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A characterization result for the existence of a two-phase material minimizing the first eigenvalue

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Abstract

Given two isotropic homogeneous materials represented by two constants $0 < \alpha < \beta$ in a smooth bounded open set $\Omega \subset \mathbb{R}^N$, and a positive number $\kappa < |\Omega|$, we consider here the problem consisting in finding a mixture of these materials $\alpha\chi_\omega + \beta(1-\chi_\omega)$, $\omega \subset \mathbb{R}^N$ measurable, with $|\omega| \le \kappa$, such that the first eigenvalue of the operator $u \in H_0^1(\Omega) \to -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u\big)$ reaches the minimum value. In a recent paper, [6], we have proved that this problem has not solution in general. On the other hand, it was proved in [1] that it has solution if Ω is a ball. Here, we show the following reciprocate result: If $\Omega \subset \mathbb{R}^N$ is smooth, simply connected and has connected boundary, then the problem has a solution if and only if Ω is a ball.

MSC: 49J20

Keywords: Two-phase material; Control in the coefficients; Eigenvalue; Non-existence

1. Introduction

We consider a bounded open set $\Omega \subset \mathbb{R}^N$, $N \ge 2$, and two constants $0 < \alpha < \beta$, representing two homogeneous isotropic materials (thermic, electric, elastic,...). A classical problem in optimal design consists in mixing these materials in order to minimize a certain functional. Such as it is proved in [17] and [18], this type of problems has not solution in general and then it is usual to deal with relaxed formulations which can be obtained by using the homogenization theory (see e.g. [2,7,19,22,23]).

Between the most studied problems of this type (see e.g. [2,6,14,15,19]) we emphasize the following one

$$\begin{cases} \min \displaystyle\int_\Omega (\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})|\nabla u|^2 dx \\ -\mathrm{div}\big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\big) = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \quad |\omega| < \kappa, \end{cases} \tag{1.1}$$

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with $f \in H^{-1}(\Omega)$ and $\kappa \in (0, |\Omega|)$ (for $\kappa \geq |\Omega|$, the solution is the trivial one $\omega = \Omega$). A special attention has been paid for $f = 1$ and $N = 2$, where it represents the optimal distribution of two materials in the cross section a beam in order to minimize the torsion. In this case, it has been proved in [19] that if Ω is simply connected and smooth and there exists a smooth solution ω, then Ω is a ball. This result has been improved in [6] by showing that the result holds true without any smoothness assumptions on ω (the case $N > 2$ is also considered). The proof is based on certain regularity results for the solution of the relaxed formulation of (1.1) also obtained in [6]. A related problem consisting in replacing the minimum in (1.1) by a maximum has been considered in [5].

It has also been observed in [6] that problem (1.1) is strongly related to another classical optimization design problem for a two-phase material. It consists in finding a measurable set $\omega \subset \Omega$ with $|\omega| \leq \kappa$ ($0 < \kappa < |\Omega|$ as above) such that the first eigenvalue of the operator

$$u \in H_0^1(\Omega) \mapsto -\mathrm{div}\big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\big) \in H^{-1}(\Omega) \tag{1.2}$$

becomes minimal. Namely, it is proved that the relaxed formulation of this problem is equivalent to solve the relaxed formulation of (1.1) for every $f \in L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} = 1$ and then to minimize in f. Thus, the regularity results proved in [6] for (1.1) also hold for the minimization of the eigenvalue. As an application, it has been shown that the problem has not solution if Ω is a rectangle or an ellipsis. On the other hand, it was proved in [1] that the eigenvalue problem has a solution in the particular case where Ω is a ball and then the optimal set ω has a radial structure. Some discussions about the exact structure of ω when Ω is a ball can be found in [8,9,16] and [20].

The purpose of the present paper is to show that, similarly to the result stated above for problem (1.1) with $f = 1$, if Ω is a smooth simply connected open set with connected boundary such that the minimization of the first eigenvalue of the operator (1.2) has a solution, then Ω is a ball. As for problem (1.1), the proof uses the results obtained in [6] but the reasoning is more involved. For problem (1.1) with $f = 1$ one has that the optimal solutions (ω, u) are such that there exist an analytic function w and a positive number μ, satisfying

$$(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u = \nabla w, \qquad \{|\nabla w| > \mu\} \subset \omega \subset \{|\nabla w| \geq \mu\}. \tag{1.3}$$

Moreover, the Laplacian of $|\nabla w|^2$ in Ω is nonnegative. For the eigenvalue problem, statement (1.3) still holds true but now w is non-analytic and the Laplacian of $|\nabla w|^2$ can change its sign in Ω. Thus, many of the ideas used in [6] (and [19]) cannot be used here.

