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Existence and asymptotic behaviour of standing waves for quasilinear Schrödinger–Poisson systems in $\mathbb{R}^3$

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Abstract

In this work we are concerned with existence and asymptotic behaviour of standing wave solutions in the whole space $\mathbb{R}^3$ for the quasilinear Schrödinger–Poisson system

$$-\frac{1}{2}\Delta u + (V + \widetilde{V})u + \omega u = 0,$$
$$-\operatorname{div}\big[\big(1 + \varepsilon^4|\nabla V|^2\big)\nabla V\big] = |u|^2 - n^*,$$

when the nonlinearity coefficient $\varepsilon > 0$ goes to zero. Under appropriate, almost optimal, assumptions on the potential $\widetilde{V}$ and the density n^* we establish existence of a ground state $(u_\varepsilon, V_\varepsilon)$ of the above system, for all ε sufficiently small, and show that $(u_\varepsilon, V_\varepsilon)$ converges to (u_0, V_0), the ground state solution of the corresponding system for $\varepsilon = 0$.

MSC: 35J50; 35Q40

Keywords: Schrödinger equation; Poisson equation; Standing wave solutions; Variational methods; Asymptotic behaviour

1. Introduction and main results

Consider the Schrödinger–Poisson system

$$\mathrm{i}\partial_t u = -\frac{1}{2}\Delta u + (V + \widetilde{V})u,$$
$$-\operatorname{div}\big[\varepsilon(\nabla V)\nabla V\big] = |u|^2 - n^*,$$
$$u(x,0) = u(x).$$

This system corresponds to a quantum mechanical model where the quantum effects are important, as in the case of microstructures (see for example P.A. Markowich, C. Ringhofer and C. Schmeiser [13]). The charge density $n(x,t)$

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derives from the Schrödinger wave function $u(x,t)$ by $n(x,t) = |u(x,t)|^2$, while n^* and $\widetilde{V}$ represent respectively a dopant-density and a real effective potential which are time-independent. More details dealing with the phenomenon may be found in R. Illner, O. Kavian and H. Lange [7], in R. Illner, H. Lange, B. Toomire and P. Zweifel [8] and references therein.

We assume, as in [1] and [6], that the field dependent dielectric constant in Poisson equation has the form

$$\varepsilon(\nabla V) = \varepsilon_0 + \varepsilon_1 |\nabla V|^2, \quad \varepsilon_i > 0.$$

For $\widetilde{V} \equiv 0$ and $n^* \in L^1$, R. Illner, H. Lange, B. Toomire and P. Zweifel have proved in [8] existence and uniqueness of global strong solutions subject to periodic boundary condition on the unit cube $Q := [0,1]^N$, $N = 1, 2, 3$. In [7] the authors have showed an existence result for an infinite number of different standing waves, i.e. solutions of the form

$$u(x,t) = e^{i\omega t} u(x); \quad \omega, u(x) \in \mathbb{R}$$

with periodic boundary conditions, assuming that $\widetilde{V}^- \in L^p([0,1]^N)$ for some $p > \frac{N}{2}$ if $N = 2, 3$ and $p = 1$ if $N = 1$.

In this paper we are mainly concerned with the existence of standing waves (actually ground states) solutions for the Schrödinger–Poisson system in the whole space $\mathbb{R}^3$ and with their asymptotic behaviour when the nonlinearity coefficient in the Poisson equation ε_1 goes to zero. For simplicity of notations, we set $\varepsilon_0 = 1$ and $\varepsilon_1 = \varepsilon^4$ for $\varepsilon > 0$. Thus we are interested in the stationary problem

$$-\frac{1}{2}\Delta u + (V + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

$$-\operatorname{div}\big[\big(1 + \varepsilon^4 |\nabla V|^2\big)\nabla V\big] = |u|^2 - n^* \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

As regards the behaviour of the system (1.1)–(1.2) when $\varepsilon \to 0^+$, the main difficulty here is the fact that Eq. (1.2) is nonlinear in V and the equation is on the unbounded domain $\mathbb{R}^3$, so that the previous techniques used in the above mentioned papers cannot be applied directly.

Let us recall the main result in a previous work [3] where one of the present authors has studied the case $\varepsilon = 0$, namely

$$-\frac{1}{2}\Delta u + (V_0 + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

$$-\Delta V_0 = |u|^2 - n^* \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

More precisely, in [3] existence of a ground state solution for the system (1.3)–(1.4) is shown, under the hypotheses (1.5)–(1.8) and (1.10) below. Recall that, for any fixed $u \in L^{12/5}(\mathbb{R}^3)$, the unique solution of the linear Poisson equation (1.4) in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, where

$$\mathcal{D}^{1,2}(\mathbb{R}^3) := \left\{ v \in L^6(\mathbb{R}^3);\ \int |\nabla v|^2 \,\mathrm{d}x < \infty \right\},$$

denoted by $V_0 := V_0(u)$ is the Newtonian potential of $|u|^2 - n^*$ and has the explicit formula (cf. [5] for instance)

$$V_0(u)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(|u|^2 - n^*)(y)}{|x-y|} \,\mathrm{d}y.$$

We should point out that here we are interested in *nontrivial solutions* of the system (1.1)–(1.2): indeed for $\varepsilon > 0$ if we denote by V_ε^* is the unique solution of

$$-\operatorname{div}\big[\big(1 + \varepsilon^4 |\nabla V_\varepsilon^*|^2\big)\nabla V_\varepsilon^*\big] = -n^* \quad \text{in } \mathbb{R}^3$$

obtained via Lemma 3.1 (see Section 3), then $(0, V_\varepsilon^*)$ is a solution of the system (1.1)–(1.2), but naturally in this paper we are interested in solutions $(u_\varepsilon, V_\varepsilon)$ so that $u_\varepsilon \not\equiv 0$.

For convenience, we shall denote throughout the paper by $\|\cdot\|$ the norm $\|\cdot\|_{L^2}$ on $L^2(\mathbb{R}^3)$. At this point we state the hypotheses which will be assumed for our main results.

Hypotheses. In the sequel we assume the following on $\widetilde{V}$ and n^*:

$$\widetilde{V} \in L^1_{\mathrm{loc}}(\mathbb{R}^3), \quad \widetilde{V}^- \in L^{q_1}(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3), \quad \text{for some } \frac{3}{2} < q_1 < \infty. \tag{1.5}$$

Here as usual $s^- := \max(-s, 0)$ for $s \in \mathbb{R}$. Concerning the way in which this assumption enters our study see below Remark 1.3 at the end of this section.

We suppose that the dopant density n^* satisfies

$$n^* \in L^{6/5}(\mathbb{R}^3) \tag{1.6}$$

and that if we denote by

$$\varrho(x) := 2\widetilde{V}(x) - \frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{n^*(y)}{|x-y|}\,\mathrm{d}y \tag{1.7}$$

we assume that the bottom of the spectrum of the linear operator $u \mapsto -\Delta u + \varrho u$ is below zero, that is

$$\inf\left\{ \int_{\mathbb{R}^3} \left(|\nabla\varphi|^2 + \varrho(x)\varphi^2\right)\mathrm{d}x;\ \varphi \in C_c^1(\mathbb{R}^3),\ \int |\varphi|^2 = 1\right\} < 0. \tag{1.8}$$

Our main results are the following. We begin by the solvability of the nonlinear Poisson equation showing that for all $\varepsilon > 0$ and any fixed u in $L^{12/5}(\mathbb{R}^3)$ Eq. (1.2) has a unique solution $V := V_\varepsilon(u)$ in the space $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ equipped with the natural norm $\|w\|_{\mathcal{D}^{1,2}\cap\mathcal{D}^{1,4}} := \|\nabla w\|_{L^2} + \|\nabla w\|_{L^4}$ where we may define $\mathcal{D}^{1,4}(\mathbb{R}^3)$ as being

$$\mathcal{D}^{1,4}(\mathbb{R}^3) := \left\{ v \in C_0(\mathbb{R}^3);\ \int |\nabla v|^4\,\mathrm{d}x < \infty\right\}.$$

Next we substitute the solution $V_\varepsilon(u)$ in the Schrödinger equation (1.1) and solve the equation thus obtained in a subset H of $H^1(\mathbb{R}^3)$, defined by

$$H := \left\{ u \in H^1(\mathbb{R}^3):\ \int_{\mathbb{R}^3} \widetilde{V}^+ u^2\,\mathrm{d}x < \infty\right\}. \tag{1.9}$$

Note that H is a Hilbert space continuously embedded in $H^1(\mathbb{R}^3)$ when endowed with its natural scalar product and norm

$$(\varphi|\psi) := \int_{\mathbb{R}^3} (\nabla\varphi\cdot\nabla\psi + \varphi\psi + \widetilde{V}^+\varphi\psi)\,\mathrm{d}x, \quad \|\varphi\|_H := (\varphi|\varphi)^{1/2}.$$

Finally, we study the asymptotic behaviour, when $\varepsilon \downarrow 0$, of the solution thus obtained of the system (1.1)–(1.2). More precisely we prove the following results:

Theorem 1.1. *Assuming* (1.5)–(1.8) *then*:

(i) *for $u \in H$ fixed and for all $\varepsilon > 0$, the nonlinear Poisson equation* (1.2) *has a unique solution $V := V_\varepsilon(u)$ in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$*;

(ii) *there exists $\omega_* > 0$ such that for any $0 < \omega < \omega_*$ and all $\varepsilon > 0$ sufficiently small, the system* (1.1)–(1.2) *has a nontrivial ground state solution u_ε which minimizes on H the functional E_ε*

$$E_\varepsilon(u_\varepsilon) = \min_{\varphi\in H} E_\varepsilon(\varphi),$$

where E_ε is defined as

$$E_\varepsilon(\varphi) := \frac{1}{4}\int |\nabla\varphi|^2 + \frac{1}{4}\int \left|\nabla V_\varepsilon(\varphi)\right|^2\mathrm{d}x + \frac{3\varepsilon^4}{8}\int \left|\nabla V_\varepsilon(\varphi)\right|^4\mathrm{d}x + \frac{1}{2}\int \widetilde{V}\varphi^2 + \frac{\omega}{2}\int \varphi^2.$$

Theorem 1.2. *If u_ε denotes the ground state solution of the system* (1.1)–(1.2) *obtained via Theorem* 1.1*, then u_ε is bounded in H and any limit point of u_ε in H-weak when $\varepsilon \downarrow 0$ is a solution of the Schrödinger–Poisson system* (1.3)–(1.4).

