



Comparison of the Calabi and Mabuchi geometries and applications to geometric flows

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Abstract

Suppose (X, ω) is a compact Kähler manifold. We introduce and explore the metric geometry of the $L^{p,q}$ -Calabi Finsler structure on the space of Kähler metrics \mathcal{H} . After noticing that the $L^{p,q}$ -Calabi and $L^{p'}$ -Mabuchi path length topologies on \mathcal{H} do not typically dominate each other, we focus on the finite entropy space \mathcal{E}^{Ent} , contained in the intersection of the L^p -Calabi and L^1 -Mabuchi completions of \mathcal{H} and find that after a natural strengthening, the L^p -Calabi and L^1 -Mabuchi topologies coincide on \mathcal{E}^{Ent} . As applications to our results, we give new convergence results for the Kähler–Ricci flow and the weak Calabi flow.

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1. Introduction and main results

Suppose (X^n, ω) is a connected compact Kähler manifold. By \mathcal{H} we denote the space of Kähler metrics ω' that are cohomologous to ω . In the 1950s Calabi initiated the study of the infinite-dimensional space \mathcal{H} , with the hopes of finding Kähler metrics with special curvature properties [9]. He introduced a Riemannian structure on \mathcal{H} and formulated many related questions, including his famous conjecture ultimately solved by Yau [35]. Addressing one of Calabi's predictions, the path length completion of Calabi's Riemannian space was computed by Clarke–Rubinstein [14], and they also found a novel relation between Calabi geometry and convergence of the Kähler–Ricci (KR) flow on Fano manifolds. One of our purposes in the present paper is to further develop this circle of ideas. The KR flow was introduced by Hamilton [24], it satisfies the equation $\frac{d}{dt}\omega_{r_t} = \omega_{r_t} - \text{Ric}\omega_{r_t}$ with $\omega \in c_1(X)$, and we refer to [6] for background, context and historic references.

In our first theorem, refining the findings of Clarke–Rubinstein ($p = 2, q = 1$) [14] and also that of Phong–Song–Sturm–Weinkove ($p = \infty, q = 1$) [28], we obtain the following convergence theorem for the KR flow, whose statement on the surface bears no connection with infinite-dimensional geometry:

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Theorem 1.1. *Suppose $1 \leq q \leq p \leq \infty$, $q \neq \infty$ and (X^n, ω) is Fano with $\omega \in c_1(X)$. Let $[0, \infty) \ni t \rightarrow \omega_{r_t} \in \mathcal{H}$ be a KR trajectory. Then a Kähler–Einstein metric in \mathcal{H} exists if and only if*

$$\int_0^\infty \|n - S_{\omega_{r_t}}\|_{L^p(X, (\omega_{r_t}^n/\omega^n)^q \omega^n)} dt < \infty, \quad (1)$$

where $S_{\omega_{r_t}}$ is the scalar curvature of ω_{r_t} . Additionally, if the above hold then $t \rightarrow r_t$ converges exponentially fast to a Kähler–Einstein metric.

As we shall see, the proof of this result requires no new a priori estimates, but instead rests on the observation that condition (1) is equivalent to saying that the KR trajectory $t \rightarrow \omega_{r_t}$ has finite length with respect to the $L^{p,q}$ -Calabi Finsler metric on \mathcal{H} , that we introduce now. By Hodge theory, if $\omega' \in \mathcal{H}$ then $\omega' = \omega_u := \omega + i\partial\bar{\partial}u$ for some $u \in C^\infty(X)$, hence \mathcal{H} can be identified with \mathcal{H}_ω , the set of normalized Kähler potentials:

$$\mathcal{H}_\omega := \{u \in C^\infty(X) : \omega_u = \omega + i\partial\bar{\partial}u > 0, \int_X u\omega^n = 0\} \simeq \mathcal{H}.$$

As our approach in this note makes heavy use of pluripotential theory, we will mostly work with potentials instead of metrics. Treating \mathcal{H}_ω as a (trivial) Fréchet manifold, one can introduce the $L^{p,q}$ -Calabi Finsler metric for arbitrary $1 \leq q \leq p < \infty$:

$$\|\beta\|_{p,q,u}^C = \left[\frac{1}{V} \int_X |\Delta_{\omega_u} \beta|^p \left[\frac{\omega_u^n}{\omega^n} \right]^q \omega^n \right]^{1/p}, \quad \beta \in T_u \mathcal{H}_\omega, \quad u \in \mathcal{H}_\omega, \quad (2)$$

where $V = \int_X \omega^n$ is the total volume. The case $p = 2$, $q = 1$ gives the Riemannian structure of Calabi [9] recently studied extensively in [12,14]. In the most important case $q = 1$, we will simply refer to the metric in (2) as the L^p -Calabi metric.