2. The characterization result

For a smooth bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and three constants $0 < \alpha < \beta$, $0 < \kappa < |\Omega|$, we consider the problem consisting in finding a measurable subset ω of Ω with $|\omega| \leq \kappa$, such that the first eigenvalue of the problem

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\big) = \lambda u \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases} \tag{2.1}$$

reaches its minimum value. This can also been formulated as

$$\begin{cases} \min \displaystyle\int_\Omega \big(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega}\big)|\nabla u|^2 dx \\ u \in H_0^1(\Omega), \quad \displaystyle\int_\Omega |u|^2 dx = 1 \\ \omega \subset \Omega \ \text{ measurable}, \quad |\omega| \leq \kappa. \end{cases} \tag{2.2}$$

We remark that if we do not assume the volume restriction $|\omega| \leq \kappa$, then the solution is the trivial one given by $\omega = \Omega$. However in the applications, the material α can be more expensive than β and thus, we can only dispose of a certain quantity κ of material α. The question then is how to distribute it in an optimal way.

As an application of (2.2) we can consider the following problem for the heat equation

$$\begin{cases} \partial_t u - \mathrm{div}\big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\big) = 0 \ \text{ in } \Omega \times (0, \infty) \\ u = 0 \ \text{ on } \partial\Omega \times (0, \infty) \\ u_{|t=0} = u_0, \end{cases}$$

where u_0 is a given function in $L^2(\Omega)$. Such as it is hoped from the physical point of view, it is well known that $u(t,x)$ tends to zero when t tends to infinity, i.e. for t large, the temperature in Ω becomes equal to the temperature outside Ω. Namely, the following estimate holds

$$\max \|u(\cdot,t)\|_{L^2(\Omega)} \le e^{-\lambda t}\|u_0\|_{L^2(\Omega)}, \quad \forall t \ge 0,$$

with λ the first eigenvalue of problem (2.1). This estimate is optimal in the sense that we can choose u_0 such that it is an equality. This means that if we want to choose the non-homogeneous material in Ω in such way that the temperature goes to zero as slowly as possible, we must take λ small. Thus, for heat conduction, problem (2.2) is equivalent to get the most insulated material in Ω by using a quantity of material α smaller or equal to κ.

Such as it has been proved in [6], problem (2.2) has not solution if Ω is a rectangle or an ellipsis. On the other hand, in the particular case where Ω is a ball, it has been proved in [1] that (2.2) has a solution and it is radial. Our purpose in the present paper is to give the following reciprocate result.

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$, simply connected, with connected boundary. If problem (2.2) has an optimal solution (ω, u) then Ω is a ball and χ_ω, u are radial functions.*

The proof of Theorem 2.1 will be given in Subsection 2.2, for this purpose we will need some previous results which expose in subsection 2.1

2.1. Preliminary results

In this subsection, we recall some previous lemmas which we need to prove Theorem 2.1.

We start by recalling some results which have been proved in [6], where it is considered the following relaxed formulation for problem (2.2)

$$\begin{cases} \min \displaystyle\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx \\ u \in H_0^1(\Omega), \quad \displaystyle\int_\Omega |u|^2 dx = 1 \\ \theta \in L^\infty(\Omega;[0,1]), \quad \displaystyle\int_\Omega \theta dx \le \kappa, \end{cases} \tag{2.3}$$

with $c = (\beta - \alpha)/\alpha$. It consists in replacing the materials of the form $\alpha\chi_\omega + \beta(1-\chi_\omega)$ by the harmonic mean value of α and β with proportions θ and $1-\theta$. These new materials are constructed via homogenization (see e.g. [2,19,23]) by using a lamination of α and β with proportions θ and $(1-\theta)$ in the direction of ∇u.

Following Theorem 4.2 in [6], we have the following regularity result for 2.2.

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^N$ is $C^{1,1}$ and consider a solution (θ, u) of problem (2.3), then we have*

- *The function u belongs to $W^{1,\infty}(\Omega)$.*
- *The function*

$$\sigma := \frac{\nabla u}{1+c\theta} \tag{2.4}$$

is in $H^1(\Omega)^N$ and denoting by ν the outward unitary normal to Ω on $\partial\Omega$, we have

$$\sigma = (\sigma\cdot\nu)\nu \ \text{ on } \partial\Omega. \tag{2.5}$$

- *The function θ satisfies*

$$\partial_i\theta\sigma_j - \partial_j\theta\sigma_i \in L^2(\Omega), \quad 1 \le i,j \le N. \tag{2.6}$$

Moreover, if θ only take the values 0 and 1, then

$$\partial_i\theta\sigma_j - \partial_j\theta\sigma_i = 0, \quad 1 \le i, j \le N \tag{2.7}$$

and

$$\mathrm{curl}(\sigma) = 0 \ \ in\ \Omega. \tag{2.8}$$

Taking into account Theorem 2.2 in [6], we also have the following optimality condition for the solutions of (2.3).

