Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 34 (2017) 1329–1354

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

On condensate of solutions for the Chern–Simons–Higgs equation

Chang-Shou Lin [a,*], Shusen Yan [b]

[a] *Taida Institute of Mathematical Sciences, Center for Advanced Study, National Taiwan University, Taipei 106, Taiwan*
[b] *Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia*

Received 13 February 2015; received in revised form 7 June 2016; accepted 20 October 2016

Available online 26 October 2016

Abstract

This is the first part of our comprehensive study on the structure of doubly periodic solutions for the Chern–Simons–Higgs equation with a small coupling constant. We first classify the bubbling type of the blow-up point according to the limit equations. Assuming that all the blow-up points are away from the vortex points, we prove the non-coexistence of different bubbling types in a sequence of bubbling solutions. Secondly, for the CS type bubbling solutions, we obtain an existence result without the condition on the blow-up set as in [4]. This seems to be the first general existence result of the multi-bubbling CS type solutions which is obtained under nearly necessary conditions. Necessary and sufficient conditions are also discussed for the existence of bubbling solutions blowing up at vortex points.

Keywords: Condensate; Chern–Simons–Higgs equation; Bubbling phenomena; Pohozaev identity; Uniqueness

1. Introduction

In the last decade, various Chern–Simons theories have been studied for their applications in different physics models, such as the relativistic Chern–Simons theory of superconductivity [11], Lozano–Marqueés–Moreno–Schaposnik model of bosonic sector of $N = 2$ super-symmetric Chern–Simons–Higgs theory [29], and Gudnason model of $N = 2$ super-symmetric Yang–Mills–Chern–Simons–Higgs theory [12], just to name a few. Those Chern–Simons systems, after a suitable ansatz, can be reduced to systems of elliptic partial differential equations with exponential nonlinearities. Although these nonlinear differential equations pose many analytically challenging problems and attract lots of attentions, there are still many problems unsolved. For the recently mathematical developments, we refer the readers to [1,2,5–8,13–16,18,22,23,27,28,30,33,32,37] and the references therein.

Among those non-trivial equations, the simplest one is the Abelian Chern–Simons–Higgs model proposed by Jackiw–Weinberg [19] and Hong–Kim–Pac [17]. The Chern–Simons–Higgs Lagrangian density is given by

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + D_\mu\phi\overline{D^\mu\phi} - \frac{1}{\kappa^2}|\phi|^2(1-|\phi|^2)^2,$$

* Corresponding author.
E-mail addresses: cslin@math.ntu.edu.tw (C.-S. Lin), syan@turing.une.edu.au (S. Yan).

http://dx.doi.org/10.1016/j.anihpc.2016.10.006
0294-1449/

where A_μ, $\mu = 0, 1, 2$, is the gauge field in $\mathbb{R}^3$, $F_{\mu\nu} = \frac{\partial}{\partial^\mu} A_\nu - \frac{\partial}{\partial^\nu} A_\mu$ is the curvature tensor, ϕ is the Higgs field in $\mathbb{R}^3$, $D_\mu = \frac{\partial}{\partial^\mu} - \mathrm{i}A_\mu$, $\mathrm{i} = \sqrt{-1}$, is the gauge covariant derivative associated with A_μ, $\epsilon_{\mu\nu\rho}$ is the skew symmetric tensor with $\epsilon_{012} = 0$ and the constant κ is the coupling constant. When the energy for the pair (ϕ, A) is saturated, in [19] and [17], the authors independently derived the following Bogomol'nyi type equations

$$(D_1 + \mathrm{i}D_2)\phi = 0, \tag{1.1}$$

and

$$F_{12} + \frac{2}{\kappa^2}|\phi|^2(1 - |\phi|^2)^2 = 0. \tag{1.2}$$

Following Jaffe and Taubes [20], we can reduce (1.1) and (1.2) to a single elliptic equation as follows. Let $p_1, \cdots, p_N$ be a set of points in $\mathbb{R}^2$. We introduce a real valued function u and θ by $\phi = e^{\frac{1}{2}(u+\mathrm{i}\theta)}$ and $\theta = 2\sum_{j=1}^N \arg(z - p_j)$, $z = x_1 + \mathrm{i}x_2 \in \mathbb{C}$. Then u satisfies

$$\Delta u + \frac{4}{\kappa^2} e^u(1 - e^u) = 4\pi \sum_{j=1}^N \delta_{p_j}, \quad \text{in } \mathbb{R}^2, \tag{1.3}$$

where $\delta_p(x)$ is the Dirac measure at p. The readers can find the details of the derivation of the above equations in [36, 38] and some recent developments of the related subjects in [3,9,24,31,35,36].

Starting with this paper, we will initiate a comprehensive study of the structure of doubly periodic solutions for (1.3). So we study the following equation

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^u(1 - e^u) = 4\pi \sum_{j=1}^N \delta_{p_j}, & \text{in } \Omega \\ u \text{ is doubly periodic on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\varepsilon = \frac{\kappa}{2} > 0$ is a small parameter, and Ω is a flat torus in $\mathbb{R}^2$.

Problem (1.4) involves Dirac measures. To eliminate them from the equation, we introduce the Green function $G(x, p)$ of $-\Delta$ in Ω with singularity at p, subject to the doubly periodic boundary condition. That is, $G(x, p)$ satisfies

$$\begin{cases} -\Delta G(x, p) = \delta_p - \frac{1}{|\Omega|}, & \int_\Omega G(x, p)\, dx = 0, \\ G(x, p) \text{ is doubly periodic on } \partial\Omega, \end{cases}$$

where $|\Omega|$ is the measure of Ω. Let

$$u_0(x) = -4\pi \sum_{j=1}^N G(x, p_j). \tag{1.5}$$

Using this function u_0, (1.4) is reduced to solving the following problem.

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^{u+u_0}(1 - e^{u+u_0}) = \frac{4N\pi}{|\Omega|}, & \text{in } \Omega, \\ u \text{ is doubly periodic on } \partial\Omega. \end{cases} \tag{1.6}$$

Using the maximum principle, we can find that any solution u_ε of (1.6) satisfies $u_\varepsilon + u_0 < 0$. On the other hand, integrating (1.6) leads to $\int_\Omega e^{u_\varepsilon+u_0}(1 - e^{u_\varepsilon+u_0}) = \frac{4N\pi\varepsilon^2}{|\Omega|}$, which implies either $u_\varepsilon \to -u_0$, or $u_\varepsilon \to -\infty$ almost everywhere in Ω as $\varepsilon \to 0$. In [10], Choe and Kim proved that (1.6) may have a sequence of solution u_ε, satisfying the following conditions: there is a finite set $\{x_{\varepsilon,1}, \cdots, x_{\varepsilon,k}\}$, $x_{\varepsilon,j} \in \Omega$, $j = 1, \cdots, k$, such that as $\varepsilon \to 0$,

$$u_\varepsilon(x_{\varepsilon,j}) + \ln\frac{1}{\varepsilon^2} \to +\infty, \quad \forall\, j = 1, \cdots, k, \tag{1.7}$$

and

$$u_\varepsilon(x) + \ln\frac{1}{\varepsilon^2} \to -\infty, \quad \text{uniformly on any compact subset of } \Omega \setminus \{q_1, \cdots, q_k\}, \tag{1.8}$$

where $q_j = \lim_{\varepsilon\to+\infty} x_{\varepsilon,j}$. Moreover,

$$\frac{1}{\varepsilon^2} e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) \to \sum_{j=1}^{k} M_j \delta_{q_j}, \quad M_j \ge 8\pi,$$

in the sense of measure.

Solution satisfying (1.7) and (1.8) is called a blow up solution, or a bubbling solution, while q_j is called a blow-up point of this bubbling solution. Let us define the local strength of a bubbling solution u_ε at q_i as follows:

$$M_{\varepsilon,i} = \frac{1}{\varepsilon^2} \int_{B_\delta(x_{\varepsilon,i})} e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}), \quad i=1,\cdots,k, \tag{1.9}$$

where $x_{\varepsilon,i} \in B_\delta(q_i)$ is a point such that $u_\varepsilon(x_{\varepsilon,i}) = \max_{y\in B_\delta(q_i)} u_\varepsilon(y)$. Note that for any $\delta > \delta_1 > 0$, it holds

$$\frac{1}{\varepsilon^2} \int_{B_\delta(x_{\varepsilon,i})\setminus B_{\delta_1}(x_{\varepsilon,i})} e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) \to 0$$

as $\varepsilon \to 0$. So $\lim_{\varepsilon\to 0} M_{\varepsilon,j}$ does not depend on $\delta > 0$.

At each blow-up point q_j, after a suitable scaling, the solutions converge to an entire solution u to either

$$\Delta u + |x|^{2m} e^u = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2m} e^u = M_j =: \lim_{\varepsilon\to 0} M_{\varepsilon,j}, \tag{1.10}$$

or

$$\Delta u + |x|^{2m} e^u(1-|x|^{2m}e^u) = 0, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2m} e^u(1-|x|^{2m}e^u) = M_j =: \lim_{\varepsilon\to 0} M_{\varepsilon,j}, \tag{1.11}$$

where $m = 0$ if q_i is not a vortex point, while $m = \#\{p_j : p_j = q_i\}$ if q_i is a vortex point. So we find that the type of the blow-up point q_i is determined by the local strength $M_{\varepsilon,i}$.

A blow-up point q_j is called the mean field type, or MF type, if the limit equation is (1.10), while it is called Chern–Simons type, or CS type, if the limit equation is (1.11). All the entire solutions of (1.10) have been classified in [34]. But for $m \ne 0$, (1.10) has non-radial solutions and this phenomenon makes the bubbling behaviors of solutions for (1.6) as $\varepsilon \to 0$ more complicated. For (1.11), however, the classification has been done only for $m = 0$, and for radial solution if $m > 0$. Indeed, for any solution u of (1.11) with $e^u \in L^1(\mathbb{R}^2)$, the question of radial symmetry has remained an open problem for many years.

In this paper, we will consider the following issues:

(i) Do the MF type blow-up point and the CS type blow-up point co-exist in a sequence of bubbling solutions?
(ii) What are the necessary and sufficient conditions for the set $\{q_1, \cdots, q_k\}$ to be a blow-up set of a sequence of bubbling solutions?

In a forthcoming paper, we will consider another two important issues.

(iii) Local uniqueness: Suppose $u_{n,i}$, $i = 1, 2$, are two sequences of blow-up solutions and they blow up at the same set $\{q_1, \cdots, q_k\}$. Is $u_{n,1} = u_{n,2}$ for large n?
(iv) The exact number of solutions for (1.6).

For problem (i), we have the following result.

Theorem 1.1. *Suppose that u_ε is a sequence of bubbling solutions for (1.6), whose blow-up set is $\{q_1, \cdots, q_k\}$ as $\varepsilon \to 0$. If $q_i \notin \{p_1, \cdots, p_N\}$, $i = 1, \cdots, k$, then $M_{\varepsilon,i} = \frac{4\pi N}{k} + o(1)$, $i = 1, \cdots, k$. Moreover, all the q_i are of mean field type if $N = 2k$, while all the q_j are of Chern–Simon type if $N > 2k$.*

The key step in the proof of Theorem 1.1 is to prove that each blow-up point q_j of u_ε, $j = 1, \cdots, k$, must be simple. Since the type of the bubble at q_j is determined by the local strength $M_{\varepsilon,j}$, a consequence of the simple blow-up is that the local strength at each q_j must be compatible, which implies the non-coexistence of different type of bubbles.

Now we discuss the existence of CS type multi-bubbling solutions for (1.6). Set $\mathbf{q} = (q_1, \cdots, q_k)$, $q_j, \in \Omega$. We assume that $q_j \notin \{p_1, \cdots, p_N\}$, $j = 1, \cdots, k$. Following [5,6], we can prove by using a Pohozaev identity that if $\{q_1, \cdots, q_k\}$ is a blow-up set of a sequence of bubbling solutions, then $\mathbf{q}$ must satisfy

$$\frac{\partial u_0(x)}{\partial x_h}\Big|_{x=q_j} = -\sum_{l\neq j,\, 1\le l\le k} M_l \frac{\partial G(q_l, x)}{\partial x_h}\Big|_{x=q_j}, \quad h = 1, 2,\ j = 1, \cdots, k,$$

where $M_i = \lim_{\varepsilon\to 0} M_{\varepsilon,i}$. See Lemma 2.2. By Theorem 1.1, $\mathbf{q}$ must satisfy

$$\nabla u_0(q_j) + \frac{4\pi N}{k} \sum_{i\neq j,\, 1\le i\le k} \nabla_{q_j} G(q_i, q_j) = 0 \tag{1.12}$$

for $j = 1, \cdots, k$. It is easy to check that any $\mathbf{q}$ satisfying (1.12) must be a critical point of the function $G_k(\mathbf{x})$ defined as follows.

$$G_k(\mathbf{x}) = \frac{2\pi N}{k} \sum_{i\neq j, 1\le i,\, j\le k} G(x_i, x_j) + \sum_{j=1}^{k} u_0(x_j), \quad \mathbf{x} = (x_1, \cdots, x_k),\ x_j \in \Omega. \tag{1.13}$$

We remark that (1.12) holds no matter whether the blow-up point is MF type, or CS type.

Our second result in this paper is the sufficient counterpart of the above result for the CS type bubbling solutions.

Theorem 1.2. *Suppose that $N > 2k$. If $\mathbf{q}$ satisfies (1.12), $deg(DG_k(\mathbf{q}), 0) \neq 0$ and $q_j \notin \{p_1, \cdots, p_N\}$, then for $\varepsilon > 0$ small, (1.6) has a CS type bubbling solution u_ε, blowing up at $\{q_1, \cdots, q_k\}$ as $\varepsilon \to 0$.*

If $N = 2k$, MF type bubbling solutions are constructed in [25] under an extra sign condition on a quantity D. In a forthcoming paper, we will prove that this sign condition on D is necessary for the existence. This shows that not all the non-degenerate critical points of $G_k(\mathbf{q})$ can generate a MF type bubbling solution for (1.6). This is a striking difference between the MF type bubbling solutions and the CS type bubbling solutions.