Using this Finsler metric it is possible to compute the length of smooth curves, and introduce the associated path length pseudometric $d_{p,q}^C$ on \mathcal{H}_ω . It turns out that $(\mathcal{H}_\omega, d_{p,q}^C)$ is a bona fide metric space and its completion and metric growth can be analytically characterized using elements from the finite energy pluripotential theory of Guedj–Zeriahi [23], generalizing [14, Theorem 5.4] in the process:

Theorem 1.2. *Suppose (X^n, ω) is Kähler and $1 \leq q \leq p < \infty$. Then $(\mathcal{H}_\omega, d_{p,q}^C)$ is a metric space whose completion is $(\mathcal{E}^{L^q}, d_{p,q}^C)$. Characterizing convergence, a sequence $\{u_j\}_j \subset \mathcal{H}_\omega$ is $d_{p,q}^C$ -Cauchy if and only if*

$$\int_X \left| \frac{\omega_{u_j}^n}{\omega^n} - \frac{\omega_{u_k}^n}{\omega^n} \right|^q \omega^n \rightarrow 0 \text{ as } j, k \rightarrow \infty. \quad (3)$$

Additionally, $(\mathcal{E}^{L^1}, d_{2,1}^C)$ is a CAT(1/4) geodesic metric space.

This theorem seems to give the first geometric characterization of the well known potential space \mathcal{E}^{L^q} . Roughly speaking, \mathcal{E}^{L^q} contains degenerate metrics with volume measure having L^q -density, and we give now the precise definition, referring to [23] for additional details. In [23], associated to any $u \in \text{PSH}(X, \omega)$ the authors introduce a non-pluripolar measure ω_u^n on X , satisfying $\int_X \omega_u^n \leq \int_X \omega^n$, generalizing the usual complex Monge–Ampère measure of Bedford–Taylor in case u is additionally bounded. By definition, $u \in \mathcal{E} \subset \text{PSH}(X, \omega)$ if u has “full volume”, i.e., $\int_X \omega_u^n = \int_X \omega^n$. Given $p, q \geq 1$, two important subclasses of potentials inside \mathcal{E} are as follows:

$$\mathcal{E}^{L^q} := \{u \in \mathcal{E}, \int_X u\omega^n = 0, \frac{\omega_u^n}{\omega^n} \in L^q(X, \omega^n)\}, \quad \mathcal{E}^p := \{u \in \mathcal{E}, \int_X u\omega^n = 0, \int_X |u|^p \omega_u^n < \infty\}.$$

Remark 1.3. By basic analysis, the $d_{p,q}^C$ -convergence characterization (3) extends to sequences inside the completion \mathcal{E}^{L^q} as well.

One can derive the equation for $L^{p,q}$ -Calabi geodesics by computing the variation of the Finsler energy along curves with fixed endpoints. Contrary to the findings of [12] in the particular case $p = 2, q = 1$, this geodesic equation does not admit smooth solutions for general p and q , and to avoid complications we omit the discussion of “weak geodesics” in this note.

A distinguishing feature of the $L^{p,q}$ -Calabi geometry is that the associated path length metric $d_{p,q}^C$ induces the same completion on \mathcal{H}_ω for all $p \geq 1$ and fixed q , even though the corresponding Finsler metrics for different p are not even fiberwise conformally equivalent. Indeed, as follows from (3), $d_{p,q}^C$ -convergence does not depend on the value of p . We are not aware of other families of infinite dimensional Finsler structures that would enjoy this same property. See [15] for a treatment of conformal deformations of the Ebin metric on the space of Riemannian metrics, where a related but different phenomenon occurs.

As we learned after the completion of this paper, in [13, Section 4], motivated by different goals, a family of Riemannian metrics was introduced and studied in detail that overlaps with our construction of $L^{p,q}$ -Calabi metrics when $p = 2$.

On top of applications to geometric flows, our motivation for studying the $L^{p,q}$ generalization of Calabi’s Riemannian structure comes from the corresponding L^p generalization of the Mabuchi geometry on \mathcal{H} [18], that led to many applications in the study of canonical Kähler metrics [1,2,19,20], and Theorem 1.2 stands in direct analogy with the findings of [17,18] that we recall now. The L^p -Mabuchi Finsler metric of \mathcal{H}_ω is defined as follows:

$$\|\phi\|_{p,u} = \left[\frac{1}{V} \int_X |\phi - \bar{\phi}_{\omega_u}|^p \omega_u^n \right]^{1/p}, \quad \phi \in T_u \mathcal{H}_\omega, \quad u \in \mathcal{H}_\omega, \tag{4}$$

where $\bar{\phi}_{\omega_u} = \frac{1}{V} \int_X \phi \omega_u^n$. The case $p = 2$ gives the Riemannian structure of Mabuchi–Semmes–Donaldson [26,31,22], a space with non-positive sectional curvature, with close ties to canonical Kähler metrics. As shown in [19,20], the case $p = 1$ gives a geometry with good compactness properties, suitable for the variational study of canonical metrics by way of infinite-dimensional convex optimization.

As in the case of the $L^{p,q}$ -Calabi metric, using the Finsler structure of (4) we can measure the length of smooth curves, and for the associated path length pseudometric d_p^M on \mathcal{H}_ω we recall now the Mabuchi analog of Theorem 1.2, concatenating the relevant parts of [17, Theorem 1] and [18, Theorem 2, Theorem 3]. We refer to [17,18] for further details.

Theorem 1.4. *Suppose (X^n, ω) is Kähler and $p \geq 1$. Then $(\mathcal{H}_\omega, d_p^M)$ is a metric space whose completion is (\mathcal{E}^p, d_p^M) . Characterizing convergence, a sequence $\{u_j\}_j \subset \mathcal{H}_\omega$ is d_p^M -Cauchy if and only if*

$$\int_X |u_j - u_k|^p \omega_{u_j}^n + \int_X |u_j - u_k|^p \omega_{u_k}^n \rightarrow 0 \text{ as } j, k \rightarrow \infty. \tag{5}$$

Additionally, (\mathcal{E}_M^2, d_2^M) is a CAT(0) geodesic metric space.