Remark 1.3. Concerning the hypothesis on $\widetilde{V}^-$ in (1.5), it is useful to point out what is actually needed. If $\widetilde{V}^- = V_1 + V_0$ with $V_1 \in L^{q_1}(\mathbb{R}^3)$ and $V_0 \in L^{3/2}(\mathbb{R}^3)$, then for $\lambda > 0$ we may write

$$\widetilde{V}^- = \{V_1 + V_0 1_{[|V_0|<\lambda^{-1}]}\} + V_0 1_{[|V_0|\geqslant\lambda^{-1}]} =: V_{1\lambda} + V_{2\lambda},$$

in such a way that $V_{1\lambda} := V_1 + V_0 1_{[|V_0|<\lambda^{-1}]} \in L^{q_1}(\mathbb{R})$ and $V_{2\lambda} \in L^{3/2}(\mathbb{R}^3)$ is small, that is $\lim_{\lambda\to 0} \|V_{2\lambda}\|_{3/2} = 0$ as $\lambda \to 0^+$. As a matter of fact what is needed in our proof of the above theorems is that $\widetilde{V} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and that for some $q_1 \in (3/2, \infty)$ and for a family $(\widetilde{V}_{1\lambda}, \widetilde{V}_{2\lambda})_\lambda$ with $\lambda \to 0^+$ one may write $\widetilde{V}^-$ as

$$\widetilde{V}^- = \widetilde{V}^-_{1\lambda} + \widetilde{V}^-_{2\lambda} \quad \text{and} \quad \widetilde{V}_{1\lambda} \in L^{q_1}(\mathbb{R}^3), \quad \lim_{\lambda\to 0} \|\widetilde{V}^-_{2\lambda}\|_{L^{3/2}} = 0. \tag{1.10}$$

Indeed the decomposition (1.10) contains a large class of potentials of practical physical interest, for example the cases in which

$$\widetilde{V}^-(x) := \sum_{j=1}^m \frac{a_j(x)}{|x - y_j|^{\alpha_j}}$$

for some $0 < \alpha_j < 2$, and $y_j \in \mathbb{R}^3$ while $a_j \in L^\infty(\mathbb{R}^3)$ is nonnegative. As a matter of fact, upon using Hardy inequality, the latter example in which one or several of the exponents α_j's satisfy $\alpha_j = 2$ can be handled provided the corresponding coefficient a_j has a sufficiently small $\|a_j\|_\infty$ norm, but we do not explore this situation thoroughly.

The remainder of this paper is organized as follows. In Section 2 we present several lemmas, useful to the sequel. In Section 3, we study the nonlinear Poisson equation (1.2). We prove existence of a unique solution $V_\varepsilon := V_\varepsilon(u)$ (for any $u \in L^{12/5}$ arbitrarily fixed and all $\varepsilon > 0$) and give its behaviour when $\varepsilon \searrow 0$. Finally, in Section 4 we conclude our study by the proofs of Theorems 1.1 and 1.2.

2. Preliminary results

In this section we recall some definitions and inequalities of Sobolev type for the spaces $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\mathcal{D}^{1,4}(\mathbb{R}^3)$, and we establish a few preliminary lemmas. As in [11] or [14], we define for $p > 1$ the space $\mathcal{D}^{1,p}(\mathbb{R}^3)$ as the completion of $C_c^\infty(\mathbb{R}^3)$ for the norm

$$\|v\|_{\mathcal{D}^{1,p}} = \left(\int_{\mathbb{R}^3} |\nabla v|^p \,\mathrm{d}x\right)^{1/p}.$$

Recall the Sobolev inequality with the best constant S_*

$$\|v\|^2_{L^6(\mathbb{R}^3)} \leqslant S_* \|\nabla v\|^2_{L^2(\mathbb{R}^3)} \tag{2.1}$$

and the Gagliardo–Nirenberg inequality

$$\|v\|_{L^\infty(\mathbb{R}^3)} \leqslant C \|v\|^{1/3}_{L^6(\mathbb{R}^3)} \|\nabla v\|^{2/3}_{L^4(\mathbb{R}^3)}. \tag{2.2}$$

Recall also that $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\mathcal{D}^{1,4}(\mathbb{R}^3)$ are continuously embedded respectively in $L^6(\mathbb{R}^3)$ and in $C_0(\mathbb{R}^3)$ (the space of continuous functions which converge to zero at infinity). Taking into account those embeddings, equivalent characterizations are

$$\mathcal{D}^{1,2}(\mathbb{R}^3) = \{v \in L^6(\mathbb{R}^3);\ |\nabla v| \in L^2(\mathbb{R}^3)\},$$
$$\mathcal{D}^{1,4}(\mathbb{R}^3) = \{v \in C_0(\mathbb{R}^3);\ |\nabla v| \in L^4(\mathbb{R}^3)\}.$$

In Section 4 we will establish and prove the expression of the energy functional E_ε corresponding to (1.1) (coupled with (1.2)), for which we require the two following lemmas

Lemma 2.1. *For any two functions $u, v \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ we have the monotonicity inequality*

$$\int \{(1 + |\nabla u|^2)\nabla u - (1 + |\nabla v|^2)\nabla v\} \cdot \nabla(u - v)\,\mathrm{d}x \geqslant \|\nabla(u - v)\|^2 + \frac{1}{4}\|\nabla(u - v)\|^4_{L^4}.$$

Proof. This is a well-known consequence of the fact that the operator

$$A_0 v := -\operatorname{div}\big(|\nabla v|^2 \nabla v\big)$$

is strongly monotone on $\mathcal{D}^{1,4}(\mathbb{R}^3)$ (see [10] for instance). However for the reader's convenience we reproduce here a complete proof. For all $x, y \in \mathbb{R}^3$ we write

$$\big\{\big(1+|x|^2\big)x - \big(1+|y|^2\big)y\big\} \cdot (x-y) = |x-y|^2 + \big(|x|^2 x - |y|^2 y\big) \cdot (x-y).$$

For the second term in the right-hand side, we set $z := x - y$ and an elementary calculation gives

$$\begin{aligned}\big(|x|^2 x - |y|^2 y\big) \cdot (x-y) &= |z|^4 + 3|z|^2(y\cdot z) + 2(y\cdot z)^2 + |z|^2|y|^2 \\ &\geqslant |z|^4 + 3|z|^2(y\cdot z) + 3(y\cdot z)^2.\end{aligned}$$

By using Young's inequality, we get

$$|z|^2(y\cdot z) \geqslant -\frac{1}{4}|z|^4 - (y\cdot z)^2$$

and therefore

$$\big(|x|^2 x - |y|^2 y\big) \cdot (x-y) \geqslant \frac{1}{4}|z|^4.$$

Consequently

$$\big\{\big(1+|x|^2\big)x - \big(1+|y|^2\big)y\big\} \cdot (x-y) \geqslant |x-y|^2 + \frac{1}{4}|x-y|^4$$

and this immediately yields the desired inequality by taking $x := \nabla u$ and $y := \nabla v$ for $u, v \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$. □

Now, in order to prove Theorem 1.1 we need the following inequality in some of our forthcoming estimates.

Lemma 2.2. *Let $\theta \in L^r(\mathbb{R}^3)$ for some $r \in [3/2, \infty]$. Then for all $\delta > 0$, there exists $C_\delta > 0$ such that for any $\varphi \in H^1(\mathbb{R}^3)$ one has*

$$\int_{\mathbb{R}^3} \theta(x)\big|\varphi(x)\big|^2 \,\mathrm{d}x \leqslant \delta\|\nabla\varphi\|^2 + C_\delta\|\varphi\|^2. \tag{2.3}$$

This is a consequence of the fact that on the one hand $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, and that on the other hand $\theta = \theta 1_{[|\theta|>\lambda]} + \theta 1_{[|\theta|\leqslant\lambda]}$ for any $\lambda > 0$ and that λ can be chosen large enough in order to have $\|\theta 1_{[|\theta|>\lambda]}\|_{3/2}$ as small as one may desire (see [4] or [2] for more details).

Remark 2.3. Note that according to assumption (1.5), taking $r = q_1$ and $r = 3/2$ in Lemma 2.2, one sees that $\widetilde{V}^-$ satisfies inequality (2.3), i.e. for all $\delta > 0$, there exists $C_\delta > 0$ such that for all $\varphi \in H^1(\mathbb{R}^3)$ one has

$$\int_{\mathbb{R}^3} \widetilde{V}^-(x)\big|\varphi(x)\big|^2 \,\mathrm{d}x \leqslant \delta\|\nabla\varphi\|^2 + C_\delta\|\varphi\|^2. \tag{2.4}$$

To end this section, we state the following lemma which will be used to show that the functional E_ε is weakly sequentially lower semi-continuous on H.

Lemma 2.4. *Let $\psi \in L^r(\mathbb{R}^3)$ for some $r > 3/2$ and assume that $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ then*

$$\int_{\mathbb{R}^3} \psi(x) v_n^2(x)\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n \to +\infty.$$

The proof is based on the fact that on the one hand $v_n \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for all $p < 6$, and that on the hand for any $\lambda > 0$, which may be chosen as small as needed, $\operatorname{meas}([|\psi| \geqslant \lambda])$ is finite and the measurable set $[|\psi| \geqslant \lambda]$ can be approximated by compact sets; the details can be found in [2].

3. Study of the nonlinear Poisson equation

In this section we shall prove existence of a unique solution V_ε of the nonlinear Poisson equation (1.2) and will give its behaviour when the coefficient of the nonlinearity ε goes to zero.

Lemma 3.1. *For all $f \in L^{6/5}$ and all $\varepsilon > 0$, there is a unique weak solution $W_\varepsilon(f) \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ for Eq.* (1.2) *in the sense that for any $\psi \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ we have*

$$\int_{\mathbb{R}^3} \left(1 + \varepsilon^4 |\nabla W_\varepsilon|^2\right) \nabla W_\varepsilon \cdot \nabla \psi \, dx = \int_{\mathbb{R}^3} f \psi \, dx. \tag{3.1}$$

Proof. By minimizing the corresponding energy functional we shall check that for all $f \in L^{6/5}$ the equation

$$-\operatorname{div}\left[\left(1 + \varepsilon^4 |\nabla W|^2\right) \nabla W\right] = f \tag{3.2}$$

has a unique weak solution $W_\varepsilon(f)$ in the Banach space $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ when equipped with the natural norm $\|\nabla v\| + \|\nabla v\|_{L^4}$. In fact, Eq. (3.2) is the Euler–Lagrange equation of the functional

$$J_\varepsilon(v) := \frac{1}{2} \int |\nabla v|^2 \, dx + \frac{\varepsilon^4}{4} \int |\nabla v|^4 \, dx - \int f v \, dx \tag{3.3}$$

which we shall minimize on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$. Let $\varepsilon = 1$ for simplicity. It is not difficult to see that J_1 is strictly convex and C^1 on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}$. Moreover, it is standard to check that

$$\begin{aligned} J_1(v) &\geqslant \frac{1}{2}\|\nabla v\|^2 + \frac{1}{4}\|\nabla v\|_{L^4}^4 - S_*^{1/2}\|f\|_{L^{6/5}}\|\nabla v\| \\ &\geqslant \frac{1}{4}\|\nabla v\|^2 + \frac{1}{4}\|\nabla v\|_{L^4}^4 - S_*\|f\|_{L^{6/5}}^2 \end{aligned}$$

where in the first inequality we have used successively Hölder's inequality and the Sobolev's inequality (2.1) while the second follows from Young's inequality. This implies that J_1 is coercive i.e. $J_1(v) \to \infty$ when $\|v\|_{\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}} \to \infty$. It is a classical result that (see for instance [9, Corollaire 3.1.4] or [15, Theorem 1.2]) J_1 achieves its minimum at a unique $W_1 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ and therefore

$$\left\langle J_1'(W_1), \psi \right\rangle = 0 \qquad \forall \psi \in \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3). \qquad \square$$

Remark that Lemma 3.1 prove the part (i) of Theorem 1.1. Indeed, if $u \in H^1(\mathbb{R}^3)$ then $|u|^2 \in L^{6/5}(\mathbb{R}^3)$. So, taking $f := |u|^2 - n^*$, and using the assumption (1.6) then Lemma 3.1 ensures that the Poisson equation (1.2) has a unique solution $V_\varepsilon(u) := W_\varepsilon(|u|^2 - n^*)$ for all $\varepsilon > 0$.

Now we shall prove the following lemma which deals with the behaviour of the solution $V_\varepsilon(u)$ of (1.2) when $\varepsilon \to 0^+$.

