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Reduced energy functionals for a three-dimensional fast rotating Bose Einstein condensates

Amandine Aftalion *, Xavier Blanc

Laboratoire Jacques-Louis Lions, Université Paris 6, 175, rue du Chevaleret, 75013 Paris, France

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Abstract

We prove that in the fast rotating regime, the three-dimensional Gross–Pitaevskii energy describing the state of a Bose Einstein condensate can be reduced to a two-dimensional problem and that the vortex lines are almost straight. Additionally, we prove that the minimum of this two-dimensional problem can be sought in a reduced space corresponding to the first eigenspace of an elliptic operator. This space is called the Lowest Landau level and is of infinite dimension

Keywords: Rotating Bose–Einstein condensates; Gross–Pitaevskii energy; Dimension reduction; Lowest Landau level; Vortex; Elliptic PDEs; Diamagnetic inequality

1. Introduction

The rotation of a Bose Einstein condensate (BEC) has been the aim of various recent experiments [1,9,21,22,25,28]. The motivation is to exhibit specific features of BEC, which are distinct from those of a classical fluid. Indeed when a quantum fluid such as a BEC is rotated, it nucleates vortices, which provides an evidence of its superfluid properties.

In a BEC, it is expected that all the atoms are in the same state and are thus described by the same complex valued wave function. The mathematical description is made using this wave function which minimizes an energy, called the Gross–Pitaevskii energy [2,24]:

$$\int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla \psi|^2 - \mathrm{i}\Omega \bar{\psi}\big((x_2\partial_{x_1}\psi - x_1\partial_{x_2})\psi\big) + \frac{1}{2}\big(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2\big)|\psi|^2 + \frac{1}{4}a|\psi|^4 \right) dx, \tag{1.1}$$

under $\int_{\mathbb{R}^3} |\psi|^2 = 1$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Because of the experimental device, the energy contains a trapping term, where ω_1, ω_2 and ω_3 are the trapping frequencies in the three directions. The presence of the rotation (Ω is the rotational frequency along the third direction) creates a term in the energy proportional to the angular momentum along the third direction. The other parameter of the problem is the scattering length a which models the interaction between the atoms. This energy is used as such in the physics literature. It has been derived mathematically as the limit of the Hamiltonian for N bosons, when N tends to infinity, first in the case of no rotation ($\Omega = 0$) by Lieb,

* Corresponding author.
E-mail addresses: aftalion@ann.jussieu.fr (A. Aftalion), blanc@ann.jussieu.fr (X. Blanc).

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Seiringer and Yngvason [19], and in the case of fixed Ω by Lieb and Seiringer [18]. The scattering length a_N of the interaction in the N-body problem is such that $Na_N \to a$, where a is the constant in (1.1).

The case of no rotation ($\Omega = 0$) amounts to a real valued problem while as soon as Ω is not zero, the minimizer is complex valued and an important issue is to derive properties about its zero sets. In general, this is a difficult question and therefore various asymptotic regimes of parameters have been sought for. For fixed a and Ω, one can study the limits where one or two of the ω_i's are large or small, so that the motion is frozen in one or two directions. Then, the system becomes lower-dimensional and the problem is reduced to a two or one-dimensional energy. This has been performed in the case $\Omega = 0$ by [19,23,26]. In particular, if ω_3 is large, the wave function is strongly confined in the third direction: it is almost in the ground state of the Hamiltonian reduced to the third direction, which provides a Gaussian, and the minimum of the 3D energy can be approximated by the minimum of a reduced 2D problem [26].

Another possibility is to fix a and ω_i and vary Ω. When the rotational velocity is small, it is expected that the wave function does not vanish. When Ω is increased, vortices are observed: they correspond to singularity lines where ψ vanishes and around which the phase has a circulation. Their three-dimensional shape is of interest, as has been described in [6,7]: they bend close to the boundary, due to the presence of the trapping potential. This has been studied in the so called Thomas Fermi regime, where the kinetic energy is much smaller than the interaction term (a is large) and vortices are sets of small measure. At high velocity, this approximation is no longer valid, the condensate expands in the transverse directions x_1 and x_2 and thus gets very flat. The experimental view from the top indicates that the vortex lines get almost straight and the number of lines increase: vortex lines arrange themselves in a lattice [1,11,14,27], referred to as an Abrikosov lattice due to the analogy with superconductors [8,16]. The description of the vortex lattice has been the focus of very recent papers in the condensed matter physics community [4,12,15], and some mathematical works [3,5]. This description relies on two ingredients: first that the condensate is very flat and the problem can be reduced to a two-dimensional problem and second, that one can study the minimization of the energy restricted to an eigenspace called the lowest Landau level. These two reductions have been assumed to be correct in the aforementioned works. The aim of this paper is to provide a rigorous analysis of these reductions.

Our regime of interest is when Ω is increased so that the effective trapping frequencies in the transverse directions become $\sqrt{\omega_1^2 - \Omega^2}$, $\sqrt{\omega_2^2 - \Omega^2}$ and are much smaller than the trapping frequency in the 3rd direction, ω_3. For simplicity, we will assume that $\omega_1 = \omega_2$. As Ω tends to $\omega_1 = \omega_2$, the condensate is much more strongly confined in the third direction than in the first two directions, and it is expected [4] that the 3D wave function is well approximated by the product of a Gaussian in the 3rd direction times a 2D wave function, so that the problem becomes almost 2D.

We will show how we can reduce the energy functional (1.1) to obtain a simpler problem. The reduction will be on the one hand a dimensional reduction from 3D to 2D and on the other hand from the full space to a reduced eigenspace.

The kinetic and rotational terms in the energy (1.1) are the beginning of the expansion of a complete square. Adding and subtracting the missing term modifies the trapping potential, creating what is called the effective trapping potential, and the energy (1.1) can be rewritten as follows (we have set $\omega_1 = \omega_2 = 1$ for simplicity):

$$E_{3\mathrm{D}}(\psi) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\psi - \mathrm{i}\Omega e_3 \times x\psi|^2 + \frac{1}{2}\left(1-\Omega^2\right)|r|^2|\psi|^2 + \frac{1}{2}\omega_3^2 x_3^2|\psi|^2 + \frac{1}{4}a|\psi|^4, \tag{1.2}$$

where $e_3 = (0,0,1)$, and $r = (x_1, x_2)$. We still minimize under the constraint $\int_{\mathbb{R}^3}|\psi|^2 = 1$. We assume that $\psi \in \mathcal{H}_{3D}$ defined by

$$\mathcal{H}_{3\mathrm{D}} = \left\{\psi \in H^1\left(\mathbb{R}^3\right),\ \text{s.t. } E_{3\mathrm{D}}(\psi) < \infty\right\}. \tag{1.3}$$

In order for the energy to be bounded below, we need to have $\Omega < 1$, which means that the trapping potential remains stronger than the rotating force. For all $\Omega < 1$, there exists a minimizer of $E_{3\mathrm{D}}$ under the constraint that $\int_{\mathbb{R}^3}|\psi|^2 = 1$: indeed, the trapping potential provides the property that a minimizing sequence strongly converges in L^2.

1.1. 2D reduction

We want to study the regime where Ω tends to 1. We expect any minimizer ψ to be close to a product $\Phi(x_1,x_2)\xi(x_3)$, where ξ is a Gaussian corresponding the ground state of the energy in the third direction, namely

$$E_1(\xi)=\int_{\mathbb{R}} \frac{1}{2}\big((\xi')^2+\omega_3^2t^2\xi^2\big)\,dt \tag{1.4}$$

and Φ is a minimizer of the two-dimensional problem corresponding to (1.2), where the coefficient of the quartic term has been modified:

$$E_{2\mathrm{D}}(\Phi)=\int_{\mathbb{R}^2} \frac{1}{2}\big|\nabla\Phi-\mathrm{i}\Omega r^{\perp}\Phi\big|^2+\frac{1}{2}\big(1-\Omega^2\big)|r|^2|\Phi|^2+\frac{1}{4}b|\Phi|^4, \tag{1.5}$$

under the constraint $\int_{\mathbb{R}^2}|\Phi|^2=1$. We have set $r=(x_1,x_2)$, $r^{\perp}=(-x_2,x_1)$, and $b=a\int_{\mathbb{R}}\xi^4$, with $\xi(x_3)=(\omega_3/(2\pi))^{1/4}\mathrm{e}^{-\omega_3x_3^2/2}$. Let us point out that ξ is the first eigenfunction of $-\frac{d^2}{dt^2}+\omega_3^2t^2$ in $\mathbb{R}$ with L^2 norm equal to 1. The corresponding eigenvalue is ω_3 and the other eigenvalues are $(2k+1)\omega_3$, $k\in\mathbb{N}$. The natural space of minimization for $E_{2\mathrm{D}}$ is

$$\mathcal{H}_{2\mathrm{D}}=\big\{\Phi\in H^1\big(\mathbb{R}^2\big),\ \text{s.t. } E_{2\mathrm{D}}(\Phi)<\infty\big\}. \tag{1.6}$$