With Theorems 1.2 and 1.4 in hand, we can answer in a much more general context the questions of Clarke–Rubinstein [14, Section 7.2], who proposed to compare the L^2 -Mabuchi and L^2 -Calabi metric structures. It follows from the proof of Theorem 1.2 that $(\mathcal{H}_\omega, d_{p,1}^C)$ has finite diameter. As $(\mathcal{H}_\omega, d_{p'}^M)$ has infinite diameter [18], it is not possible for $d_{p,1}^C$ to globally dominate $d_{p'}^M$. Even in the absence of global metric domination, one may still hope that domination holds on the level of the induced topologies. In case $q > 1$, thanks to strong estimates of Kolodziej [25, p. 668], the $d_{p,q}^C$ -topology dominates the C^0 -topology hence implicitly also the $d_{p'}^M$ -topology. However the case $q = 1$ is much more delicate and does not allow any kind of domination (see [3, Theorem 1.3] for related a priori estimates), as we summarize in the following result that fully characterizes the relationship between the $L^{p,q}$ -Calabi and $L^{p'}$ -Mabuchi topologies:

Theorem 1.5. *Suppose (\mathcal{H}, ω) is Kähler, $1 \leq q \leq p < \infty$ and $1 \leq p'$. The following hold:*

- (i) *There exists a sequence $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega$ that is $d_{p'}^M$ -Cauchy but does not contain any $d_{p,q}^C$ -Cauchy subsequences.*

- (ii) If $q > 1$, then any $d_{p,q}^C$ -Cauchy sequence inside \mathcal{H}_ω is $d_{p'}^M$ -Cauchy as well.
- (iii) There exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega$ that is $d_{p,1}^C$ -Cauchy but does not contain any $d_{p'}^M$ -Cauchy subsequences.

For the rest of the introduction, let us focus on the case $q = 1$, most important from the point of view of geometric applications. An important step in the proof of the previous theorem is noticing that the completion of \mathcal{H}_ω with respect to the L^p -Calabi and $L^{p'}$ -Mabuchi metrics cannot contain each other. Though containment is not possible, it is natural to search for interesting subspaces of the intersection $(\mathcal{H}_\omega, d_{p'}^M) \cap (\mathcal{H}_\omega, d_{p,1}^C) = \mathcal{E}^{p'} \cap \mathcal{E}^{L^1}$.

Given the importance of the L^1 -Mabuchi metric in applications to existence/uniqueness of canonical Kähler metrics [20,19,1], we further restrict attention to the case $p \geq 1$ and $p' = 1$. A natural subspace of the intersection $\mathcal{E}^1 \cap \mathcal{E}^{L^1}$ is the space of potentials with finite entropy, studied in [7,2] in connection with canonical Kähler metrics:

$$\mathcal{E}^{\text{Ent}} := \{u \in \mathcal{E} : \text{Ent}(\omega^n, \omega_u^n) < \infty, \int_X u \omega^n = 0\},$$

where $\text{Ent}(\omega^n, \omega_u^n) = \infty$ if ω_u^n is not absolutely continuous with respect to ω^n , and is equal to $\int_X \log \left(\frac{\omega_u^n}{\omega^n} \right) \frac{\omega_u^n}{\omega^n} \omega^n$ otherwise. By definition, $\mathcal{E}^{\text{Ent}} \subset \mathcal{E}^{L^1}$ and it is well known that also $\mathcal{E}^{\text{Ent}} \subset \mathcal{E}^1$ (see [7,2]), hence as proposed

$$\mathcal{E}^{\text{Ent}} \subset \mathcal{E}^1 \cap \mathcal{E}^{L^1}.$$

It follows that \mathcal{E}^{Ent} can be endowed with two different non-complete topologies induced by $d_{p,1}^C$ and d_1^M . A natural way to make these topologies complete on \mathcal{E}^{Ent} is to strengthen them enough to make the map $u \rightarrow \text{Ent}(\omega^n, \omega_u^n)$ continuous. It turns out this procedure gives equivalent topologies and in fact much more is true:

Theorem 1.6. *Suppose $u_j, u \in \mathcal{E}^{\text{Ent}}$ satisfy $\text{Ent}(\omega^n, \omega_{u_j}^n) \rightarrow \text{Ent}(\omega^n, \omega_u^n)$. Then the following are equivalent:*

- (i) $u_j \rightarrow u$ in $L^1(X, \omega^n)$.
- (ii) $\omega_{u_j}^n \rightarrow \omega_u^n$ in the weak sense of measures.
- (iii) $d_1^M(u_j, u) \rightarrow 0$.
- (iv) $d_{p,1}^C(u_j, u) \rightarrow 0$.

