chanillo, R.L. Wheeden, Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions, Amer. J. Math. 107 (1985) 1191–1226.
[5] S. Chanillo, R.L. Wheeden, Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations, Comm. Partial Differential Equations 11 (1986) 1111–1134.

[6] R.R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974) 241–250.
[7] E. Fabes, C. Kenig, R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7 (1982) 77–116.
[8] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss., vol. 224, Springer, Berlin, Heidelberg, New York, 1998.
[9] S. Hildebrandt, K.O. Widman, On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order, Ann. Sc. Norm. Super. Pisa IV (1977) 145–178.
[10] S. Hildebrandt, K.O. Widman, Sätze vom Liouvilleschen Typ für quasilineare elliptische Gleichungen und Systeme, Nachr. Akad. Wiss. Göttingen II. Math.-Phys. Klasse 4 (1979) 41–59.
[11] M. Meier, On quasilinear elliptic systems with quadratic growth, preprint 1984.
[12] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972) 207–226.
[13] B. Muckenhoupt, The equivalence of two conditions for weight functions, Studia Math. 49 (1974) 101–106.
[14] B. Muckenhoupt, R.L. Wheeden, Weighted bounded mean oscillation and the Hilbert transform, Studia Math. 54 (1976) 221–237.
[15] M. Pingen, Zur Regularitätstheorie elliptischer Systeme und harmonischer Abbildungen, Thesis, University of Duisburg-Essen, 2006.
[16] N.S. Trudinger, On the regularity of generalized solutions of linear non-uniformly elliptic equations, Arch. Rat. Mech. Anal. 42 (1971) 51–62.
[17] M. Wiegner, Ein optimaler Regularitätssatz für schwache Lösungen gewisser elliptischer Systeme, Math. Z. 147 (1976) 21–28.
[18] M. Wiegner, A-priori Schranken für Lösungen gewisser elliptischer Systems, Manuscripta Math. 18 (1976) 279–297.