Lemma 3.2. *For $\varepsilon > 0$, consider f_ε and f in $L^{6/5}(\mathbb{R}^3)$ and let $W_0(f)$ be the unique solution of $-\Delta W_0 = f$ in $\mathbb{R}^3$. Then*:

(i) *if $f_\varepsilon \rightharpoonup f$ weakly in $L^{6/5}(\mathbb{R}^3)$ then $W_\varepsilon(f_\varepsilon) \rightharpoonup W_0(f)$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\varepsilon \to 0$.*
(ii) *If $f_\varepsilon \to f$ strongly in $L^{6/5}(\mathbb{R}^3)$ then*:

$$W_\varepsilon(f_\varepsilon) \longrightarrow W_0(f) \quad \textit{strongly in } \mathcal{D}^{1,2}(\mathbb{R}^3),$$
$$\varepsilon W_\varepsilon(f_\varepsilon) \longrightarrow 0 \quad \textit{strongly in } \mathcal{D}^{1,4}(\mathbb{R}^3).$$

Proof. Denote by $W_\varepsilon(f_\varepsilon)$ and $W_\varepsilon(f)$ the solutions to (3.2) respectively for f_ε and f. Multiplying (3.2), with f replaced with f_ε, by the corresponding solution $W_\varepsilon(f_\varepsilon)$ and integrating by parts we get

$$\int \left|\nabla W_\varepsilon(f_\varepsilon)\right|^2 dx + \varepsilon^4 \int \left|\nabla W_\varepsilon(f_\varepsilon)\right|^4 dx = \int f_\varepsilon W_\varepsilon(f_\varepsilon) \, dx. \tag{3.4}$$

By Hölder's and Sobolev's inequalities we obtain

$$\int f_\varepsilon W_\varepsilon(f_\varepsilon)\,\mathrm{d}x \leqslant S_*^{1/2}\|f_\varepsilon\|_{L^{6/5}}\big\|\nabla W_\varepsilon(f_\varepsilon)\big\|.$$

Thus, it follows from (3.4) that, successively

$$\big\|\nabla W_\varepsilon(f_\varepsilon)\big\| \leqslant S_*^{1/2}\|f_\varepsilon\|_{L^{6/5}} \quad \text{and} \quad \varepsilon^4\big\|\nabla W_\varepsilon(f_\varepsilon)\big\|_{L^4}^4 \leqslant S_*\|f_\varepsilon\|_{L^{6/5}}^2. \tag{3.5}$$

Consider now $W_0(f) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ the unique solution of $-\Delta W_0 = f$ (see for instance [2, Lemma 2.1]). We will prove successively that $W_\varepsilon(f_\varepsilon)$ converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ to $W_0(f)$ (using only the fact that $f_\varepsilon \rightharpoonup f$ weakly in $L^{6/5}$) and that this convergence is in fact strong.

• **Step 1.** In a first step we show property (i) of our lemma. Indeed, since $W_\varepsilon(f_\varepsilon)$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ there exists $W^* \in \mathcal{D}^{1,2}$ and a subsequence denoted again by $W_\varepsilon(f_\varepsilon)$ so that $W_\varepsilon(f_\varepsilon) \rightharpoonup W^*$ weakly in $\mathcal{D}^{1,2}$. Using (3.1) we have for any $\psi \in C_c^\infty(\mathbb{R}^3)$

$$\int \nabla W_\varepsilon(f_\varepsilon)\cdot\nabla\psi\,\mathrm{d}x + \varepsilon^4\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^2\nabla W_\varepsilon(f_\varepsilon)\cdot\nabla\psi\,\mathrm{d}x = \int f_\varepsilon\psi\,\mathrm{d}x. \tag{3.6}$$

Knowing by (3.5) that $(\varepsilon\nabla W_\varepsilon(f_\varepsilon))$ is bounded in $L^4(\mathbb{R}^3)$ we estimate

$$\left|\varepsilon^4\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^2\nabla W_\varepsilon(f_\varepsilon)\cdot\nabla\psi\right| \leqslant \varepsilon\big\|\varepsilon\nabla W_\varepsilon(f_\varepsilon)\big\|_{L^4}^3\|\nabla\psi\|_{L^4} \leqslant \varepsilon C$$

for some constant $C > 0$ independent of ε. Passing to the limit in (3.6) and using only the weak convergence of f_ε to f in $L^{6/5}(\mathbb{R}^3)$, we obtain

$$\int \nabla W^*\cdot\nabla\psi\,\mathrm{d}x = \int f\psi\,\mathrm{d}x \quad \forall\psi \in C_c^\infty(\mathbb{R}^3).$$

Hence W^* satisfies $-\Delta W^* = f$ in $\mathcal{D}'(\mathbb{R}^3)$. Thanks to the uniqueness of the solution of $-\Delta W_0 = f$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, we conclude that $W^* = W_0(f)$ and that *all* the sequence $(W_\varepsilon(f_\varepsilon))_\varepsilon$ verifies

$$W_\varepsilon(f_\varepsilon) \rightharpoonup W_0(f) \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^3). \tag{3.7}$$

• **Step 2.** Now we assert that $W_\varepsilon(f_\varepsilon) \to W_0(f)$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ if $f_\varepsilon \to f$ strongly in $L^{6/5}$.

Indeed, on account of the first step it suffices to show that $\int|\nabla W_\varepsilon(f_\varepsilon)|^2$ converges to $\int|\nabla W_0(f)|^2$. Since $W_\varepsilon(f_\varepsilon) \rightharpoonup W_0(f)$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ we have in particular

$$\int\big|\nabla W_0(f)\big|^2 \leqslant \liminf_{\varepsilon\to 0}\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^2. \tag{3.8}$$

On the other hand, consider $(W_0^j)_j \subset C_c^\infty(\mathbb{R}^3)$ so that $W_0^j \to W_0(f)$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $j \to \infty$. Knowing that $W_\varepsilon(f_\varepsilon)$ is a minimizer on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ of the functional $\widetilde{J}_\varepsilon$ defined, as in (3.3) taking f_ε instead of f, by

$$\tilde{J}_\varepsilon(v) := \frac{1}{2}\int|\nabla v|^2\,\mathrm{d}x + \frac{\varepsilon^4}{4}\int|\nabla v|^4\,\mathrm{d}x - \int f_\varepsilon v\,\mathrm{d}x$$

we may write $\tilde{J}_\varepsilon(W_\varepsilon(f_\varepsilon)) \leqslant \tilde{J}_\varepsilon(W_0^j)$ and consequently we have

$$\begin{aligned}\frac{1}{2}\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^2\,\mathrm{d}x &= \tilde{J}_\varepsilon\big(W_\varepsilon(f_\varepsilon)\big) - \frac{\varepsilon^4}{4}\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^4\,\mathrm{d}x + \int f_\varepsilon W_\varepsilon(f_\varepsilon)\,\mathrm{d}x \\ &\leqslant \tilde{J}_\varepsilon(W_0^j) + \int f_\varepsilon W_\varepsilon(f_\varepsilon)\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\int|\nabla W_0^j|^2\,\mathrm{d}x + \frac{\varepsilon^4}{4}\int|\nabla W_0^j|^4\,\mathrm{d}x - \int f_\varepsilon W_0^j\,\mathrm{d}x + \int f_\varepsilon W_\varepsilon(f_\varepsilon)\,\mathrm{d}x.\end{aligned}$$

Since $W_\varepsilon(f_\varepsilon) \rightharpoonup W_0(f)$ weakly in L^6 and f_ε converges strongly to f in $L^{6/5}$, then

$$\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int\big|\nabla W_\varepsilon(f_\varepsilon)\big|^2\,\mathrm{d}x\right) \leqslant \frac{1}{2}\int|\nabla W_0^j|^2\,\mathrm{d}x - \int fW_0^j\,\mathrm{d}x + \int fW_0(f)\,\mathrm{d}x.$$

Letting j go to infinity, we obtain

$$\limsup_{\varepsilon\to 0}\int\left|\nabla W_\varepsilon(f_\varepsilon)\right|^2\mathrm{d}x\leqslant\int\left|\nabla W_0(f)\right|^2\mathrm{d}x.$$

This inequality, combined with (3.8), gives

$$\int\left|\nabla W_\varepsilon(f_\varepsilon)\right|^2\mathrm{d}x\longrightarrow\int\left|\nabla W_0(f)\right|^2\mathrm{d}x\quad\text{as }\varepsilon\searrow 0.$$

Weak convergence and convergence of norms imply that $W_\varepsilon(f_\varepsilon)\to W_0(f)$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Finally to see that $\varepsilon\nabla W_\varepsilon(f_\varepsilon)\to 0$ strongly in $L^4(\mathbb{R}^3)$, we write from (3.4)

$$\varepsilon^4\int\left|\nabla W_\varepsilon(f_\varepsilon)\right|^4\mathrm{d}x=\int f_\varepsilon W_\varepsilon(f_\varepsilon)\,\mathrm{d}x-\int\left|\nabla W_\varepsilon(f_\varepsilon)\right|^2\mathrm{d}x.$$

Passing to the limit as $\varepsilon\to 0$ we conclude easily that

$$\varepsilon^4\int\left|\nabla W_\varepsilon(f_\varepsilon)\right|^4\mathrm{d}x\longrightarrow\int fW_0(f)\,\mathrm{d}x-\int\left|\nabla W_0(f)\right|^2\mathrm{d}x=0$$

which completes the proof of Lemma 3.2 □

4. The nonlinear Schrödinger equation

Our purpose in this last section is to study the Schrödinger equation after having showed existence and uniqueness of the solution for the nonlinear Poisson's equation (1.2). Precisely we solve Eq. (1.1) for ε small enough and establish the behaviour of the solution thus obtained when $\varepsilon\downarrow 0$. In other words we will prove successively Theorems 1.1 and 1.2.

4.1. Proof of Theorem 1.1

(i) The first point in Theorem 1.1, is a direct consequence of Lemma 3.1 taking $f:=|u|^2-n^*$ and setting $V_\varepsilon(u):=W_\varepsilon(|u|^2-n^*)$, as it was pointed out in the remark after the proof of Lemma 3.1.

(ii) After solving the nonlinear Poisson's equation (1.2), we plug its unique solution $V_\varepsilon(u):=W_\varepsilon(|u|^2-n^*)$ for any fixed $u\in H$ (the space defined by (1.9)) into the Schrödinger equation (1.1). Hence we will prove existence of a ground state for the equation

$$-\frac{1}{2}\Delta u+\left(V_\varepsilon(u)+\widetilde{V}\right)u+\omega u=0\quad\text{in }\mathbb{R}^3.\tag{4.1}$$

To this end we shall minimize the functional

$$E_\varepsilon(\varphi):=\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}x+I_\varepsilon(\varphi)+\frac{1}{2}\int_{\mathbb{R}^3}\widetilde{V}\varphi^2\,\mathrm{d}x+\frac{\omega}{2}\int_{\mathbb{R}^3}\varphi^2\,\mathrm{d}x\tag{4.2}$$

on the space H, where

$$I_\varepsilon(\varphi):=\frac{1}{4}\int_{\mathbb{R}^3}\left|\nabla V_\varepsilon(\varphi)\right|^2\mathrm{d}x+\frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3}\left|\nabla V_\varepsilon(\varphi)\right|^4\mathrm{d}x.\tag{4.3}$$

However, we must first prove that the expression of E_ε in (4.2)–(4.3) corresponds exactly to the energy functional of Eq. (4.1). More precisely, we just have to prove that the derivative of I_ε gives exactly the term $V_\varepsilon(u)u$ in Eq. (4.1). So we state the following

Proposition 4.1. *For any $\varepsilon>0$ the functional $\varphi\mapsto I_\varepsilon(\varphi)$ is C^1 on $H^1(\mathbb{R}^3)$ and its Fréchet-derivative satisfies*

$$\left\langle I_\varepsilon'(\varphi),\psi\right\rangle=\int V_\varepsilon(\varphi)\varphi\psi\,\mathrm{d}x\quad\forall\varphi,\psi\in H^1(\mathbb{R}^3).\tag{4.4}$$

Proof. For the sake of simplicity, and without loss of generality, we may set $\varepsilon := 1$. We shall follow the same steps of the proof of [7, Lemma 2.2] i.e. we will show that $\varphi \mapsto I_1(\varphi)$ is Gâteaux-differentiable on $H^1(\mathbb{R}^3)$ namely

$$\lim_{t\to 0^+} \frac{I_1(\varphi + t\psi) - I_1(\varphi)}{t} = \int V_1(\varphi)\varphi\psi \, dx \quad \forall \varphi, \psi \in H^1(\mathbb{R}^3)$$

and that the Gâteaux differential is continuous from $H^1(\mathbb{R}^3)$ into its dual space $H^{-1}(\mathbb{R}^3)$. However due to the fact that here we are dealing with $H^1(\mathbb{R}^3)$ instead of $H^1((0,1)^3)$, the final step uses a slightly different argument (see in particular below the proof of Lemma 4.10).