In view of Theorem 1.1 and (1.12), all the conditions in Theorem 1.2, except the non-degeneracy condition $deg(DG_k(\mathbf{q}), 0) \neq 0$, are necessary. If $k = 1$, Theorem 1.2 was proved in [26]. By Theorem 1.1, the strength of the bubble at q_j is close to $\frac{4\pi N}{k}$. In the construction of the bubbling solutions in Theorem 1.2, if we take the strength of the bubble at q_j exactly $\frac{4\pi N}{k}$ as in [4], then the following condition needs to be imposed:

$$u_0(q_j) + \frac{4\pi}{k}\Big(\gamma(q_j, q_j) + \sum_{i\neq j,\, 1\le i\le k} G(q_i, q_j)\Big) \quad \text{is independent of } j, \tag{1.14}$$

where $\gamma(y, x)$ is the regular part of the Green function $G(y, x)$. See [4] for a similar construction of bubbling solutions in $\mathbb{R}^2$. Let us point out that (1.14) comes from the uniqueness of entire solutions of the limit problem in $\mathbb{R}^2$. It seems that (1.12) and (1.14) can not hold true simultaneously for most of the configuration $\{p_1, \cdots, p_N\}$. The question whether (1.14) is really needed for the existence of *doubly periodic* bubbling solutions has puzzled us for long time. We realize now that fortunately (1.11) has a solution for all $M_j > 8\pi$ if $m = 0$. Therefore we can choose the strength $M_{\varepsilon,i}$ at each point q_i suitably close to $\frac{4\pi N}{k}$ so that the balance condition away from the blow-up set in the construction of an approximate solution for (1.6) can be matched and thus obtain the existence result without condition (1.14). Let us point out that by doing so, the condition "$\sum_{k\neq j} \ln|p_j - p_k|$ is independent of j" used in [4] is not needed to obtain the existence result there. In fact, the following result can be proved by using the same method as in Section 3.

Theorem 1.3. *For any $p_{i_j} \in \{p_1, \cdots, p_N\}$, $j = 1, \cdots, k$, such that the weight of each p_{i_j} is one, (1.3) has a non-topological solution u_κ in $\mathbb{R}^2$ for $\kappa > 0$ small, satisfying*

$$\frac{4}{\kappa^2} e^{u_\kappa}(1 - e^{u_\kappa}) \to \frac{\beta}{k} \sum_{j=1}^{k} \delta_{p_{i_j}}$$

as $\kappa \to 0$ provided $\beta > 16\pi k$.

The readers can compare Theorem 1.3 with Theorem 3.2 obtained in section 3.

In this paper, we also discuss the necessary and sufficient conditions for the existence of bubbling solutions whose blow-up set contains vortex points. Let

$$q_i \in \{p_1, \cdots, p_N\},\ i = 1, \cdots t, \quad q_j \notin \{p_1, \cdots, p_N\},\ j = t+1, \cdots k, \tag{1.15}$$

where $0 < t \le k$ is an integer. Define

$$\begin{aligned} G_k^*(\mathbf{x}) = &\frac{2\pi N - 2\pi t}{k} \sum_{i \neq j, t+1 \le i,\ j \le k} G(x_i, x_j) \\ &+ \frac{4\pi N + 4\pi(k-t)}{k} \sum_{i=1}^{t} \sum_{j=t+1}^{k} G(q_i, x_j) + \sum_{j=t+1}^{k} u_0(x_j). \end{aligned} \tag{1.16}$$

Then, we have

Theorem 1.4. *Suppose the weight m_i of the vortex q_i is one for $i = 1, \cdots, t$, and $\lim_{\varepsilon\to 0} M_{\varepsilon,i} > 16\pi$, $i = 1, \cdots, t$. Let u_ε be a solution of (1.6) whose blow-up set is given in (1.15). Then $M_{\varepsilon,i} = \frac{4\pi N + 4\pi(k-t)}{k} + o(1)$, $i = 1, \cdots, t$, $M_{\varepsilon,i} = \frac{4\pi N - 4\pi t}{k} + o(1)$, $i = t+1, \cdots, k$, and each q_j is CS type. Moreover, $DG_k^* = 0$ at $(q_{t+1}, \cdots, q_k)$. Conversely, if $\mathbf{q} = (q_{t+1}, \cdots, q_k)$ satisfies $DG_k^*(\mathbf{q}) = 0$, $deg(DG_k^*(\mathbf{q}), 0) \neq 0$ and $k < \frac{1}{3}(N-t)$, then for $\varepsilon > 0$ small, (1.6) has a solution u_ε whose blow-up set is given by (1.15) as $\varepsilon \to 0$.*

When a blow-up point q_j is a vortex point, the asymptotic behavior of u_ε as $\varepsilon \to 0$ near q_j becomes more complicated. For example, the simple blow-up property of the solution near a blow-up point may not hold true in general. This complication will cause the problem of non-coexistence of bubbles more difficult to study. Theorem 1.4 only gives a result for the non-coexistence of bubbles if the weight of the vortex point is one. On the other hand, in the construction of bubbling solutions whose blow-up set contains some vortex points, what we really need is the non-degeneracy of the radial solution of the corresponding limit problem at vortex point. This non-degeneracy condition is proved in [4] if the weight of the vortex point is one. But it is still an open problem if the weight of the vortex point is bigger than one.

This paper is organized as follows. In section 2, we will discuss the simple blow-up problem for the bubbling solutions and thus prove Theorem 1.1 and the necessary part of Theorem 1.4. Theorem 1.2 and the sufficient part of Theorem 1.4 are proved in section 3.

2. Non-coexistence of different bubbles

In the section, we will study the non-coexistence of different bubbles for the bubbling solutions u_ε of (1.6), and prove Theorem 1.1 and the necessary part in Theorem 1.4.

Lemma 2.1. *Let u_ε be a bubbling solution of (1.6) satisfying (1.7) and (1.8). Then for any small $\theta > 0$,*

$$u_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \to \sum_{i=1}^{k} M_i G(q_i, x), \quad in\ C^1(\Omega \setminus \cup_{i=1}^k B_\theta(q_i)), \tag{2.1}$$

as $\varepsilon \to 0$, where $M_i = \lim_{\varepsilon\to 0} M_{\varepsilon,i}$.

Proof. From (1.8), we find that for any $\theta \in (0, \delta)$,

$$\frac{1}{\varepsilon^2} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}) = o(1), \quad \text{in } \Omega \setminus \cup_{i=1}^k B_\theta(q_i). \tag{2.2}$$

So,

$$\frac{1}{\varepsilon^2} \int_{B_\theta(q_i)} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}) = M_{\varepsilon,i} + o(1) = M_i + o(1), \quad i = 1, \cdots, k. \tag{2.3}$$

As a result,

$$
\begin{aligned}
&u_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon = \frac{1}{\varepsilon^2}\int_\Omega G(y,x)e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0})\,dy \\
&= \sum_{i=1}^k M_i G(q_i,x) + \frac{1}{\varepsilon^2}\sum_{i=1}^k \int_{B_\delta(q_i)} \big(G(y,x)-G(q_i,x)\big)e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0})\,dy + o(1) \\
&= \sum_{i=1}^k M_i G(q_i,x) + o(1),
\end{aligned} \tag{2.4}
$$

uniformly in $\Omega \setminus \cup_{i=1}^k B_\theta(q_i)$ as $\varepsilon \to 0$.

Similarly, we can prove that

$$
Du_\varepsilon(x) \to \sum_{i=1}^k M_i DG(q_i,x), \tag{2.5}
$$

uniformly in $\Omega \setminus \cup_{i=1}^k B_\theta(q_i)$ as $\varepsilon \to 0$. □

Lemma 2.2. *Let u_ε be a bubbling solution of (1.6) whose blow-up set is $\{q_1,\cdots,q_k\}$, $q_j \notin \{p_1,\cdots,p_N\}$, $j=1,\cdots,k$. Then*

$$
\frac{\partial u_0(x)}{\partial x_l}\Big|_{x=q_j} = -\sum_{t\neq j,\,1\le t\le k} M_t \frac{\partial G(q_t,x)}{\partial x_l}\Big|_{x=q_j}, \quad l=1,2,\ j=1,\cdots,k. \tag{2.6}
$$

Proof. For $j=1,\cdots,k$, let

$$
\bar u_\varepsilon(x) = u_\varepsilon - \frac{\pi N|x-q_j|^2}{|\Omega|}.
$$

Then

$$
\Delta \bar u_\varepsilon + \frac{1}{\varepsilon^2}h(x)e^{\bar u_\varepsilon+u_0}(1-h(x)e^{\bar u_\varepsilon+u_0}) = 0, \tag{2.7}
$$

where $h(x) = e^{\frac{\pi N|x-q_j|^2}{|\Omega|}}$. For $l=1,2$, we have the following Pohozaev identity for $\bar u_\varepsilon$:

$$
\begin{aligned}
&\int_{\partial B_r(q_j)} \langle \nu, D\bar u_\varepsilon\rangle D_l \bar u_\varepsilon - \frac{1}{2}\int_{\partial B_r(q_j)} |D\bar u_\varepsilon|^2 \nu_l \\
&= -\frac{1}{\varepsilon^2}\int_{\partial B_r(q_j)} \Big(e^{u_\varepsilon+u_0} - \frac{1}{2}e^{2(u_\varepsilon+u_0)}\Big)\nu_l + \frac{1}{\varepsilon^2}\int_{B_r(q_j)} e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0})\Big(D_l u_0 + \frac{D_l h(x)}{h(x)}\Big),
\end{aligned} \tag{2.8}
$$

where ν is the outward unit normal of $\partial B_r(q_j)$.

By (1.8), noting that $Dh(q_j)=0$, we can prove that

$$
\text{RHS of (2.8)} \to M_j D_l u_0(q_j). \tag{2.9}
$$

Using (2.1), we find

$$
\begin{aligned}
\bar u_\varepsilon &= \frac{1}{|\Omega|}\int_\Omega u_\varepsilon + \sum_{t=1}^k M_t G(q_t,x) - \frac{N\pi|x-q_j|^2}{|\Omega|} + o_\varepsilon(1) \\
&:= F(x,q) + o_\varepsilon(1), \quad x \in \partial B_r(q_j),
\end{aligned} \tag{2.10}
$$

and

$$\Delta F(x,q)=0, \quad x\in B_r(q_j)\setminus\{q_j\}. \tag{2.11}$$

So, for any small $\theta>0$,

$$\begin{aligned}
\text{LHS of (2.8)} &= \int_{\partial B_r(q_j)} \langle \nu, DF(x,q)\rangle D_l F(x,q) - \frac{1}{2}\int_{\partial B_r(q_j)} |DF(x,q)|^2\nu_l + o_\varepsilon(1)\\
&= \int_{\partial B_\theta(q_j)} \langle \nu, DF(x,q)\rangle D_l F(x,q) - \frac{1}{2}\int_{\partial B_\theta(q_j)} |DF(x,q)|^2\nu_l + o_\varepsilon(1)\\
&= \int_{\partial B_\theta(q_j)} \langle \nu, D\sum_{t=1}^k M_t G(q_t,x)\rangle D_l \sum_{t=1}^k M_t G(q_t,x) - \frac{1}{2}\int_{\partial B_\theta(q_j)} |D\sum_{t=1}^k M_t G(q_t,x)|^2\nu_l + o_\theta(1) + o_\varepsilon(1)\\
&= -M_j \sum_{t\neq j,\, 1\le t\le k} M_t D_l G(q_t,q_j) + o_\theta(1)+o_\varepsilon(1),
\end{aligned} \tag{2.12}$$

which, together with (2.9), gives (2.6). □

Our next result shows that if the weight $m=\#\{p_j : p_j=p\}$ of a vortex point p is large, then p can not be a blow-up point. This is due to the energy constraint. In fact we have

Proposition 2.3. *Suppose that the weight m_j of the vortex point p_j satisfies $2(m_j+1)>N$. Then, p_j can not be a blow-up point of a bubbling solution u_ε of (1.6).*

Proof. We argue by contradiction. Suppose that a vortex point p, whose weight is m, is a blow-up point of a sequence of bubbling solution u_ε of (1.6). Let

$$\bar u_\varepsilon(x) = u_\varepsilon - \frac{\pi N|x-p|^2}{|\Omega|}.$$

Then

$$\Delta \bar u_\varepsilon + \frac{1}{\varepsilon^2} h(x) e^{\bar u_\varepsilon + u_0}(1-h(x)e^{\bar u_\varepsilon+u_0}) = 0, \tag{2.13}$$

where $h(x)=e^{\frac{\pi N|x-q|^2}{|\Omega|}}$.

We have the following Pohozaev identity in a small disk $B_r(p)$ for (2.13):

$$\begin{aligned}
&\frac{1}{2}\int_{\partial B_r(p)} \langle x-p,\nu\rangle |D\bar u_\varepsilon|^2 - \int_{\partial B_r(p)} \langle x-p, D\bar u_\varepsilon\rangle\langle \nu, D\bar u_\varepsilon\rangle\\
&= -\frac{1}{\varepsilon^2}\sum_{i=1}^2 \int_{B_r(p)} \Big(\frac{\partial((x_i-p_i)h(x)e^{u_0})}{\partial x_i} e^{\bar u_\varepsilon} - \frac{1}{2}\frac{\partial((x_i-p_i)h^2(x)e^{2u_0})}{\partial x_i} e^{2\bar u_\varepsilon}\Big)\\
&\quad + \frac{1}{\varepsilon^2}\int_{\partial B_r(p)} \langle x-p,\nu\rangle\Big(e^{u_0+u_\varepsilon} - \frac{1}{2}e^{2(u_0+u_\varepsilon)}\Big),
\end{aligned} \tag{2.14}$$

where ν is the outward unit normal of $\partial B_r(p)$.