An immediate application of the equivalence between (iii) and (iv) in the last theorem and [2, Theorem 1.2, Theorem 1.11] is a new convergence result for the weak Calabi flow. This weak flow is a generalization of the classical smooth Calabi flow [10,11] (that is governed by the equation $\frac{d}{dt}c_t = S_{\omega_{c_t}} - \bar{S}$) and was initially introduced and studied by Streets in the context of the abstract metric completion (\mathcal{H}, d_2^M) [32,33]. A better understanding of this latter space in [17] led to more precise long time convergence and asymptotics results for the weak Calabi flow in [2] and for more details, related terminology and historic references we refer to this last paper. We note the following corollary, which is a direct consequence of Theorem 1.6 above and [2, Theorem 1.5, Theorem 1.11]:

Corollary 1.7. *Suppose $p \geq 1$ and there exists a constant scalar curvature metric in \mathcal{H} . Then, given any weak Calabi flow trajectory $[0, \infty) \ni t \rightarrow c_t \in \mathcal{E}^2$, there exists a constant scalar curvature potential $c_\infty \in \mathcal{H}_\omega$ such that $d_{p,1}^C(c_t, c_\infty) \rightarrow 0$, in particular $\int_X \left| \frac{\omega_{c_t}^n}{\omega^n} - \frac{\omega_{c_\infty}^n}{\omega^n} \right| \omega^n \rightarrow 0$.*

2. The $L^{p,q}$ -Calabi geometry of \mathcal{H}_ω

As we will see, \mathcal{H}_ω equipped with the $L^{p,q}$ -Calabi Finsler structure can be embedded isometrically into $L^p(X, \omega^n)$. For this reason, we focus on the “flat” geometry of $L^p(X, \omega^n)$ in the next short subsection.

2.1. L^p -geometry on $L^{p/q}$ -spheres

Let $1 \leq q \leq p < \infty$ and X be a compact manifold with a positive Borel measure μ satisfying $\mu(X) = \int_X \mu < \infty$. As a Fréchet manifold, $C^\infty(X)$ can be equipped with the trivial L^p -Finsler structure:

$$\|\psi\|_{p,f} = \left(\frac{1}{\mu(X)} \int_X |\psi|^p \mu\right)^{1/p}, \quad \psi \in T_f C^\infty(X) \simeq C^\infty(X), \quad f \in C^\infty(X). \tag{6}$$

It is a classical fact that straight segments joining various points of $C^\infty(X)$ are geodesics for this metric. The $L^{p/q}$ -sphere with radius r is denoted by:

$$\mathbb{S}_{L^{p/q}}(\mu, r) = \{f \in C^\infty(X) \text{ s.t. } \frac{1}{\mu(X)} \int_X |f|^{p/q} \mu = r^{p/q}\}.$$

Unfortunately, for most p and q , the $L^{p/q}$ -sphere is not even a smooth submanifold of $C^\infty(X)$, however if we restrict to the ‘‘octant’’ $\mathbb{S}_{L^{p/q}}^+(\mu, r) = \mathbb{S}_{L^{p/q}}(\mu, r) \cap \{f > 0\}$, we do get a smooth submanifold. As such, one can pullback the L^p -Finsler metric of (6) to $\mathbb{S}_{L^{p/q}}^+(\mu, r)$, and study the resulting path length metric space $(\mathbb{S}_{L^{p/q}}^+(\mu, r), d_{p,q}^{\mathbb{S}^+})$. We will need the following basic result in this direction, which roughly says that the ‘‘chordal metric’’ is equivalent to the ‘‘round metric’’ on $\mathbb{S}_{L^{p/q}}^+(\mu, r)$:

Proposition 2.1. Fix $f \in \mathbb{S}_{L^{p/q}}^+(\mu, r)$. Then there exists $C := C(\mu, p, q, f, r) > 0$ such that for any $f_0, f_1 \in \mathbb{S}_{L^{p/q}}^+(\mu, r)$ the following holds:

$$\frac{C}{d_{p,q}^{\mathbb{S}^+}(f, f_0) + d_{p,q}^{\mathbb{S}^+}(f, f_1) + 1} d_{p,q}^{\mathbb{S}^+}(f_0, f_1) \leq \left(\frac{1}{\mu(X)} \int_X |f_0 - f_1|^p \mu\right)^{1/p} \leq d_{p,q}^{\mathbb{S}^+}(f_0, f_1). \tag{7}$$

Proof. The inequality $(\frac{1}{\mu(X)} \int_X |f_0 - f_1|^p \mu)^{1/p} \leq d_{p,q}^{\mathbb{S}^+}(f_0, f_1)$ follows from the fact that $[0, 1] \ni t \rightarrow f_t := f_0 + t(f_1 - f_0) \in C^\infty(X)$ is a geodesic in $C^\infty(X)$ with respect to the L^p -Finsler metric and has length equal to $(\frac{1}{\mu(X)} \int_X |f_0 - f_1|^p \mu)^{1/p}$. Any curve joining f_0, f_1 inside $\mathbb{S}_{L^{p/q}}^+(\mu, r)$ has to have length less than $t \rightarrow f_t$.

