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Second-order elliptic integro-differential equations: viscosity solutions' theory revisited

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Abstract

The aim of this work is to revisit viscosity solutions' theory for second-order elliptic integro-differential equations and to provide a general framework which takes into account solutions with arbitrary growth at infinity. Our main contribution is a new Jensen–Ishii's lemma for integro-differential equations, which is stated for solutions with no restriction on their growth at infinity. The proof of this result, which is of course a key ingredient to prove comparison principles, relies on a new definition of viscosity solution for integro-differential equation (equivalent to the two classical ones) which combines the approach with test-functions and sub-superjets.

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0. Introduction

In this paper, we revisit viscosity solutions' theory for second-order elliptic integro-differential equations. We first present in a general framework three (equivalent) definitions of viscosity solutions, each of them having its own utility; in particular, one of them is new. We also present stability results and we discuss comparison principles on a model example.

The core of the paper lies in an analogue of the celebrated Jensen–Ishii's lemma in the framework of second-order elliptic integro-differential equations. This nonsmooth analysis lemma is the keystone of the proofs of comparison principles in viscosity solution theory for *local* second-order fully nonlinear elliptic equations but, because of some particular features of *nonlocal* equations, it needs to be reformulated in this context.

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The first statement of the Jensen–Ishii's lemma is due to Ishii [12] and relies on ideas developed by Jensen [14] when adapting the viscosity solutions approach to second-order elliptic equations. Let us recall that comparison principles themselves are, by many ways, the cornerstone of this theory since they provide not only uniqueness but also existence of continuous solutions when coupled with the Perron's method adapted by Ishii [12] to the framework of fully nonlinear, possibly degenerate, elliptic equations. We refer to the "Users' guide" [8] for an introduction and a general presentation of the whole viscosity solutions theory.

Motivated by applications to finance but also by an increasing number of other ones (physical sciences, mechanics, biological models, etc.), the theory has been almost immediately extended (1986) to the context of partial integro-differential equations (PIDE for short), i.e. partial differential equations involving nonlocal operators such as Lévy ones

$$\mathcal{I}_L[u](x) = \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_B(z)\big)\mu(dz) \tag{1}$$

where μ is a singular measure and B is some ball centered at 0. To the best of our knowledge, the first paper devoted to this extension is the one by Soner [19] in the context of stochastic control of jump diffusion processes. Since then, a huge literature has grown up and it would be difficult (and irrelevant with respect to our goals) to cite all papers. Instead, let us describe the difficulties that were successively overcome. Following Soner's work, a quite general class of integro-differential equations, nonlinear with respect to the nonlocal operators, were developed by Sayah [17]. At that time, it was not possible to deal with equations involving second-order derivatives of u such as

$$\lambda u + H(x, u, Du) - \Delta u - \mathcal{I}_L[u](x) = 0 \quad \text{in } \mathbb{R}^d.$$

In the case of bounded measures, Alvarez and Tourin [2] obtained quite general results for parabolic equations. In [5,16] for instance, several comparison results were obtained in special cases for singular measures. A first attempt is made by Jakobsen and Karlsen [13] to give general results applicable to second-order elliptic equations. In order to get an analogous of Jensen–Ishii's lemma, the authors have to assume that solutions are subquadratic. This assumption is not always relevant since, roughly speaking, the restriction of the behavior of solutions at infinity is related to the integrability of the singular measure away from the origin. For instance, in [5], solutions with arbitrary polynomial growth are considered and even for a system of PIDE.

To sum it up, the difficulties involved by elliptic nonlinear PIDE are

- the coupling of second-order derivatives and nonlocal terms,
- the singularity of the measure appearing in the nonlocal operator,
- the behavior at infinity of solutions.

The third difficulty is studied in details in [1] in the special case of semilinear parabolic PIDE and we refer the reader to it for a detailed discussion.

