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Global convergence toward traveling fronts in nonlinear parabolic systems with a gradient structure

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Abstract

We consider nonlinear parabolic systems of the form $u_t = -\nabla V(u) + u_{xx}$, where $u \in \mathbb{R}^n$, $n \geqslant 1$, $x \in \mathbb{R}$, and the potential V is coercive at infinity. For such systems, we prove a result of global convergence toward bistable fronts which states that invasion of a stable homogeneous equilibrium (a local minimum of the potential) necessarily occurs via a traveling front connecting to another (lower) equilibrium. This provides, for instance, a generalization of the global convergence result obtained by Fife and McLeod [P. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to traveling front solutions, Arch. Rat. Mech. Anal. 65 (1977) 335–361] in the case $n = 1$. The proof is based purely on energy methods, it does not make use of comparison principles, which do not hold any more when $n > 1$.

Keywords: Nonlinear parabolic system; Gradient structure; Global convergence; Bistable traveling front; Energy functional; Maximum principle

1. Introduction

We consider a nonlinear parabolic system of the form

$$u_t = -\nabla V(u) + u_{xx}, \tag{1}$$

where the space variable x belongs to $\mathbb{R}$ and $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, $n \geqslant 1$. Our main assumption is that the nonlinearity in (1) is the gradient of a scalar function $V : \mathbb{R}^n \to \mathbb{R}$, which will be referred to as the *potential*. We assume that V is of class $\mathcal{C}^k$, $k \geqslant 3$, and that $V(u) \to +\infty$ when $\|u\| \to +\infty$.

The aim of this paper is to provide some insight into the global dynamics of this class of systems. Our approach is based on the (formal) gradient structure of system (1). If $u(x,t)$ is a solution of (1), let us consider the *energy functional*

$$E\big[u(.,t)\big] = \int_{\mathbb{R}} \left(\frac{u_x(x,t)^2}{2} + V\big(u(x,t)\big)\right) \mathrm{d}x \tag{2}$$

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where $u_x^2 = (u_1)_x^2 + \cdots + (u_n)_x^2$. Then, at least formally,

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big[u(.,t)\big] = -\int_{\mathbb{R}} u_t(x,t)^2\,\mathrm{d}x \leqslant 0 \tag{3}$$

(and the system (1) can formally be rewritten in the form: $u_t(.,t) = -\frac{\delta}{\delta u} E[u(.,t)]$).

If the system (1) was considered on a bounded domain, then the integrals in (2) and (3) would converge, and the decrease of the energy would enable to show that any solution converges toward the set of stationary solutions. But since we consider an unbounded domain and since we are interested in global perturbations of traveling wave solutions, we cannot restrict ourselves to solutions of finite energy, therefore we will have to deal with the formal character of the gradient structure.

The key point, on which the whole proof relies, is that this formal gradient structure exists not only in the laboratory frame, but also in any frame traveling at a constant velocity [7]. Indeed, for any $c \in \mathbb{R}$, if we let $x = ct + y$ and $v(y,t) = u(ct + y, t)$, then system (1) becomes

$$v_t - c v_y = -\nabla V(v) + v_{yy}, \tag{4}$$

and if we consider the energy

$$E_c\big[v(.,t)\big] = \int_{\mathbb{R}} e^{cy}\left(\frac{v_y(y,t)^2}{2} + V\big(v(y,t)\big)\right)\mathrm{d}y, \tag{5}$$

then, at least formally,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_c\big[v(.,t)\big] = -\int_{\mathbb{R}} e^{cy} v_t(y,t)^2\,\mathrm{d}y \tag{6}$$

(and the system (4) can formally be rewritten in the form: $v_t(.,t) = -e^{-cy}\frac{\delta}{\delta v} E_c[v(.,t)]$).

This formal gradient structure indicates that, in any frame traveling with constant velocity c, any solution should (in a certain sense) converge toward solutions which are stationary in this frame, or equivalently toward waves traveling at velocity c in the laboratory frame. More precisely, one should find convergence toward traveling fronts (i.e. traveling waves connecting homogeneous equilibria), since those are essentially the only bounded waves traveling at nonzero velocity (see Proposition 3 below). In short, this formal gradient structure seems to be sufficient to yield by itself global convergence toward traveling fronts. Nevertheless, it seems that none of the known results concerning global convergence toward traveling fronts for systems of the form (1) was ever obtained as a consequence of this formal gradient structure only.

Global stability of traveling fronts of dissipative systems is an old question that gave rise to a considerable amount of work and results (see [29] and references therein), but mostly devoted to systems satisfying a comparison principle (also called maximum principle). This hypothesis states that a certain order among solutions is preserved by the semi-flow, a constraint that can be strong enough to yield global stability results.

Two cases can be naturally distinguished, called *monostable* or *bistable*, depending on the stability of the equilibrium which is “invaded” by the front (the equilibrium behind the front is usually stable). In the monostable case, global convergence results go back to the seminal work of Kolmogorov, Petrovskii, and Piskunov [19], and in the bistable case, they go back to the work of Fife and McLeod in the late seventies [7]. Both are concerned with parabolic equations of the form (1) when the variable u is scalar. In this case the equation admits the formal gradient structure recalled above and also satisfies a comparison principle (which does not hold any more when u is higher-dimensional). Whereas in [19] the proof does not make use of the gradient structure, in [7] both ingredients, the formal gradient structure and the maximum principle are used. Those results gave rise to numerous generalizations to more general cases [29,27,4,23,24,2] where in general there exists no gradient structure but still a comparison principle.

In the present paper we shall consider in some sense the opposite case, that is the case of systems with arbitrary number of components, for which in general no maximum principle is available, but we choose the nonlinearity in such a way that a gradient structure exists. Our main purpose will be to prove a result of global convergence toward traveling fronts invading a stable equilibrium (bistable case). In the absence of comparison principle we shall use only energy arguments derived from the formal gradient structure recalled above. In particular, this will provide a proof of

Fife and McLeod's result which does not make use of any comparison principle. Unfortunately we were not able to treat the monostable case by similar techniques (see [22] for another attempt).

Notations and preliminary results

Let us denote by X the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R},\mathbb{R}^n)$ (its definition is recalled in Section 2 below). We are going to study the semi-flow of the parabolic system (1) in this space, which is the most appropriate for our approach. However, due to the smoothing properties of the system, the choice of the functional framework is not crucial, and other spaces could be used as well. For instance, all statements in this introduction remain true if we suppose that X denotes the more familiar Banach space $\mathcal{C}^1_{\mathrm{b}}(\mathbb{R},\mathbb{R}^n)$ of functions in $\mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ which are uniformly bounded, together with their first derivative.

We assume that the potential function V is strictly coercive at infinity in the following sense:

(H1) there exist constants $\varepsilon_V >$ and $C_V > 0$ such that, for any $u\in\mathbb{R}^n$, we have $u\cdot\nabla V(u)\geqslant \varepsilon_V u^2 - C_V$.

System (1) defines a local semi-flow on X, and due to hypothesis (H1) this semi-flow is actually global (see Section 2 and Lemma 6 in Appendix A). Let us denote by $(S_t)_{t\geqslant 0}$ this semi-flow (in other words $u(x,t)=(S_t u_0)(x)$ denotes the solution of (1) with initial data $u(x,0)=u_0(x)$).

We are interested in the long time behavior of solutions which are close, for x large positive, to a stable homogeneous equilibrium (a local minimum of V) which, without loss of generality, we assume to be at 0:

(H2) $V(0)=0$, $\nabla V(0)=0$, and $D^2V(0)>0$ (the Hessian of V at 0 is positive definite).

The class of solutions will shall consider is the following (invariant) subset of X:

$$\mathcal{A}=\Big\{u_0\in X \mid \limsup_{t\to+\infty}\limsup_{x\to+\infty}\big|(S_t u_0)(x)\big|=0\Big\}.$$

This class is of course nonempty (it contains the homogeneous stationary solution $u\equiv 0$), actually it contains any initial data that are sufficiently close to 0 for x large positive, as is shown by the following proposition.

Proposition 1. *Assume that V satisfies* (H1) *and* (H2)*. Then there exists $\delta>0$ such that any $u_0\in X$ satisfying*

$$\limsup_{x\to+\infty}\int_x^{x+1}\big(u_0(x)^2+u_0'(x)^2\big)\,\mathrm{d}x\leqslant\delta$$

belongs to $\mathcal{A}$.

It follows in particular from this proposition that $\mathcal{A}$ is open in X (see Corollary 2 in Section 3). Let us consider the following (invariant) subset of $\mathcal{A}$:

$$\mathcal{A}_{\mathrm{inv}}=\Big\{u_0\in\mathcal{A}\mid \text{there exists } \varepsilon>0 \text{ such that } \limsup_{t\to+\infty}\sup_{x\geqslant\varepsilon t}\big|(S_t u_0)(x)\big|>0\Big\}.$$

Roughly speaking, the bulk of a solution belonging to $\mathcal{A}_{\mathrm{inv}}$ travels to the right at a nonzero mean velocity, and therefore "invades" the domain, far to the right in space, where this solution is very close to 0. For instance traveling waves, connecting to 0 at $+\infty$ and to any other equilibrium at $-\infty$ and with positive velocity, belong to this class.

The following proposition (due to Thierry Gallay) provides a sufficient condition in order initial data to belong to $\mathcal{A}_{\mathrm{inv}}$.

Proposition 2. *Assume that V satisfies* (H1) *and* (H2)*. Then any $u_0\in\mathcal{A}$ such that*

$$\int_{-L}^{0}\bigg(\frac{u_0'(x)^2}{2}+V\big(u_0(x)\big)\bigg)\,\mathrm{d}x\to-\infty\quad\textit{when } L\to+\infty$$

belongs to $\mathcal{A}_{\mathrm{inv}}$.

To state our results, the following notations and preliminary result on traveling waves will be required.

Let $\lambda_{\min}$ (resp. $\lambda_{\max}$) denote the smallest (resp. the largest) of the eigenvalues of the Hessian $D^2V(0)$. We have $0 < \lambda_{\min} \leqslant \lambda_{\max}$. For the remaining of this paper, we choose and fix $r_0 > 0$ sufficiently small so that, for any $v \in \mathbb{R}^n$ satisfying $|v| \leqslant r_0$, any eigenvalue λ of $D^2V(v)$ satisfies

$$\frac{\lambda_{\min}}{2} \leqslant \lambda \leqslant 2\lambda_{\max}. \tag{7}$$

Take any $c > 0$. A function $(x,t) \mapsto \phi(x - ct)$ (i.e. a wave traveling at velocity c) is a solution of the system (1) if and only if $\phi(\cdot)$ is a solution of the differential system

$$\phi''(y) = -c\phi'(y) + \nabla V\big(\phi(y)\big), \quad y \in \mathbb{R} \tag{8}$$

(equation of motion of a particle of unit mass moving in potential $-V(\cdot)$ with viscous damping c).

For any $\nu \in \mathbb{R}^n \times \mathbb{R}^n$, let $y \mapsto \phi_{c,\nu}(y)$ denote the maximal solution of the differential system (8) with initial data $(\phi_{c,\nu}(0), \phi'_{c,\nu}(0)) = \nu$. Let $S(r_0) = \{v \in \mathbb{R}^n \mid |v| = r_0\}$, and let

$$\mathcal{W}_c = \Big\{\nu \in S(r_0) \times \mathbb{R}^n \mid \phi_{c,\nu}(\cdot) \text{ is defined up to } +\infty \text{ and } \sup_{y \geqslant 0}\big|\phi_{c,\nu}(y)\big| \leqslant r_0\Big\},$$

$$\mathcal{W}_c^{\mathrm{b}} = \Big\{\nu \in \mathcal{W}_c \mid \phi_{c,\nu}(\cdot) \text{ is defined on } \mathbb{R} \text{ and } \sup_{y \in \mathbb{R}}\big|\phi_{c,\nu}(y)\big| < +\infty\Big\}.$$

The set $\mathcal{W}_c$ thus provides us with a parametrization of the trajectories on the stable manifold of 0 for the differential system (8), for the value c of the velocity. The next proposition states some properties of the sets $\mathcal{W}_c$ and $\mathcal{W}_c^{\mathrm{b}}$, in particular it shows that $\mathcal{W}_c$ can be parametrized by $S(r_0)$ and that $\mathcal{W}_c^{\mathrm{b}}$ provides us with a parametrization of the fronts traveling at velocity c and connecting to 0 at $+\infty$.

Proposition 3. *Take $c > 0$.*

(1) *For any $\nu \in \mathcal{W}_c$, we have $\phi_{c,\nu}(x) \to 0$ and $\phi'_{c,\nu}(x) \to 0$ when $x \to +\infty$.*
(2) *The set $\mathcal{W}_c$ is the graph of a $\mathcal{C}^1$-map: $S(r_0) \to \mathbb{R}^n$ (and consequently a $(n-1)$-dimensional compact $\mathcal{C}^1$-submanifold of $\mathbb{R}^{2n}$).*
(3) *The set $\mathcal{W}_c^{\mathrm{b}}$ is a compact subset of $\mathbb{R}^{2n}$ (possibly empty), and, for any $\nu \in \mathcal{W}_c^{\mathrm{b}}$, there exists $h \in\,]-\infty; 0[$ such that the set*

$$\Sigma_{\mathrm{crit},h} = \big\{u \in \mathbb{R}^n \mid V(u) = h \text{ and } \nabla V(u) = 0\big\}$$

is nonempty and such that

$$\operatorname{dist}\big(\phi_{c,\nu}(x), \Sigma_{\mathrm{crit},h}\big) \to 0 \quad \text{and} \quad \phi'_{c,\nu}(x) \to 0 \quad \text{when } x \to -\infty.$$

Main results

Using the above notations, our main result is the following (Fig. 1).

Theorem 1. *Assume that V satisfies* (H1) *and* (H2). *Then, for any $u_0 \in \mathcal{A}_{\mathrm{inv}}$, there exists $c > 0$ such that $\mathcal{W}_c^{\mathrm{b}} \neq \emptyset$, and there exists a $\mathcal{C}^1$-function $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \bar{x}(t)$ and a $\mathcal{C}^1$-map $\mathbb{R}_+ \to \mathcal{W}_c$, $t \mapsto \nu(t)$ such that the following statements hold*:

(i) $\bar{x}'(t) \to c$, $\nu'(t) \to 0$, *and* $\operatorname{dist}(\nu(t), \mathcal{W}_c^{\mathrm{b}}) \to 0$ *when $t \to +\infty$,*
(ii) *for any $L > 0$,*

$$\sup_{y \in [-L; +\infty[} \big|(S_t u_0)\big(\bar{x}(t) + y\big) - \phi_{c,\nu(t)}(y)\big| \to 0 \quad \text{when } t \to +\infty.$$

Remarks. (a) This result means that, if the domain, far to the right in space, where the solution is close to 0 is "invaded" at a nonzero mean velocity (i.e. if initial data belong to $\mathcal{A}_{\mathrm{inv}}$), then this "invasion" necessarily occurs at a constant asymptotic velocity c, and the solution around the interface is close, for t large, to a front traveling at

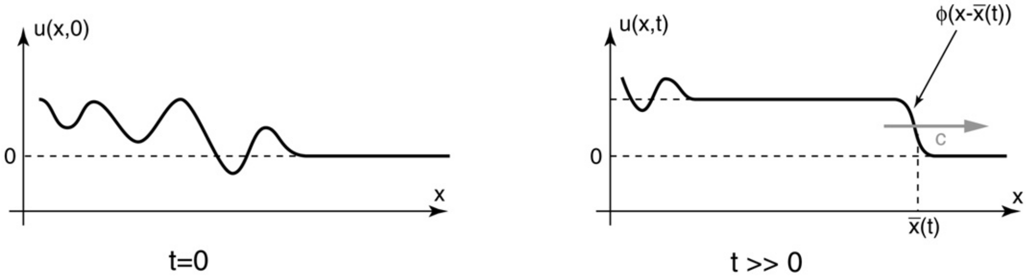

Fig. 1. Illustration of the main result.

velocity c. However, our assumptions do not imply convergence toward a single front. With the notations of the theorem, let us denote by $\mathcal{L}(u_0)$ the ω-limit set of the function $t \mapsto \nu(t)$, i.e.

$$\mathcal{L}(u_0) = \bigcap_{t>0} \overline{\nu\big([t;+\infty[\big)}.$$

This set is a nonempty compact connected subset of $\mathbb{R}^{2n}$, it is included in $\mathcal{W}_c^{\mathrm{b}}$, and it depends only on u_0, not on a particular choice of the function $\nu(\cdot)$. If $\mathcal{L}(u_0)$ is reduced to a singleton $\{\nu_0\}$, then, using the notations of Theorem 1, the following more precise conclusion holds: for any $L > 0$,

$$\sup_{y\in[-L;+\infty[} \big|(S_t u_0)\big(\bar{x}(t) + y\big) - \phi_{c,\nu_0}(y)\big| \to 0 \quad \text{when } t \to +\infty$$

(in other words the solution converges, around the interface, toward the profile ϕ_{c,ν_0} of a single traveling front).

Transversality arguments (see Section A.4 in Appendix A) show that the following property holds generically (i.e. for V in a G_δ-dense subset of the set of functions of class $\mathcal{C}^k$ satisfying (H1) and (H2)):

(G) "for any $c > 0$, the set $\mathcal{W}_c^{\mathrm{b}}$ is either empty or totally disconnected (i.e. its connected components are reduced to singletons)".

If property (G) holds, then, for any $u_0 \in \mathcal{A}_{\mathrm{inv}}$, the set $\mathcal{L}(u_0)$ is necessarily reduced to a singleton. It would be interesting to construct examples where $\mathcal{L}(u_0)$ is not reduced to a singleton, and on the other hand to provide sufficient conditions under which $\mathcal{L}(u_0)$ must be reduced to a singleton ("V(.) analytic" might be such a sufficient condition, see [16]).

(b) This result, together with Propositions 1 and 2, furnishes the essential step in order to generalize to systems the result of global convergence toward a traveling front in a bistable potential proved by Fife and MacLeod [7] in the scalar case $\dim_u = 1$. We will indicate (Section A.6 in Appendix A), how, in the particular case of a bistable potential, the conclusions of Theorem 1 can be strengthened to obtain global convergence toward traveling fronts uniformly on $\mathbb{R}$, and not only on semi-infinite intervals of the form $[-L;+\infty[$ as in Theorem 1. Another purely variational proof of Fife and McLeod's result (in the simplest possible case) can be found in [11].

We refer to [26] (see also [25]) for a more complete result, describing the asymptotic behavior of all *bistable* solutions (i.e. solutions which are close to local minima of the potential both at $-\infty$ and at $+\infty$ in space), and generalizing the other global convergence results proved by Fife and McLeod, namely the result of global convergence toward a cascade of traveling fronts [8] and the results about the global behavior in a bistable potential [6].

(c) As a direct consequence of Theorem 1 we have the following corollary.

Corollary 1. *If* 0 *is not a global minimum of* V (*namely if* $\min_{u\in\mathbb{R}^n} V(u) < V(0)$) *then there exists* $c > 0$ *such that* $\mathcal{W}_c^{\mathrm{b}}$ *is nonempty.*

In other words there exists a nonconstant bounded solution $y \mapsto \phi(y)$ to the differential system (8) satisfying $\phi(y) \to 0$ when $y \to +\infty$ (i.e. a wave traveling at a positive velocity and connecting to 0 at $+\infty$). If moreover we make the generic hypothesis that the set of critical points of V is finite, then according to Proposition 3 above $\phi(y)$ necessarily converges toward one of these critical points when $y \to -\infty$ (heteroclinic connection). This result has to be compared to classical results of existence of homoclinic or heteroclinic connections that can be obtained by calculus of variation techniques (see for instance [1]); the novelty is that the equilibria that are connected here do not belong to the same level set of V. Similar results of existence of traveling waves have recently been obtained by variational methods (S. Heinze, [14]). And of course, many other results of existence of multidimensional traveling waves have been obtained in other contexts by other methods, see for instance [3,30,28].

(d) For $u_0 \in \mathcal{A}_{\text{inv}}$, let us denote by $c[u_0]$ the velocity c defined by Theorem 1, and, for $u_0 \in \mathcal{A} \setminus \mathcal{A}_{\text{inv}}$, let us write $c[u_0] = 0$. The techniques developed thereafter in the proof of Theorem 1 will provide us with all required tools in order to establish the following result (compare to the approach developed by C.B. Muratov in [22]).

Theorem 2. *The function $c : \mathcal{A} \to \mathbb{R}_+$, $u_0 \mapsto c[u_0]$ is lower semi-continuous (for the usual topology on $\mathbb{R}_+$ and the topology induced by $\|\ldots\|_{H^1_{\text{ul}}(\mathbb{R})}$ on $\mathcal{A}$).*

As a consequence, for any $c_0 \geqslant 0$, the set $\{u_0 \in \mathcal{A} \mid c[u_0] > c_0\}$ is open (in $\mathcal{A}$, or equivalently in $H^1_{\text{ul}}(\mathbb{R})$). In particular $\mathcal{A}_{\text{inv}}$ is open.

Now take a traveling front $\phi_{c,\nu}$, $\nu \in \mathcal{W}^{\text{b}}_c$, $c > 0$, and assume that it is unstable, more precisely that there exists a globally defined solution $(x,t) \mapsto u(x,t)$, $(x,t) \in \mathbb{R}^2$, of Eq. (1), satisfying

$$\sup_{x\in\mathbb{R}} \bigl|u(x,t) - \phi_{c,\nu}(x - x_0 - ct)\bigr| \to 0 \quad \text{when} \quad t \to -\infty \text{ for some } x_0 \in \mathbb{R},$$

and which differs from a translate of the front (namely such that $(x,t) \mapsto u(x,t)$ and $(x,t) \mapsto \phi_{c,\nu}(x - x_0 - ct)$ are not equal). Observe that, if $\phi_{c,\nu}$ is linearly unstable for the semi-flow of equation (1), then any solution in its unstable manifold satisfies these conditions. Then, if we write $u_0(x) = u(x,0)$, $x \in \mathbb{R}$, we have $u_0 \in \mathcal{A}$, and according to Theorem 2, we must have $c[u_0] \geqslant c$.

In such a case, we actually expect that $c[u_0] > c$ (indeed, the energy $E_c[\phi_{c,\nu}]$ equals zero, see [11], therefore $E_c[u_0]$ should be negative). If this could be proved, this would show the existence of at least one front traveling at the velocity $c[u_0]$ (thus different from $\phi_{c,\nu}$). As a consequence, this would yield that among the traveling fronts invading 0, the "fastest" ones cannot be unstable in the sense above (and probably that they must be – locally – stable). This would be in high contrast with respect to the "KPP" monostable case (where the "slowest" traveling front is in a certain sense the only stable one, [19]).

(e) A natural generalization concerns the hyperbolic system

$$\alpha u_{tt} + u_t + \nabla V(u) = u_{xx}, \quad \alpha > 0, \tag{9}$$

obtained by adding some inertia to the initial parabolic system. In this case there still exists a (formal) decreasing energy, but no comparison principle holds as soon as α is above a certain value, even when $\dim_u = 1$. This case was recently studied by Thierry Gallay and Romain Joly, and similar results of global convergence toward bistable fronts were obtained [9,10]. Despite formal similarity, this case presents significant differences (no regularization, finite speed of propagation) with respect to the "parabolic case" considered here.

On the other hand, we do not know if our results can be extended to systems of the form

$$u_t = -\nabla V(u) + D u_{xx},$$

where D is a positive definite symmetric matrix, since in this case we could not find gradient structures in frames traveling at nonzero velocity.