Let us define

$$I_{3\mathrm{D}}(\Omega)=\inf\bigg\{E_{3\mathrm{D}}(\psi),\ \psi\in\mathcal{H}_{3\mathrm{D}},\ \int_{\mathbb{R}^3}|\psi|^2=1\bigg\}, \tag{1.7}$$

$$I_{2\mathrm{D}}(\Omega)=\inf\bigg\{E_{2\mathrm{D}}(\Phi),\ \Phi\in\mathcal{H}_{2\mathrm{D}},\ \int_{\mathbb{R}^2}|\Phi|^2=1\bigg\}. \tag{1.8}$$

Both $I_{2\mathrm{D}}(\Omega)$ and $I_{3\mathrm{D}}(\Omega)$ depend on Ω, but also on a. Since a is fixed, we have not written explicitly the dependence. Moreover, $I_{2\mathrm{D}}(\Omega)$ also depends on ξ through b. We are going to prove that, as Ω tends to 1, the expansion for $E_{2\mathrm{D}}(\psi)$ is given by

$$E_{2\mathrm{D}}(\Phi)+E_1(\xi). \tag{1.9}$$

This is not an exact decoupling since ξ is included in $E_{2\mathrm{D}}$ through b. Though these two terms are of order 1, at leading order, we prove that they are minimized separately, due to their behaviour at the next order.

In [3], we have studied properties of $E_{2\mathrm{D}}$. In particular, we have derived an upper and lower bound for $I_{2\mathrm{D}}(\Omega)$ which provides that, as Ω tends to 1,

$$I_{2\mathrm{D}}(\Omega)-1=\mathrm{O}\big(\sqrt{1-\Omega}\big). \tag{1.10}$$

This behaviour and the fact that the eigenvalues of $-\frac{d^2}{dt^2}+\omega_3^2t^2$ are $(2k+1)\omega_3$ will allow us to show that $I_{2\mathrm{D}}(\Omega)-1$ is a perturbation of the energy E_1 along the third axis. This implies that, for each x_1,x_2, a minimizer of $E_{3\mathrm{D}}$ is confined on the lowest eigenstate of $-\frac{d^2}{dx_3^2}+\omega_3^2x_3^2$ and that the energy contributions decouple. This is what we prove:

Theorem 1.1. *For each Ω, the minima $I_{3\mathrm{D}}(\Omega)$ and $I_{2\mathrm{D}}(\Omega)$ are achieved and, as Ω tends to* 1,

$$I_{3\mathrm{D}}(\Omega)-\bigg(\frac{\omega_3}{2}+I_{2\mathrm{D}}(\Omega)\bigg)=\mathrm{o}\big(\sqrt{1-\Omega}\big). \tag{1.11}$$

Moreover, if $I_{3\mathrm{D}}(\Omega)$ is achieved for some ψ then, there exists $\phi(x_1,x_2)\in\mathcal{H}_{2\mathrm{D}}\cap C^{0,1/2}(\mathbb{R}^2)$ such that $w=\psi-\phi(x_1,x_2)\xi(x_3)$ has the property that $|w|$ tends to 0 *in $H^1(\mathbb{R}^3)\cap C^{0,1/2}(\mathbb{R}^3)$ as Ω tends to* 1.

Let us insist on the fact that (1.10) and (1.11) provide an expansion for $I_{3\mathrm{D}}$ at the second order and justifies the detailed study of the 2D Gross–Pitaevskii energy $E_{2\mathrm{D}}$. At leading order, $I_{3\mathrm{D}}(\Omega)$ behaves like $1+\omega_3/2$, or rather, in the initial scaling as $(\omega_1+\omega_2+\omega_3)/2$. Though the leading order term of $I_{2D}(\Omega)$ is of the same order as ω_3, the 1D energy along the third axis, we can prove that a minimizer ψ is almost confined on the first eigenfunction in the third direction, because the next term in the expansion of $I_{2D}(\Omega)$ is small.

The convergence in $C^{0,1/2}$ implies that the zero sets of ψ are almost straight lines, located close to the lines passing through the zeroes of ϕ, which is the projection of ψ on the space spanned by ξ. We expect that ϕ is close to a minimizer Φ of E_{2D} though we are not able to prove it. We only know that its energy is close to the minimal 2D energy I_{2D}.

The proof of the theorem consists in two steps: first we derive an upper bound with an appropriate test function, namely $\phi(x_1,x_2)\xi(x_3)$, where ϕ is a solution of I_{2D} and ξ is the normalized Gaussian. Then we derive a lower bound as follows: we project a minimizer ψ of I_{3D} onto the space spanned by $\xi(x_3)$, so that $\phi(x_1,x_2)$ is the projection and $w=\psi-\phi\xi$ is the rest, orthogonal to ξ. We prove that w is small, first in L^2 norm, then in energy, so that at leading order $E_{3D}(\psi)-E_{3D}(\phi\xi)=\mathrm{o}(\sqrt{1-\Omega})$. Since $E_{3D}(\phi\xi)$ decouples into $E_{2D}(\phi)+E_1(\xi)$, and the last one is equal to $\omega_3/2$, this allows us to derive that $E_{2D}(\phi)-I_{2D}$ is small, and (1.11) holds. The Euler–Lagrange equation for ψ provides an improvement of the smallness of w. This relies on elliptic estimates for the operator $-\nabla_\Omega^2=-(\nabla-\mathrm{i}\Omega r^\perp)^2$ and yields the convergence in $C^{0,1/2}$. We expect that ϕ is close to a minimizer Φ of E_{2D}; the difficulties to prove it are on the one hand the degeneracy of the minimizer of E_{2D} (invariance by rotation of the energy though the minimizer is not rotationally symmetric) and the fact that at the limit $\Omega=1$, we cannot identify any asymptotic problem and we do not expect any asymptotic uniqueness.

As we have mentioned, a related problem is the minimization of (1.1) when ω_3 is fixed and $\omega_1=\omega_2$ tend to 0. When $\Omega=0$, this has been studied by Schnee and Yngvason [26]. The dimensional reduction holds in this case with $\Omega=0$ but with a very different scaling: the confining energy in the third direction is still of order ω_3 but the transverse 2D energy, the equivalent of E_{2D}, is small: of order ω_1. Indeed it is equal to

$$\int_{\mathbb{R}^2}\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}\omega_1^2r^2|\phi|^2+\frac{b}{4}|\phi|^4 \tag{1.12}$$

and the scaling $\phi(r)=\sqrt{\omega_1}u(\sqrt{\omega_1}r)$ yields that the energy of ϕ is equal to $\omega_1E(u)$ where E is independent of ω_1. As soon as Ω is not zero, such an estimate of the 2D energy no longer holds and the result is open as far as we know. The difference between this case and our limit Ω close to 1, is that here a limiting problem at $\omega_1=\omega_2=0$ can be identified. In our case, the energy $I_{2D}(\Omega)$ is at leading order of the same magnitude as the confinement in the third direction. It is thanks to the next term that we are able to get the decoupling but no limiting problem has been identified yet.

We have treated the case where $\omega_1=\omega_2$, but we expect that this two-dimensional reduction holds if $\omega_1=\omega_2+\varepsilon$, with ε much smaller than ω_3. This would require a precise estimate for I_{2D} in terms of ε, which does not exist in the literature for the moment. Our proof applies as such if ε is much smaller than ω_3 and $\sqrt{1-\Omega}$.

1.2. LLL reduction

Once we have made the 2D reduction to the Gross–Pitaevskii energy (1.5), we can still perform another reduction to the first eigenspace of the operator $-\nabla_\Omega^2=-(\nabla-\mathrm{i}\Omega r^\perp)^2$, which corresponds to the first term in the energy. The eigenvalues of $-\nabla_\Omega^2$ are well known: $2(2k+1)\Omega$, $k\in\mathbb{N}$. The eigenspace corresponding to the first eigenvalue 2Ω is of infinite dimension and called the lowest Landau level (LLL). It is made up of functions

$$\phi(x_1,x_2)=f(z)\mathrm{e}^{-\Omega|z|^2/2}\quad\text{with } z=x_1+\mathrm{i}x_2, \tag{1.13}$$

where f is a holomorphic function.

For some reason due to our energy decoupling, that we will explain below, instead of using the space defined by (1.13), we will use a space independent of Ω, where the Gaussian is $\mathrm{e}^{-|z|^2/2}$, that is, the space (1.13) for $\Omega=1$:

$$L=\left\{u(x_1,x_2)\in L^2(\mathbb{R}^2)\text{ s.t. }u(x_1,x_2)=f(z)\mathrm{e}^{-|z|^2/2},\text{ with } f \text{ holomorphic}\right\}. \tag{1.14}$$

The property of this space is to be the first eigenspace for $-(\nabla-\mathrm{i}r^\perp)^2$, but also to be stable by the action of $H_\Omega=-(\nabla-\mathrm{i}\Omega r^\perp)^2+(1-\Omega^2)|r|^2$, while (1.13) is invariant under $-(\nabla-\mathrm{i}\Omega r^\perp)^2$ only. This is the key ingredient to get the decoupling of the energy.

