Structural stability of the inverse limit of endomorphisms

Stabilité structurelle de l'extension naturelle des endomorphismes

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Abstract

We prove that every endomorphism which satisfies Axiom A and the strong transversality conditions is C^1-inverse limit structurally stable. These conditions were conjectured to be necessary and sufficient. This result is applied to the study of unfolding of some homoclinic tangencies. This also achieves a characterization of C^1-inverse limit structurally stable covering maps.
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Résumé

Nous montrons qu'un endomorphisme a son extension naturelle qui est C^1-structurellement stable s'il vérifie l'axiome A et la condition de transversalité forte. Ces conditions étaient conjecturées nécessaires et suffisantes. Ce résultat est appliqué à l'étude des déploiements des tangences homoclines. Aussi, cela accomplit la description des recouvrements dont l'extension naturelle est C^1-structurellement stable.
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1. Introduction

Following Smale [19], a diffeomorphism f is C^r-structurally stable if any C^r-perturbation f' of f is conjugate to f via a homeomorphism h of M:

$$f \circ h = h \circ f'.$$

A great work was done by many authors to provide a satisfactory description of C^1-structurally stable diffeomorphisms, which starts with Anosov, Smale, Palis (see [13,10]) and finishes with Robinson [18] and Mañé [6]. Such diffeomorphisms are those which satisfy Axiom A and the strong transversality condition.

The descriptions of the structurally stable maps for smoother topologies (C^r, C^ω, holomorphic...) remain some of the hardest, fundamental and open questions in dynamics.

Hence the description of C^r-structurally stable endomorphisms (C^r-maps of a manifold not necessarily bijective) with critical points (points at which the differential is not surjective) is even harder.

Indeed, this implies that the critical set must be stable (i.e. the map must be equivalent to its perturbations via homeomorphisms) and so that r must be at least 2. We recall that the description of critical sets which are stable is still an open problem [7].

It is not the case when we consider the structural stability of the inverse limit. We recall that the *inverse limit set* of a C^1-endomorphism f is the space of the full orbits $(x_i)_i \in M^{\mathbb{Z}}$ of f. The dynamics induced by f on its inverse limit set is the shift. The endomorphism f is C^1-*inverse limit stable* (or equivalently inverse limit of f is C^1-structurally stable) if for every C^1 perturbation f' of f, the inverse limit set of f' is homeomorphic to the one of f via a homeomorphism which conjugates both induced dynamics and which is C^0-close to the canonical inclusion into $M^{\mathbb{Z}}$.

When the dynamics f is a diffeomorphism, the inverse limit set $\overleftarrow{M}_f$ is homeomorphic to the manifold M. The C^1-inverse limit stability of f is then equivalent to the C^1-*structural stability* of f: every C^1-perturbation of f is conjugated to f via a homeomorphism of M C^0-close to the identity.

The concept of inverse limit stability is an area of great interest for semi-flows given by PDEs, although still at its infancy [16,4].

There were many works giving sufficient conditions for an endomorphism to be structurally stable [9,12,2]. The latter work generalized Axiom A and the strong transversality condition to differentiable endomorphisms of manifolds, and conjectured these conditions to be equivalent to C^1-inverse limit stability. A main point of this work was to give evidences that the notion of inverse stability should be independent to the nature of the critical set (stable or not for instance). A similar conjecture was sketched in [15].

We prove here one direction of this conjecture, generalizing [9,12,2,17,18]:

Theorem 1.1 (Main result). *Every C^1-endomorphism of a compact manifold which satisfies Axiom A and the strong transversality condition is C^1-inverse limit structurally stable.*

The definitions of Axiom A and the strong transversality condition will be recalled in §2.3.

Joint with the works of [1] and [2], this proves that C^1-inverse limit stable covering maps of manifolds are exactly the C^1 covering maps which satisfy Axiom A and strong transversality conditions (see §3.1).

On the other hand, our main result applies to the dynamical studies of homoclinic tangencies unfolding as seen in section (see §3.2).

The proof of the main result is done by generalizing Robbin–Robinson proof of the structural stability with two new difficulties. We will have to handle the geometrical and analytical part of the argument on the inverse limit space which is in general not a manifold as it is the case for diffeomorphisms (see §6). Also we will have to take care of the critical set in the plane fields constructions and in the inverse of the operator considered (see §7, 8 and 9).

2. Notations and definitions

Along this article M will denote a smooth Riemannian compact manifold without boundary. The distance on M induced by the Riemannian structure will be simply denoted by d. For any $r \in \mathbb{N}$, we denote by $\mathrm{End}^r(M)$ the space of C^r endomorphisms of M. By C^r *endomorphism of M*, we mean a C^r map f of M into M, which is possibly non surjective and can have a non-empty *critical set*:

$$C_f := \{x \in M : T_x f \text{ not surjective}\}.$$

We endow $\mathrm{End}^r(M)$ with the topology of uniform convergence of the first r derivatives.

Given any $f \in \mathrm{End}^r(M)$, a subset $\Lambda \subset M$ is *forward invariant* whenever $f(\Lambda) \subset \Lambda$, and *totally invariant* when $f^{-1}(\Lambda) = \Lambda$. Note that totally invariance implies forward invariance.

The set of periodic points of f is denoted by $\mathrm{Per}(f)$ and we write $\Omega(f)$ for the set of non-wandering points. Observe that $f(\mathrm{Per}(f)) = \mathrm{Per}(f)$ and $f(\Omega(f)) = \Omega(f)$, but in general they are not totally invariant.

Now, let K be a compact metric space and $E \to K$ a finite dimensional vector bundle over K. If $F \subset E$ is a sub-vector-bundle of $E \to K$, we denote by E/F the quotient bundle. Note that any (Riemannian) norm $\|\cdot\|_E$ on E naturally induces a (Riemannian) norm on E/F defining

$$\|v_x + F_x\|_{E/F} := \inf_{w_x \in F_x} \|v_x + w_x\|_E, \quad \forall x \in K, \; \forall v_x \in E_x.$$

On the other hand, observe that any bundle map $T \colon E \to E$ that leaves invariant F (i.e. F is forward invariant for T) naturally induces a bundle map $[T] \colon E/F \to E/F$.

2.1. Inverse limits

Given any set X and an arbitrary map $f \colon X \to X$, we define its *global attractor* by $X_f := \bigcap_{n \geq 1} f^n(X)$ and its *inverse limit* by

$$\overleftarrow{X}_f := \left\{ \underline{x} = (x_n)_n \in X^{\mathbb{Z}} : f(x_n) = x_{n+1}, \; \forall n \in \mathbb{Z} \right\}. \tag{1}$$

Observe that $f(X_f) = X_f$, but in general X_f is not totally invariant ($X_f \neq f^{-1}(X_f)$), and $\overleftarrow{X}_f \subset (X_f)^{\mathbb{Z}}$. Moreover, f acts coordinate-wise on $\overleftarrow{X}_f$. In fact, we can define $\overleftarrow{f} \colon \overleftarrow{X}_f \to \overleftarrow{X}_f$ by $\overleftarrow{f}(\underline{x}) := (f(x_n))_n = (x_{n+1})_n$, and in this way, $\overleftarrow{f}$ turns out to be a bijection and f a factor of it. Indeed, for every $j \in \mathbb{Z}$ we can define the jth-projection $\pi_j \colon \overleftarrow{X}_f \to X_f$ by $\pi_j(\underline{x}) = x_j$ and then we have

$$\pi_{j+1} = \pi_j \circ \overleftarrow{f} = f \circ \pi_j.$$

Whenever X is a topological space and f is continuous, we shall consider $X^{\mathbb{Z}}$ endowed with the product topology. In this case, $\overleftarrow{X}_f$ turns out to be closed in $X^{\mathbb{Z}}$ and $\overleftarrow{f}$ a homeomorphism. Of course, $\mathrm{Per}(f)$ and $\Omega(f)$ are contained in X_f and $\overleftarrow{X}_f$ is compact whenever X_f is compact itself.

Finally, when X is endowed with a finite distance d, we shall consider $X^{\mathbb{Z}}$ equipped with the distance d_1 given by

$$d_1(\underline{x}, \underline{y}) := \sum_{n \in \mathbb{Z}} \frac{d(x_n, y_n)}{2^{|n|}}. \tag{2}$$

The metric space $(X^{\mathbb{Z}}, d_1)$ is compact if and only if X is compact itself.

2.2. Structural and inverse limit stability

Two endomorphisms $f, g \in \mathrm{End}^r(M)$ are *conjugate* when there exists a homeomorphism $h \in \mathrm{Homeo}(M)$ satisfying $h \circ f = g \circ h$. More generally, the endomorphisms f and g are *inverse limit conjugate* whenever there exists a homeomorphism $H \colon \overleftarrow{M}_f \to \overleftarrow{M}_g$ such that $H \circ \overleftarrow{f} = \overleftarrow{g} \circ H$. Remark that the conjugacy relation implies the inverse limit conjugacy one.

A C^r-endomorphism f is C^s-*structurally stable* (with $0 \leq s \leq r$) when there exists a C^s-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is conjugate to f. Analogously, f is C^s-*inverse limit stable* when every $g \in \mathcal{U}$ is inverse limit conjugate to f.

2.3. Axiom A endomorphisms

Let $f \in \mathrm{End}^1(M)$ and let $\Lambda \subset M$ be a compact forward invariant set. The set Λ is *hyperbolic* whenever there exists a continuous sub-bundle $E^s \subset T_\Lambda M$ satisfying the following properties:

1. E^s is forward invariant by Tf, i.e.

$$T_x f(E^s_x) \subset E^s_{f(x)}, \quad \forall x \in \Lambda;$$

2. the induced linear map $[T_x f]: T_x M / E^s \to T_{f(x)} M / E^s$ is an isomorphism, for every $x \in \Lambda$; (see §2 for notation of quotient bundles and induced maps)

3. $\forall x \in \Lambda$, $\left\| T_x f \big|_{E_x^s} \right\| < 1$ and $\left\| [T_x f]^{-1} \right\| < 1$, where the first operator norm is induced by the Riemannian structure of M, and the second by its quotient.

Remark 2.1. Notice that despite E^s is contained in $T_\Lambda M$, in general we cannot define E^u as a sub-bundle of the tangent bundle.

However, using a classical cone field argument, we show:

Proposition 2.2. *There exists a continuous family* $(E_{\underline{x}}^u)_{\underline{x} \in \overleftarrow{\Lambda}_f}$ *of subspaces of* $T_\Lambda M$ *such that:*

1. *for every* $\underline{x} \in \overleftarrow{\Lambda}_f$, $E_{\underline{x}}^u \subset T_{\pi_0(\underline{x})} M$ *and* $T f (E_{\underline{x}}^u) = E_{\overleftarrow{f}(\underline{x})}^u$,

2. *for every* $\underline{x} \in \overleftarrow{\Lambda}_f$, *the restriction* $T f : E_{\underline{x}}^u \to E_{\overleftarrow{f}(\underline{x})}^u$ *is invertible and*

$$\left\| (T f \big|_{E_{\underline{x}}^u})^{-1} \right\| < 1.$$

Given any (small) $\varepsilon > 0$ and $\underline{x} \in \overleftarrow{\Lambda}_f$, we define the ε-*local stable set* of $\underline{x}$ by

$$W_\varepsilon^s(\underline{x}, f) := \left\{ \underline{y} \in \overleftarrow{M}_f : \varepsilon \geq d_1(\overleftarrow{f}^n(\underline{x}), \overleftarrow{f}^n(\underline{y})) \xrightarrow{n \to +\infty} 0, \ \forall n \geq 0 \right\},$$

where d_1 denotes the distance given by (2); and the ε-*local unstable set* of $\underline{x}$ is defined analogously by

$$W_\varepsilon^u(\underline{x}, f) := \left\{ \underline{y} \in \overleftarrow{M}_f : \varepsilon \geq d_1(\overleftarrow{f}^{-n}(\underline{x}), \overleftarrow{f}^{-n}(\underline{y})) \xrightarrow{n \to +\infty} 0, \ \forall n \geq 0 \right\}.$$

The geometry of these sets was described in [2]. Let us recall that $\pi_0\big(W_\varepsilon^u(\underline{x}, f)\big)$ and $\pi_0\big(W_\varepsilon^s(\underline{x}, f)\big)$ are submanifolds of M (for ϵ sufficiently small).

The endomorphism f satisfies *Axiom A* when $\Omega(f)$ is hyperbolic and coincides with the closure of $\mathrm{Per}(f)$.

An Axiom A endomorphism satisfies the *strong transversality condition* if for every $\underline{x}, \underline{y} \in \Omega(\overleftarrow{f})$ and every $n \geq 0$, the map $f^n \big|_{\pi_0(W_\epsilon^u(\underline{x}))}$ is transverse to $\pi_0(W_\epsilon^s(\underline{y}))$. This means that for every $z \in \pi_0(W_\epsilon^u(\underline{x})) \cap f^{-n}(\pi_0(W_\epsilon^s(\underline{y})))$ the following holds:

$$T f^n \big(T_z \pi_0(W_\epsilon^u(\underline{x})) \big) + T_{f^n(z)} \pi_0(W_\epsilon^s(\underline{y})) = T_{f^n(z)} M$$

An endomorphism which satisfies Axiom A and the strong transversality condition is called an *AS-endomorphism*. This notion generalizes the one of diffeomorphism. Let us recall some other examples.

Examples 2.3.

– The action of any hyperbolic linear matrix in $\mathcal{M}_n(\mathbb{Z})$ in the n-dimensional torus is hyperbolic and so is AS. It is not structurally stable whenever the matrix is not in $SL_n(\mathbb{Z})$ nor expanding [11].
– The constant map $\mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ satisfies Axiom A and the strong transversality condition.
– The map $\mathbb{R}^n \ni x \mapsto x^2 + c$ satisfies Axiom A and the strong transversality condition whenever c is such that a (possibly super) attracting periodic orbit exists.

Remark 2.4. Let us notice that if two endomorphisms $f_1 \in C^1(M_1, M_1)$ and $f_2 \in C^1(M_2, M_2)$ satisfy Axiom A and the strong transversality condition, then the product dynamics $f_1 \times f_2 \in C^1(M_1 \times M_2, M_1 \times M_2)$ also satisfy Axiom A and the strong transversality condition.

As an endomorphism is AS iff its one of its iterates is AS, it follows that the following delay dynamics is AS if $f \in C^1(M, M)$ is AS:

$$M^n \ni (x_i)_i \mapsto (f(x_m), x_1, \ldots, x_{m-1}, 0, \ldots, 0) \in M^n.$$

From the latter example and remark, we have the following.

Example 2.5. For every $c \in \mathbb{R}$ such that $x^2 + c$ has an attracting periodic orbit, the following map is AS:

$$(x_i)_i \in \mathbb{R}^n \mapsto (x_m^2 + c, x_1, \ldots, x_{m-1}, 0, \ldots, 0) \in \mathbb{R}^n.$$

We will see that this example appears in the unfolding of generics homoclinic tangency in §3.2.

3. Applications of main Theorem 1.1

3.1. Description of C^1-inverse limit stable covering maps

A C^1-*covering map* of a compact, connected manifold M is a surjective C^1 endomorphism f of M without critical points. Then, every point of M has the same number p of preimages under f. We remark that every distinguish neighborhood $U \subset M$ has its preimage $\pi_0^{-1}(U)$ in the inverse limit $\overleftarrow{M}_f$ which is homeomorphic to $U \times \mathcal{Z}_p$ where $\mathcal{Z}_p$ is a Cantor set labeling the different f-preorbits of U. These homeomorphisms endow $\overleftarrow{M}_f$ with a structure of lamination called the Sullivan solenoid [20].

It follows immediately from a theorem due to Aoki, Moriyasu and Sumi [1] that: if an endomorphism f is C^1-inverse limit stable and has no critical point in the non-wandering set, then f satisfies Axiom A. By Theorem 2.4 of [2], if f is C^1-inverse limit stable and satisfies Axiom A, then f satisfies the strong transversality condition. Together with Main Theorem 1.1, it comes the following description of C^1-inverse stable covering maps.