1.1. Idea of the proof and organization of the paper

As sketched at the beginning of the introduction, the formal scheme of the proof is quite simple: in any frame traveling at a constant velocity $c \in \mathbb{R}$, the formal gradient structure (6) indicates that any solution should converge toward solutions which are stationary in this frame, or equivalently toward waves traveling at velocity c in the laboratory frame, and this is roughly speaking what we want to prove. Nevertheless, in order to transform this scheme into a proof, we have to deal with the following two issues.

First, in contrast to the scalar case treated by Fife and McLeod [7], the velocity is not a priori known. There might indeed exist several fronts, traveling with different velocities, and invading the same homogeneous equilibrium, and our hypotheses do not tell us a priori to which of these fronts our solution will converge. On the other hand, Theorem 1 shows a posteriori that the asymptotic velocity at which invasion occurs is unique; thus if we place ourselves in a frame traveling at another velocity, there is no hope to get from the formal relaxation scheme sketched above more than convergence toward "trivial" traveling waves, that is spatially homogeneous equilibria. Our strategy will be to adapt the velocity of the moving frame in order to keep track of the position of the "interface" (the domain of space where the solution escapes from the invaded equilibrium), and therefore to obtain at end convergence toward "nontrivial" traveling waves.

Second, in order the energy integral (5) to converge, we have to replace the weight e^{cy} by a function belonging to $L^1(\mathbb{R})$. This induces new terms in the right-hand side of (6). These new terms correspond to "fluxes of energy" through the domains of space where the weight differs from e^{cy}. The sign of these terms is arbitrary, and because of them the energy functional is not always decreasing. In order to recover some decrease, it is thus necessary to control these terms. To be more precise, assume for instance that $c > 0$. In this case we can choose the weight function as equal to e^{cy} at the left of some point y_0; the "fluxes of energy" are thus zero on $]-\infty; y_0]$, but we cannot avoid nonzero fluxes somewhere between y_0 and $+\infty$. In other words the issue actually consists in controlling fluxes of energy "far to the right" in space.

In the scalar case $n = 1$, the comparison principle provides a powerful tool in order to deal with both issues, as was proved by Fife and McLeod. Their method consists in constructing appropriate sub- and super-solutions, converging toward translates of the front (which, in their case, is unique). This trivially solves the first issue mentioned above (keep track of the interface), but this also solves the second issue, since the bounds provided by the sub- and the super-solution give a sufficiently nice control of the solution "far to the right", and therefore of the above mentioned fluxes of energy.

In the vector case $n \geqslant 2$, systems of the form $u_t = F(u) + u_{xx}$, $F:\mathbb{R}^n \to \mathbb{R}^n$, satisfy a comparison principle if the following conditions hold:

$$\partial F_i/\partial u_j \geqslant 0, \quad 1 \leqslant i, j \leqslant n,\ i \neq j \tag{10}$$

(see [29]). For functions F of the form $F = -\nabla V$, these conditions read: $\partial^2 V/\partial u_i \partial u_j \leqslant 0$, $i \neq j$, and are thus clearly not satisfied in general.

As a consequence, no comparison principle exists in general for systems of the form (1) (when $n \geqslant 2$). Nevertheless, we are going to show that, for such systems, both issues mentioned above can be tackled by "purely energetic" methods.

Our starting point will be to introduce a quantity $\tilde{x}(t)$, called the *invasion point*, which is defined roughly speaking as the first point starting from the right end of space where the solution reaches a certain distance from 0 (its definition is roughly similar to that of $\bar{x}(t)$). This invasion point is used in the sequel in order to materialize the position of the interface we want to keep track of. Then, our purpose throughout the proof is to win on two counts – the control of this invasion point, and the control of the fluxes of energy far to the right in space – any progress on one of these counts providing us with the opportunity to win on the other.

An introduction to this proof can be found in the shorter paper [11], where the same variational scheme is applied in the simplest possible case, namely the scalar case considered by Fife and McLeod with additional assumptions on initial data. These assumptions enable to get rid of the second issue (the fluxes of energy) and to apply the variational scheme (including the control of an invasion point) with much less technicalities.

The paper is organized as follows: Section 2 is devoted to preliminary material about existence of solutions and smoothing properties. In Section 3 we get a preliminary rough control of the invasion point $\tilde{x}(t)$, namely we prove that its mean velocity is finite (more precisely bounded from above by a bound depending only on V), and at the same time we prove Proposition 1. For this purpose only energy functionals in the laboratory frame are required.

The proof of Theorem 1 really begins in the short Section 4, where the invasion point $\tilde{x}(t)$ is defined, and where the scene is set up.

The crucial step – computations on weighted energy functionals in a traveling frame – is carried out in Section 5. There we choose a weight function, write down the approximate decrease (decrease up to "fluxes of energy") of the so-defined *localized energy*, and we introduce a *firewall* functional which, thanks to the rough preliminary control of the invasion point obtained in Section 3, enables us to get some control of these fluxes of energy.

These computations are applied in Section 6, in order to get some better control of the behavior of the invasion point $\tilde{x}(t)$. There we prove that the limit $\lim_{t\to+\infty} \tilde{x}(t)/t$ exists – in other words the mean velocity of invasion exists and is unique (a more uniform control, necessary for the next step, is actually reached). In the proof this velocity is denoted by $\tilde{c}$ but it corresponds to the velocity denoted by c in Theorem 1.

At this point we have at our disposal enough information – enough control of the behavior of the invasion point, and enough control of the fluxes of energy – in order to apply the formal relaxation scheme sketched above, and we do this in Section 7. The conclusion is that, in a frame traveling at velocity $\tilde{c}$, the solution is asymptotically stationary on any bounded interval around $\tilde{x}(t)$. The proof of Theorem 1 then follows naturally, and is completed in Section 8.

Appendix A is made of several sections. In Section A.1 we prove the existence of an attracting ball in $H^1_{\mathrm{ul}}(\mathbb{R})$ for the semi-flow of (1). Proposition 2 (sufficient condition in order invasion to occur) is proved in Section A.2. Section A.3 is devoted to the proof of some properties of the differential system (8) (governing the profiles of waves traveling at constant velocity), in particular those stated in Proposition 3. Counting arguments on the phase space of (8) are given in Section A.4. Section A.5 is devoted to the proof of Theorem 2 (lower semi-continuity of the function $u_0 \mapsto c[u_0]$). Finally, in Section A.6, the particular case of a bistable potential is considered, and a statement strengthening the conclusions of Theorem 4 and generalizing Fife and McLeod's global convergence result is proved.

2. Preliminaries

Uniformly local Sobolev spaces

For sake of generality and clarity, following [21,5,12], we shall study the semi-flow of the parabolic system (1) in the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R})$ introduced by Kato [17], which is the natural "energy space" containing solutions of physical interest such as traveling waves. Of course, due to smoothing properties of the semi-flow recalled below, this framework is by no means essential, and the presentation can be easily adapted in order to avoid any reference to $H^1_{\mathrm{ul}}(\mathbb{R})$ or $L^2_{\mathrm{ul}}(\mathbb{R})$ (it suffices to systematically replace H^1 by $\mathcal{C}^1$, H^1_{ul} by $\mathcal{C}^1_{\mathrm{b}}$, L^2 by $\mathcal{C}^0$, and L^2_{ul} by $\mathcal{C}^0_{\mathrm{b}}$). However these uniformly local Sobolev spaces are the most appropriate for our approach based on energy functionals, and we believe the arguments are more clearly expressed using them. They are also the most appropriate for further generalization to equations having no smoothing properties [9,10]. Their definition and basic properties of the semi-flow are recalled now (we refer to [15,21,13,20,12] for more details).

For $s \in \mathbb{N}$, the uniformly local Sobolev space $H^s_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^n)$ is defined as the set:

$$\left\{u : \mathbb{R} \to \mathbb{R}^n \mid u \in H^s_{\mathrm{loc}}\left(\mathbb{R}, \mathbb{R}^n\right),\ \|u\|_{H^s_{\mathrm{ul}}(\mathbb{R},\mathbb{R}^n)} < +\infty,\ \lim_{x\to 0} \|T_x u - u\|_{H^s_{\mathrm{ul}}(\mathbb{R},\mathbb{R}^n)} = 0\right\},$$

where $T_x u(y) = u(y - x)$ and

$$\|u\|_{H^s_{\mathrm{ul}}(\mathbb{R},\mathbb{R}^n)} = \sup_{x\in\mathbb{R}} \|u_{|[x,x+1]}\|_{H^s([x,x+1],\mathbb{R}^n)}.$$

This space $H^s_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^n)$ is a Banach space, and the space $\mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}, \mathbb{R}^n)$ (the space of $\mathcal{C}^\infty$-functions that are bounded together with all their derivatives) is dense in $H^s_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^n)$.

For simplicity, we shall simply write $H^s_{\mathrm{ul}}(\mathbb{R})$ instead of $H^s_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^n)$. We shall write $L^2_{\mathrm{ul}}(\mathbb{R})$ for $H^0_{\mathrm{ul}}(\mathbb{R})$.

We shall denote by $\mathcal{C}^s_{\mathrm{b}}(\mathbb{R})$ the Banach space of functions: $\mathbb{R} \to \mathbb{R}^n$, of class $\mathcal{C}^s$, and which are uniformly bounded, together with their s first derivatives, and by $\|\ldots\|_{\mathcal{C}^s_{\mathrm{b}}(\mathbb{R})}$ the usual norm on this space (the norm $\|\ldots\|_{\mathcal{C}^s_{\mathrm{b}}([-L;+\infty[)}$, used in remark (b) following Theorem 1, is defined similarly).

Existence of solutions and regularity

Since V is assumed to be of class $\mathcal{C}^k$, $k \geqslant 3$, the map $v \in \mathbb{R}^n \mapsto \nabla V(v)$ is of class at least $\mathcal{C}^2$, and therefore the nonlinearity $u(.) \mapsto -\nabla V(u(.))$ in (1) is locally Lipschitz in $H^1_{\mathrm{ul}}(\mathbb{R})$. Thus local existence of solutions in that space follows from general results [15]. More precisely, for any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$, the system (1) has a unique (mild) solution in $\mathcal{C}^0([0; T_{\max}[, H^1_{\mathrm{ul}}(\mathbb{R}))$ with initial data u_0. This solution depends continuously on u_0 and is defined up to a (unique) maximal time of existence $T_{\max} = T_{\max}(u_0) \in\ ]0; +\infty]$.

Moreover, hypothesis (H1) yields the existence of an attracting ball in $H^1_{\mathrm{ul}}(\mathbb{R})$ for the so-defined semi-flow, as stated in Lemma 6 (Section A.1 in Appendix A). As a consequence, this semi-flow $(S_t)_{t\geqslant 0}$ is actually global, i.e. for any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$, $T_{\max}(u_0) = +\infty$ and we have

$$\sup_{t\geqslant 0} \|S_t u_0\|_{H^1_{\mathrm{ul}}(\mathbb{R})} < +\infty. \tag{11}$$

In addition, the system (1) has smoothing properties [15]. Due to these properties, since V is of class $\mathcal{C}^k$, $k \geqslant 3$, any solution $t \mapsto S_t u_0$ in $\mathcal{C}^0([0;+\infty[, H^1_{\mathrm{ul}}(\mathbb{R}))$ actually belongs to

$$\mathcal{C}^0\big(]0;+\infty[\,, \mathcal{C}^{k+1}_{\mathrm{b}}(\mathbb{R})\big) \cap \mathcal{C}^1\big(]0;+\infty[, \mathcal{C}^{k-1}_{\mathrm{b}}(\mathbb{R})\big),$$

and, for any $\varepsilon > 0$, we have

$$\sup_{t\geqslant \varepsilon} \|S_t u_0\|_{\mathcal{C}^{k+1}_{\mathrm{b}}(\mathbb{R})} < +\infty \quad \text{and} \quad \sup_{t\geqslant \varepsilon} \left\| \frac{\mathrm{d}(S_t u_0)}{\mathrm{d}t}(t) \right\|_{\mathcal{C}^{k-1}_{\mathrm{b}}(\mathbb{R})} < +\infty. \tag{12}$$

The following weaker properties will be specifically required:

$$\sup_{t\geqslant \varepsilon} \|S_t u_0\|_{\mathcal{C}^{2}_{\mathrm{b}}(\mathbb{R})} < +\infty, \tag{13}$$

$$\sup_{t\geqslant \varepsilon} \left\| \frac{\mathrm{d}(S_t u_0)}{\mathrm{d}t}(t) \right\|_{\mathcal{C}^{1}_{\mathrm{b}}(\mathbb{R})} < +\infty, \tag{14}$$

and, according to Lemma 6 in Section A.1 (in Appendix A), there exists $R_0 > 0$, depending only on V, and, for any $R > 0$, there exists $T(R) > 0$, depending only on V and R, such that, if $\|u_0\|_{H^1_{\mathrm{ul}}(\mathbb{R})} \leqslant R$, then

$$\sup_{t\geqslant T(R)} \|S_t u_0\|_{\mathcal{C}^{0}_{\mathrm{b}}(\mathbb{R})} \leqslant R_0. \tag{15}$$

Compactness

The following compactness argument will be used several times. Take any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$, and let $u(x,t) = (S_t u_0)(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ denote the solution of the system (1) with initial data u_0. Take any sequences $(x_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$, $t_n \geqslant 0$, $t_n \to +\infty$ when $n \to +\infty$, and let us define functions w_n and $\hat{w}_n$ by

$$w_n(y) = u(x_n + y, t_n) \quad \text{and} \quad \hat{w}_n(y) = u_t(x_n + y, t_n), \quad n \in \mathbb{N},\ y \in \mathbb{R}.$$

According to the regularity properties above, by compactness and a diagonal extraction procedure, there exists $w_\infty \in H^2_{\mathrm{ul}}(\mathbb{R})$ and $\hat{w}_\infty \in L^2_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, for any $L > 0$,

$$w_n \to w_\infty \text{ in } H^2\big([-L;L]\big) \quad \text{and } \hat{w}_n \to \hat{w}_\infty \quad \text{in } L^2\big([-L;L]\big), \tag{16}$$

and passing to the limit in (1) we have

$$\hat{w}_\infty = -\nabla V(w_\infty) + w''_\infty.$$

Of course due to the regularity properties above, the same compactness results hold for stronger norms, but this formulation will be sufficient for our purpose.

Notations

We shall denote by "·" (resp. by "$|\dots|$") the usual Euclidean scalar product (resp. Euclidean norm) in $\mathbb{R}^n$.

Small constants will be denoted by denoted by $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$, and large constants either by $C, C_1, C_2, \dots$, or by $K_1, K_2, \dots$. We used the two letters C and K in order to shed light on important constants – those denoted by $K_1, K_2, \dots$ – that will be used at several places in the paper.

Let us mention here that, according to assertion (7) corresponding to the choice of r_0, for all $v \in \mathbb{R}^n$ satisfying $|v| \leqslant r_0$, we have

$$\frac{\lambda_{\min}}{2} v^2 \leqslant v \cdot \nabla V(v) \leqslant 2\lambda_{\max} v^2 \quad \text{and} \quad \frac{\lambda_{\min}}{4} v^2 \leqslant V(v) \leqslant \lambda_{\max} v^2. \tag{17}$$

3. Upper bound on the invasion speed

We assume that V satisfies hypotheses (H1) and (H2), we give ourselves $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$, and we note $u(x,t) = (S_t u_0)(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ the solution of the system (1) for initial data u_0. According to (11), the quantity $\sup_{x\in\mathbb{R}, t\geqslant 0} |u(x,t)|$ is finite. We give ourselves a constant $R > 0$ such that

$$\sup_{x\in\mathbb{R}, t\geqslant 0} \left|u(x,t)\right| \leqslant R. \tag{18}$$

For any function $x \mapsto \eta(x) \in W^{2,1}(\mathbb{R},\mathbb{R})$, a direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta(x)\left(\frac{u_x(x,t)^2}{2} + V\big(u(x,t)\big)\right)\mathrm{d}x = -\int_{\mathbb{R}} \eta u_t^2\,\mathrm{d}x - \int_{\mathbb{R}} \eta' u_x \cdot u_t\,\mathrm{d}x, \quad t > 0, \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta(x)\frac{u(x,t)^2}{2}\,\mathrm{d}x = -\int_{\mathbb{R}} \eta u \cdot \nabla V(u)\,\mathrm{d}x - \int_{\mathbb{R}} \eta u_x^2\,\mathrm{d}x + \int_{\mathbb{R}} \eta'' \frac{u^2}{2}\,\mathrm{d}x, \quad t > 0. \tag{20}$$

According to hypotheses (H1) and (H2), for $\alpha_0 > 0$ sufficiently small, we have

$$\alpha_0 V(v) \geqslant -\frac{v^2}{4}, \quad v \in \mathbb{R}^n. \tag{21}$$

Let us fix $\alpha_0 \in]0;1]$ satisfying this property, and let $\beta_0 = \min(1, \sqrt{\lambda_{\min}/2})$. Let $\psi_0(x) = e^{-\beta_0|x|}$, and, for $\xi \in \mathbb{R}$, let us denote by $T_\xi\psi_0$ the map $x \mapsto \psi_0(x-\xi)$. Let

$$\Psi_0(\xi,t) = \int_{\mathbb{R}} T_\xi\psi_0(x)\left(\alpha_0\left(\frac{u_x(x,t)^2}{2} + V\big(u(x,t)\big)\right) + \frac{u(x,t)^2}{2}\right)\mathrm{d}x, \quad \xi \in \mathbb{R}, \quad t \geqslant 0.$$

According to (21), Ψ_0 is coercive in the following sense:

$$\Psi_0(\xi,t) \geqslant \min(\alpha_0/2, 1/4)\int_{\mathbb{R}} T_\xi\psi_0\big(u_x^2 + u^2\big)\,\mathrm{d}x \geqslant 0, \quad \xi \in \mathbb{R},\ t \geqslant 0. \tag{22}$$

Since $|\psi_0'| \leqslant \beta_0\psi_0$ and $\psi_0'' \leqslant \beta_0^2\psi_0$ (indeed ψ_0'' equals $\beta_0^2\psi_0$ plus a Dirac mass of negative weight), we have, according to (19) and (20),

$$\frac{\partial\Psi_0}{\partial t}(\xi,t) \leqslant \int_{\mathbb{R}} T_\xi\psi_0\left(\left(\frac{\alpha_0\beta_0^2}{4} - 1\right)u_x^2 + \frac{\beta_0^2}{2}u^2 - u\cdot\nabla V(u)\right)\mathrm{d}x, \quad t > 0,$$

which yields (just adding and subtracting the same quantity),

$$\begin{aligned}\frac{\partial\Psi_0}{\partial t}(\xi,t) \leqslant &-\int_{\mathbb{R}} T_\xi\psi_0\left(\left(1 - \frac{\alpha_0\beta_0^2}{4}\right)u_x^2 + \frac{\lambda_{\min}}{4(\alpha_0\lambda_{\max} + 1/2)}\left(\alpha_0 V(u) + \frac{u^2}{2}\right)\right)\mathrm{d}x \\ &+ \int_{\mathbb{R}} T_\xi\psi_0\left(\frac{\lambda_{\min}}{4(\alpha_0\lambda_{\max} + 1/2)}\left(\alpha_0 V(u) + \frac{u^2}{2}\right) + \frac{\beta_0^2}{2}u^2 - u\cdot\nabla V(u)\right)\mathrm{d}x. \end{aligned} \tag{23}$$

For $t \geqslant 0$, let

$$S_{\mathrm{far}}(t) = \left\{x \in \mathbb{R} \mid \left|u(x,t)\right| > r_0\right\}.$$

According to (17), and since $\beta_0^2/2 \leqslant \lambda_{\min}/4$, we see that the expression below the last integral of the right-hand side of inequality (23) is nonpositive when $x \in \mathbb{R}\setminus S_{\mathrm{far}}(t)$. Thus we deduce from (23) and (18) that

$$\frac{\partial\Psi_0}{\partial t}(\xi,t) \leqslant -\varepsilon_1\Psi_0(\xi,t) + K_1\int_{S_{\mathrm{far}}(t)} T_\xi\psi_0\,\mathrm{d}x, \quad \xi \in \mathbb{R},\ t > 0, \tag{24}$$

where $\varepsilon_1 > 0$ is a constant depending only on V, namely

$$\varepsilon_1 = \min\left(\frac{2}{\alpha_0}\left(1-\frac{\alpha_0\beta_0^2}{4}\right), \frac{\lambda_{\min}}{4(\alpha_0\lambda_{\max}+1/2)}\right)$$

and $K_1 > 0$ is a constant depending only on V and R, namely

$$K_1 = \max_{|v|\leqslant R}\left(\frac{\lambda_{\min}}{4(\alpha_0\lambda_{\max}+1/2)}\left(\alpha_0 V(v)+\frac{v^2}{2}\right)+\frac{\beta_0^2}{2}v^2 - v\cdot\nabla V(v)\right).$$

For any $v \in H^1_{\mathrm{ul}}(\mathbb{R})$, we have

$$v(0)^2 = \psi_0(0)v(0)^2 \leqslant \frac{1}{2}\int_{\mathbb{R}}\left|\left(\psi_0 v^2\right)'(x)\right|\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\left|\psi_0' v^2 + 2\psi_0 v\cdot v'\right|\mathrm{d}x$$
$$\leqslant \frac{1}{2}\int_{\mathbb{R}}\psi_0\left(v^2+2|v\cdot v'|\right)\mathrm{d}x \leqslant \int_{\mathbb{R}}\psi_0\left(v^2+v'^2\right)\mathrm{d}x \tag{25}$$

(recall that $\beta_0 \leqslant 1$). According to (22), this shows that

$$\min(\alpha_0/2, 1/4)\left|u(x,t)\right|^2 \leqslant \Psi_0(x,t), \quad x\in\mathbb{R},\ t\geqslant 0.$$

In order to state Lemma 1 below, which is the essential step in the proof of Proposition 1, the following notations are required. Let

$$\varepsilon_2 = \min(\alpha_0/2, 1/4)r_0^2 \quad \left(\text{thus } \left|u(x,t)\right| \leqslant \sqrt{\Psi_0(x,t)/\varepsilon_2}\,r_0\right), \tag{26}$$

$$L = \frac{2}{\beta_0}\log\frac{2K_1}{\varepsilon_1\varepsilon_2\beta_0} \quad \left(\text{equivalently } K_1\int_{-\infty}^{-\frac{L}{2}}\psi_0(x)\,\mathrm{d}x = \frac{\varepsilon_1\varepsilon_2}{2}\right),$$

and let us define the function $\chi : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ by:

$$\chi(y) = +\infty \quad \text{for } y<0, \qquad \chi(y) = \varepsilon_2\left(1-\frac{y}{2L}\right) \quad \text{for } 0\leqslant y\leqslant L,$$
$$\chi(y) = \frac{\varepsilon_2}{2} \quad \text{for } y \geqslant L$$

(see Fig. 2). For $x\in\mathbb{R}$ and $t\geqslant 0$, let us consider the property $\mathcal{P}(x,t)$ defined as follows:

$$\mathcal{P}(x,t) \text{ holds} \quad \Leftrightarrow \quad \Psi_0(y,t) \leqslant \chi(y-x) \text{ for all } y\in\mathbb{R}.$$

In other words $\mathcal{P}(x,t)$ holds when the solution is, at time t_0, sufficiently close to 0 at the right of x in space.

Lemma 1. *There exists a constant $c_{\max}(R) > 0$, depending only on V, R, such that, for any $x_0\in\mathbb{R}$ and $t_0 \geqslant 0$,*

$$\mathcal{P}(x_0,t_0) \textit{ holds} \quad \Rightarrow \quad \mathcal{P}\left(x_0 + c_{\max}(R)(t-t_0), t\right) \textit{ holds for all } t > t_0.$$

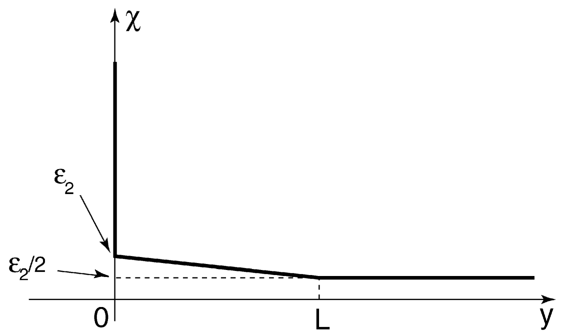

Fig. 2. Graph of function χ.