The space L is related to the Fock Bargmann space $\mathcal{F}$, see [5]:

$$\mathcal{F} = \left\{ f \in L^2\left(\mathbb{C}, \mathrm{e}^{-|z|^2}\, dz\right),\ \text{s.t. } f \text{ entire} \right\} \quad \text{with} \tag{1.15}$$

$$\| f \|_{\mathcal{F}}^2 = \int_{\mathbb{C}} \left| f(z) \right|^2 \mathrm{e}^{-|z|^2}\, dz \tag{1.16}$$

where dz denotes the Lebesgue measure $dx_1\, dx_2$. It is a Hilbert space and the orthogonal projection from $L^2(\mathbb{C}, \mathrm{e}^{-|z|^2}\, dz)$ onto $\mathcal{F}$ is explicit (see [5]), which provides the expression for the orthogonal projection from $L^2(\mathbb{R}^2)$ onto L, that we call Π_L:

$$[\Pi_L u](z) = \frac{\mathrm{e}^{-|z|^2/2}}{\pi} \int_{\mathbb{C}} \mathrm{e}^{z\bar{z}'} \mathrm{e}^{-|z'|^2/2} u(z')\, dz' \tag{1.17}$$

for all $u \in L^2(\mathbb{R}^2)$, with $z = x_1 + \mathrm{i}x_2$ and $u(z) = u(x_1, x_2)$.

In [3,5], we have minimized the energy (1.5) on the space (1.14), and in particular understood the location of the zeroes or vortices of a minimizer ϕ. We have also studied the averaged behaviour of ϕ on large disks.

Our aim is to prove that $I_{2\mathrm{D}}(\Omega)$ is well approximated by the minimization of $E_{2\mathrm{D}}$ restricted to L. If $u \in L$, then as we will prove in Lemma 3.1 below, the energy (1.5) simplifies into

$$E_{LLL}(u) := \Omega + \int_{\mathbb{R}^2} (1-\Omega)|r|^2|u|^2 + \frac{b}{4}|u|^4. \tag{1.18}$$

Let us define

$$\begin{aligned} I_{LLL}(\Omega) &= \inf\left\{ E_{LLL}(u),\ u \in L \cap \mathcal{H}_{2\mathrm{D}},\ \int_{\mathbb{R}^2} |u|^2 = 1 \right\} \\ &= \inf\left\{ E_{2\mathrm{D}}(u),\ u \in L \cap \mathcal{H}_{2\mathrm{D}},\ \int_{\mathbb{R}^2} |u|^2 = 1 \right\}. \end{aligned} \tag{1.19}$$

We have proved in [3] that as Ω tends to 1,

$$I_{LLL}(\Omega) - 1 = \mathrm{O}\left(\sqrt{1-\Omega}\right). \tag{1.20}$$

Theorem 1.2. *For each Ω, the minimum $I_{LLL}(\Omega)$ is achieved and, as Ω tends to* 1,

$$I_{2\mathrm{D}}(\Omega) - I_{LLL}(\Omega) = \mathrm{o}\left(\sqrt{1-\Omega}\right). \tag{1.21}$$

Moreover, if $I_{2\mathrm{D}}(\Omega)$ is achieved for some ϕ then, there exists $u(x_1, x_2)$ in L such that $w = \phi - u$ is such that $|w|$ tends to 0 *in $H^1(\mathbb{R}^2) \cap C^{0,\alpha}(\mathbb{R}^2)$, for all $\alpha \in (0,1)$, as Ω tends to* 1.

As in the previous theorem, we are not able to prove that u is close to a minimizer of E_{LLL}, but we prove that the energy of its projection onto L is almost minimizing. The $C^{0,\alpha}$ estimate implies that the zeroes of ϕ and u are close.

The proof again consists in an upper bound and a lower bound. The upper bound is obvious because the space of minimization is restricted. For the lower bound, the key ingredient is to be able to decouple the energy into the contribution of u, the projection of the minimizer ϕ of $I_{2\mathrm{D}}$ onto L and that of $w = \phi - u$. The orthogonality of u and w is not sufficient. This is where we use that L is stable by $H_\Omega = -\nabla_\Omega^2 + (1-\Omega^2)|r|^2$, so that $H_\Omega u$ and w are orthogonal. This provides the smallness of w in energy. Then we obtain the estimate $E_{2\mathrm{D}}(\phi) - E_{2\mathrm{D}}(u) = \mathrm{o}(\sqrt{1-\Omega})$. This estimate is significant since both terms behave like $1 + \mathrm{O}(\sqrt{1-\Omega})$. It means that the energy of w is a lower order term. As before, we use the Euler Lagrange equation and elliptic estimates for H_Ω to derive the smallness of w in appropriate norms.

1.3. Open directions

As we have mentioned above, we are not able to prove that $\phi(x_1, x_2)$ given by Theorem 1.1 is close to a minimizer of E_{2D} though its energy is almost minimizing and it is an almost solution of the Euler Lagrange equation. We do not know how to tackle the nonuniqueness due to the rotation of the variable. A first step would be to prove that, for fixed Ω, all the minimizers of E_{2D} are $e^{i\alpha}\Phi(e^{i\beta}r)$, where Φ is a given minimizer, and α, β are real numbers. Then, one would need to find how α and β should be related to ϕ, the projection of the 3D minimizer. Similar issues arise for the reduction to the Lowest Landau level. Another major difficulty is the lack of limiting problem at $\Omega = 1$: indeed, it is expected that as Ω tends to 1, the disappearance of the trapping potential turns the problem into a periodic one. The good limiting problem would be to minimize the L^4 norm with prescribed L^2 norm, not on the whole space $\mathbb{R}^2$, but on a lattice, with periodic boundary conditions. Such a limit is not proved and would contain invariance under the rotation of the variable.

Related questions to the ones addressed in this paper are the ones of the time dependent problems, namely

$$i\partial_t\psi = -(\nabla - i\Omega e_3 \times x)^2\psi + (1-\Omega^2)|r|^2\psi + x_3^2\psi + a|\psi|^2\psi. \tag{1.22}$$

One issue would be to prove that if the initial condition is $\psi_0 = \phi(x_1, x_2)\xi(x_3)$ where ξ is the Gaussian defined above and ϕ is any function in $\mathcal{H}_{2D}$, then the decoupling holds for all time, namely $\psi(x,t) = \phi(x_1, x_2, t)\xi(x_3)$ where ϕ is a solution of

$$i\partial_t\phi = -(\nabla - i\Omega r^\perp)^2\phi + (1-\Omega^2)|r|^2\psi + b|\psi|^2\psi, \tag{1.23}$$

with the same relation as before between a and b. Another issue would be to prove that if the initial condition satisfies the energy estimate

$$E_{3D}(\psi_0) = 1 + \frac{\omega_3}{2} + O\big(\sqrt{1-\Omega}\big),$$

then for all times $\psi(x,t)$ is close to a function $\phi(x_1, x_2, t)\xi(x_3)$. Similar questions hold for the reduction to the Lowest Landau level.

The framework of the Gross–Pitaevskii energy (1.2), known as the mean field Quantum Hall regime, is acceptable only if the number of vortices is much smaller than the number of atoms in the condensate, which is the case of the present experiments. The number of vortices is proportional to $1/\sqrt{1-\Omega}$. The validity of the mean field model is thus that $N\sqrt{1-\Omega}$ is large, with N large and Ω close to 1. If ever the asymptotic regime is $N\sqrt{1-\Omega}$ of order 1, one can no longer consider the Gross Pitaevskii energy and has to go back to the Hamiltonian for N bosons [13]. The wave function can no longer be approximated by the product of N identical functions but is strongly correlated. The behaviour is that of the fractional Quantum Hall regime.

2. 2D reduction

In this section, we prove Theorem 1.1. We first deal with the proof of (1.11) in Subsection 2.1, then with the convergence of solutions in Subsection 2.2. In all the proofs below, C stands for a generic constant independent of Ω.

2.1. Convergence of the energy

In this subsection, we prove the convergence (1.11), which we recall here for the convenience of the reader:

Proposition 2.1. *Let $I_{3D}(\Omega)$ and $I_{2D}(\Omega)$ be defined by* (1.7) *and* (1.8) *respectively. Then, as Ω tends to* 1*,*

$$I_{3D}(\Omega) = \frac{\omega_3}{2} + I_{2D}(\Omega) + o\big(\sqrt{1-\Omega}\big). \tag{2.1}$$

Proof. For simplicity, we set $\omega_3 = 1$.