Theorem 3.1. *A C^1-covering map of a compact manifold is C^1-inverse limit stable if and only if it is an AS-endomorphism.*

3.2. Application to dynamical study of unfolding homoclinic tangencies

Let M be a manifold of dimension m and let $(f_\mu)_\mu$ be a smooth family of diffeomorphisms of M which has a hyperbolic fixed point p with unstable and stable directions of dimensions $u \geq 1$ and $s \geq 1$ respectively. Hence $m = u + s$. The following Theorem has been proven in the general case as in [8] (Proposition 1). For more restricted cases see [14] when $(s, u) = (1, 1)$ and Theorem 1 [21] when $(s, u) = (1, 2)$.

Theorem 3.2 (L. Mora). *There exist $h \geq u$, an open set of families $(f_\mu)_{\mu \in \mathbb{R}^h}$ of smooth diffeomorphisms of M, which exhibit at $\mu_0 \in \mathbb{R}^h$ an unfolding of a homoclinic tangency at $q \in W^s(p) \cap W^u(p)$, such that there exists a small neighborhood $N_q \subset M$ of q, there exists a small neighborhood $N_\mu \subset \mathbb{R}^h$ of μ_0 covered by submanifolds $\mathcal{L}$ of dimension u, satisfying for every n large:*

- *μ_0 belongs to every submanifold $\mathcal{L}$ and the intersection of two different such manifolds $\mathcal{L}$ is the single point μ_0,*
- *for every $\mathcal{L}$, there is parametrization γ_n of $\mathcal{L}$ by $\mathbb{R}^u$, such that for every $\mu = \gamma_n(\underline{b}) \in \mathcal{L} \setminus \{\mu_0\}$, there is a chart ϕ_μ of N_q, such that the rescaled first return map has the form:*

$$\phi_\mu \circ f_{\gamma(\underline{b})}^n \circ \phi_\mu^{-1} \colon \mathbb{R}^u \times \mathbb{R}^s \longrightarrow \mathbb{R}^u \times \mathbb{R}^s = \mathbb{R}^m$$

$$(\underline{x}, \underline{y}) \mapsto (x_u^2 + b_u + \sum_{i=1}^{u-1} b_i x_i, x_1, \ldots, x_{u-1}, 0, \ldots, 0) + E_\mu(\underline{x}, \underline{y}),$$

with $\underline{b} = b_1, \ldots b_u$, $\underline{x} = (x_1, \ldots, x_u)$, $\underline{y} = (y_1, \ldots, y_s)$ and $E_\mu \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ small in the compact-open C^r-topology for every r when n is large.

In particular, near the curve $\{\mu_n(a) := \gamma_n(0, \ldots, 0, a), a \in \mathbb{R}\}$, the rescaled first return map $f_{\mu_n(a)}$ is C^r close to the endomorphism:

$$F_a := (x_1, \ldots, x_u, y_1, \ldots, y_s) \mapsto (x_u^2 + a, x_1, \ldots, x_{u-1}, 0, \ldots, 0).$$

For an open and dense set of parameters a, the map $x \mapsto x^2 + a$ has an attracting periodic orbit from [5,3]. Then its non-wandering set is the union of an attracting periodic orbit with an expanding compact set. By Example 2.5, we know that F_a is AS. Moreover we can extend F_a to the n-torus which is the product of n-times the one point compactification of $\mathbb{R}$. Its extension is analytic and AS.

Hence by Theorem 1.1, the inverse limit of F_a restricted to bounded orbits is conjugate to an invariant compact set of $f^n_{\mu_n(a)}|U_n$ with n large.

In particular, if such an a is fixed and then n is taken large, then there exist an open set V_n of M and a neighborhood W_n of $\mu_n(a)$ such that for every $a' \in W_n$, $f^{nu}_{a'}|V_n$ has its maximal invariant compact set conjugated to the product of u-times the inverse limit dynamics of $x^2 + a$ (restricted to the bounded orbits).

For instance, when $a = 0$, then the non-wandering set of $x \mapsto x^2$ consists of the attracting fixed point 0 and the repelling fixed point 1. On the other hand the non-wandering set of F_0 is $\{0, 1\}^u \times \{0\}$. We remark also that the set of points for which the orbit is bounded is homeomorphic to the square $[0, 1]^u \times \{0\}$ via the first coordinate projection. Hence for a small, the maximal invariant of F_a is a topological u-cube bounded by the stable and unstable manifolds of the hyperbolic continuation of the non-wandering points.

It will be useful to keep this example in mind because, even though the most interesting case is when $x \mapsto x^2 + a$ exhibits positive entropy, this one (with the parameter a sufficiently close to 0) already presents most of the geometric difficulties that appear in the proof of Theorem 1.1.

4. Proof of Main Theorem 1.1

4.1. Sufficient conditions for the existence of a conjugacy

We want to find, for every g which is C^1-close to f, a continuous map $h \colon \overleftarrow{M}_f \to \overleftarrow{M}_g$ which is close to the canonical injection $\overleftarrow{M}_f \hookrightarrow M^{\mathbb{Z}}$ and satisfies $h \circ \overleftarrow{f} = \overleftarrow{g} \circ h$. This is equivalent to find a continuous map $h_0 \colon \overleftarrow{M}_f \to M$ satisfying

$$h_0 \circ \overleftarrow{f} = g \circ h_0, \tag{$\mathring{C}_1$}$$

and which is C^0-*close* to the zeroth coordinate projection π_0. This means that for every $\eta > 0$ small and every g sufficiently C^1-close to f, h_0 satisfies

$$\sup_{\underline{x} \in \overleftarrow{M}_f} d(h_0(\underline{x}), \pi_0(\underline{x})) \le \eta. \tag{$\mathring{C}_2$}$$

Indeed, one can construct such an h_0 from such an h and *vice versa* writing:

$$h_0 := \pi_0 \circ h, \quad h := (h_0 \circ \overleftarrow{f}^n)_{n \in \mathbb{Z}}.$$

Let us suppose the existence of such an h. We would like h to be a homeomorphism, so let us find sufficient conditions to ensure its injectiveness.

In the Anosov case this follows easily from $(\mathring{C}_1)$ and $(\mathring{C}_2)$. In fact, if two points $\underline{x}$ and $\underline{y}$ have the same image by h, then these two f-orbits must be uniformly close by $(\mathring{C}_1)$ and $(\mathring{C}_2)$, and so they are equal by expansiveness. In the wider case of AS dynamical systems, we shall consider the following Robbin metric on $\overleftarrow{M}_f$:

$$d_\infty(\underline{x}, \underline{y}) = \sup_{i \in \mathbb{Z}} d(x_i, y_i).$$

For every $\underline{x}, \underline{y} \in \overleftarrow{M}_f$, let us observe that:

$$d_1(\underline{x}, \underline{y}) = \sum_i \frac{d(x_i, y_i)}{2^{|i|}} \le \sum_i \frac{d_\infty(\underline{x}, \underline{y})}{2^{|i|}} = 3 d_\infty(\underline{x}, \underline{y}). \tag{3}$$

This metric enabled Robbin [17] to find a sufficient condition on h to guarantee its injectiveness. We adapt it to our context.

Proposition 4.14 of [2] gives a geometric interpretation of the metric d_∞. After showing that $\overleftarrow{M}_f$ is a finite union of laminations, the leaves of which are intersection of stable sets with unstable manifolds of points in $\overleftarrow{\Omega}_f$, we proved that d_∞-distance between any two of these leaves is positive. Moreover the restriction of d_∞ to each leaf is equivalent to a Riemannian metric on its manifold structure.

Choosing $\eta > 0$ small, any continuous map $h_0 \colon \overleftarrow{M}_f \to M$ satisfying $(\mathring{C}_2)$ can be written as a perturbation of π_0 via the exponential map $\exp \colon TM \to M$ associated to the Riemannian metric of M. For the sake of simplicity, let us fix a bundle trivialization $TM \subset M \times \mathbb{R}^N$, for some positive integer N. As h_0 satisfies $(\mathring{C}_2)$ (with η small), there exists $w \colon \overleftarrow{M}_f \to \mathbb{R}^N$ such that $(\pi_0(\underline{x}), w(\underline{x})) \in TM \subset M \times \mathbb{R}^N$ and

$$h_0(\underline{x}) = \exp_{\pi_0(\underline{x})}(w(\underline{x})). \tag{4}$$

Now let us extend the Riemannian metric $(\langle \cdot, \cdot \rangle_x)_{x \in M}$ of M to an Euclidean norm $(\| \cdot \|_x)_{x \in M}$ on the bundle $M \times \mathbb{R}^N \to M$. Let us denote by $\Lambda w \in [0, \infty]$ the d_∞-Lipschitz constant of w, i.e.

$$\Lambda w := \sup_{\underline{x}, \underline{x}' \in \overleftarrow{M}_f} \frac{\| w(\underline{x}) - w(\underline{x}') \|_{\pi_0(\underline{x})}}{d_\infty(\underline{x}, \underline{x}')}. \tag{5}$$

Here is the Robbin condition:

$$\Lambda w \leq \eta. \tag{C_3}$$

Proposition 4.1 (*Robbin [17]*). *There exists $\eta > 0$ which depends only on the Riemannian metric of M, such that for every pair f and g of C^1-endomorphisms of M, if there exists $h \colon \overleftarrow{M}_f \to \overleftarrow{M}_g$ satisfying $(\mathring{C}_1)$, $(\mathring{C}_2)$ and (C_3), then h is injective.*

Proof. Let $\underline{x}, \underline{x}' \in \overleftarrow{M}_f$ be such that $h(\underline{x}) = h(\underline{x}')$. Note that by $(\mathring{C}_1)$ and $(\mathring{C}_2)$, the point $\pi_i \circ h(\underline{x})$ is η-close to $\pi_i(\underline{x})$, for every i. Thus $\pi_i(\underline{x})$ and $\pi_i(\underline{x}')$ are 2η-close for every $i \in \mathbb{Z}$.

Let $i \in \mathbb{Z}$ be such that $d_\infty(\underline{x}, \underline{x}') \leq 2d(x_i, x_i')$. We recall that $\pi_i \circ h(\underline{x}) = h_0 \circ \overleftarrow{f}^i(\underline{x}) = \exp_{x_i}(w \circ \overleftarrow{f}^i(\underline{x}))$, and so:

$$\exp_{x_i}(w \circ \overleftarrow{f}^i(\underline{x})) = \exp_{x_i'}(w \circ \overleftarrow{f}^i(\underline{x}'))$$

The exponential maps $\exp_{x_i}$ and $\exp_{x_i'}$ produce two charts centered at x_i and x_i', and modeled on the vector subspaces $T_{x_i}M$ and $T_{x_i'}M$ of $\mathbb{R}^N$. The coordinates change of these charts is the translation by the vector $\exp_{x_i}^{-1} x_i'$ plus a linear map L bounded by a constant K times $d(x_i, x_i')$, where K depends only on the curvature of M. Thus, in $\mathbb{R}^N$, it holds:

$$\exp_{x_i}^{-1} x_i' + (id + L) \circ w \circ \overleftarrow{f}^i(\underline{x}) = w \circ \overleftarrow{f}^i(\underline{x}') + o(d(x_i, x_i')). \tag{6}$$

We recall that $d_\infty(\underline{x}, \underline{x}') \leq 2d(x_i, x_i') \leq 4\eta$ is small.

On the other hand by (C_3):

$$\| w \circ \overleftarrow{f}^i(\underline{x}) - w \circ \overleftarrow{f}^i(\underline{x}') \|_{x_i} \leq \eta d_\infty(\underline{x}, \underline{x}')$$

Thus, replacing each term of equality (6) by these estimates, it holds:

$$\frac{d_\infty(\underline{x}, \underline{x}')}{2} \leq d(x_i, x_i') = \| \exp_{x_i}^{-1} x_i' \|$$
$$\leq \| w \circ \overleftarrow{f}^i(\underline{x}) - w \circ \overleftarrow{f}^i(\underline{x}') \|_{x_i} + \| L \| \eta + o(d(x_i, x_i'))$$
$$\leq \eta d_\infty(\underline{x}, \underline{x}') + K d_\infty(\underline{x}, \underline{x}') \eta + o(d_\infty(\underline{x}, \underline{x}'))$$

This implies $d_\infty(\underline{x}, \underline{x}') = 0$ and so $\underline{x} = \underline{x}'$. $\square$

On the other hand, in Proposition 5.4 of [2] it is showed the following:

Proposition 4.2. *For every AS C^1-endomorphism f of M, there exists $\eta > 0$ such that for every endomorphism g sufficiently close to f, if there exists a d_1-continuous and injective $h \colon \overleftarrow{M}_f \to \overleftarrow{M}_g$ satisfying $(\mathring{C}_1)$ and $(\mathring{C}_2)$ with f, g and η, then h is surjective onto $\overleftarrow{M}_g$.*

Hence if we prove that for every g C^1-close to an AS endomorphism f, there exists a continuous map h_0 satisfying $(\mathring{C}_1)$, $(\mathring{C}_2)$ and (C_3), then Propositions 4.1 and 4.2 imply that g is inverse limit conjugate to f, and so that f is C^1-inverse limit stable. In other words, to prove Theorem 1.1 it remains only to prove the following:

Proposition 4.3. *Let f be a C^1-AS endomorphism. For every $\eta > 0$ and for every endomorphism g sufficiently C^1-close f, there exists a continuous map $h_0 \colon \overleftarrow{M}_f \to M$ satisfying $(\mathring{C}_1)$, $(\mathring{C}_2)$ and (C_3) with f, g and η.*

Therefore the remaining part of this manuscript is devoted to the proof of this proposition, by using the contraction mapping Theorem.

4.2. A contracting map on a functional space

Let Γ be the space of functions $w \colon \overleftarrow{M}_f \to \mathbb{R}^N$ which are continuous for d_1 and d_∞-Lipschitz (i.e. they satisfy $\Lambda(w) < \infty$, where $\Lambda(w)$ is defined as in (5)). We endow Γ with the uniform norm:

$$\|v\|_{C^0} := \max_{\underline{x} \in \overleftarrow{M}_f} \|v(\underline{x})\|_{\pi_0(\underline{x})}.$$

Equivalent conditions in the space Γ We recall that $M \times \mathbb{R}^N \supset TM$ is a trivialization. Moreover we have already fixed an Euclidean structure $(\|\cdot\|_x)_{x \in M}$ on $M \times \mathbb{R}^N$ which extends the Riemannian metric $(\langle \cdot, \cdot \rangle_x)_{x \in M}$ on TM. Let $p_x \colon \mathbb{R}^N \to T_x M$ be the orthogonal projection given by $\|\cdot\|_x$.

Any g sufficiently C^0-close to f induces the following map from a neighborhood N_Γ of $0 \in \Gamma$ into Γ:

$$\Phi_f^g(w) := \underline{x} \mapsto \exp_{x_0}^{-1}\left(g \circ \exp_{x_{-1}}\left(p_{x_{-1}} \circ w \circ \overleftarrow{f}^{-1}(\underline{x})\right)\right), \quad \forall w \in N_\Gamma,$$

where $x_i = \pi_i(\underline{x})$ for every i, as defined in §2.1.

For every η small, and for every g sufficiently close to f, to find h_0 satisfying conditions $(\mathring{C}_1)$, $(\mathring{C}_2)$ and (C_3) is equivalent to find $w \in \Gamma$ satisfying

$$\Phi_f^g(w) = w \tag{C_1}$$

$$\|w\|_{C^0} \le \eta \tag{C_2}$$

$$\Lambda(w) \le \eta \tag{C_3}$$

Indeed, by (C_1) any such w satisfies $w(\underline{x}) \in T_{x_0} M$, for every $\underline{x} \in \overleftarrow{M}_f$. It is then easy to remark that $h_0 \colon \underline{x} \mapsto \exp_{x_0}(w(\underline{x}))$ satisfies $(\mathring{C}_1)$ and $(\mathring{C}_2)$.