Proof. Let $c > 0$ to be chosen below, take any $x_0 \in \mathbb{R}$ and $t_0 \geqslant 0$, and let us assume that $\mathcal{P}(x_0, t_0)$ holds and that the set

$$\{t > t_0 \mid \mathcal{P}(x_0 + c_{\max}(R)(t - t_0), t) \text{ does not hold}\}$$

is nonempty. Let $t_1 \geqslant t_0$ denote the infimum of this set, and let us write $x_1 = x_0 + c(t_1 - t_0)$.

According to (24) (or to the continuity of $\Psi_0(x,t)$ with respect to t), $\mathcal{P}(x_1, t_1)$ holds. We are going to show that, if c is sufficiently large, then there exists $\varepsilon > 0$ such that, for $s \in [0; \varepsilon]$, $\mathcal{P}(x_1 + cs, t_1 + s)$ holds; since this last statement is contradictory to the definition of t_1, this will prove the lemma.

Let us write $\tilde{\chi}(x,s) = \chi(x - x_1 - cs)$, $x \in \mathbb{R}$, $s \geqslant 0$. By definition of $\mathcal{P}(.,.)$, for any $s \geqslant 0$,

$$\mathcal{P}(x_1 + cs, t_1 + s) \text{ holds} \quad \Longleftrightarrow \quad \sup_{x \geqslant x_1 + cs} \Psi_0(x, t_1 + s) - \tilde{\chi}(x,s) \leqslant 0.$$

Since $\mathcal{P}(x_1, t_1)$ holds, we have $\sup_{x \geqslant x_1 + L/2} \Psi_0(x, t_1) \leqslant 3\varepsilon_2/4$, and thus, according to (26),

$$\sup_{x \geqslant x_1 + L/2} |u(x, t_1)| \leqslant \frac{\sqrt{3}}{2} r_0 < r_0.$$

Thus, by continuity of the semi-flow in $H^1_{\mathrm{ul}}(\mathbb{R})$, there exists $\varepsilon > 0$ such that, for all $s \in [0; \varepsilon]$,

$$\sup_{x \geqslant x_1 + L/2} |u(x, t_1 + s)| \leqslant r_0.$$

As a consequence, according to (24) and to the choice of L, we have, for all $x \geqslant x_1 + L$,

$$\frac{\partial \Psi_0}{\partial t}(x, t_1 + s) \leqslant -\varepsilon_1\big(\Psi_0(x, t_1 + s) - \varepsilon_2/2\big), \quad 0 \leqslant s \leqslant \varepsilon,$$

and since

$$\Psi_0(x, t_1) - \varepsilon_2/2 \leqslant 0,$$

this shows that

$$\Psi_0(x, t_1 + s) - \varepsilon_2/2 \leqslant 0 \quad \text{thus} \quad \Psi_0(x, t_1 + s) - \tilde{\chi}(x,s) \leqslant 0 \tag{27}$$

for $0 \leqslant s \leqslant \varepsilon$ and $x \geqslant x_1 + L$. On the other hand, according to (24) (and since $\int_{\mathbb{R}} \psi_0(x)\,\mathrm{d}x = \frac{2}{\beta_0}$), we have

$$\frac{\partial \Psi_0}{\partial t}(x,t) \leqslant \frac{2K_1}{\beta_0}, \quad x \in \mathbb{R},\ t \geqslant 0,$$

whereas

$$\frac{\partial \tilde{\chi}}{\partial s}(x,s) = \frac{c\varepsilon_2}{2L}, \quad x \in]x_1 + cs; x_1 + cs + L[,\ s \geqslant 0.$$

This shows that, if we choose $c = \frac{4LK_1}{\varepsilon_2 \beta_0}$, then we have, for all $x \in [x_1; x_1 + L]$,

$$\Psi_0(x, t_1 + s) \leqslant \tilde{\chi}(x,s), \quad s \in [0; \varepsilon].$$

In view of (27), this shows that $\mathcal{P}(x_1 + cs, t_1 + s)$ holds for $s \in [0; \varepsilon]$ (and this is contradictory to the definition of s_1). Thus the conclusions of the lemma hold with

$$c_{\max}(R) = \frac{4LK_1}{\varepsilon_2 \beta_0}. \qquad \square$$

Proposition 1 follows from the next corollary.

Corollary 2. *The three following assertions are equivalent.*

(i) *There exists $t_0 > 0$ and $x_0 \in \mathbb{R}$ such that $\mathcal{P}(x_0, t_0)$ holds.*
(ii) *For any $c > c_{\max}(R_0)$, $\lim_{t \to +\infty} \limsup_{x \geqslant ct} |u(x,t)| = 0$.*
(iii) $\lim_{t \to +\infty} \limsup_{x \to +\infty} |u(x,t)| = 0$ (*i.e.* $u_0 \in \mathcal{A}$).

Proof. (ii) ⇒ (iii) is obvious and (iii) ⇒ (i) follows from (13) (bound on $\|x \mapsto u(x,t)\|_{\mathcal{C}^2_b(\mathbb{R})}$).

It remains to prove (i) ⇒ (ii). For this purpose, let us assume that (i) holds. Then, according to the previous lemma and to (15), up to changing the values of x_0 and t_0, we can suppose that $\sup_{x\in\mathbb{R}, t\geqslant t_0} |u(x,t)| \leqslant R_0$. Then, according to (29), we have, for any $t \geqslant t_0$,

$$S_{\text{far}}(t) \subset \left]-\infty; x_0 + c_{\max}(R_0)(t-t_0)\right].$$

Take any $c' > c_{\max}(R_0)$. According to (24), for any $t \geqslant t_0$ and $x \geqslant x_0 + c'(t-t_0)$, we have

$$\frac{\partial \Psi_0(x,t)}{\partial t} \leqslant -\varepsilon_1 \Psi_0(x,t) + K_1 \beta_0^{-1} e^{-\beta_0(c'-c_{\max}(R_0))(t-t_0)},$$

and this shows that

$$\sup_{x \geqslant x_0 + c'(t-t_0)} \Psi_0(x,t) \to 0 \quad \text{when } t \to +\infty.$$

This proves (i) and thus finishes the proof. □

In view of this corollary (or of Proposition 1) the set $\mathcal{A}$ is, on one hand, obviously nonempty and on the other hand, according to (13) (bound on $\|x \mapsto u(x,t)\|_{\mathcal{C}^2_b(\mathbb{R})}$), open in X.

Notation. In the following, we shall simply write $c_{\max}$ for $c_{\max}(R_0)$, where R_0 is the constant introduced in (15) (Section 2).

4. Setup

The aim of this section is to setup the frame for the proof of Theorem 1 (this section and the next Sections 5, 6, 7, and 8 are devoted to this proof). We assume that V satisfies hypotheses (H1) and (H2), we give ourselves $u_0 \in \mathcal{A}_{\text{inv}}$, and we note $u(x,t) = (S_t u_0)(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ the solution of the system (1) with initial data u_0.

Up to changing the origin of time, we can assume (according to (15)) that, for any $t \geqslant 0$,

$$\sup_{x\in\mathbb{R},\ t\geqslant 0} \left|u(x,t)\right| \leqslant R_0. \tag{28}$$

Moreover, according to Corollary 2 (and still up to changing the origin of time), we can assume (using the notations of the previous section) that there exists $x_0 \in \mathbb{R}$ such that $\mathcal{P}(x_0, 0)$ holds. Then, according to Lemma 1, for any $t \geqslant 0$, $\mathcal{P}(x_0 + c_{\max} t, t)$ holds. Remark that, if $\mathcal{P}(x,t)$ holds, then $\mathcal{P}(y,t)$ holds for all $y \geqslant x$, so that, for any $t \geqslant 0$, the set

$$\left\{x \in \mathbb{R} \mid \mathcal{P}(x,t) \text{ holds}\right\}$$

is a (nonempty) interval, unbounded from above, and which cannot be equal to $\mathbb{R}$ (or else, because of Lemma 1, it would remain equal to $\mathbb{R}$ for all times $t' \geqslant t$, and, in view of (22) and (24), this would be contradictory to $u_0 \in \mathcal{A}_{\text{inv}}$).

Let us denote by $\tilde{x}(t)$ the infimum of this set. This point represents the first point, starting from $+\infty$ in space, where the solution reaches a certain distance from 0 (in the sense corresponding to $\mathcal{P}(.,.)$). In the following, we will refer to this point as to the "invasion point". According to Lemma 1, we have

$$\tilde{x}(t+s) \leqslant \tilde{x}(t) + c_{\max} s \quad t \geqslant 0,\ s \geqslant 0. \tag{29}$$

By continuity of $\Psi_0(x,t)$ with respect to x, the property $\mathcal{P}(\tilde{x}(t),t)$ holds, and as a consequence, according to (26), we have

$$\left|u(x,t)\right| \leqslant r_0 \quad \text{for all } x \geqslant \tilde{x}(t),\ t \geqslant 0. \tag{30}$$

For any $s \geqslant 0$, let

$$\tilde{X}(s) = \sup_{t\geqslant 0} \tilde{x}(t+s) - \tilde{x}(t).$$

According to (29), we have $\tilde{X}(s) \leqslant c_{\max} s$, $s \geqslant 0$. Let us write

$$\tilde{c}_- = \liminf_{t\to+\infty} \tilde{x}(t)/t, \quad \tilde{c}_+ = \limsup_{t\to+\infty} \tilde{x}(t)/t, \quad \text{and} \quad c^* = \limsup_{s\to+\infty} \tilde{X}(s)/s.$$

We have $\tilde{c}_- \leqslant \tilde{c}_+ \leqslant c^* \leqslant c_{\max}$. Moreover, since the initial data u_0 belong to $\mathcal{A}_{\text{inv}}$, we have

$$\tilde{c}_+ > 0.$$

Indeed, if conversely we had $\tilde{c}_+ \leqslant 0$, then, for any $\varepsilon > 0$, we would have, according to (24) (and proceeding as in the proof of Corollary 2),

$$\sup_{x \geqslant \varepsilon t} \Psi_0(x,t) \to 0 \quad \text{when } t \to +\infty,$$

and, in view of (22), this would contradict $u_0 \in \mathcal{A}_{\text{inv}}$.

Our first task will be to prove that the three mean velocities $\tilde{c}_-$, $\tilde{c}_+$, and c^* are equal.

5. Weighted functionals in a traveling frame

We keep the notations and hypotheses of the previous section and we give ourselves four parameters:

$$t_{\text{init}} \geqslant 0, \quad x_{\text{init}} \in \mathbb{R}, \quad 0 < c \leqslant c_{\max}, \quad \text{and} \quad y_0 \in \mathbb{R}.$$

The following computations will be used several times in the following (four times actually, in Sections 6, 7, A.2, and A.5), for various choices of these parameters. Let us consider the function $v(y,s) = u(x,t)$, where the variables y and s are defined as follows:

$$x = x_{\text{init}} + cs + y, \qquad t = t_{\text{init}} + s$$

(in other words, we place ourselves in a frame traveling at velocity c, with t_{init} as the origin of times and x_{init} as the origin of space). The parameter y_0 will be used thereafter.

According to (1), $v(y,s)$ satisfies the following differential system

$$v_s - cv_y = -\nabla V(v) + v_{yy}.$$

The aim of this section is to give a concrete meaning to the following formal expression of "decrease of energy":

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}} e^{cy}\left(\frac{v_y(y,s)^2}{2} + V\big(v(y,s)\big)\right)\mathrm{d}y = -\int_{\mathbb{R}} e^{cy} v_s(y,s)^2\,\mathrm{d}y.$$

For any function $(y,s) \mapsto \eta(y,s)$, such that, for any $s \in \mathbb{R}$, $y \mapsto \eta(y,s) \in W^{2,1}(\mathbb{R},\mathbb{R})$ and $y \mapsto \eta_s(y,s) \in L^1(\mathbb{R},\mathbb{R})$ (where $\eta_s = \partial_s \eta$), a direct calculation shows that, for all $s > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}} \eta(y,s)\left(\frac{v_y(y,s)^2}{2} + V\big(v(y,s)\big)\right)\mathrm{d}y$$
$$= -\int_{\mathbb{R}} \eta v_s^2\,\mathrm{d}y + \int_{\mathbb{R}} \eta_s\left(\frac{v_y^2}{2} + V(v)\right)\mathrm{d}y + \int_{\mathbb{R}} (c\eta - \eta_y) v_y \cdot v_s\,\mathrm{d}y \tag{31}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}} \eta(y,s)\frac{v(y,s)^2}{2}\,\mathrm{d}y = -\int_{\mathbb{R}} \eta\big(v_y^2 + v\cdot\nabla V(v)\big)\,\mathrm{d}y + \int_{\mathbb{R}} (\eta_s + \eta_{yy} - c\eta_y)\frac{v^2}{2}\,\mathrm{d}y. \tag{32}$$

Let us introduce the following three constants:

$$\alpha = \min\left(1, \alpha_0, \frac{1}{(c_{\max}+1)^2}\right), \quad \beta = \min\left(1, \frac{\lambda_{\min}}{8(c_{\max}+1)}\right), \quad \gamma = \min\left(\frac{\lambda_{\min}}{8\lambda_{\max}}, \frac{\lambda_{\min}}{8(c_{\max}+1)}\right),$$

where $\alpha_0 > 0$ is the constant introduced in Section 3.

Let us consider the function $\varphi(y,s)$ defined by:

$$\varphi(y,s) = e^{cy} \quad \text{for } y \leqslant y_0 + \gamma s \quad \text{and} \quad \varphi(y,s) = e^{-\beta y} e^{(c+\beta)(y_0+\gamma s)} \quad \text{for } y \geqslant y_0 + \gamma s$$

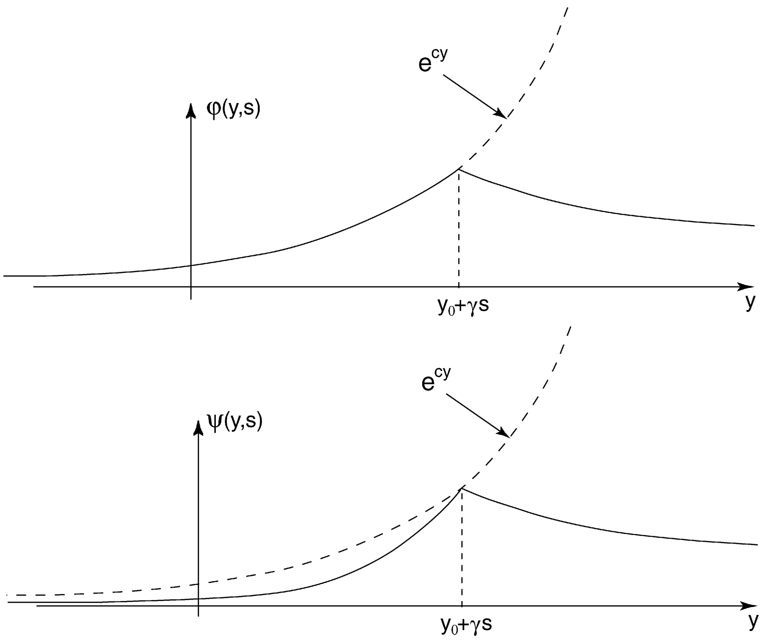

Fig. 3. Graphs of the weight functions $\varphi(y,s)$ and $\psi(y,s)$.

(see Fig. 3) and let us define the *energy functional*

$$\Phi(s)=\int_{\mathbb{R}}\varphi(y,s)\left(\frac{v_y(y,s)^2}{2}+V\big(v(y,s)\big)\right)\mathrm{d}y,\quad s\geqslant 0,$$

and the corresponding *energy dissipation functional*

$$\mathcal{D}(s)=\int_{\mathbb{R}}\varphi(y,s)v_s(y,s)^2\,\mathrm{d}y,\quad s\geqslant 0.$$

We have

$$\varphi_s=c\varphi-\varphi_y=0,\quad y<y_0+\gamma s,$$

and

$$\varphi_s=\gamma(c+\beta)\varphi,\quad c\varphi-\varphi_y=(c+\beta)\varphi,\quad y>y_0+\gamma s.$$

Thus we see from (31) that

$$\Phi'(s)\leqslant-\int_{\mathbb{R}}\varphi v_s^2\,\mathrm{d}y+\int_{y_0+\gamma s}^{+\infty}\varphi\left(\gamma(c+\beta)\left(\frac{v_y^2}{2}+V(v)\right)+\frac{v_s^2}{2}+\frac{(c+\beta)^2}{2}v_y^2\right)\mathrm{d}y.$$

Let $C_1=\sup_{0<|w|\leqslant R_0}V(w)/w^2$ (we have $0<C_1<+\infty$). The last inequality yields

$$\Phi'(s)\leqslant-\frac{1}{2}\mathcal{D}(s)+C_2\int_{y_0+\gamma s}^{+\infty}\varphi\big(v_y^2+v^2\big)\,\mathrm{d}y,\tag{33}$$

where $C_2>0$ is a constant depending only on V, namely (since $c\leqslant c_{\max}$, and according to (28))

$$C_2=\max\left(\frac{\gamma(c_{\max}+\beta)}{2}+\frac{(c_{\max}+\beta)^2}{2},\gamma(c_{\max}+\beta)C_1\right).$$

In order to get some control of the second term of the right-hand side of inequality (33), we now introduce another functional. Let us consider the function $\psi(y,s)$ defined by:

$$\psi(y,s) = e^{(c+\beta)y} e^{-\beta(y_0+\gamma s)} \quad \text{for } y \leqslant y_0 + \gamma s,$$

and

$$\psi(y,s) = \varphi(y,s) = e^{-\beta y} e^{(c+\beta)(y_0+\gamma s)} \quad \text{for } y \geqslant y_0 + \gamma s,$$

and let us define the *firewall functional*

$$\Psi(s) = \int_{\mathbb{R}} \psi(y,s)\left(\alpha\left(\frac{v_y(y,s)^2}{2} + V\big(v(y,s)\big)\right) + \frac{v(y,s)^2}{2}\right) \mathrm{d}y, \quad s \geqslant 0.$$

Since $\alpha \leqslant \alpha_0$, the following coercivity property holds:

$$\Psi(s) \geqslant \min(\alpha/2, 1/4) \int_{\mathbb{R}} \psi\big(v_y^2 + v^2\big)\,\mathrm{d}y, \tag{34}$$

thus inequality (33) gives

$$\Phi'(s) \leqslant -\frac{1}{2}\mathcal{D}(s) + K_2 \Psi(s), \quad s \geqslant 0, \tag{35}$$

where

$$K_2 = \frac{C_2}{\min(\alpha/2, 1/4)}$$

(the firewall functional $\Psi(s)$ provides us with a bound on the "pollution" term of (33)). Inequality (35) is a key ingredient that will be used extensively in the following sections.

The aim of the next computations is to provide some control of $\Psi(s)$. We have

$$|\psi_s| \leqslant \gamma(c+\beta)\psi, \quad |c\psi - \psi_y| \leqslant (c+\beta)\psi, \quad \text{and} \quad \psi_{yy} - c\psi_y \leqslant \beta(c+\beta)\psi$$

(indeed $\psi_{yy} - c\psi_y$ equals $\beta(c+\beta)\psi$ plus a Dirac mass of negative weight at $y = y_0 + \gamma s$). Thus, according to (31) and (32), we have

$$\Psi'(s) \leqslant \int_{\mathbb{R}} \psi\left(v_y^2\left(\frac{\alpha\gamma(c+\beta)}{2} + \frac{\alpha(c+\beta)^2}{4} - 1\right) + \alpha\gamma(c+\beta)\big|V(v)\big| - v\cdot\nabla V(v) + \frac{(\gamma+\beta)(c+\beta)}{2}v^2\right)\mathrm{d}y.$$

According to the values of α, β, and γ, this inequality yields

$$\Psi'(s) \leqslant \int_{\mathbb{R}} \psi\left(-\frac{v_y^2}{2} + \frac{\lambda_{\min}}{8\lambda_{\max}}\big|V(v)\big| - v\cdot\nabla V(v) + \frac{\lambda_{\min}}{8}v^2\right)\mathrm{d}y,$$

and thus (as in Section 3 we just add and subtract the same quantity)

$$\begin{aligned}\Psi'(s) \leqslant{}& \int_{\mathbb{R}} \psi\left(-\frac{v_y^2}{2} - \frac{\lambda_{\min}}{4(\alpha\lambda_{\max}+1/2)}\left(\alpha V(v) + \frac{v^2}{2}\right)\right)\mathrm{d}y \\ &+ \int_{\mathbb{R}} \psi\left(\frac{\lambda_{\min}}{4(\alpha\lambda_{\max}+1/2)}\left(\alpha V(v) + \frac{v^2}{2}\right) + \frac{\lambda_{\min}}{8\lambda_{\max}}\big|V(v)\big| - v\cdot\nabla V(v) + \frac{\lambda_{\min}}{8}v^2\right)\mathrm{d}y.\end{aligned} \tag{36}$$

Let

$$S_{\mathrm{far}}(s) = \big\{y \in \mathbb{R} \mid \big|v(y,s)\big| > r_0\big\}, \quad s \geqslant 0.$$

According to (17), we see that the expression below the second integral of the right-hand side of (36) is nonpositive when $y \in \mathbb{R} \setminus S_{\mathrm{far}}(s)$. Thus we deduce from (36) that

$$\Psi'(s) \leqslant -\varepsilon_3 \Psi(s) + C_3 \Theta(s), \quad s \geqslant 0, \tag{37}$$

where

$$\Theta(s)=\int_{S_{\mathrm{far}}(s)}\psi(y,s)\,\mathrm{d}y,\qquad \varepsilon_3=\min\left(\frac{1}{\alpha},\frac{\lambda_{\min}}{4(\alpha\lambda_{\max}+1/2)}\right),$$

and $C_3>0$ is a constant depending only on V, namely, according to (28)),

$$C_3=\max_{|w|\leqslant R_0}\left(\frac{\lambda_{\min}}{4(\alpha\lambda_{\max}+1/2)}\left(\alpha V(v)+\frac{v^2}{2}\right)+\frac{\lambda_{\min}}{8\lambda_{\max}}\left|V(v)\right|-v\cdot\nabla V(v)+\frac{\lambda_{\min}}{8}v^2\right).$$

Let us write

$$\tilde{y}(s)=\tilde{x}(t)-x_{\mathrm{init}}-cs,\quad s\geqslant 0.$$

According to (30), we have $S_{\mathrm{far}}(s)\subset\,]{-\infty};\tilde{y}(s)]$. Since (by definition of $\psi(.,.)$) we have $\psi(y,s)\leqslant e^{(c+\beta)y}\times e^{-\beta(y_0+\gamma s)}$, we get

$$\Theta(s)\leqslant\beta^{-1}e^{-\beta y_0}\exp\bigl((c+\beta)\tilde{y}(s)-\beta\gamma s\bigr),\quad s\geqslant 0. \tag{38}$$

Thus, if we moreover assume that

$$x_{\mathrm{init}}=\tilde{x}(t_{\mathrm{init}}), \tag{39}$$

then we have $\tilde{y}(s)\leqslant\tilde{X}(s)-cs$, and thus (38) yields

$$\Theta(s)\leqslant\beta^{-1}e^{-\beta y_0}e^{-\frac{\beta\gamma}{2}s}\exp\left((c+\beta)\bigl(\tilde{X}(s)-c^*s\bigr)+\left((c+\beta)(c^*-c)-\frac{\beta\gamma}{2}\right)s\right),\quad s\geqslant 0.$$

Thus, if we moreover assume that

$$(c+\beta)(c^*-c)\leqslant\frac{\beta\gamma}{4}, \tag{40}$$

then the previous inequality yields

$$\Theta(s)\leqslant C_4e^{-\beta y_0}e^{-\frac{\beta\gamma}{2}s},\quad s\geqslant 0,$$

where $C_4>0$ is a constant depending only on V and the function $\tilde{X}(.)$, namely

$$C_4=\beta^{-1}\exp\left(\sup_{s\geqslant 0}(c_{\max}+\beta)\bigl(\tilde{X}(s)-c^*s\bigr)-\frac{\beta\gamma}{4}s\right)<+\infty.$$

Thus, provided that conditions (39) and (40) are fulfilled, (37) yields, for any $\bar{s}\geqslant 0$,

$$\Psi(\bar{s})\leqslant-\varepsilon_3\int_0^{\bar{s}}\Psi(s)\,\mathrm{d}s+\Psi(0)+\frac{2C_3C_4}{\beta\gamma}e^{-\beta y_0},$$

and thus, since $0\leqslant\Psi(\bar{s})$,

$$\int_0^{+\infty}\Psi(s)\,\mathrm{d}s\leqslant\varepsilon_3^{-1}\Psi(0)+K_3e^{-\beta y_0},\quad K_3=\frac{2C_3C_4}{\varepsilon_3\beta\gamma} \tag{41}$$

(this constant K_3 depends on V and the function $\tilde{X}(.)$, but not on the parameters t_{init}, x_{init}, c, y_0).