Denote by $\mathcal{A}$ the operator defined on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ by

$$\mathcal{A}W := -\operatorname{div}\big[\big(1 + |\nabla W|^2\big)\nabla W\big]$$

and set for convenience $V^t := V_1(\varphi + t\psi)$ and $V^0 := V_1(\varphi)$ which are respectively the unique solution in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ of

$$\mathcal{A}V^t = |\varphi + t\psi|^2 - n^* \quad \text{and} \quad \mathcal{A}V^0 = |\varphi|^2 - n^*. \tag{4.5}$$

Let us remark first that multiplying $(\mathcal{A}V^t - \mathcal{A}V^0)$ by $(V^t - V^0)$ and using Lemma 2.1, we obtain

$$\langle \mathcal{A}V^t - \mathcal{A}V^0, V^t - V^0 \rangle \geqslant \big\|\nabla(V^t - V^0)\big\|^2 + \frac{1}{4}\big\|\nabla(V^t - V^0)\big\|_{L^4}^4. \tag{4.6}$$

On the other hand we have

$$\langle \mathcal{A}V^t - \mathcal{A}V^0, V^t - V^0 \rangle = \int (2t\varphi\psi + t^2\psi^2)(V^t - V^0)\, dx.$$

Hence, setting finally

$$Z^t := \frac{V^t - V^0}{t} \quad \text{and} \quad m^t := \frac{|\varphi + t\psi|^2 - |\varphi|^2}{t} = 2\varphi\psi + t\psi^2,$$

inequality (4.6) becomes

$$\|\nabla Z^t\|^2 + \frac{t^2}{4}\|\nabla Z^t\|_{L^4}^4 \leqslant \int m^t Z^t \, dx. \tag{4.7}$$

Note that since $\varphi, \psi \in H^1(\mathbb{R}^3)$ then $\varphi, \psi \in L^{12/5}(\mathbb{R}^3)$ and therefore $\|m^t\|_{L^{6/5}}$ is uniformly bounded for $0 < t < 1$. Hence by Hölder and Sobolev inequalities we have $\int m^t Z^t \, dx \leqslant C\|\nabla Z^t\|$ and taking into account (4.7) we get successively

$$\|\nabla Z^t\| \leqslant C \quad \text{and} \quad \|\nabla Z^t\|_{L^4} \leqslant C t^{-1/2} \tag{4.8}$$

for some constant C independent of t. Going back to the Gâteaux-derivative of I_1, we may write

$$\frac{I_1(\varphi + t\psi) - I_1(\varphi)}{t} = \frac{1}{4t}\int \big(|\nabla V^t|^2 - |\nabla V^0|^2\big)\, dx + \frac{3}{8t}\int \big(|\nabla V^t|^4 - |\nabla V^0|^4\big)\, dx.$$

Replacing V^t by $V^0 + tZ^t$ and after elementary calculations, we obtain

$$\frac{I_1(\varphi + t\psi) - I_1(\varphi)}{t} = \frac{1}{2}\int (\nabla V^0 \cdot \nabla Z^t) + \frac{3}{2}\int |\nabla V^0|^2 (\nabla V^0 \cdot \nabla Z^t) + tR(t) \tag{4.9}$$

where we have set

$$R(t) := R_1(t) + R_2(t) + R_3(t),$$
$$R_1(t) := \frac{1}{4}\int |\nabla Z^t|^2 \, dx + \frac{3}{2}\int (\nabla V^0 \cdot \nabla Z^t)^2 \, dx + \frac{3}{4}\int |\nabla V^0|^2 |\nabla Z^t|^2 \, dx,$$
$$R_2(t) := \frac{3t}{2}\int |\nabla Z^t|^2 (\nabla V^0 \cdot \nabla Z^t)\, dx,$$
$$R_3(t) := \frac{3}{8}t^2 \int |\nabla Z^t|^4 \, dx.$$

The rest of the proof will be done in two steps: we prove in a first lemma that $R(t)$ is uniformly bounded for $t > 0$ small enough. In a second lemma we show that Z^t converges weakly in a certain subspace of $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Lemma 4.2. *The term $R(t)$ in* (4.9) *is uniformly bounded for $t > 0$ small enough.*

Proof. According to (4.8), the term $R_3(t)$ is obviously bounded for $0 < t < 1$. On the other hand, multiplying $\mathcal{A}V^t - \mathcal{A}V^0$ by $V^t - V^0$ and replacing $V^t - V^0$ by tZ^t we get after some calculations

$$\int |\nabla Z^t|^2 + \int |\nabla V^0|^2 |\nabla Z^t|^2 + 2\int (\nabla V^0 \cdot \nabla Z^t)^2 + 2R_2(t) + \frac{8}{3}R_3(t) = \int m^t Z^t. \tag{4.10}$$

Now, using successively Hölder and Young's inequalities we get

$$\begin{aligned}
\left|2R_2(t)\right| &\leqslant 3t\|\nabla Z^t\|_{L^4}^2 \left(\int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x\right)^{1/2} \\
&\leqslant C_1 t^2 \|\nabla Z^t\|_{L^4}^4 + \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x \\
&\leqslant \frac{8}{3} C_1 R_3(t) + \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x \\
&\leqslant C_2 + \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x,
\end{aligned} \tag{4.11}$$

since we know already that $R_3(t)$ is bounded. It follows from inequality (4.11) that

$$2R_2(t) \geqslant -C_2 - \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x$$

which, plugged into (4.10), gives

$$\int \left(1 + |\nabla V^0|^2\right)|\nabla Z^t|^2 \,\mathrm{d}x + \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x \leqslant C_2 + \int m^t Z^t.$$

Knowing that $\|m^t\|_{L^{6/5}}$ and $\|Z^t\|_{L^6}$ are bounded for $0 < t < 1$, then

$$\int \left(1 + |\nabla V^0|^2\right)|\nabla Z^t|^2 \,\mathrm{d}x + \int (\nabla V^0 \cdot \nabla Z^t)^2 \,\mathrm{d}x \leqslant C_3. \tag{4.12}$$

Note that all constants C_1, C_2 and C_3 are positive and independent of t. The last inequality (4.12) shows that $R_1(t)$ is bounded for $0 < t < 1$. Therefore $R_2(t)$ is bounded too, since inequalities (4.11) and (4.12) imply in particular that

$$\left|2R_2(t)\right| \leqslant C_2 + C_3 \tag{4.13}$$

and so the lemma is proved. □

Lemma 4.3. *Let X be the Hilbert space, continuously embedded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, obtained upon completion of $C_c^\infty(\mathbb{R}^3)$ with the norm*

$$\|\Psi\|_X^2 := \int_{\mathbb{R}^3} \left(1 + |\nabla V^0|^2\right)|\nabla \Psi|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} (\nabla V^0 \cdot \nabla \Psi)^2 \,\mathrm{d}x.$$

Then there exists $Z^\star \in X$ such that all the sequence $(Z^t)_t$ satisfies

$$Z^t \rightharpoonup Z^\star \quad \text{weakly in } X \text{ as } t \searrow 0.$$

Proof. Multiplying both equalities in (4.5) by $u \in C_c^\infty(\mathbb{R}^3)$ and subtracting, we get

$$\int \left(1 + |\nabla V^0|^2\right)(\nabla Z^t \cdot \nabla u) \,\mathrm{d}x + 2\int (\nabla V^0 \cdot \nabla Z^t)(\nabla V^0 \cdot \nabla u) \,\mathrm{d}x + S(t) = \int m^t u \,\mathrm{d}x \tag{4.14}$$

where $S(t) := S_1(t) + S_2(t)$ and

$$S_1(t) := 2t \int (\nabla V^0 \cdot \nabla Z^t)(\nabla Z^t \cdot \nabla u) \,\mathrm{d}x + t \int |\nabla Z^t|^2 (\nabla V^0 \cdot \nabla u) \,\mathrm{d}x,$$

$$S_2(t) := t^2 \int |\nabla Z^t|^2 (\nabla Z^t \cdot \nabla u) \,\mathrm{d}x.$$

On the other hand, inequality (4.12) means that $(Z^t)_t$ is bounded in X. Hence there exists $Z^\star \in X$ and a subsequence $(Z^{t_n})_n$ such that

$$Z^{t_n} \rightharpoonup Z^\star \quad \text{weakly in } X \text{ as } n \to \infty.$$

Now, we want to show that $Z^\star$ verifies a certain equation by passing to the limit in (4.14). To this end, we will prove first that in (4.14) the term $S(t)$ converges to 0 as $t \to 0^+$. Indeed, we may estimate easily

$$\begin{aligned}\left|S_1(t)\right| &\leqslant 3t\|\nabla u\|_\infty \int |\nabla V^0||\nabla Z^t|^2\,\mathrm{d}x \\ &\leqslant 3t\|\nabla u\|_\infty \left(\int |\nabla V^0|^2|\nabla Z^t|^2\,\mathrm{d}x\right)^{1/2}\|\nabla Z^t\| \\ &\leqslant Ct,\end{aligned}$$

for a positive constant C independent of t. Note that the last inequality holds thanks to (4.12) (for instance). Regarding $S_2(t)$, using Hölder inequality and (4.8) we have

$$\begin{aligned}\left|S_2(t)\right| &\leqslant t^2 \int |\nabla Z^t|^3|\nabla u|\,\mathrm{d}x \\ &\leqslant t^2\|\nabla Z^t\|_{L^4}^3\|\nabla u\|_{L^4} \\ &\leqslant C_0 t^{1/2}.\end{aligned}$$

Consequently $\lim_{t\to 0^+} S(t) = 0$. Letting n go to ∞ in Eq. (4.14) written for $Z^t = Z^{t_n}$ and knowing that $Z^{t_n} \rightharpoonup Z^\star$ weakly in the Hilbert space X and that $m^{t_n} \to 2\varphi\psi$ strongly in $L^{6/5}$ as $n \to +\infty$, then we obtain for all $u \in C_c^\infty(\mathbb{R}^3)$:

$$\int \left(1 + |\nabla V^0|^2\right)(\nabla Z^\star \cdot \nabla u)\,\mathrm{d}x + 2\int (\nabla V^0 \cdot \nabla Z^\star)(\nabla V^0 \cdot \nabla u)\,\mathrm{d}x = 2\int \varphi\psi u\,\mathrm{d}x. \tag{4.15}$$

Furthermore, the variational problem (4.15) has a unique solution in X according to the Lax–Milgram theorem. Hence, *all* the sequence $(Z^t)_t$ converges to $Z^\star$ weakly in X as $t \searrow 0$. □