Lemma 2.3. *Assume that (θ, u) is a solution of problem (2.3) and define σ by (2.5), then,*

$$\int_\Omega \theta\, dx = \kappa \tag{2.9}$$

and there exists a constant $\mu > 0$ such that

$$\theta(x) = \begin{cases} 1 & \text{if } |\sigma(x)| > \mu \\ 0 & \text{if } |\sigma(x)| < \mu. \end{cases} \tag{2.10}$$

Remark 2.4. In the previous Lemma we have asserted that μ is strictly positive which is a sufficient condition in Theorem 2.2 in [6] in order to have (2.9). Indeed, if $\mu = 0$, then by (2.10) and the restriction of the integral of θ in (2.3) we get

$$|\Omega| > \kappa \ge \int_\Omega \theta\, dx \ge |\{x \in \Omega : |\sigma(x)| > 0\}|,$$

and therefore σ vanishes on a set of positive measure but then, taking into account that σ is in $H^1(\Omega)^N$ by (2.2) we get that $\lambda u = -\mathrm{div}\,\sigma$ vanishes on a set of positive measure, which is in contradiction with the maximum principle.

In the present paper, we are interested in the existence of a solution for problem (1.1), which is equivalent to the existence of a solution (θ, u) for problem (2.3), such that $\theta = \chi_\omega$ with ω a measurable subset of Ω. In this case, Lemmas 2.2 and 2.3 imply

Corollary 2.5. *Assume $\Omega \in C^{1,1}$, simply connected, with connected boundary. We suppose there exist $u \in H_0^1(\Omega)$ and $\omega \subset \Omega$ measurable such that (χ_ω, u) is a solution for problem (2.3), then, u belongs to $W^{1,\infty}(\Omega)$ and defining w as the solution of*

$$\begin{cases} -\Delta w = \lambda u \ \ in\ \Omega \\ w = 0 \ \ on\ \partial\Omega, \end{cases} \tag{2.11}$$

with

$$\lambda = \int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx, \tag{2.12}$$

we have

$$\big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)\nabla u = \nabla w. \tag{2.13}$$

Moreover,

$$|\omega| = \kappa, \tag{2.14}$$

and there exists $\mu > 0$ such that

$$\{x \in \Omega : |\nabla w(x)| > \mu\} \subset \omega \subset \{x \in \Omega : \ |\nabla w(x)| \ge \mu\} \tag{2.15}$$

Proof. Lemma 2.3, provides $u \in W^{1,\infty}(\Omega)$, and since $\theta = \chi_\omega$, it also provides

$$\sigma := \frac{\nabla u}{1+c\theta} = \left(\frac{\alpha}{\beta}\chi_\omega + \chi_{\Omega\setminus\omega}\right)\nabla u$$

in $H^1(\Omega)^N$ with vanishing curl. Since Ω is simply connected, this implies the existence of $w \in H^2(\Omega)$ such that $\beta\sigma = \nabla w$. Now, using (2.5) we get that w is constant on $\partial\Omega$, but w is defined up to a constant. Therefore, we can take w vanishing on $\partial\Omega$. Using then that u gives the minimum in

$$\min_{v\in H^1_0(\Omega)} \int_\Omega \frac{|\nabla v|^2}{1+c\theta}dx = \min_{v\in H^1_0(\Omega)} \int_\Omega \left(\frac{\alpha}{\beta}\chi_\omega + \chi_{\Omega\setminus\omega}\right)|\nabla v|^2 dx$$

and the definition of σ, we get

$$-\Delta w = -\beta\,\mathrm{div}\sigma = \lambda u,$$

with λ defined by (2.12). This proves that w is the solution of (2.11).

To finish the proof we use (2.9) and (2.10), which imply that (2.14) and (2.15) hold, with the constant μ which appears in Corollary 2.5 replaced by $\beta\mu$. $\square$

In order to prove Theorem 2.1, we will also need the following result.

Lemma 2.6. *Assume $\Omega \subset \mathbb{R}^N$ open, $K \subset \Omega$ compact and connected and $c \in \mathbb{R}$. We consider two functions $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$ and $w \in C^1(\Omega)$ such that $w = c$ in K, ∇w does not vanish in K and ∇u is proportional to ∇w a.e. in Ω. Then, there exist a neighborhood U of K contained in Ω, $\tau > 0$ and $h \in W^{1,p}(c-\tau, c+\tau)$ such that*

$$w(U) = (c-\tau, c+\tau), \qquad u(x) = h(w(x)),\ \forall x \in U, \qquad \nabla w \ne 0 \ \text{ in } U.$$

Proof. We start assuming that K reduces to a point $x_0 \in \Omega$. In this case, the result is well known, at least if u is also in $C^1(\Omega)$. If $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$ the reasoning is similar to the classical one but for more clarity, let us detail it.