Step 1: upper bound. We consider a solution ϕ of $I_{2D}(\Omega)$, and use the following test function ψ to get an upper bound for $I_{3D}(\Omega)$:

$$\psi(x_1, x_2, x_3) = \phi(x_1, x_2)\xi(x_3), \quad \text{where } \xi(t) = \frac{e^{-t^2/2}}{(2\pi)^{1/4}}$$

is the first eigenvector of the operator $-\frac{d^2}{dt^2}+t^2$, in $\mathbb{R}$ with the eigenvalue 1. We compute the energy of ψ, using the property

$$|\nabla\psi - \mathrm{i}\Omega e_3\times x\psi|^2 = |\nabla'\psi - \mathrm{i}\Omega e_3\times x\psi|^2 + |\partial_3\psi|^2 \tag{2.2}$$

where ∇' stands for the projection of the gradient on the first two directions: $\nabla'\psi = (\partial_1\psi, \partial_2\psi, 0)$. We exploit the decoupling of the integrals in the transverse (x_1,x_2) and longitudinal (x_3) directions and the property $\int_{\mathbb{R}}\xi^2 = \int_{\mathbb{R}^2}|\phi|^2 = 1$ to find

$$\begin{aligned}
E_{3\mathrm{D}}(\psi) &= \int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \frac{1}{2}\int_{\mathbb{R}}(\xi')^2 + \frac{1}{2}\left(1-\Omega^2\right)\int_{\mathbb{R}^2}|r|^2|\phi|^2 + \frac{1}{2}\int_{\mathbb{R}}x_3^2\xi(x_3)^2\,dx_3 + \frac{1}{4}a\int_{\mathbb{R}}\xi^4\int_{\mathbb{R}^2}|\phi|^4\\
&= E_{2\mathrm{D}}(\phi) + \frac{1}{2}\int_{\mathbb{R}}\left(\xi'(t)^2 + t^2\xi(t)^2\right)\\
&= I_{2\mathrm{D}}(\Omega) + \frac{1}{2},
\end{aligned}$$

The last equality uses that ξ is the first eigenfunction of $-\frac{d^2}{dt^2}+t^2$. Hence

$$I_{3\mathrm{D}}(\Omega) \leqslant \frac{1}{2} + I_{2\mathrm{D}}(\Omega). \tag{2.3}$$

Step 2: lower bound. We denote by ψ a solution of $I_{3\mathrm{D}}(\Omega)$, and define $\phi(x_1,x_2)$ as its projection onto $\xi(x_3)$ in $L^2(\mathbb{R})$, that is $\phi(x_1,x_2) = \int_{\mathbb{R}}\psi(x)\xi(x_3)\,dx_3$. This implies

$$\psi(x) = \phi(x_1,x_2)\xi(x_3) + w(x), \quad \text{where } \forall (x_1,x_2)\in\mathbb{R}^2,\ \int_{\mathbb{R}}\xi(x_3)w(x_1,x_2,x_3)\,dx_3 = 0. \tag{2.4}$$

The orthogonality condition implies in particular

$$1 = \int_{\mathbb{R}^3}|\psi|^2 = \int_{\mathbb{R}^2}|\phi|^2 + \int_{\mathbb{R}^3}|w|^2. \tag{2.5}$$

Note that the definition of ϕ provides that $\phi\in\mathcal{H}_{2\mathrm{D}}$. We are going to decouple the energy into the contribution of ϕ and that of w to derive first that w is small in L^2 norm, and then improve the norm. Thus, we compute the energy of ψ, and find, using (2.2)

$$\begin{aligned}
E_{3\mathrm{D}}(\psi) &= \int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \int_{\mathbb{R}^3}\frac{1}{2}|\nabla' w - \mathrm{i}\Omega e_3\times xw|^2 + \frac{1}{2}|\partial_3 w|^2 + \frac{1}{2}|\xi'|^2|\phi|^2\\
&\quad + \frac{1}{2}\xi'(\phi\partial_3\bar{w} + \bar{\phi}\partial_3 w) + \int_{\mathbb{R}^2}\frac{1}{2}\left(1-\Omega^2\right)|r|^2|\phi|^2 + \int_{\mathbb{R}^3}\frac{1}{2}\left(1-\Omega^2\right)|r|^2|w|^2\\
&\quad + \int_{\mathbb{R}^3}\frac{1}{2}x_3^2|w|^2 + \frac{1}{2}x_3^2|\phi|^2\xi^2 + \frac{1}{2}x_3^2\xi(w\bar{\phi} + \bar{w}\phi) + \frac{1}{4}a\int_{\mathbb{R}^3}|\psi|^4.
\end{aligned}$$

Let us point out that the expansions of $|\nabla'\psi - \mathrm{i}\Omega e_3\times x\psi|^2$ and $|r|^2|\psi|^2$ do not produce any cross terms because of the orthogonality condition between w and ξ. In order to get rid of the other cross terms, we use an integration by part on the third variable, as well as the equation for ξ and the orthogonality condition between w and ξ. Finally, we use that $\int\xi'^2 + x_3^2\xi^2 = 1$ and obtain

$$\begin{aligned}
I_{3\mathrm{D}}(\Omega) &= \int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \frac{1}{2}\left(1-\Omega^2\right)|r|^2|\phi|^2 + \frac{1}{2}|\phi|^2\\
&\quad + \int_{\mathbb{R}^3}\frac{1}{2}|\nabla' w - \mathrm{i}\Omega e_3\times xw|^2 + \frac{1}{2}\left(1-\Omega^2\right)|r|^2|w|^2 + \frac{1}{4}a|\psi|^4 + \int_{\mathbb{R}^3}\frac{1}{2}\left(|\partial_3 w|^2 + x_3^2|w|^2\right).
\end{aligned} \tag{2.6}$$

Let us point out that (2.6) provides that $I_{3D}(\Omega)$ is almost $\int_{\mathbb{R}^2}\frac{1}{2}|\phi|^2 + E_{2D}(\phi) + E_{3D}(w)$ except for the quartic term $\int|\psi|^4$ that we have not expanded in terms of w and ϕ. Our computations aim at deriving the smallness of w from this identity.

Since the second eigenvalue of $-\frac{d^2}{dx_3^2} + x_3^2$ is equal to 3, (2.5) and (2.6) imply

$$I_{3D}(\Omega) \geqslant \int_{\mathbb{R}^2} \frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \int_{\mathbb{R}^2}\frac{1}{2}|\phi|^2 + \frac{3}{2}\int_{\mathbb{R}^3}|w|^2 + \int_{\mathbb{R}^3}\frac{1}{2}|\nabla' w - \mathrm{i}\Omega e_3\times xw|^2$$
$$= \frac{1}{2} + \int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \int_{\mathbb{R}^3}\frac{1}{2}|\nabla' w - \mathrm{i}\Omega e_3\times xw|^2 + \int_{\mathbb{R}^3}|w|^2.$$

We now use the fact that the operator $-(\nabla - \mathrm{i}\Omega r^\perp)^2$ defined on $\mathbb{R}^2$ has a first eigenvalue equal to 2Ω. Hence, because of (2.5), we have $I_{3D}(\Omega) \geqslant 1/2 + \Omega + \|w\|_{L^2}^2$. We finally use (2.3) and the property (1.10), and find

$$\int_{\mathbb{R}^3}|w|^2 \leqslant C\sqrt{1-\Omega}. \tag{2.7}$$

We are now going to improve this estimate on the smallness of w using the energy. We start with the inequality

$$\int_{\mathbb{R}^3}\frac{1}{2}|\nabla w - \mathrm{i}\Omega e_3\times xw|^2 \geqslant \frac{1}{3}\int_{\mathbb{R}^3}|\nabla w - \mathrm{i}\Omega e_3\times xw|^2 + \frac{1}{6}\int_{\mathbb{R}^3}|\partial_3 w|^2.$$

We use that w is orthogonal to ξ and that the second eigenvalue of $-\frac{d^2}{dx_3^2} + x_3^2$ is equal to 3 to get from (2.6):

$$I_{3D}(\Omega) \geqslant \int_{\mathbb{R}^2} \frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \int_{\mathbb{R}^2}\frac{1}{2}|\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^3}|w|^2 + \int_{\mathbb{R}^3}\frac{1}{3}|\nabla' w - \mathrm{i}\Omega e_3\times xw|^2$$
$$= \frac{1}{2} + \int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\phi - \mathrm{i}\Omega r^\perp\phi\right|^2 + \int_{\mathbb{R}^3}\frac{1}{3}|\nabla w - \mathrm{i}\Omega e_3\times xw|^2. \tag{2.8}$$