The present paper is focused on the first two difficulties we listed above and so let us be a bit more specific about them. To do so, we recall that when proving a comparison principle for a standard local equation, the definition of viscosity solutions with test-functions has to be completed with the equivalent definition in terms of so-called sub and superjets (see next section for a definition); and the classical Jensen–Ishii's lemma allows to build elements of limiting semi-jets which play the role of first and second-order derivatives at the maximum point (after the doubling of variables).

But, on one hand, this method relies on the fact that one can pass to the limit in the equation in order to write it for this larger class of generalized derivatives and, on the other hand, such a technic cannot be directly applied in the context of PIDE since test-functions are used not only to give a weak sense to the first and second derivatives of the solution but also to give a sense to the nonlocal operator around the singularity.

In the present work, we try to present a general framework to deal with degenerate elliptic integro-differential equations. In particular, we give different equivalent definitions of solutions, we state and prove a general stability result and we propose a nonlocal version of Jensen–Ishii's lemma for these equations in order to prove comparison principles. We provide a comparison result which will be for us the occasion to discuss the assumptions one has to impose on the nonlocal term(s).

Finding a proper notion of viscosity solution for degenerate elliptic integro-differential equation is an important problem, especially when second-order terms are involved. See for instance the recent paper of Arisawa [3]. The first two equivalent definitions of viscosity solutions (Definitions 1 and 2) we give are quite classical, even if we present them in an original general framework in order to deal with solutions with arbitrary growth at infinity (not only polynomial growth). Freely speaking, the first one consists in replacing the solution by the test-function on the whole space while the second one consists in replacing it only around the singularity of the measure in the nonlocal operator. A third definition (Definition 4) is given and it is new. It combines the use of semi-jets and test-functions. We also prove on a special case (even if such a result holds true in a general setting) that one can in fact, in some way, use the function u on the whole space and thus obtain a definition that only relies on semi-jets. A whole section is devoted to examples in order to illustrate and justify the general framework we introduce.

We next explain how to pass to the limit in PIDE. In the viscosity solution context, the proper limits are the half-relaxed ones. As remarked in [6] in a sublinear setting, dealing with nonlocal operators involve specific technical difficulties. We explain how to overcome them without restricted the behavior at infinity of solutions.

The nonlocal version of Jensen–Ishii's lemma relies on an adapted inf-convolution procedure: the slope of the test-function is taken into account and the infimum is localized (in order to deal with functions with arbitrary growth). Since the statement of the lemma is quite technical, we immediately derive two corollaries that can be used in most examples of comparison principles to derive proper viscosity inequalities.

Keeping in mind our illustrative purpose, we state and prove a comparison principle under quite general assumptions focusing our attention on the coupling between the nonlocal term and the x-dependence of the Hamiltonian.

In a forthcoming work, we would like to continue our investigation by studying in details nonlocal operators on bounded domains, in order to clarify for instance what are the equivalent conditions of Dirichlet and Neumann conditions for nonlocal operators (several interpretations are already given in the literature). We refer to Arisawa [3,4] for results in this direction.

The paper is organized as follows. In Section 1, we recall two equivalent definitions of viscosity solutions for PIDE and we show how to combine semi-jets and test-functions in order to get a third equivalent definition. In Section 2, we give examples of singular measures, nonlocal operators and PIDE. The general stability result is presented in Section 3. In Section 4, we state our nonlocal version of Jensen–Ishii's lemma. In Section 5, we apply this lemma to proving a model comparison principle.

Notation. The scalar product of $x, y \in \mathbb{R}^N$ is denoted by $x \cdot y$ and the Euclidean norm of x is denoted by $|x|$. The unit ball of $\mathbb{R}^N$ (with $N = d$ most of the time) is denoted by B. A ball of radius r centered at the origin is denoted by B_r. The Hessian matrix of a twice differentiable function u is denoted by D^2u. The $N \times N$ (real) identity matrix is simply denoted by I. The space of $N \times N$ symmetric matrices with real entries is denoted by $\mathbb{S}_N$.