Consider $N-1$ vectors $\xi_2, \dots, \xi_N \in \mathbb{R}^N$ such that $\{\nabla w(x_0), \xi_2, \cdots, \xi_N\}$ is a basis in $\mathbb{R}^N$ and define $F = (F_1, \cdots, F_N) : \Omega \to \mathbb{R}^N$ by

$$F_1(x) = w(x), \quad F_i(x) = \xi_i \cdot x,\ 2 \le i \le N, \qquad \forall x \in \Omega.$$

By the inverse function theorem, there exists $\delta > 0$ with $B(x_0, \delta) \Subset \Omega$, and $V \subset \mathbb{R}^N$ open, bounded, such that $F : B(x_0,\delta) \to V$ is invertible and F^{-1} belongs to $C^1(V)^N \cap W^{1,\infty}(V)^N$. In particular, since $DF(x)$ is invertible for $x \in B(x_0,\delta)$, we have that its first row, ∇w does not vanish in $B(x_0,\delta)$. Defining $h = u \circ F^{-1} \in W^{1,p}(V)$, we have that $u(x) = h(F(x))$ for every $x \in B(x_0,\delta)$ and then

$$\nabla u(x) = \partial_{y_1} h(F(x))\nabla w(x) + \sum_{i=2}^N \partial_{y_i} h(F(x))\xi_i,$$

but by assumption, $\nabla u(x)$ is proportional to $\nabla w(x)$ and therefore, recalling that $\nabla w(x)$, $\xi_2, \cdots, \xi_N$ are linearly independent, the previous inequality shows

$$\partial_{y_i} h(F(x)) = 0, \quad 2 \le i \le N. \tag{2.16}$$

Using that F is a isomorphism from $B(x_0,\delta)$ into V, we then get

$$\partial_{y_i} h = 0 \ \text{ in } V, \quad 2 \le i \le N.$$

Taking τ small enough to have the cube

$$Q := (w(x_0)-\tau, w(x_0)+\tau) \times (\xi_2\cdot x_0 - \tau, \xi_2 \cdot x_0 + \tau) \times \ldots \times (\xi_N \cdot x_0 - \tau, \xi_N \cdot x_0 + \tau),$$

contained in V, we deduce from (2.16) that h only depends on the first variable in Q and therefore, using the definition of F, we have

$$u(x) = h(F(x)) = h(w(x)), \quad \forall x \in F^{-1}(Q).$$

Defining $U = F^{-1}(Q)$ which is an open set contained in $B(x_0, \delta)$ and observing that by definition of F and Q, w is onto from Q into $(w(x_0) - \tau, w(x_0) + \tau)$, we get the result for $K = \{x_0\}$.

Let us now consider the general case: By the above proved, for every $z \in K$, there exists a neighborhood U_z of z, $\tau_z > 0$ and $h_z \in W^{1,p}(c - \tau_z, c + \tau_z)$ such that

$$w(U_z) = (c - \tau_z, c + \tau_z), \qquad u(x) = h(w(x)), \ \forall x \in U_z, \qquad \nabla w \neq 0 \ \text{ in } U_z.$$

Since K is compact, we can take the open sets U_z such that there exists $\varepsilon > 0$ satisfying that $B(z, 2\varepsilon) \subset U_z$, for every $z \in K$. Thus

$$B(z_1, \varepsilon) \cup B(z_2, \varepsilon) \subset U_{z_1} \cap U_{z_2}, \quad \forall z_1, z_2 \in K, \ |z_1 - z_2| < \varepsilon,$$

and then

$$h_{z_1}(w(x)) = u(x) = h_{z_2}(w(x)), \quad \forall z_1, z_2 \in K, \ |z_1 - z_2| < \varepsilon, \ \forall x \in B(z_1, \varepsilon) \cup B(z_2, \varepsilon). \tag{2.17}$$

Now, we use that K compact, $w = c$ in K, $w \in C^1(\Omega)$ and $\nabla w \neq 0$ in K, imply the existence of $\tau > 0$ such that $(c - \tau, c + \tau) \subset w(B(z, \varepsilon))$, for every $z \in K$, and then (2.17) shows

$$h_{z_1} = h_{z_2} \ \text{ in } (c - \tau, c + \tau), \quad \forall z_1, z_2 \in K, \quad |z_1 - z_2| < \varepsilon.$$

This allows us to show that for $\hat{z} \in K$ fixed, the set of $z \in K$ such that

$$h_z = h_{\hat{z}} \ \text{ in } (c - \tau, c + \tau)$$

is open and closed in the topology relative to K, and thus, since K is connected

$$h_z = h_{\hat{z}} \ \text{ in } (c - \tau, c + \tau), \quad \forall z \in K.$$

Defining then

$$h = h_{\hat{z}}, \qquad U = w^{-1}(c - \tau, c + \tau) \bigcap \left(\bigcup_{z \in K} B(z, \varepsilon) \right),$$

we get the result. $\square$

To finish this subsection we recall the coarea formula (see e.g. Theorem 2 section 3.4.3 in [11]) and as Corollary the version we use in the proof of Theorem 2.1. The definition of Jacobian of a derivable function $f : \mathbb{R}^N \to \mathbb{R}^M$, $J(f)$ is given in [11], section 3.2.