Finally, observe that

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}s}=\int_{\mathbb{R}}\varphi_s\frac{v_s^2}{2}\,\mathrm{d}y-\int_{\mathbb{R}}\varphi v_s\cdot D^2V(v)v_s\,\mathrm{d}y-\int_{\mathbb{R}}\varphi v_{ys}^2\,\mathrm{d}y+\int_{\mathbb{R}}(c\varphi-\varphi_y)v_s\cdot v_{ys}\,\mathrm{d}y,$$

and thus

$$\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}s}\leqslant C\mathcal{D}(s),\quad s\geqslant 0, \tag{42}$$

where C is a constant depending only on V, namely, according to (28),

$$C=\gamma(c_{\max}+\beta)+2\bar{\lambda}+\frac{c_{\max}+\beta}{2},\quad \bar{\lambda}=\sup\{\lambda \mid \lambda \text{ eigenvalue of } D^2V(w),\ |w|\leqslant R_0\}.$$

Inequalities (35), (41), and (42) are the basic ingredients for the arguments that will be used in the following.

6. Control of the invasion point

The aim of this section is to prove the following proposition, which provides a control of the invasion point that will be useful for the relaxation argument in the next section.

Proposition 4. *We have* $\tilde{c}_-=\tilde{c}_+=c^*$.

Proof. Let us proceed by contradiction and suppose that $\tilde{c}_- < c^*$.

Take any $c>0$ satisfying

$$0<c\leqslant c_{\max},\quad \tilde{c}_-<c<c^*<c+\gamma,\quad \text{and}\quad (c+\beta)(c^*-c)\leqslant\frac{\beta\gamma}{4} \tag{43}$$

(any c smaller than c^* but sufficiently close to c^* is convenient, recall that $c^*>0$ since $\tilde{c}_+>0$).

Take a sequence $(s_n)_{n\in\mathbb{N}}$, satisfying $s_n\to+\infty$ and $\tilde{X}(s_n)/s_n\to c^*$ when $n\to+\infty$. For each n, by definition of $\tilde{X}(.)$, one can find a time $t_n\geqslant 0$ such that $\tilde{x}(t_n+s_n)-\tilde{x}(t_n)\geqslant\tilde{X}(s_n)-1$.

The strategy of the proof is to exploit the fact that, in certain frames traveling at the velocity c, the invasion point makes large excursions to the right (between times t_n and t_n+s_n, indeed $\tilde{x}(t_n+s_n)-\tilde{x}(t_n)\sim c^*s_n\gg cs_n$) followed by returns (indeed there are arbitrarily large values of t such that $\tilde{x}(t)\sim c_-t\ll ct$). We are going to show that these large excursions followed by returns are incompatible with the approximate decrease of the energy functional Φ established in (35). Roughly speaking, a large amount of dissipation must occur during an excursion far to the right, whereas Φ must be bounded from below (in view of its definition) when the invasion point is to the left of the origin of the traveling frame, and, provided that the total amount of the flux of energy is bounded, this is contradictory to (35).

Take and fix $n\in\mathbb{N}$. We are going to apply the computations of Section 5 with the following set of parameters (see Fig. 4):

$$t_{\text{init}}=t_n,\quad x_{\text{init}}=\tilde{x}(t_{\text{init}}),\quad c \text{ (chosen above)},\quad \text{and}\quad y_0=0.$$

Let us denote by $v^{(n)}(y,s)$, $\Phi^{(n)}(s)$, $\mathcal{D}^{(n)}(s)$, $\Psi^{(n)}(s)$, and $\tilde{y}^{(n)}(s)$ the quantities defined in Section 5 (with the same notations except the "(n)" exponent), for this set of parameters.

Since $y_0=0$, and according to the bound (11) on $\|x\mapsto u(x,t)\|_{H^1_{\text{ul}}(\mathbb{R})}$, the quantity $\Phi^{(n)}(0)$ is bounded from above, uniformly with respect to $n\in\mathbb{N}$.

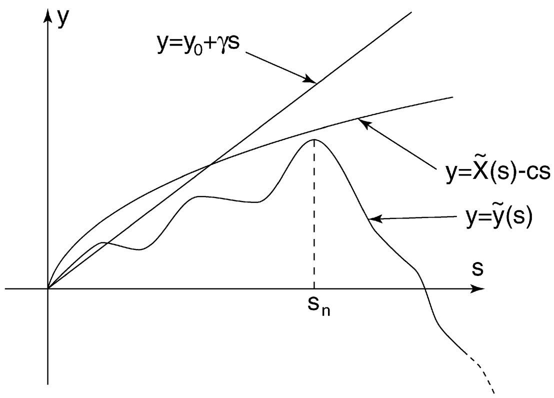

Fig. 4. Proof of Proposition 4.

Since $\tilde{y}^{(n)}(s) = \tilde{x}(t_n + s) - \tilde{x}(t_n) - cs$, we have $\liminf_{s\to+\infty} \tilde{y}^{(n)}(s)/s = \tilde{c}_- - c < 0$, thus

$$\liminf_{s\to+\infty} \tilde{y}^{(n)}(s) = -\infty.$$

According to the definition of $\tilde{y}^{(n)}(.)$ and to (30), we have $V(v(n)(y,s)) \geqslant 0$ for $y \geqslant \tilde{y}^{(n)}(s)$, and this shows that

$$\Phi^{(n)}(s) \geqslant \Big(\min_{|w|\leqslant R_0} V(w)\Big) \int_{-\infty}^{\tilde{y}^{(n)}(s)} e^{cy}\, \mathrm{d}y, \tag{44}$$

and finally that $\limsup_{s\to+\infty} \Phi^{(n)}(s) \geqslant 0$.

Since $y_0 = 0$ and according to (11) the quantity $\Psi^{(n)}(0)$ is bounded from above, uniformly with respect to $n \in \mathbb{N}$. Moreover, since $x_{\mathrm{init}} = \tilde{x}(t_{\mathrm{init}})$ and according to (43), inequality (41) holds, and it shows that the quantity $\int_0^{+\infty} \Psi^{(n)}(s)\,\mathrm{d}s$ is bounded from above, uniformly with respect to $n \in \mathbb{N}$.

According to (35), this shows that the quantity

$$\int_0^{+\infty} \mathcal{D}^{(n)}(s)\,\mathrm{d}s$$

is finite, and bounded from above uniformly with respect to $n \in \mathbb{N}$. In view of (42), this shows that the quantity $\mathcal{D}^{(n)}(s)$ itself is bounded, uniformly with respect to $s \in [1, +\infty[$ and to $n \in \mathbb{N}$.

On the other hand, we have, by definition of $\mathcal{D}^{(n)}(s)$,

$$\mathcal{D}^{(n)}(s_n) \geqslant \int_{-\infty}^{\gamma s_n} e^{cy} v_s^{(n)}(y, s_n)^2\, \mathrm{d}y,$$

which becomes, writing $y = \tilde{y}^{(n)}(s_n) + z$,

$$\mathcal{D}^{(n)}(s_n) \geqslant e^{c\tilde{y}^{(n)}(s_n)} \int_{-\infty}^{\gamma s_n - \tilde{y}^{(n)}(s_n)} e^{cz} v_s^{(n)}\big(\tilde{y}^{(n)}(s_n) + z, s_n\big)^2\, \mathrm{d}z. \tag{45}$$

We have $\tilde{y}^{(n)}(s_n) = \tilde{x}(t_n + s_n) - \tilde{x}(t_n) - cs_n$, thus, according to the choice of t_n,

$$\tilde{X}(s_n) - 1 - cs_n \leqslant \tilde{y}^{(n)}(s_n) \leqslant \tilde{X}(s_n) - cs_n,$$

and thus $\tilde{y}^{(n)}(s_n) \sim (c^* - c)s_n$ when $n \to +\infty$; in particular, according to (43),

$$e^{c\tilde{y}^{(n)}(s_n)} \to +\infty \quad \text{and} \quad \gamma s_n - \tilde{y}^{(n)}(s_n) \to +\infty \quad \text{when } n \to +\infty.$$

According to the above mentioned bound on $\mathcal{D}^{(n)}(s)$, (45) shows that, for any $L > 0$,

$$\big\| z \mapsto v_s^{(n)}\big(\tilde{y}^{(n)}(s_n) + z, s_n\big)\big\|_{L^2([-L;L])} \to 0 \quad \text{when } n \to +\infty. \tag{46}$$

We have

$$v_s^{(n)}\big(\tilde{y}^{(n)}(s_n) + z, s_n\big) = u_t\big(\tilde{x}(t_n + s_n) + z, t_n + s_n\big) + cu_x\big(\tilde{x}(t_n + s_n) + z, t_n + s_n\big), \quad z \in \mathbb{R}.$$

For $n \in \mathbb{N}$, let us define the functions w_n and $\hat{w}_n$ by

$$w_n(z) = u\big(\tilde{x}(t_n + s_n) + z, t_n + s_n\big), \quad \hat{w}_n(z) = u_t\big(\tilde{x}(t_n + s_n) + z, t_n + s_n\big), \quad z \in \mathbb{R}.$$

By compactness (see Section 2), there exists $w_\infty \in H^2_{\mathrm{ul}}(\mathbb{R})$ and $\hat{w}_\infty \in L^2_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, we have, for any $L > 0$, $w_n \to w_\infty$ in $H^2([-L;L])$ and $\hat{w}_n \to \hat{w}_\infty$ in $L^2([-L;L])$ when $n \to +\infty$.

Assertion (46) shows that $\hat{w}_\infty + cw'_\infty = 0$. But observe that the sequences w_n, $\hat{w}_n$, and therefore their limits w_∞, $\hat{w}_\infty$, do not depend on c, and thus, that the identity $\hat{w}_\infty + cw'_\infty = 0$ holds not only for one value of c, but for a whole interval of values (actually for any c satisfying (43)). As a consequence, we get $\hat{w}_\infty = 0$ and $w'_\infty = 0$, thus w_∞ is constant, and passing to the limit in (1), we get $\nabla V(w_\infty) \equiv 0$. On the other hand, according to (30) we have

$|w_\infty(0)| \leqslant r_0$, and according to (17) this finally yields $w_\infty \equiv 0$. Finally, the fact that $w_\infty \equiv 0$ and $w'_\infty \equiv 0$ is, for n sufficiently large, contradictory to the definition of $\tilde{x}(t_n + s_n)$, and this finishes the proof. □

In the following, the velocity $\tilde{c}_- = \tilde{c}_+ = c^*$ will be denoted by $\tilde{c}$ (recall that, since $u_0 \in \mathcal{A}_{\mathrm{inv}}$, we must have $\tilde{c}_+ > 0$, thus $\tilde{c} > 0$).

7. Relaxation

We keep the notations of the previous section. The aim of this section is to prove the following proposition.

Proposition 5. *For any $L > 0$, we have*

$$\big\| y \mapsto u_t\big(\tilde{x}(t) + y, t\big) + \tilde{c} u_x\big(\tilde{x}(t) + y, t\big) \big\|_{L^2([-L;L])} \to 0 \quad \textit{when } t \to +\infty.$$

Proof. Let us proceed by contradiction and suppose that we can find $L_0 > 0$, $\varepsilon_0 > 0$, and a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$ when $n \to +\infty$, such that, for any $n \in \mathbb{N}$,

$$\big\| y \mapsto u_t\big(\tilde{x}(t_n) + y, t_n\big) + \tilde{c} u_x\big(\tilde{x}(t_n) + y, t_n\big) \big\|_{L^2([-L_0;L_0])} \geqslant \varepsilon_0. \tag{47}$$

Let us write $w_n(y) = u(\tilde{x}(t_n) + y, t_n)$, $n \in \mathbb{N}$, $y \in \mathbb{R}$.

By compactness (see Section 2), one can find $w_\infty \in H^2_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, we have, for any $L > 0$, $w_n \to w_\infty$ in $H^2([-L; L])$.

For any fixed $n \in \mathbb{N}$, we have $t_p - t_n \to +\infty$ when $p \to +\infty$ and

$$\frac{\tilde{x}(t_p) - \tilde{x}(t_n)}{t_p - t_n} \to \tilde{c} \quad \text{when } p \to +\infty.$$

For $n \in \mathbb{N}^*$, let

$$p(n) = \min\left\{ p \in \mathbb{N} \mid t_p - t_n \geqslant n \text{ and } \left| \frac{\tilde{x}(t_p) - \tilde{x}(t_n)}{t_p - t_n} - \tilde{c} \right| \leqslant \frac{1}{n} \right\}$$

(any sequence $n \mapsto p(n)$ satisfying $t_{p(n)} - t_n \to +\infty$ and $\frac{\tilde{x}(t_{p(n)}) - \tilde{x}(t_n)}{t_{p(n)} - t_n} \to \tilde{c}$ when $n \to +\infty$ would be convenient).

For $n \in \mathbb{N}$, let $s_n = t_{p(n)} - t_n$, and let

$$c_n = \frac{\tilde{x}(t_{p(n)}) - \tilde{x}(t_n)}{s_n}.$$

We have $s_n \to +\infty$ and $c_n \to \tilde{c}$ when $n \to +\infty$, and, according to (29), we have $c_n \leqslant c_{\max}$ for all $n \in \mathbb{N}^*$. Let us take $n \in \mathbb{N}^*$ sufficiently large so that

$$\tilde{c}/2 < c_n \leqslant c_{\max} \quad \text{and} \quad (c_n + \beta)(\tilde{c} - c_n) \leqslant \beta\gamma/4. \tag{48}$$

We are going to apply the computations of Section 5 with the following set of parameters (see Fig. 5):

$$t_{\mathrm{init}} = t_n, \quad x_{\mathrm{init}} = \tilde{x}(t_{\mathrm{init}}), \quad c = c_n, \quad \text{and} \quad y_0 \geqslant 0 \quad \text{to be chosen below.}$$

Let us denote by $v^{(n)}(y,s)$, $\varphi^{(n)}(y,s)$, $\Phi^{(n)}(s)$, $\mathcal{D}^{(n)}(s)$, $\psi^{(n)}(y,s)$, $\Psi^{(n)}(s)$, and $\tilde{y}^{(n)}(s)$ the quantities that were defined in Section 5 (with the same notations except the "(n)" exponent), for this set of parameters. Observe that

$$v^{(n)}(y,0) = w_n(y), \quad v^{(n)}(y,s_n) = w_{p(n)}(y), \quad \text{and} \quad \tilde{y}^{(n)}(s_n) = 0.$$

Claim 1. *There exists $\varepsilon_{\mathrm{dissip}} > 0$ such that, for any n sufficiently large, we have, uniformly with respect to $y_0 \geqslant 0$,*

$$\frac{1}{2}\int_0^{s_n} \mathcal{D}^{(n)}(s)\,\mathrm{d}s \geqslant \varepsilon_{\mathrm{dissip}}.$$

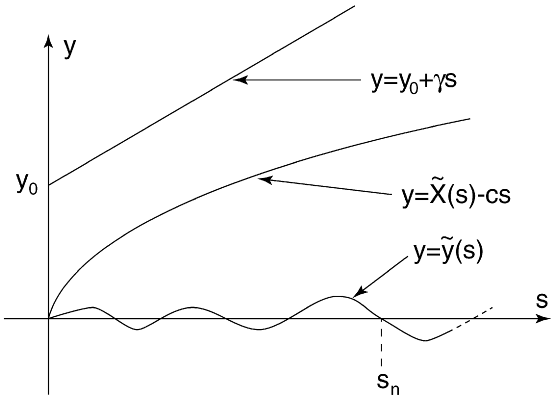

Fig. 5. Proof of Proposition 5.

Remark that the quantity $\mathcal{D}^{(n)}(s)$ is, by definition, increasing with y_0. As a consequence, it is sufficient to prove this claim in the case $y_0 = 0$. Thus, let us assume, just for the proof of this claim, that $y_0 = 0$. If this claim was not true, then there would exist a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, $n_k \to +\infty$ when $k \to +\infty$, such that

$$\int_0^{s_{n_k}} \mathcal{D}^{(n_k)}(s)\,\mathrm{d}s \to 0 \quad \text{when } k \to +\infty.$$

According to (42), this would yield

$$\mathcal{D}^{(n_k)}(s_{n_k}) \to 0 \quad \text{when } k \to +\infty,$$

and, since

$$\mathcal{D}^{(n_k)}(s_{n_k}) \geqslant \int_{-\infty}^{\gamma s_{n_k}} e^{c_{n_k} y} v_s^{(n_k)}(y, s_{n_k})^2\,\mathrm{d}y,$$

this would show that the function $y \mapsto v_s^{(n_k)}(y, s_{n_k})$ converges toward 0 in $L^2([-L; L])$ when $k \to +\infty$, for any $L > 0$. Since

$$v_s^{(n_k)}(y, s_{n_k}) = u_t\big(\tilde{x}(t_{p(n_k)}) + y, t_{p(n_k)}\big) + c_{n_k} u_x\big(\tilde{x}(t_{p(n_k)}) + y, t_{p(n_k)}\big)$$

this would be contradictory to (47). Claim 1 is proved.

The conclusions of Proposition 5 will immediately follow from the four following claims, which are proved thereafter.

Claim 2. *The integral*

$$\int_{\mathbb{R}} e^{\tilde{c}y}\left(\frac{w_\infty'(y)^2}{2} + V\big(w_\infty(y)\big)\right)\mathrm{d}y$$

converges (*let us denote by* $\Phi^{(\infty)}$ *its value*).

Claim 3. *For* n *sufficiently large* (*depending on* y_0), *we have* $\Phi^{(n)}(0) \leqslant \Phi^{(\infty)} + \varepsilon_{\mathrm{dissip}}/4$.

Claim 4. *For* n *sufficiently large, we have, uniformly with respect to* $y_0 \geqslant 0$, $\Phi^{(n)}(s_n) \geqslant \Phi^{(\infty)} - \varepsilon_{\mathrm{dissip}}/4$.

Claim 5. *For* y_0 *sufficiently large, and for* n *sufficiently large* (*depending on* y_0), *we have*

$$K_2 \int_0^{s_n} \Psi^{(n)}(s)\,\mathrm{d}s \leqslant \varepsilon_{\mathrm{dissip}}/4,$$

(*the constant* K_2 *was introduced in inequality* (35)).

It is thus possible to choose y_0 sufficiently large, and n sufficiently large depending on the choice of y_0, in such a way that the conclusions of Claims 1, 3, 4, and 5 be simultaneously satisfied. These conclusions are contradictory to (35), and this proves Proposition 5.

Proof of Claim 2. Let us assume, just for the proof of this claim, that $y_0 = 0$. Then, since $c_n \geqslant \tilde{c}/2$, and according to (11), the quantities $\Phi^{(n)}(0)$ and $\Psi^{(n)}(0)$ are bounded from above, uniformly with respect to n, and, in view of (41), we see that $\int_0^{s_n} \Psi^{(n)}(s)\,\mathrm{d}s$ is bounded from above, uniformly with respect to n. According to (35), this shows that there exists a constant $C > 0$ (independent of n), such that, for n sufficiently large, we have $C \geqslant \Phi^{(n)}(s_n)$. Besides, since $\tilde{y}^{(n)}(s_n) = 0$, we have

$$\Phi^{(n)}(s_n) \geqslant \int_{-\infty}^{\gamma s_n} e^{c_n y}\left(\frac{w'_{p(n)}(y)^2}{2} + V\big(w_{p(n)}(y)\big)\right)\mathrm{d}y,$$

thus, for any $L > 0$, we have, for n sufficiently large (depending on L),

$$C \geqslant \int_{-\infty}^{L} e^{c_n y}\left(\frac{w'_{p(n)}(y)^2}{2} + V\big(w_{p(n)}(y)\big)\right)\mathrm{d}y,$$

and, passing to the limit when $n \to +\infty$,

$$C \geqslant \int_{-\infty}^{L} e^{\tilde{c} y}\left(\frac{w'_{\infty}(y)^2}{2} + V\big(w_{\infty}(y)\big)\right)\mathrm{d}y.$$

Since L is any and C does not depend on L, Claim 2 is proved. $\square$

Proof of Claim 3. Let us recall that

$$\varphi^{(n)}(y,s) = e^{c_n y} \quad \text{for } y \leqslant y_0 + \gamma s \quad \text{and} \quad \varphi^{(n)}(y,s) = e^{-\beta y} e^{(c_n+\beta)(y_0+\gamma s)} \quad \text{for } y \geqslant y_0 + \gamma s.$$

Let us write

$$\varphi^{(\infty)}(y) = e^{\tilde{c} y} \quad \text{for } y \leqslant y_0 \quad \text{and} \quad \varphi^{(\infty)}(y) = e^{-\beta y} e^{(\tilde{c}+\beta) y_0} \quad \text{for } y \geqslant y_0,$$

and, for $y \in \mathbb{R}$,

$$f^{(n)}(y) = \varphi^{(n)}(y,0)\left(\frac{w'_n(y)^2}{2} + V\big(w_n(y)\big)\right),$$
$$f^{(\infty)}(y) = \varphi^{(\infty)}(y)\left(\frac{w'_\infty(y)^2}{2} + V\big(w_\infty(y)\big)\right),$$
$$\bar{f}^{(\infty)}(y) = e^{\tilde{c} y}\left(\frac{w'_\infty(y)^2}{2} + V\big(w_\infty(y)\big)\right).$$

We have

$$\Phi^{(n)}(0) = \int_{\mathbb{R}} f^{(n)}(y)\,\mathrm{d}y, \quad \Phi^{(\infty)} = \int_{\mathbb{R}} \bar{f}^{(\infty)}(y)\,\mathrm{d}y \geqslant \int_{\mathbb{R}} f^{(\infty)}(y)\,\mathrm{d}y$$

(indeed, according to (30), we have $|w_\infty(y)| \leqslant r_0$ and thus $V(w_\infty(y)) \geqslant 0$ for $y \geqslant 0$).