1. Nonlocal operators and viscosity solutions of PIDE

In order to emphasize the common features of the (monotone) PIDE to which viscosity solution theory applies in a natural way, we are going to consider the case of general equations written under the form

$$F\big(x, u, \nabla u, D^2u, \mathcal{I}[x,u]\big) = 0 \quad \text{in } \mathbb{R}^d, \tag{2}$$

where F is a continuous function satisfying the local and nonlocal degenerate ellipticity conditions (E) (see below).

Unfortunately, this simple, general model equation does not cover all the interesting cases: in particular, the cases of the Bellman equation arising in stochastic control (cf. (16) below) or the system studied in [5] cannot be written in this way. But the ideas described in the present paper can be extended and used readily in this more complex framework. A more general model equation could be the following one

$$F\big(x, u, \nabla u, D^2u, \{\mathcal{I}_\alpha[x,u]\}_{\alpha\in A}\big) = 0 \quad \text{in } \mathbb{R}^d, \tag{3}$$

where F is continuous and $\{\mathcal{I}[x,u]\}_{\alpha\in A}$ is a family of nonlocal terms.

1.1. Assumptions

We first recall the classical definitions of semicontinuous envelopes and half-relaxed limits. For a locally bounded function u, its lower semicontinuous (lsc for short) envelope u_* and its upper semicontinuous (usc for short) one u^* are defined as follows

$$u_*(x) = \liminf_{y\to x} u(y), \qquad u^*(x) = \limsup_{y\to x} u(y).$$

For a sequence $(z^\varepsilon)_\varepsilon$ of uniformly locally bounded functions in some space $\mathbb{R}^m$

$$\operatorname{liminf}_* z^\varepsilon(x) = \liminf_{\substack{y\to x\\ \varepsilon\to 0}} z^\varepsilon(y), \qquad \operatorname{limsup}^* z^\varepsilon(x) = \limsup_{\substack{y\to x\\ \varepsilon\to 0}} z^\varepsilon(y).$$

In order to be more specific on the assumptions we are going to use for the nonlocal term, we first introduce a space of functions $\mathcal{C}$ which is, typically, a set of functions satisfying a suitable growth condition. We use the following type of assumptions.

Assumption (C). Given an upper semicontinuous function $R:\mathbb{R}^d \to \mathbb{R}$, $\mathcal{C}$ is the space of functions u such that there exists a constant $\bar{c} > 0$ such that, for any $x \in \mathbb{R}^d$

$$\left|u(x)\right| \leqslant \bar{c}\bigl(1 + R(x)\bigr).$$

We remark that functions u of $\mathcal{C}$ are locally bounded, the set $\mathcal{C}$ is stable by the supremum and infimum operations, and by passing to lower and upper envelopes and even to half-relaxed limits. Moreover, if $K \subset \mathbb{R}^d$ is a compact set and if $\phi \in C^2(K)$, there exists a function $\psi \in \mathcal{C} \cap C^2(K)$ such that $\psi = \phi$ on the interior of K. All these properties will be used throughout the paper.

Example 1. The simplest example of set $\mathcal{C}$ is the space of bounded functions $C_b(\mathbb{R}^d)$. Another example is the space of sublinear functions, i.e. functions u for which there exists a constant $\bar{c}$ such that

$$\left|u(x)\right| \leqslant \bar{c}\bigl(1 + |x|\bigr).$$

We will see below that this kind of growth conditions is mainly related to the behavior at infinity of the measure μ appearing in the nonlocal term. If this measure has a compact support, then we can even deal with $\mathcal{C} = C(\mathbb{R}^d)$.

Our assumptions on the nonlocal term are the following.