Theorem 2.7. *Let $f \in W^{1,\infty}(\mathbb{R}^N)^M$ be, $N \geq M$. Then, for every measurable function $g \in L^1(\mathbb{R}^N)$, we have that $g_{|f^{-1}(y)}$ is integrable with respect to the Hausdorf $(N - M)$-dimensional measure, H_{N-M}, for a.e. $y \in \mathbb{R}^M$, the function*

$$y \in \mathbb{R}^M \mapsto \int_{f^{-1}(y)} g \, dH_{N-M}$$

is integrable in $\mathbb{R}^M$ and

$$\int_{\mathbb{R}^N} g J(f) \, dx = \int_{\mathbb{R}^M} \int_{f^{-1}(y)} g \, dH_{N-M} \, dy. \tag{2.18}$$

Corollary 2.8. *Assume $U \subset \mathbb{R}^N$ open and $w \in W^{1,\infty}(U)$ such that $\nabla w(x) \neq 0$, then, for any $g \in L^1(U)$, we have that $(g/|\nabla w|)_{|w^{-1}(s)}$ is integrable with respect to H_{N-1}, for a.e. $s \in \mathbb{R}$, the function*

$$s \in \mathbb{R} \mapsto \int_{f^{-1}(s)} g\, dH_{N-1}$$

is integrable in $w(U)$ and

$$\int_U g\, dx = \int_{w(U)} \int_{w^{-1}(s)} \frac{g}{|\nabla w|}\, dH_{N-1}\, ds. \tag{2.19}$$

Proof. For $\varepsilon > 0$, we define

$$U_\varepsilon = \left\{ x \in U : \ |\nabla w(x)| \geq \varepsilon \right\}.$$

Thanks to Kirszbraun's Theorem ([11], chapter 3.1.1, [12], chapter 2.10.43), we know there exists $w_\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$, such that $w_\varepsilon = w$ in U_ε. Applying (2.18) to

$$g_\varepsilon = \frac{g}{|\nabla w|}\chi_{U_\varepsilon}, \quad f = w_\varepsilon,$$

and taking into account that $J(w_\varepsilon) = |\nabla w_\varepsilon|$, we deduce that $(g_\varepsilon)_{|w^{-1}(s)}$ is integrable with respect to the H_{N-1}, for a.e. $s \in \mathbb{R}$, the function

$$s \mapsto \int_{w^{-1}(s)} g_\varepsilon\, dH_{N-1}$$

is integrable in $\mathbb{R}$ and

$$\int_{U_\varepsilon} g\, dx = \int_{\mathbb{R}^N} \frac{g}{|\nabla w|}\chi_{U_\varepsilon} |\nabla w_\varepsilon|\, dx = \int_{\mathbb{R}} \int_{w_\varepsilon^{-1}(s)} \frac{g}{|\nabla w|}\chi_{U_\varepsilon}\, dH_{N-1}\, ds$$
$$= \int_{w(U_\varepsilon)} \int_{w^{-1}(s)} \frac{g}{|\nabla w|}\, dH_{N-1}\, ds.$$

Thanks to the Lebesgue dominated convergence theorem, we can now pass to the limit in this equality when ε tends to zero to deduce the result. □

2.2. *Proof of the main result*

In the present subsection, let us use the results of the previous one in order to prove Theorem 2.1, which is the aim of the present paper.

Proof of Theorem 2.1. By Corollary 2.5, we know that if there exists a solution (ω, u) of (2.2) and λ is the corresponding eigenvalue, then u belongs to $W^{1,\infty}(\Omega)$ and (2.13) holds, with w, the solution of (2.11) and λ given by (2.12). Moreover, (2.14) and (2.15) are satisfied, with μ strictly positive

Using the classical smoothness results for elliptic equations and $\Omega \in C^{1,1}$, we also have w in $W^{2,p}(\Omega) \cap W^{3,p}_{loc}(\Omega)$, for every $p \geq 1$. On the other hand, it is well known that u and then w can be chosen strictly positive in Ω. The following three steps are devoted to show that these properties of ω, u and w imply that Ω is a ball and that χ_ω, u and w are radial functions.

Step 1. Assume $x_0 \in \Omega$ such that

$$|B(x_0, r) \cap \omega| > 0, \quad |B(x_0, r) \cap (\Omega \setminus \omega)| > 0, \qquad \forall r > 0. \tag{2.20}$$

Let us prove that there exists a connected open set $O \Subset \Omega$ of class $W^{3,p}$ for every $p \in [1, \infty)$, with connected boundary such that

$$x_0 \in \partial O, \quad |\nabla w| = \mu \ \text{ on } \partial O, \quad w = w(x_0) \text{ on } \partial O. \tag{2.21}$$

Moreover, all the points in ∂O satisfy condition (2.20) and there exists a connected neighborhood U of ∂O, $\tau > 0$ and $h \in W^{1,\infty}(w(x_0) - \tau, w(x_0) + \tau)$ such that

$$w(U) = (w(x_0) - \tau, w(x_0) + \tau), \quad u(x) = h(w(x)), \quad \forall x \in U. \tag{2.22}$$

For this purpose, we first observe that (2.15), (2.20) and ∇w continuous imply $|\nabla w(x_0)| = \mu \neq 0$ and thus, the implicit function theorem, $w \in W^{3,p}_{\rm loc}(\Omega)$ for every $p \in [1, \infty)$, and $w(x_0) \neq 0$ prove that the set

$$\hat{\Upsilon} = \left\{x \in \Omega : \quad w(x) = w(x_0), \quad |\nabla w(x)| > \frac{\mu}{2}\right\} \tag{2.23}$$

is a $(N-1)$-dimensional submanifold of $\mathbb{R}^N$ strictly contained in Ω of class $W^{3,p}$ for every $p \in [1, \infty)$, containing x_0.