Since $c_n \geqslant \tilde{c}/2$, we see that, for any fixed $y_0 \geqslant 0$, the functions $\varphi^{(n)}(y,0)$ converge exponentially toward 0 when $y \to \pm\infty$, and this convergence is uniform with respect to n. Thus, for any fixed $y_0 \geqslant 0$, we have

$$f^{(n)}(.) \to f^{(\infty)}(.) \quad \text{in } L^1(\mathbb{R}) \text{ when } n \to +\infty,$$

and Claim 3 follows. $\square$

Proof of Claim 4. Let

$$g^{(n)}(y) = \varphi^{(n)}(y, s_n)\left(\frac{w'_{p(n)}(y)^2}{2} + V\big(w_{p(n)}(y)\big)\right).$$

We have

$$\Phi^{(n)}(s_n) = \int_{\mathbb{R}} g^{(n)}(y)\,\mathrm{d}y.$$

Take $L > 0$ such that

$$\int_{-\infty}^{L} \bar{f}^{(\infty)}(y)\,\mathrm{d}y \geqslant \Phi^{(\infty)} - \frac{\varepsilon_{\text{dissip}}}{8}.$$

Since $\varphi^{(n)}(y, s_n) = e^{c_n y}$ for $y \leqslant y_0 + \gamma s_n$, thus in particular for $y \leqslant \gamma s_n$, and since $c_n \geqslant \tilde{c}/2$, we see that

$$g^{(n)}(.) \to \bar{f}^{(\infty)}(.) \quad \text{in } L^1\big(]{-\infty}; L]\big) \text{ when } n \to +\infty, \text{ uniformly with respect to } y_0 \geqslant 0.$$

Since $\Phi^{(n)}(s_n) \leqslant \int_{-\infty}^{L} g^{(n)}(y)\,\mathrm{d}y$ (indeed, according to (30), we have $|w_{p(n)}(y)| \leqslant r_0$ and thus $V(w_{p(n)}(y)) \geqslant 0$ for $y \geqslant 0$), this proves Claim 4. □

Proof of Claim 5. Since $x_{\text{init}} = \tilde{x}(t_{\text{init}})$ and according to (48), estimate (41) holds, and according to this estimate, we have

$$K_2 \int_0^{s_n} \Psi^{(n)}(s)\,\mathrm{d}s \leqslant K_2 \varepsilon_3^{-1} \Psi^{(n)}(0) + K_2 K_3 e^{-\beta y_0}.$$

For y_0 sufficiently large, we have

$$K_2 K_3 e^{-\beta y_0} \leqslant \frac{\varepsilon_{\text{dissip}}}{8}.$$

We are going to prove that, for y_0 sufficiently large, and for n sufficiently large (depending on y_0), we have

$$\Psi^{(n)}(0) \leqslant \varepsilon_4, \quad \text{where } \varepsilon_4 = \frac{\varepsilon_3 \varepsilon_{\text{dissip}}}{8K_2},$$

and this will prove Claim 5. Recall that

$$\psi^{(n)}(y, 0) = e^{(c_n+\beta)y} e^{-\beta y_0} \quad \text{for } y \leqslant y_0 \quad \text{and} \quad \psi^{(n)}(y, 0) = e^{-\beta y} e^{(c_n+\beta)y_0} \quad \text{for } y \geqslant y_0.$$

Let us write

$$\psi^{(\infty)}(y) = e^{(\tilde{c}+\beta)y} e^{-\beta y_0} \quad \text{for } y \leqslant y_0 \quad \text{and} \quad \psi^{(\infty)}(y) = e^{-\beta y} e^{(\tilde{c}+\beta)y_0} \quad \text{for } y \geqslant y_0,$$

and, for $y \in \mathbb{R}$,

$$h^{(n)}(y) = \psi^{(n)}(y, 0)\left(\alpha\left(\frac{w'_n(y)^2}{2} + V\big(w_n(y)\big)\right) + \frac{w_n(y)^2}{2}\right),$$

$$h^{(\infty)}(y) = \psi^{(\infty)}(y)\left(\alpha\left(\frac{w'_\infty(y)^2}{2} + V\big(w_\infty(y)\big)\right) + \frac{w_\infty(y)^2}{2}\right),$$

and

$$\bar{h}^{(\infty)}(y) = e^{\tilde{c}y}\left(\alpha\left(\frac{w'_\infty(y)^2}{2} + V\big(w_\infty(y)\big)\right) + \frac{w_\infty(y)^2}{2}\right).$$

According to the choice of α, these three functions are nonnegative. We have

$$\Psi^{(n)}(0) = \int_{\mathbb{R}} h^{(n)}(y)\,\mathrm{d}y.$$

Let

$$\Psi^{(\infty)} = \int_{\mathbb{R}} h^{(\infty)}(y)\,\mathrm{d}y, \qquad \bar{\Psi}^{(\infty)} = \int_{\mathbb{R}} \bar{h}^{(\infty)}(y)\,\mathrm{d}y.$$

According to Claim 2, the integral $\int_{\mathbb{R}} \bar{h}^{(\infty)}(y)\,\mathrm{d}y$ converges (indeed, for $y \geqslant 0$, we have $|w_\infty(y)| \leqslant r_0$, and thus, according to (17), $w_\infty(y)^2$ and $V(w_\infty(y))$ are of the same order of magnitude).

We have $h^{(\infty)} \leqslant \bar{h}^{(\infty)}$, and, for $y \leqslant y_0/2$, we have $h^{(\infty)}(y) \leqslant e^{-\beta y_0/2}\bar{h}^{(\infty)}(y)$. Thus,

$$\Psi^{(\infty)} \leqslant e^{-\beta y_0/2}\bar{\Psi}^{(\infty)} + \int_{y_0/2}^{+\infty} \bar{h}^{(\infty)}(y)\,\mathrm{d}y.$$

This shows that, for y_0 sufficiently large, we have

$$\Psi^{(\infty)} \leqslant \varepsilon_4/2.$$

Since $c_n \geqslant \tilde{c}/2$, we see that, for any fixed $y_0 \geqslant 0$, the functions $\psi^{(n)}(y,0)$ converge exponentially toward 0 when $y \to \pm\infty$, and this convergence is uniform with respect to n. Thus, for any fixed $y_0 \geqslant 0$, we have

$$h^{(n)}(.) \to h^{(\infty)}(.) \quad \text{in } L^1(\mathbb{R}) \text{ when } n \to +\infty,$$

and this shows that, for y_0 sufficiently large, and for n sufficiently large depending on y_0, we have $\Psi^{(n)}(0) \leqslant \varepsilon_4$, which is the desired bound. This finishes the proof of Claim 5, and thus of Proposition 5. □

8. Convergence

We keep the notations of the previous section. For $t \geqslant 0$ let $\bar{x}(t) \in [-\infty; +\infty]$ denote the supremum of the set

$$\{x \in \mathbb{R} \mid |u(x,t)| > r_0\}$$

(with the convention that $\bar{x}(t) = -\infty$ if this set is empty). According to (30) we have $\bar{x}(t) \leqslant \tilde{x}(t)$, and thus $\bar{x}(t) < +\infty$, for all $t \geqslant 0$.

The conclusions of Theorem 1 will follow naturally from Proposition 5, through the four following lemmas.

Lemma 2. *For t sufficiently large $\tilde{x}(t) - \bar{x}(t)$ is bounded from above.*

Proof. Suppose by contradiction that the converse is true, i.e. that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $t_n \to +\infty$ and $\tilde{x}(t_n) - \bar{x}(t_n) \to +\infty$ when $n \to +\infty$. For $n \in \mathbb{N}$, let us write

$$w_n(y) = u(\tilde{x}(t_n) + y, t_n), \quad \hat{w}_n(y) = u_t(\tilde{x}(t_n) + y, t_n), \quad y \in \mathbb{R}.$$

By compactness (Section 2), there exists $w_\infty \in H^2_{\mathrm{ul}}(\mathbb{R})$ and $\hat{w}_\infty \in L^2_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, for any $L > 0$,

$$w_n \to w_\infty \quad \text{in } H^2([-L;L]) \quad \text{and} \quad \hat{w}_n \to \hat{w}_\infty \quad \text{in } L^2([-L;L]).$$

According to Proposition 5, we have $\hat{w}_\infty + \tilde{c}w'_\infty = 0$, and passing to the limit in (1), we see that w_∞ is a solution of

$$w'' + \tilde{c}w' - \nabla V(w) = 0. \tag{49}$$

Moreover, according to the definition of $\bar{x}(.)$, we have $|w_\infty(y)| \leqslant r_0$ for all $y \in \mathbb{R}$. According to Lemma 9 in Section A.3, this yields $w_\infty \equiv 0$, which is contradictory to the definition of $\tilde{x}(.)$. □

We use the notations $S(r_0)$, $\mathcal{W}_{\tilde{c}}$, $\mathcal{W}^{\mathrm{b}}_{\tilde{c}}$, and $\phi_{\tilde{c},\nu}(.)$ (for $\nu \in \mathbb{R}^{2n}$) of the introduction. According to the above lemma, we have, for t sufficiently large, $-\infty < \bar{x}(t)$, and, by definition of $\bar{x}(t)$, $|u(\bar{x}(t),t)| \equiv r_0$ (in other words $u(\bar{x}(t),t) \in S(r_0)$).

According to Proposition 3, the set $\mathcal{W}_{\tilde{c}}$ is the graph of a map: $S(r_0) \to \mathbb{R}^n$, of class $\mathcal{C}^1$. Let us denote by $f_{\tilde{c}}$ this map. For t large enough, let

$$\nu(t) = \big(u(\bar{x}(t),t), f_{\tilde{c}}(u(\bar{x}(t),t))\big) \in \mathcal{W}_{\tilde{c}} \tag{50}$$

(this choice is convenient, but, as mentioned in introduction, not the only possible one).

Lemma 3. *We have*

$$\operatorname{dist}\big(\nu(t), \mathcal{W}_{\tilde c}^{\mathrm{b}}\big) \to 0 \quad \textit{when } t \to +\infty,$$

and, for any $L > 0$

$$\sup_{y\in[-L;L]} \big|u\big(\bar x(t) + y, t\big) - \phi_{\tilde c,\nu(t)}(y)\big| \to 0 \quad \textit{when } t \to +\infty.$$

Proof. Take any sequence $t_n \to +\infty$, and, for $n \in \mathbb{N}$, let us write

$$w_n(y) = u\big(\bar x(t_n) + y, t_n\big), \quad y \in \mathbb{R}.$$

Proceeding as in the proof of the previous lemma, we see that there exists a solution w_∞ of (49) such that, up to extracting a subsequence, for any $L > 0$, $w_n \to w_\infty$ in $H^2([-L; L])$. Moreover, according to (28), we have $\sup_{y\in\mathbb{R}} |w_\infty(y)| \leqslant R_0$, and, according to the definition of $\bar x(.)$, we have $|w_\infty(0)| = r_0$ and $|w_\infty(y)| \leqslant r_0$ for $y \geqslant 0$. This shows that, if we write $\nu_\infty = (w_\infty(0), w'_\infty(0))$, then we have

$$\nu_\infty \in \mathcal{W}_{\tilde c}^{\mathrm{b}}, \quad w'_\infty(0) = f_{\tilde c}\big(w_\infty(0)\big), \quad w_\infty = \phi_{\tilde c,\nu_\infty}, \tag{51}$$

and of course $\nu(t_n) \to \nu_\infty$ when $n \to +\infty$.

The proof by contradiction of the lemma follows from these observations. Indeed, if the first assertion did not hold, there would exist $\varepsilon_0 > 0$ and a sequence $t_n \to +\infty$ such that $\operatorname{dist}(\nu(t_n), \mathcal{W}_{\tilde c}^{\mathrm{b}}) \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$, and in view of the above conclusions this is impossible.

Similarly, if the second assertion did not hold, there would exist $\varepsilon_0 > 0$, $L_0 > 0$, and a sequence $t_n \to +\infty$ such that, for all $n \in \mathbb{N}$,

$$\sup_{y\in[-L_0;L_0]} \big|u\big(\bar x(t_n) + y, t_n\big) - \phi_{\tilde c,\nu(t_n)}(y)\big| \geqslant \varepsilon_0.$$

After extracting a subsequence as above, this would yield

$$\sup_{y\in[-L_0;L_0]} \big|w_\infty(y) - \phi_{\tilde c,\nu_\infty}(y)\big| \geqslant \varepsilon_0$$

which is contradictory to (51). □

Lemma 4. *For t sufficiently large the function $t \mapsto \bar x(t)$ and the map $t \mapsto \nu(t)$ are of class $\mathcal{C}^1$ and we have $\bar x'(t) \to \tilde c$ and $\nu'(t) \to 0$ when $t \to +\infty$.*

Proof. According to (13), we see that the conclusions of Lemma 3 yield

$$u\big(\bar x(t), t\big) \cdot u_x\big(\bar x(t), t\big) - \phi_{\tilde c,\nu(t)}(0) \cdot \phi'_{\tilde c,\nu(t)}(0) \to 0 \quad \text{when } t \to +\infty.$$

According to Lemma 9 (Section A.3) and since $\mathcal{W}_c$ is compact (Proposition 3), this shows that there exists $\varepsilon > 0$ such that, for t sufficiently large, we have

$$u\big(\bar x(t), t\big) \cdot u_x\big(\bar x(t), t\big) \leqslant -\varepsilon. \tag{52}$$

Let us write $G(x, t) = (u(x, t)^2 - r_0^2)/2$, $x \in \mathbb{R}$, $t \geqslant 0$. According to the regularity of the solution (Section 2), the so-defined function G is of class $\mathcal{C}^1$ with respect to x and t, and we have $G(\bar x(t), t) \equiv 0$ and $\partial G/\partial x(\bar x(t), t) = u(\bar x(t), t) \cdot u_x(\bar x(t), t)$. Thus, according to (52) and to the implicit function theorem, for t sufficiently large the function $t \mapsto \bar x(t)$ is of class $\mathcal{C}^1$ and we have

$$\bar x'(t) = -\frac{u(\bar x(t), t) \cdot u_t(\bar x(t), t)}{u(\bar x(t), t) \cdot u_x(\bar x(t), t)}.$$

On the other hand, according to Proposition 5 and to the regularity properties (13) and (14), we must have $u_t(\bar x(t), t) + \tilde c u_x(\bar x(t), t) \to 0$ when $t \to +\infty$, and this shows that

$$\bar x'(t) \to \tilde c \quad \text{when} \quad t \to +\infty.$$

As a consequence, the function $t \mapsto u(\bar x(t), t)$ is, for t sufficiently large, also of class $\mathcal{C}^1$, and its derivative goes to 0 when $t \to +\infty$. In view of the expression (50) of $\nu(t)$, and since the map $f_{\tilde c}$ is of class $\mathcal{C}^1$, the same properties hold for the function $t \mapsto \nu(t)$, and this finishes the proof. □

Lemma 5. *For any $L > 0$, we have*

$$\sup_{y\in[-L;+\infty[} \left|u\big(\bar{x}(t)+y,t\big)-\phi_{\tilde{c},\nu(t)}(y)\right| \to 0 \quad \textit{when } t \to +\infty.$$

Proof. Let us proceed by contradiction and suppose that there exists $\varepsilon > 0$ and $L_0 > 0$ and a sequence $t_n \to +\infty$ such that, for any $n \in \mathbb{N}$,

$$\sup_{y\in[-L_0;+\infty[} \left|u\big(\bar{x}(t_n)+y,t_n\big)-\phi_{\tilde{c},\nu(t_n)}(y)\right| \geqslant \varepsilon.$$

On the other hand, according to Lemma 3, for any $L > 0$,

$$\sup_{y\in[-L;L]} \left|u\big(\bar{x}(t_n)+y,t_n\big)-\phi_{\tilde{c},\nu(t_n)}(y)\right| \to 0 \quad \text{when } n \to +\infty.$$

This shows that, for any $L \geqslant L_0$ and for n sufficiently large (depending on L),

$$\sup_{y\in]L;+\infty[} \left|u\big(\bar{x}(t_n)+y,t_n\big)-\phi_{\tilde{c},\nu(t_n)}(y)\right| \geqslant \varepsilon. \tag{53}$$

According to Lemmas 9 and 11 in Section A.3, for any $\nu \in \mathcal{W}_{\tilde{c}}$, we have

$$\left|\phi_{\tilde{c},\nu}(y)\right| \to 0 \quad \text{when } y \to +\infty,$$

and this convergence is uniform with respect to $\nu \in \mathcal{W}_{\tilde{c}}$. Thus (53) shows that there exists a sequence $y_n \to +\infty$ such that, for n large enough, $|u(\bar{x}(t_n)+y_n,t_n)| \geqslant \varepsilon/2$.

Using the notations of Section 4, this shows that there exists $\varepsilon' > 0$ such that, for n large enough, $\Psi_0(\bar{x}(t_n)+y_n,t_n) \geqslant \varepsilon'$. Let us write

$$\Psi_{0,n}(s) = \Psi_0\big(\bar{x}(t_n)+y_n,t_n-s\big), \quad 0 \leqslant s \leqslant t_n.$$

Inequality (24) yields

$$\Psi'_{0,n}(s) = -\frac{\partial \Psi_0}{\partial t}\big(\bar{x}(t_n)+y_n,t_n-s\big) \geqslant \varepsilon_1 \Psi_{0,n}(s) - K_1 \int_{S_{\text{far}}(t_n-s)} T_{\bar{x}(t_n)+y_n}\psi_0(x)\,\mathrm{d}x, \quad 0 \leqslant s \leqslant t_n.$$

According to Lemma 4, for n large enough, the function $t \mapsto \bar{x}(t)$ is of class $\mathcal{C}^1$ and satisfies $\bar{x}'(t) > 0$ for $t \in [t_n/2; t_n]$. Thus, for $s \in [0; t_n/2]$, we have $S_{\text{far}}(t_n - s) \subset]-\infty; \bar{x}(t_n)]$, and the last term in the above inequality is arbitrarily small if n is sufficiently large. Finally, for n large enough, we see that, since $\Psi_{0,n}(0) \geqslant \varepsilon'$, $\Psi_{0,n}(.)$ grows exponentially on $[0; t_n/2]$, in particular $\Psi_{0,n}(t_n/2)$ is arbitrarily large if n is large, which is in contradiction with (11), and proves the lemma. □

The proof of Theorem 1 is complete.

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Appendix A

A.1. Attracting ball in $H^1_{\mathrm{ul}}(\mathbb{R})$

Lemma 6. *Assume that V satisfies hypothesis* (H1). *Then there exists a constant $R_1 > 0$, depending only on V, such that, for any $R > 0$, there exists $T(R) > 0$ such that, for any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$ satisfying $\|u_0\|_{H^1_{\mathrm{ul}}(\mathbb{R})} \leqslant R$, the solution $t \mapsto S_t u_0$ of the parabolic system* (1) *with initial data u_0 is defined up to $+\infty$ in time, and, for any $t \geqslant T(R)$, we have*

$$\|S_t u_0\|_{H^1_{\mathrm{ul}}(\mathbb{R})} \leqslant R_1.$$

Proof. Recall that hypothesis (H1) asserts the existence of positive constants ε_V and C_V such that $u \cdot \nabla V(u) \geqslant \varepsilon_V u^2 - C_V$ for all $u \in \mathbb{R}^n$. Remark that, according to inequality (20), this hypothesis immediately shows that the L^2_{ul}-norm of any global solution is eventually bounded by a constant depending only on V. The existence of an attracting ball in $H^1_{\mathrm{ul}}(\mathbb{R})$ requires more care.

Hypothesis (H1) guarantees that V is bounded from below on $\mathbb{R}^n$; let us write $\overline{V}(u) = V(u) - \min_{v\in\mathbb{R}^n} V(v) \geqslant 0$, $u \in \mathbb{R}^n$.

Take any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$, and let $T_{\max} \in \,]0; +\infty]$ denote the upper bound of the maximal time interval where the solution $t \mapsto S_t u_0$ with initial data u_0 is defined. Let $u(x,t) = (S_t u_0)(x)$, $t \in [0; T_{\max}[$, $x \in \mathbb{R}$.

Let $\beta_1 = \min(1, \sqrt{\varepsilon_V})$, and let us write $\psi_1(x) = e^{-\beta_1 |x|}$, and $T_\xi \psi_1(x) = \psi_1(x - \xi)$, $(\xi, x) \in \mathbb{R}^2$. For $0 \leqslant t < T_{\max}$ and $\xi \in \mathbb{R}$, let

$$\Psi_1(\xi,t) = \int_{\mathbb{R}} T_\xi \psi_1(x) \left(\frac{u_x(x,t)^2}{2} + \overline{V}\big(u(x,t)\big) + \frac{u(x,t)^2}{2} \right) \mathrm{d}x,$$

$$H(\xi,t) = \int_{\mathbb{R}} T_\xi \psi_1(x) \big(u_x(x,t)^2 + u(x,t)^2\big)\, \mathrm{d}x.$$

Take any $t \in \,]0; T_{\max}[$. According to (19), (20), and (H1), we have

$$\frac{\partial \Psi_1}{\partial t}(\xi,t) \leqslant -\varepsilon H(\xi,t) + 2C_V/\beta_1, \quad \xi \in \mathbb{R},$$

where $\varepsilon = \min(3/4, \varepsilon_V/2)$ (we have used the fact that $\int_{\mathbb{R}} \psi_1(x)\,\mathrm{d}x = 2/\beta_1$). Let us write $C_5 = \varepsilon^{-1}(1 + 2C_V/\beta_1)$. Thus, if $H(\xi,t) \geqslant C_5$, then $\partial\Psi_1/\partial t(\xi,t) \leqslant -1$.

Let $L > 0$ to be chosen below. There exists a constant $C_6 > 0$, depending on $V(.)$, L, C_5, such that, if $H(\xi,t) \leqslant C_5$, then

$$\int_{\xi-L}^{\xi+L} T_\xi \psi_1 \left(\frac{u_x^2}{2} + \overline{V}(u) + \frac{u^2}{2} \right) \mathrm{d}x \leqslant C_6$$

and thus, at least one of the two following inequalities holds:

$$\int_{-\infty}^{\xi-L} T_\xi \psi_1 \left(\frac{u_x^2}{2} + \overline{V}(u) + \frac{u^2}{2} \right) \mathrm{d}x \geqslant \big(\Psi_1(\xi,t) - C_6\big)/2,$$

$$\int_{\xi+L}^{+\infty} T_\xi \psi_1 \left(\frac{u_x^2}{2} + \overline{V}(u) + \frac{u^2}{2} \right) \mathrm{d}x \geqslant \big(\Psi_1(\xi,t) - C_6\big)/2.$$

Let us suppose for instance that the first of these two last inequalities holds. Observe that $T_{\xi-L}\psi_1 = e^{\beta_1 L} T_\xi \psi_1$ on $]-\infty; x - L]$. Thus, since the expression below the integral is nonnegative, we have

$$\Psi_1(\xi - L, t) \geqslant e^{\beta_1 L}\big(\Psi_1(\xi,t) - C_6\big)/2.$$

Let us choose $L = \beta_1^{-1} \log 3$, thus $e^{\beta_1 L} = 3$. The last inequality shows that, if $\Psi_1(\xi, t) \geqslant 3C_6 + 2$, then $\Psi_1(\xi - L, t) \geqslant \Psi_1(\xi, t) + 1$.

In short, we have shown that, for any $\xi \in \mathbb{R}$ and $t \in [0; T_{\max}[$, if $\Psi_1(\xi, t) \geqslant 3C_6 + 2$, then either $\partial\Psi_1/\partial t(\xi, t) \leqslant -1$ or $\sup_{x\in\mathbb{R}} \Psi_1(x, t) \geqslant \Psi_1(\xi, t) + 1$. This shows that, as long as $\sup_{x\in\mathbb{R}} \Psi_1(x, t)$ is larger than $3C_6 + 2$, this supremum decreases with time (at least at speed 1). In view of the coercivity of Ψ_1, this finishes the proof. □

A.2. A sufficient condition for invasion to occur

The aim of this paragraph is to prove Proposition 2. The idea of this proof is due to Thierry Gallay. First we need the following preliminary lemma (this result was explained to me by J.-F. Burnol, to whom I am grateful for interesting discussions on Tauberian theorems).