This together with the diamagnetic inequality (see [17])

$$|\nabla w - \mathrm{i}\Omega e_3\times xw|^2 \geqslant |\nabla|w||^2 \quad \text{almost everywhere} \tag{2.9}$$

provides that $\|\nabla|w|\|_{L^2}^2 \leqslant C\sqrt{1-\Omega}$. With (2.7), this yields

$$\big\||w|\big\|_{H^1(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{2.10}$$

Sobolev embeddings then imply that

$$\forall p \leqslant 6, \quad \|w\|_{L^p(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{2.11}$$

We go back to (2.6) and expand the quartic term to get

$$I_{3D}(\Omega) = E_{2D}(\phi) + E_{3D}(w) + \frac{1}{2}\int_{\mathbb{R}^2}|\phi|^2 - \frac{a}{4}\int_{\mathbb{R}^3}|w|^4$$
$$+ \frac{a}{4}\int_{\mathbb{R}^3} 2|\phi|^2\xi^2|w|^2 + 4\left(\Re(\phi\xi\bar{w}) + \frac{1}{2}|w|^2\right)^2 + 4|\phi|^2\xi^2\Re(\phi\xi\bar{w}), \tag{2.12}$$

where $\Re(z)$ denotes the real part of the complex number z. We bound (2.12) from below using the same trick as in (2.8) for the L^2 norm to obtain

$$I_{3D}(\Omega) \geqslant E_{2D}(\phi) + \frac{1}{2}\int_{\mathbb{R}^2}|\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^3}|w|^2 - a\int_{\mathbb{R}^3}|\phi|^3\xi^3|w|. \tag{2.13}$$

We have

$$\int_{\mathbb{R}^3} |\phi|^3\xi^3|w| \leqslant \frac{1}{8}\int_{\mathbb{R}^3} |\phi|^4\xi^4 + 54\int_{\mathbb{R}^3} |w|^4. \tag{2.14}$$

Estimates (2.11), (2.13) and (2.14) imply

$$I_{3\mathrm{D}}(\Omega) \geqslant E_{2\mathrm{D}}(\phi) + \frac{1}{2} - \frac{1}{8}b\int_{\mathbb{R}^3} |\phi|^4 - C(1-\Omega).$$

Since $E_{2\mathrm{D}}$ contains the quartic term with a coefficient $b/4$, this, together with (2.3), implies

$$\int_{\mathbb{R}^2} |\phi|^4 \leqslant C\sqrt{1-\Omega}. \tag{2.15}$$

This in turn allows to improve (2.14):

$$\int_{\mathbb{R}^3} |\phi|^3\xi^3|w| \leqslant \biggl(\int_{\mathbb{R}^3} |\phi|^4\xi^4\biggr)^{3/4}\biggl(\int_{\mathbb{R}^3} |w|^4\biggr)^{1/4} \leqslant C(1-\Omega)^{5/8},$$

which provides a better lower bound for (2.13), namely

$$I_{3\mathrm{D}}(\Omega) \geqslant I_{2\mathrm{D}}(\Omega) + \frac{1}{2} - C(1-\Omega)^{5/8}. \tag{2.16}$$

Inequalities (2.3) and (2.16) imply (2.1). □

Remark 2.2. This proof implies in particular that ϕ defined by (2.4) has an almost minimizing energy as Ω tends to 1, since

$$E_{2\mathrm{D}}(\phi) = I_{2\mathrm{D}}(\Omega) + \mathrm{o}(\sqrt{1-\Omega}\,),$$

while $I_{2\mathrm{D}}(\Omega) = 1 + \mathrm{O}(\sqrt{1-\Omega}\,)$. However, as pointed out in the introduction, this does not prove that ϕ is close to a solution of $I_{2\mathrm{D}}(\Omega)$.

2.2. Convergence of the minimizers

In this subsection, we prove the second part of Theorem 1.1, that is:

Proposition 2.3. *Let ψ be a minimizer of* (1.7)*, and let $\xi(t) = \mathrm{e}^{-\omega_3 t^2/2}(\omega_3/2\pi)^{1/4}$. Then, there exists $\phi \in \mathcal{H}_{2\mathrm{D}} \cap C^{0,1/2}$ such that*

$$\bigl\| |\psi - \phi(x_1,x_2)\xi(x_3)| \bigr\|_{H^1(\mathbb{R}^3)\cap C^{0,1/2}(\mathbb{R}^3)} \underset{\Omega\to 1}{\longrightarrow} 0. \tag{2.17}$$

The proof of this proposition consists in improving the estimates on w obtained earlier. It relies on a careful use of the Euler Lagrange equation together with elliptic estimates detailed in the following lemma, which is an adaptation of the estimates of Lu and Pan [20] to the present three-dimensional setting:

Lemma 2.4. *Consider a vector-valued map $A \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^3)$ such that* $\operatorname{div} A = 0$. *Let $\mathcal{D}$ be a domain of $\mathbb{R}^3$, and let $g \in L^2(\mathcal{D})$. Let w be a solution of*

$$-(\nabla - \mathrm{i}A)^2 w = g \quad \text{in } \mathcal{D}. \tag{2.18}$$

Then, for any $R > 0$ such that $B_{2R} \subset \mathcal{D}$, there exists a constant $C > 0$ depending only on $\|\Delta A\|_{L^\infty(B_{2R})}$, $\|\operatorname{curl} A\|_{L^\infty(B_{2R})}$ and R, such that

$$\sum_{j,k=1}^{3}\int_{B_R} \bigl|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\bigr|^2 \leqslant C\biggl(\int_{B_{2R}} |g|^2 + \int_{B_{2R}} |w|^2\biggr). \tag{2.19}$$

Moreover, C remains bounded as R goes to infinity.

Proof. We first prove the following estimate:

$$\int_{B_R} \left|(\nabla - \mathrm{i}A)w\right|^2 \leqslant \frac{1}{2}\int_{B_{2R}} |g|^2 + \frac{1}{2}\int_{B_{2R}} |w|^2 + \frac{4}{R^2}\int_{B_{2R}} |w|^2. \tag{2.20}$$

For this purpose, we define a cut-off function $\eta \in \mathcal{D}(\mathbb{R}^3)$ such that

$$0 \leqslant \eta \leqslant 1, \quad \eta = 1 \quad \text{in } B_R, \qquad \eta = 0 \quad \text{in } B_{2R}, \qquad |\nabla \eta| \leqslant \frac{2}{R}. \tag{2.21}$$

Multiplying (2.18) by $\eta^2 \bar{w}$, integrating by parts, computing $\int |(\nabla - \mathrm{i}A)(\eta w)|^2$ and taking the real part of the equation yields (2.20).

We next prove the following estimate:

$$\sum_{j,k=1}^{3} \int_{B_R} \left|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\right|^2 \leqslant 2\int_{B_{2R}} |g|^2 + \|\Delta A\|_{L^\infty(B_{2R})} \|w\|_{L^2(B_{2R})} \left\|(\nabla - \mathrm{i}A)w\right\|_{L^2(B_R)} + \|\operatorname{curl} A\|_{L^\infty(B_{2R})} \int_{B_{2R}} \left|(\nabla - \mathrm{i}A)w\right|^2 + \frac{8}{R^2}\int_{B_{2R}} \left|(\nabla - \mathrm{i}A)w\right|^2. \tag{2.22}$$

We first set $w_j = (\partial_j - \mathrm{i}A_j)w$, and write down the equation satisfied by w_j:

$$(\nabla - \mathrm{i}A)^2 w_j = (\partial_j - \mathrm{i}A_j)(\nabla - \mathrm{i}A)^2 w - \mathrm{i}\Delta A_j w + 2\mathrm{i}(\partial_j A - \nabla A_j)(\nabla - \mathrm{i}A)w.$$

Hence, we have

$$-(\nabla - \mathrm{i}A)^2\big((\nabla - \mathrm{i}A)w\big) = (\nabla - \mathrm{i}A)g + \mathrm{i}(\Delta A)w - 2\mathrm{i}\operatorname{curl} A \times (\nabla - \mathrm{i}A)w. \tag{2.23}$$

We next multiply (2.23) by $\eta^2\overline{(\nabla - \mathrm{i}A)w}$, where $\eta \in \mathcal{D}(\mathbb{R}^3)$ satisfies (2.21), and integrate. Integrating by parts and taking the real part of the result, we find

$$\sum_{j,k=1}^{3} \int \left|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)(\eta w)\right|^2 = \int |\nabla\eta|^2 \left|(\nabla - \mathrm{i}A)w\right|^2 + \Re\left[\int \eta^2 \overline{(\nabla - \mathrm{i}A)w}\,(\nabla - \mathrm{i}A)g + \mathrm{i}\int (\Delta A)\overline{(\nabla - \mathrm{i}A)w}\, w\eta^2 + 2\mathrm{i}\int \big(\operatorname{curl}(A) \times (\nabla - \mathrm{i}A)w\big)\overline{(\nabla - \mathrm{i}A)w}\,\eta^2\right].$$