From (1.8), we see

$$\frac{1}{\varepsilon^2}\int_{\partial B_r(p)} \langle x-p,\nu\rangle\Big(e^{u_0+u_\varepsilon} - \frac{1}{2}e^{2(u_0+u_\varepsilon)}\Big) = o_\varepsilon(1). \tag{2.15}$$

Recall that the function $F(x,q)$ is defined in (2.10). Using (2.11), we find

$$\begin{aligned}
&\frac{1}{2}\int_{\partial B_r(p)}\langle x-p,\nu\rangle|Du_\varepsilon|^2-\int_{\partial B_r(p)}\langle x-p,Du_\varepsilon\rangle\langle\nu,Du_\varepsilon\rangle\\
&=\frac{1}{2}\int_{\partial B_r(p)}\langle x-p,\nu\rangle|DF(x,p)|^2-\int_{\partial B_r(p)}\langle x-p,DF(x,p)\rangle\langle\nu,DF(x,p)\rangle+o_\varepsilon(1)\\
&=\frac{1}{2}\int_{\partial B_\theta(p)}\langle x-p,\nu\rangle|DF(x,p)|^2-\int_{\partial B_\theta(p)}\langle x-p,DF(x,p)\rangle\langle\nu,DF(x,p)\rangle+o_\varepsilon(1)\\
&=-\frac{M_j^2}{4\pi}+o_\theta(1)+o_\varepsilon(1).
\end{aligned}\tag{2.16}$$

Write $e^{u_0}=|x-p|^{2m}g(x)$. Then $g(x)$ is a smooth function satisfying $0<g_1\le g(x)\le g_2<+\infty$ in $B_r(p)$ if $r>0$ is small. So, we have

$$\sum_{i=1}^{2}\frac{\partial[(x_i-p_i)h(x)e^{u_0}]}{\partial x_i}=(2m+2)h(x)e^{u_0}+\sum_{i=1}^{2}(x_i-p_i)|x-p|^{2m}\frac{\partial[g(x)h(x)]}{\partial x_i}$$

and

$$\sum_{i=1}^{2}\frac{\partial[(x_i-p_i)h^2(x)e^{2u_0}]}{\partial x_i}=(4m+2)h^2(x)e^{2u_0}+\sum_{i=1}^{2}(x_i-p_i)|x-p|^{4m}\frac{\partial[g^2(x)h^2(x)]}{\partial x_i}.$$

By Proposition 4.1 in [10], there exists a constant $c>0$, such that $u_\varepsilon+u_0\le -c$. So, we find

$$\frac{1}{\varepsilon^2}\int_\Omega e^{2(u_\varepsilon+u_0)}\le\frac{1}{\varepsilon^2}\int_\Omega e^{u_\varepsilon+u_0}\le\frac{C}{\varepsilon^2}\int_\Omega e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0})\le C',\tag{2.17}$$

which implies

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\sum_{i=1}^{2}\int_{B_r(p)}\frac{\partial((x_i-p_i)h(x)e^{u_0})}{\partial x_i}e^{\bar u_\varepsilon}\\
&=-\frac{2m+2}{\varepsilon^2}\int_{B_r(p)}e^{u_0+u_\varepsilon}+O\Big(\frac{1}{\varepsilon^2}\int_{B_r(p)}|x-p|e^{u_0+u_\varepsilon}\Big)\\
&=-\frac{2m+2}{\varepsilon^2}\int_{B_r(p)}e^{u_0+u_\varepsilon}+o_r(1),
\end{aligned}\tag{2.18}$$

and

$$\frac{1}{2\varepsilon^2}\sum_{i=1}^{2}\int_{B_r(p)}\frac{\partial((x_i-p_i)h^2(x)e^{2u_0})}{\partial x_i}e^{2\bar u_\varepsilon}=\frac{2m+1}{\varepsilon^2}\int_{B_r(p)}e^{2(u_0+u_\varepsilon)}+o_r(1).\tag{2.19}$$

So we obtain from (2.14)–(2.19) that

$$\begin{aligned}
\frac{M_j^2}{4\pi}&=\frac{2m+2}{\varepsilon^2}\int_{B_r(p)}e^{u_0+u_\varepsilon}-\frac{2m+1}{\varepsilon^2}\int_{B_r(p)}e^{2(u_0+u_\varepsilon)}+o_r(1)+o_\varepsilon(1)\\
&\ge\frac{2m+2}{\varepsilon^2}\int_{B_r(p)}e^{u_0+u_\varepsilon}\big(1-e^{u_0+u_\varepsilon}\big)+o_r(1)+o_\varepsilon(1)\\
&=2(m+1)M_j+o_r(1)+o_\varepsilon(1),
\end{aligned}\tag{2.20}$$

which implies $M_j\ge 8\pi(m+1)$. This is a contradiction to $M_j\le 4N\pi$ and the assumption $2(m+1)>N$. □

Now we discuss the local behavior of a bubbling solution near a blow-up point $q_j \notin \{p_1, \cdots, p_N\}$. We will prove that q_j must be simple. That is, the bubbling solution u_ε is controlled by a single bubble near q_j. Define

$$\beta_{\varepsilon,j} = \max_{B_\delta(q_j)} u_\varepsilon. \tag{2.21}$$

Let $x_{\varepsilon,j} \in B_\delta(q_j)$ be a point satisfying

$$u_\varepsilon(x_{\varepsilon,j}) = \max_{B_\delta(q_j)} u_\varepsilon. \tag{2.22}$$

Then, $x_{\varepsilon,j} \to q_j$ as $\varepsilon \to 0$.

Lemma 2.4. *We have*

$$\beta_{\varepsilon,j} \le C < +\infty. \tag{2.23}$$

Proof. Suppose that there exists $\varepsilon_n \to 0$, such that $\beta_n = \beta_{\varepsilon_n,j} \to +\infty$. Let

$$\bar{u}_n(x) = u_n\Big(\frac{\varepsilon_n}{e^{\beta_n}}x + x_n\Big) - \max_{B_\delta(q_j)} u_n,$$

where $x_n = x_{\varepsilon_n,j}$. Then, $\bar{u}_n \le 0$ in $B_{\frac{\delta e^{\beta_n}}{\varepsilon_n}}(0)$. It is easy to see that $\bar{u}_n$ satisfies

$$-\Delta \bar{u}_n = \frac{1}{e^{\beta_n}} e^{\bar{u}_n + u_0(x_n+\varepsilon_n e^{-\beta_n}x)} - e^{2(\bar{u}_n+u_0(x_n+\varepsilon_n e^{-\beta_n}x))} - \frac{4\pi N\pi\varepsilon_n^2}{e^{2\beta_n}|\Omega|}. \tag{2.24}$$

Since $\bar{u}_n \le 0$ in $B_{\frac{\delta e^{\beta_n}}{\varepsilon_n}}(0)$, we may assume that $\bar{u}_n \to u$ in $C^1_{loc}(\mathbb{R}^2)$, and u satisfies

$$-\Delta u + e^{2u_0(q_j)}e^{2u} = 0, \quad \text{in } \mathbb{R}^2, \tag{2.25}$$

and $u \le 0$. But (2.25) has no non-positive solution. This is a contradiction. □

It follows from Lemma 2.4 that there are two different cases: (i) $\beta_{\varepsilon,j} \to -\infty$; (ii) $\beta_{\varepsilon,j} \ge C > -\infty$. In case (i), by (1.7), $\frac{1}{\varepsilon^2}e^{\beta_{\varepsilon,j}} \to +\infty$. Let

$$\tilde{u}_{\varepsilon,j}(x) = u_\varepsilon\Big(\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}}x + x_{\varepsilon,j}\Big) - \max_{B_\delta(q_j)} u_\varepsilon. \tag{2.26}$$

Then,

$$-\Delta \tilde{u}_{\varepsilon,j} = e^{\tilde{u}_{\varepsilon,j}+u_0(x_{\varepsilon,j}+\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}x)} - e^{\beta_{\varepsilon,j}}e^{2(\tilde{u}_{\varepsilon,j}+u_0(x_{\varepsilon,j}+\varepsilon e^{-\beta_{\varepsilon,j}}x))} - \frac{4\pi N\pi\varepsilon^2}{e^{\beta_{\varepsilon,j}}|\Omega|}. \tag{2.27}$$

Lemma 2.5. *Suppose that $\beta_{\varepsilon,j} \to -\infty$ as $\varepsilon \to 0$. Then $\tilde{u}_{\varepsilon,j}(x) \to u_j$ in $C^1_{loc}(\mathbb{R}^2)$ and u_j satisfies*

$$\begin{cases} -\Delta u_j = e^{u_0(q_j)}e^{u_j}, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_j} < +\infty, & u_j(0) = 0. \end{cases} \tag{2.28}$$

Moreover,

$$\frac{1}{\varepsilon^2}\int_{B_{R\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j})} e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) = 8\pi + o_R(1) + o_\varepsilon(1), \tag{2.29}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ and $o_R(1) \to 0$ as $R \to +\infty$.

Proof. Since $\tilde{u}_{\varepsilon,j}(x) \le 0$, $\tilde{u}_{\varepsilon,j}(0) = 0$, and $\tilde{u}_{\varepsilon,j}$ satisfies (2.27), we can assume that $\tilde{u}_{\varepsilon,j}(x) \to u_j$ in $C^1_{loc}(\mathbb{R}^2)$. From (2.27), we find that u_j satisfies (2.28). From $\int_{\mathbb{R}^2} e^{u_0(q_j)}e^{u_j} = 8\pi$, we obtain (2.29). □

Lemma 2.6. *Suppose that $\beta_{\varepsilon,j} \ge C > -\infty$ as $\varepsilon \to 0$. Then $\tilde{u}_{\varepsilon,j}(x) \to u_j$ in $C^1_{loc}(\mathbb{R}^2)$ and u_j satisfies*

$$\begin{cases} -\Delta u_j = e^{u_0(q_j)} e^{u_j} - e^{\beta_j + 2u_0(q_j)} e^{2u_j}, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} \big(e^{u_j} - e^{\beta_j + u_0(q_j)} e^{2u_j}\big) < +\infty, & u_j(0) = 0, \end{cases} \tag{2.30}$$

where $\beta_j = \lim_{\varepsilon \to 0} \beta_{\varepsilon,j}$. Moreover, if $\varepsilon > 0$ is small and $R > 0$ is large,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int\limits_{B_{R\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j})} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}) > 8\pi. \tag{2.31}$$

Proof. The proof is similar to that of Lemma 2.5. □

Lemma 2.7. *For $R > 0$ large and y satisfying $\delta \ge |y - x_{\varepsilon,j}| > R\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}$, it holds*

$$\frac{1}{\varepsilon^2} \int\limits_{B_{\frac{1}{2}|y - x_{\varepsilon,j}|}(y)} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}) = o_R(1) + o_\varepsilon(1), \tag{2.32}$$

where $o_R(1) \to 0$ as $R \to +\infty$.

Proof. It is obvious that (2.32) holds if $|y - x_{\varepsilon_n,j}| \ge c' > 0$. We argue by contradiction. Suppose that there are $\varepsilon_n \to 0$, $R_n \to +\infty$ and x_n with $|x_n - x_{\varepsilon_n,j}| > R_n \varepsilon_n e^{-\frac{1}{2}\beta_{\varepsilon_n,j}}$, $|x_n - x_{\varepsilon_n,j}| \to 0$, satisfying

$$\frac{1}{\varepsilon_n^2} \int\limits_{B_{\frac{1}{2}|x_n - x_{\varepsilon,j}|}(x_n)} e^{u_{\varepsilon_n} + u_0}(1 - e^{u_{\varepsilon_n} + u_0}) \ge c_0 > 0. \tag{2.33}$$

Let $u_n^*(x) = u_{\varepsilon_n}(\delta_n x + x_{\varepsilon_n,j})$, $\delta_n = |x_n - x_{\varepsilon_n,j}|$. Then, u_n^* satisfies

$$\Delta u_n^* + \frac{\delta_n^2}{\varepsilon_n^2} e^{u_n^* + u_0^*}(1 - e^{u_n^* + u_0^*}) = \frac{4N\pi \delta_n^2}{|\Omega|}, \tag{2.34}$$

where $u_0^*(x) = u_0(\delta_n x + x_{\varepsilon_n,j})$. Then, we can do the blow-up analysis for the sequence u_n^* as in [10] and prove that there is a finite set $S^* = \{z_1, \cdots, z_l\}$, such that

$$\frac{\delta_n^2}{\varepsilon_n^2} e^{u_n^* + u_0^*}(1 - e^{u_n^* + u_0^*}) \to \sum_{j=1}^{l} m_j^* \delta_{z_j}, \quad m_j^* \ge 8\pi, \tag{2.35}$$

in the sense of measure. By (2.33), S^* contains at least two points. Moreover, using the Pohozaev identity as in Lemma 2.2, we can prove that z_j satisfies

$$\sum_{i \ne j} \frac{m_j^*(z_j - z_i)}{|z_j - z_i|^2} = 0, \quad j = 1, \cdots, l. \tag{2.36}$$

It is easy to see that (2.36) can not hold true for z_j with $|z_j| = \max_i |z_i|$. So the lemma is proved. □

Next, we will prove the following result, which shows that the blow up must be simple.

Proposition 2.8. *Suppose that $q_j \notin \{p_1, \cdots, p_N\}$. Let $\delta > 0$ be a small constant. Then there exists a constant $C > 0$, such that*

$$\Big|u_\varepsilon(x) - u_\varepsilon(x_{\varepsilon,j}) - u_{\varepsilon,j}\Big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}(x - x_{\varepsilon,j})\Big)\Big| \le C, \quad \forall\, x \in B_\delta(x_{\varepsilon,j}), \tag{2.37}$$

where $u_{\varepsilon,j}$ is the solution of

$$\begin{cases} \Delta u + e^{u_0(x_{\varepsilon,j})} e^u (1 - e^{\beta_{\varepsilon,j} + u_0(x_{\varepsilon,j})} e^u) = 0, & \text{in } \mathbb{R}^2; \\ \int_{\mathbb{R}^2} e^{u_0(x_{\varepsilon,j})} e^u (1 - e^{\beta_{\varepsilon,j} + u_0(x_{\varepsilon,j})} e^u) = M_{\varepsilon,j}, & \lim_{\varepsilon \to 0} M_{\varepsilon,j} > 8\pi, \end{cases} \tag{2.38}$$

if $\beta_{\varepsilon,j} \ge -c_0 > -\infty$; while $u_{\varepsilon,j}$ is the solution of

$$\begin{cases} \Delta u + e^{u_0(x_{\varepsilon,j})} e^u = 0, & \text{in } \mathbb{R}^2; \\ \int_{\mathbb{R}^2} e^{u_0(x_{\varepsilon,j})} e^u = 8\pi, & u(0) = 0, \end{cases} \tag{2.39}$$

if $\beta_{\varepsilon,j} \to -\infty$.