We define Υ as the connected component of $\hat{\Upsilon}$ containing x_0. By (2.13), we can apply Lemma 2.6 to deduce that for every compact set $K \subset \Upsilon$ which contains x_0, there exists an open neighborhood $U \subset \Omega$ of K, $\tau > 0$, and a function $h \in W^{1,\infty}(w(x_0) - \tau, w(x_0) + \tau)$ such that

$$w(U) = (w(x_0) - \tau, w(x_0) + \tau), \qquad u(x) = h(w(x)), \quad \forall x \in U, \qquad \nabla w \neq 0 \ \text{ in } U. \tag{2.24}$$

In particular, we have

$$\nabla u(x) = h'(w(x))\nabla w(x), \ \text{ a.e. } x \in U. \tag{2.25}$$

Taking into account (2.13) and (2.25), we can apply (2.19) to deduce

$$0 = \int_U \big|(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u - \nabla w\big| dx = \int_{w(x_0)-\tau}^{w(x_0)+\tau} \int_{U\cap w^{-1}(s)} \frac{\big|(\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u - \nabla w\big|}{|\nabla w|} dH_{N-1} ds,$$

$$0 = \int_U \big|\nabla u - h'(w)\nabla w\big| dx = \int_{w(x_0)-\tau}^{w(x_0)+\tau} \int_{U\cap w^{-1}(s)} \frac{\big|\nabla u - h'(s)\nabla w\big|}{|\nabla w|} dH_{N-1} ds,$$

and therefore

$$\int_{w(x_0)-\tau}^{w(x_0)+\tau} \int_{U\cap w^{-1}(s)} \left|h'(s) - \left(\frac{1}{\alpha}\chi_\omega + \frac{1}{\beta}\chi_{\Omega\setminus\omega}\right)\right| dH_{N-1}\, ds = 0.$$

Thus, there exists a null measure set $\mathcal{N} \subset (w(x_0) - \tau, w(x_0) + \tau)$ such that

$$s \in (w(x_0) - \tau, w(x_0) + \tau) \setminus N \Longrightarrow \exists\, h'(s) \in \{\alpha^{-1}, \beta^{-1}\}$$

and

$$\begin{cases} h'(s) = \alpha^{-1} \Longrightarrow H_{N-1}((w^{-1}(s) \cap U) \setminus \omega) = 0 \\ h'(s) = \beta^{-1} \Longrightarrow H_{N-1}(w^{-1}(s) \cap U \cap \omega) = 0. \end{cases} \tag{2.26}$$

On the other hand, we observe that for every $r > 0$ with $B(x_0, r) \subset U$ formula (2.19) gives

$$|B(x_0, r) \cap \omega| = \int_{w(x_0)-\tau}^{w(x_0)+\tau} \int_{B(x_0,r)\cap\omega\cap w^{-1}(s)} \frac{1}{|\nabla w|} dH_{N-1} ds, \tag{2.27}$$

$$|B(x_0, r) \cap (\Omega \setminus \omega)| = \int_{w(x_0)-\tau}^{w(x_0)+\tau} \int_{B(x_0,r)\cap(\Omega\setminus\omega)\cap w^{-1}(s)} \frac{1}{|\nabla w|} dH_{N-1} ds, \tag{2.28}$$

which thanks to (2.20) shows the existence of $s_n^1, s_n^2 \notin \mathcal{N}$, converging to $w(x_0)$ when n tends to infinity such that

$$H_{N-1}\big(B(x_0, 1/n) \cap \omega \cap w^{-1}(s_n^1)\big) > 0, \quad H_{N-1}\big(B(x_0, 1/n) \cap (\Omega \setminus \omega) \cap w^{-1}(s_n^2)\big) > 0.$$

Therefore, by (2.26), we get

$$H_{N-1}\big((w^{-1}(s_n^1)\cap U)\setminus\omega\big)=0,\qquad H_{N-1}\big(w^{-1}(s_n^2)\cap U\cap\omega\big)=0$$

and then, by (2.15)

$$|\nabla w|\geq\mu\ \text{ in } w^{-1}(s_n^1)\cap U,\qquad |\nabla w|\leq\mu\ \text{ in } w^{-1}(s_n^2)\cap U.$$