In addition, we have

$$\int \eta^2 \overline{(\nabla - \mathrm{i}A)w}\,(\nabla - \mathrm{i}A)g = \int \eta^2 |g|^2 - 2\int \eta\nabla\eta\overline{(\nabla - \mathrm{i}A)w}\, g.$$

Hence,

$$\sum_{j,k=1}^{3} \int \left|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)(\eta w)\right|^2 \leqslant \int_{B_{2R}} |g|^2 + \int |\nabla\eta|^2 \left|(\nabla - \mathrm{i}A)w\right|^2 + 2\int \eta|\nabla\eta|\left|(\nabla - \mathrm{i}A)w\right||g| + \int \eta^2 \left|(\nabla - \mathrm{i}A)w\right||w||\Delta A| + \int \left|\operatorname{curl}(A)\right|\left|\big(\nabla - \mathrm{i}A\big)w\right|^2 \eta^2,$$

which implies (2.22). We then apply successively (2.22) and (2.20), finding (if $B_{4R} \subset \mathcal{D}$):

$$\sum_{j,k=1}^{3} \int_{B_R} \left|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\right|^2 \leqslant \left[2 + \frac{\|\Delta A\|_{L^\infty(B_{2R})}}{4} + \frac{\|\operatorname{curl} A\|_{L^\infty(B_{2R})}}{2} + \frac{4}{R^2}\right]\int_{B_{4R}} |g|^2 + \left[\left(\frac{3}{4} + \frac{1}{2R^2}\right)\|\Delta A\|_{L^\infty(B_{2R})} + \left(\frac{1}{2} + \frac{1}{R^2}\right)\|\operatorname{curl} A\|_{L^\infty(B_{2R})}\right]\int_{B_{4R}} |w|^2.$$

This concludes the proof of Lemma 2.4. □

Proof of Proposition 2.3. For simplicity, we set $\omega_3 = 1$. Let ψ be a minimizer of (1.7), ξ the one-dimensional normalized Gaussian, and let ϕ be the projection of ψ onto the space spanned by ξ so that $w = \psi - \xi\phi$ and ϕ satisfy (2.4). According to the proof of Proposition 2.1, w satisfies (2.10), namely the convergence in $H^1(\mathbb{R}^3)$. We define $A(x) = \Omega e_3 \times x$, which satisfies the hypotheses of Lemma 2.4. The Euler Lagrange equation satisfied by ψ is

$$-(\nabla - \mathrm{i}A)^2\psi + \left(1-\Omega^2\right)|r|^2\psi + x_3^2\psi + a|\psi|^2\psi = \lambda\psi, \tag{2.24}$$

where λ is the Lagrange multiplier associated with the L^2 constraint in (1.7). Multiplying the equation by ψ and integrating, we see that $I_{3\mathrm{D}}(\Omega) \leqslant \lambda \leqslant 2I_{3\mathrm{D}}(\Omega)$. Moreover, multiplying (2.24) by ξ and integrating with respect to x_3, yields

$$-\nabla_\Omega^2\phi + \left(1-\Omega^2\right)|r|^2\phi + \phi + a\int_{\mathbb{R}} |\psi|^2\psi\xi = \lambda\phi, \tag{2.25}$$

where the operator ∇_Ω^2 is equal to $\nabla_\Omega^2 = (\nabla - \mathrm{i}\Omega r^\perp)^2$ in two dimensions. Subtracting (2.25), multiplied by $\xi(x_3)$, to (2.24), we thus find an equation satisfied by w:

$$-(\nabla - \mathrm{i}A)^2 w = \lambda w + a|\psi|^2\psi - a\xi\int_{\mathbb{R}} |\psi|^2\psi\xi - \left(1-\Omega^2\right)|r|^2 w - x_3^2 w. \tag{2.26}$$

We are now going to prove that the right-hand side of (2.26) is suitably bounded in $L^2(B_R)$ for any ball B_R of radius $R \geqslant 1$.

We first write

$$\|\lambda w\|_{L^2(B_R)} \leqslant \lambda\|w\|_{L^2(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{2.27}$$

Next, we need to bound $|\psi|^2\psi$. For this purpose, we set $f = |\psi|^2$ and write down the equation satisfied by f:

$$-\Delta f + 2\mathrm{i}\Omega e_3 \times x(\nabla\psi\bar{\psi} - \overline{\nabla\psi}\psi) + 2|\nabla\psi|^2 + 2|r|^2 f + 2x_3^2 f + 2af^2 = 2\lambda f,$$

which implies

$$-\Delta f + 2\left(1-\Omega^2\right)|r|^2 f + 2x_3^2 f + 2af^2 \leqslant 2\lambda f. \tag{2.28}$$

Using suitable supersolutions which blow up as in [10], we get

$$|\psi|^2 = f \leqslant \frac{\lambda}{a}. \tag{2.29}$$

Hence, $|\psi|$ is bounded in $L^\infty(\mathbb{R}^3)$, independently of Ω. We use (2.11) and (2.15) to get

$$\big\||\psi|^2\psi\big\|_{L^2(\mathbb{R}^3)} \leqslant C\left(\int_{\mathbb{R}^3} |\psi|^4\right)^{1/2} \leqslant C(1-\Omega)^{1/4}. \tag{2.30}$$

The same kind of method applies to the third term of the right-hand side of (2.26), we thus turn to the last two terms. In order to estimate them, we write down the equation satisfied by $g = |w|^2$, which implies:

$$-\Delta g + 2\left(1-\Omega^2\right)|r|^2 g + 2x_3^2 g \leqslant 2\lambda g + 2|\psi|^2\Re(\psi\bar{w}) - 2\xi\int_{\mathbb{R}} |\psi|^2\xi\Re(\psi\bar{w}). \tag{2.31}$$

We multiply this equation by $|r|^2$ and integrate over $\mathbb{R}^3$, finding

$$-4\int_{\mathbb{R}^3} g + 2\left(1-\Omega^2\right)\int_{\mathbb{R}^3} |r|^4 g + 2\int_{\mathbb{R}^3} x_3^2|r|^2 g \leqslant 2\lambda\int_{\mathbb{R}^3} |r|^2 g + 2C\int_{\mathbb{R}^3} |r|^2|w||\psi|,$$

where we have used the inequality (2.29). We thus have, using (2.10),

$$\left(1-\Omega^2\right)\int_{\mathbb{R}^3} |r|^4 g \leqslant C\sqrt{1-\Omega} + C\int_{\mathbb{R}^3} |r|^2 g + C\int_{\mathbb{R}^3} |r|^2|\psi|^2.$$

Due to the energy estimates $I_{3\mathrm{D}}(\Omega) \leqslant 1 + C\sqrt{1-\Omega}$, we also have $(1-\Omega)\int |r|^2|\psi|^2 \leqslant C\sqrt{1-\Omega}$. In addition, multiplying (2.26) by $\bar{w}$ and integrating, we also have $(1-\Omega)\int |r|^2 g \leqslant C\sqrt{1-\Omega}$. Hence,

$$(1-\Omega)\big\||r|^2 w\big\|_{L^2(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{2.32}$$

We treat the last term of (2.26) by the same kind of method, multiplying (2.31) by $x_3^2 g$ and integrating. This gives $\int x_3^4 g \leqslant C(1-\Omega)^{1/2}$, hence

$$\big\|x_3^2 w\big\|_{L^2(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{2.33}$$

Hence, collecting (2.7), (2.27), (2.30), (2.32) and (2.33) and applying Lemma 2.4, we have

$$\sum_{j,k=1}^{3} \int_{B_R} \big|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\big|^2 \leqslant C\sqrt{1-\Omega},$$

for any $R \geqslant 1$, where the constant C does not depend on R, nor on the center of the ball B_R. Hence, using the diamagnetic inequality (2.9) twice, we infer

$$\big\|\nabla|w|\big\|_{H^1(B_R)} \leqslant C(1-\Omega)^{1/4},$$

with C independent of R and of the center of the ball B_R. This estimate and (2.10) imply

$$\big\||w|\big\|_{H^2(B_R)} \leqslant C(1-\Omega)^{1/4}.$$

Then, using the Sobolev embeddings, we have

$$\big\||w|\big\|_{C^{0,1/2}(B_R)} \leqslant C(1-\Omega)^{1/4}.$$

Here again, the constant C does not depend on the center of the ball B_R. □

3. LLL reduction

This section is devoted to the proof of Theorem 1.2. We need a preliminary lemma which states that the Gross–Pitaevskii energy has a simplified expression in the reduced space L. We prove (1.21) in Subsection 3.1, and then prove the convergence of minimizers in Subsection 3.2. Here again, C is a generic constant independent of Ω.