Lemma 7. *For any $h \in L^\infty(\mathbb{R})$, if*

$$\int_{-L}^{0} h(x)\,\mathrm{d}x \to -\infty \quad \textit{when } L \to +\infty, \quad \textit{then} \int_{-\infty}^{0} e^{cx} h(x)\,\mathrm{d}x \to -\infty \quad \textit{when } c \to 0^+.$$

Proof. Let

$$H(x) = \int_{x}^{0} h(y)\,\mathrm{d}y, \quad \text{and} \quad I(c) = \int_{-\infty}^{0} e^{cx} h(x)\,\mathrm{d}x.$$

We have $H'(x) = -h(x)$. Integrating by parts, we get

$$I(c) = \int_{-\infty}^{0} c e^{cx} H(x)\,\mathrm{d}x.$$

If $H(x) \leqslant -M < 0$ for $x \leqslant -L < 0$, then we obtain

$$I(c) \leqslant -Me^{-cL} + \left(1 - e^{-cL}\right) \max_{x\in[-L;0]} H(x).$$

The right-hand side converges toward $-M$ when $c \to 0^+$, which proves the result. □

Remark. The same result still holds for a function $h \in L^1_{\mathrm{ul}}(\mathbb{R})$.

Now let us prove Proposition 2. Let us give ourselves $u_0 \in \mathcal{A}$ satisfying the hypothesis of this proposition, and let $u(x, t) = (S_t u_0)(x)$, $t \geqslant 0$, $x \in \mathbb{R}$, denote the solution of (1) with initial data u_0.

Lemma 8. *For any $t \geqslant 0$, we have*

$$\int_{-L}^{0} \left(\frac{u_x(x, t)^2}{2} + V\big(u(x, t)\big)\right) \mathrm{d}x \to -\infty \quad \textit{when } L \to +\infty.$$

Proof. For $L \geqslant 0$, let us consider the function $\varphi_L : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_L(x) = e^{x+L} \quad \text{for } x \leqslant -L, \quad \varphi_L(x) = 1 \quad \text{for } -L \leqslant x \leqslant 0, \quad \text{and} \quad \varphi_L(x) = e^{-x} \quad \text{for } 0 \leqslant x.$$

Let

$$\Phi_L(t) = \int_{\mathbb{R}} \varphi_L(x) \left(\frac{u_x(x, t)^2}{2} + V\big(u(x, t)\big)\right) \mathrm{d}x, \quad t \geqslant 0.$$

According to (19), we have

$$\Phi_L'(t) \leqslant \int_{]-\infty;-L]\cup[L;+\infty[} \varphi_L(x)\frac{u_x(x,t)^2}{4}\,\mathrm{d}x, \quad t>0.$$

Thus, according to (11), we see that $\Phi_L'(t)$ is bounded from above, uniformly with respect to $t \geqslant 0$ and $L \geqslant 0$. The result follows. □

We pursue the proof of Proposition 2. For $t \geqslant 0$, let us define $\bar{x}(t) \in [-\infty;+\infty]$ as in Section 8. Since $u_0 \in \mathcal{A}$, we can suppose, up to changing the origin of time, that $\bar{x}(t) < +\infty$ for all $t \geqslant 0$, and, according to (15), we can suppose that

$$\left|u(x,t)\right| \leqslant R_0, \quad x \in \mathbb{R},\ t \geqslant 0.$$

According to the above lemmas, we have

$$\int_{-\infty}^{0} e^{cx}\left(\frac{u_x(x,0)^2}{2} + V\bigl(u(x,0)\bigr)\right)\mathrm{d}x \to -\infty \quad \text{when } c \to 0^+. \tag{54}$$

In order to prove the proposition, it is sufficient to prove that $\limsup_{t\to+\infty} \bar{x}(t)/t > 0$. Let us proceed by contradiction and suppose the converse. Let us consider the following set of parameters:

$$t_{\text{init}} = 0, \quad x_{\text{init}} = 0, \quad 0 < c \leqslant c_{\max} \text{ (to be chosen later)}, \quad \text{and} \quad y_0 = 0.$$

As in Section 5, we can define quantities $\Phi(s)$, $\Psi(s)$, and $\Theta(s)$, corresponding to this set of parameters. Let $\bar{y}(s) = \bar{x}(s) - cs$, $s \geqslant 0$. We have $\bar{y}(s) \leqslant \bar{x}(s)$ and thus (compare to (38) in Section 5)

$$\Theta(s) \leqslant \beta^{-1}\exp\bigl(-\beta\gamma s + (c+\beta)\bar{x}(s)\bigr)$$

which yields

$$\Theta(s) \leqslant C_7 e^{-\frac{\beta\gamma}{2}s}$$

where $C_7 > 0$ is a constant which does not depend on c, namely

$$C_7 = \beta^{-1}\sup_{s\geqslant 0}\exp\left((c_{\max}+\beta)\bar{x}(s) - \frac{\beta\gamma}{2}s\right).$$

According to (37), and proceeding as in Section 5, this yields

$$\int_0^{+\infty}\Psi(s)\,\mathrm{d}s \leqslant \varepsilon_3^{-1}\Psi(0) + \frac{2C_3C_7}{\varepsilon_3\beta\gamma}. \tag{55}$$

According to (11) and since $y_0 = 0$, $\Psi(0)$ is bounded from above by a constant which does not depend on c, and, in view of (55), the same is true for the quantity $\int_0^{+\infty}\Psi(s)\,\mathrm{d}s$.

Now, on one hand, since $\bar{y}(s) \to -\infty$ when $t \to +\infty$, we see that $\liminf_{s\to+\infty}\Phi(s) \geqslant 0$ (see (44) in the proof of Proposition 4), and on the other hand, (54) shows that $\Phi(0) \to -\infty$ when $c \to 0^+$. This shows that, for $c > 0$ sufficiently close to 0, and for $s > 0$ sufficiently large (depending on c), we have

$$\Phi(s) \geqslant \Phi(0) + K_2\int_0^s \Psi(s)\,\mathrm{d}s,$$

and this is contradictory to (35). Proposition 2 is proved.

A.3. The profiles of traveling wave solutions

The aim of this section is to prove some properties satisfied by the solutions of Eq. (58) governing the profiles of fronts traveling at constant velocity; these properties are used throughout the paper.

We assume that V satisfies (H1) and (H2). Let r_0 be as in introduction. For technical reasons, it is convenient to take and fix $r_0' > r_0$, with similar properties, i.e. such that, for any $v \in \mathbb{R}^n$ satisfying $|v| \leqslant r_0'$, any eigenvalue λ of $D^2V(v)$ satisfies

$$\frac{\lambda_{\min}}{4} \leqslant \lambda. \tag{56}$$

As a consequence, for any $v \in \mathbb{R}^n$ satisfying $|v| \leqslant r_0'$, we have

$$\frac{\lambda_{\min}}{4} v^2 \leqslant v \cdot \nabla V(v). \tag{57}$$

Take any $c > 0$, and let us consider the differential system (governing the profiles of fronts traveling at the velocity c):

$$\phi'' = -c\phi' + \nabla V(\phi). \tag{58}$$

Lemma 9. *Let $x \mapsto \phi(x)$ be any solution of* (58)*, defined on a maximal interval I of the form $]x_-; +\infty[$, $-\infty \leqslant x_- < 0$, and satisfying $|\phi(x)| < r_0'$ for all $x \geqslant 0$, and $\phi(.) \not\equiv 0$. Then we have*

- $\phi(x) \cdot \phi'(x) < 0$ *for all $x \geqslant 0$, and*
- $(\phi(x), \phi'(x)) \to (0, 0)$ *when $x \to +\infty$.*

If moreover $x_- = -\infty$ and $\sup_{x\in\mathbb{R}} |\phi(x)| < +\infty$, then we have

$$\sup_{x\in\mathbb{R}} |\phi(x)| \leqslant \sqrt{2C_V/\varepsilon_V} \tag{59}$$

(*where C_V and ε_V are the constants of hypothesis* (H1)), *and there exists $h < 0$ such that*

$$\operatorname{dist}\big(\phi(x), \Sigma_{\text{crit},h}\big) \to 0 \quad \text{and} \quad \phi'(x) \to 0 \quad \text{when } x \to +\infty,$$

where $\Sigma_{\text{crit},h} = \{v \in \mathbb{R}^n \mid V(v) = h \text{ and } \nabla V(v) = 0\}$ (in particular $\sup_{x\in\mathbb{R}} |\phi(x)| > r_0'$).

Proof. Let $x \mapsto \phi(x)$ be any solution of the differential system (58), defined on a maximal interval I of the form $]x_-; +\infty[$, $-\infty \leqslant x_- < 0$, and satisfying $|\phi(x)| < r_0'$ for all $x \geqslant 0$, and $\phi(.) \not\equiv 0$. Let us write

$$E(x) = \frac{\phi'(x)^2}{2} - V\big(\phi(x)\big), \quad Q(x) = \frac{\phi(x)^2}{2}, \quad x \in I. \tag{60}$$

We have

$$E' = -c\phi'^2 \quad \text{and} \quad Q'' + cQ' = \phi'^2 + \phi \cdot \nabla V(\phi). \tag{61}$$

Thus $E(x)$ is decreasing with x. Since $|\phi(.)| < r_0'$ on $\mathbb{R}_+$, $E(.)$ is bounded from below on $\mathbb{R}_+$, thus it converges toward a finite limit when $x \to +\infty$, and this shows that $\phi'(.)$ is square integrable on $\mathbb{R}_+$. On the other hand, $E(.)$ is bounded from above on $\mathbb{R}_+$ (because it is decreasing), and as a consequence the same is true for $|\phi'(.)|$, and thus for $|\phi''(.)|$. Thus $\phi'(x) \to 0$ when $x \to +\infty$. In view (58) this shows that $\phi(x)$ must converge toward the set of critical points of V, thus (in view of (57)) toward 0, when $x \to +\infty$.

According to (61) and (57), we have

$$Q'' + cQ' \geqslant \frac{\lambda_{\min}}{2} Q \quad \text{on } \mathbb{R}_+. \tag{62}$$

Since $Q(x) \to 0$ and $Q'(x) \to 0$ when $x \to +\infty$, integrating this inequality between any $x \in \mathbb{R}_+$ and $+\infty$ yields

$$-Q'(x) - cQ(x) \geqslant \frac{\lambda_{\min}}{2} \int_x^{+\infty} Q(y)\,\mathrm{d}y,$$

and since $Q \geqslant 0$ and $\phi \not\equiv 0$, this shows that $Q'(x) < 0$, and thus finishes the proof of the first assertion.

Now let us assume that $x_- = -\infty$ and that $\sup_{x\in\mathbb{R}} |\phi(x)| < +\infty$. According to (61) and to hypothesis (H1), we have

$$Q'' + cQ' \geqslant \varepsilon_V Q - C_V + \phi'^2 \quad \text{on } \mathbb{R}. \tag{63}$$

Integrating this equation between any $x \in \mathbb{R}_-$ and 0 yields

$$Q'(0) - Q'(x) + c\big(Q(0) - Q(x)\big) \geqslant \int_x^0 \big(\varepsilon_V Q(y) - C_V + \phi'^2(y)\big)\,\mathrm{d}y. \tag{64}$$

Since $E(x)$ decreases with x and converges toward 0 when $x \to +\infty$, there exists $E_{-\infty} \in [0; +\infty]$ such that $E(x) \to E_{-\infty}$ when $x \to -\infty$ (actually, since $\phi \not\equiv 0$, we must have $E_{-\infty} > 0$).

If we had $E_{-\infty} = +\infty$, then, since V is bounded from below, we would have $|\phi'(x)| \to +\infty$ when $x \to -\infty$, and, in view of (64), and since $Q \geqslant 0$, this would yield $Q'(x) \to -\infty$ when $x \to -\infty$, and thus $Q(x) \to +\infty$ when $x \to -\infty$, which is impossible since $|\phi(.)|$ was supposed to be bounded.

Thus $E_{-\infty} < +\infty$, and since $|\phi(.)|$ is uniformly bounded, this shows that $|\phi'(.)|$ is also uniformly bounded. Thus, in view of (58), $|\phi''(.)|$ is uniformly bounded, and since according to (61) $\phi'(.)$ is square-integrable on $\mathbb{R}$, we must have $\phi'(x) \to 0$ when $x \to -\infty$, and, in view of (58), $\phi(.)$ necessarily converges toward the set of critical points of V belonging to the level set $\{v \in \mathbb{R}^n | V(v) = -E_{-\infty}\}$.

It remains to prove that $Q(x) \leqslant C_V/\varepsilon_V$ for all $x \in \mathbb{R}$. Let us proceed by contradiction and suppose the converse, i.e. that there exists $x_0 \in \mathbb{R}$ such that $Q(x_0) > C_V/\varepsilon_V$. Let us write $q(x) = Q(x_0 + x) - C_V/\varepsilon_V$, $x \in \mathbb{R}$. We have $q(0) > 0$, and, according to (63), we have

$$q'' + cq' \geqslant \varepsilon_V q \quad \text{on } \mathbb{R}. \tag{65}$$

Claim. *If $q'(0) \geqslant 0$ (resp. $q'(0) < 0$) then $q(x) \to +\infty$ when $x \to +\infty$ (resp. when $x \to -\infty$).*

Let us assume first that $q'(0) \geqslant 0$. In this case let $x \mapsto p(x)$ denote the solution of the differential equation $p'' + cp' = \varepsilon_V p$ with initial data $p(0) = q(0)/2$ and $p'(0) = q'(0) \geqslant 0$, and let $r = q - p$. According to (65) we have

$$r'' + cr' \geqslant \varepsilon_V r, \quad \text{on } \mathbb{R}. \tag{66}$$

Clearly $p(x) \to +\infty$ when $x \to +\infty$. We have $r(0) = q(0)/2 > 0$ and $r'(0) = 0$, and we claim that $r(x) > 0$ for all $x \in \mathbb{R}_+$. If the converse was true there would exist $x_0 > 0$ such that $r(x_0) = 0$ and $r(x) > 0$ for $x \in [0; x_0[$. Then, integrating (66) between 0 and x_0, we would get

$$r'(x_0) - cr(0) \geqslant \varepsilon_V \int_0^{x_0} r(y)\,\mathrm{d}y$$

and thus $r'(x_0) > 0$, which is impossible. Thus $r(x) > 0$ for all $x \in \mathbb{R}_+$, and thus $q(x) \to +\infty$ when $x \to +\infty$.

Now let us assume that $q'(0) < 0$. In this case we claim that $q'(x) < q'(0)$ for all $x < 0$. Indeed, according to (65), we have $q''(0) < 0$, thus $q'(x) < q'(0)$ holds at least for $x < 0$ sufficiently close to 0. If this did not hold for all $x < 0$ then there would exist $x_0 < 0$ such that $q'(x_0) = q'(0)$ and $q'(x) < q'(0)$ for all $x \in\,]x_0; 0[$. Then, integrating (65) between x_0 and 0 would yield

$$c\int_{x_0}^0 q'(y)\,\mathrm{d}y \geqslant \varepsilon_V \int_{x_0}^0 q(y)\,\mathrm{d}y > 0,$$

which is impossible. Thus $q(x) \to +\infty$ when $x \to -\infty$, and the claim is proved.

Since this claim is contradictory to the hypothesis $\sup_{x\in\mathbb{R}} |\phi(x)| < +\infty$, the lemma is proved. □

Let $\phi_{c,\nu}(.)$ (for $\nu \in \mathbb{R}^{2n}$), $\mathcal{W}_c$, and $\mathcal{W}_c^{\mathrm{b}}$ be as in introduction. Let $B(r_0') = \{v \in \mathbb{R}^n \mid |v| < r_0'\}$, and let

$$W_c^{\mathrm{s,loc}}(0) = \big\{\nu \in \mathbb{R}^{2n} \mid \phi_{c,\nu}(.) \text{ is defined up to } +\infty \text{ and } \phi_{c,\nu}(x) \in B(r_0') \text{ for all } x \geqslant 0\big\}.$$

Lemma 10. *The set* $W_c^{\mathrm{s,loc}}(0)$ *is the graph of a* $\mathcal{C}^1$*-map* $f_c : B(r'_0) \to \mathbb{R}^n$.

Proof. For $r > 0$, let $B(r) = \{u \in \mathbb{R}^n \mid |u| < r\}$, and let $S(r) = \{u \in \mathbb{R}^n \mid |u| = r\}$.

Linearizing the differential system (58) at $(0, 0)$, we get

$$\phi'' = -c\phi' + D^2V(0)\phi \Leftrightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ D^2V(0) & -c \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{67}$$

To any eigenvector ϕ_0 of $D^2V(0)$, corresponding to an eigenvalue μ, correspond two eigenvectors $(\phi_0, \lambda_\pm\phi_0)$ for (67), corresponding to the two eigenvalues $\lambda_\pm = (-c \pm \sqrt{c^2 + 4\mu})/2$. Since $\mu > 0$, we have $\lambda_- < 0 < \lambda_+$. Thus $(0, 0)$ is a hyperbolic equilibrium of (58), and its stable and unstable manifold both have dimension n. Let us denote by $W_c^{\mathrm{s}}(0)$ and $W_c^{\mathrm{u}}(0)$ these manifolds. Since V is of class $\mathcal{C}^k$, $k \geqslant 3$, then these manifolds are of class (at least) $\mathcal{C}^{k-1}$, thus at least $\mathcal{C}^1$ ([18]).

Claim 1. *The set* $W_c^{\mathrm{s,loc}}(0)$ *is an open subset of* $W_c^{\mathrm{s}}(0)$.

Indeed, according to the local stable manifold theorem, for $\varepsilon > 0$ small enough, the set

$$\left\{\nu \in \mathbb{R}^{2n} \mid \phi_{c,\nu}(.) \text{ is defined up to } +\infty \text{ and satisfies } |\phi_{c,\nu}(x)| < \varepsilon \text{ for all } x \geqslant 0\right\}$$

is an open neighborhood of $(0, 0)$ in $W_c^{\mathrm{s}}(0)$, and Claim 1 immediately follows.

For $\nu \in W_c^{\mathrm{s}}(0)$, let $T_\nu W_c^{\mathrm{s}}(0)$ denote the (n-dimensional) tangent space to $W_c^{\mathrm{s}}(0)$ at ν.

Claim 2. *For all* $\nu \in W_c^{\mathrm{s,loc}}(0)$*, the space* $T_\nu W_c^{\mathrm{s}}(0)$ *is transverse to* $\{0\} \times \mathbb{R}^n$.

Indeed, take any $\nu \in W_c^{\mathrm{s,loc}}(0)$. If $\nu = (0, 0)$, then the conclusion of the claim follows from the above expression of the eigenvectors of (67). Let us assume that $\nu \neq (0, 0)$, let us take any pair $(\phi_0, \phi'_0) \in \mathbb{R}^{2n}$, and let $x \mapsto \phi(x)$ denote the solution of the linear differential system

$$\phi''(x) = -c\phi'(x) + D^2V\big(\phi_{c,\nu}(x)\big)\phi(x) \tag{68}$$

with initial data $(\phi(0), \phi'(0)) = (\phi_0, \phi'_0)$. The pair (ϕ_0, ϕ'_0) belongs to $T_\nu W_c^{\mathrm{s}}(0)$ if and only if $(\phi(x), \phi'(x)) \to (0, 0)$ when $x \to +\infty$. Let us assume that we are in this case, and let us write $Q(x) = \phi(x)^2/2$. Since $|\phi_{c,\nu}(x)| < r'_0$ for all $x \geqslant 0$, (68) yields

$$Q''(x) + cQ'(x) \geqslant \frac{\lambda_{\min}}{2} Q(x), \quad x \geqslant 0.$$

Since $Q(x) \to 0$ and $Q'(x) \to 0$ when $x \to +\infty$, integrating this equation between any $x \in \mathbb{R}_+$ and $+\infty$ yields

$$-Q'(x) - cQ(x) \geqslant \frac{\lambda_{\min}}{2} \int_x^{+\infty} Q(y)\,\mathrm{d}y,$$

and thus, since $Q(.) \geqslant 0$ and $\phi \not\equiv 0$, $Q'(x) < 0$, and this proves the claim.

Let us consider the map

$$\pi_1 : W_c^{\mathrm{s,loc}}(0) \to B(r'_0), \quad (u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto u.$$

According to Claim 2 above, for any $(u, v) \in W_c^{\mathrm{s,loc}}(0)$, π_1 defines a local diffeomorphism between a neighborhood of (u, v) in $W_c^{\mathrm{s,loc}}(0)$ and a neighborhood of u in $B(r'_0)$. In particular, the image set of π_1 is open. It remains to prove that this diffeomorphism is global.

Claim 3. *The map* π_1 *is surjective, i.e.* $\pi_1(W_c^{\mathrm{s,loc}}(0)) = B(r'_0)$.

As mentioned above, π_1 defines a local diffeomorphism between a neighborhood $\mathcal{V}$ of $(0, 0)$ in $W_c^{\mathrm{s,loc}}(0)$ and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$. Let us denote by $f : \mathcal{U} \to \mathcal{V}$ the inverse of this local diffeomorphism, and take $\varepsilon > 0$ small enough so that $S(\varepsilon) \subset \mathcal{U}$.

Take any $u \in S(\varepsilon)$ and let us write $Q_u(x) = \phi_{c,f(u)}(x)^2/2$. According to Lemma 9, we have $Q_u'(0) < 0$, and, according to (62), we see that there exists $x_u(r_0') < 0$ such that the function

$$x \mapsto \left|\phi_{c,f(u)}(x)\right|$$

is strictly decreasing on the interval $[x_u(r_0'); 0]$, and defines a diffeomorphism (of class $\mathcal{C}^{k+1}$ since V is of class $\mathcal{C}^k$) between $[x_u(r_0'); 0]$ and $[\varepsilon; r_0']$. Let $r \mapsto x_u(r)$ denote its the inverse. Then, for any $r \in [\varepsilon; r_0'[$, we have

$$\left(\phi_{c,f(u)}\left(x_u(r)\right), \phi_{c,f(u)}'\left(x_u(r)\right)\right) \in W_c^{\mathrm{s,loc}}(0), \quad \text{thus} \quad \phi_{c,f(u)}\left(x(u,r)\right) \in \pi_1\left(W_c^{\mathrm{s,loc}}(0)\right).$$

Let us consider the one-parameter family $(g_r)_{r\in[\varepsilon;r_0'[}$ of maps: $S(1) \to S(1)$, defined by

$$g_r(v) = \frac{1}{r}\phi_{c,f(\varepsilon v)}\left(x_v(r)\right).$$

We have $g_\varepsilon = \mathrm{Id}_{S(1)}$, so that, for any $r \in [\varepsilon; r_0'[$, the map g_r is isotopic to $\mathrm{Id}_{S(1)}$, thus surjective (because otherwise we could construct a retraction of the $n-1$-dimensional sphere $S(1)$ to a point, and as is well known this is impossible). This shows that $S(r) \in \pi_1(W_c^{\mathrm{s,loc}}(0))$ for all $r \in [\varepsilon; r_0'[$, and thus proves Claim 3.

Thus π_1 defines a covering of $B(r_0')$ by $W_c^{\mathrm{s,loc}}(0)$, and since $W_c^{\mathrm{s,loc}}(0)$ is connected and $B(r_0')$ is simply connected, this covering must be one to one. If we denote by $\bar{f}$ its inverse, and by π_2 the map: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(u, v) \mapsto v$, then we see that $W_c^{\mathrm{s,loc}}(0)$ is the graph of the map $\pi_2 \circ \bar{f} : B(r_0') \to \mathbb{R}^n$. The lemma is proved. □

It immediately follows from this lemma that $\mathcal{W}_c$ is a compact subset of $\mathbb{R}^{2n}$.