Strategy To solve this (implicit) problem, let us regard the partial derivative of Φ_f^f at $0 \in \Gamma$ with respect to $w \in \Gamma$:

$$D_0 \Phi_f^f = w \mapsto \left[\underline{x} \mapsto T_{x_{-1}} f(p_{x_{-1}} \circ w \circ \overleftarrow{f}^{-1}(\underline{x}))\right].$$

The first difficulty that appears is the following: if f is only C^1, in general the map $D\Phi_f^f$ does not leave invariant the space Γ. In the C^2-case, Robbin's strategy in [17] consists in solving (C_1)–(C_2)–(C_3) by finding a right inverse for $D\Phi_f^f - id$, and then by following a classical proof of the implicit function theorem which uses the contraction mapping Theorem.

A second difficulty which will appear is that Tf is possibly non-invertible, and this will give us many difficulties to construct this right inverse with bounded norm. To eliminate some of these, in Lemma 4.4, we will suppose N twice larger than necessary to embed TM into $M \times \mathbb{R}^N$.

Robinson trick When f is just C^1 and not C^2, by classical argument of convoluting f with a smooth mollifier we can construct a family $(f_\delta)_\delta$ such that $f_0 = f$, $f_\delta \in C^\infty(M, M)$ for every $\delta > 0$ and such that the map $\delta \mapsto f_\delta \in C^1(M, M)$ is continuous.

Then we can consider a continuous family of C^0-maps $(x \mapsto F_x^\delta)_\delta$ from M into the space of linear maps of $\mathbb{R}^N$ given by $F_x^\delta = T_x f_\delta \circ p_x$.

In such a way, for each δ we get a linear bundle morphism F^δ given by

$$F^\delta: \quad \overleftarrow{M}_f \times \mathbb{R}^N \quad \to \quad \overleftarrow{M}_f \times \mathbb{R}^N \qquad\qquad (7)$$
$$(\underline{x}, v) \quad \mapsto \quad (\overleftarrow{f}(\underline{x}), F_{x_0}^\delta(v))$$

Note that F^δ is still over $\overleftarrow{f}$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
\overleftarrow{M}_f \times \mathbb{R}^N & \xrightarrow{\;F^\delta\;} & \overleftarrow{M}_f \times \mathbb{R}^N \\
\downarrow & & \downarrow \\
\overleftarrow{M}_f & \xrightarrow{\;\overleftarrow{f}\;} & \overleftarrow{M}_f
\end{array}
$$

Lemma 4.4. *If N is large enough, then we can suppose moreover that F_x^δ is invertible for every $\delta > 0$ and $x \in M$.*

Proof. Put $N' = 2N$. Let f_δ be a smooth endomorphism C^1-close f when δ is small. We extend the projection $p_x: \mathbb{R}^N \to T_x M$ to $\mathbb{R}^{N'} = \mathbb{R}^N \times \mathbb{R}^N$ by

$$p_x: \mathbb{R}^N \times \mathbb{R}^N \ni (v_1, v_2) \mapsto p_x(v_1) \in \mathbb{R}^N.$$

Also we identify $\mathbb{R}^N$ to $\mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N'}$. Let us regard:

$$F_x^\delta: \mathbb{R}^{N'} \ni v = (v_1, v_2) \mapsto (T_x f_\delta \circ p_x(v_1) + \delta v_2, \delta v_1) = T_x f_\delta \circ p_x(v) + \delta(v_2, v_1) \in \mathbb{R}^{N'}.$$

For every $x \in M$ and $\delta > 0$, such a map is invertible, depends smoothly on x and is δ-close to $\mathbb{R}^{N'} \ni v \to T_x f_\delta \circ p_x(v) \in \mathbb{R}^{N'}$. $\quad\square$

Corollary 4.5. *The map F^δ is a homeomorphism of $\overleftarrow{M}_f \times \mathbb{R}^N$, for every $\delta > 0$.*

For every $v \in \Gamma$, the following map is well defined:

$$F^\delta{}_\star(v) := \underline{x} \in \overleftarrow{M} \mapsto F^\delta(v(\overleftarrow{f}^{-1}(\underline{x}))).$$

This map is continuous, linear and for $\delta > 0$ it is bijective.

Moreover, we remark that $F^\delta{}_\star(v)$ belongs to Γ.

Now, let us suppose the existence of a right inverse J of $F^\delta{}_\star - id$. This means:

$$(F^\delta{}_\star - id)J = id.$$

Notice that the existence of a $w \in \Gamma$ satisfying (C_1), (C_2) and (C_3) is equivalent to find a fixed point $\phi \in \Gamma$ of the operator

$$\left[(F^\delta{}_\star - id) - (\Phi_f^g - id)\right] \circ J = id - (\Phi_f^g - id) \circ J$$

such that $w = J(\phi)$ satisfies (C_2) and (C_3).

We construct J in §5. From its construction we get

Proposition 4.6. *For every $\epsilon > 0$ and every $\eta > 0$ sufficiently small w.r.t. ϵ, there exists $\delta > 0$ small enough such that for every g C^1-close enough to f, the operator*

$$\left[(F^\delta{}_\star - id) - (\Phi_f^g - id)\right] \circ J = (F^\delta{}_\star - \Phi_f^g) \circ J$$

is well defined on a 2η-neighborhood of 0 in Γ and is C^0-contracting.

Moreover, $\left\|(F^\delta{}_\star - \Phi_f^g)Jv\right\|_{C^0} \le \eta$ and $\Lambda\big((F^\delta{}_\star - \Phi_f^g)Jv\big) \le \epsilon$, whenever $\|v\|_{C^0} \le \eta$ and $\Lambda(v) \le \epsilon$.

Together with Proposition 4.3, this implies the inverse limit structural stability of AS-endomorphisms.

5. Construction of the right inverse J of $F^\delta_\star - id$

We recall that f denotes an AS-endomorphism of a compact manifold M. Let $\Omega(\overleftarrow{f})$ be the non-wandering set of $\overleftarrow{f}$. It is shown in [2] that:

$$\Omega(\overleftarrow{f}) = \overleftarrow{M}_f \cap \Omega(f)^{\mathbb{Z}}.$$

Moreover, the non-wandering set $\Omega(\overleftarrow{f})$ is the disjoint union of compact, transitive subsets $(\overleftarrow{\Omega}_i)_i$, called *basic pieces*. The family of all basic pieces is finite and called the *spectral decomposition* of $\Omega(\overleftarrow{f})$.

For every basic piece $\overleftarrow{\Omega}_i$, we define the *stable* and *unstable sets* of $\overleftarrow{\Omega}_i$, respectively, by

$$W^s(\overleftarrow{\Omega}_i) = \{\underline{x} \in \overleftarrow{M} : d(\overleftarrow{f}^n(\underline{x}), \overleftarrow{\Omega}_i) \to 0, \text{ as } n \to +\infty\}$$
$$W^u(\overleftarrow{\Omega}_i) = \{\underline{x} \in \overleftarrow{M} : d(\overleftarrow{f}^n(\underline{x}), \overleftarrow{\Omega}_i) \to 0, \text{ as } n \to -\infty\}$$

The geometry of these sets is studied in [2].

Given two basic pieces $\overleftarrow{\Omega}_i$ and $\overleftarrow{\Omega}_j$, we write $\overleftarrow{\Omega}_i \succ \overleftarrow{\Omega}_j$ if $W^u(\overleftarrow{\Omega}_i)$ intersects $W^s(\overleftarrow{\Omega}_j) \setminus \overleftarrow{\Omega}_j$. In [2] it is shown that for any AS-endomorphism f, the relation $\succ$ is an order relation. This enables us to enumerate the spectral decomposition $(\overleftarrow{\Omega}_i)_{i=1}^q$ of $\Omega(\overleftarrow{f})$ in such a way that $\overleftarrow{\Omega}_i \succ \overleftarrow{\Omega}_j$ implies $i > j$.

We recall that a *filtration adapted to* $(\overleftarrow{\Omega}_i)_i$ is an increasing sequence of compact sets

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots \subset M_q = \overleftarrow{M}_f$$

such that for $q \geq i \geq 1$:

$$\bigcap_{n \in \mathbb{Z}} \overleftarrow{f}^n(M_i \setminus M_{i-1}) = \overleftarrow{\Omega}_i \quad \text{and} \quad \overleftarrow{f}(M_i) \subset int(M_i).$$

The existence of such a filtration is shown in Corollary 4.7 of [2].

The following proposition is formally similar to the one used by Robbin [17] or Robinson [18], but it is technically much more complicated and its proof requires to be handled very carefully. New ideas will be needed. The proof will be done in §7, 8, 9 and will use §6.

Proposition 5.1. *There exist $\lambda \in (0, 1)$, $K > 0$ and an open cover $(W_i)_{i=1}^q$ of $\overleftarrow{M}_f$, where each W_i is a neighborhood of $\overleftarrow{\Omega}_i$, and such that for every $\delta > 0$, there exist vector subbundles E_i^u and E_i^s of the trivial bundle $W_i \times \mathbb{R}^N \to W_i$ satisfying the following properties:*

(i) *For every $\underline{x} \in W_i \cap \overleftarrow{f}^{-1}(W_i)$, the map F^δ sends $E_{i\underline{x}}^s$ and $E_{i\underline{x}}^u$ onto $E_{i\overleftarrow{f}(\underline{x})}^s$ and $E_{i\overleftarrow{f}(\underline{x})}^u$ respectively.*

(ii) *For any $k \geq j$ and every $\underline{x} \in W_k \cap \overleftarrow{f}^{-1}(W_j)$, the following inclusions hold:*

$$F^\delta(E_{k\underline{x}}^s) \subset E_{j\overleftarrow{f}(\underline{x})}^s, \quad F^\delta(E_{k\underline{x}}^u) \supset E_{j\overleftarrow{f}(\underline{x})}^u.$$

(iii) *$E_{i\underline{x}}^s \oplus E_{i\underline{x}}^u = \mathbb{R}^N$, for any $i \in \{1, \ldots, q\}$ and every $\underline{x} \in W_i$; the angle between E_i^s and E_i^u is bounded from below by K^{-1}.*

(iv) *The subbundles E_i^u and E_i^s are d_1-continuous and d_∞-Lipschitz.*

(v) *For every i and any $\underline{x} \in W_i$, it holds*

$$\|F^\delta(v^u)\| \geq \|v^u\|/K, \quad \forall v^u \in E_{i\underline{x}}^u.$$

(vi) *For every q', if $\underline{x} \in W_{q'}$ then $\overleftarrow{f}(\underline{x}) \notin \cup_{j>q'} W_j$ and $\cap_{n \in \mathbb{Z}} \overleftarrow{f}^n(W_{q'}) = \overleftarrow{\Omega}_{q'}$.*

(vii) *For every i, any $\underline{x}$ in a neighborhood of $\overleftarrow{\Omega}_i$ which does not depend on δ, and for all $v^s \in E_{i\underline{x}}^s$ and $v^u \in E_{i\underline{x}}^u$, it holds:*

$$\|F^\delta(v^s)\| \leq \lambda\|v^s\| \quad \text{and} \quad \|F^\delta(v^u)\| \geq \|v^u\|/\lambda.$$

The subbundles E_i^u and E_i^s can be considered as functions from W_i to the Grassmannian G_N of $\mathbb{R}^N$. Property (iv) means that they are d_1-continuous and d_∞-Lipschitz. The Grassmannian G_N is a manifold with as many connected components as possible dimension for $\mathbb{R}^N$-subspaces, i.e. $N + 1$.

Remark 5.2. A main difficulty in this proposition is that K does not depend on δ, whereas the norm of the inverse of F^δ blows up as δ approaches 0 whenever f has critical points. Hence the proof of this proposition will not be symmetric in u and s.

At this point is important to remark that the vector subbundles E_i^s and E_i^u depend on δ.

We will prove in Corollary 6.2 the existence of a partition of the unity $(\gamma_i)_i$ subordinated to $(W_i)_{i=1}^q$, where each γ_i is d_1-continuous and d_∞-Lipschitz.

Given any $\underline{x} \in W_i$, let $\pi_{i\underline{x}}^s : \mathbb{R}^N \to E_{i\underline{x}}^s$ denote the projection parallely to $E_{i\underline{x}}^u$ and $\pi_{i\underline{x}}^u : \mathbb{R}^N \to E_{i\underline{x}}^u$ the projection parallely to $E_{i\underline{x}}^s$.

For $v \in \Gamma$ and $\sigma \in \{s, u\}$, put:

$$v_i^\sigma := \pi_i^\sigma(\gamma_i \cdot v), \quad J_{is}(v) := -\sum_{n=0}^\infty F_\star^{\delta^n}(v_i^s) \quad \text{and} \quad J_{iu}(v) := \sum_{n=-\infty}^{-1} F_\star^{\delta^n}(v_i^u). \tag{8}$$

We can now define:

$$J := \sum_{i,\sigma} J_{i\sigma}. \tag{9}$$

Let us define

$$C := \sup_{1 \le i \le q} \sup_{\underline{x} \in \text{supp}\gamma_i} \left\{ \|\pi_{i\underline{x}}^s\|, \|\pi_{i\underline{x}}^u\| \right\}.$$

By Property (iii) the constant C is bounded from above independently of δ.

Lemma 5.3. *There exists a constant D independent of δ such that for every j, for all $\underline{x} \in W_j$ and $u \in E_{j\underline{x}}^s$ (resp. $u \in E_{j\underline{x}}^u$), for every $n \ge 0$ (resp. $n \le 0$):*

$$\|(F^\delta)^n(u)\| \le D\lambda^{|n|}\|u\|$$

Proof. For every $\underline{x} \in W_j$, since $(W_k)_k$ is a cover of $\overleftarrow{M}_f$, there exists a sequence $(n_i)_i$ such that $\overleftarrow{f}^i(\underline{x}) \in W_{n_i}$ for every i. From property (vi), the sequence $(n_i)_i$ must be decreasing.

As $\underline{x} \in W_j$, we can suppose that $n_0 = j$. By property (ii), for every $k \ge 0$, $F^{\delta^{-k}}$ sends $E_{j\underline{x}}^u$ into $E_{n_k}^u$ and F^{δ^k} sends $E_{j\underline{x}}^s$ into $E_{n_k}^s$.

Let $(V_k)_k$ be the neighborhoods of respectively $(\overleftarrow{\Omega}_k)_k$ on which (vii) holds. Since the non-wandering set contains the limit set and $\overleftarrow{M}_f$ is compact, there exists $m \ge 0$ such that there is no $\underline{x} \in \overleftarrow{M}_f$ such that $(\underline{x}, \overleftarrow{f}(\underline{x}), \cdots \overleftarrow{f}^{m-1}(\underline{x}))$ are all outside of $\cup_k V_k$. We can suppose V_k included in W_k for every k. Consequently, for every $\underline{x}$, all the terms $f^i(\underline{x})$ of the sequence $(f^i(\underline{x}))_i$ but qm are in V_{n_i}. From (vii), it follows that for all $\underline{x} \in W_i$ and $u \in E_{i\underline{x}}^s$ (resp. $u \in E_{i\underline{x}}^u$), for every $n \ge 0$ (resp. $n \le 0$):

$$\|(F^\delta)^n(u)\| \le D\lambda^{|n|}\|u\|,$$

with $D = \max(\|F^\delta\|, \|(F^\delta|E_{i\underline{x}}^u)^{-1}\|)^{qm}$, which is bounded by a constant independent of δ by (v). $\square$

From Lemma 5.3, it holds that for every $v \in \Gamma$:

$$\|Jv\|_{C^0} \le \frac{2CDq}{1-\lambda}\|v\|_{C^0}, \tag{10}$$

where C, D and λ are independent of $\delta > 0$ small.

Moreover we easily compute the following.

Proposition 5.4. *The map J is the right inverse of $F^\delta{}_\star - id$:*

$$(F^\delta{}_\star - id) \circ J = id.$$

To prove main Theorem 1.1, it remains only to prove Propositions 4.6 and 5.1.

To show Proposition 5.1, we will develop some analytical tools in the next section. On the other hand, we are ready to prove Proposition 4.6.