For the other inequality, we will estimate the length of the curve

$$[0, 1] \ni t \rightarrow \alpha_t := \frac{rf_t}{\|f_t\|_{p/q}} = \frac{r(f_0 + t(f_1 - f_0))}{\|f_0 + t(f_1 - f_0)\|_{p/q}} \in \mathbb{S}_{L^{p/q}}^+(\mu, r),$$

joining f_0, f_1 . Note that the denominator of the expression above is nonzero, as $f_0, f_1 > 0$. Using that

$$\dot{\alpha}_t = \frac{r(f_1 - f_0)}{\|f_t\|_{p/q}} - \frac{rf_t}{\|f_t\|_{p/q}^{p/q+1}} \int_X (f_1 - f_0) f_t^{p/q-1} d\mu,$$

we have the following sequence of estimates:

$$\begin{aligned} \int_0^1 \|\dot{\alpha}_t\|_p dt &\leq \int_0^1 \frac{r\|f_1 - f_0\|_p}{\|f_t\|_{p/q}} dt + \int_0^1 \frac{r\|f_t\|_p}{\|f_t\|_{p/q}^{p/q+1}} \int_X |f_1 - f_0| f_t^{p/q-1} d\mu dt \\ &\leq \int_0^1 \frac{r\|f_1 - f_0\|_p}{\|f_t\|_{p/q}} dt + \int_0^1 \frac{r\|f_t\|_p}{\|f_t\|_{p/q}^{p/q+1}} \|f_1 - f_0\|_{p/q} \|f_t\|_{p/q}^{p/q-1} dt \\ &\leq r\|f_1 - f_0\|_p \int_0^1 \frac{1}{\|f_t\|_{p/q}} dt + C'(\mu, p, q) \|f_1 - f_0\|_p \int_0^1 \frac{\|f_t\|_p}{\|f_t\|_{p/q}^2} dt \\ &\leq r\|f_1 - f_0\|_p \int_0^1 \frac{1}{\|f_t\|_{p/q}} dt + C'(\mu, p, q) \frac{(1-t)\|f - f_0\|_p + t\|f - f_1\|_p + \|f\|_p}{\|f_t\|_{p/q}^2} dt \\ &\leq C(\mu, p, q, r, f) (d_{p,q}^{\mathbb{S}^+}(f, f_0) + d_{p,q}^{\mathbb{S}^+}(f, f_1) + 1) \|f_1 - f_0\|_p, \end{aligned}$$

where to obtain the second line we have used the Hölder inequality with exponents $p/q \geq 1$ and $(p/q)/(p/q - 1) \geq 1$ in the last integrand. To get the third line, we have used that $\|f_0 - f_1\|_{p/q} \leq C'(\mu, p, q) \|f_0 - f_1\|_p$. For the fourth line,

we have used the triangle inequality for the L^p -norm. To get the last line, we have used that $\|f - f_0\|_p \leq d_{p,q}^{\mathbb{S}^+}(f, f_0)$, $\|f - f_1\|_p \leq d_{p,q}^{\mathbb{S}^+}(f, f_1)$, $f_t \geq f_0/2 \geq 0$ for $t \in [0, 1/2]$ and $f_t \geq f_1/2 \geq 0$ for $t \in [1/2, 1]$, hence $\|f_t\|_{p/q} \geq r/2$ for all $t \in [0, 1]$. To finish the proof, we conclude:

$$d_{p,q}^{\mathbb{S}^+}(f_0, f_1) \leq \int_0^1 \|\dot{\alpha}_t\|_p dt \leq C(\mu, p, q, f)(d_{p,q}^{\mathbb{S}^+}(f, f_0) + d_{p,q}^{\mathbb{S}^+}(f, f_1) + 1)\|f_1 - f_0\|_p. \quad \square$$

2.2. Proof of Theorem 1.2 and Theorem 1.1

Proof of Theorem 1.2. To start the proof, we notice that for arbitrary $1 \leq q \leq p < \infty$ the infinite-dimensional Finsler manifolds $(\mathcal{H}_\omega, \|\cdot\|_{p,q,(\cdot)}^C)$ and $(\mathbb{S}_{L^{p/q}}^+(\omega^n, p/q), \|\cdot\|_{p,(\cdot)})$ are isometric via the map $F : \mathcal{H}_\omega \rightarrow \mathbb{S}_{L^{p/q}}^+(\omega^n, p/q)$, given by the formula

$$F(u) := \frac{p}{q} \left(\frac{\omega_u^n}{\omega^n} \right)^{\frac{q}{p}}.$$

By the Calabi–Yau theorem, the map F is bijective. As $F(u)_*(\delta u) = (\omega_u^n/\omega^n)^{q/p} \Delta_{\omega_u} \delta u$, by inspection we see that $F^* \|\cdot\|_{p,F(\cdot)} = \|\cdot\|_{p,q,(\cdot)}^C$. All this implies that

$$d_{p,q}^C(u_0, u_1) = d_{p,q}^{\mathbb{S}^+}(F(u_0), F(u_1)),$$

in particular, (7) gives that $d_{p,q}^C$ is indeed a metric on \mathcal{H}_ω . From (7) it also follows that $\{u_j\}_j \subset \mathcal{H}_\omega$ is $d_{p,q}^C$ -Cauchy if and only if

$$\int_X \left| \left(\frac{\omega_{u_j}^n}{\omega^n} \right)^{q/p} - \left(\frac{\omega_{u_k}^n}{\omega^n} \right)^{q/p} \right|^p \omega^n \rightarrow 0, \quad j, k \rightarrow \infty.$$

Using this, Lemma 2.2 below implies that $\{u_j\}_j$ is $d_{p,q}^C$ -Cauchy if and only if (3) holds. The identification $(\mathcal{H}_\omega, d_{p,q}^C) = \mathcal{E}^{L^q}$ readily follows as well.