Estimate (2.37) is proved in [21] for equation $\Delta u + V(x)e^u = 0$ by using a moving plane method. But it seems that the moving plane method used in [21] does not work for (2.24). Here, we will give a more direct proof of Proposition 2.8. We first prove some lemmas. Recall that

$$\tilde{u}_{\varepsilon,j}(x) = u_\varepsilon\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}} x + x_{\varepsilon,j}\big) - \max_{B_\delta(q_j)} u_\varepsilon.$$

Lemma 2.9. *For any small $\theta > 0$, there exists a constant $C > 0$, such that*

$$\tilde{u}_{\varepsilon,j}(x) \le -(4-\theta)\ln|x| + C, \quad \forall x \in B_{\frac{\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0) \setminus B_1(0).$$

Proof. It follows from (2.27) that

$$\tilde{u}_{\varepsilon,j}(x) - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \tilde{u}_{\varepsilon,j} = \int_{\Omega_\varepsilon} G_\varepsilon(y,x) \Big(e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} \Big)\, dy, \tag{2.40}$$

where $G_\varepsilon(y,x)$ is the Green function in $\Omega_\varepsilon = \big\{ y : \varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}} y + x_{\varepsilon,j} \in \Omega \big\}$ subject to the doubly periodic boundary condition, and $\tilde{u}_0(x) = u_0\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}} x + x_{\varepsilon,j}\big)$.

From $\tilde{u}_{\varepsilon,j}(0) = 0$, we obtain from (2.40)

$$-\frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \tilde{u}_{\varepsilon,j}(x) = \int_{\Omega_\varepsilon} G_\varepsilon(y,0) \Big(e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} \Big)\, dy. \tag{2.41}$$

In view of (2.17), we obtain from (2.40) and (2.41) that

$$\tilde{u}_{\varepsilon,j}(x) = \frac{1}{2\pi} \int_{\Omega_\varepsilon} \ln \frac{|y|}{|y-x|} \Big(e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} \Big)\, dy + O(1). \tag{2.42}$$

From (2.29) and (2.31), for any $\theta' > 0$, we can find a large $R > 0$, such that

$$\frac{1}{2\pi} \int_{B_R(0)} \Big(e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} \Big) \ge 4 - \theta'. \tag{2.43}$$

Suppose now $|x| \ge 2R$. Note that we have

$$e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} = e^{-\beta_{\varepsilon,j}} e^{u_\varepsilon + u_0} (1 - e^{u_\varepsilon + u_0}) > 0.$$

So for any set S, in which $\ln \frac{|y|}{|y-x|} \le C$, we find

$$\int_S \ln \frac{|y|}{|y-x|} \Big(e^{\tilde{u}_{\varepsilon,j} + \tilde{u}_0} - e^{\beta_{\varepsilon,j}} e^{2(\tilde{u}_{\varepsilon,j} + \tilde{u}_0)} \Big)\, dy \le C'. \tag{2.44}$$

It is easy to see for $R \le |y| \le \frac{1}{2}|x|$, $\frac{|y|}{|y-x|} \le 1$. If $y \in \Omega_\varepsilon \setminus \big(B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x)\big)$, we can check $\frac{|y|}{|y-x|} \le C$. So, we obtain from (2.42)

$$\begin{aligned}\tilde u_{\varepsilon,j}(x) \le& \frac{1}{2\pi}\int\limits_{B_R(0)} \ln\frac{|y|}{|y-x|}\Big(e^{\tilde u_{\varepsilon,j}+\tilde u_0} - e^{\beta_{\varepsilon,j}}e^{2(\tilde u_{\varepsilon,j}+\tilde u_0)}\Big)\,dy \\ &+ \frac{1}{2\pi}\int\limits_{B_{\frac{|x|}{2}}(x)} \ln\frac{|y|}{|y-x|}\Big(e^{\tilde u_{\varepsilon,j}+\tilde u_0} - e^{\beta_{\varepsilon,j}}e^{2(\tilde u_{\varepsilon,j}+\tilde u_0)}\Big)\,dy + C.\end{aligned} \tag{2.45}$$

Let $\sigma > 0$ be a small constant. Noting $\tilde u_\varepsilon \le 0$ in $B_\sigma(x) \subset B_{\frac{\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0)$, using (2.32), we find

$$\begin{aligned}&\int\limits_{B_{\frac{|x|}{2}}(x)} \ln\frac{|y|}{|y-x|}\Big(e^{\tilde u_{\varepsilon,j}+\tilde u_0} - e^{\beta_{\varepsilon,j}}e^{2(\tilde u_{\varepsilon,j}+\tilde u_0)}\Big)\,dy \\ &\le C\int\limits_{B_\sigma(x)} \ln\frac{|y|}{|y-x|} + \ln\frac{C|x|}{\sigma}\int\limits_{B_{\frac{|x|}{2}}(x)\setminus B_\sigma(x)} \Big(e^{\tilde u_{\varepsilon,j}+\tilde u_0} - e^{\beta_{\varepsilon,j}}e^{2(\tilde u_{\varepsilon,j}+\tilde u_0)}\Big)\,dy \\ &= (o_\sigma(1) + o_\varepsilon(1) + o_R(1))\ln|x|.\end{aligned} \tag{2.46}$$

Combining (2.45) and (2.46), we are led to

$$\tilde u_{\varepsilon,j}(x) \le \frac{1}{2\pi}\int\limits_{B_R(0)} \ln\frac{|y|}{|y-x|}\Big(e^{\tilde u_{\varepsilon,j}+\tilde u_0} - e^{\beta_{\varepsilon,j}}e^{2(\tilde u_{\varepsilon,j}+\tilde u_0)}\Big)\,dy + (o_\sigma(1) + o_\varepsilon(1) + o_R(1))\ln|x| + C. \tag{2.47}$$

For any $y \in B_R(0)$, if $|x| >> R$,

$$\ln\frac{|y|}{|y-x|} = \ln\frac{1}{|x|} + o_{|x|}(1),$$

which, together with (2.47) and (2.43), gives

$$\tilde u_{\varepsilon,j}(x) \le (4-\theta)\ln\frac{1}{|x|} + C. \qquad \square \tag{2.48}$$

Lemma 2.10. *For any $\Lambda > 0$ large, we have*

$$\frac{1}{\varepsilon^2}\int\limits_{B_{\Lambda\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j})} e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big) = M_{\varepsilon,j} + O(\Lambda^{-2+\theta}),$$

where $\theta > 0$ is any small fixed constant.

Proof. It follows from Lemma 2.9 that

$$\begin{aligned}M_{\varepsilon,j} &= \frac{1}{\varepsilon^2}\int\limits_{B_\delta(x_{\varepsilon,j})} e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big) \\ &= \frac{1}{\varepsilon^2}\int\limits_{B_{\Lambda\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j})} e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big) + O\Big(\int\limits_{B_{\delta\varepsilon^{-1}e^{\frac{1}{2}\beta_{\varepsilon,j}}}(0)\setminus B_\Lambda(0)} \frac{1}{|y|^{4-\theta}}\Big) \\ &= \frac{1}{\varepsilon^2}\int\limits_{B_{\Lambda\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j})} e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big) + O(\Lambda^{-2+\theta}). \qquad \square\end{aligned}$$

Lemma 2.11. *We have*

$$u_\varepsilon(x) - u_\varepsilon(x_{\varepsilon,j}) = \frac{M_{\varepsilon,j}}{2\pi} \ln \frac{1}{|\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}(x - x_{\varepsilon,j})|} + O(1), \quad \forall x \in B_\delta(x_{\varepsilon,j}) \setminus B_{R\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}}(x_{\varepsilon,j}). \tag{2.49}$$

Proof. For any $x \in B_\delta(x_{\varepsilon,j})$, by Lemma 2.9, it holds

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(x_{\varepsilon,j}) &= \frac{1}{\varepsilon^2}\int_\Omega \big(G(y,x) - G(y,x_{\varepsilon,j})\big)e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big)\,dy \\ &= \frac{1}{2\pi\varepsilon^2}\int_{B_{2\delta}(x_{\varepsilon,j})} \ln\frac{|y - x_{\varepsilon,j}|}{|y-x|} e^{u_\varepsilon+u_0}\big(1 - e^{u_\varepsilon+u_0}\big)\,dy + O(1) \\ &= \frac{1}{2\pi}\int_{B_{\frac{2\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0)} \ln\frac{|y|}{|y-\bar{x}_\varepsilon|} e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big(1 - e^{\beta_{\varepsilon,j}}e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big)\,dy + O(1) \\ &= \frac{1}{2\pi}\int_{B_{\frac{2\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0)} \ln\frac{1}{|y-\bar{x}_\varepsilon|} e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big(1 - e^{\beta_{\varepsilon,j}}e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big)\,dy + O(1), \end{aligned} \tag{2.50}$$

where $\bar{x}_\varepsilon = \frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}(x - x_{\varepsilon,j})$.

Let $\sigma > 0$ be a small fixed constant. Then

$$\begin{aligned} &\Big|\int_{B_{\frac{2\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0)\setminus B_{\sigma|\bar{x}_\varepsilon|}(0)} \ln\frac{1}{|y-\bar{x}_\varepsilon|} e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big(1 - e^{\beta_{\varepsilon,j}}e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big)\,dy\Big| \\ &\le C\int_{B_{\frac{2\delta e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}}(0)\setminus B_{\sigma|\bar{x}_\varepsilon|}(0)} |\ln|y-\bar{x}_\varepsilon||\Big(\frac{1}{|y|^{4-\theta}} + \frac{1}{|y|^{2(4-\theta)}}\Big) \le \frac{C}{|\bar{x}_\varepsilon|^{2-2\theta}}. \end{aligned} \tag{2.51}$$

But

$$\begin{aligned} &\frac{1}{2\pi}\int_{B_{\sigma|\bar{x}_\varepsilon|}(0)} \ln\frac{1}{|y-\bar{x}_\varepsilon|} e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big(1 - e^{\beta_{\varepsilon,j}}e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big)\,dy \\ &= \frac{1}{2\pi}\int_{B_{\sigma|\bar{x}_\varepsilon|}(0)} e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big(1 - e^{\beta_{\varepsilon,j}}e^{\tilde{u}_\varepsilon+\tilde{u}_0}\big)\,dy \ln\frac{1}{|\bar{x}_\varepsilon|} + O(1) \end{aligned} \tag{2.52}$$

since $|\ln\frac{|\bar{x}_\varepsilon|}{|y-\bar{x}_\varepsilon|}| \le C$ for $y \in B_{\sigma|\bar{x}_\varepsilon|}(0)$. So the result follows from (2.50)–(2.52) and Lemma 2.10. □

Before we prove Proposition 2.8, we will use Lemma 2.11 to improve the estimates in Lemma 2.1.

Lemma 2.12. *For any $\delta_0 > 0$,*

$$u_\varepsilon(x) = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon + \sum_{j=1}^k M_{\varepsilon,j}G(x_{\varepsilon,j},x) + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big), \quad \text{in } C^1(\Omega \setminus \cup_{j=1}^k B_{\delta_0}(x_{\varepsilon,j})).$$

Proof. Fix $\delta \in (0, \delta_0)$. It follows from Lemma 2.11 that

$$u_\varepsilon(x) = \beta_{\varepsilon,j} + \frac{M_{\varepsilon,j}}{2\pi}\ln\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}} + O(1), \quad x \in \partial B_\delta(x_{\varepsilon,j}). \tag{2.53}$$

Using Lemma 2.1, we find

$$\frac{1}{|\Omega|}\int_\Omega u_\varepsilon = u_\varepsilon(x) + O(1), \quad x \in \partial B_\delta(x_{\varepsilon,j}), \tag{2.54}$$

which, together with (2.53), gives

$$\frac{1}{|\Omega|}\int_\Omega u_\varepsilon = \beta_{\varepsilon,j} + \frac{M_{\varepsilon,j}}{2\pi}\ln\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}} + O(1). \tag{2.55}$$

It is easy to deduce from Lemma 2.1 and (2.55) that

$$u_\varepsilon(x) = \beta_{\varepsilon,j} + \frac{M_{\varepsilon,j}}{2\pi}\ln\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}} + O(1), \quad x \in \Omega\setminus\cup_{j=1}^k B_\delta(x_{\varepsilon,j}). \tag{2.56}$$

As a result

$$\frac{1}{\varepsilon^2}e^{u_\varepsilon+u_0}(1-e^{u_\varepsilon+u_0}) = O\Big(\big(\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}}\big)^{\frac{M_{\varepsilon,j}}{2\pi}-2}\Big), \quad x \in \Omega\setminus\cup_{j=1}^k B_\delta(x_{\varepsilon,j}). \tag{2.57}$$

Similar to (2.4), using Lemma 2.9 and (2.57), we obtain

$$\begin{aligned} &u_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \\ &= \sum_{j=1}^k M_{\varepsilon,j}G(x_{\varepsilon,j},x) + O\Big(\frac{1}{\varepsilon^2}\int_{B_\delta(x_{\varepsilon,j})}|y-x_{\varepsilon,j}|e^{u_\varepsilon} + \big(\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}}\big)^{\frac{M_{\varepsilon,j}}{2\pi}-2}\Big) \\ &= \sum_{j=1}^k M_{\varepsilon,j}G(x_{\varepsilon,j},x) + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big). \end{aligned} \tag{2.58}$$

The estimate for Du_ε is similar. □

Proof of Proposition 2.8. If the limit problem is given by (2.38), then Proposition 2.8 follows from Lemma 2.11.