Proof of Proposition 4.6. Let us start by computing $(F^\delta{}_\star - \Phi^g_f)$ at $0 \in \Gamma$:

$$
\begin{aligned}
(F^\delta{}_\star - \Phi^g_f)(0)(\underline{x}) &= F^\delta_{x_{-1}}(0) - \exp^{-1}_{x_0} \circ g \circ \exp_{x_{-1}}(p_{x_{-1}}(0)) \\
&= -\exp^{-1}_{x_0}(g(x_{-1})).
\end{aligned}
\tag{11}
$$

In particular, this implies $\left\| (F^\delta{}_\star - \Phi^g_f)(0) \right\|_{C^0} = d_{C^0}(f, g)$.

On the other hand, $(F^\delta{}_\star - \Phi^f_f)$ is a C^1 map defined on a neighborhood of $0 \in \Gamma$. Thus we can compute its derivative at the origin:

$$
D_0\big(F^\delta{}_\star - \Phi^f_f\big)(v)(\underline{x}) = \left(F^\delta_{x_{-1}} - T_{x_{-1}} f \right) (p_{x_{-1}} \circ v \circ \overleftarrow{f}^{-1}(\underline{x}))
\tag{12}
$$

for every $v \in \Gamma$ and every $\underline{x} \in \overleftarrow{M}_f$. In particular, the operator norm subordinate to the C^0 norm satisfies:

$$
\left\| D_0\big(F^\delta{}_\star - \Phi^f_f\big) \right\|_{C^0} \to 0, \text{ as } \delta \to 0.
$$

At a neighborhood of 0, the derivative of $F^\delta_\star$ is constant, whereas the one of ϕ^f_f is continuous. Furthermore, for g C^1-close to f, $D\phi^g_f$ is close to $D\phi^f_f$.

Hence, for every $\mu > 0$, there exists a small $\eta(\mu) > 0$ such that for any g sufficiently close to f in the C^1-topology and any $w \in \Gamma$ with $\|w\|_{C^0} \le \eta(\mu)$ and $\delta \le \eta(\mu)$, it holds

$$
\left\| D_w\big(F^\delta{}_\star - \Phi^g_f\big) \right\|_{C^0} \le \mu.
\tag{13}
$$

Then, putting together (11), (13), we get

$$
\begin{aligned}
\left\| (F^\delta{}_\star - \Phi^g_f) J v \right\|_{C^0} &\le d_{C^0}(f, g) + \mu \|J v\|_{C^0} \\
&\le d_{C^0}(f, g) + \|J\|_{C^0} \mu \|v\|_{C^0} .
\end{aligned}
$$

By (10), $\|J\|_{C^0}$ is bounded independently of δ, we put

$$
\mu_0 = \inf_{\delta \text{ small}} \min \left(\frac{1}{2}, \frac{1}{2\|J\|_{C^0}} \right) > 0.
\tag{14}
$$

Hence for every $\eta, \delta < \eta(\mu_0)$, for every g moreover $\eta/2$-C^0-close to f it holds for $v \in \Gamma$:

$$
\|v\|_{C^0} \le \eta \Rightarrow \left\| (F^\delta{}_\star - \Phi^g_f) J v \right\|_{C^0} \le \frac{\eta}{2} + \frac{1}{2} \|v\|_{C^0} < \eta.
$$

Which is the second statement of the Proposition. Also inequalities (13) and (14) implies that $(F^\delta{}_\star - \Phi^g_f)J$ contracts the C^0-norm by a small factor when η, δ are small and g is close to f, which is the first statement of the Proposition. We remark that as far as $\delta \le \eta(\mu_0)$, which does not depend on η, we can suppose $\eta \le \eta(\mu_0)$ as small as we want which satisfies the same property, if g is sufficiently close to f.

It remains only to estimate $\Lambda((F^\delta{}_\star - \Phi^g_f)(Jv))$ for $v \in \Gamma$. To do that, we prove the following lemma similar to the Robin's computation §6 of [17]:

Lemma 5.5. *For $\sigma \in \{s, u\}$, there exist a constant A which depends on f but not on δ, and a constant B_δ which depends on δ such that for every $v \in \Gamma$, for any i and $\sigma = s, u$:*

$$\Lambda(J_{i\sigma} v_i^\sigma) \leq A\Lambda(v_i^\sigma) + B_\delta \left\| v_i^\sigma \right\|_{C^0}. \tag{15}$$

As the norm of $\Lambda(v_i^\sigma)$ is dominated by $\Lambda(v)$ times a constant independent of δ, it holds by taking the constants A and B_δ larger:

$$\Lambda(Jv) \leq A\Lambda(v) + B_\delta \|v\|_{C^0}. \tag{16}$$

Put $L_{\underline{x}} := F_{x_{-1}}^\delta - \exp_{x_0}^{-1} \circ g \circ \exp_{x_{-1}} \circ p_{x_{-1}}$. We have:

$$(F_\star^\delta - \Phi_f^g)J(v)(\underline{x}) - (F_\star^\delta - \Phi_f^g)J(v)(\underline{y}) = L_{\underline{x}}(J(v)(\underline{x}) - J(v)(\underline{y})) + (L_{\underline{x}} - L_{\underline{y}})J(v)(\underline{y})$$

Hence:

$$\Lambda((F_\star^\delta - \Phi_f^g)J(v)) \leq \|L\|_{C^0}(A\Lambda(v) + B_\delta \|v\|_{C^0}) + \Lambda(L)\|J(v)\|_{C^0},$$

where $\Lambda(L)$ depends on δ.

By (13), we can suppose g sufficiently close to f and δ small enough so that $\|L\|_{C^0}$ is $1/(2A)$ contracting on a small neighborhood of 0. From this:

$$\Lambda((F_\star^\delta - \Phi_f^g)J(v)) \leq \frac{\Lambda(v)}{2} + \left(\frac{B_\delta}{2A} + \Lambda(L)\|J\|_{C^0}\right)\|v\|_{C^0}.$$

Hence for every $\epsilon > 0$, for every η such that:

$$\eta \leq \left(\frac{B_\delta}{2A} + \Lambda(L)\|J\|_{C^0}\right)^{-1}\frac{\epsilon}{2}$$

If $\|v\|_{C^0} \leq \eta$ and $\Lambda(v) \leq \epsilon$ and g sufficiently close to f, it holds:

$$\Lambda((F_\star^\delta - \Phi_f^g)J(v)) \leq \epsilon \qquad \square$$

Proof of Lemma 5.5. We prove the case $\sigma = s$, since the other case $\sigma = u$ is similar. For $n \geq 0$, we evaluate:

$$\|F^{\delta^n}(\underline{x}, v_i^s(\underline{x})) - F^{\delta^n}(\underline{y}, v_i^s(\underline{y}))\|$$

$$\leq \|F^{\delta^n} \circ \pi_i^s(\underline{x}, v_i^s(\underline{x}) - v_i^s(\underline{y}))\| + \|F^{\delta^n} \circ \pi_i^s(\underline{x}, v_i^s(\underline{y})) - F^{\delta^n} \circ \pi_i^s(\underline{y}, v_i^s(\underline{y}))\|$$

By Remark 5.3, there exists a constant D which does not depend on n nor δ such that:

$$\|F^{\delta^n}|E_i^s\| \leq D\lambda^n$$

Hence:

$$\|F^{\delta^n} \circ \pi_i^s(\underline{x}, v_i^s(\underline{x}) - v_i^s(\underline{y}))\| \leq DC\lambda^n C\Lambda(v_i^s)d_\infty(\underline{x}, \underline{y})$$

On the other hand,

$$\|F^{\delta^n} \circ \pi_i^s(\underline{x}, \cdot) - F^{\delta^n} \circ \pi_i^s(\underline{y}, \cdot)\|$$

$$\leq \sum_k \|F^{\delta^{n-k-1}}|E_{n_k \overleftarrow{f}^{k+1}(\underline{x})}^s\| \cdot \|F^\delta(\overleftarrow{f}^k(\underline{x}), \cdot) - F^\delta(\overleftarrow{f}^k(\underline{y}), \cdot)\| \cdot \|Tf^k|E_{\underline{y}}^s\|,$$

where n_k is such that $\overleftarrow{f}^{k+1}(\underline{x}) \in W_{n_k}$. This is less than:

$$\sum_k CD\lambda^{n-k-1} \cdot \|F^\delta(\overleftarrow{f}^k(\underline{x}), \cdot) - F^\delta(\overleftarrow{f}^k(\underline{y}), \cdot)\| \cdot CD\lambda^k.$$

Hence there exists a constant $K(\delta)$ which depends only on f and δ such that:

$$\|F^{\delta^n} \circ \pi_i^s(\underline{x}, \cdot) - F^{\delta^n} \circ \pi_i^s(\underline{y}, \cdot)\| \leq n \cdot K(\delta)\lambda^n d_\infty(\underline{x}, \underline{y})$$

Consequently:

$$\|F^{\delta^n}(\underline{x}, v_i^s(\underline{x})) - F^{\delta^n}(\underline{y}, v_i^s(\underline{y}))\| \leq (DC^2\lambda^n \Lambda(v_i^s) + n \cdot K(\delta)\lambda^n \|v\|_{C^0})d_\infty(\underline{x}, \underline{y})$$

Summing over n we conclude. $\square$

6. Analysis on M_f

Let us introduce a few notations. Let N be an arbitrary Riemannian manifold. We recall that $C^0(\overleftarrow{M}_f, N)$ denotes the space of d_1-continuous maps $\phi : \overleftarrow{M}_f \to N$. Also $\mathrm{Lip}^\infty(\overleftarrow{M}_f, N)$ denotes the space of d_∞-Lipschitz maps $\phi : \overleftarrow{M}_f \to N$. Let us define:

$$\mathrm{Mor}_0^\infty(\overleftarrow{M}_f, N) := C^0(\overleftarrow{M}_f, N) \cap \mathrm{Lip}^\infty(\overleftarrow{M}_f, N).$$

We endow $C^0(\overleftarrow{M}_f, N)$ with the uniform distance given by the Riemannian metric of N. Note that $C^0(\overleftarrow{M}_f, N)$ is a Banach manifold. Actually its topology does not depend on the Riemannian metric of N. The aim of this section is to prove the denseness of $\mathrm{Mor}_0^\infty(\overleftarrow{M}_f, N)$ in $C^0(\overleftarrow{M}_f, N)$. To do this, we will use a new technique based on convolutions.

Let $\rho \in C^\infty(\mathbb{R})$ be a non-negative bump function with support in $(-1, 1)$. Let μ be any Lebesgue measure on M such that $\mu(M) = 1$, and let $\tilde\mu = \bigotimes_{\mathbb{Z}} \mu$ be the induced probability on $M^{\mathbb{Z}}$.

For every map g from $\overleftarrow{M}_f$ into $\mathbb{R}^n$, for every $r > 0$, we define g_r by:

$$g_r : \overleftarrow{M}_f \ni \underline{x} \mapsto \int_{M^{\mathbb{Z}}} g(\underline{y}) \cdot \rho\left(\frac{d_1(\underline{x}, \underline{y})}{r}\right) d\tilde\mu(\underline{y}).$$

The following result plays a key role:

Lemma 6.1. *Let $\phi : \overleftarrow{M}_f \to \mathbb{R}^n$ be a continuous function with respect to the distance d_1. Let $\tilde\phi$ be a continuous extension to $(M^{\mathbb{Z}}, d_1)$. Let $\mathbb{1}$ be the function on $M^{\mathbb{Z}}$ constantly equal to $1 \in \mathbb{R}$. For every $r > 0$, the functions $\mathbb{1}_r$ and $\tilde\phi_r$ (defined as above) satisfy:*

   (i) *$\tilde\phi_r$ and $\tilde\phi_r/\mathbb{1}_r$ are well defined.*
  (ii) *$\tilde\phi_r$ is d_1-continuous and d_∞-Lipschitz, i.e. it belongs to $\mathrm{Mor}_0^\infty(\overleftarrow{M}_f, \mathbb{R}^n)$.*
 (iii) *The function $\tilde\phi_r/\mathbb{1}_r$ is C^0-close to $\tilde\phi$ whenever r is small.*
 (iv) *The support of $\tilde\phi_r$ is included in the r-neighborhood of the support of ϕ.*

The following are immediate corollaries of this lemma:

Corollary 6.2. *For every open cover $(U_i)_i$ of $\overleftarrow{M}_f$, there exists a partition of unity $(\rho_i)_i \subset \mathrm{Mor}_0^\infty(\overleftarrow{M}_f, \mathbb{R})$ subordinate to it.*

Corollary 6.3. *The subset $\mathrm{Mor}_0^\infty(\overleftarrow{M}_f, N)$ is dense in $C^0(\overleftarrow{M}_f, N)$.*

Remark 6.4. Both above corollaries are also true if we replace $\overleftarrow{M}_f$ by any compact subset E of it.

Proof of Lemma 6.1. Let us start by proving (i). As $\tilde\phi$ and ρ are continuous on a compact space, they are bounded. As $\tilde\mu(M^{\mathbb{Z}}) = 1$, the functions $\tilde\phi_r$ and $\mathbb{1}_r$ are well defined. Let $\underline{x} \in M^{\mathbb{Z}}$. There exists $\delta > 0$ such that $\rho|B_{d_1}(0, \delta/r)$ is greater than δ. For every $\underline{x} \in M^{\mathbb{Z}}$, the $\tilde\mu$-volume of the ball $B_{d_1}(\underline{x}, \delta/r)$ is greater than:

$$\prod_{-N}^{N} \mu\left(B\left(x_i, \frac{\delta}{6r}\right)\right) > 0,$$

where N is any natural number satisfying $\sum_{|n| \geq N} 2^{-|n|} diam(M) \leq \frac{\delta}{2r}$.

Thus, $m := \inf\{\tilde\mu(B_{d_1}(\underline{x}, \delta/r)) : \underline{x} \in \overleftarrow{M}\}$ is positive and $\mathbb{1}_r > m\delta$. Consequently, $\phi_r/\mathbb{1}_r$ is everywhere well defined.

Let us proof (iii). As $(M^{\mathbb{Z}}, d_1)$ is compact, the function $\tilde\phi$ is uniformly continuous: for every $\delta > 0$, there exists $r > 0$ such that the image by $\tilde\phi$ of any d_1-ball of radius r has diameter less than δ. Thus for every $\underline{x} \in \overleftarrow{M}_f$:

$$|\tilde{\phi}_r(\underline{x}) - \tilde{\phi}(\underline{x}) \cdot \mathbb{1}_r(\underline{x})| \le \int\limits_{M^{\mathbb{Z}}} \delta \cdot \rho\left(\frac{d_1(\underline{x}, \underline{y})}{r}\right) \tilde{\mu}(\underline{y}) \le \delta \cdot \mathbb{1}_r(\underline{x})$$

Let us proof (ii). We remark that if a function is d_1-Lipschitz, then it is d_∞-Lipschitz, and so it belongs to $\mathrm{Mor}_0^\infty(\overleftarrow{M}_f, \mathbb{R}^m)$. Then, let us prove $\tilde{\phi}_r$ is d_1-Lipschitz. For every $\underline{x}' \in \overleftarrow{M}_f$:

$$\tilde{\phi}_r(\underline{x}) - \tilde{\phi}_r(\underline{x}') = \int\limits_{M^{\mathbb{Z}}} \tilde{\phi}(\underline{y}) \cdot \left(\rho\left(\frac{d_1(\underline{x}, \underline{y})}{r}\right) - \rho\left(\frac{d_1(\underline{x}', \underline{y})}{r}\right)\right) d\tilde{\mu}(\underline{y})$$

As ρ is smooth, its derivative is bounded by some L, and so:

$$\left|\rho\left(\frac{d_1(\underline{x}, \underline{y})}{r}\right) - \rho\left(\frac{d_1(\underline{x}', \underline{y})}{r}\right)\right| \le \frac{L}{r}|d_1(\underline{x}, \underline{y}) - d_1(\underline{x}', \underline{y})| \le \frac{L}{r}d_1(\underline{x}, \underline{x}')$$

Consequently:

$$|\tilde{\phi}_r(\underline{x}) - \tilde{\phi}_r(\underline{x}')| \le \frac{L}{r}d_1(\underline{x}, \underline{x}') \int\limits_{M^{\mathbb{Z}}} |\tilde{\phi}(\underline{y})| d\tilde{\mu}(\underline{y})$$

Thus, since $\tilde{\phi}$ is bounded and $\tilde{\mu}$ is a probability, we get that $\tilde{\phi}_r$ is d_1-Lipschitz as desired. $\quad\square$

7. Proof of Proposition 5.1

Let f be an AS endomorphism of a compact manifold M.