Recall that the potential function V is assumed to be of class $\mathcal{C}^k$, $k \geqslant 3$.

Lemma 11. 1) *The solutions of the differential system* (58) *are of class $\mathcal{C}^{k+1}$, and we have*

$$\sup_{\nu\in\mathcal{W}_c} \left\|x \geqslant 0 \mapsto \phi_{c,\nu}(x)\right\|_{\mathcal{C}_{\mathrm{b}}^{k+1}([0;+\infty[)} < +\infty.$$

2) *The convergence*

$$\left|\phi_{c,\nu}(x)\right| \to 0 \quad \textit{when } x \to +\infty,\ \nu \in \mathcal{W}_c$$

is uniform with respect to $\nu \in \mathcal{W}_c$.

3) *The set $\mathcal{W}_c^{\mathrm{b}}$ is compact.*

Proof. Since V is of class $\mathcal{C}^k$, the solutions of (58) are clearly of class $\mathcal{C}^{k+1}$. By definition of $\mathcal{W}_c$ we have $\sup_{\nu\in\mathcal{W}_c} \sup_{x\geqslant 0} |\phi_{c,\nu}(x)| < +\infty$ and, according to Lemma 10, $\sup_{\nu\in\mathcal{W}_c} |\phi_{c,\nu}'(0)| < +\infty$. Thus, if we write $E_{c,\nu}(x) = \phi_{c,\nu}'(x)^2/2 - V(\phi_{c,\nu}(x))$, we have, since these quantities are decreasing with x, $\sup_{\nu\in\mathcal{W}_c} \sup_{x\geqslant 0} E_{c,\nu}(x) < +\infty$, and therefore $\sup_{\nu\in\mathcal{W}_c} \sup_{x\geqslant 0} |\phi_{c,\nu}'(x)| < +\infty$. In view (58), this yields $\sup_{\nu\in\mathcal{W}_c} \sup_{x\geqslant 0} |\phi_{c,\nu}''(x)| < +\infty$, and the proof of assertion 1 follows by differentiation of the system (58) and by induction.

Assertion 2 immediately follows from the compactness of $\mathcal{W}_c$ and from the fact that, according to the first assertion of Lemma 9, for any $\nu \in \mathcal{W}_c$, the function $x \mapsto |\phi_{c,\nu}(x)|$ is (strictly) decreasing on $\mathbb{R}_+$.

Assertion 3 immediately follows from the compactness of $\mathcal{W}_c$ and from the uniform bound (59) stated in Lemma 9. □

In view of these three lemmas, Proposition 3 is proved.

A.4. Transversality arguments

Consider the following hypotheses:

(G1) the set $\{c > 0 \mid \mathcal{W}_c^{\mathrm{b}} \neq \emptyset\}$ is discrete (it has no accumulation point in $\mathbb{R}_+^*$),

(G2) for any $c \in \{c > 0 \mid \mathcal{W}_c^{\mathrm{b}} \neq \emptyset\}$, the set $\mathcal{W}_c^{\mathrm{b}}$ is reduced to a singleton $\{\nu_0\}$, and the corresponding traveling front ϕ_{c,ν_0} is bistable (i.e. $\phi_{c,\nu_0}(x)$ converges toward a stable homogeneous equilibrium when $x \to -\infty$).

These hypotheses hold generically with respect to the potential function V. To formulate more rigorously this statement, let us introduce the following notations.

Let $\mathcal{C}^k_{\mathrm{b}}(\mathbb{R}^n,\mathbb{R})$ denote the Banach space of functions: $\mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^k$, $k \geqslant 3$, which are uniformly bounded together with their derivatives up to order k. Observe that, for any $W \in \mathcal{C}^k_{\mathrm{b}}(\mathbb{R}^n,\mathbb{R})$, the potential function $V(u) = u^2/2 + W(u)$ satisfies hypothesis (H1) (it is quadratic at infinity). Let us consider the affine Banach space

$$\mathcal{V} = \left\{ V \in \mathcal{C}^k\big(\mathbb{R}^n,\mathbb{R}\big) \mid \text{the function } W(u) = V(u) - \frac{u^2}{2} \text{ belongs to } \mathcal{C}^k_{\mathrm{b}}\big(\mathbb{R}^n,\mathbb{R}\big)\right\}$$

equipped with the distance induced by the usual norm on $\mathcal{C}^k_{\mathrm{b}}(\mathbb{R}^n,\mathbb{R})$ and with the topology induced by this distance. Let

$$\tilde{\mathcal{V}} = \big\{ V \in \mathcal{V} \mid \nabla V(0) = 0 \text{ and } D^2V(0) > 0 \big\}$$

(the subset of $\mathcal{V}$ made of potentials satisfying hypothesis (H2)). This set is a Baire space (for the above mentioned topology) i.e. any countable intersection of dense open subsets of $\tilde{\mathcal{V}}$ is still dense in $\tilde{\mathcal{V}}$ (such an intersection is called a *residual* subset of $\tilde{\mathcal{V}}$).

The following result is based on classical transversality arguments (Sard–Smale theorem), we refer to [26] for its proof.

Theorem 3. *The set $\tilde{\mathcal{V}}_G$ is residual in $\tilde{\mathcal{V}}$.*

In other words hypotheses (G1) and (G2) hold generically with respect to V (observe that, according to the a priori bound (59) on traveling fronts, the constraint on the "quadraticity at infinity" of the potentials considered in this theorem – which enables to parametrize them by a Banach space – does not weaken this statement). As a consequence, the weaker hypothesis (G) stated in introduction (see remark (a) following the statement of Theorem 1) also holds generically.

The next counting arguments provide a rough justification of Theorem 3.

Let $\Sigma_{\mathrm{crit}} = \{u \in \mathbb{R}^n \mid \nabla V(u) = 0\}$. Take any $c > 0$ and $u_0 \in \Sigma_{\mathrm{crit}}$. Linearizing the differential system (58) at $(u_0, 0)$ gives

$$\phi'' = -c\phi' + D\phi \Leftrightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ D & -c \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{69}$$

where $D = D^2V(u_0)$. To any eigenvector ϕ_0 of D, corresponding to an eigenvalue μ, correspond two eigenvectors $(\phi_0, \lambda_\pm\phi_0)$ for (69), corresponding to the two eigenvalues $\lambda_\pm = (-c \pm \sqrt{c^2+4\mu})/2$ (or, if $c^2 + 4\mu = 0$, a Jordan Block $\mathrm{Vect}\{(\phi_0, -(c/2)\phi_0), (0,\phi_0)\}$ corresponding to the double eigenvalue $-c/2$). In particular,

- if $\mu > 0$, then $\lambda_+ \in \mathbb{R}^*_+$ and $\lambda_- \in \mathbb{R}^*_-$,
- if $\mu < 0$ and $c > 0$, then $\lambda_\pm$ (which can be real or complex conjugate) have strictly negative real parts.

Let us assume that the Hessian $D^2V(u_0)$ is not degenerate (i.e. 0 is not an eigenvalue of $D^2V(u_0)$). Let us denote by $i(u_0)$ the number of negative eigenvalues of $D^2V(u_0)$, and let $W^{\mathrm{s}}_c(u_0)$ and $W^{\mathrm{u}}_c(u_0)$ denote the stable and the unstable manifolds of the equilibrium $(u_0, 0)$ for the flow (in $\mathbb{R}^{2n}$) of (58). In view of the above computations we have

$$\dim W^{\mathrm{s}}_c(u_0) = n + i(u_0) \quad \text{and} \quad \dim W^{\mathrm{u}}_c(u_0) = n - i(u_0).$$

Let us assume that $u_0 \neq 0$, and let us consider the intersection $I(c) = W^{\mathrm{u}}_c(u_0) \cap W^{\mathrm{s}}_c(0)$. The generic dimension of the intersection of $I(c)$ with an hyper-surface transverse to the flow is:

$$\big(n - i(u_0) - 1\big) + (n-1) - (2n-1) = -i(u_0) - 1.$$

As a consequence the generic dimension of the intersection of $\bigcup_{c>0} I(c)$ with an hyper-surface transverse to the flow is $-i(u_0)$. Generically, this intersection should thus be empty if $i(u_0) > 0$, and made of isolated points if $i(u_0) = 0$.

A.5. Lower semi-continuity of the velocity $c[u_0]$

The aim of this paragraph is to prove Theorem 2, which states that the function: $\mathcal{A} \to \mathbb{R}$, $u_0 \mapsto c[u_0]$ (with $c[u_0] = 0$ if $u_0 \in \mathcal{A} \setminus \mathcal{A}_{\mathrm{inv}}$) is lower semi-continuous.

The proof is rather similar to that of Proposition 4 in Section 6. The idea is, once again, to proceed by contradiction, and to exhibit "large excursions to the right followed by returns" for the invasion point of certain solutions (see the proof of Proposition 4 for more explanatory comments).

Thus let us proceed by contradiction and suppose that the above function is not lower semi-continuous. Then there exists $u_{\infty,0} \in \mathcal{A}$ and a sequence $(u_{n,0})_{n\in\mathbb{N}}$, $u_{n,0} \in \mathcal{A}$, such that $\|u_{n,0} - u_{\infty,0}\|_{H^1_{\mathrm{ul}}(\mathbb{R})} \to 0$ when $n \to +\infty$ and

$$\limsup_{n\to+\infty} c[u_{n,0}] < c[u_{\infty,0}].$$

Let us write $\bar{c}_\infty = c[u_{\infty,0}]$. Since (by definition of $c[.]$) $c[u_{n,0}] \geqslant 0$ for all $n \in \mathbb{N}$, we necessarily have $\bar{c}_\infty > 0$ (in other words $u_{\infty,0} \in \mathcal{A}_{\mathrm{inv}}$). Up to extracting a subsequence, we can suppose that there exists $c_\infty \in [0; \bar{c}_\infty[$ such that

$$c[u_{n,0}] \to c_\infty \quad \text{when } n \to +\infty. \tag{70}$$

For $n \in \mathbb{N} \cup \{\infty\}$, let $u_n(x,t) = (S_t u_{n,0})(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ denote the solution of the parabolic system (1) with initial data $u_{n,0}$. According to Theorem 1, there exists a function $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \bar{x}_\infty(t)$ and a map $\mathbb{R}_+ \to \mathcal{W}_{\bar{c}_\infty}$, $t \mapsto \nu_\infty(t)$, both of class $\mathcal{C}^1$, such that the following statements hold:

$$\bar{x}'_\infty(t) \to \bar{c}_\infty, \quad \nu'_\infty(t) \to 0, \quad \text{and} \quad \mathrm{dist}\big(\nu_\infty(t), \mathcal{W}^{\mathrm{b}}_{\bar{c}_\infty}\big) \to 0 \quad \text{when} \quad t \to +\infty, \tag{71}$$

and, for any $L \geqslant 0$,

$$\sup_{y\in[-L;+\infty[} \big|u_\infty\big(\bar{x}_\infty(t) + y, t\big) - \phi_{\bar{c}_\infty,\nu_\infty(t)}(y)\big| \to 0 \quad \text{when } t \to +\infty. \tag{72}$$

Take any μ in the limit set $\mathcal{L}(u_{\infty,0}) = \bigcap_{t\geqslant 0} \overline{\nu_\infty([t;+\infty[)} \subset \mathcal{W}^{\mathrm{b}}_{\bar{c}_\infty}$, and take any sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \geqslant 0$, $t_k \to +\infty$ when $k \to +\infty$, such that $\nu_\infty(t_k) \to \mu$ when $k \to +\infty$.

According to (71), (72), (11), and assertion 2 of Lemma 11 (the convergence $\phi_{\bar{c}_\infty,\nu}(x) \to 0$ when $x \to +\infty$ is uniform with respect to $\nu \in \mathcal{W}_{\bar{c}_\infty}$), for any $L \geqslant 0$, the function

$$t \mapsto \sup_{y\in[-L;+\infty[} \big|u_\infty\big(\bar{x}_\infty(t_k) + \bar{c}_\infty t + y, t_k + t\big) - \phi_{\bar{c}_\infty,\mu}(y)\big|$$

converges toward 0 when $k \to +\infty$, uniformly on compact subsets of $\mathbb{R}_+$.

According to the continuity of the semi-flow in $H^1_{\mathrm{ul}}(\mathbb{R})$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers, $n_k \to +\infty$ when $k \to +\infty$, such that

$$\big\|x \mapsto u_{n_k}(x, t_k) - u_\infty(x, t_k)\big\|_{H^1_{\mathrm{ul}}(\mathbb{R})} \to 0 \quad \text{when } k \to +\infty.$$

Let us write

$$u_{1,k}(x,t) = u_{n_k}\big(\bar{x}_\infty(t_k) + x, t_k + t\big), \quad k \in \mathbb{N},\ x \in \mathbb{R},\ t \geqslant 0.$$

The two last assertions show that, for any $L \geqslant 0$, the function

$$t \mapsto \sup_{y\in[-L;+\infty[} \big|u_{1,k}(\bar{c}_\infty t + y, t) - \phi_{\bar{c}_\infty,\mu}(y)\big| \tag{73}$$

converges toward 0 when $k \to +\infty$, uniformly on compact subsets of $\mathbb{R}_+$.

For $k \in \mathbb{N}$ and $t \geqslant 0$, let $\bar{x}_k(t) \in [-\infty; +\infty]$ denote the supremum of the set $\{x \in \mathbb{R} \mid |u_{1,k}(x,t)| > r_0\}$ (with the convention that $\bar{x}_k(t) = -\infty$ if this set is empty). According to Lemma 9, we see from (73) that the function

$$t \mapsto \bar{x}_k(t) - \bar{c}_\infty t \tag{74}$$

converges toward 0 when $k \to +\infty$, uniformly on compact subsets of $\mathbb{R}_+$.

For $k \in \mathbb{N}$, let

$$c_k = c[u_{n_k,0}], \qquad c^*_k = \sup_{t\geqslant 1} \frac{\bar{x}_k(t)}{t}.$$

According to Theorem 1 we have

$$\frac{\bar{x}_k(t)}{t} \to c_k \quad \text{when } t \to +\infty, \tag{75}$$

and, in view of (74), we have

$$\liminf_{k\to+\infty} c_k^* \geqslant \bar{c}_\infty. \tag{76}$$

Claim 1. *We have* $\limsup_{k\to+\infty} c_k^* \leqslant c_{\max}$.

For $k \in \mathbb{N}$, let us define a function $\mathbb{R}_+ \to [-\infty; +\infty]$, $t \mapsto \tilde{x}_k(t)$, corresponding to the solution $(x,t) \mapsto u_{1,k}(x,t)$ of the system (1), as the function $t \mapsto \tilde{x}(t)$ was defined in Section 4, for the solution $(x,t) \mapsto u(x,t)$ considered there. We have $\bar{x}_k(t) \leqslant \tilde{x}_k(t)$, $t \geqslant 0$.

According to (73), we have $\limsup_{k\to+\infty} \tilde{x}_k(0) < +\infty$, and the results of Sections 3 and 4 apply to this function $t \mapsto \tilde{x}_k(t)$. According to (15), for k sufficiently large, we have, for all $t \geqslant 0$,

$$\sup_{x\in\mathbb{R}} \left|u_{1,k}(x,t)\right| \leqslant R_0, \tag{77}$$

and in this case, according to (29), we have

$$\tilde{x}_k(t) \leqslant \tilde{x}_k(0) + c_{\max} t, \quad t \geqslant 0.$$

Take any $\varepsilon > 0$. According to these assertions, there exists $T > 0$ such that, for any k sufficiently large,

$$\sup_{t\geqslant T} \frac{\bar{x}_k(t)}{t} \leqslant c_{\max} + \varepsilon,$$

and, according to (74), there exists $k_0 \in \mathbb{N}$, depending on T, such that, for any $k \geqslant k_0$,

$$\sup_{1\leqslant t\leqslant T} \frac{\bar{x}_k(t)}{t} \leqslant \bar{c}_\infty + \varepsilon \leqslant c_{\max} + \varepsilon,$$

and this proves Claim 1.

According to this claim and to (76), there exists $c_\infty^* \in [\bar{c}_\infty; c_{\max}]$ such that, up to extracting a subsequence,

$$c_k^* \to c_\infty^* \quad \text{when } k \to +\infty. \tag{78}$$

Claim 2. *There exists a sequence* $(s_k)_{k\in\mathbb{N}}$, $s_k \geqslant 0$, $s_k \to +\infty$ *when* $k \to +\infty$, *such that*

$$\frac{\bar{x}_k(s_k)}{s_k} \to c_k^* \quad \textit{when } k \to +\infty. \tag{79}$$

Indeed, according to (74), for any $q \in \mathbb{N}^*$, the set

$$\left\{k \in \mathbb{N} \mid \sup_{1\leqslant t\leqslant q} \left|\frac{\bar{x}_k(t)}{t} - \bar{c}_\infty\right| > \frac{1}{q} \quad \text{and} \quad |c_k^* - c_\infty^*| > \frac{1}{q}\right\}$$

is finite. Let $m_q \in \mathbb{N}$ denote the maximum of this set (with the convention that $m_q = 0$ is this set is empty), and let $m'_q = m_q + q$. The sequence $(m'_q)_{q\in\mathbb{N}^*}$ is strictly increasing, and $m'_q \to +\infty$ when $q \to +\infty$.

Take any $q \in \mathbb{N}^*$ and any $k \in \mathbb{N}$ satisfying $m'_q \leqslant k < m'_{q+1}$. By definition of m'_q, we have

$$\frac{\bar{x}_k(q)}{q} \geqslant \bar{c}_\infty - \frac{1}{q} \geqslant \sup_{1\leqslant t\leqslant q} \frac{\bar{x}_k(t)}{t} - \frac{2}{q}$$

and thus

$$\sup_{q\leqslant t} \frac{\bar{x}_k(t)}{t} \geqslant \sup_{1\leqslant t} \frac{\bar{x}_k(t)}{t} - \frac{2}{q} = c_k^* - \frac{2}{q} \geqslant c_\infty^* - \frac{3}{q}.$$

This shows that there exists $s_k \geqslant q$ such that $\bar{x}_k(s_k)/s_k \geqslant c_\infty^* - 4/q$, and proves Claim 2.

The remaining of the proof is similar to the end of the proof of Proposition 4. Take any $c > 0$ satisfying

$$0 < c \leqslant c_{\max}, \quad c_\infty < c < c_\infty^* < c + \gamma, \quad \text{and} \quad (c+\beta)(c_\infty^* - c) \leqslant \frac{\beta\gamma}{4} \tag{80}$$

(any c smaller than c_∞^* but sufficiently close to c_∞^* is convenient).

Take $k \in \mathbb{N}^*$ sufficiently large so that

$$c_k < c \quad \text{and} \quad (c+\beta)(c_k^* - c) \leqslant \frac{\beta\gamma}{2}, \tag{81}$$

and such that (77) holds. We are going to apply the computations of Section 5 to the solution $(x,t) \mapsto u_{1,k}(x,t)$, with the following set of parameters:

$$t_{\text{init}} = 0, \quad x_{\text{init}} = 0, \quad c \text{ (chosen above)}, \quad \text{and} \quad y_0 = 0.$$

Let us denote by $v^{(k)}(y,s)$, $\Phi^{(k)}(s)$, $\mathcal{D}^{(k)}(s)$, $\Psi^{(k)}(s)$, and $\Theta^{(k)}(s)$ the quantities corresponding to those defined in Section 5 (with the same notations except the "(k)" exponent), for the solution $(x,t) \mapsto u_{1,k}(x,t)$ and for this set of parameters (according to (77) and, since according to (80) $0 < c \leqslant c_{\max}$, the hypotheses required to apply these computations are satisfied).

According to the existence of an attracting ball in $H^1_{\text{ul}}(\mathbb{R})$ (Lemma 6) and since $y_0 = 0$, $\Phi^{(k)}(0)$ is bounded from above, independently of k.

For $s \geqslant 0$, let $\bar{y}^{(k)}(s) = \bar{x}_k(x) - cs$. Since (according to (81)) $c_k < c$, we have $\bar{y}^{(k)}(s) \to -\infty$ when $s \to +\infty$, and, since $|v^{(k)}(y,s)| \leqslant r_0$ for $y \geqslant \bar{y}^{(k)}(s)$, this yields $\liminf_{s\to+\infty} \Phi^{(k)}(s) \geqslant 0$ (see (44) in the proof of Proposition 4).

Again, since $|v^{(k)}(y,s)| \leqslant r_0$ for $y \geqslant \bar{y}^{(k)}(s)$, we have (compare to (38) in Section 5)

$$\Theta^{(k)}(s) \leqslant \beta^{-1} e^{(c+\beta)\bar{y}^{(k)}(s) - \beta\gamma s}.$$

We have $\bar{y}^{(k)}(s) = (\bar{x}_k(s) - c_k^* s) + (c_k^* - c)s$, thus, according to the definition of c_k^*,

$$\bar{y}^{(k)}(s) \leqslant \sup_{0 \leqslant s \leqslant 1} \bar{x}_k(s) + (c_k^* - c)s \leqslant C + (c_k^* - c)s, \quad s \geqslant 0,$$

where $C > 0$ is a constant independent of k. In view of (81), this yields

$$\Theta^{(k)}(s) \leqslant C_8 e^{-\frac{\beta\gamma}{2}s}, \quad s \geqslant 0,$$

where $C_8 > 0$ is a constant independent of k. According to (37), and proceeding as in Section 5, this yields

$$\int_0^{+\infty} \Psi^{(k)}(s)\,\mathrm{d}s \leqslant \varepsilon_3^{-1} \Psi^{(k)}(0) + \frac{2C_3C_8}{\beta\gamma}.$$

Since, according to Lemma 6 and to the fact that $y_0 = 0$, $\Psi^{(k)}(0)$ is bounded from above uniformly with respect to k, this inequality shows that the same is true for the quantity $\int_0^{+\infty} \Psi^{(k)}(s)\,\mathrm{d}s$.

According to (35), this shows that the quantity

$$\int_0^{+\infty} \mathcal{D}^{(k)}(s)\,\mathrm{d}s$$

is also bounded from above uniformly with respect to k. As a consequence, according to (42), $\mathcal{D}^{(k)}(s)$ must in turn be bounded, uniformly with respect to k and to $s \geqslant 1$.