Then, passing to the limit in n, we deduce

$$|\nabla w|=\mu\ \text{ in } w^{-1}(w(x_0))\cap U\supset K. \tag{2.29}$$

Now, we observe that (2.27), (2.28), (2.26) and (2.20), imply that for every $\delta>0$ there exist two disjoint measurable sets $S_\delta^1, S_\delta^2\subset(w(x_0)-\delta, w(x_0)+\delta)$ such that

$$\begin{gathered}0<|S_\delta^1|,\quad 0<|S_\delta^2|,\quad |S_\delta^1|+|S_\delta^2|=2\delta,\\ H_{N-1}((w^{-1}(s)\cap U)\setminus\omega)=0,\ \forall s\in S_\delta^1,\qquad H_{N-1}(w^{-1}(s)\cap U\cap\omega)=0,\ \forall s\in S_\delta^2,\end{gathered}$$

while $w=w(x_0)$ in K implies that for every $r>0$, there exists $\delta>0$ such that

$$H_{N-1}(B(x,r)\cap w^{-1}(s))>0,\ \text{ a.e. } s\in(w(x_0)-\delta, w(x_0)+\delta).$$

Then, using again (2.19), we have for every $r>0$, with $B(x,r)\subset U$

$$|B(x,r)\cap\omega|=\int_{w(x_0)-\tau}^{w(x_0)+\tau}\int_{B(x,r)\cap\omega\cap w^{-1}(s)}\frac{1}{|\nabla w|}dH_{N-1}ds\geq\int_{S_\delta^1}\int_{B(x,r)\cap w^{-1}(s)}\frac{1}{|\nabla w|}dH_{N-1}ds>0,$$

$$|B(x,r)\setminus\omega|=\int_{w(x_0)-\tau}^{w(x_0)+\tau}\int_{B(x,r)\cap(\Omega\setminus\omega)\cap w^{-1}(s)}\frac{1}{|\nabla w|}dH_{N-1}ds\geq\int_{S_\delta^2}\int_{B(x,r)\cap w^{-1}(s)}\frac{1}{|\nabla w|}dH_{N-1}ds>0.$$

This proves that every point in K satisfies (2.20).

We have thus proved that every compact and connected set $K\subset\Upsilon$ which contains x_0 is such that $|\nabla w|=\mu$ in K and that every point in K satisfies (2.20). Since Υ is connected and locally connected by paths, because it is a manifold, it is connected by paths (see e.g. [10], chapter 5.5). Therefore, for every point x in Υ there exists a compact set $K\subset\Upsilon$ containing x and x_0. This proves

$$|\nabla w|=\mu\ \text{ in }\Upsilon, \tag{2.30}$$

and that every point in Υ satisfies condition (2.20).

Let us use (2.30) to show that Υ is closed in $\mathbb{R}^N$. In particular this will imply that we can take $K=\Upsilon$ above and then that there exists a neighborhood U of Υ, $\tau>0$ and $h\in W^{1,\infty}(w(x_0)-\tau, w(x_0)+\tau)$ satisfying (2.24).

To prove Υ closed, we take $x_n\in\Upsilon$ converging to a certain point $x\in\bar\Omega$. Then, since $w\in C^1(\bar\Omega)$, we have $w(x)=w(x_0)$, $|\nabla w(x)|=\mu$. This shows in particular that x belongs to $\hat\Upsilon$ but since Υ is a connected component of $\hat\Upsilon$, it is closed (and open) for the topology of Υ. Therefore, $x_n\in\Upsilon$ converging to $x\in\hat\Upsilon$ implies that x belongs to Υ, and so Υ is closed in $\mathbb{R}^N$.

The above proved shows that Υ is a connected compact submanifold of class $W^{3,p}$, for every $p\in[1,\infty)$ and dimension $N-1$. By the Jordan–Brouwer theorem (see e.g. [13] Chapter 5.2) and Υ strictly contained in Ω (because $w(x)=w(x_0)>0$ in Υ), we have that Υ is the boundary of a bounded connected open set $O\Subset\Omega$ of class $W^{3,p}$, for every $p\in[1,\infty)$.

Step 2. Let us prove that u and w are radial in a ball contained in Ω.

First we observe that a point x_0 satisfying (2.20) always exists because in another case, the sets

$$\{x\in\Omega:\ \exists r>0\ \text{ with } |B(x_0,r)\cap\omega|=0\},\qquad \{x\in\Omega:\ \exists r>0\ \text{ with } |B(x_0,r)\cap(\Omega\setminus\omega)|=0\},$$

are two disjoint nonempty open sets with positive measure (because $0<|\omega|<|\Omega|$) whose union agrees with Ω, which is a contradiction with Ω connected. Therefore, we can consider a bounded open set O in the conditions of Step 1.

Using that w is constant on ∂O and $-\Delta w = \lambda u > 0$ in O, we get that the maximum of w in O is attained in an interior point $\bar{x}$ and then $\nabla w(\bar{x}) = 0$. By (2.15) and $\mu > 0$, this shows that there exists a neighborhood of $\bar{x}$ contained in $O \setminus \omega$.