7.1. Preliminaries

Distance on Grassmannian bundles   We endow the space of linear endomorphisms of $\mathbb{R}^N$ with the operator norm $\|\cdot\|$ induced by the Euclidean one of $\mathbb{R}^N$. We recall that the Grassmannian G_N of $\mathbb{R}^N$ is the space of d-planes of $\mathbb{R}^N$, for $0 \le d \le N$. Given two planes $P, P' \in G_N$ let π_P and π'_P be their associated orthogonal projections. The metric d_G on G_N is defined by:

$$d_G(P, P') = \|\pi_P - \pi_{P'}\|$$

Angle between planes   Two planes P and P' of $\mathbb{R}^n$ make an angle greater than η if for all $u \in P \setminus \{0\}$ and $v \in P' \setminus \{0\}$, the angle between u and v is greater than η (for the Euclidean norm), in particular they are in direct sum.

Definition of E^s   We recall that for any $(\underline{x}, a) \in \overleftarrow{M}_f \times \mathbb{R}^N$, $F^\delta(\underline{x}, a) = (\overleftarrow{f}(\underline{x}), F^\delta_{x_0}(a))$, where $x_0 = \pi_0(\underline{x})$.

The stable direction $E^s_{\underline{x}}$ of F^0 at $\underline{x}$ is given by

$$E^s_{\underline{x}} := \mathrm{Ker}\ p_{x_0} \oplus T_{x_0} W^s(x_0, f), \tag{17}$$

where $W^s(x_0, f)$ is the stable set of x_0; its intersection with a neighborhood of $\pi_0(\overleftarrow{M}_f)$ is an immersed manifold (see Proposition 4.11 [2]).

We remark that $E^s_{\underline{x}}$ depends only on x_0, for every $\underline{x} \in \overleftarrow{M}_f$.

In order to construct the plane fields of Proposition 5.1, we will have to take care of the critical points of f. The unique control that we have on them is the strong transversality condition. This condition implies, in particular, that for every $\underline{x} \in \overleftarrow{M}_f$ it holds

$$E^s_{\overleftarrow{f}(\underline{x})} + Tf(T_{x_0}M) = \mathbb{R}^N. \tag{18}$$

Therefore we shall construct the distributions $(E^s_i)_{i=1}^q$ "close" to E^s in G_N. Let us explain how we will proceed, and what does it mean.

Topology on plane fields of nested domains of definition   For a subset $C \subset \overleftarrow{M}_f$, we denote by $C^0(C, G_N)$ the space of d_1-continuous maps from C into G_N. When C is compact, we endow this space with the uniform metric:

$$d(g, g') = \max_{\underline{x} \in C} d(g(\underline{x}), g'(\underline{x})).$$

Given a plane field $E \in C^0(C, G_N)$ and $\eta > 0$, we denote by $B(E, \eta)$ (resp. $\bar{B}(E, \eta)$) the open (resp. closed) ball centered at E and radius η.

Let W be subset of $\overleftarrow{M}_f$ and V a neighborhood of W. Let $E_W \in C^0(W, G_N)$ and $E_V \in C^0(V, G_N)$ be two plane fields. We say that E_W is *compact-open close* to E_V if for any compact subset $C \subset W$, there exists a small compact neighborhood N of C in V such that the graph of $E_V | N$ is close to the graph of $E_W | C$ for the Hausdorff distance on compact subsets of $M_f \times G_N$ induced by $d_1 + d_G$. This will be explained in greater details for its application case in Remark 7.2.

7.2. Splitting *Proposition 5.1* into the stable and unstable fields

We are going to illustrate the geometrical part of the proof of Proposition 5.1 by depicting the construction for the following example. Let $f : (x, y, z) \in \mathbb{R}^3 \mapsto (x^2, y^2, 0)$. This map is AS and can be extended to an AS endomorphism of the compactification $(\mathbb{R} \cup \{\infty\})^3$ of $\mathbb{R}^3$ equal to the 3-torus. On this compact manifold, its inverse limit is homeomorphic to $[0, \infty]^3$, via the projection π_0. Since this map is invariant via the symmetries $(x, y, z) \mapsto (x^{\delta_x}, y^{\delta_y}, z^{\delta_z})$, $(\delta_x, \delta_y, \delta_z) \in \{-1, 1\}^3$, we will focus only on the restricted dynamics on $\pi_0^{-1}([0, 1]^3)$ which is the inverse limit of f restricted to the set of points with bounded orbit. The restricted non-wandering set $\overleftarrow{\Omega}_f$ is formed by 4 fixed points $(0, 0, 0)^{\mathbb{Z}}, (0, 1, 0)^{\mathbb{Z}}, (1, 0, 0)^{\mathbb{Z}}, (1, 1, 0)^{\mathbb{Z}}$.

Let us split Proposition 5.1 into two propositions.

Proposition 7.1. *There exist neighborhoods $(V_i)_{i=1}^q$ of respectively $(W^s(\overleftarrow{\Omega}_i))_{i=1}^q$ in $\overleftarrow{M}_f$, and for every small δ, there are functions*

$$E_i^s : V_i \to G_N$$

satisfying the following properties for every i:

(i) *for every $\underline{x} \in V_i \cap \overleftarrow{f}^{-1}(V_i)$ the following inclusion holds:*

$$F^\delta(E_{i\underline{x}}^s) \subset E_{i\overleftarrow{f}(\underline{x})}^s.$$

(ii) *for every $k \geq j$, for every $\underline{x} \in V_k \cap \overleftarrow{f}^{-1}(V_j)$ the following inclusion holds:*

$$F^\delta(E_{k\underline{x}}^s) \subset E_{j\overleftarrow{f}(\underline{x})}^s.$$

(iii) E_i^s *is compact-open close to $E^s | W^s(\overleftarrow{\Omega}_i)$, when δ is small.*
(iv) E_i^s *is of constant dimension, d_1-continuous, and locally Lipschitz for the metric d_∞.*

Fig. 1 depicts an example of such plane fields.

Remark 7.2. From the definition given in §7.1, Property (iii) means that for every i, for every compact subset C of $W^s(\overleftarrow{\Omega}_i)$, for every $\epsilon > 0$, there exists a compact neighborhood U of C in V_i such that for every δ sufficiently small

$$d_H(Graph(E_i^s | U), Graph(E^s | C)) \leq \epsilon,$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff distance of compact subsets of $\overleftarrow{M}_f \times G_N$ induced by the distance $d_1 + d_G$. We notice that U depends on C and ϵ but not on δ small enough.
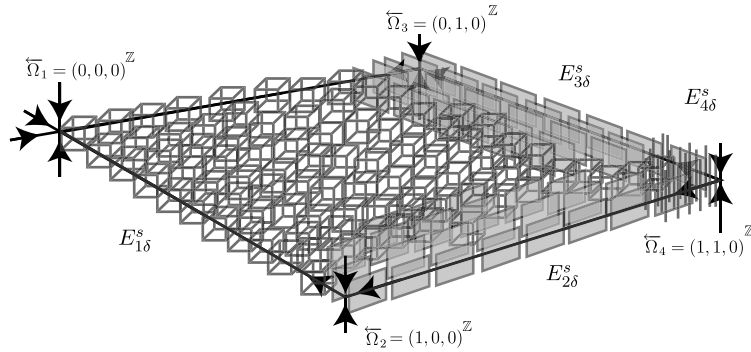
Fig. 1. Plane fields E_i^s in the example given by $f : (x, y, z) \mapsto (x^2, y^2, 0)$.

Remark 7.3. Property (iv) means that E_i^s is of constant dimension, d_1-continuous, and that for every compact subset C of V_i there exists a constant L_C^δ such that:

$$d_G(E_{i\underline{x}}^s, E_{i\underline{y}}^s) \le L_C^\delta d_\infty(\underline{x}, \underline{y}), \quad \forall \underline{x}, \underline{y} \in C.$$

In the diffeomorphism case, to obtain the existence of $(E_{i\delta}^u)_i$ it suffices to first push forward by F^δ each of the plane field E_i^s on $\cup_n \overleftarrow{f}^n(V_i)$ (which is a neighborhood of $W^u(\overleftarrow{\Omega}_i)$), and then to apply the same proposition to $\overleftarrow{f}^{-1}$. In our case, even though $\overleftarrow{f}$ is invertible, the bundle map F^0 is not. However, in Lemma 4.4, we saw that F^δ is invertible for every $\delta > 0$. Nonetheless, the norm of the inverse of this map depends on δ, and so the angle between E_i^s and E_i^u as well. However in Proposition 5.1 such an angle must be bounded by a constant which is independent of δ (and this is necessary in the proof of Proposition 4.6).

Hence we must redo a similar construction, still in a neighborhood of each $W^s(\overleftarrow{\Omega}_i)$ since it is the only place where we control the singularities.

Another difference in the construction of E_i^u is the following: to construct the plane field E_i^u we will not be allowed to pull back, since the critical set might intersect $W^s(\overleftarrow{\Omega}_i)$, and a pull back by F^0 would contain critical vectors which belong to E^s, this would contradict the angle condition (iii) for $\delta > 0$. Hence the construction of E_i^u must be done in compact set in a small neighborhood of $W^s(\overleftarrow{\Omega}_i)$ via push forward.

Proposition 7.4. *There exist $K > 0$, an open cover $(W_i)_{i=1}^q$ of $\overleftarrow{M}_f$, where each W_i contains $\overleftarrow{\Omega}_i$ and is included in V_i, such that for every $\delta > 0$, there exists a subbundle E_i^u of $W_i \times \mathbb{R}^N \to W_i$ satisfying the following properties for $i \in [1, q]$:*

(i) *For every $\underline{x} \in W_i \cap \overleftarrow{f}^{-1}(W_i)$, the map F^δ sends $E_{i\underline{x}}^u$ into $E_{i\overleftarrow{f}(\underline{x})}^u$.*

(ii) *For every $j \ge i$, for every $\underline{x} \in W_i \cap \overleftarrow{f}^{-1}(W_j)$ the following inclusion holds:*

$$F^\delta(E_{j\underline{x}}^u) \supset E_{i\overleftarrow{f}(\underline{x})}^u.$$

(iii) *$E_{i\underline{x}}^s \oplus E_{i\underline{x}}^u = \mathbb{R}^N$, for every $\underline{x} \in W_i$, the angle between E_i^s and E_i^u is bounded from below by K^{-1}.*

(iv) *The subbundle E_i^u is of constant dimension, d_1-continuous and d_∞-Lipschitz.*

(v) *For every $\underline{x} \in W_i$, it holds*

$$\|F^\delta(v^u)\| \ge \|v^u\|/K, \quad \forall v^u \in E_{i\underline{x}}^u.$$

(vi) *For every q', if $\underline{x} \in W_{q'}$ then $\overleftarrow{f}(\underline{x}) \notin \cup_{j > q'} W_j$ and $\cap_{n \in \mathbb{Z}} \overleftarrow{f}^n(W_{q'}) = \overleftarrow{\Omega}_{q'}$.*
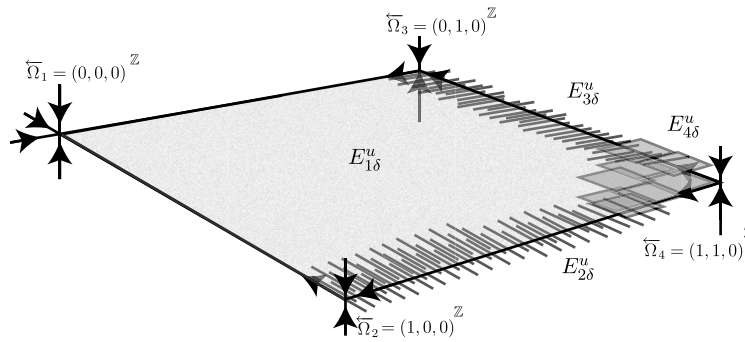
Fig. 2 depicts an example of such plane fields.

Fig. 2. Plane fields E_i^u in the example given by $f \colon (x, y, z) \mapsto (x^2, y^2, 0)$.
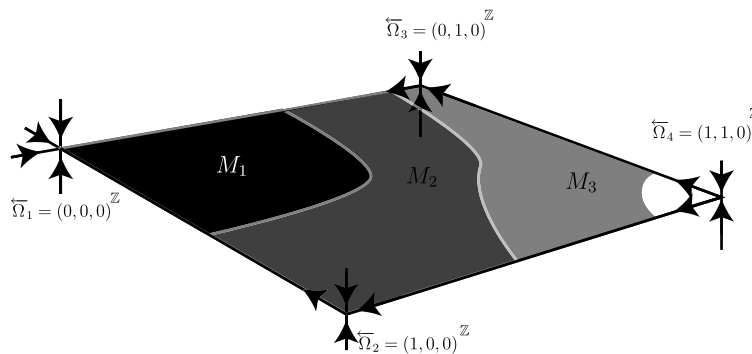

Fig. 3. A filtration for the example given by $f \colon (x, y, z) \mapsto (x^2, y^2, 0)$.

From the two latter propositions, we easily deduce:

Proof of Proposition 5.1. By Propositions 7.1 and 7.4, we have immediately properties (i)–(ii)–(iii)–(iv)–(v)–(vi) of Proposition 5.1. To prove property (vii), we remark that by Proposition 7.1 (i) and (iii) together with the hyperbolicity of $\overleftarrow{\Omega}$, the bundle E_i^s is contracted by F^δ over a neighborhood of $\overleftarrow{\Omega}_i$, for every i. Moreover, by properties (i)–(iii)–(iv) of Proposition 7.4, the bundle E_i^u is close to $E^u | \overleftarrow{\Omega}_i$, and so expanded by F^δ on a neighborhood of $\overleftarrow{\Omega}_i$ (see Proposition 2.2). $\square$

8. Proof of Proposition 7.1

Let us recall that $(M_j)_{j=1}^q$ is a filtration adapted $(\overleftarrow{\Omega}_j)_{j=1}^q$ (see §5 for details). An example of such a filtration is depicted Fig. 3.

We are going to construct $(E_i^s)_i$ by (increasing) induction on i. Here it is the induction hypothesis at the step i:

For every $N_i \leq 0$, there exist neighborhoods $(V_j^i)_{j=1}^i$ of respectively $(W^s(\overleftarrow{\Omega}_j) \cap \overleftarrow{f}^{N_i}(M_i))_{j=1}^i$ in $\overleftarrow{f}^{N_i}(M_i)$, there are functions

$$E_j^s \colon V_j^i \to G_N$$

which satisfy the following properties for every $j \leq i$:

(i) for every $\underline{x} \in V_j^i \cap \overleftarrow{f}^{-1}(V_j^i)$ the following inclusion holds:

$$F^\delta(E_{j\underline{x}}^s) \subset E_{j\overleftarrow{f}(\underline{x})}^s.$$

(ii) for every $k \geq j$, for every $\underline{x} \in V_k^i \cap V_j^i$ the following inclusion holds:

$$E^s_{k\underline{x}} \subset E^s_{j\underline{x}}.$$

(iii) E^s_j is compact-open close to $E^s|W^s(\Omega_j) \cap V^i_j$, when δ-is small.

(iv) E^s_j is of constant dimension, d_1-continuous, and locally Lipschitz for the metric d_∞.