Lemma 3.1. *If $u \in L$, where L is defined by* (1.14)*, and if $\|u\|_{L^2(\mathbb{R}^2)} = 1$, then the Gross–Pitaevskii energy* (1.5) *has a simplified expression given by* (1.18) *namely,*

$$E_{2\mathrm{D}}(u) = \Omega + \int_{\mathbb{R}^2} (1-\Omega)|r|^2|u|^2 + \frac{b}{4}|u|^4 = E_{LLL}(u). \tag{3.1}$$

Proof. Consider $u \in L$ such that $\|u\|_{L^2(\mathbb{R}^2)} = 1$: according to (1.14), we have

$$u(x_1,x_2) = f(x_1 + \mathrm{i}x_2)\mathrm{e}^{-(x_1^2+x_2^2)/2},$$

where f is a holomorphic function. One of the properties of L is that such a u is an eigenfunction for $-\nabla_1^2 = -(\nabla - \mathrm{i}r^\perp)^2$ with eigenvalue 1. We use this property to compute the first term of the energy:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla u - \mathrm{i}\Omega r^\perp u\big|^2 &= \frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla u - \mathrm{i}r^\perp u\big|^2 + \frac{(1-\Omega)^2}{2}\int_{\mathbb{R}^2} |r|^2|u|^2 + (1-\Omega)\int_{\mathbb{R}^2} \Re\big((\nabla u - \mathrm{i}r^\perp u)\mathrm{i}r^\perp \bar{u}\big)\\ &= 1 + \frac{\Omega^2-1}{2}\int_{\mathbb{R}^2} |r|^2|u|^2 + (1-\Omega)\int_{\mathbb{R}^2} \Re\big(\mathrm{i}r^\perp \nabla u \bar{u}\big).\end{aligned} \tag{3.2}$$

We have $\nabla u = (\nabla f - rf)\mathrm{e}^{-|r|^2/2}$, which implies $r^\perp\nabla u = r^\perp \nabla f \mathrm{e}^{-|r|^2/2}$. We now set $z = x_1 + \mathrm{i}x_2$, and $\partial_z = \frac{1}{2}(\partial_{x_1} - \mathrm{i}\partial_{x_2})$. Since f is holomorphic, $\partial_{\bar{z}} f = 0$ and we find

$$r^\perp \nabla u = \mathrm{i}z\partial_z f \mathrm{e}^{-|r|^2/2} = \mathrm{i}z\partial_z f \mathrm{e}^{-|z|^2/2}.$$

Hence, integrating by parts, we get

$$\int_{\mathbb{R}^2} \Re\big(\mathrm{i}r^\perp \nabla u \bar{u}\big) = \Re\bigg(\int_{\mathbb{R}^2} z\partial_z f \bar{f} \mathrm{e}^{-|z|^2}\bigg)$$

$$= \Re\bigg(-\int_{\mathbb{R}^2} \big|f(z)\big|^2 \mathrm{e}^{-|z|^2} + \int_{\mathbb{R}^2} |z|^2 \big|f(z)\big|^2 \mathrm{e}^{-|z|^2}\bigg)$$

$$= -1 + \int_{\mathbb{R}^2} |r|^2 |u|^2.$$

Inserting this equality in (3.2), we find

$$\frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla u - \mathrm{i}\Omega r^\perp u\big|^2 = \Omega + \frac{(1-\Omega)^2}{2}\int_{\mathbb{R}^2} |r|^2|u|^2,$$

which implies (3.1). □

3.1. Convergence of the energy

We prove (1.21), namely:

Proposition 3.2. *For any $\Omega \in (0,1)$, let $I_{2\mathrm{D}}(\Omega)$ and $I_{LLL}(\Omega)$ be defined by* (1.8) *and* (1.19)*, respectively. Then*

$$I_{2\mathrm{D}}(\Omega) = I_{LLL}(\Omega) + \mathrm{o}(\sqrt{1-\Omega}\,), \tag{3.3}$$

as Ω tends to 1.

Proof. We clearly have

$$I_{2\mathrm{D}}(\Omega) \leqslant I_{LLL}(\Omega), \tag{3.4}$$

since, in order to compute $I_{LLL}(\Omega)$, we reduce the space of minimization. In order to bound $I_{2\mathrm{D}}(\Omega)$ from below, we denote by ϕ one of its minimizers, u its projection onto L in $L^2(\mathbb{R}^2)$, that is given by (1.17), and w their difference:

$$E_{2\mathrm{D}}(\phi) = I_{2\mathrm{D}}(\Omega), \quad u = \Pi_L(\phi), \quad \phi = u + w.$$

The operators $-\frac{1}{2}\Delta + \frac{1}{2}|r|^2$ and $\mathrm{i}r^\perp\nabla$ have the same eigenvectors, namely

$$\phi_{j,k} = \mathrm{e}^{|r|^2/2}(\partial_{x_1} + \mathrm{i}\partial_{x_2})^j (\partial_{x_1} - \mathrm{i}\partial_{x_2})^k \big(\mathrm{e}^{-|r|^2}\big).$$

The corresponding eigenvalues are $1 + j + k$ for $-\frac{1}{2}\Delta + \frac{1}{2}|r|^2$ and $j - k$ for $\mathrm{i}r^\perp\nabla$. Hence, each $\phi_{j,k}$ is also an eigenvector for

$$H_\Omega = -\frac{1}{2}\Delta + \frac{1}{2}|r|^2 - \mathrm{i}\Omega r^\perp\nabla = -\frac{1}{2}\big(\nabla - \mathrm{i}\Omega r^\perp\big)^2 + \frac{1}{2}\big(1-\Omega^2\big)|r|^2,$$

with eigenvalue

$$\lambda_{j,k} = 1 + (1-\Omega)j + (1+\Omega)k.$$

Since L is spanned by $\phi_{j,0}$ for $j \in \mathbb{N}$, it is stable under the action of H_Ω. Hence, we have $\langle H_\Omega u|w\rangle = 0$. We thus have a decoupling of the energy, exactly as in the proof of Proposition 2.1 (see (2.6)):

$$E_{2\mathrm{D}}(\phi) = \frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla u - \mathrm{i}\Omega r^\perp u\big|^2 + \frac{1-\Omega^2}{2}\int_{\mathbb{R}^2} |r|^2|u|^2 + \frac{b}{4}\int_{\mathbb{R}^2} |\phi|^4$$
$$+ \frac{1}{2}\int_{\mathbb{R}^2} \big|\nabla w - \mathrm{i}\Omega r^\perp w\big|^2 + \frac{1-\Omega^2}{2}\int_{\mathbb{R}^2} |r|^2|w|^2. \tag{3.5}$$

The lowest eigenvalue of H_Ω is equal to 1. When restricted to the orthogonal of L the lowest eigenvalue is $2+\Omega$, so that

$$I_{2\mathrm{D}}(\Omega) = E_{2\mathrm{D}}(\phi) \geqslant \int_{\mathbb{R}^2} |u|^2 + (2+\Omega)\int_{\mathbb{R}^2} |w|^2 + \frac{b}{4}\int_{\mathbb{R}^2} |\phi|^4$$
$$\geqslant 1 + (1+\Omega)\int_{\mathbb{R}^2} |w|^2.$$

We thus have

$$\int_{\mathbb{R}^2} |w|^2 \leqslant C\sqrt{1-\Omega}, \tag{3.6}$$

where we have used (1.10). Going back to (3.5), we use the same remark about the eigenvalues of H_Ω to infer:

$$I_{2\mathrm{D}}(\Omega) = E_{2\mathrm{D}}(\phi) \geqslant 1 + \frac{1}{6}\int_{\mathbb{R}^2} \left|\nabla w - \mathrm{i}\Omega r^\perp w\right|^2.$$

The diamagnetic inequality (2.9) (see [17]) thus implies $\int |\nabla |w||^2 \leqslant C\sqrt{1-\Omega}$. With the help of (3.6), we deduce

$$\big\| |w| \big\|_{H^1(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{3.7}$$

Hence, using Sobolev embeddings,

$$\forall p < \infty, \quad \|w\|_{L^p(\mathbb{R}^2)} \leqslant C_p(1-\Omega)^{1/4}, \tag{3.8}$$

where C_p depends on p but not on Ω. Next, we expand the quartic term of the energy, and find

$$E_{2\mathrm{D}}(\phi) = E_{LLL}(u) + E_{2\mathrm{D}}(w) - \frac{b}{4}\int_{\mathbb{R}^3} |w|^4 + \frac{b}{4}\int_{\mathbb{R}^3} 2|u|^2|w|^2 + 4\left(\Re(u\bar{w}) + \frac{1}{2}|w|^2\right)^2 + 4|u|^2\Re(u\bar{w}),$$
$$\geqslant E_{LLL}(u) + E_{2\mathrm{D}}(w) - \frac{b}{4}\int_{\mathbb{R}^3} |w|^4 - b\int_{\mathbb{R}^2} |u|^3|w|. \tag{3.9}$$