Now, we observe that the boundaries of two open sets in the conditions of Step 1 cannot intersect. Thus, we have

$$O_1, O_2 \text{ in the conditions of Step 1} \Longrightarrow \text{ one of these conditions hold } \begin{cases} O_1 \subset O_2 \\ O_2 \subset O_1 \\ O_1 \cap O_2 = \emptyset. \end{cases} \tag{2.31}$$

Take

$$\hat{O} = \bigcap_{\substack{\bar{x} \in O \\ O \text{ in the conditions of Step 1}}} O$$

and observe that if x belongs to $\partial \hat{O}$ then, there exists a decreasing sequence O_m of open sets in the conditions of Step 1 and a sequence $x_m \in \partial O_m$ converging to x. Since the points x_m satisfy condition (2.20), we have that x also satisfies this condition and then by Step 1, we can construct a set O in the conditions of this step such that $x \in \partial O$. By (2.31) we necessarily have $O = \hat{O}$.

We have then constructed an open set O in the conditions of Step 1 containing $\bar{x}$ which does not contain any other set in these conditions. From Step 1, we have that there is no point in O satisfying condition (2.20), which combined with the fact that a neighborhood of $\bar{x}$ is contained in the complementary of ω, implies $|\omega \cap O| = 0$. Then, O connected, (2.13), (2.11) and w, $|\nabla w|$ constants on ∂O, imply the existence of a constant $c \in \mathbb{R}$, such that

$$\beta u = w + c \text{ a.e. in } O,$$

and that w satisfies

$$\begin{cases} -\Delta w = \dfrac{\lambda}{\beta}(w + c) \text{ in } O \\ w, \dfrac{\partial w}{\partial n} \text{ constant in } O. \end{cases}$$

Since O is of class C^2 we can then apply Serrin's theorem ([21]) to deduce that O is a ball of center $\bar{x}$ and that w and u are radial functions in O.

Step 3. Let us now finish the proof of Theorem 2.1.

For $\bar{x}$ the center of a ball in the conditions of Step 2, we define R by

$$B(\bar{x}, R) = \bigcup_{w \text{ radial in } B(\bar{x}, r)} B(\bar{x}, r),$$

or equivalently, as the maximum of $r > 0$ such that w is radial in $B(\bar{x}, r)$. If $w = 0$ on $\partial B(\bar{x}, R)$, we get $\Omega = \partial B(\bar{x}, R)$, and then the proof of Theorem 2.1 is finished. So, we assume $w > 0$ on $\partial B(\bar{x}, R)$. By Hopf's Lemma applied to the equation

$$-\Delta w = \lambda u \text{ in } B(\bar{x}, R),$$

we get that $\frac{\partial w}{\partial n}$ is strictly negative on $\partial B(\bar{x}, R)$, and then, thanks to (2.13) we can apply Lemma 2.6 to deduce the existence of a neighborhood U of $\partial B(\bar{x}, R)$, $\tau > 0$ and $h \in W^{1,\infty}(U)$ satisfying (2.22). From (2.11), we have that w is a solution of

$$-\Delta w = \lambda h(w) \text{ in } U. \tag{2.32}$$

Now, we introduce ϕ as the solution of the Cauchy problem

$$\begin{cases} -(r^{N-1}\phi')' = \lambda r^{N-1} h(\phi) \\ \phi(R) = w_{|\partial B(\bar{x}, R)}, \quad \phi'(R) = \dfrac{\partial w}{\partial n}_{|\partial B(\bar{x}, R)}, \end{cases} \tag{2.33}$$

which is defined in an open set $(R-\varepsilon, R+\varepsilon)$, where $\varepsilon > 0$ can be chosen small enough to have $B(\bar{x}, R+\varepsilon) \setminus \bar{B}(\bar{x}, R-\varepsilon) \subset U$. Then, we define the functions z and k in $B(\bar{x}, R+\varepsilon) \setminus \bar{B}(\bar{x}, R-\varepsilon)$ by

$$z(x) = w(x) - \phi(|x-\bar{x}|),$$
$$k(x) = \begin{cases} -\lambda \dfrac{h(w(x)) - h(\phi(|x-\bar{x}|))}{z(x)} & \text{if } z(x) \neq 0 \\ 0 & \text{if } z(x) = 0. \end{cases}$$

Using that h is Lipschitz, we get that k is in $L^\infty(B(\bar{x}, R+\varepsilon) \setminus \bar{B}(\bar{x}, R-\varepsilon))$, while w radial in $B(\bar{x}, R) \setminus \bar{B}(\bar{x}, R-\varepsilon)$, (2.32) and (2.33) imply that z satisfies

$$\begin{cases} -\Delta z + k\, z = 0 \ \text{ in } B(\bar{x}, R+\varepsilon) \setminus \bar{B}(\bar{x}, R-\varepsilon) \\ z = 0 \ \text{ in } B(\bar{x}, R) \setminus \bar{B}(\bar{x}, R-\varepsilon) \end{cases}$$

From Carleman's unique continuation theorem, (see e.g. [3,4]), the function z vanishes on $B(\bar{x}, R+\varepsilon) \setminus B(\bar{x}, R-\varepsilon)$ and then u is radial in $B(\bar{x}, R+\varepsilon)$, in contradiction to the definition of R. □

Conflict of interest statement

There is no conflict.

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