Remark 8.1. The induction hypothesis at step $i \geq 0$ states that for any $N_i \leq 0$ there exists a covering of $(V^i_j)_{j \leq i}$ of $\overleftarrow{f}^{N_i}(M_i)$ which satisfies the above properties. We recall that M_i is a compact included in $\cup_{j \leq i} W^s(\overleftarrow{\Omega}_j)$. But in general, for $i < q$, it does not contain $W^s(\overleftarrow{\Omega}_j)$ (see Fig. 3). Thus V^i_j is in general not a neighborhood of $W^s(\overleftarrow{\Omega}_j)$, but only a neighborhood of $W^s(\overleftarrow{\Omega}_j) \cap \overleftarrow{f}^{N_i}(M_i)$ in the space $\overleftarrow{f}^{N_i}(M_i)$ endowed with the topology induced by $\overleftarrow{M}_f$.

Nevertheless, when $i = q$, then $M_q = \overleftarrow{M}_f$, and so V^q_j is a neighborhood of $W^s(\overleftarrow{\Omega}_j)$ for every $j \leq q$.

Hence the step $i = q$ gives the statement of Proposition 7.1 with for every j, $V_j := \overleftarrow{f}^{-1}(V^q_j) \cap V^q_j$. Indeed, V_j is a neighborhood of $W^s(\overleftarrow{\Omega}_j)$ since V^q_j is so.

We recall that each E^s_j depends on δ. During the induction, several parameters will be fixed.

The order is the following at the step i. First an arbitrary negative integer N_i is given. Then $\eta > 0$ is chosen. Depending on N_i and η, we will suppose δ small. The induction hypothesis is used with δ and N_{i-1} chosen large in function of N_i and η.

Step $i = 1$ Let $N_1 \leq 0$, and put $K_1 := \overleftarrow{f}^{N_1}(M_1)$. We notice that $W^s(\overleftarrow{\Omega}_1)$ is an open set of $\overleftarrow{M}_f$. Hence we put $V^1_1 := K_1$. Note that $\overleftarrow{f}(K_1) \subset K_1$.

Let $K_1 \ni \underline{x} \mapsto E'_{\underline{x}}$ be the restriction to K_1 of a smooth approximation of the continuous map $E^s|W^s(\overleftarrow{\Omega}_1)$: $W^s(\overleftarrow{\Omega}_1) \to G_N$ given by Corollary 6.3.

Observe that E' is uniformly close to $E^s|K_1$. Moreover it is d_1-continuous and d_∞-Lipschitz. We recall that the Banach manifold $C^0(K_1, G_N)$ was defined in §7.1.

For all $\eta > 0$ and $\delta > 0$, the following is well defined on the closed ball $\bar{B}_{C^0}(E', \eta) \subset C^0(K_1, G_N)$ with image in $C^0(K_1, G_N)$:

$$F^{\delta\#} := \bar{B}_{C^0}(E', \eta) \ni P \mapsto \left[\underline{x} \mapsto F^{\delta^{-1}}_{x_0}(P_{\overleftarrow{f}(\underline{x})})\right], \quad \text{with } x_0 := \pi_0(\underline{x}).$$

By hyperbolicity, for δ small enough and E' sufficiently close to E^s, there exists some $k \in \mathbb{N}$ such that $F^{\delta\#k}$ is λ-contracting and sends the closed ball $\bar{B}_{C^0}(E', \eta)$ into itself.

Let E^s_1 be the unique fixed point of $F^{\delta\#}$ in $B_{C^0}(E', \eta)$. By definition, condition (i) is satisfied.

Condition (iii) follows from the fact that η can be taken small when δ is small.

It remains only to show that (iv) holds. First let us recall that $E^s_1 \in B_{C^0}(E', \eta)$ and $x_0 \in M \mapsto F^\delta_{x_0} \in L_N(\mathbb{R})$ is of class C^1, and so, $K_1 \ni \underline{x} \mapsto F^\delta_{x_0}$ is d_∞-Lipschitz. Since F^δ is moreover invertible, there exists $L_{\delta,k}$ such that for all $\underline{x}, \underline{y} \in K_1$ and $P \in B_{C^0}(E', \eta)$ it holds

$$d\left(F^{\delta k^{-1}}_{\underline{x}}(P_{\overleftarrow{f}^k(\underline{y})}), F^{\delta k^{-1}}_{\underline{y}}(P_{\overleftarrow{f}^k(\underline{y})})\right) \leq L_{\delta,k} d_\infty(\underline{x}, \underline{y}), \tag{19}$$

where $F^{\delta k}_{\underline{x}} := F^\delta_{x_{k-1}} \circ \cdots \circ F^\delta_{x_0}$ and $x_i := \pi_i(\underline{x})$ the ith coordinate of $\underline{x}$.

On the other hand, the map $F^{\delta\#k}$ is pointwise λ-contracting:

$$d\left(F^{\delta k^{-1}}_{\underline{x}}(P_{\overleftarrow{f}^k(\underline{x})}), F^{\delta k^{-1}}_{\underline{x}}(P_{\overleftarrow{f}^k(\underline{y})})\right) \leq \lambda d(P_{\overleftarrow{f}^k(\underline{x})}, P_{\overleftarrow{f}^k(\underline{y})}). \tag{20}$$

Consequently, adding (19) and (20) we get

$$d\left(F^{\delta^{-k}}_{\underline{x}}(P_{\overleftarrow{f}^k(\underline{x})}), F^{\delta^{-k}}_{\underline{y}}(P_{\overleftarrow{f}^k(\underline{y})})\right) \leq L_{\delta,k} d_\infty(\underline{x}, \underline{y}) + \lambda d(P_{\overleftarrow{f}^k(\underline{x})}, P_{\overleftarrow{f}^k(\underline{y})}). \tag{21}$$

For every d_∞-Lipschitz distribution P let us denote by $\Lambda(P)$ its Lipschitz constant. It holds for every k:
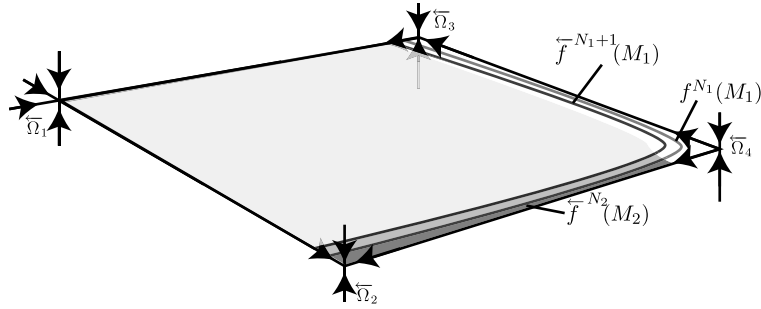
Fig. 4. Construction of $Z_2 = f^{N_2}(M_2) \setminus int\, f^{N_1+1}(M_1)$ in the example given by $f : (x, y, z) \mapsto (x^2, y^2, 0)$.

$$d(P_{\overleftarrow{f}^k(\underline{x})}, P_{\overleftarrow{f}^k(\underline{y})}) \leq \Lambda(P)d_\infty(\underline{x}, \underline{y}). \tag{22}$$

Thus by (21) and (22):

$$d\big(F^{\delta k^{-1}}_{\underline{x}}(P_{\overleftarrow{f}^k(\underline{x})}), F^{\delta k^{-1}}_{\underline{y}}(P_{\overleftarrow{f}^k(\underline{y})})\big) \leq (L_{\delta,k} + \lambda\Lambda(P))d_\infty(\underline{x}, \underline{y})$$

Consequently the closed subset of $B_{C^0}(E', \eta)$ formed by sections with d_∞-Lipschitz constant smaller or equal than any $\Lambda \geq (1 - \lambda)^{-1}L_{\delta,k}$ is forward invariant under $F^{\delta\#k}$. We recall that E' is d_∞-Lipschitz. Hence if $\Lambda \geq \Lambda(E')$, this subset is non empty (it contains E'), thus there exists a fixed point d_∞-Lipschitz in $\bar{B}_{C^0}(E', \eta)$. By uniqueness, the fixed point $K_1 \ni \underline{x} \mapsto E^s_{1\underline{x}} \in G_N$ is d_∞-Lipschitz. □

Step $i - 1 \to i$ Let N_i be an arbitrary negative integer. Put:

$$K_i := W^s(\overleftarrow{\Omega}_i) \cap \overleftarrow{f}^{N_i}(M_i).$$

Let us begin as in the step $i = 1$.

We can extend d_1-continuously the section $E^s|K_i : K_i \to G_N$ to an open neighborhood of K_i. Let $\underline{x} \mapsto E'_{\underline{x}}$ be a smooth approximation given by Corollary 6.3 of such a continuous extension.

The section E' is well defined on a small neighborhood Z_i of K_i in $\overleftarrow{f}^{N_i}(M_i)$ of the form:

$$Z_i := \overleftarrow{f}^{N_i}(M_i) \setminus int\, \overleftarrow{f}^{N_{i-1}+1}(M_{i-1}), \quad N_{i-1} \leq 0$$

Indeed note that $K_i = \overleftarrow{f}^{N_i}(M_i) \setminus \cup_{n \leq 0} \overleftarrow{f}^n(M_{i-1})$, so Z_i is close to K_i whenever $-N_{i-1}$ is large enough (see Fig. 4).

Observe that $E'|K_i$ is C^0-close to $E^s|K_i$, d_1-continuous and d_∞-Lipschitz.

Hence, for every η small, for every Z_i and δ small enough, for E' sufficiently close to E^s, the following is well defined on the ball $\bar{B}_{C^0}(E', \eta) \subset C^0(Z_i, G_N)$ with image in $C^0(Z_i \cap \overleftarrow{f}^{-1}(Z_i), G_N)$:

$$F^{\delta\#} := \bar{B}_{C^0}(E'|Z_i, \eta) \ni P \mapsto \Big[Z_i \cap \overleftarrow{f}^{-1}(Z_i) \ni \underline{x} \mapsto F^{\delta^{-1}}_{x_0}(P(\overleftarrow{f}(\underline{x})))\Big].$$

By hyperbolicity, for η small and then for Z_i and δ small enough and E' sufficiently close to E^s, there exist some $k \in \mathbb{N}$, such that the following map:

$$F^{\delta\#k} := \bar{B}_{C^0}(E'|Z_i, \eta) \ni P \mapsto \Bigg[\bigcap_{0 \leq l \leq k} \overleftarrow{f}^{-l}(Z_i) \ni \underline{x} \mapsto F^{\delta^{-k}}_{x_0}(P(\overleftarrow{f}^k(\underline{x})))\Bigg],$$

is contracting and sends the closed ball $\bar{B}_{C^0}(E'|Z_i, \eta)$ into $\bar{B}_{C^0}(E'|\cap_{l \leq k} \overleftarrow{f}^{-l}(Z_i), \eta)$.

As the target space is not the same as the source space, we cannot conclude to the existence of a fixed point. We are going to extend the sections in the image of $F^{\delta\#}$ by a section constructed by the following lemma shown below:

Lemma 8.2. *There exist a sequence of negative integers $(N_j)_{j<i-1}$ and a section $\tilde{E} \in B_{C^0}(E'|Z_i, \eta/2)$ which is d_1-continuous and d_∞-Lipschitz such that for every $j < i$:*

(1) $Z_j := \overleftarrow{f}^{N_j}(M_j) \setminus int \, \overleftarrow{f}^{N_{j-1}+1}(M_{j-1}) \subset int \, V_j^{i-1}$.

(2) $\forall \underline{x} \in Z_j \cap Z_i, \quad \tilde{E}(\underline{x}) \subset E_j^s(\underline{x})$.

Gluing $\tilde{E}$ to $F^{\delta \#}\tilde{E}$ and definition of E_i^s   We remark that $\tilde{E}$ and $F^{\delta \#}\tilde{E}$ are well defined on:

$$V_i^i := \overleftarrow{f}^{N_i}(M_i) \setminus \overleftarrow{f}^{N_{i-1}}(M_{i-1}).$$

By [Corollary 6.2](#), there exists a partition of the unity $(\rho, 1 - \rho) \in \mathrm{Mor}_0^\infty(\overline{M}_f)^2$ subordinated to the cover $(\overleftarrow{f}^{N_{i-1}-1}(int \, M_{i-1}), \overline{M}_f \setminus \overleftarrow{f}^{N_{i-1}-1}(M_{i-1}))$.

For $\underline{x} \in V_i^i$, let $p_{\underline{x}}$ and $p'_{\underline{x}}$ be the orthogonal projections of $\mathbb{R}^N$ onto respectively $\tilde{E}(\underline{x})$ and $F^\# \tilde{E}(\underline{x})$. Put

$$E^0(\underline{x}) := \{(\rho(\underline{x})p_{\underline{x}} + (1 - \rho(\underline{x}))p'_{\underline{x}})(u); \ u \in \tilde{E}(\underline{x})\}.$$

We notice that $V_i^i \ni \underline{x} \mapsto E^0(\underline{x}) \in G_N$ is d_1-continuous and d_∞-Lipschitz. Furthermore, for $\underline{x} \in V_i^i$ close to $\overleftarrow{f}^{N_{i-1}}(M_{i-1})$, the plane $E^0(\underline{x})$ is equal to $\tilde{E}(\underline{x})$ and for $\underline{x} \in V_i^i \setminus \overleftarrow{f}^{N_{i-1}-1}(int \, M_{i-1})$, the plane $E^0(\underline{x})$ is equal to $F^{\delta \#}\tilde{E}_{\underline{x}}$.

We define

$$\Pi_\eta := \{P \in C^0(V_i^i, G_N) : \ P = E^0 \text{ on } V_i^i \setminus \overleftarrow{f}^{N_{i-1}-1}(M_{i-1}) \text{ and}$$

$$\text{the restriction } P \text{ and } E' \text{ to } \cap_{j=0}^k \overleftarrow{f}^j(V_i^i) \text{ are } \eta - C^0 - \text{close}\}.$$

We remark that the following map is continuous:

$$F^{\delta \star} := P \in \Pi_\eta \mapsto \left[V_i^i \ni \underline{x} \mapsto \begin{cases} E^0(\underline{x}) & \text{if } \underline{x} \in V_i^i \setminus \overleftarrow{f}^{N_{i-1}-1}(M_{i-1}) \\ F^{\delta-1}_{\underline{x}}(P(\overleftarrow{f}(x))) & \text{otherwise.} \end{cases} \right]$$

As $F^{\delta \# k}$ is contracting and sends $\bar{B}_{C^0}(E'|Z_i, \eta)$ into $\bar{B}_{C^0}(E'| \cap_j \overleftarrow{f}^j(Z_i), \eta)$, the map $F^{\delta \star^k}$ is contracting and sends Π_η into itself.

Let E_i^s be the fixed point of $F^{\delta \star}$.

By definition, E_i^s satisfies property (i). Similarly to the step $i = 1$, the section E_i^s satisfies Properties (iii) and (iv). However all the sections $(E_j^s)_{j \leq i}$ need to be extended from $(V_j^{i-1})_{j \leq i}$ to $(V_j^i)_{j \leq i}$ which remains to be constructed.