Suppose now that the limit problem is given by (2.39). We use the Pohozaev identity (2.14) with p replaced by $x_{\varepsilon,j}$. By Lemma 2.12, we find

$$\text{LHS of (2.14)} = -\frac{M_{\varepsilon,j}^2}{4\pi} + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big). \tag{2.59}$$

Using Lemma 2.9, we can deduce

$$\text{RHS of (2.14)} = -2M_{\varepsilon,j} + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big). \tag{2.60}$$

From

$$-\frac{M_{\varepsilon,j}^2}{4\pi} = -2M_{\varepsilon,j} + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big), \tag{2.61}$$

we obtain $M_{\varepsilon,j} = 8\pi + O\big(\varepsilon e^{-\frac{1}{2}\beta_{\varepsilon,j}}\big)$. So the result follows from Lemma 2.11. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Proposition 2.8 and Lemma 2.1 that for each j,

$$u_\varepsilon(x_{\varepsilon,j}) + u_{\varepsilon,j}\big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}\delta e_1\big) = \frac{1}{|\Omega|}\int_\Omega u_\varepsilon + O(1), \quad j = 1,\cdots,k, \tag{2.62}$$

where $e_1 = (1, 0)$. So, we obtain

$$u_\varepsilon(x_{\varepsilon,j}) + u_{\varepsilon,j}\big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}\delta e_1\big) = u_\varepsilon(x_{\varepsilon,i}) + u_{\varepsilon,i}\big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,i}}}{\varepsilon}\delta e_1\big) + O(1), \quad \text{for all } i, j. \tag{2.63}$$

Note that if $\beta_{\varepsilon,j} \geq c_0 > -\infty$, then

$$u_\varepsilon(x_{\varepsilon,j}) + u_{\varepsilon,j}\big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}\delta e_1\big) = \frac{M_{\varepsilon,j}}{2\pi}\ln\varepsilon + O(1); \tag{2.64}$$

while if $\beta_{\varepsilon,j} \to -\infty$, then $M_{\varepsilon,j} = 8\pi + o(1)$, and

$$u_\varepsilon(x_{\varepsilon,j}) + u_{\varepsilon,j}\big(\frac{e^{\frac{1}{2}\beta_{\varepsilon,j}}}{\varepsilon}\delta e_1\big) = \beta_{\varepsilon,j} + 4\ln\frac{\varepsilon}{e^{\frac{1}{2}\beta_{\varepsilon,j}}} + O(1) = 4\ln\varepsilon - \beta_{\varepsilon,j} + O(1). \tag{2.65}$$

Note that the relation

$$\frac{M_{\varepsilon,i}}{2\pi}\ln\varepsilon = 4\ln\varepsilon - \beta_{\varepsilon,j} + O(1)$$

can not be true, if $\frac{M_{\varepsilon,i}}{2\pi} > 4$ and $\beta_{\varepsilon,j} \to -\infty$. So, we have proved that either $\beta_{\varepsilon,j} \geq c_0 > -\infty$ for all j, or $\beta_{\varepsilon,j} \to -\infty$ for all j. In the first case, we obtain from (2.63) and (2.64) that $M_i = M_j > 8\pi$. In the second case, $M_j = 8\pi$, $j = 1, 2, \cdots, k$. □

Before we close this section, let us briefly discuss the bubbling solutions, whose blow-up set contains some vortex points.

Proposition 2.13. *Suppose that the blow-up set of a bubbling solution u_ε contains a vortex point p_j, whose weight is 1, and the following relation holds*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int\limits_{B_\delta(p_j)} e^{u_\varepsilon+u_0}(1 - e^{u_\varepsilon+u_0}) > 16\pi, \tag{2.66}$$

where $\delta > 0$ is a small constant. Then

$$u_\varepsilon(x) = \frac{M_{\varepsilon,j}}{2\pi}\ln\frac{\varepsilon}{|x - x_{\varepsilon,j}|} - 2\ln\varepsilon + O(1), \quad \forall\, x \in B_\delta(x_{\varepsilon,j}) \setminus B_{\varepsilon R}(x_{\varepsilon,j}). \tag{2.67}$$

Proof. Without loss of generality, we assume that $p_j = 0$.

Step 1. We claim that $|x_{\varepsilon,j}| \leq C\varepsilon$. Suppose we have (up to a subsequence) that $\varepsilon^{-1}|x_{\varepsilon,j}| \to +\infty$. Define $u_\varepsilon^{**}(x) = u_\varepsilon(\delta_\varepsilon x) + 2\ln\delta_\varepsilon$, where $\delta_\varepsilon = |x_{\varepsilon,j}|$. Then u_ε^{**} satisfies

$$\Delta u_\varepsilon^{**} + \frac{\delta_\varepsilon^2}{\varepsilon^2}e^{u_\varepsilon^{**}+u_0^*}(1 - e^{u_\varepsilon^{**}+u_0^*}) = \frac{4N\pi\delta_\varepsilon^2}{|\Omega|}, \tag{2.68}$$

where $u_0^*(x) = u_0(\delta_\varepsilon x) - 2\ln\delta_\varepsilon$. We can do the blow-up analysis for the sequence u_ε^{**} and prove that there is a non-empty finite set $S^* = \{z_1, \cdots, z_l\}$, such that

$$\frac{\delta_\varepsilon^2}{\varepsilon^2}e^{u_\varepsilon^{**}+u_0^*}(1 - e^{u_\varepsilon^{**}+u_0^*}) \to \sum_{j=1}^{l} m_j^*\delta_{z_j}, \quad m_j^* \geq 8\pi, \tag{2.69}$$

in the sense of measure.

Suppose that $0 \in S^*$. Then $l \geq 2$. We assume that $z_1 = 0$ and use the Pohozaev identity to find that z_j satisfies

$$\big(\frac{m_1^*}{2\pi} - 2\big)\frac{z_j}{|z_j|^2} + \sum_{i\neq j}\frac{m_j^*(z_j - z_i)}{2\pi|z_j - z_i|^2} = 0, \quad j = 2, \cdots, l. \tag{2.70}$$

Since $\frac{m_1^*}{2\pi} - 2 > 0$, (2.70) can not hold true for z_j with $|z_j| = \max_i |z_i|$. So $0 \notin S^*$. Using the Pohozaev identity again, we find that z_j satisfies

$$\frac{2z_j}{|z_j|^2} = \sum_{i \neq j} \frac{m_j^*(z_j - z_i)}{2\pi |z_j - z_i|^2}, \quad j = 1, \cdots, l. \tag{2.71}$$

From (2.71), we find $l \geq 2$. Let us assume that $|z_1| = \max_i |z_i|$. By dividing (2.71) with $|z_1|$, we can assume that $|z_1| = 1$. By rotating suitably, we can make $z_1 = (1, 0)$. Let $z_j = (r\cos\theta, r\sin\theta)$, $j > 1$. Then $|z_j - z_1|^2 = 1 + r^2 - 2r\cos\theta$ and the x_1 component of $z_1 - z_j$ is $1 - r\cos\theta$. Define $f(r,\theta) = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}$. We claim $\min_{\overline{B_1(0)}} f(r,\theta) = \frac{1}{2}$ and $f(r,\theta) > \frac{1}{2}$ if $r < 1$. Assume this at the moment, then from (2.71), we obtain

$$2 = \sum_{j=2}^{l} \frac{m_j^*(z_1 - z_j)_{x_1}}{2\pi |z_1 - z_j|^2} \geq \sum_{j=2}^{l} \frac{m_j^*}{4\pi} > 2, \tag{2.72}$$

if $l \geq 3$, which is a contradiction. But if $l = 2$ and $|z_2| < 1$, then the first inequality in (2.72) must be strict and we obtain a contradiction. From $|z_2| = 1$ and (2.71), we find $z_1 = -z_2$. So (2.72) can not hold true since one of $m_j^* > 8\pi$ due to (2.66). So we have proved $|x_\varepsilon| \leq C\varepsilon$. To prove $\min_{B_1(0)} f(r,\theta) = \frac{1}{2}$, we first know that $f(r,\theta)$ is smooth in $\overline{B_1(0)} \setminus \{(1,0)\}$. But as $r \to 1$ and $\theta \to 0$,

$$f(r,\theta) = \frac{1 - r + \frac{\theta^2}{2} + O(\theta^3 + (1-r)\theta^2)}{(1-r)^2 + \theta^2 + O(\theta^3 + (1-r)\theta^2)} \geq \frac{\frac{(1-r)^2}{2} + \frac{\theta^2}{2} + O(\theta^3 + (1-r)\theta^2)}{(1-r)^2 + \theta^2 + O(\theta^3 + (1-r)\theta^2)} \geq \frac{1}{2}.$$

On the other hand, $f_r(r,\theta) = \frac{(1+r^2)\cos\theta - 2r}{(1+r^2-2r\cos\theta)^2}$ if $r < 1$, which implies that the minimum of $f(r,\theta)$ can only be attained at $r = 0$ or $r = 1$. But for $r = 1$ and $\theta \neq 0$, it is easy to see that $f(1,\theta) = \frac{1}{2}$, and $f(0,\theta) = 1$. So we find $\min_{B_1(0)} f(r,\theta) = \frac{1}{2}$.

Step 2. As in Lemma 2.7, we claim that for $R > 0$ large and y satisfying $|y - x_{\varepsilon,j}| > R\varepsilon$, it holds

$$\frac{1}{\varepsilon^2} \int_{B_{\frac{1}{2}|y - x_{\varepsilon,j}|}(y)} e^{u_\varepsilon + u_0}(1 - e^{u_\varepsilon + u_0}) = o_R(1) + o_\varepsilon(1), \tag{2.73}$$

where $o_R(1) \to 0$ as $R \to +\infty$.

In fact, similar to Lemma 2.4, we can prove that $u_\varepsilon + 2\ln\varepsilon \leq C < +\infty$. If (2.73) was not true, by doing the blow-up analysis for $u_\varepsilon + 2\ln\varepsilon \leq C < +\infty$, we would find that $u_\varepsilon + 2\ln\varepsilon$ has a blow-up set $\{0, z_2, \cdots, z_l\}$ with z_j satisfying (2.70). This is a contradiction.

Step 3. From Steps 1 and 2, we can prove (2.67) in the same way as in Lemma 2.11. □

Proof of the necessary part of Theorem 1.4. By Proposition 2.13, we find

$$M_{\varepsilon,i} = M_{\varepsilon,1} + o(1),\ i = 1, \cdots, t; \quad M_{\varepsilon,i} = M_{\varepsilon,t+1} + o(1),\ i = t+1, \cdots, k,$$

and

$$M_{\varepsilon,t+1} = M_{\varepsilon,1} - 4\pi + o(1), \quad M_{\varepsilon,1} + \cdots + M_{\varepsilon,k} = 4N\pi.$$

So the first claim follows. The second claim can be proved by using the Pohozaev identity. □

3. Existence of bubbling solutions

In this section, we will prove Theorem 1.2 and the existence part of Theorem 1.4. More precisely, we will prove the following theorems.

Theorem 3.1. *Suppose that k is a positive integer with $k < \frac{N}{2}$. Let $\mathbf{q} = (q_1, \cdots, q_k)$ with $q_j \notin \{p_1, \cdots p_N\}$, $j = 1, \cdots, k$. Assume that $\mathbf{q}$ satisfies (1.12) and $deg(DG_k(\mathbf{q}), 0) \neq 0$. Then there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.6) has a solution u_ε, satisfying*

$$\frac{1}{\varepsilon^2}e^{u_\varepsilon+u_0}\big(1-e^{u_\varepsilon+u_0}\big)\to\frac{4\pi N}{k}\sum_{j=1}^{k}\delta_{q_j}$$

as $\varepsilon\to 0$.

Theorem 3.2. *Let* $\{q_1,\cdots,q_t,q_{t+1},\cdots,q_k\}$ *satisfy* (1.15). *Suppose that* k *and* t *satisfy* $k<\frac{1}{3}(N-t)$ *and the weight* m_i *of each vertex point* q_i *is one. Assume that* $\mathbf{q}=(q_{t+1},\cdots,q_k)$ *satisfies* $DG_k^*(\mathbf{q})=0$ *and* $deg(DG_k^*(\mathbf{q}),0)\neq 0$. *Then there is an* $\varepsilon_0>0$, *such that for any* $\varepsilon\in(0,\varepsilon_0)$, (1.6) *has a solution* u_ε, *satisfying*

$$\frac{1}{\varepsilon^2}e^{u_\varepsilon+u_0}\big(1-e^{u_\varepsilon+u_0}\big)\to\frac{4\pi N+4\pi(k-t)}{k}\sum_{i=1}^{t}\delta_{q_i}+\frac{4\pi N-4\pi t}{k}\sum_{j=t+1}^{k}\delta_{q_j}$$

as $\varepsilon\to 0$.

Let us point out that in Theorem 3.2, $\{q_{t+1},\cdots,q_k\}=\emptyset$ is allowed. Early results on the existence of single bubbling solutions of CS type can be found in [10,24,26]. To prove Theorems 3.2 and 3.1, it is essential to construct a good approximate solution for (1.6) without (1.14). This will be carried out in details in section 3.1. Once this is done, we can use a reduction argument to finish the proof. This part is quite standard, so we just sketch it. We refer to [24–26] for the details.

In the following, we only give the proof of Theorem 3.2, since the proof of Theorem 3.1 is very similar.

3.1. The approximate solutions

Without loss of generality, we assume that $|\Omega|=1$. Firstly, we construct an approximate solution.

We want to construct an approximate solution for (1.6), whose blow-up set is

$$\{q_1,\cdots,q_t,x_{\varepsilon,t+1},\cdots x_{\varepsilon,k}\},\tag{3.1}$$

where $x_{\varepsilon,j}$ is close to q_j, $j=t+1,\cdots,k$, and q_j is a vortex point, $j=1,\cdots,t$.

For bubble at $x_{\varepsilon,j}$, we consider

$$\begin{cases}\Delta V+e^V(1-e^V)=0,\ \ V \text{ is radial}, \quad \text{in } \mathbb{R}^2;\\ \int_{\mathbb{R}^2}e^V(1-e^V)=M_{\varepsilon,j},\end{cases}\tag{3.2}$$

where $M_{\varepsilon,j}$ is a constant satisfying $M_{\varepsilon,j}>8\pi$. By Theorem 2.1 of [4], (3.2) has a solution $V_{\varepsilon,j}(|x|)$, which has the following expansion:

$$V_{\varepsilon,j}(|x|)=-\frac{M_{\varepsilon,j}}{2\pi}\ln|x|+I_{\varepsilon,j}+O\Big(\frac{1}{|x|^2}\Big),\quad \text{as } |x|\to+\infty,\tag{3.3}$$

where $I_{\varepsilon,j}$ is a constant, which depends on $M_{\varepsilon,j}$ smoothly, and

$$V'_{\varepsilon,j}(|x|)=-\frac{M_{\varepsilon,j}}{2\pi|x|}+O\Big(\frac{1}{|x|^2}\Big),\quad \text{as } |x|\to+\infty.\tag{3.4}$$

The solution $V_{\varepsilon,j}(|x|)$ forms the major part of the bubble near a regular point q_j.