Now, for the remainder of the proof of Proposition 4.1, we will pass to the limit in equality (4.9) as $t \searrow 0$. To this end, note that in the Hilbert space X defined in Lemma 4.3, the mapping $\Psi \mapsto \int |\nabla V^0|^2(\nabla V^0 \cdot \nabla\Psi)\,\mathrm{d}x$ is a continuous linear form on X. Hence, using Lemma 4.3 we infer that

$$\int |\nabla V^0|^2(\nabla V^0 \cdot \nabla Z^t)\,\mathrm{d}x \longrightarrow \int |\nabla V^0|^2(\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x \quad \text{as } t \searrow 0. \tag{4.16}$$

Knowing that X is continuously embedded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, we infer also that

$$\int (\nabla V^0 \cdot \nabla Z^t)\,\mathrm{d}x \longrightarrow \int (\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x \quad \text{as } t \searrow 0. \tag{4.17}$$

Letting t go to 0^+ in (4.9) and using (4.16), (4.17) and Lemma 4.2, we get

$$\lim_{t\to 0^+} \frac{I_1(\varphi + t\psi) - I_1(\varphi)}{t} = \frac{1}{2}\int (\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x + \frac{3}{2}\int |\nabla V^0|^2(\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x.$$

On the other hand, taking $u = V^0$ in (4.15) we obtain

$$\int (\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x + 3\int |\nabla V^0|^2(\nabla V^0 \cdot \nabla Z^\star)\,\mathrm{d}x = 2\int \varphi\psi V^0\,\mathrm{d}x.$$

Consequently we have

$$\lim_{t\to 0^+} \frac{I_1(\varphi + t\psi) - I_1(\varphi)}{t} = \int \varphi\psi V^0\,\mathrm{d}x.$$

Recall that $V^0 := V_1(\varphi)$, so we infer that the Gâteaux-derivative of I_1 satisfies (4.4).

In order to finish our proof of Proposition 4.1, we just have to verify that $\varphi \mapsto \varphi V_\varepsilon(\varphi)$ is continuous from $H^1(\mathbb{R}^3)$ into $H^{-1}(\mathbb{R}^3)$. Indeed, it is clear that $\varphi \mapsto (|\varphi|^2 - n^*)$ is continuous from $H^1(\mathbb{R}^3)$ into $L^{6/5}(\mathbb{R}^3)$. Moreover, the same argument leading to (3.5) by using Lemma 2.1, yields that for any $f, g \in L^{6/5}(\mathbb{R}^3)$ and all $\varepsilon > 0$ we have the two following estimates

$$\left\|\nabla W_\varepsilon(f) - \nabla W_\varepsilon(g)\right\| \leqslant S_*^{1/2}\|f-g\|_{L^{6/5}}, \tag{4.18}$$

$$\varepsilon\left\|\nabla W_\varepsilon(f) - \nabla W_\varepsilon(g)\right\|_{L^4} \leqslant \sqrt{2}S_*^{1/4}\|f-g\|_{L^{6/5}}^{1/2}. \tag{4.19}$$

This means that $f \mapsto W_\varepsilon(f)$ is continuous from $L^{6/5}(\mathbb{R}^3)$ into $\mathcal{D}^{1,2}\cap\mathcal{D}^{1,4}(\mathbb{R}^3)$. We conclude easily by remarking that $\varphi V_\varepsilon(\varphi) := \varphi W_\varepsilon(|\varphi|^2 - n^*) \in L^2$ for all $\varphi \in H^1(\mathbb{R}^3)$. The proof of Proposition 4.1 is thus complete and consequently the energy functional E_ε corresponding to (4.1) is exactly the expression (4.2)–(4.3). □

Now, we are in a position to prove part (ii) of Theorem 1.1 by minimizing the functional E_ε. For $\varepsilon > 0$ and $c \in \mathbb{R}$ we denote by $[E_\varepsilon \leqslant c]$ the subset of the Hilbert space H defined as

$$[E_\varepsilon \leqslant c] := \left\{\varphi \in H:\ E_\varepsilon(\varphi) \leqslant c\right\}.$$

The proof will be done in four propositions as follows

Proposition 4.4. *Let $\varepsilon \geqslant 0$, $\omega \geqslant 0$, $c \in \mathbb{R}$ and $R > 0$ given. There exists $R_\star > 0$ such that if the set $[E_\varepsilon \leqslant c]$ is contained in the ball $B_{L^2}(0, R)$ then it is also contained in $B_H(0, R_\star)$ with a constant $R_\star$ depending only on R, c and $\widetilde{V}^-$.*

Proof. This is analogous to the a priori estimate obtained by P.L. Lions [12] for the quadratic case. For any $\varepsilon > 0$, the inequality $E_\varepsilon(\varphi) \leqslant c$ gives in particular

$$\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\widetilde{V}^+\varphi^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\widetilde{V}^-\varphi^2\,\mathrm{d}x \leqslant c.$$

On account of (2.4) with $\delta = 1/4$, there exists a constant $K > 0$, depending only on $\widetilde{V}^-$, such that

$$\frac{1}{8}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\widetilde{V}^+\varphi^2\,\mathrm{d}x \leqslant K\|\varphi\|^2 + c. \qquad \square$$

Proposition 4.5. *For all $\omega > 0$, $\varepsilon > 0$ and $c \in \mathbb{R}$, there is $R = R(\omega, \varepsilon, c) > 0$ such that the set $[E_\varepsilon \leqslant c]$ is contained in the ball $B_{L^2}(0, R)$.*

Proof. Suppose by contradiction that there exist $\omega > 0$, $\varepsilon_0 > 0$, $c \in \mathbb{R}$ and $(\varphi_k)_k \subset H$ such that $E_{\varepsilon_0}(\varphi_k) \leqslant c$ and $\|\varphi_k\| \to \infty$. Set

$$\alpha_k := \|\varphi_k\|, \qquad \psi_k := \frac{\varphi_k}{\|\varphi_k\|}$$

then we have $\alpha_k \to \infty$ as $k \to \infty$ and $\|\psi_k\| = 1$ for all k. Using the expression of E_{ε_0} given in (4.2), it follows from $E_{\varepsilon_0}(\varphi_k) \leqslant c$ that ψ_k satisfies for all k

$$\frac{1}{4}\int|\nabla\psi_k|^2\,\mathrm{d}x + \frac{1}{\alpha_k^2}I_{\varepsilon_0}(\alpha_k\psi_k) - \frac{1}{2}\int\widetilde{V}^-\psi_k^2\,\mathrm{d}x + \frac{\omega}{2} \leqslant \frac{c}{\alpha_k^2}. \tag{4.20}$$

We want to show, by the next two lemmas, that $(\psi_k)_k$ contains a subsequence, noted again by $(\psi_k)_k$, such that $\psi_k \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ and that this implies $\omega \leqslant 0$ (which contradicts our hypothesis on ω).

Lemma 4.6. *Let ψ_k be as above, then up to a subsequence, $\psi_k \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ when $k \to \infty$.*

Proof. According to (2.4) with $\delta = 1/4$, we get for some constant C_0

$$\int\widetilde{V}^-(x)\left|\psi_k(x)\right|^2\mathrm{d}x \leqslant \frac{1}{4}\|\nabla\psi_k\|^2 + C_0$$

and therefore the inequality (4.14) implies

$$\frac{1}{8}\int |\nabla \psi_k|^2 \,\mathrm{d}x + \frac{1}{\alpha_k^2} I_{\varepsilon_0}(\alpha_k \psi_k) \leqslant \frac{c}{\alpha_k^2} + C_0. \tag{4.21}$$

This yields in particular that ψ_k is bounded in $H^1(\mathbb{R}^3)$. Consider $\psi \in H^1(\mathbb{R}^3)$ and a subsequence, denoted again ψ_k, such that $\psi_k \rightharpoonup \psi$ weakly in $H^1(\mathbb{R}^3)$. We must show that $\psi \equiv 0$. To this end, fix arbitrarily k and consider $V_k := V_{\varepsilon_0}(\alpha_k \psi_k) = W_{\varepsilon_0}(\alpha_k^2 |\psi_k|^2 - n^*)$ the unique solution of (3.2) with $f := \alpha_k^2 |\psi_k|^2 - n^*$ which minimizes on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ the functional

$$J^k(\varphi) := \frac{1}{2}\int |\nabla \varphi|^2 \,\mathrm{d}x + \frac{\varepsilon_0^4}{4}\int |\nabla \varphi|^4 \,\mathrm{d}x - \int \left(\alpha_k^2 |\psi_k|^2 - n^*\right)\varphi \,\mathrm{d}x. \tag{4.22}$$

Note that J^k is the same functional J_{ε_0} in (3.3) taking $f := \alpha_k^2 |\psi_k|^2 - n^*$. The solution V_k verifies then

$$-\operatorname{div}\left[\left(1 + \varepsilon_0^4 |\nabla V_k|^2\right)\nabla V_k\right] = \alpha_k^2 |\psi_k|^2 - n^*.$$

Multiplying this equation by V_k and integrating by parts we find that

$$\begin{aligned} J^k(V_k) &= -\frac{1}{2}\int |\nabla V_k|^2 \,\mathrm{d}x - \frac{3}{4}\varepsilon_0^4 \int |\nabla V_k|^4 \,\mathrm{d}x \\ &= -2 I_{\varepsilon_0}(\alpha_k \psi_k), \end{aligned} \tag{4.23}$$

and therefore, in view of (4.21), we have the following estimate

$$-\frac{1}{2}\alpha_k^{-2} J^k(V_k) \leqslant \frac{c}{\alpha_k^2} + C_0 \leqslant C_1. \tag{4.24}$$

Consider on the other hand $\overline{V}_k := W_{\varepsilon_0}(|\psi_k|^2)$ the unique solution in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ of (3.2) with $f := |\psi_k|^2$. Then by (3.1), $\overline{V}_k$ satisfies

$$\int |\psi_k|^2 \overline{V}_k \,\mathrm{d}x = \int |\nabla \overline{V}_k|^2 \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x. \tag{4.25}$$

Since V_k is the minimizer of J^k on $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ and using successively (4.22) and (4.25) we may write

$$\begin{aligned} J^k(V_k) &\leqslant J^k(\alpha_k^{2/3} \overline{V}_k) \\ &\leqslant \frac{\alpha_k^{4/3}}{2}\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x + \frac{\alpha_k^{8/3}}{4}\varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x - \alpha_k^{2/3}\int \left(\alpha_k^2 |\psi_k|^2 - n^*\right)\overline{V}_k \,\mathrm{d}x \\ &\leqslant \left(\frac{\alpha_k^{4/3}}{2} - \alpha_k^{8/3}\right)\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x - \frac{3}{4}\alpha_k^{8/3}\varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x + \alpha_k^{2/3}\int n^* \overline{V}_k \,\mathrm{d}x. \end{aligned}$$

By a standard argument using Hölder, Sobolev and Young inequalities we can estimate

$$\begin{aligned} \alpha_k^{2/3}\int n^* \overline{V}_k \,\mathrm{d}x &\leqslant \alpha_k^{2/3} \|n^*\|_{L^{6/5}} \|\overline{V}_k\|_{L^6} \\ &\leqslant C_2 \alpha_k^{2/3} \|n^*\|_{L^{6/5}} \|\nabla \overline{V}_k\| \\ &\leqslant \frac{\alpha_k^{4/3}}{2} \|\nabla \overline{V}_k\|^2 + C_3, \end{aligned}$$

for some positive constants C_2 and C_3 independent of k. Hence, we get

$$J^k(V_k) \leqslant (\alpha_k^{4/3} - \alpha_k^{8/3})\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x - \frac{3}{4}\alpha_k^{8/3}\varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x + C_3.$$