Lastly, we focus on the case $p = 2, q = 1$ extensively treated in [12,14]. As observed in [12, Theorem 1.1] (see also the discussion following [14, Remark 4.2]), the Riemannian space $(\mathcal{H}_\omega, \|\cdot\|_{2,1,(\cdot)}^C)$ has positive constant sectional curvature equal to $1/4$, what is more, any two points of \mathcal{H}_ω can be joined by a Riemannian geodesic. Roughly, this follows from the fact that $(\mathcal{H}_\omega, \|\cdot\|_{2,1,(\cdot)}^C)$ is isometric to $(\mathbb{S}_{L^2}^+(\omega^n, 2), \|\cdot\|_{2,(\cdot)})$, which is a totally geodesic open subset of an infinite-dimensional sphere with radius 2.

Given $u, v, w \in \mathcal{H}_\omega$, let $U = F(u), V = F(v), W = F(w) \in \mathbb{S}_{L^2}^+(\omega^n)$. Also let $\mathcal{V} \subset L^2(X, \omega^n)$ be the 3 dimensional subspace spanned by U, V, W and $\mathbb{S}_{UVW} = \mathcal{V} \cap \mathbb{S}_{L^2}^+(\omega^n, 2)$. Together with the induced Riemannian metric, \mathbb{S}_{UVW} is isometric to an open subset of the 2-dimensional round sphere with its round metric, hence it has constant sectional curvature $1/4$. The geodesic triangle UVW of $\mathbb{S}_{L^2}^+(\omega^n, 2)$, with edges at U, V, W lies inside \mathbb{S}_{UVW} [12, Theorem 1.4]. As \mathbb{S}_{UVW} is a model space with constant scalar curvature equal to $1/4$, the geodesic triangle UVW (lying inside it) has to satisfy the CAT(1/4) inequality [8], ultimately giving that $(\mathcal{H}_\omega, d_2^C)$ is a CAT(1/4) space.

To finish the proof, we can use [8, Corollary 3.11, p. 187] to conclude that the metric completion $(\overline{\mathcal{H}_\omega}, d_2^C) = (\mathcal{E}^{L^1}, d_2^C)$ is a CAT(1/4) geodesic metric space as well. \square

As promised in the above proof, let us state the following measure theoretic lemma, whose proof uses the classical Vitali convergence theorem [29, Theorem 8.5.14], and is exactly the same as the argument of [14, Lemma 5.3]:

Lemma 2.2. *Suppose $f_j, f \in L^q(X, \omega^n)$ with $f_j, f \geq 0$. Then $\|f_j - f\|_{L^q} \rightarrow 0$ if and only if $\|f_j^{q/p} - f^{q/p}\|_{L^p} \rightarrow 0$.*

Lastly, we provide the following theorem, which contains Theorem 1.1 as a particular case. We refer to [18,19,27] for analogous results on the $L^{p'}$ -Mabuchi convergence of the Kähler–Ricci flow.

Theorem 2.3. *Suppose (X^n, ω) is Fano with $[\omega] = -c_1(K_X)$ and $1 \leq q \leq p \leq \infty$. Suppose $[0, \infty) \ni t \rightarrow r_t \in \mathcal{H}_\omega$ is a Kähler–Ricci trajectory. Then the following are equivalent:*

- (i) *There exists a Kähler–Einstein potential inside \mathcal{H}_ω .*
- (ii) *$t \rightarrow r_t$ converges C^∞ -exponentially fast to some Kähler–Einstein potential $r_\infty \in \mathcal{H}_\omega$.*
- (iii) *$t \rightarrow r_t$ has finite $d_{p,1}^C$ -length, i.e., $\int_0^\infty \|n - S_{\omega_{r_t}}\|_{L^p(X, (\omega_{r_t}^n/\omega^n)^q \omega^n)} < \infty$.*
- (iv) *$\{r_t\}_{t \geq 0} \subset \mathcal{H}_\omega$ forms a $d_{p,q}^C$ -Cauchy sequence.*

Proof. Let us first assume that $p \neq \infty$. If (i) holds then by results of Perelman, Tian–Zhu, Phong–Song–Sturm–Weinkove and Collins–Székelyhidi [34,28,16] imply that the Kähler–Ricci trajectory $t \rightarrow r_t$ converges C^∞ exponentially fast to some Kähler–Einstein potential $r_\infty \in \mathcal{H}_\omega$, hence (ii) holds. C^∞ -exponential convergence of $t \rightarrow r_t$ implies the finiteness of the integral in (iii). Condition (iii) implies (iv) trivially.

We are left to show that (iv) implies (i). The ideas of [14, Corollary 6.7] apply again, but we give here a slightly different argument. Suppose (iv) holds but (i) does not. Let $r_\infty \in \mathcal{E}^{L^q}$ be the $d_{p,q}^C$ -limit of r_t . It follows from the convergence criterion of (3) and Remark 3.2 below that we also have weak convergence on the level of potentials, namely $r_t \rightarrow_{L^1(X, \omega^n)} r_\infty$. On the other hand, by [30, Theorem 1.3], there exists $t_j \rightarrow \infty$ and $\psi \in \text{PSH}(X, \omega)$ such that ψ has proper multiplier ideal sheaf, in particular by Skoda’s theorem ψ has non-zero Lelong number at some $x \in X$. By [23, Corollary 1.8] this implies that $\psi \notin \mathcal{E}$. But by uniqueness of $L^1(X, \omega^n)$ -limits, we have $\psi = r_\infty \in \mathcal{E}^{L^q} \subset \mathcal{E}$, a contradiction.