For bubble at the vortex point q_i, we need to consider

$$\begin{cases}\Delta V+|x|^2e^V(1-|x|^2e^V)=0,\ \ V \text{ is radial}, \quad \text{in } \mathbb{R}^2;\\ \int_{\mathbb{R}^2}|x|^2e^V(1-|x|^2e^V)=M_{\varepsilon,i}.\end{cases}\tag{3.5}$$

By Theorem 2.1 of [4], if $M_{\varepsilon,i}>16\pi$, (3.2) has a solution $V_{\varepsilon,j}(|x|)$, which has the same expansions as in (3.3) and (3.4).

We will construct an approximate solution for (1.6) whose blow-up set is given by (3.1). For simplicity of the notations, we denote $x_{\varepsilon,j}=q_j$, $j=1,\cdots,t$. For $j=t+1,\cdots,k$, we let $V_{\varepsilon,j}$ be the solution of (3.2). We define the approximate solution for (1.6) near $x_{\varepsilon,j}$ as follows.

$$\varphi_{\varepsilon,j}(x) = V_{\varepsilon,j}\Big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon}\Big) + M_{\varepsilon,j}\big(\gamma(x,x_{\varepsilon,j}) - \gamma(x_{\varepsilon,j},x_{\varepsilon,j})\big) + \sum_{i\neq j,\,1\le i\le k} M_{\varepsilon,i}\big(G(x,x_{\varepsilon,i}) - G(x_{\varepsilon,i},x_{\varepsilon,j})\big) - u_0(x_{\varepsilon,j}), \quad j=t+1,\cdots,k. \tag{3.6}$$

For $j=1,\cdots,t$, we let $V_{\varepsilon,j}$ be the solution of (3.5). We define the approximate solution for (1.6) near $x_{\varepsilon,j}$ by

$$\varphi_{\varepsilon,j}(x) = V_{\varepsilon,j}\Big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon}\Big) + 2\ln\frac{1}{\varepsilon} + M_{\varepsilon,j}\big(\gamma(x,x_{\varepsilon,j}) - \gamma(x_{\varepsilon,j},x_{\varepsilon,j})\big) + \sum_{i\neq j,\,1\le i\le k} M_{\varepsilon,i}\big(G(x,x_{\varepsilon,i}) - G(x_{\varepsilon,i},x_{\varepsilon,j})\big) - u_j(x_{\varepsilon,j}), \quad j=1,\cdots,t, \tag{3.7}$$

where $u_j(x) = u_0(x) - 2\ln|x-x_{\varepsilon,j}|$.

For $x\in\Omega\setminus\{x_{\varepsilon,1},\cdots,x_{\varepsilon,k}\}$, we define the approximate solution as

$$\varphi_{\varepsilon,0}(x) = \sum_{i=1}^{k} M_{\varepsilon,i}G(x,x_{\varepsilon,i}) + L_\varepsilon, \tag{3.8}$$

where L_ε is a very negative constant, which is to be determined later.

To glue all the $\varphi_{\varepsilon,j}$ together to form an approximate solution for (1.6), we need to make $\varphi_{\varepsilon,j} = \varphi_{\varepsilon,0}$ on $\partial B_\delta(x_{\varepsilon,j})$ up to a small term. By (3.3), we let $M_{\varepsilon,j}$ satisfy the following equations:

$$-M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j}) - \sum_{i\neq j,\,1\le i\le k} M_{\varepsilon,i}G(x_{\varepsilon,j},x_{\varepsilon,i}) - \frac{M_{\varepsilon,j}}{2\pi}\ln\frac{1}{\varepsilon} + I_{\varepsilon,j} - u_0(x_{\varepsilon,j}) = L_\varepsilon, \quad j=t+1,\cdots,k, \tag{3.9}$$

and

$$-M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j}) - \sum_{i\neq j,1\le i\le k} M_{\varepsilon,i}G(x_{\varepsilon,j},x_{\varepsilon,i}) - \Big(\frac{M_{\varepsilon,j}}{2\pi} - 2\Big)\ln\frac{1}{\varepsilon} + I_{\varepsilon,j} - u_j(x_{\varepsilon,j}) = L_\varepsilon, \quad j=1,\cdots,t. \tag{3.10}$$

Therefore, for $j=t+1,\cdots,k$,

$$\begin{aligned} &M_{\varepsilon,j} - M_{\varepsilon,t+1} \\ &+ \frac{2\pi}{|\ln\varepsilon|}\Big(M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j}) - M_{\varepsilon,t+1}\gamma(x_{\varepsilon,t+1},x_{\varepsilon,t+1}) - I_{\varepsilon,j} + I_{\varepsilon,t+1}\Big) \\ &+ \frac{2\pi}{|\ln\varepsilon|}\Big(\sum_{i\neq j,1\le i\le k} M_{\varepsilon,i}G(x_{\varepsilon,j},x_{\varepsilon,i}) - \sum_{i\neq t+1,1\le i\le k}^{k} M_{\varepsilon,i}G(x_{\varepsilon,t+1},x_{\varepsilon,i})\Big) \\ =& \frac{2\pi}{|\ln\varepsilon|}\Big(u_0(x_{\varepsilon,t+1}) - u_0(x_{\varepsilon,j})\Big), \quad j=t+1,\cdots,k. \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned} &M_{\varepsilon,j} - M_{\varepsilon,1} \\ &+ \frac{2\pi}{|\ln\varepsilon|}\Big(M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j}) - M_{\varepsilon,1}\gamma(x_{\varepsilon,1},x_{\varepsilon,1}) - I_{\varepsilon,j} + I_{\varepsilon,1}\Big) \\ &+ \frac{2\pi}{|\ln\varepsilon|}\Big(\sum_{i\neq j,\,1\le i\le k} M_{\varepsilon,i}G(x_{\varepsilon,j},x_{\varepsilon,i}) - \sum_{i=2}^{k} M_{\varepsilon,i}G(x_{\varepsilon,1},x_{\varepsilon,i})\Big) \\ =& \frac{2\pi}{|\ln\varepsilon|}\Big(u_1(x_{\varepsilon,1}) - u_j(x_{\varepsilon,j})\Big), \quad j=1,\cdots,t. \end{aligned} \tag{3.12}$$

On the other hand, it is easy to find

$$\begin{aligned}&M_{\varepsilon,j}-M_{\varepsilon,1}+4\pi\\&+\frac{2\pi}{|\ln\varepsilon|}\Big(M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j})-M_{\varepsilon,1}\gamma(x_{\varepsilon,1},x_{\varepsilon,1})-I_{\varepsilon,j}+I_{\varepsilon,1}\Big)\\&+\frac{2\pi}{|\ln\varepsilon|}\Big(\sum_{i\neq j,1\le i\le k}M_{\varepsilon,i}G(x_{\varepsilon,j},x_{\varepsilon,i})-\sum_{i=2}^{k}M_{\varepsilon,i}G(x_{\varepsilon,1},x_{\varepsilon,i})\Big)\\=&\frac{2\pi}{|\ln\varepsilon|}\Big(u_1(x_{\varepsilon,1})-u_0(x_{\varepsilon,j})\Big),\quad j=t+1,\cdots,k.\end{aligned}\tag{3.13}$$

Furthermore, we take

$$M_{\varepsilon,1}+\cdots+M_{\varepsilon,k}=4\pi N.\tag{3.14}$$

Noting that $I_{\varepsilon,j}$ depends on $M_{\varepsilon,j}$ smoothly, we can solve (3.11)–(3.14) to find

$$M_{\varepsilon,j}=\frac{4\pi N+4\pi(k-t)}{k}+O\big(\frac{1}{|\ln\varepsilon|}\big),\quad j=1,\cdots,t,\tag{3.15}$$

and

$$M_{\varepsilon,j}=\frac{4\pi N-4\pi t}{k}+O\big(\frac{1}{|\ln\varepsilon|}\big),\quad j=t+1,\cdots,k.\tag{3.16}$$

It is easy to check that the constant L_ε determined by (3.9) is very negative.

We are now ready to construct an approximate solution for (1.6). Let $\chi(t)\in C^\infty(\mathbb{R}^1)$ be a function satisfying $\chi=1$ in $[0,d]$, $\chi=0$ in $[2d,+\infty)$, and $0\le\chi\le1$, where $d>0$ is a small constant. Define

$$\varphi_{\varepsilon,\mathbf{x}}=\sum_{j=1}^{k}\chi(|x-x_{\varepsilon,j}|)\varphi_{\varepsilon,j}+\big(1-\sum_{j=1}^{k}\chi(|x-x_{\varepsilon,j}|)\big)\varphi_{\varepsilon,0}.\tag{3.17}$$

Using (3.14), we see that for $x\in B_d(x_{\varepsilon,j})$, $j=t+1,\cdots,k$,

$$\Delta\varphi_{\varepsilon,\mathbf{x}}-4\pi N=\frac{1}{\varepsilon^2}e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big(e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}-1\Big).\tag{3.18}$$

On the other hand, from (3.6),

$$\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)=V_{\varepsilon,j}\big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon}\big)+O\big(|x-x_{\varepsilon,j}|\big),\tag{3.19}$$

which implies

$$\begin{aligned}&\frac{1}{\varepsilon^2}e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big(e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}-1\Big)\\=&\frac{1}{\varepsilon^2}e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}-1\Big)+O\Big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon^2}e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big),\quad x\in B_d(x_{\varepsilon,j}).\end{aligned}\tag{3.20}$$

Combining (3.18) and (3.20), we are led to

$$\begin{aligned}\Delta\varphi_{\varepsilon,\mathbf{x}}-4\pi N=&\frac{1}{\varepsilon^2}e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}-1\Big)\\&+O\Big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon^2}e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big),\quad x\in B_d(x_{\varepsilon,j}),\ j=t+1,\cdots,k.\end{aligned}\tag{3.21}$$

On the other hand, for $x\in B_{2d}(x_{\varepsilon,j})\setminus B_d(x_{\varepsilon,j})$, from (3.3) and (3.9), we obtain

$$\begin{aligned}\varphi_{\varepsilon,j}-\varphi_{\varepsilon,0}=&V_{\varepsilon,j}\big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon}\big)-\frac{M_{\varepsilon,j}}{2\pi}\ln\frac{1}{|x-x_{\varepsilon,j}|}-M_{\varepsilon,j}\gamma(x_{\varepsilon,j},x_{\varepsilon,j})\\&-\sum_{i\neq j,\ 1\le i\le k}M_{\varepsilon,i}G(x_{\varepsilon,i},x_{\varepsilon,j})-u_0(x_{\varepsilon,j})-L_\varepsilon=O(\varepsilon^2),\end{aligned}\tag{3.22}$$

and from (3.4), we also obtain

$$D\varphi_{\varepsilon,j} - D\varphi_{\varepsilon,0} = O(\varepsilon^2). \tag{3.23}$$

Therefore,

$$\begin{aligned}&\Delta\varphi_{\varepsilon,\mathbf{x}} - 4\pi N = \Delta\varphi_{\varepsilon,j} - 4\pi N + O(\varepsilon^2)\\ &= \frac{1}{\varepsilon^2} e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)} - 1\Big)\\ &\quad + O\Big(\frac{|x-x_{\varepsilon,j}|}{\varepsilon^2} e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big), \quad x \in B_{2d}(x_{\varepsilon,j}) \setminus B_d(x_{\varepsilon,j}),\ j = t+1,\cdots,k.\end{aligned} \tag{3.24}$$

Similarly, we have

$$\begin{aligned}&\Delta\varphi_{\varepsilon,\mathbf{x}} - 4\pi N = \frac{1}{\varepsilon^2}\frac{|x-x_{\varepsilon,j}|^2}{\varepsilon^2} e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big(\frac{|x-x_{\varepsilon,j}|^2}{\varepsilon^2} e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})} - 1\Big)\\ &= \frac{1}{\varepsilon^2} e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)} - 1\Big)\\ &\quad + \frac{|x-x_{\varepsilon,j}|}{\varepsilon^2} O\Big(\frac{|x-x_{\varepsilon,j}|^2}{\varepsilon^2} e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big), \quad x \in B_{2d}(x_{\varepsilon,j}) \setminus B_d(x_{\varepsilon,j}),\ j = 1,\cdots,t,\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}&\Delta\varphi_{\varepsilon,\mathbf{x}} - 4\pi N\\ &= \frac{1}{\varepsilon^2} e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)} - 1\Big)\\ &\quad + \frac{|x-x_{\varepsilon,j}|}{\varepsilon^2} O\Big(\frac{|x-x_{\varepsilon,j}|^2}{\varepsilon^2} e^{V_{\varepsilon,j}(\frac{|x-x_{\varepsilon,j}|}{\varepsilon})}\Big), \quad x \in B_d(x_{\varepsilon,j}),\ j = 1,\cdots,t.\end{aligned} \tag{3.26}$$

Moreover, using (3.14), we obtain

$$\begin{aligned}&\Delta\varphi_{\varepsilon,\mathbf{x}} - 4\pi N = 0\\ &= \frac{1}{\varepsilon^2} e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)}\Big(e^{\varphi_{\varepsilon,\mathbf{x}}(x)+u_0(x)} - 1\Big) + O(\varepsilon^2), \quad x \in \Omega \setminus \cup_{m=1}^k B_{2d}(x_{\varepsilon,m}).\end{aligned} \tag{3.27}$$

3.2. *The reduction*

Our objective is to find a solution for (1.6) near $\varphi_{\varepsilon,\mathbf{x}}$. Let $u_\varepsilon = \varphi_{\varepsilon,\mathbf{x}} + \omega_\varepsilon$ be a solution of (1.6). Then ω_ε satisfies

$$L_\varepsilon \omega_\varepsilon := \Delta\omega_\varepsilon - f_\varepsilon(x)\omega_\varepsilon = g_\varepsilon(x,\omega_\varepsilon), \tag{3.28}$$

where

$$\begin{aligned}f_\varepsilon(x) &= \frac{1}{\varepsilon^2}\sum_{i=1}^t \chi(|x-x_{\varepsilon,i}|)\Big(2\frac{|x-x_{\varepsilon,i}|^4}{\varepsilon^4} e^{2V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})} - \frac{|x-x_{\varepsilon,i}|^2}{\varepsilon^2} e^{V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})}\Big)\\ &\quad + \frac{1}{\varepsilon^2}\sum_{i=t+1}^k \chi(|x-x_{\varepsilon,i}|)\Big(2e^{2V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})} - e^{V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})}\Big),\end{aligned} \tag{3.29}$$

and

$$g_\varepsilon(x,t) = -f_\varepsilon(x)t + \frac{1}{\varepsilon^2}\big(e^{2(\varphi_{\varepsilon,\mathbf{x}}+u_0+t)} - e^{\varphi_{\varepsilon,\mathbf{x}}+u_0+t}\big) - \Delta\varphi_{\varepsilon,\mathbf{x}} + 4\pi N. \tag{3.30}$$

Let us introduce two function spaces $X_{\alpha,\varepsilon}$ and $Y_{\alpha,\varepsilon}$. Define

$$\rho(x) = (1+|x|)^{1+\frac{\alpha}{2}}, \quad \hat{\rho}(x) = \frac{1}{(1+|x|)\big(\ln(2+|x|)\big)^{1+\frac{\alpha}{2}}},$$

where $\alpha > 0$ is a fixed small constant.