It is clear that $(\alpha_k^{4/3} - \alpha_k^{8/3}) \leqslant -\frac{3}{4}\alpha_k^{8/3}$ for k large enough since $\alpha_k \to +\infty$ when $k \to \infty$. Therefore we infer that

$$J^k(V_k) \leqslant -\frac{3}{4}\alpha_k^{8/3}\left[\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x\right] + C_3$$

which, combined with (4.24), implies that for k sufficiently large

$$\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x \leqslant \frac{4}{3}\left(2\alpha_k^{-2/3} C_1 + \alpha_k^{-8/3} C_3\right).$$

So

$$\int |\nabla \overline{V}_k|^2 \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^4 \,\mathrm{d}x \to 0 \quad \text{as } k \to \infty. \tag{4.26}$$

Now, multiplying equation (3.2), with $f := |\psi_k|^2$, by $u \in C_c^1(\mathbb{R}^3)$ we have

$$\int (\nabla \overline{V}_k \cdot \nabla u) \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^2 (\nabla \overline{V}_k \cdot \nabla u) \,\mathrm{d}x = \int |\psi_k|^2 u. \tag{4.27}$$

Knowing that ψ_k converges to ψ weakly in $H^1(\mathbb{R}^3)$ and therefore strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ then

$$\int |\psi_k|^2 u \longrightarrow \int |\psi|^2 u$$

as $k \to \infty$. Moreover, (4.26) implies that when $k \to \infty$

$$\int (\nabla \overline{V}_k \cdot \nabla u) \,\mathrm{d}x + \varepsilon_0^4 \int |\nabla \overline{V}_k|^2 (\nabla \overline{V}_k \cdot \nabla u) \,\mathrm{d}x \longrightarrow 0.$$

Consequently, passing to the limit in (4.27) we infer that $\int |\psi|^2 u = 0$ for any $u \in C_c^1(\mathbb{R}^3)$ which means that $\psi \equiv 0$. □

Next, we show the following lemma.

Lemma 4.7. *The subsequence* $(\psi_k)_k$ *satisfies*

$$\int \widetilde{V}^- \psi_k^2 \,\mathrm{d}x \longrightarrow 0 \quad \textit{as } k \to \infty.$$

Proof. Using the fact that $\widetilde{V}^-$ has a decomposition (1.10) (see Remark 1.3) we may estimate

$$\begin{aligned}\int \widetilde{V}^- \psi_k^2 \,\mathrm{d}x &\leqslant \int \widetilde{V}_{2\lambda}^- \psi_k^2 \,\mathrm{d}x + \int \widetilde{V}_{1\lambda}^- \psi_k^2 \,\mathrm{d}x \\ &\leqslant C \|\widetilde{V}_{2\lambda}^-\|_{L^{3/2}} + \int \widetilde{V}_{1\lambda}^- \psi_k^2 \,\mathrm{d}x\end{aligned}$$

where we have used the fact that $(\psi_k)_k$ is bounded in $H^1(\mathbb{R}^3)$ and thus in $L^6(\mathbb{R}^3)$ by the Sobolev embedding. Now for an arbitrary $\delta > 0$, we fix at first λ sufficiently small so that $C\|\widetilde{V}_{2\lambda}^-\|_{L^{3/2}} < \delta/2$. Next knowing that $\psi_k \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ we use Lemma 2.4 with $\psi = \widetilde{V}_{1\lambda}^-$ to deduce the desired convergence. □

In conclusion, by (4.20) we have in particular

$$-\frac{1}{2} \int \widetilde{V}^- \psi_k^2 \,\mathrm{d}x + \frac{\omega}{2} \leqslant \frac{c}{\alpha_k^2}.$$

Letting $k \to \infty$ in this inequality, it follows by Lemma 4.7 that $\omega \leqslant 0$ which contradicts our assumption. The proof of Proposition 4.5 is thus complete. □

Proposition 4.8. *For all* $\omega > 0$ *and any* $\varepsilon > 0$ *the functional* E_ε *is weakly lower semicontinuous on* H *and attains its minimum in* H *at* u_ε *which is a weak solution of* (4.1).

Proof. Let $\varepsilon = 1$ for convenience. In order to show that E_1 is weakly lower semicontinuous on H we just have to prove it for the mapping

$$\varphi \longmapsto I_1(\varphi) - \frac{1}{2} \int_{\mathbb{R}^3} \widetilde{V}^- \varphi^2 \,\mathrm{d}x$$

since the other terms in (4.2) are continuous and convex (therefore weakly lower semicontinuous). The rest of the proof will be done in two steps as follows:
• **Step 1.** At first, we claim

Lemma 4.9. *If $\varphi_j \rightharpoonup \varphi_\star$ weakly in H as $j \to \infty$ then we have*

$$\int \widetilde{V}^- \varphi_j^2 \,\mathrm{d}x \longrightarrow \int \widetilde{V}^- \varphi_\star^2 \,\mathrm{d}x.$$

Proof. We write

$$\int_{\mathbb{R}^3} \widetilde{V}^- \varphi_j^2 - \int_{\mathbb{R}^3} \widetilde{V}^- \varphi_\star^2 = \int_{\mathbb{R}^3} \widetilde{V}^- (\varphi_j - \varphi_\star)^2 + 2 \int_{\mathbb{R}^3} \widetilde{V}^- \varphi_\star (\varphi_j - \varphi_\star). \tag{4.28}$$

As in the proof of Lemma 4.7, taking $(\varphi_j - \varphi_\star)$ instead of ψ_k, we can show easily that the first term in the right-hand side of (4.28) converges to zero.

For the second term, note first that as observed in Remark 1.3, the decomposition (1.10) yields in particular that $\widetilde{V}^- \in L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$. Consequently, we may write

$$\widetilde{V}^- = \Theta_1 + \Theta_2, \quad \Theta_1 \in L^\infty(\mathbb{R}^3) \text{ and } \Theta_2 \in L^{3/2}(\mathbb{R}^3), \tag{4.29}$$

and therefore

$$\int_{\mathbb{R}^3} \widetilde{V}^- \varphi_\star (\varphi_j - \varphi_\star) \,\mathrm{d}x = \int_{\mathbb{R}^3} \Theta_1 \varphi_\star (\varphi_j - \varphi_\star) \,\mathrm{d}x + \int_{\mathbb{R}^3} \Theta_2 \varphi_\star (\varphi_j - \varphi_\star) \,\mathrm{d}x. \tag{4.30}$$

Since $H^1(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for all $2 \leqslant p \leqslant 6$ then $\varphi_j \rightharpoonup \varphi_\star$ weakly in $L^2(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$. Furthermore as $\varphi_\star \in L^2 \cap L^6$ then $\Theta_1 \varphi_\star \in L^2$ and $\Theta_2 \varphi_\star \in L^{6/5} = (L^6)'$. Hence both integrals in the right-hand side of (4.30) converge to zero. □

• **Step 2.** Next, we prove that the mapping

$$\varphi \longmapsto I_1(\varphi) := \frac{1}{4} \int \left|\nabla V_1(\varphi)\right|^2 + \frac{3}{8} \int \left|\nabla V_1(\varphi)\right|^4$$

is weakly lower semicontinuous on H. In fact, it suffices to show that if $\varphi_j \rightharpoonup \varphi_\star$ weakly in H then $V_1(\varphi_j) \rightharpoonup V_1(\varphi_\star)$ weakly in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ as $j \to \infty$. To this end we claim

Lemma 4.10. *Let $(f_j)_{j \geqslant 1} \subset L^{6/5}(\mathbb{R}^3)$ such that $\|f_j\|_{6/5} \leqslant C$ for some positive constant C independent of j. Assume that there exists $f \in L^{6/5}(\mathbb{R}^3)$ so that when $j \to \infty$ we have*

$$f_j \longrightarrow f \quad \text{strongly in } L^{6/5}_{\mathrm{loc}}.$$

Consider v_j and v respectively the unique solution in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ of $\mathcal{A} v_j = f_j$ and $\mathcal{A} v = f$ where $\mathcal{A}$ denotes the operator

$$\mathcal{A} u := -\operatorname{div}\left[\left(1 + |\nabla u|^2\right) \nabla u\right].$$

Then as $j \to \infty$ we have

$$v_j \longrightarrow v \quad \text{strongly in } \mathcal{D}^{1,2}_{\mathrm{loc}} \cap \mathcal{D}^{1,4}_{\mathrm{loc}}.$$

Proof of Lemma 4.10. Note that v_j and v are in fact respectively $v_j := W_1(f_j)$ and $v := W_1(f)$ obtained via Lemma 3.1. In the same way, since $\|f_j\|_{6/5} \leqslant C$ then v_j is bounded in $\mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}(\mathbb{R}^3)$ and therefore there is $C_0 > 0$ such that

$$\|v_j\|_{L^6} + \|v_j\|_\infty \leqslant C_0 \quad \forall j \geqslant 1.$$

Let $\xi_0 \in C_c^1(\mathbb{R})$ such that $0 \leqslant \xi_0 \leqslant 1$, $\xi_0 \equiv 1$ on $[0, 1]$ and $\xi_0(t) = 0$ for $t \geqslant 1 + R$ for some $R > 0$ and consider φ_n defined by

$$\varphi_n(x) := \xi_0\left(\frac{|x|}{n}\right)$$

for $n \geqslant 1$. Multiplying the equation $\mathcal{A}v - \mathcal{A}v_j = f - f_j$ by $(v - v_j)\varphi_n^2$ and integrating by parts we get

$$\int_{\mathbb{R}^3} \left[\left(1 + |\nabla v|^2\right)\nabla v - \left(1 + |\nabla v_j|^2\right)\nabla v_j\right] \cdot (\nabla v - \nabla v_j)\varphi_n^2 \,\mathrm{d}x = G_1 + G_2 \tag{4.31}$$

where we have set

$$G_1 := \int_{\mathbb{R}^3} (f - f_j)\varphi_n^2 (v - v_j)\,\mathrm{d}x,$$

$$G_2 := -2\int_{\mathbb{R}^3} (v - v_j)\left[\left(1 + |\nabla v|^2\right)\nabla v - \left(1 + |\nabla v_j|^2\right)\nabla v_j\right]\varphi_n \nabla\varphi_n \,\mathrm{d}x.$$

We denote also the left-hand side in (4.31) by

$$G_0 := \int_{\mathbb{R}^3} \left[\left(1 + |\nabla v|^2\right)\nabla v - \left(1 + |\nabla v_j|^2\right)\nabla v_j\right] \cdot (\nabla v - \nabla v_j)\varphi_n^2 \,\mathrm{d}x$$

and as in the proof of Lemma 2.1 we obtain

$$\begin{aligned} G_0 &\geqslant \int_{\mathbb{R}^3} \left(\frac{1}{4}|\nabla v - \nabla v_j|^4 + |\nabla v - \nabla v_j|^2\right)\varphi_n^2 \,\mathrm{d}x \\ &\geqslant \frac{1}{4} \int_{|x| \leqslant n} |\nabla v - \nabla v_j|^4 \,\mathrm{d}x + \left[\int_{\mathbb{R}^3} \left|\nabla\left(\varphi_n (v - v_j)\right)\right|^2 \mathrm{d}x \right. \\ &\quad \left. - 2\int_{\mathbb{R}^3} \varphi_n \nabla\varphi_n (v - v_j)\nabla(v - v_j)\,\mathrm{d}x - \int_{\mathbb{R}^3} (v - v_j)^2 |\nabla\varphi_n|^2 \,\mathrm{d}x\right], \end{aligned} \tag{4.32}$$

since $\varphi_n \equiv 1$ on $[|x| \leqslant n]$. On the other hand, we have

$$\begin{aligned} |G_1| &\leqslant \left\|(f - f_j)1_{[|x| \leqslant (1+R)n]}\right\|_{L^{6/5}} \|v - v_j\|_{L^6} \\ &\leqslant C_1 \left\|(f - f_j)1_{[|x| \leqslant (1+R)n]}\right\|_{L^{6/5}} \end{aligned} \tag{4.33}$$

for some positive constant C_1 independent of n, j, R. Furthermore we decompose G_2 as where

$$\begin{aligned} G_2 &:= G_{21} + G_{22}, \\ G_{21} &:= -2\int_{\mathbb{R}^3} (v - v_j)(\nabla v - \nabla v_j)\varphi_n \nabla\varphi_n \,\mathrm{d}x, \\ G_{22} &:= -2\int_{\mathbb{R}^3} (v - v_j)\left(|\nabla v|^2 \nabla v - |\nabla v_j|^2 \nabla v_j\right)\varphi_n \nabla\varphi_n \,\mathrm{d}x. \end{aligned}$$