Now we deal with the case $p = \infty$. Clearly, the directions (i) \rightarrow (ii) \rightarrow (iii) \rightarrow (iv) still hold. To prove that (iv) implies (i) we just need to notice that $d_{\infty,1}^C$ -convergence trivially implies $d_{p,q}^C$ -convergence for any $1 \leq q \leq p < \infty$. \square

3. $L^{p,q}$ -Calabi vs. $L^{p'}$ -Mabuchi geometry

3.1. Proof of Theorem 1.5

To prove (i), we recall first that $\mathcal{E}^{p'} \not\subset \mathcal{E}^{L^q}$. Indeed, we can choose $v_0, v_1 \in \mathcal{H}_\omega$ such that the level set $\{v_0 - v_1 = 0\}$ does not contain critical points of $v_0 - v_1$. Then a basic calculation yields that the bounded potential $u = \max(v_0, v_1) - \int_X \max(v_0, v_1) \omega^n$ satisfies $u \in \mathcal{E}^{p'} \setminus \mathcal{E}^{L^q}$, because ω_u^n charges the hypersurface $\{v_0 = v_1\}$, a set of Lebesgue measure zero.

Now let $u \in \mathcal{E}^{p'} \setminus \mathcal{E}^{L^q}$ arbitrary. By Theorem 1.4 there exists $u_j \in \mathcal{H}_\omega$ such that $d_{p'}^M(u_j, u) \rightarrow 0$. This in particular gives that $\omega_{u_j}^n \rightarrow \omega_u^n$ weakly [18, Theorem 5(i)]. We claim that $\{u_j\}_j$ cannot contain a $d_{p,q}^C$ -Cauchy subsequence. Indeed, if this were the case, then by Theorem 1.2 above, for some subsequence of u_j , again denoted by u_j , the densities $\omega_{u_j}^n/\omega^n$ would converge in $L^q(X, \omega^n)$ to some $f \in L^q(X, \omega)$. But as $\omega_{u_j}^n \rightarrow \omega_u^n$ weakly, this would imply that $\omega_u^n/\omega^n = f \in L^q(X, \omega^n)$, a contradiction with $u \in \mathcal{E}^{p'} \setminus \mathcal{E}^{L^q}$.

To argue (iii), we first show that $\mathcal{E}^{L^1} \not\subset \mathcal{E}^{p'}$. This is again likely known to experts, however we could not find an exact reference, so we give a construction allowing a great amount of flexibility. Let $u \in \mathcal{E}^{p'}$, $u \leq -1$ and unbounded such that for each set $U_k = \{k < |u|^{p'} \leq k + 1\}$ we have $\omega^n(U_k) > 0$, $k \geq 1$. By the construction in [23, Example 2.14] (see also [4, Proposition 5]), such u can be found. We introduce $f \in L^1(X, \omega^n)$:

$$f(x) = \sum_{k \geq 1} \frac{6V}{(\pi k)^2 \omega^n(U_k)} \mathbb{1}_{U_k}(x).$$

Clearly $f \in L^1(X)$ with $\int_X f \omega^n = V$, hence by [23, Theorem A] there exists $v \in \mathcal{E}^{L^1}$ such that $\omega_v^n = f \omega^n$. We claim that $v \notin \mathcal{E}^{p'}$. Indeed, if this were not true, then [23, Theorem C] would give that

$$\infty = \frac{6V}{\pi^2} \sum_{k \geq 1} \frac{1}{k} \leq \int_X |u|^{p'} \omega_v^n < \infty,$$

a contradiction. Finally, as $v \in \mathcal{E}^{L^1} \setminus \mathcal{E}^{p'}$, the same argument as in the previous step yields now a $d_{p,1}^C$ -Cauchy sequence $\{v_j\}_j \subset \mathcal{H}_\omega$ for which $d_{p,1}^C(v_j, v) \rightarrow 0$, without any $d_{p'}^M$ -Cauchy subsequences.

Finally, to argue (ii), we have to use jointly the $d_{p,q}^C$ -convergence criteria (3) and the estimates of Kolodziej [25, p. 668], according to which $d_{p,q}^C$ -convergence implies C^0 -convergence of potentials. According to the $d_{p'}^M$ -convergence criteria (5), C^0 -convergence in turn implies $d_{p'}^M$ -convergence, finishing the proof.