Let $\Omega' = \cup_{i=1}^k B_d(x_{\varepsilon,i})$. We say a function ξ is in $X_{\alpha,\varepsilon}$ if

$$\|\xi\|_{X_{\alpha,\varepsilon}}^2 = \sum_{i=1}^k \left(\|\Delta\tilde{\xi}_i \rho\|_{L^2(B_{2d/\varepsilon})}^2 + \|\tilde{\xi}_i \hat{\rho}\|_{L^2(B_{2d/\varepsilon})}^2\right) + \|\Delta\xi\|_{L^2(\Omega\setminus\Omega')}^2 + \|\xi\|_{L^2(\Omega\setminus\Omega')}^2 < +\infty, \tag{3.31}$$

where $\tilde{\xi}_i(y) = \xi(\varepsilon y + x_{\varepsilon,i})$, $B_t = B_t(0)$. On the other hand, we say $\xi \in Y_{\alpha,\varepsilon}$ if

$$\|\xi\|_{Y_{\alpha,\varepsilon}}^2 = \varepsilon^4 \sum_{i=1}^k \|\tilde{\xi}_i \rho\|_{L^2(B_{2d/\varepsilon})}^2 + \|\xi\|_{L^2(\Omega\setminus\Omega')}^2 < +\infty. \tag{3.32}$$

Define

$$Z_{\varepsilon,i,h} = -\Delta\Big(\chi(|x - x_{\varepsilon,i}|)\frac{\partial V_{\varepsilon,i}\left(\frac{x-x_{\varepsilon,i}}{\varepsilon}\right)}{\partial x_h}\Big) + \frac{1}{\varepsilon^2} e^{V_{\varepsilon,i}\left(\frac{x-x_{\varepsilon,i}}{\varepsilon}\right)} \chi(|x - x_{\varepsilon,i}|)\frac{\partial V_{\varepsilon,i}\left(\frac{x-x_{\varepsilon,i}}{\varepsilon}\right)}{\partial x_h}, \tag{3.33}$$

for $h = 1, 2$, $i = t+1, \cdots, k$,

$$E_\varepsilon = \Big\{\omega : \omega \in X_{\alpha,\varepsilon,}, \int_\Omega Z_{\varepsilon,i,h}\omega = 0,\ h = 1, 2,\ i = t+1, \cdots, k\Big\}, \tag{3.34}$$

and

$$F_\varepsilon = \Big\{\omega : \omega \in Y_{\alpha,\varepsilon}, \int_\Omega \chi(|x - x_{\varepsilon,i}|)\frac{\partial V_{\varepsilon,i}\left(\frac{x-x_{\varepsilon,i}}{\varepsilon}\right)}{\partial x_h}\omega = 0,\ h = 1, 2,\ i = t+1, \cdots, k\Big\}. \tag{3.35}$$

We define the following projection operator from $Y_{\alpha,\varepsilon}$ to F_ε:

$$Q_\varepsilon u = u - \sum_{i=t+1}^k \sum_{h=1}^2 c_{ih} Z_{\varepsilon,i,h}, \tag{3.36}$$

where the constants c_{ih} are chosen in such a way that $Q_\varepsilon u \in F_\varepsilon$. Then it is easy to check that

$$\|Q_\varepsilon u\|_{Y_{\alpha,\varepsilon}} \le C\|u\|_{Y_{\alpha,\varepsilon}}. \tag{3.37}$$

We have

Proposition 3.3. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$ and $(x_{\varepsilon,t+1}, \cdots, x_{\varepsilon,k})$ near $\mathbf{q}$, there exists $\omega_\varepsilon \in E_\varepsilon$, satisfying*

$$Q_\varepsilon\big(L_\varepsilon\omega_\varepsilon - g_\varepsilon(x, \omega_\varepsilon)\big) = 0. \tag{3.38}$$

Moreover, ω_ε is a C^1 map of $(x_{\varepsilon,t+1}, \cdots, x_{\varepsilon,k})$ in $X_{\alpha,\varepsilon}$, and

$$\|\omega_\varepsilon\|_{L^\infty(\Omega)} + \|\omega_\varepsilon\|_{X_{\alpha,\varepsilon}} \le C\varepsilon \ln\frac{1}{\varepsilon}. \tag{3.39}$$

Proof. By Theorem A.1, (3.38) can be rewritten as

$$\omega = B_\varepsilon\omega =: (Q_\varepsilon L_\varepsilon)^{-1} Q_\varepsilon g_\varepsilon(x, \omega),$$

and

$$\|B_\varepsilon\omega\|_{L^\infty(\Omega)} + \|B_\varepsilon\omega\|_{X_{\alpha,\varepsilon}} \le C\ln\frac{1}{\varepsilon}\|g_\varepsilon(x, \omega)\|_{Y_{\alpha,\varepsilon}}.$$

Fix a small constant $\theta > 0$. Let

$$S_\varepsilon = \big\{\omega :\ \omega \in E_\varepsilon,\ \|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{X_{\alpha,\varepsilon,}} \le \varepsilon^{1-\theta}\big\}.$$

We will prove that B_ε is a contraction map from S_ε to S_ε.

To prove that B_ε maps S_ε to S_ε, we use the definition of $g_\varepsilon(x,\omega)$ in (3.30) to obtain

$$\|g_\varepsilon(x,\omega)\|_{Y_{\alpha,\varepsilon}} \le C\varepsilon, \tag{3.40}$$

which gives

$$\|B_\varepsilon\omega\|_{L^\infty(\Omega)} + \|B_\varepsilon\omega\|_{X_{\alpha,\varepsilon}} \le C\ln\frac{1}{\varepsilon}\|g_\varepsilon(x,\omega)\|_{Y_{\alpha,\varepsilon}} \le C\varepsilon\ln\frac{1}{\varepsilon} \le \varepsilon^{1-\theta}. \tag{3.41}$$

So, B_ε maps S_ε to S_ε.

To show that B_ε is a contraction map, for any $\omega, \eta \in S_\varepsilon$, we note

$$\|B_\varepsilon(\omega) - B_\varepsilon(\eta)\|_{L^\infty(\Omega)} + \|B_\varepsilon(\omega) - B_\varepsilon(\eta)\|_{X_{\alpha,\varepsilon}} \le C\ln\frac{1}{\varepsilon}\|g_\varepsilon(x,\omega) - g_\varepsilon(x,\eta)\|_{Y_{\alpha,\varepsilon}}. \tag{3.42}$$

On the other hand, it is easy to check that

$$\|g_\varepsilon(x,\omega) - g_\varepsilon(x,\eta)\|_{Y_{\alpha,\varepsilon}} \le C\varepsilon^{1-\theta}\|\omega-\eta\|_{X_{\alpha,\varepsilon}},$$

which, together with (3.42), gives

$$\|B_\varepsilon(\omega) - B_\varepsilon(\eta)\|_{L^\infty(\Omega)} + \|B_\varepsilon(\omega) - B_\varepsilon(\eta)\|_{X_{\alpha,\varepsilon}} \le \frac{1}{2}\|\omega-\eta\|_{X_{\alpha,\varepsilon}}. \tag{3.43}$$

So, we have proved that B_ε is a contraction map.

By the contraction mapping theorem, there is a unique $\omega_\varepsilon \in S_\varepsilon$, such that $\omega_\varepsilon = B_\varepsilon\omega_\varepsilon$. Moreover, it follows from (3.41) that

$$\|\omega_\varepsilon\|_{L^\infty(\Omega)} + \|\omega_\varepsilon\|_{X_{\alpha,\varepsilon}} = \|B_\varepsilon\omega_\varepsilon\|_{L^\infty(\Omega)} + \|B_\varepsilon\omega_\varepsilon\|_{X_{\alpha,\varepsilon}} \le C\varepsilon\ln\frac{1}{\varepsilon}.$$

From the uniqueness, it is standard to prove ω_ε is a C^1 map of $(x_{\varepsilon,t+1},\cdots,x_{\varepsilon,k})$ in $X_{\alpha,\varepsilon}$. □

3.3. Existence of bubbling solutions

By Proposition 3.3, there is $\omega_\varepsilon \in S_\varepsilon$, satisfying

$$L_\varepsilon\omega_\varepsilon - g_\varepsilon(x,\omega_\varepsilon) = \sum_{i=t+1}^{k}\sum_{h=1}^{2} c_{\varepsilon,i,h}Z_{\varepsilon,i,h}, \tag{3.44}$$

for some constants $c_{\varepsilon,i,h}$. If $t=k$, Proposition 3.3 gives the existence of a solution for (1.6), whose blow-up set consists of vortex points only. In the case $t<k$, we need to choose $(x_{\varepsilon,t+1},\cdots,x_{\varepsilon,k})$ in $X_{\alpha,\varepsilon}$ suitably, such that the corresponding $c_{\varepsilon,i,h}$ are zero. So, $\varphi_{\varepsilon,\mathbf{x}}+\omega_\varepsilon$ is a true solution of (1.6). It is well known now that we just need to make $(x_{\varepsilon,t+1},\cdots,x_{\varepsilon,k})$ satisfy the following equations.

$$\int_\Omega \big(L_\varepsilon\omega_\varepsilon - g_\varepsilon(x,\omega_\varepsilon)\big)\chi(|x-x_{\varepsilon,j}|)\frac{\partial V_{\varepsilon,j}\big(\frac{x-x_{\varepsilon,j}}{\varepsilon}\big)}{\partial x_l} = 0, \quad l=1,2,\ j=t+1,\cdots,k. \tag{3.45}$$

Proof of Theorem 3.2. We just need to solve (3.45). Similar to the calculations in [24], it is not difficult to show that there is a constant $a_j \ne 0$, such that

$$\begin{aligned}&\int_\Omega \big(L_\varepsilon\omega_\varepsilon - g_\varepsilon(x,\omega_\varepsilon)\big)\chi(|x-x_{\varepsilon,j}|)\frac{\partial V_{\varepsilon,j}\big(\frac{x-x_{\varepsilon,j}}{\varepsilon}\big)}{\partial x_l}\\ &= a_j\Big(D_l u_0(x_{\varepsilon,j}) + \sum_{i\ne j} M_{\varepsilon,i}D_lG(x_{\varepsilon,i},x_{\varepsilon,j})\Big) + O(\varepsilon), \quad j=t+1,\cdots,k.\end{aligned} \tag{3.46}$$

So, by (3.15) and (3.16), we find that (3.45) is equivalent to

$$\begin{aligned} &D_l u_0(x_{\varepsilon,j}) + \frac{4\pi N + 4\pi(k-t)}{k}\sum_{i=1}^{t} D_l G(x_{\varepsilon,i}, x_{\varepsilon,j}) \\ &+ \frac{4\pi N - 4\pi t}{k}\sum_{i\neq j} D_l G(x_{\varepsilon,i}, x_{\varepsilon,j}) = o(1), \quad l=1,2,\ j=t+1,\cdots,k. \end{aligned} \tag{3.47}$$

By the assumption $deg(DG_k^*(\mathbf{q}),0)\neq 0$, (3.47) has a solution $(x_{\varepsilon,t+1},\cdots,x_{\varepsilon,k})$ near $\mathbf{q}$ if $\varepsilon>0$ is small. Thus the theorem follows. □

Conflict of interest statement

There is no conflict of interest.

Acknowledgements

The authors would like to thank the referee for carefully reading this paper. The work of the second author is partially supported by ARC (DP130102773), and National Science Foundation of China (11629101).

Appendix A. A linear equation

Consider the following linear operator L_ε defined in (3.28). In this appendix, we will prove the following result.

Theorem A.1. *Suppose that $w_\varepsilon\in E_\varepsilon$ and $h_\varepsilon\in F_\varepsilon$ satisfy*

$$L_\varepsilon w_\varepsilon = h_\varepsilon + \sum_{i=t+1}^{k}\sum_{h=1}^{2} c_{ih} Z_{\varepsilon,i,h}, \tag{A.1}$$

for some constants c_{ih}. Then there is a constant $C>0$, independent of ε, such that

$$\|w_\varepsilon\|_{L^\infty(\Omega)} + \|w_\varepsilon\|_{X_{\alpha,\varepsilon}} \le C\big(\ln\frac{1}{\varepsilon}\big)\|h_\varepsilon\|_{Y_{\alpha,\varepsilon}}. \tag{A.2}$$

Moreover, $Q_\varepsilon L_\varepsilon$ is an isomorphism from E_ε to F_ε.