Then we may estimate, with a constant $C_2 > 0$ independent of n, j, R,

$$\begin{aligned} |G_{22}| &\leqslant 2\|v - v_j\|_\infty \left(\|\nabla v\|_{L^4}^3 + \|\nabla v_j\|_{L^4}^3\right)\|\nabla\varphi_n\|_{L^4} \\ &\leqslant C_2 \|\nabla\varphi_n\|_{L^4}. \end{aligned}$$

However a standard calculation gives

$$\begin{aligned}\|\nabla\varphi_n\|_{L^4}^4 &= \int\limits_{n\leqslant|x|\leqslant n(1+R)} \left|\nabla\varphi_n(x)\right|^4 \mathrm{d}x \\ &= \frac{1}{n^4}\int\limits_{n\leqslant|x|\leqslant n(1+R)} \left|\xi_0'\left(\frac{|x|}{n}\right)\right|^4 \mathrm{d}x \\ &= \frac{1}{n}\int\limits_{1\leqslant|y|\leqslant(1+R)} \left|\xi_0'(|y|)\right|^4 \mathrm{d}y = \frac{1}{n}\int_1^{1+R} \left|\xi_0'(t)\right|^4 t^2\,\mathrm{d}t \\ &= \frac{1}{n}C_3(R,\xi_0'),\end{aligned}$$

and therefore we get, for a constant $C_4(R,\xi_0')$ depending only on R and ξ_0',

$$|G_{22}| \leqslant \frac{C_4(R,\xi_0')}{n^{1/4}}. \tag{4.34}$$

Now, inequality (4.32) can be written as

$$G_0 \geqslant \frac{1}{4}\int\limits_{|x|\leqslant n} |\nabla v-\nabla v_j|^4 + \int\limits_{\mathbb{R}^3}\left|\nabla\big(\varphi_n(v-v_j)\big)\right|^2 - \int\limits_{\mathbb{R}^3}(v-v_j)^2|\nabla\varphi_n|^2\,\mathrm{d}x + G_{21}$$

which implies that

$$\begin{aligned}\frac{1}{4}\int\limits_{|x|\leqslant n} |\nabla v-\nabla v_j|^4 + \int\limits_{|x|\leqslant n}\left|\nabla\big(\varphi_n(v-v_j)\big)\right|^2 &\leqslant G_0 - G_{21} + \int\limits_{\mathbb{R}^3}(v-v_j)^2|\nabla\varphi_n|^2\,\mathrm{d}x \\ &\leqslant G_1 + G_{22} + \int\limits_{\mathbb{R}^3}(v-v_j)^2|\nabla\varphi_n|^2\,\mathrm{d}x,\end{aligned}$$

since $G_0 = G_1 + (G_{21}+G_{22})$. Using (4.33), (4.34) and Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{4}\int\limits_{|x|\leqslant n} |\nabla v-\nabla v_j|^4 + \int\limits_{|x|\leqslant n}\left|\nabla\big(\varphi_n(v-v_j)\big)\right|^2 &\leqslant C_1\left\|(f-f_j)1_{[|x|\leqslant(1+R)n]}\right\|_{L^{6/5}} \\ &\quad + \frac{C_4(R,\xi_0')}{n^{1/4}} + \|v-v_j\|_{L^6}^2\left(\int\limits_{\mathbb{R}^3}|\nabla\varphi_n|^3\,\mathrm{d}x\right)^{2/3}.\end{aligned}$$

By an elementary calculus, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^3}|\nabla\varphi_n|^3\,\mathrm{d}x &= \frac{1}{n^3}\int\limits_{n\leqslant|x|\leqslant n(1+R)}\left|\xi_0'\left(\frac{|x|}{n}\right)\right|^3\mathrm{d}x \\ &= \int_1^{1+R}\left|\xi_0'(t)\right|^3 t^2\,\mathrm{d}t,\end{aligned}$$

and therefore for a constant C_5 independent of n, j, R we have

$$\begin{aligned}&\frac{1}{4}\int\limits_{|x|\leqslant n} |\nabla v-\nabla v_j|^4\,\mathrm{d}x + \int\limits_{|x|\leqslant n}|\nabla v-\nabla v_j|^2\,\mathrm{d}x \\ &\quad\leqslant C_1\left\|(f-f_j)1_{[|x|\leqslant(1+R)n]}\right\|_{L^{6/5}} + \frac{C_4(R,\xi_0')}{n^{1/4}} + C_5\left(\int_1^{1+R}\left|\xi_0'(t)\right|^3 t^2\,\mathrm{d}t\right)^{2/3}.\end{aligned} \tag{4.35}$$

To conclude we use the following lemma

Lemma 4.11. *Let β be the nonnegative real number defined by*

$$\beta := \inf\left\{ \int_1^\infty \left|\xi'(t)\right|^3 t^2 \, dt;\ \xi \in C_c^1\big([1,\infty[\big), \xi(1) = 1, \xi'(1) = 0, 0 \leqslant \xi \leqslant 1 \right\}$$

then $\beta = 0$.

Before showing this lemma, we finish the proof of Lemma 4.10 as follows. Let $\varepsilon > 0$:

- First, according to Lemma 4.11, we may choose $\xi_* = \xi_{*\varepsilon} \in C_c^1([1,\infty[)$ such that $\xi_*(1) = 1$, $\xi_*'(1) = 0$, $0 \leqslant \xi_* \leqslant 1$ and

$$\left(\int_1^\infty \left|\xi_*'(t)\right|^3 t^2 \, dt \right)^{2/3} \leqslant \frac{\varepsilon}{3C_5}.$$

  Let $R > 0$ be so that $(\xi_*) \subset]1, 1+R[$ and $\xi_0 \in C_c^1([0,\infty[)$ such that $\xi_0 = 1$ on $[0,1]$ and $\xi_0 = \xi_*$ on $[1,\infty[$. Then we have

$$C_5 \left(\int_1^{1+R} \left|\xi_0'(t)\right|^3 t^2 \, dt \right)^{2/3} \leqslant \frac{\varepsilon}{3}.$$

- Next, we take $n_0 \in \mathbb{N}^*$ so that

$$\frac{C_4(R, \xi_0')}{n_0^{1/4}} \leqslant \frac{\varepsilon}{3}$$

  and we fix some $n \geqslant n_0$.
- Finally, we consider $j_0 \in \mathbb{N}^*$ such that for all $j \geqslant j_0$

$$C_1 \left\| (f - f_j) 1_{[|x| \leqslant (1+R)n]} \right\|_{L^{6/5}} \leqslant \frac{\varepsilon}{3}.$$

Consequently for all $\varepsilon > 0$ there are $n_0 \in \mathbb{N}^*$ such that for any $n \geqslant n_0$ fixed we can find $j_0 \in \mathbb{N}^*$ such that $j \geqslant j_0$ we have (according to (4.35))

$$\int_{|x| \leqslant n} \left(\frac{1}{4} |\nabla v - \nabla v_j|^4 + |\nabla v - \nabla v_j|^2 \right) dx \leqslant \varepsilon.$$

This means that $v_j \to v$ strongly in $\mathcal{D}^{1,2}_{\mathrm{loc}} \cap \mathcal{D}^{1,4}_{\mathrm{loc}}$, and so the proof of Lemma 4.10 is complete. □

Proof of Lemma 4.11. Consider $\xi_\alpha(t) := \exp(1 - t^\alpha)$ for $t \geqslant 1$ and $\alpha > 0$. We have

$$\int_1^\infty \left|\xi_\alpha'(t)\right|^3 t^2 \, dt = \alpha^3 e^3 \int_1^\infty \exp(-3t^\alpha) t^{3\alpha - 1} \, dt$$

$$= \alpha^2 e^3 \int_1^\infty s^2 \exp(-3s) \, ds$$

where we have used the substitution $s := t^\alpha$. It is clear that

$$\lim_{\alpha \to 0^+} \int_1^\infty \left|\xi_\alpha'(t)\right|^3 t^2 \, dt = 0,$$

and upon considering $\xi(t) := \xi_\alpha(t)\chi(t)$, where $\chi \in C_c^\infty([1,\infty))$ is a truncation function, one sees that $\beta = 0$. □

Next, we finish the proof of Proposition 4.8. According to Lemma 4.10 with

$$f_j = |\varphi_j|^2 - n^*, \quad f = |\varphi_\star|^2 - n^*, \quad v_j = V_1(\varphi_j), \quad v = V_1(\varphi_\star)$$

and using the uniqueness of the solution of $\mathcal{A}u := |\varphi_\star|^2 - n^*$ in $\mathcal{D}^{1,2}$, we infer that *all* the sequence $V_1(\varphi_j)$ verifies

$$V_1(\varphi_j) \rightharpoonup V_1(\varphi_\star) \quad \text{weakly in } \mathcal{D}^{1,2} \cap \mathcal{D}^{1,4}. \tag{4.36}$$

Since $I_1(\varphi)$ can be written as

$$I_1(\varphi) = \frac{1}{4}\|V_1(\varphi)\|^2_{\mathcal{D}^{1,2}} + \frac{3}{8}\|V_1(\varphi)\|^4_{\mathcal{D}^{1,4}},$$

the weak convergence (4.36) implies

$$I_1(\varphi_\star) \leqslant \liminf_{j\to\infty} I_1(\varphi_j).$$

We conclude that E_1 is weakly lower semicontinuous on H.