We then use the same argument as in the proof of Proposition 2.1, proving that

$$\int_{\mathbb{R}^2} |u|^3|w| \leqslant \frac{1}{8}\int_{\mathbb{R}^2} |u|^4 + 54\int_{\mathbb{R}^2} |w|^4,$$

and finding, using (3.8),

$$E_{2\mathrm{D}}(\phi) \geqslant E_{LLL}(u) - \frac{b}{8}\int_{\mathbb{R}^2} |u|^4 - C(1-\Omega).$$

We thus have, using the energy estimates (1.10) and (1.20),

$$\int_{\mathbb{R}^2} |u|^4 \leqslant C\sqrt{1-\Omega}.$$

Hence, using Hölder's inequality,

$$\int_{\mathbb{R}^2} |u|^3|w| \leqslant \left(\int_{\mathbb{R}^2} |u|^4\right)^{3/4}\left(\int_{\mathbb{R}^2} |w|^4\right)^{1/4} \leqslant C(1-\Omega)^{5/8}.$$

Inserting this inequality in (3.9), we thus have

$$I_{2\mathrm{D}}(\Omega) \geqslant E_{LLL}(u) - C(1-\Omega)^{5/8}. \tag{3.10}$$

Inequalities (3.4) and (3.10) imply (3.3). □

3.2. Convergence of the minimizers

In this subsection, we prove the second part of Theorem 1.2, that is:

Proposition 3.3. *Let ϕ be a minimizer of* (1.8), *and let $\Pi_L\phi$ be the projection of ϕ on L. Then, for any $\alpha < 1$, we have, as Ω tends to 1,*

$$\big\| |\phi - \Pi_L\phi| \big\|_{H^1(\mathbb{R}^3)\cap C^{0,\alpha}(\mathbb{R}^2)} \longrightarrow 0. \tag{3.11}$$

Let us first recall a result of Lu and Pan [20, Theorem 4.1], which will be useful in the sequel:

Lemma 3.4. (Lu and Pan, [20]) *Consider a vector-valued map $A \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ such that* $\operatorname{div} A = 0$. *Let $\mathcal{D}$ be an open domain of $\mathbb{R}^2$, and let $g \in L^2(\mathcal{D})$. Let w be a solution of*

$$-(\nabla - \mathrm{i}A)^2 w = g \quad \text{in } \mathcal{D}. \tag{3.12}$$

Then, for any $R > 0$ such that $B_{2R} \subset \mathcal{D}$, there exists a constant $C > 0$ depending only on $\|\Delta A\|_{L^\infty(B_{2R})}$, $\|\operatorname{curl} A\|_{L^\infty(B_{2R})}$ and R, such that

$$\sum_{j,k} \int_{B_R} \big|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\big|^2 \leqslant C\bigg(\int_{B_{2R}} |g|^2 + \int_{B_{2R}} |w|^2\bigg). \tag{3.13}$$

Moreover, C remains bounded as R goes to infinity.

Proof of Proposition 3.3. We use the same strategy as in the proof of Proposition 2.3. We first write down the equation satisfied by the minimizer ϕ of (1.8), namely

$$H_\Omega\phi + b|\phi|^2\phi = \lambda\phi, \tag{3.14}$$

where $H_\Omega = -(\nabla - \mathrm{i}\Omega r^\perp)^2 + (1-\Omega^2)|r|^2$, and $\lambda \geqslant 0$ is the Lagrange multiplier associated with the L^2 constraint. Multiplying (3.14) and integrating, one finds $I_{2\mathrm{D}}(\Omega) \leqslant \lambda \leqslant 2I_{2\mathrm{D}}(\Omega)$, so that λ is bounded independently of Ω. In addition, as pointed out in the proof of Proposition 3.1, H_Ω commutes with Π_L, the projector onto L. Hence, applying Π_L to (3.14), we get:

$$H_\Omega\Pi_L\phi + b\Pi_L\big(|\phi|^2\phi\big) = \lambda\Pi_L\phi. \tag{3.15}$$

Subtracting (3.14) and (3.15), we thus find that $w = \phi - \Pi_L\phi$ is a solution of

$$-\big(\nabla - \mathrm{i}\Omega r^\perp\big)^2 w = \lambda w + b|\phi|^2\phi - b\Pi_L\big(|\phi|^2\phi\big) - \big(1-\Omega^2\big)|r|^2 w. \tag{3.16}$$

We bound the right-hand side of (3.16) and apply Lemma 3.4 on a ball of radius $R > 1$. We first have:

$$\|\lambda w\|_{L^2(B_R)} \leqslant \lambda\|w\|_{L^2(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{3.17}$$

Next, we deal with $|\phi|^2\phi$. For this purpose, we argue exactly as in the proof of Proposition 2.3: we write down the equation satisfied by $|\phi|^2$, and use the maximum principle to prove that $|\phi| \leqslant \lambda/b$. This implies

$$\big\||\phi|^2\phi\big\|_{L^2(\mathbb{R}^2)} \leqslant C\bigg(\int_{\mathbb{R}^2} |\phi|^4\bigg)^{1/2} \leqslant C(1-\Omega)^{1/4}. \tag{3.18}$$

Since Π_L is a projector, the same estimate holds for the term $\Pi_L(|\phi|^2\phi)$. We then bound the last term of (3.16). For this purpose, we set $g = |w|^2$, and write down the equation satisfied by g. We have in particular:

$$-\Delta g + 2\big(1-\Omega^2\big)|r|^2 g \leqslant 2\lambda g + 2b\Re\big(|\phi|^2\phi\bar{w}\big) - 2b\Re\big(\Pi_L\big(|\phi|^2\phi\big)\bar{w}\big). \tag{3.19}$$

We multiply this equation by $|r|^2$ and integrate, finding

$$-4\int_{\mathbb{R}^2} g + 2\big(1-\Omega^2\big)\int_{\mathbb{R}^2} |r|^4 g \leqslant 2\lambda\int_{\mathbb{R}^2} |r|^2 g + 2b\int_{\mathbb{R}^2} |\phi|^3|w||r|^2 g + 2b\bigg|\int_{\mathbb{R}^2} \Pi_L\big(r|\phi|^2\phi\big)\overline{rwg}\bigg|. \tag{3.20}$$

The last two terms of this inequality are bounded in the same way as follows:

$$\left|\int_{\mathbb{R}^2} \Pi_L\big(r|\phi|^2\phi\big)\overline{rwg}\right| \leqslant \frac{1}{2}\big\|\Pi_L\big(r|\phi|^2\phi\big)\big\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|rwg\|^2_{L^2(\mathbb{R}^2)}$$
$$\leqslant \frac{1}{2}\int_{\mathbb{R}^2}|r|^2|\phi|^6 + \frac{1}{2}\int_{\mathbb{R}^2}|r|^2|w|^6$$
$$\leqslant C\int_{\mathbb{R}^2}|r|^2|\phi|^2 + C\int_{\mathbb{R}^2}|r|^2|w|^2$$
$$\leqslant \frac{C}{\sqrt{1-\Omega}},$$

where the last inequality has been obtained by multiplying (3.15) by $\bar\phi$ and integrating on the one hand, and by multiplying (3.16) by $\bar w$ and integrating on the other hand. Finally, inserting this estimate in (3.20), we have

$$(1-\Omega)\big\||r|^2 w\big\|_{L^2(\mathbb{R}^3)} \leqslant C(1-\Omega)^{1/4}. \tag{3.21}$$

Hence, collecting (3.17), (3.18) and (3.21), we see that (3.16) implies

$$\big\|-\big(\nabla - \mathrm{i}\Omega r^\perp\big)^2 w\big\|_{L^2(\mathbb{R}^2)} \leqslant C(1-\Omega)^{1/4}.$$

Hence, applying Lemma 3.4 with $A = \Omega r^\perp$, we find that

$$\sum_{j,k}\int_{B_R}\big|(\partial_j - \mathrm{i}A_j)(\partial_k - \mathrm{i}A_k)w\big|^2 \leqslant C\sqrt{1-\Omega},$$

for any $R \geqslant 1$, where the constant C does not depend on R, nor on the center of the ball B_R. Hence, using the diamagnetic inequality (2.9), we infer

$$\big\|\big|\big(\nabla - \mathrm{i}\Omega r^\perp\big)w\big|\big\|_{H^1(B_R)} \leqslant C(1-\Omega)^{1/4},$$

with C independent of R and of the center of the ball B_R. Using the Sobolev embeddings, we thus have $\|(\nabla - \mathrm{i}\Omega r^\perp)w\|_{L^p(\mathbb{R}^3)} \leqslant C_p(1-\Omega)^{1/4}$, for all $p < \infty$. Using the diamagnetic inequality and Sobolev embeddings once again, we finally get, for any $\alpha \in (0,1)$,

$$\big\||w|\big\|_{C^{0,\alpha}(B_R)} \leqslant C_\alpha(1-\Omega)^{1/4}.$$

Here again, the constant C_α does not depend on the center of the ball B_R, so (3.11) is proved. □

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