Proof. To prove (A.2), we argue by contradiction. Suppose that there are $\varepsilon_n\to 0$, $w_n\in E_{\varepsilon_n}$, $h_n\in F_{\varepsilon_n}$, satisfying

$$L_{\varepsilon_n} w_n = h_n + \sum_{i=t+1}^{k}\sum_{h=1}^{2} c_{ih} Z_{\varepsilon_n,i,h}, \tag{A.3}$$

$$\|w_n\|_{L^\infty(\Omega)} + \|w_n\|_{X_{\alpha,\varepsilon_n}} = 1, \tag{A.4}$$

and

$$\|h_n\|_{Y_{\alpha,\varepsilon_n}} = o\big(\frac{1}{|\ln\varepsilon_n|}\big). \tag{A.5}$$

Step 1. We claim

$$|c_{ih}| \le C\varepsilon_n^2|\ln\varepsilon_n|. \tag{A.6}$$

This can be proved by solving a linear system of c_{ih}, which is obtained by multiplying (A.3) by $\chi(|x-x_{\varepsilon,j}|)\frac{\partial V_{\varepsilon,j}(\frac{x-x_{\varepsilon,j}}{\varepsilon})}{\partial x_{jl}}$ and integrating this relation on Ω.

Step 2. For any $R>0$, we have

$$\max_{x\in B_{\varepsilon R}(x_{\varepsilon,i})} |w_n(x)| \to 0, \quad i=t+1,\cdots,k. \tag{A.7}$$

Define $\tilde{w}_n(y) = w_n(\varepsilon_n y + x_{\varepsilon_n,i})$. Then $|\tilde{w}_n| \le 1$. Moreover, $\tilde{w}_n$ satisfies

$$\Delta\tilde{w}_n - \varepsilon^2 f_\varepsilon(\varepsilon_n y + x_{\varepsilon_n,i})\tilde{w}_n = \varepsilon_n^2 h_n(\varepsilon_n y + x_{\varepsilon_n,i}) + \varepsilon_n^2 \sum_{l=t+1}^{k}\sum_{h=1}^{2} c_{lh} Z_{\varepsilon_n,t,h}(\varepsilon_n y + x_{\varepsilon_n,i}).$$

Using Step 1 and $w_n \in E_{\varepsilon_n}$, we can prove that $\tilde{w}_n \to 0$ uniformly in any compact subset of $\mathbb{R}^2$.

Step 3. We claim that there is a constant b_0, such that for any small $c' > 0$,

$$w_n(x) = b_0 + o(1), \quad \forall\, x \in \Omega \setminus \cup_{i=1}^k B_{c'}(q_i). \tag{A.8}$$

Since $|w_n| \le 1$, we find that $w_n \to w_0$ on any compact subset of $\Omega \setminus \{q_1 \cdots, q_k\}$ and $\Delta w_0 = 0$. As a result, $w_0 = b_0$ for some constant b_0.

Step 4. Let $w^*_{n,i}(r) = \int_0^{2\pi} w_n(r,\theta)\,d\theta$, $r = |x - x_{\varepsilon,i}|$. We claim

$$w^*_{n,i}(r) = o(1), \quad r \le d,\ i = 1, \cdots, k. \tag{A.9}$$

Note that $w^*_{n,i}(r)$ satisfies the equation

$$\Delta\omega_\varepsilon + \frac{1}{\varepsilon^2} f_{\varepsilon,i}(x)\omega_\varepsilon = h^*_{n,i}(r), \quad r = |x - x_{\varepsilon,i}| \le d, \tag{A.10}$$

where $h^*_{n,i}(r) = \int_0^{2\pi} h_n(r,\theta)\,d\theta$

$$f_{\varepsilon,i}(x) = \begin{cases} \chi(|x - x_{\varepsilon,i}|)\Big(2\frac{|x-x_{\varepsilon,i}|^4}{\varepsilon^4} e^{2V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})} - \frac{|x-x_{\varepsilon,i}|^2}{\varepsilon^2} e^{V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})}\Big), & 1 \le i \le t; \\ \chi(|x - x_{\varepsilon,i}|)\Big(2e^{2V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})} - e^{V_{\varepsilon,i}(\frac{|x-x_{\varepsilon,i}|}{\varepsilon})}\Big), & t+1 \le i \le k. \end{cases} \tag{A.11}$$

For $i = 1, \cdots, k$, let $\psi_{n,i}(r)$, $r = |x - x_{\varepsilon_n,i}|$, be the solution of

$$-\Delta v = \frac{1}{\varepsilon_n^2} f_{\varepsilon,i}(x) v \tag{A.12}$$

satisfying $\psi_{n,i}(0) = 1$. Then, $\psi_{n,i}(r) = -M_i \ln\frac{|x - x_{\varepsilon_n,i}|}{\varepsilon_n} + O(1)$ for some $M_i > 0$, if $|x - x_{\varepsilon,i}| \ge \varepsilon R$, where $R > 0$ is a large constant. See Remark 2.5 and Lemma 2.2 in [4]. On the other hand, (A.12) has another solution which is given by $\phi_{n,i}(r) = \psi_{n,i}(r)\int_0^r \frac{1}{s\psi^2_{n,i}(s)}\,ds$. Note that $\phi_{n,i}(r) \sim \ln r$ as $r \to 0$. So we have the following relation

$$w^*_{n,i}(r) = w^*_{n,i}(0)\psi_{n,i}(r) + U_{n,i}(r), \tag{A.13}$$

where

$$U_{n,i}(r) = \psi_{n,i}(r)\int_0^r s\phi_{n,i}(s)h^*_{n,i}(s)\,ds - \phi_{n,i}(r)\int_0^r s\psi_{n,i}(s)h^*_{n,i}(s)\,ds. \tag{A.14}$$

It is easy to show that

$$|U_{n,i}(r)| \le C\ln(1 + \frac{r}{\varepsilon_n})\|h_n\|_{Y_{\alpha,\varepsilon}}. \tag{A.15}$$

Integrating (A.1) over Ω, we find

$$\sum_{i=1}^k w^*_{n,i}(0)\int_\Omega \frac{1}{\varepsilon^2} f_{\varepsilon,i}(x)\psi_{n,i}(r) = O\big(\|h_n\|_{Y_{\alpha,\varepsilon_n}} + \varepsilon_n^2|\ln\varepsilon_n|\big). \tag{A.16}$$

Noting that $|\int_\Omega \frac{1}{\varepsilon^2} f_{\varepsilon,i}(x)\psi_{n,i}(r)| \ge c' > 0$ and all $\int_\Omega f_{\varepsilon,i}(x)\psi_{n,i}(r)$ have the same sign, we obtain from (A.16)

$$|w^*_{n,i}(0)| = o\Big(\frac{1}{|\ln\varepsilon_n|}\Big), \tag{A.17}$$

which, together with (A.13) and (A.15), gives (A.9).

Step 5. It follows from Step 4, the constant b_0 in (A.8) must be zero.

Step 6. Let x_n^* be a maximum point of w_n. Then from Steps 2 and 5, we have

$$x_n^* \to q_j, \quad \varepsilon^{-1}|x_n^* - x_{\varepsilon_n,j}| \to +\infty, \tag{A.18}$$

for some $j = 1, \cdots, k$.

It is easy to check that $\|w_n\|_{L^\infty(\Omega)} \to C_0 > 0$. Otherwise, we can deduce from (A.3) and (A.5) that $\|w_n\|_{X_{\alpha,\varepsilon_n}} \to 0$. This will contradict (A.4).

Let $s_n = |x_n^* - x_{\varepsilon_n,j}|$ and $\bar{w}_n = w_n(s_n x + x_{\varepsilon_n,j})$. Then $\bar{w}_n\big(s_n^{-1}(x_n^* - x_{\varepsilon_n,j}\big) = \|w_n\|_{L^\infty(\Omega)}$ and $\bar{w}_n \to \bar{w}_0$ in any compact subset of $\mathbb{R}^2 \setminus \{0\}$. In view of (A.18), $\bar{w}_0$ satisfies $\Delta \bar{w}_0 = 0$. Thus, $\bar{w}_0 = C_0 > 0$. In particular,

$$w_n(x) \ge \frac{1}{2}C_0, \quad s_n \le |x - x_{\varepsilon_n,j}| \le 2s_n.$$

This is a contradiction to (A.9). So we complete the proof of Theorem A.1. □

References

[1] W. Ao, C.-S. Lin, J. Wei, On the non-topological solutions of the A_2 and B_2 Chern–Simons system, preprint.

[2] D. Bartolucci, Y. Lee, C.-S. Lin, M. Onodera, Asymptotic analysis of solutions to a gauged $O(3)$ sigma model, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (3) (2015) 651–685.

[3] L. Caffarelli, Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, Commun. Math. Phys. 168 (2) (1995) 321–336.

[4] H. Chan, C.-C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation, Commun. Math. Phys. 231 (2) (2002) 189–221.

[5] C.-C. Chen, C.-S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, Commun. Pure Appl. Math. 55 (2002) 728–771.

[6] C.-C. Chen, C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, Commun. Pure Appl. Math. 56 (2003) 1667–1727.

[7] S. Chen, Y. Yang, Existence of multiple vortices in supersymmetric gauge field theory, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 468 (2148) (2012) 3923–3946.

[8] J.-L. Chern, Z.-Y. Chen, C.-S. Lin, Uniqueness of topological solutions and the structure of solutions for the Chern–Simons system with two Higgs particles, Commun. Math. Phys. 296 (2010) 323–351.

[9] K. Choe, Asymptotic behavior of condensate solutions in the Chern–Simons–Higgs theory, J. Math. Phys. 48 (2007) 103501.

[10] K. Choe, N. Kim, Blow-up solutions of the self-dual Chern–Simons–Higgs vertex equation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 25 (2008) 313–338.

[11] G.V. Dunne, Aspects of Chern–Simons theory, in: Aspects topologiques de la physique en basse dimension (Topological Aspects of Low Dimensional Systems), Les Houches, 1998, EDP Sci., Les Ulis, 1999, pp. 177–263.

[12] S.B. Gudnason, Non-Abelian Chern–Simons vertices with generic groups, Nucl. Phys. B (2009) 151–169.

[13] S.B. Gudnason, Fractional and semi-local non-Abelian Chern–Simons vortices, Nucl. Phys. B 840 (2010) 160–185.

[14] X. Han, C.-S. Lin, Existence of non-Abelian vortices with product gauge groups, Nucl. Phys. B 878 (2014) 117–149.

[15] X. Han, C.-S. Lin, G. Tarantello, Y. Yang, Chern–Simons vertices in the Gudnason model, preprint.

[16] X. Han, G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern–Simons model, Calc. Var. Partial Differ. Equ. 49 (2014) 1149–1176.

[17] J. Hong, Y. Kim, P. Pac, Multivortex solutions of the abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 64 (1990) 2230–2233.

[18] H. Huang, C.-S. Lin, On the entire radial solutions of the Chern–Simons SU(3) system, Commun. Math. Phys. 327 (3) (2014) 815–848.

[19] R. Jackiw, E. Weinberg, Self-dual Chern–Simons vortex, Phys. Rev. Lett. 64 (1990) 2234–2237.

[20] A. Jaffe, C. Taubes, Vortices and Monopoles, Prog. Phys., vol. 2, Birkhäuser Boston, Boston, MA, 1990.

[21] Yan Yan Li, Harnack type inequality: the method of moving planes, Commun. Math. Phys. 200 (1999) 421–444.

[22] C.-S. Lin, C. Ponce, Y. Yang, A system of elliptic equations arising in Chern–Simons field theory, J. Funct. Anal. 247 (2) (2007) 289–350.

[23] C.-S. Lin, J.V. Prajapat, Vortex condensates for relativistic Abelian Chern–Simons model with two Higgs scalar fields and two gauge fields on a torus, Commun. Math. Phys. 288 (2009) 311–347.

[24] C.-S. Lin, S. Yan, Bubbling solutions for relativistic Abelian Chern–Simons model on a torus, Commun. Math. Phys. 297 (3) (2010) 733–758.

[25] C.-S. Lin, S. Yan, Existence of bubbling solutions for Chern–Simons model on a torus, Arch. Ration. Mech. Anal. 207 (2) (2013) 353–392.

[26] C.-S. Lin, S. Yan, Bubbling solutions for the SU(3) Chern–Simons model on a torus, Commun. Pure Appl. Math. 66 (2013) 991–1027.

[27] C.-S. Lin, Y. Yang, Non-Abelian multiple vortices in supersymmetric field theory, Commun. Math. Phys. 304 (2011) 433–457.

[28] C.-S. Lin, Y. Yang, Sharp existence and uniqueness theorems for non-Abelian multiple vortex solutions, Nucl. Phys. B 846 (3) (2011) 650–676.

[29] G. Lozano, D. Marqueés, E. Moreno, F. Schaposnik, Non-Abelian Chern–Simons vortices, Phys. Lett. B 654 (2007) 27–34.

[30] A. Malchiodi, C. Ndiaye, Some existence results for the Toda system on closed surfaces, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. Rend. Lincei (9) Mat. Appl. 18 (4) (2007) 391–412.

[31] M. Nolasco, G. Tarantello, Double vortex condensates in the Chern–Simons–Higgs theory, Calc. Var. Partial Differ. Equ. 9 (1999) 31–94.

[32] M. Nolasco, G. Tarantello, Vortex condensates for the SU(3) Chern–Simons theory, Commun. Math. Phys. 213 (3) (2000) 599–639.

[33] A. Poliakovsky, G. Tarantello, On a planar Liouville-type problem in the study of selfgravitating strings, J. Differ. Equ. 252 (2012) 3668–3693.
[34] J. Prajapat, G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results, Proc. R. Soc. Edinb., Sect. A 131 (4) (2001) 967–985.
[35] G. Tarantello, Uniqueness of selfdual periodic Chern–Simons vortices of topological-type, Calc. Var. Partial Differ. Equ. 29 (2) (2007) 191–217.
[36] G. Tarantello, Self-Dual Gauge Field Vortices: An Analytical Approach, Springer, 2007.
[37] Y. Yang, The relativistic non-abelian Chern–Simons equations, Commun. Math. Phys. 186 (1997) 199–218.
[38] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer Monogr. Math., Springer-Verlag, New York, 2001.