On the other hand, we have, by definition of $\mathcal{D}^{(k)}(s)$,

$$\mathcal{D}^{(k)}(s_k) \geqslant \int_{-\infty}^{\gamma s_k} e^{cy} v_s^{(k)}(y, s_k)^2\,\mathrm{d}y,$$

which becomes, writing $y = \bar{y}^{(k)}(s_k) + z$,

$$\mathcal{D}^{(k)}(s_k) \geqslant e^{c\bar{y}^{(k)}(s_k)} \int_{-\infty}^{\gamma s_k - \bar{y}^{(k)}(s_k)} e^{cz} v_s^{(k)}\big(\bar{y}^{(k)}(s_k) + z, s_k\big)^2\,\mathrm{d}z. \tag{82}$$

We have

$$\bar{y}^{(k)}(s_k) = s_k\left(\frac{\bar{x}_k(s_k)}{s_k} - c\right) \quad \text{and} \quad \gamma s_k - \bar{y}^{(k)}(s_k) \geqslant \bigl(\gamma - (c_k^* - c)\bigr)s_k$$

thus, according to (78), (79), and (80),

$$\bar{y}^{(k)}(s_k) \to +\infty \quad \text{and} \quad \gamma s_k - \bar{y}^{(k)}(s_k) \to +\infty \quad \text{when } k \to +\infty.$$

In view of (82), and according to the bound from above (uniform with respect to $s \geqslant 1$ and to k) on $\mathcal{D}^{(k)}(s)$, this shows that, for any $L > 0$,

$$\bigl\| z \mapsto v_s^{(k)}\bigl(\bar{y}^{(k)}(s_k) + z\bigr)\bigr\|_{L^2([-L;L])} \to 0 \quad \text{when } k \to +\infty. \tag{83}$$

We have

$$v_s^{(k)}\bigl(\bar{y}^{(k)}(s_k) + z\bigr) = \partial_t u_{1,k}\bigl(\bar{x}_k(s_k) + z, s_k\bigr) + c\partial_x u_{1,k}\bigl(\bar{x}_k(s_k) + z, s_k\bigr), \quad z \in \mathbb{R}.$$

Let us define the functions w_k and $\hat{w}_k$ by

$$w_k(z) = u_{1,k}\bigl(\bar{x}_k(s_k) + z, s_k\bigr), \quad \hat{w}_k(z) = \partial_t u_{1,k}\bigl(\bar{x}_k(s_k) + z, s_k\bigr), \quad z \in \mathbb{R}.$$

By compactness (see Section 2), there exists $w_\infty \in H^2_{\mathrm{ul}}(\mathbb{R})$ and $\hat{w}_\infty \in L^2_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, we have, for any $L > 0$, $w_k \to w_\infty$ in $H^2([-L;L])$ and $\hat{w}_k \to \hat{w}_\infty$ in $L^2([-L;L])$, when $k \to +\infty$.

Assertion (83) shows that $\hat{w}_\infty + cw'_\infty = 0$, and we can make the same remark as at the end of the proof of Proposition 4, namely that the sequences w_k, $\hat{w}_k$, and therefore their limits w_∞, $\hat{w}_\infty$, do not depend on c, and thus that the identity $\hat{w}_\infty + cw'_\infty = 0$ actually holds for a whole interval of values of c (namely for any c satisfying (80)). This yields $w'_\infty \equiv 0$, thus $w_\infty \equiv 0$, which is in contradiction with the definition of $\bar{x}_k(s_k)$. This finishes the proof of Theorem 2.

A.6. Case of a bistable potential

The aim of this paragraph is to consider the particular case of a bistable potential (more precisely, of a potential satisfying hypotheses (H4–H6) below), and, in this case, to reinforce the conclusions of Theorem 1, namely to obtain convergence toward a traveling front uniformly on $\mathbb{R}$ (in contrast with Theorem 1, where convergence was stated on a semi-infinite interval of the form $[-L; +\infty[$).

The result (Theorem 4 below) furnishes a generalization of one of the global convergence results proved by Fife and McLeod in the case where $\dim_u = 1$, namely their result of global convergence toward a single bistable front in a bistable potential [7]. Again, our proof does not make use of any comparison principle (which by the way does not exist in general under the assumed hypotheses).

We assume that the potential V satisfies hypotheses (H1) and (H2) stated in introduction, and we make the following supplementary hypotheses.

(H3) There exists $m \in \mathbb{R}^n$ such that $V(m) < 0$, $\nabla V(m) = 0$, $D^2V(m)$ is positive definite, and such that

$$\bigl\{u \in \mathbb{R}^n \mid V(u) < 0 \text{ and } \nabla V(u) = 0\bigr\} = \{m\}$$

(in other words m is the unique global minimum of V, and is the only critical point of V where V takes a strictly negative value).

(H4) Any solution $x \mapsto v(x)$ of the differential system $v'' = \nabla V(v)$, satisfying $(v(x), v'(x)) \to (m, 0)$ when $x \to +\infty$ and $v \not\equiv m$, is unbounded.

Remarks. (a) In the scalar case $n = 1$ (as in [7]) hypothesis (H4) is a consequence of (H3).

(b) The result stated below (Theorem 4) remains true if we replace hypothesis (H4) by the weaker hypothesis:

(H4′) there exists no solution $x \mapsto v(x)$ of the differential system $v'' = \nabla V(v)$ satisfying $(v(x), v'(x)) \to (m, 0)$ when $x \to \pm\infty$ (i.e. homoclinic to the equilibrium $(m, 0)$) except the trivial solution $v \equiv m$.

We have chosen to adopt the more restrictive hypothesis (H4) for convenience, because this slightly simplifies the proof (we refer to [25] and [26] for a treatment under the weaker hypothesis (H4′)).

Take any $u_0 \in H^1_{\mathrm{ul}}(\mathbb{R})$ and let $u(x,t) = (S_t u_0)(x)$, $t \geqslant 0$, $x \in \mathbb{R}$ denote the solution of the parabolic system (1) with initial data u_0. We make the following hypotheses:

$$\limsup_{x\to+\infty} \bigl|u(x,t)\bigr| \to 0 \quad \text{when } t \to +\infty \tag{84}$$

(namely $u_0 \in \mathcal{A}$), and

$$\limsup_{x\to-\infty} \bigl|u(x,t) - m\bigr| \to 0 \quad \text{when } t \to +\infty. \tag{85}$$

As shown by Proposition 1, there exists $\delta > 0$ such that hypothesis (84) (resp. (85)) is implied by the (stronger) statement

$$\limsup_{x\to+\infty} \int_x^{x+1} \bigl(u_0(y)^2 + u_0'(y)^2\bigr)\,\mathrm{d}y \leqslant \delta \quad \left(\text{resp. } \limsup_{x\to-\infty} \int_x^{x+1} \bigl(\bigl(u_0(y) - m\bigr)^2 + u_0'(y)^2\bigr)\,\mathrm{d}y \leqslant \delta\right).$$

According to Proposition 2, and since $V(m) < 0$, hypothesis (85) shows that $u_0 \in \mathcal{A}_{\mathrm{inv}}$. As a consequence, the conclusions of Theorem 1 hold: there exists $c > 0$ such that $\mathcal{W}_c^{\mathrm{b}} \neq \emptyset$, and there exists a function: $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \bar{x}(t)$, and a map: $\mathbb{R}_+ \to \mathcal{W}_c$, $t \mapsto \nu(t)$, both of class $\mathcal{C}^1$, such that

$$\bar{x}'(t) \to c, \quad \nu'(t) \to 0, \quad \operatorname{dist}\bigl(\nu(t), \mathcal{W}_c^{\mathrm{b}}\bigr) \to 0 \quad \text{when} \quad t \to +\infty$$

and, for any $L \geqslant 0$,

$$\sup_{y\in[-L;+\infty[} \bigl|u\bigl(\bar{x}(t) + y, t\bigr) - \phi_{c,\nu(t)}(y)\bigr| \to 0 \quad \text{when } t \to +\infty.$$

Let us consider the set $\mathcal{L} = \bigcap_{t\geqslant 0} \overline{\nu([t;+\infty[)}$ (the limit set of $\nu(.)$). Since $\mathcal{L}$ is compact, there exists a map $\pi : \mathcal{W}_c \to \mathcal{L}$, such that, for any $\nu \in \mathcal{W}_c$,

$$\operatorname{dist}\bigl(\nu, \pi(\nu)\bigr) = \operatorname{dist}(\nu, \mathcal{L})$$

(in particular $\pi(\nu) = \nu$ for $\nu \in \mathcal{L}$). Let us choose any such map π (this requires the Axiom's choice).

Since $\operatorname{dist}(\nu(t), \mathcal{L}) \to 0$ when $t \to +\infty$, and according to assertion 2 of Lemma 11 (the convergence $\phi_{c,\nu}(x) \to 0$ when $x \to +\infty$ is uniform with respect to $\nu \in \mathcal{W}_c$), we see that, for any $L \geqslant 0$,

$$\sup_{y\in[-L;+\infty[} \bigl|u\bigl(\bar{x}(t) + y, t\bigr) - \phi_{c,\pi(\nu(t))}(y)\bigr| \to 0 \quad \text{when } t \to +\infty.$$

The aim of this paragraph is to prove the following result.

Theorem 4. *We have*

$$\sup_{y\in\mathbb{R}} \bigl|u\bigl(\bar{x}(t) + y, t\bigr) - \phi_{c,\pi(\nu(t))}(y)\bigr| \to 0 \quad \textit{when } t \to +\infty.$$

Remark. If moreover $\mathcal{L}$ is reduced to a singleton $\{\nu_0\}$, and if the front ϕ_{c,ν_0} is linearly stable for the semi-flow of (1), then the conclusion can be made more precise: the solution converges toward a well-defined translate of this front (i.e. we can choose $\bar{x}(t) = x_0 + ct$, for a certain $x_0 \in \mathbb{R}$) and the convergence is exponential. If $\dim_u = 1$, then hypothesis (H3) guarantees that we are in this case (see [7]).

Proof. Up to changing the origin of time, we can suppose that

$$\sup_{x\in\mathbb{R}} \bigl|u(x,t)\bigr| \leqslant R_0, \quad t \geqslant 0 \tag{86}$$

(for the constant R_0 introduced in Section 2). Proceeding as in Section 3, we see that there exists $c_- > 0$ such that

$$\sup_{x\leqslant -c_- t} \bigl|u(x,t) - m\bigr| \to 0 \quad \text{when} \quad t \to +\infty. \tag{87}$$

Let $\hat{x}_-(t) = -(c_- + 1)t$, $t \geqslant 0$. For any $L > 0$, we have, according to (13),

$$\left\| y \mapsto u\big(\hat{x}_-(t) + y, t\big) - m \right\|_{H^1([-L;L])} \to 0 \quad \text{when } t \to +\infty. \tag{88}$$

Observe that, according to assertion 3 of Proposition 3, for any $\nu \in \mathcal{W}_c^{\mathrm{b}}$, we have

$$\phi_{c,\nu}(x) \to m \quad \text{when } x \to -\infty.$$

Claim. *This convergence is uniform with respect to* $\nu \in \mathcal{W}_c^{\mathrm{b}}$.

Indeed, for $\nu \in \mathcal{W}_c^{\mathrm{b}}$, let $E_{c,\nu}(x) = \phi'_{c,\nu}(x)^2/2 - V(\phi_{c,\nu}(x))$. We have $E'_{c,\nu}(x) = -c\phi'_{c,\nu}(x)^2 \leqslant 0$ and $E_{c,\nu}(x) \to -V(m)$ when $x \to -\infty$, and, according to the monotonicity of $E_{c,\nu}(.)$ and to the compactness of $\mathcal{W}_c^{\mathrm{b}}$, this convergence is uniform with respect to $\nu \in \mathcal{W}_c^{\mathrm{b}}$. On the other hand, since (according to Lemma 9) $\phi_{c,\nu}(.)$ is uniformly bounded and since $-V(m) \geqslant E_{c,\nu}(.) \geqslant 0$, we see that $\phi'_{c,\nu}(.)$ and thus $\phi''_{c,\nu}(.)$ are uniformly bounded. Since the convergence

$$\int_{-\infty}^{x} \phi'_{c,\nu}(x)^2 \, \mathrm{d}x \to 0 \quad \text{when } x \to -\infty$$

is uniform with respect to $\nu \in \mathcal{W}_c^{\mathrm{b}}$, the same must be true for the convergence $|\phi'_{c,\nu}(x)| \to 0$ when $x \to -\infty$, and thus also for the convergence $V(\phi_{c,\nu}(x)) \to V(m)$ when $x \to -\infty$, and this proves the claim.

Let us consider the sequence of times $(t_n)_{n\in\mathbb{N}}$ defined as follows: $t_0 = 0$, and, for any $n \in \mathbb{N}^*$,

$$t_n = \max\left(t_{n-1} + n, \sup\left\{t \geqslant 0 \,\Big|\, \sup_{y\in[-2n;0]} \left|u\big(\bar{x}(t) + y, t\big) - \phi_{c,\pi(\nu(t))}(y)\right| \geqslant \frac{1}{n}\right\}\right).$$

Take any smooth function $\theta : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leqslant \theta \leqslant 1$, $0 \leqslant \theta'$, $\theta \equiv 0$ on $]-\infty; 0]$, and $\theta \equiv 1$ on $[1; +\infty[$. Let us define a function $\mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \hat{x}_+(t)$ as follows: for any $n \in \mathbb{N}^*$,

$$\hat{x}_+(t) = \bar{x}(t) - (n-1) - \theta\left(\frac{t - t_{n-1}}{t_n - t_{n-1}}\right), \quad t_{n-1} \leqslant t \leqslant t_n.$$

This function is of class $\mathcal{C}^1$ on $\mathbb{R}_+$ (it is as smooth as $t \mapsto \bar{x}(t)$), and, for any $n \in \mathbb{N}^*$ and $t \in [t_{n-1}; t_n]$, we have

$$-\frac{C}{n} \leqslant \hat{x}'_+(t) - \bar{x}'(t) \leqslant 0, \quad C = \max_{x\in\mathbb{R}} \theta'(x) < +\infty,$$

and this shows that $\hat{x}'_+(t) \to c$ when $t \to +\infty$. Moreover, for any $n \in \mathbb{N}^*$, we have

$$\bar{x}(t) - n \leqslant \hat{x}_+(t) \leqslant \bar{x}(t) - n + 1 \quad \text{for } t \in [t_{n-1}; t_n],$$

and, according to the definition of t_n, to the above claim, and to (13), this shows that, for any $L > 0$,

$$\left\| y \mapsto u\big(\hat{x}_+(t) + y, t\big) - m \right\|_{H^1([-L;L])} \to 0 \quad \text{when } t \to +\infty. \tag{89}$$

Let

$$\overline{V}(u) = V(u) - V(m), \quad u \in \mathbb{R}$$

(we have $\min_{u\in\mathbb{R}} \overline{V}(u) = 0$). Let us define a function $\varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by:

- $\varphi(x,t) = e^{x - \hat{x}_-(t)}$ for $x \leqslant \hat{x}_-(t)$,
- $\varphi(x,t) = 1$ for $\hat{x}_-(t) \leqslant x \leqslant \hat{x}_+(t)$,
- $\varphi(x,t) = e^{\hat{x}_+(t) - x}$ for $\hat{x}_+(t) \leqslant x$,

and let us write

$$\Phi(t) = \int_{\mathbb{R}} \varphi(x,t) \left(\frac{u_x(x,t)^2}{2} + \overline{V}\big(u(x,t)\big)\right) \mathrm{d}x, \quad t \geqslant 0.$$

Take $T > 0$ sufficiently large so that, for $t \geqslant T$, we have $-(c_- + 1) \leqslant \hat{x}_-'(t) \leqslant \hat{x}_+'(t) \leqslant c + 1$. According to (19), we have, for $t \geqslant T$,

$$\Phi'(t) \leqslant -\int_{\mathbb{R}} \varphi u_t^2 \,\mathrm{d}x + \int_{]-\infty;\hat{x}_-(t)]\cup[\hat{x}_+(t);+\infty[} \varphi\left(\max(c_- + 1, c + 1)\left(\frac{u_x^2}{2} + \overline{V}(u)\right) + \frac{u_t^2}{2} + \frac{u_x^2}{2}\right)\mathrm{d}x,$$

and thus

$$\Phi'(t) \leqslant -\frac{1}{2}\mathcal{D}(t) + \Psi(t), \tag{90}$$

where

$$\mathcal{D}(t) = \int_{\mathbb{R}} \varphi(x,t) u_t(x,t)^2 \mathrm{d}x,$$

$$\Psi(t) = C \int_{]-\infty;\hat{x}_-(t)]\cup[\hat{x}_+(t);+\infty[} \varphi(x,t)\left(\left(u(x,t) - m\right)^2 + u_x(x,t)^2\right)\mathrm{d}x,$$

and $C > 0$ is a constant which depends only on V, namely (according to (86))

$$C = \max\left(\frac{\max(c_- + 1, c + 1) + 1}{2}, \max(c_- + 1, c + 1)\left(\max_{|v|\leqslant R_0} \frac{\overline{V}(v)}{(v - m)^2}\right)\right).$$

According to (88) and (89), to the definition of φ, and to (11), we see that $\Psi(t) \to 0$ when $t \to +\infty$. According to (90) and since $\Phi \geqslant 0$, we see that

$$\liminf_{t\to+\infty} \mathcal{D}(t) = 0. \tag{91}$$

Let $\mu_{\min}$ (resp. $\mu_{\max}$) denote the smallest (resp. the largest) of the eigenvalues of $D^2V(m)$ (we have $0 < \mu_{\min} \leqslant \mu_{\max}$), and take $r_m > 0$ sufficiently small so that, for any $v \in \mathbb{R}^n$ satisfying $|v - m| \leqslant r_m$, any eigenvalue λ of $D^2V(v)$ satisfies

$$\frac{\mu_{\min}}{2} \leqslant \lambda \leqslant 2\mu_{\max}. \tag{92}$$

According to (91) there exists a sequence $(t_n')_{n\in\mathbb{N}}$, $t_n' \geqslant 0$, $t_n' \to +\infty$ when $n \to +\infty$, such that $\mathcal{D}(t_n') \to 0$ when $n \to +\infty$.

Lemma 12. *We have*

$$\sup_{x\in[\hat{x}_-(t_n');\hat{x}_+(t_n')]} \left|u(x,t_n') - m\right| \to 0 \quad \textit{when } n \to +\infty.$$

Proof. Let us proceed by contradiction and suppose that the converse holds. Then there exists $\varepsilon > 0$ such that, up to extracting a subsequence, $\sup_{x\in[\hat{x}_-(t_n');\hat{x}_+(t_n')]} |u(x,t_n') - m| > \varepsilon$ for all $n \in \mathbb{N}$. Moreover, we can suppose that $\varepsilon \leqslant r_m$.

For $n \in \mathbb{N}$, let x_n denote the supremum of the (nonempty) set

$$\left\{x \in \left[\hat{x}_-(t_n'); \hat{x}_+(t_n')\right] \,\middle|\, \left|u(x,t_n') - m\right| > \varepsilon\right\}.$$

According to (89), for n sufficiently large we have $|u(x_n,t_n') - m| = \varepsilon$ and $\hat{x}_+(t_n') - x_n \to +\infty$ when $n \to +\infty$. For $n \in \mathbb{N}$, let us write

$$w_n(x) = u(x_n + x, t_n') \quad \text{and} \quad \hat{w}_n(x) = u_t(x_n + x, t_n').$$

By compactness (Section 2), there exists $w_\infty \in H^1_{\mathrm{ul}}(\mathbb{R})$ such that, up to extracting a subsequence, for any $L > 0$, $w_n \to w_\infty$ in $H^1([-L;L])$. According to (91), we see that, for any $L > 0$, $\hat{w}_n \to 0$ in $L^2([-L;L])$. Passing to the limit in (1), we see that w_∞ is a solution of the differential system

$$w'' = \nabla V(w) \tag{93}$$

satisfying $|w_\infty(x) - m| \leqslant \varepsilon \leqslant r_m$ for all $x \geqslant 0$.

Claim. *We have* $(w_\infty(x), w'_\infty(x)) \to (m, 0)$ *when* $x \to +\infty$.

Let

$$E(x) = \frac{w'_\infty(x)^2}{2} - \overline{V}\big(w_\infty(x)\big), \quad Q(x) = \frac{(w_\infty(x) - m)^2}{2}, \quad x \geqslant 0.$$

According to (92), we have

$$E' = 0 \quad \text{and} \quad Q'' = w'^2_\infty + (w_\infty - m) \cdot \nabla \overline{V}(w_\infty) \geqslant \mu_{\min} Q \geqslant 0 \quad \text{on } \mathbb{R}_+.$$

Thus $Q'(x)$ admits a limit (finite or infinite) when $x \to +\infty$, and since Q is bounded on $\mathbb{R}_+$ we must have $Q'(x) \to 0$ when $x \to +\infty$, thus $Q' \leqslant 0$ on $\mathbb{R}_+$. In turn, $Q(x)$ (which is nonnegative) must converge when $x \to +\infty$, and in view of the above inequality, we must have $Q(x) \to 0$ and thus $w_\infty(x) \to m$ when $x \to +\infty$. Eq. (93) then shows that $w''_\infty(x) \to 0$ when $x \to +\infty$, and this finally yields $w'_\infty(x) \to 0$ when $x \to +\infty$, which proves the claim.

According to hypothesis (H4), this claim shows that w_∞ is unbounded, and this is contradictory to the a priori bound (86) on the solution. The lemma is proved. □

Let $\beta_2 = \min(1, \sqrt{\mu_{\min}/2})$, let $\psi_2(x) = e^{-\beta_2|x|}$, let

$$\Psi_2(\xi, t) = \int T_\xi(x)\psi_2\left(\frac{u_x(x,t)^2}{2} + \overline{V}\big(u(x,t)\big) + \frac{(u(x,t) - m)^2}{2}\right)\mathrm{d}x, \quad \xi \in \mathbb{R},\ t \geqslant 0,$$

and let

$$S'_{\text{far}}(t) = \big\{x \in \mathbb{R} \mid \big|u(x,t) - m\big| > r_m\big\}, \quad t \geqslant 0.$$

Proceeding as in Section 3, we see that

$$\frac{\partial \Psi_2}{\partial t}(\xi, t) \leqslant -\varepsilon_5 \Psi_2(\xi, t) + C \int_{S'_{\text{far}}(t)} T_\xi \psi_2(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R},\ t \geqslant 0, \tag{94}$$

where

$$\varepsilon_5 = \min\left(3/2, \frac{\mu_{\min}}{4(\mu_{\max} + 1/2)}\right),$$

and $C > 0$ is a constant depending only on V.

According to (91), Lemma 12, and (11), we have

$$\sup_{x \in [\hat{x}_-(t'_n); \hat{x}_+(t'_n)]} \Psi_2(x, t'_n) \to 0 \quad \text{when } n \to +\infty.$$

In view of (87), the following claim completes the proof of Theorem 4.

Claim. *We have*

$$\sup_{x \in [\hat{x}_-(t); \hat{x}_+(t)]} \Psi_2(x, t) \to 0 \quad \textit{when } t \to +\infty.$$

Indeed, let us proceed by contradiction and suppose that the converse is true. Then there exists $\varepsilon > 0$ such that

$$\limsup_{t \to +\infty} \sup_{x \in [\hat{x}_-(t); \hat{x}_+(t)]} \Psi_2(x, t) > \varepsilon.$$

Since $\overline{V} \geqslant 0$, we have (see (25))

$$\Psi_2(x, t) \leqslant \frac{r_m^2}{2} \Rightarrow \big|u(x,t) - m\big| \leqslant r_m.$$

We can suppose that $\varepsilon \leqslant r_m^2/2$.

For $n \in \mathbb{N}$, let t_n'' denote the infimum of the (nonempty) set

$$\Big\{t \geqslant t_n' \mid \sup_{x\in[\hat{x}_-(t);\hat{x}_+(t)]} \Psi_2(x,t) \geqslant \varepsilon\Big\}.$$

By continuity of $\Psi_2(x,t)$ with respect to x and t, for $n \in \mathbb{N}$ sufficiently large, we have

$$t_n'' > t_n', \quad \text{and} \quad \sup_{x\in[\hat{x}_-(t_n'');\hat{x}_+(t_n'')]} \Psi_2(x,t_n'') = \varepsilon,$$

and there exists $x_n'' \in [\hat{x}_-(t_n''); \hat{x}_+(t_n'')]$ such that $\Psi_2(x_n'', t_n'') = \varepsilon$. Moreover, by definition of t_n'' (as an infimum), we must have

$$\frac{\partial \Psi_2}{\partial t}(x_n'', t_n'') \geqslant 0, \tag{95}$$

and, according to (88) and (89), we must have $\hat{x}_+(t_n'') - x_n'' \to +\infty$ and $x_n'' - \hat{x}_-(t_n'') \to +\infty$ when $n \to +\infty$, and as a consequence we see that, for n sufficiently large, (95) is contradictory to (94). This finishes the proof of the claim, and thus of Theorem 4. □

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