Finally, let

$$\mu := \inf\{E_1(\varphi);\, \varphi \in H\}$$

and let $(u_j)_j$ be a minimizing sequence in H. According to Propositions 4.4 and 4.5, $(u_j)_j$ is bounded in H. Since H is a Hilbert space, we may assume that $u_j \rightharpoonup u$ weakly in H for some $u \in H$. But E_1 is weakly lower semicontinuous on H then

$$E_1(u) \leqslant \liminf_{j\to+\infty} E_1(u_j) = \mu,$$

and therefore $E_1(u) = \mu$. As E_1 is C^1 on H then $E_1'(u) = 0$. This implies, in view of (4.4), that u satisfies Eq. (4.1) in the weak sense and the proof of Proposition 4.8 is thus complete. □

Proposition 4.12. *There exist $\omega_* > 0$ and $\varepsilon_* > 0$ such that if $0 < \omega < \omega_*$ then $E_\varepsilon(u_\varepsilon) < E_\varepsilon(0)$ and therefore $u_\varepsilon \not\equiv 0$ for all $0 < \varepsilon < \varepsilon_*$.*

Proof. Remark first that by Lemma 3.2, with $f_\varepsilon = f := |\varphi|^2 - n^*$ for all $\varepsilon > 0$, we can easily check that when $\varepsilon \to 0$ we have

$$E_\varepsilon(\varphi) \longrightarrow E_0(\varphi) \quad \forall \varphi \in H^1(\mathbb{R}^3). \tag{4.37}$$

On the other hand, according to assumption (1.8) there is $\mu_1 < 0$ and $\varphi_1 \in C_c^1(\mathbb{R}^3)$ such that $\int |\varphi_1|^2 = 1$ and

$$\int_{\mathbb{R}^3} |\nabla\varphi_1|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} \varrho(x)\varphi_1^2(x)\,\mathrm{d}x < \mu_1 < 0 \tag{4.38}$$

where ϱ is given by (1.7). As it is proved in [2] (or [3]), we may show that for all $0 < \omega < \omega_* := -\mu_1/2$ there is $t_* > 0$ sufficiently small such that $E_0(t_*\varphi_1) < E_0(0)$. Indeed, following to Lemma 2.2 in [2], the energy functional E_0 corresponding to Eq. (1.3) (coupled with (1.4)) is written explicitly as $E_0(\varphi) = E_{01}(\varphi) - E_{02}(\varphi) + E_{03}(\varphi) + E_0(0)$ where

$$E_{01}(\varphi) := \frac{1}{4}\int |\nabla\varphi|^2\,\mathrm{d}x + \frac{1}{2}\int \widetilde{V}^+\varphi^2\,\mathrm{d}x + \frac{\omega}{2}\int \varphi^2\,\mathrm{d}x,$$

$$E_{02}(\varphi) := \frac{1}{2}\int \widetilde{V}^-\varphi^2\,\mathrm{d}x + \frac{1}{8\pi}\iint \frac{n^*(y)}{|x-y|}\varphi^2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$E_{03}(\varphi) := \frac{1}{16\pi}\iint \frac{\varphi^2(x)\varphi^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y,$$

$$E_0(0) := \frac{1}{16\pi}\iint \frac{n^*(x)n^*(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

Then we may write

$$\int_{\mathbb{R}^3} |\nabla\varphi_1|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \varrho(x)\varphi_1^2(x)\,\mathrm{d}x = 4E_{01}(\varphi_1) - 4E_{02}(\varphi_1) - 2\omega,$$

and therefore inequality (4.38) gives

$$E_{01}(\varphi_1) - E_{02}(\varphi_1) - \frac{\omega}{2} < \frac{\mu_1}{4}.$$

For $t > 0$ we verify easily that

$$\begin{aligned} E_0(t\varphi_1) - E_0(0) &= t^2 E_{01}(\varphi_1) - t^2 E_{02}(\varphi_1) + t^4 E_{03}(\varphi_1) \\ &< \frac{t^2}{4}\big[(\mu_1 + 2\omega) + 4t^2 E_{03}(\varphi_1)\big]. \end{aligned}$$

Hence, if $(\mu_1 + 2\omega) < 0$ there exists $t_* > 0$ small enough such that

$$(\mu_1 + 2\omega) + 4t_*^2 E_{03}(\varphi_1) < 0.$$

In other words, setting $\omega_* := -\mu_1/2$ then if $0 < \omega < \omega_*$ we have $E_0(t_*\varphi_1) < E_0(0)$. Now, from (4.37) it follows that $E_\varepsilon(t_*\varphi_1)$ and $E_\varepsilon(0)$ converge respectively to $E_0(t_*\varphi_1)$ and $E_0(0)$ as $\varepsilon \to 0$. Hence, there exists $\varepsilon_* > 0$ small enough such that for all $0 < \varepsilon < \varepsilon_*$

$$E_\varepsilon(t_*\varphi_1) < E_\varepsilon(0).$$

Since E_ε attains its minimum on H at u_ε, it suffices to observe that $\varphi_1 \in H$ and hence

$$E_\varepsilon(u_\varepsilon) \leqslant E_\varepsilon(t_*\varphi_1) < E_\varepsilon(0) \tag{4.39}$$

for all $0 < \varepsilon < \varepsilon_*$, which implies that $u_\varepsilon \not\equiv 0$. This completes the proof of Proposition 4.12 and consequently that of Theorem 1.1. □

Remark 4.13. We would like to point out that if n^* is nonnegative then we may replace the assumption (1.8) by the following

$$\inf\left\{\int |\nabla\varphi|^2 \,\mathrm{d}x + 2\int \widetilde{V}(x)\varphi^2 \,\mathrm{d}x;\ \int |\varphi|^2 = 1\right\} < 0$$

which does not depend on n^* and implies obviously (1.8).

4.2. Proof of Theorem 1.2

Here we conclude our study by the proof of Theorem 1.2 which deals with the behaviour of the solution u_ε obtained via Theorem 1.1.

First of all, we claim that $(u_\varepsilon)_\varepsilon$ is bounded in H. Indeed, using (4.39) and convergence (4.37) for $\varphi \equiv 0$, there exists $c \in \mathbb{R}$ independent of ε such that $E_\varepsilon(u_\varepsilon) \leqslant c$. Following to Proposition 4.4, we just have to show that $(u_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^3)$ in order to assert that it is also bounded in H. Now to prove that $\|u_\varepsilon\|$ is bounded, assume by contradiction that there exists a subsequence, noted again by u_ε, satisfying

$$E_\varepsilon(u_\varepsilon) \leqslant c \quad \text{and} \quad \|u_\varepsilon\| \longrightarrow \infty.$$

Set $\alpha_\varepsilon := \|u_\varepsilon\|$ and $\psi_\varepsilon := u_\varepsilon/\alpha_\varepsilon$. Then $\|\psi_\varepsilon\| = 1$ for all ε and from $E_\varepsilon(u_\varepsilon) \leqslant c$ it follows

$$\frac{1}{4}\int |\nabla\psi_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\alpha_\varepsilon^2} I_\varepsilon(\alpha_\varepsilon\psi_\varepsilon) - \frac{1}{2}\int \widetilde{V}^-\psi_\varepsilon^2 \,\mathrm{d}x + \frac{\omega}{2} \leqslant \frac{c}{\alpha_\varepsilon^2}.$$

An argument similar to the one used in the proof of Proposition 4.5 allows us to conclude that the last inequality implies $\omega \leqslant 0$ even if here I_ε depends on ε which goes to 0 (instead of the fixed ε_0). Indeed, a simple inspection of the proof of Lemma 4.6 shows that all the constants C_1, C_2 and C_3 are also independent of ε_0, provided for instance $\varepsilon \leqslant 1$.

But $\omega \leqslant 0$ contradicts our assumption on ω. Hence $\|u_\varepsilon\|$ is necessarily bounded and therefore by Proposition 4.4, $(u_\varepsilon)_\varepsilon$ is also bounded in the Hilbert space H. Therefore there exists $u \in H$ such that, up to a subsequence, we have

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H \text{ as } \varepsilon \downarrow 0.$$

Let $\psi \in C_c^\infty(\mathbb{R}^3)$ and a compact subset K such that $\mathrm{supp}(\psi) \subset K$ then according to (4.1) we have

$$a(u_\varepsilon, \psi) + \int_K V_\varepsilon(u_\varepsilon) u_\varepsilon \psi \, \mathrm{d}x - \int_{\mathbb{R}^3} \widetilde{V}^- u_\varepsilon \psi \, \mathrm{d}x = 0 \tag{4.40}$$

where we have set

$$a(u_\varepsilon, \psi) := \frac{1}{2} \int_{\mathbb{R}^3} (\nabla u_\varepsilon \cdot \nabla \psi) \, \mathrm{d}x + \int_{\mathbb{R}^3} \widetilde{V}^+ u_\varepsilon \psi \, \mathrm{d}x + \omega \int_{\mathbb{R}^3} u_\varepsilon \psi \, \mathrm{d}x.$$

We would like to pass to the limit in Eq. (4.40), so we observe:

- Since $u_\varepsilon \rightharpoonup u$ weakly in H then

$$a(u_\varepsilon, \psi) \longrightarrow \frac{1}{2} \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \psi) \, \mathrm{d}x + \int_{\mathbb{R}^3} \widetilde{V}^+ u \psi \, \mathrm{d}x + \omega \int_{\mathbb{R}^3} u \psi \, \mathrm{d}x \quad \text{as } \varepsilon \to 0^+. \tag{4.41}$$

- For the second term in the left-hand side of (4.40), recall that $V_\varepsilon(u_\varepsilon) := W_\varepsilon(|u_\varepsilon|^2 - n^*)$ and remark at first that we may see that $|u_\varepsilon|^2 \rightharpoonup |u|^2$ weakly in $L^{6/5}(\mathbb{R}^3)$. Indeed let $\theta \in L^6 = (L^{6/5})'$ and write

$$\int_{\mathbb{R}^3} \theta u_\varepsilon^2 \, \mathrm{d}x - \int_{\mathbb{R}^3} \theta u^2 \, \mathrm{d}x = \int_{\mathbb{R}^3} \theta (u_\varepsilon - u)^2 \, \mathrm{d}x + 2 \int_{\mathbb{R}^3} \theta u (u_\varepsilon - u) \, \mathrm{d}x.$$

Since $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ it follows from Lemma 2.4 that $\int_{\mathbb{R}^3} \theta(u_\varepsilon - u)^2 \, \mathrm{d}x$ converges to 0. On the other hand, having $\theta \in L^6$ and $u \in L^3$, we have $\theta u \in L^2(\mathbb{R}^3)$, and the weak convergence in $H^1(\mathbb{R}^3)$ of $(u_\varepsilon - u)$ implies that $\int_{\mathbb{R}^3} \theta u (u_\varepsilon - u) \, \mathrm{d}x$ converges also to zero. We infer that $\int_{\mathbb{R}^3} \theta u_\varepsilon^2 \, \mathrm{d}x \to \int_{\mathbb{R}^3} \theta u^2 \, \mathrm{d}x$ as $\varepsilon \to 0$. Next, since $|u_\varepsilon|^2 \rightharpoonup |u|^2$ weakly in $L^{6/5}(\mathbb{R}^3)$ and using property (i) of Lemma 3.2, we assert in particular that

$$V_\varepsilon(u_\varepsilon) \rightharpoonup V_0(u) \quad \text{weakly in } L^6(\mathbb{R}^3).$$

Furthermore, knowing that $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ then by Sobolev embedding we have

$$u_\varepsilon \longrightarrow u \qquad \text{strongly in } L^{12/5}(K).$$

Hence we infer that

$$\int_K V_\varepsilon(u_\varepsilon) u_\varepsilon \psi \, \mathrm{d}x \longrightarrow \int_K V_0(u) u \psi \, \mathrm{d}x. \tag{4.42}$$

- Finally by the same argument as in (4.29), (4.30) and knowing that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$ we verify easily that

$$\int_{\mathbb{R}^3} \widetilde{V}^- u_\varepsilon \psi \, \mathrm{d}x \longrightarrow \int_{\mathbb{R}^3} \widetilde{V}^- u \psi \, \mathrm{d}x. \tag{4.43}$$

Consequently, letting ε go to 0 in (4.40) and according to (4.41), (4.42) and (4.43), u satisfies for all $\psi \in C_c^\infty(\mathbb{R}^3)$

$$\frac{1}{2} \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \psi) \, \mathrm{d}x + \int_{\mathbb{R}^3} V_0(u) u \psi \, \mathrm{d}x + \int_{\mathbb{R}^3} \widetilde{V} u \psi \, \mathrm{d}x + \omega \int_{\mathbb{R}^3} u \psi \, \mathrm{d}x = 0.$$

We conclude that u_ε converges weakly in H to u which is a solution of the system (1.1)–(1.2) with $\varepsilon = 0$. □

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