3.2. Proof of Theorem 1.6

The following basic consequence of the dominated convergence theorem will be used shortly:

Lemma 3.1. *Suppose $f \geq 0$ such that $\int_X f \log(f)\omega^n < \infty$. Then there exists $\tilde{f}_k \in C^\infty(X)$ such that $\tilde{f}_k > 0$, $\int_X |f - \tilde{f}_k|\omega^n \rightarrow 0$ and $\int_X f(\log(f) - \log(\tilde{f}_k))\omega^n \rightarrow 0$.*

Proof. Let $f^m = \max\{\min\{f, m\}, 1/m\}$. By the dominated convergence theorem every sequence $\{g_j\}_j \subset C^\infty(X)$ satisfying $m > g_j > 1/m$ and $\int_X |f^m - g_j|\omega^n \rightarrow 0$ contains an element $\tilde{f}_m := g_{j_m}$ with $\int_X f(\log(f^m) - \log(\tilde{f}_m))\omega^n \leq 1/n$ and $\int_X |f^m - \tilde{f}_m|\omega^n \leq 1/n$. By the absolute continuity of the Lebesgue integral, it follows that $\{\tilde{f}_k\}_k$ satisfies the properties of the lemma. \square

Proof of Theorem 1.6. First we show the equivalence between (i), (ii) and (iii). From [18, Theorem 5(i)] it follows that (iii) \rightarrow (i) and (iii) \rightarrow (ii).

The proof of (i) \rightarrow (iii) and (ii) \rightarrow (iii) are almost the same and we only carry out the latter. It follows from the compactness theorem [7, Theorem 2.17] (for a statement most suitable for our purposes see [20, Theorem 5.6]) that any subsequence of $\{u_j\}$ contains a subsubsequence $\{u_{j_k}\}$ such that $d_1(u_{j_k}, v) \rightarrow 0$ for some $v \in \mathcal{E}^{\text{Ent}}$. If we can show that $v = u$ then we are done. By [18, Theorem 5(i)] again, we have $\omega_{u_{j_k}}^n \rightarrow \omega_v^n$ weakly, hence by the assumption we get $\omega_u^n = \omega_v^n$. As $u, v \in \mathcal{E}^1$, by uniqueness [23, Theorem B] (see [21] for a more general result), we conclude that $v = u$.

The direction (iv) \rightarrow (ii) is trivial and the main step is to argue that (ii) \rightarrow (iv). Let $f_j = \omega_{u_j}^n / \omega^n$ and $f = \omega_u^n / \omega^n$. By the $d_{p,1}^C$ -convergence criteria (3), we have to show that $\int_X |f - f_j|\omega^n \rightarrow 0$. Let $\tilde{f}_j \in C^\infty(X)$ be the sequence from the previous lemma. For fixed k we have

$$\lim_j \int_X |f - f_j|\omega^n \leq \limsup_j \int_X |\tilde{f}_k - f_j|\omega^n + \int_X |f - \tilde{f}_k|\omega^n,$$

hence we only need to check that the first term on the right hand side goes to zero as $k \rightarrow \infty$. Using the classical Küllback–Pinsker inequality (see [7, Proposition 2.10(ii)] for statement tailored to our setting) we have the following sequence of estimates:

$$\begin{aligned} \limsup_j \left(\int_X |f_j - \tilde{f}_k|\omega^n \right)^2 &\leq \limsup_j \int f_j \log \left(\frac{f_j}{\tilde{f}_k} \right) \omega^n \\ &\leq \limsup_j \int f_j \log f_j \omega^n - \liminf_j \int f_j \log \tilde{f}_k \omega^n \\ &= \int f \log f \omega^n - \int f \log \tilde{f}_k \omega^n, \end{aligned} \tag{8}$$

where to get the last line we have used that $\text{Ent}(\omega_{u_j}, \omega) \rightarrow \text{Ent}(\omega_u, \omega)$ and that $\omega_{u_j}^n = f_j \omega^n$ converges weakly to ω_u^n . By the previous lemma, the expression in (8) tends to zero as $k \rightarrow \infty$, hence we are done. \square

It is perhaps worth noting that (iv) implies (ii) in Theorem 1.6 without the assumption on the convergence of entropy, as we elaborate now. Suppose $u_j, u \in \mathcal{E}^1$ with $d_{p,1}^M(u_j, u) \rightarrow 0$. As $\{u_j\}_j$ is $L^1(X, \omega^n)$ -compact (since $\int_X u_j \omega^n = 0$), we have to argue that any $L^1(X, \omega)$ -convergent subsequence of $\{u_j\}$ L^1 -converges to u . Suppose $u_{j_k} \rightarrow_{L^1} v \in \text{PSH}(X, \omega)$. By [5, Proposition 2.10(i)] it follows that $v \in \mathcal{E}^1$, in particular ω_v^n has full mass ($u \in \mathcal{E}$). As a consequence of [5, Corollary 2.21] we now obtain that $\omega_v^n \geq \omega_u^n$. As both of these last measures have the same total volume we have in fact $\omega_v^n = \omega_u^n$, hence $v = u$ as desired (here we used again [23, Theorem B]).

For $q > 1$, $d_{p,q}^C$ -convergence implies C^0 -convergence (hence also $L^1(X, \omega^n)$ -convergence of potentials) as was noted in the proof of [Theorem 1.5](#), and we summarize our findings in the next remark, obtaining a partial analog of [[18, Theorem 5\(i\)](#)] for the $d_{p,q}^C$ metric in the process:

Remark 3.2. Suppose $1 \leq q \leq p < \infty$ and $u_j, u \in \mathcal{E}^{L^q}$. Then $d_{p,q}^C(u_j, u) \rightarrow 0$ implies that $u_j \rightarrow_{L^1(X, \omega^n)} u$.

Conflict of interest statement

The author certifies that they have no affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

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