Construction of $(V_j^i)_{j \leq i}$ and extension of (E_j^s)   For every $j < i$, we recall that for every $\underline{x} \in Z_j \cap Z_i$, the plane $\tilde{E}_{\underline{x}}$ is included in $E_{j\underline{x}}^s$. By induction hypothesis (i) and since $V_i^i = Z_i \cap \overleftarrow{f}^{-1}(Z_i)$, for every $\underline{x} \in Z_j \cap V_i^i$, the plane $F^{\delta \#}\tilde{E}_{\underline{x}}$ is included in $E_{j\underline{x}}^s$. Hence, for every $\underline{x} \in Z_j \cap V_i^i$, the plane $E_{\underline{x}}^0$ is included in $E_{j\underline{x}}^s$. Again by induction hypothesis (i), since the fixed point $E_{i\underline{x}}^s$ of $F^\star$ is obtained by iterating it, for every $\underline{x} \in Z_j \cap V_i^i$, the plane $E_{i\underline{x}}^s$ is included in $E_{j\underline{x}}^s$. Put

$$\tilde{V}_j := V_j^{i-1} \cap \cup_{l=j}^{i-1} Z_l$$

By induction hypothesis (ii), for every $\underline{x} \in \tilde{V}_j \cap V_i^i$, the plane $E_{i\underline{x}}^s$ is included in $E_{j\underline{x}}^s$. Using that $Z_j = \overleftarrow{f}^{N_j}(M_j) \setminus int \, \overleftarrow{f}^{N_{j-1}+1}(M_{j-1})$, we remark that:

$$\tilde{V}_j \supset V_j^{i-1} \cap \overleftarrow{f}^{N_{i-1}}(M_{i-1}) \setminus int \, \overleftarrow{f}^{N_{j-1}+1}(M_{j-1}) = V_j^{i-1} \setminus int \, \overleftarrow{f}^{N_{j-1}+1}(M_{j-1}).$$

Hence $\tilde{V}_j$ is a neighborhood of $W^s(\overleftarrow{\Omega}_j) \cap \overleftarrow{f}^{N_{i-1}}(M_{i-1})$ since $W^s(\overleftarrow{\Omega}_j)$ does not intersect $\overleftarrow{f}^{N_{j-1}+1}(M_{j-1})$. Put

$$V_j^i := \cup_{n \geq 0}(\overleftarrow{f}|V_i^i)^{-n}(\tilde{V}_j) \quad \text{where} \quad V_i^i = \overleftarrow{f}^{N_i}(M_i) \setminus \overleftarrow{f}^{N_{i-1}}(M_{i-1})$$

Lemma 8.3. *For every $j \leq i$, the set V_j^i is a neighborhood of $W^s(\Omega_j) \cap f^{N_i}(M_i)$ in $f^{N_i}(M_i)$.*

Proof. As the case $i = j$ is obvious, we suppose $j < i$. For every $\underline{x} \in f^{N_i}(M_i) \cap W^s(\overleftarrow{\Omega}_j)$ there exists n such that $\overleftarrow{f}^n(\underline{x}) \in int(f^{N_{i-1}}(M_{i-1}))$. Consequently, for every $\underline{y} \in \overleftarrow{f}^{N_i}(M_i)$ nearby $\underline{x}$, there exists m such that $\overleftarrow{f}^m(\underline{y}) \in \overleftarrow{f}^{N_{i-1}}(M_{i-1})$. Let us consider such an m minimal. Since $\tilde{V}_j$ is a neighborhood of $W^s(\overleftarrow{\Omega}_j) \cap f^{N_{i-1}}(M_{i-1})$ in $f^{N_{i-1}}(M_{i-1})$, the point $\overleftarrow{f}^m(\underline{y})$ belongs to $\tilde{V}_j$ for $\underline{y}$ sufficiently close to $\underline{x}$. Also for every $k < m$ the point $\overleftarrow{f}^k(\underline{y})$ belongs to the complement of $f^{N_{i-1}}(M_{i-1})$. On the other hand, $\overleftarrow{f}^k(\underline{y})$ belongs to $f^{N_i}(M_i)$ for every k. Thus for every $k < m$, the point $\overleftarrow{f}^k(\underline{y})$ belongs to $V_i^i = \overleftarrow{f}^{N_i}(M_i) \setminus \overleftarrow{f}^{N_{i-1}}(M_{i-1})$. As $\overleftarrow{f}^m(\underline{y})$ belongs to $\tilde{V}_j$, it follows that $\underline{y}$ belongs to V_j^i. $\quad\square$

We extend E_j^s on V_j^i by:

$$\forall x \in V_j^i, \forall n \geq 0 \text{ minimal such that } \overleftarrow{f}^n(\underline{x}) \in \tilde{V}_j, \ E_{j\underline{x}}^s = F_{\underline{x}}^{\delta - n}(E_{j\overleftarrow{f}^n(\underline{x})}^s).$$

Induction hypotheses (i), (iii) and (iv) for $j < i$ imply properties (i), (iii) and (iv) for E_j^s on V_j^i.

Let us check property (ii). As this property is invariant by pull back, property (i) implies that property (ii) holds for every $k \leq j$ both less than i. Let $\underline{x} \in V_i^i \cap V_j^i$. Let n be such that $\underline{x}$ belongs to $\overleftarrow{f}^{-n}(\tilde{V}_j) \cap V_i^i$. We recall that $V_i^i = \overleftarrow{f}^{N_i}(M_i) \setminus \overleftarrow{f}^{N_{i-1}}(M_{i-1})$, hence $\overleftarrow{f}^n(\underline{x})$ belongs to $\overleftarrow{f}^{N_i+n}(M_i) \subset \overleftarrow{f}^{N_i}(M_i)$. Also $\overleftarrow{f}^n(\underline{x})$ belongs to $\tilde{V}_j \subset V_j^{i-1} \subset f^{N_{i-1}}(M_{i-1}^c)$. Thus $\overleftarrow{f}^n(\underline{x}) \in \tilde{V}_j \cap V_i^i$. Consequently, Property (ii) holds at $\overleftarrow{f}^n(\underline{x})$. By pull back invariance and property (i), Property (ii) holds at $\underline{x}$.

Proof of Lemma 8.2. We are going to project E' onto each E_j^s, $j \leq i$. The following is a consequence of the Lambda-lemma and the strong transversality condition.

Claim 8.1. *For Z_i small enough (that is $-N_{i-1}$ large enough), for δ small enough, for every $j < i$, every $\underline{x} \in W^s(\overleftarrow{\Omega}_j) \cap Z_i$, if $q_j(\underline{x})$ denotes the orthogonal projection of $\mathbb{R}^N$ onto $E_{\underline{x}}^s$, the distance between $q_j(\underline{x})(E'_{\underline{x}})$ and $E'_{\underline{x}}$ is less than $\eta/4i$.*

Proof. By the strong transversality condition, on $W_\epsilon^u(\overleftarrow{\Omega}_i) \setminus \overleftarrow{\Omega}_i$ the stable direction E^s is transverse to $TW_\epsilon^u(\overleftarrow{\Omega}_i)$. This is true in particular on $W_\epsilon^u(\overleftarrow{\Omega}_i) \cap W^s(\overleftarrow{\Omega}_j)$. By hyperbolicity and the strong transversality condition, for every $(\underline{x}_n)_n$ in $W_\epsilon^u(\overleftarrow{\Omega}_i) \cap W^s(\overleftarrow{\Omega}_j)$ approaching $\underline{x} \in \Omega_i$, every accumulation plane P of $(E_{\underline{x}_n}^s)_n$ contains the plane $E_{\underline{x}}^s$. This implies that for every $(\underline{x}_n)_n$ in $W^s(\overleftarrow{\Omega}_j)$ approaching $\underline{x} \in K_i = \overleftarrow{f}^{N_i}(M_i) \cap W^s(\overleftarrow{\Omega}_j)$, every accumulation plane P of $(E_{\underline{x}_n}^s)_n$ contains the plane $E_{\underline{x}}^s$. The claim follows since E' is close to $E^s|K_i$ for δ small and Z_i small. $\quad\square$

We will perform orthogonal projections of E_i^s on compact subsets of each $Z_i \cap W^s(\overleftarrow{\Omega}_j)$.
Let us implement these compact subsets.
First let us notice that $cl(Z_i \setminus \overleftarrow{f}^{-1}(Z_i))$ is a compact subset of $\cup_{j<i} W^s(\Omega_j)$:

$$Z_i \setminus \overleftarrow{f}^{-1}(Z_i) = \overleftarrow{f}^{N_i}(M_i) \setminus int\, \overleftarrow{f}^{N_{i-1}+1}(M_{i-1}) \setminus \left(\overleftarrow{f}^{N_i-1}(M_i) \setminus int\, \overleftarrow{f}^{N_{i-1}}(M_{i-1})\right).$$

As $\overleftarrow{f}^{N_i}(M_i)$ is included in $\overleftarrow{f}^{N_i-1}(M_i)$ it comes:

$$Z_i \setminus \overleftarrow{f}^{-1}(Z_i) = \overleftarrow{f}^{N_i}(M_i) \cap int\, \overleftarrow{f}^{N_{i-1}}(M_{i-1}) \setminus int\, \overleftarrow{f}^{N_{i-1}+1}(M_{i-1}) \subset \cup_{j<i} W^s(\overleftarrow{\Omega}_j).$$

Let $(V_k^{i-1})_{k\leq i-1}$ be the neighborhoods given by the induction hypothesis at step $i-1$ for the integer N_{i-1} defined above.

By decreasing induction we construct $(N_k)_{k=1}^{i-2} \in \mathbb{Z}^-$ such that, the following holds.

Claim 8.2. *For every $k \leq i - 1$, the set $Z_k := \overleftarrow{f}^{N_k}(M_k) \setminus \overleftarrow{f}^{N_{k-1}+1}(M_{k-1})$ has its closure included in the interior of V_k^{i-1}. Moreover, for δ small enough, the distance between the orthogonal projection p_k onto E_k^s satisfies:*

$$\|p_k(\underline{x})(E_{\underline{x}}') - E_{\underline{x}}'\| \leq \eta/3i, \quad \forall \underline{x} \in Z_k. \tag{23}$$

Proof. For $k \leq i - 1$, suppose N_k constructed. Then for $-N_{k-1}$ large, the set Z_k is close to the compact set $W^s(\overleftarrow{\Omega}_k) \cap f^{N_k}(M_k)$, and so it is included in V_k. Moreover, by Remark 7.3 and Claim 8.1 for $-N_{k-1}$ large and δ small, inequality (23) holds. $\square$

Let $\hat{Z}_k$ be a neighborhood of Z_k in V_k^{i-1} such that for all $\underline{x} \in \hat{Z}_k$,

$$\|p_k(\underline{x})(E_{\underline{x}}') - E_{\underline{x}}'\| \leq \frac{\eta}{2i}.$$

By Corollary 6.2, there exists a dump function $\rho \in Mor_0^\infty(\overleftarrow{f}^{N_{i-1}}(M_{i-1}), [0, 1])$ equal to 1 on Z_k and to 0 on $\hat{Z}_k^c$. We construct $(P_{\underline{x}}^j)_{j<i}$ by induction. Put $P^0 = E'$, and for $j \in [1, i-1]$ put

$$P_{\underline{x}} = \{\rho_j(\underline{x}) \cdot q_j(\underline{x})(u) + (1 - \rho_j(\underline{x})) \cdot u : u \in P_{\underline{x}}^{j-1}\}$$

Let $\hat{E} = P_{\underline{x}}^{i-1}$. By induction hypothesis (ii), for every $\underline{x} \in Z_i \cap Z_j$, it holds $\tilde{E}_{\underline{x}} \subset E_{j\underline{x}}^s$.

By definition of $\hat{Z}_k$, the section $\tilde{E}$ is in $B_{C^0}(E'|Z_i, \eta/2)$ and is d_∞-Lipschitz. $\square$

9. Proof Proposition 7.4

The proof of Proposition 7.4 is done by decreasing induction on $q' \in [1, q]$. We recall that Proposition 7.1 constructed sections $(E_j^s)_j$ on neighborhoods $(V_j)_j$ of respectively $(W_j^s(\overleftarrow{\Omega}_j))_j$, which satisfy properties (i)–(ii)–(iii) and (iv). Here is the induction hypothesis.

For every $q' \leq q$, there exist $K > 0$ and an open cover $(W_i)_{i=q'}^q$ of $\cup_{j \geq q'} W^s(\overleftarrow{\Omega}_j)$, where each W_i is a neighborhood of $\overleftarrow{\Omega}_i$ included in V_i^i and such that for every $\delta > 0$, there exists a function $E_i^u \in C^0(W_i, G_N)$ satisfying the following properties for $i \in [q', q]$:

(i) For every $\underline{x} \in W_i \cap \overleftarrow{f}^{-1}(W_i)$, the map F^δ sends $E_{i\underline{x}}^u$ into $E_{i\overleftarrow{f}(\underline{x})}^u$.

(ii) For every $j \geq i$, for every $\underline{x} \in W_i \cap \overleftarrow{f}^{-1}(W_j)$ the following inclusion holds:

$$E_{i\underline{x}}^u \subset E_{j\overleftarrow{f}(\underline{x})}^u.$$

(iii) $E_{i\underline{x}}^s \oplus E_{i\underline{x}}^u = \mathbb{R}^N$, for every $\underline{x} \in W_i$; the angle between E_i^s and E_i^u is bounded from below by K^{-1}.

(iv) The subbundle E_i^u is of constant dimension, d_1-continuous and d_∞-Lipschitz.

(v) For any $\underline{x} \in W_i$, it holds

$$\|F^\delta(v^u)\| \geq \|v^u\|/K, \quad \forall v^u \in E_{i\underline{x}}^u.$$

(vi) It holds $cl(\overleftarrow{f}^{-1}(\cup_{q'}^q W_i)) \subset \cup_{q'}^q W_i$. Moreover, for every $j \geq i$, if $\underline{x} \in W_j$ then $\overleftarrow{f}(\underline{x}) \notin \cup_{k>j} W_k$ and $\cap_{n \in \mathbb{Z}} \overleftarrow{f}^n(W_j) = \overleftarrow{\Omega}_j$.

We continue to denote by $(M_j)_{j=1}^q$ a filtration adapted to $(\overleftarrow{\Omega}_j)_{j=1}^q$ (see §5 for details and Fig. 3).

At each step q' of the induction we will work with a small η and we will suppose an integer $-N_{q'}$ large and δ small both depending on η.

Step $q' = q$ The subset $\overleftarrow{\Omega}_q = W^s(\overleftarrow{\Omega}_q)$ is compact. Moreover there exists an arbitrarily small compact neighborhood W_q of $\overleftarrow{\Omega}_q$ which satisfies (vi) for $i = q'$. Indeed, consider W_q of the form $\overleftarrow{M}_f \setminus \overleftarrow{f}^{-N}(M_{q-1})$. Hence we can suppose that W_q is included in V_q.

Let $\eta > 0$ be small, in particular smaller than the angle between $E^s | \Omega(\overleftarrow{f})$ and $E^u | \Omega(\overleftarrow{f})$.

Let $\underline{x} \mapsto E'_{\underline{x}}$ be the restriction to W_q of a smooth approximation of a continuous extension of the continuous map $E^u | \overleftarrow{\Omega}_q : \overleftarrow{\Omega}_q \to G_N$ given by Corollary 6.3. This means that on the one hand, E' is d_1-continuous and d_∞-Lipschitz, and that for every ϵ small, if W_q is sufficiently small then for every $\underline{x} \in W_q$ there exists $\underline{y} \in \overleftarrow{\Omega}_q$ ϵ-close to $\underline{x}$ such that the distance between $E'_{\underline{x}}$ and $E^u_{\underline{y}}$ is η small.

By hyperbolicity of $\overleftarrow{\Omega}_q$, the angle between $E^s_{\underline{y}}$ and $E^u_{\underline{y}}$ is uniformly bounded from below on $\underline{y} \in \overleftarrow{\Omega}_q$ and $Tf | E^u_{\underline{y}}$ is bijective. By property (iii) of Proposition 7.1 and Remark 7.2, there exists K large such that for every $\eta > 0$ small, for every W_q sufficiently small, for all $\delta \geq 0$ small, and for every $\underline{y} \in W_q$ the following holds:

(a) the angle between $E^s_{q \underline{y}}$ and $E'_{\underline{y}}$ is greater than K^{-1},
(b) for every plane P making an angle with $E'_{\underline{y}}$ smaller than η, it holds:

$$\forall u \in P, \quad \|F^\delta(u)\| \geq \|u\|/K.$$

Indeed, for $\underline{x} \in \overleftarrow{\Omega}_q$, every vector u in E^u is expanded by F^0.

We can now proceed as in the step $i = 1$ of the proof of Proposition 7.1.

Since for every $\delta > 0$, the map F^δ is bijective, the following is well defined

$$F_\# := \bar{B}_{C^0}(E', \eta) \ni P \mapsto \left[\underline{x} \mapsto F^\delta_{x_0}(P_{\overleftarrow{f}^{-1}(\underline{x})}) \right] \in C^0(W_q, G_N).$$

Moreover, for δ, W_q small enough and E' close enough to E^u, there exists some $k \in \mathbb{N}$ such that $F^k_\#$ is contracting and sends the closed ball $\bar{B}_{C^0}(E', \eta)$ into itself.

Let E^s_q be the unique fixed point of $F_\#$ in $B_{C^0}(E', \eta)$. In this way, condition (i) is clearly satisfied.

Properties (iii) and (v) follow from respectively Properties (a) and (b) above. Property (ii) is empty.

To prove property (iv), we proceed as in the proof of Proposition 7.1, step $i = 1$.   □

Step $q' + 1 \to q'$ Let us suppose the neighborhoods $(W_i)_{i=q'+1}^q$ constructed so that

– property (vi) holds,
– W_i is a neighborhood of $\overleftarrow{\Omega}_i$,

Let us proceed again as in the proof of Proposition 7.1 step $i - 1 \to i$.

We remark that $C_{q'} := W^s(\overleftarrow{\Omega}_{q'}) \setminus \overleftarrow{f}^{-2}(O_{q'})$ is compact, with $O_{q'} := \cup_{i=q'+1}^q W_i$. Moreover for every $N_{q'} \leq 0$, the following is a compact set containing $C_{q'}$:

$$Y_{q'} := \overleftarrow{f}^{N_{q'}}(M^c_{q'-1}) \setminus \overleftarrow{f}^{-2}(O_{q'}).$$

Moreover, when $-N_{q'}$ is large, $Y_{q'}$ is close to $C_{q'}$ for the Hausdorff metric. By $Y_{q'}$ *small* we mean $-N_{q'}$ large.

First, we assume $-N_{q'}$ large enough so that the set $Y_{q'}$ is included in $V_{q'}$.

By strong transversality and property (iii) of Proposition 7.1, for every $\eta > 0$, there exists K large such that for every δ and $Y_{q'}$ small, it holds:

$$\forall \underline{x} \in Y_{q'}, \quad \forall u \in \mathbb{R}^N \setminus \{0\}: \quad |\angle(u, E^s_{q'\underline{x}})| > \eta \Rightarrow \|F^\delta(u)\| \geq \|u\|/K. \tag{24}$$

Indeed, if $\underline{x} \in C_{q'}$, a unit vector u making an angle at least η with $E^s_{\underline{x}}$ has its image by F^0 not in $E^s_{\overleftarrow{f}(x)}$, by the strong tranversality condition. Hence the norm if its image is bounded from below by a certain $1/2K$. Consequently for δ and $Y_{q'}$ small inequality (24) holds.

For $\eta > 0$, let U_η be the closed subset of $C^0(Y_{q'}, G_N)$ made by sections P such that for every $\underline{x} \in Y_{q'}$ the angle between $P_{\underline{x}}$ and $E^s_{q'\underline{x}}$ is at least η.

For all η, δ and $Y_{q'}$, the following is well defined with image in $C^0(Y_{q'} \cap \overleftarrow{f}(Y_{q'}), G_N)$:

$$F_\# := U_\eta \ni P \mapsto \left[\underline{x} \in Y_{q'} \cap \overleftarrow{f}(Y_{q'}) \mapsto F^\delta_{x_0}(P_{\overleftarrow{f}^{-1}(\underline{x})}) \right].$$

Similarly, for every $k \geq 0$, for all η, δ and $Y_{q'}$, the following is well defined with image in $C^0(\cap_{i=0}^k \overleftarrow{f}^i(Y_{q'}), G_N)$:

$$F_\#^k := U_\eta \ni P \mapsto \left[\underline{x} \in \cap_{i=0}^k \overleftarrow{f}^i(Y_{q'}) \mapsto F^{\delta\,k}_{x_0}(P_{\overleftarrow{f}^{-k}(\underline{x})}) \right].$$

We remark that $\cap_{i=0}^k \overleftarrow{f}^i(Y_{q'}) = f^{N_{q'}}(M^c_{q'-1}) \setminus \overleftarrow{f}^{-2+k}(O_{q'})$ is close to $\overleftarrow{\Omega}_{q'}$, when k is large and $Y_{q'}$ is small (that is $-N_{q'}$ large). We assume $\eta > 0$ smaller than the angle between $E^s|\Omega_{q'}$ and $E^u|\Omega_{q'}$. Hence by hyperbolicity, for $Y_{q'}$ and δ sufficiently small, there exists k such that $F_\#^k$ is contracting for the C^0-metric. Note that k does not depend on η. Moreover, by hyperbolicity, if k is large enough and δ small enough, for every $\underline{x} \in \overleftarrow{f}^k(C_{q'})$, for every $P_{\underline{x}}$ η-close to $E^s_{q'\underline{x}}$, the plane $F^{\delta-k}_{\underline{x}}(P_{\underline{x}})$ is η-close to $E^s_{\overleftarrow{f}^{-k}(\underline{x})}$. Hence every $\underline{x} \in C_p$, every P making an angle greater than η with $E^s_{\underline{x}}$, the plane $F^{\delta k}_{\underline{x}}(P_{\underline{x}})$ makes an angle greater than η with $E^s_{q'\overleftarrow{f}^k(\underline{x})}$.

Consequently, for $-N_{q'}$ large enough, $F_\#^k$ takes its values in the subspace of $C^0(\cap_{i=0}^k f^i(Y_{q'}), G_N)$ formed by sections P such that for every $\underline{x} \in \cap_{i=0}^k f^i(Y_{q'})$, $P_{\underline{x}}$ makes an angle with $E^s_{q'\underline{x}}$ greater than $\eta > 0$.

However, the target space of $F_\#$ is not the same as the source space. So we cannot conclude to a fixed point. We are going to complement the sections in the image of $F_\#$ by sections obtained by the following lemma shown below:

Lemma 9.1. *For δ and $Y_{q'}$ small enough, there exists a d_1-continuous and d_∞-Lipschitz section $\tilde{E} \in U_\eta$ such that for every $j > q'$:*

$$\underline{x} \in Y_{q'} \cap \overleftarrow{f}(W_j), \quad \tilde{E}_{\underline{x}} \subset F^\delta(E^u_{j\overleftarrow{f}^{-1}(\underline{x})}).$$

Gluing $\tilde{E}$ to $F_\#\tilde{E}$ and definition of E_i^u   We remark that $\tilde{E}$ and $F_\#\tilde{E}$ are well defined on:

$$W_{q'} := \overleftarrow{f}^{N_{q'}}(int\, M^c_{q'-1}) \setminus cl(\overleftarrow{f}^{-1}(O_{q'})) \subset Y_{q'} \cap \overleftarrow{f}(Y_{q'}).$$

We remark that induction hypothesis (vi) is satisfied.

By Corollary 6.2, there exists a partition of the unity $(\rho, 1 - \rho) \in Mor_0^\infty(\overline{M}_f)^2$ subordinated to the cover $(O_{q'}, int\, \overleftarrow{f}^{-1}(O_{q'})^c)$.

Let $p_{\underline{x}}$ and $p'_{\underline{x}}$ be the orthogonal projections of $\mathbb{R}^N$ onto respectively $\tilde{E}$ and $F_\#\tilde{E}$. For $\underline{x} \in W_{q'}$, put

$$E^0_{\underline{x}} := \{(\rho(\underline{x})p_{\underline{x}} + (1 - \rho(\underline{x}))p'_{\underline{x}})(u); \ u \in \tilde{E}_{\underline{x}}\}.$$

We notice that $\underline{x} \in W_{q'} \mapsto E^0_{\underline{x}} \in G_N$ is d_1-continuous and d_∞-Lipschitz. Furthermore, for $\underline{x} \in W_{q'}$ close to $\overleftarrow{f}^{-1}(O_{q'})$, the plane $E^0_{\underline{x}}$ is equal to $\tilde{E}_{\underline{x}}$ and for $\underline{x} \in W_{q'} \setminus O_{q'}$, the plane $E^0_{\underline{x}}$ is equal to $F_\#\tilde{E}_{\underline{x}}$.

We put

$$\Pi_\eta := \{P \in C^0(W_{q'}, G_N): \ P = E^0 \text{ on } W_{q'} \setminus O_{q'} \text{ and}$$
$$P \text{ and } E^s_{q'} \text{ makes an angle greater than } \eta \text{ on } W_{q'} \setminus \overleftarrow{f}^{-2+k}(O_{q'})\}.$$

We put:

$$F^\delta_\star := P \in \Pi_\eta \mapsto \left[\underline{x} \in W_{q'} \mapsto \begin{cases} E^0_{\underline{x}} & \text{if } \underline{x} \in W_{q'} \setminus \overleftarrow{f}^{-1}(O_{q'}) \\ F^\delta_{\underline{x}}(P_{\overleftarrow{f}^{-1}(x)}) & \text{otherwise.} \end{cases} \right]$$

We notice that the map $F_\star^\delta$ takes its values in $C^0(W_{q'}, G_N)$. Moreover, from the properties of $F_\#^\delta$, the map $F_\star^\delta$ is λ-contracting and takes its values in Π_η.

Let $E_{q'}^u$ be the fixed point of $F_\star^\delta$.

By definition, $E_{q'}^u$ satisfies property (i). Similarly to the step $i = 1$, the section E_i^s satisfies Property (iv).

Also, Property (iii) is satisfied for every $P \in F_\star^{\delta^k} \Pi_\eta$, if $-N_{q'}$ is large enough and δ small enough. Hence it holds for $E_{q'}^u$. Likewise by (24), if $-N_{q'}$ is large enough and δ small enough, property (v) holds for $E_{q'}^u$. This gives a bound on K. Such a bound at this step does not depend on δ small enough.

Let us check Property (ii). We only need to check that for $j > q'$, for $\underline{x} \in \overleftarrow{f}^{-1}(W_{q'}) \cap W_j$ it holds:

$$E_{q'\overleftarrow{f}(\underline{x})}^u \subset F^\delta(E_{j\underline{x}}^u).$$

That is for every $\underline{x} \in W_{q'} \cap \overleftarrow{f}(W_j)$ it holds:

$$E_{q'\underline{x}}^u \subset F^\delta(E_{j\overleftarrow{f}^{-1}(\underline{x})}^u).$$

Let $\underline{x} \in W_{q'} \cap \overleftarrow{f}(W_i) = \overleftarrow{f}^{N_{q'}}(int\, M_{q'-1}^c) \setminus cl(\overleftarrow{f}^{-1}(O_{q'})) \cap \overleftarrow{f}(W_j)$. In particular $\underline{x}$ belongs to $W_{q'} \cap \overleftarrow{f}(O_{q'}) \setminus \overleftarrow{f}^{-1}(O_{q'})$.

If $\underline{x}$ belongs to $W_{q'} \cap O_{q'} \setminus \overleftarrow{f}^{-1}(O_{q'})$, then $E_{q'\underline{x}}^u$ is a linear sum of vectors included in $\tilde{E}_{\underline{x}}$ and $F^\delta(\tilde{E}_{\overleftarrow{f}^{-1}(\underline{x})})$. We recall that $\tilde{E}_{\underline{x}}$ is included in $F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$ by Lemma 9.1. Also $\tilde{E}_{\overleftarrow{f}^{-1}(\underline{x})}$ is included in $F^\delta(E_{j\overleftarrow{f}^{-2}(\underline{x})}^u)$ with $j \geq i$ such that $\overleftarrow{f}^{-2}(\underline{x}) \in W_j$. By (ii), $F^\delta(\tilde{E}_{\overleftarrow{f}^{-1}(\underline{x})})$ is included in $F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$. Hence $E_{q'\underline{x}}^u$ is included in $F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$.

If $\underline{x}$ belongs to $W_{q'} \cap \overleftarrow{f}(O_{q'}) \setminus O_{q'}$, then $E_{q'\underline{x}}^u$ is a linear sum of vectors included in $F^\delta(\tilde{E}_{\overleftarrow{f}^{-1}(\underline{x})})$ and $F^{\delta^2}(\tilde{E}_{\overleftarrow{f}^{-2}(\underline{x})})$. As in the previous case, $F^\delta(\tilde{E}_{\overleftarrow{f}^{-1}(\underline{x})})$ is included in $F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$. Similarly, $\tilde{E}_{\overleftarrow{f}^{-2}(\underline{x})}$ is included in $F^\delta(E_{j\overleftarrow{f}^{-3}(\underline{x})}^u)$ with $j \geq i$ such that $\overleftarrow{f}^{-3}(\underline{x}) \in W_j$. By (ii), $F^{\delta^2}(\tilde{E}_{\overleftarrow{f}^{-2}(\underline{x})})$ is included in $F^{\delta^3}(E_{j\overleftarrow{f}^{-3}(\underline{x})}^u) \subset F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$. Hence $E_{q'\underline{x}}^u$ is included in $F^\delta(E_{i\overleftarrow{f}^{-1}(\underline{x})}^u)$.

Proof of Lemma 9.1. We want to construct $\tilde{E} \in U_\eta$ which is d_1-continuous and d_∞-Lipschitz and such that for every $j > q'$:

$$\underline{x} \in Y_{q'} \cap \overleftarrow{f}(W_j), \quad E_{q'\underline{x}}^u \subset F^\delta(E_{j\overleftarrow{f}^{-1}(\underline{x})}^u).$$

By property (v), the section

$$\underline{x} \in \overleftarrow{f}(W_j) \mapsto F^\delta(E_{j\overleftarrow{f}^{-1}(\underline{x})}^u)$$

is d_1-continuous and d_∞-Lipschitz.

For every $j > q'$, let W_j' be an open neighborhood of $\overleftarrow{\Omega}_j$ with closure in W_j and such that $\cup_{j > q''} W_j'$ contains $\cup_{j > q''} W^s(\overleftarrow{\Omega}_j)$ for every $q'' \geq q'$ and (vi) is satisfied.

For every $j > q'$, let ρ_j be a dump function equal to 1 on $\overleftarrow{f}(W_j') \cup W_j'$ and with support in $\overleftarrow{f}(W_j) \cup W_j$. We remark that $(\rho_j, 1 - \rho_j)$ is a partition of the unity subordinate to the cover $(\overleftarrow{f}(W_j) \cup W_j, \overleftarrow{f}(W_j')^c \cap W_j'^c)$.

Let $p_j(\underline{x})$ be the projection of $\mathbb{R}^N$ onto $F^\delta(E_{j\overleftarrow{f}^{-1}(\underline{x})}^u)$ parallely to $E_{j\underline{x}}^s$.

Let $\tilde{E}_{q'}$ be the orthogonal of $E_{q'}^s | Y_{q'}$.

By Claim 8.1, for every j, for $Y_{q'}$ and δ-small enough, the plane $\tilde{E}_{q'}$ makes an angle greater than a certain $\eta > 0$ with $E_{q'\delta}^s(\underline{x})$. Hence its projection by p_j remains of constant dimension and so is d_1-continuous and d_∞-Lipschitz.

We now construct inductively $(\tilde{E}_j)_{j \geq q'}$.

Let $j \geq q'$ and let us suppose the section $\tilde{E}_j \in U_\eta$, d_1-continuous and d_∞-Lipschitz, constructed so that for every $i \in (q', j)$:

– for every $\underline{x} \in \overleftarrow{f}(W_i')$, the plane $\tilde{E}_{j\underline{x}}$ is included in $F^\delta(E^u_{i\overleftarrow{f}^{-1}(\underline{x})})$.

– for every $\underline{x} \in W_i'$, the plane $\tilde{E}_{j\underline{x}}$ is included in $E^u_{i\underline{x}}$.

Put:

$$\tilde{E}_{j+1\underline{x}} := \left\{ \rho_{j+1}(\underline{x}) \cdot p_{j+1}(\underline{x})(u) + (1 - \rho_{j+1}(\underline{x})) \cdot (u) : u \in \tilde{E}_{j\underline{x}} \right\}.$$

We define $\tilde{E} = \tilde{E}_q$.

We remark that by (ii):

$$\forall \underline{x} \in Y_{q'} \cap \overleftarrow{f}(W_{j+1}'), \quad E^u_{q'\underline{x}} \subset F^\delta(E^u_{j\overleftarrow{f}^{-1}(\underline{x})}).$$

Hence by replacing $(W_j)_j$ by $(W_j')_j$, Lemma 9.1 is proved. $\quad\square$

Conflict of interest statement

There are no known conflicts of interest associated with this publication and the manuscript has been read and approved by both authors.

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