Assumption (NLT). For any $\delta > 0$, there exist operators $\mathcal{I}^{1,\delta}[x,\phi]$, $\mathcal{I}^{2,\delta}[x,p,\phi]$ which are well-defined for any $x, p \in \mathbb{R}^d$ and $\phi \in \mathcal{C} \cap C^2(\mathbb{R}^d)$ and which satisfy

- For any $x \in \mathbb{R}^d$ and $\phi \in \mathcal{C} \cap C^2(\mathbb{R}^d)$, $\mathcal{I}[x,\phi] = \mathcal{I}^{1,\delta}[x,\phi] + \mathcal{I}^{2,\delta}[x,\nabla\phi(x),\phi]$. Moreover, for any $a \in \mathbb{R}$, $\mathcal{I}^{1,\delta}[x,\phi+a] = \mathcal{I}^{1,\delta}[x,\phi]$ and $\mathcal{I}^{2,\delta}[x,\nabla\phi(x),\phi+a] = \mathcal{I}^{2,\delta}[x,\nabla\phi(x),\phi]$.
- There exists $R_\delta > 0$ with $R_\delta \to 0$ as $\delta \to 0$, such that if $\phi_1 = \phi_2$ on $B(x,R_\delta)$ (resp. on $\mathbb{R}^d \setminus B(x,R_\delta)$), then $\mathcal{I}^{1,\delta}[x,\phi_1] = \mathcal{I}^{1,\delta}[x,\phi_2]$ (resp. $\mathcal{I}^{2,\delta}[x,p,\phi_1] = \mathcal{I}^{2,\delta}[x,p,\phi_2]$).
- For any $\phi \in C^2(\mathbb{R}^d)$ and $u \in \mathcal{C}$ such that $u - \phi$ attains a maximum at x on $B(x,R_\delta)$, there exists $\phi_k \in \mathcal{C} \cap C^2(\mathbb{R}^d)$ such that:

$$\left.\begin{aligned} &u - \phi_k \text{ attains a global maximum at } x,\\ &\mathcal{I}^{1,\delta}[x,\phi_k] \to \mathcal{I}^{1,\delta}[x,\phi],\\ &\mathcal{I}^{2,\delta}\bigl[x,\nabla\phi_k(x),\phi_k\bigr] \to \mathcal{I}^{2,\delta}\bigl[x,\nabla\phi(x),u\bigr] \end{aligned}\right\} \quad \text{as } k \to +\infty.$$

- The operator $\mathcal{I}^{1,\delta}[x,\phi]$ is well-defined for any $x \in \mathbb{R}^d$ and $\phi \in C^2(B(x,r)) \cap \mathcal{C}$ for any $r \in (0,R_\delta)$; moreover $\mathcal{I}^{1,\delta}[x,\phi] \to 0$ when $\delta \to 0$ and $\mathcal{I}^{1,\delta}[x_k,\phi_k] \to \mathcal{I}^{1,\delta}[x,\phi]$ if $x_k \to x$ and $\phi_k \to \phi$ in $C^2(B(x,r)) \cap C(\overline{B(x,R_\delta)})$.

- The operator $\mathcal{I}^{2,\delta}[x,p,\phi]$ is defined for any $x, p \in \mathbb{R}^d$ and $\phi \in \mathcal{C}$. Moreover, if $x_k \to x$, $p_k \to p$ and $(\phi_k)_k$ is a sequence of uniformly locally bounded functions such that $|\phi_k| \leqslant \psi$ with $\psi \in \mathcal{C}$,

$$\limsup_{k\to+\infty} \mathcal{I}^{2,\delta}[x_k,p_k,\phi_k] \leqslant \mathcal{I}^{2,\delta}[x,p,\overline{\phi}] \quad \Big(\text{resp. } \liminf_{k\to+\infty} \mathcal{I}^{2,\delta}[x_k,p_k,\phi_k] \geqslant \mathcal{I}^{2,\delta}[x,p,\underline{\phi}]\Big)$$

where $\overline{\phi} := \operatorname{limsup}^* \phi_k$ (resp. $\underline{\phi} := \operatorname{liminf}_* \phi_k$).

Remark 1. In the general case, i.e. as far as (3) is concerned, Assumption (NLT) must be satisfied by all the nonlocal operators $\mathcal{I}_\alpha$ with the same space $\mathcal{C}$. Such an assumption is natural in many examples since only one singular measure appears in the different nonlocal operators.

Example 2. Consider the Lévy operator appearing in (1) with $\mu(dz) = \frac{dz}{|z|^{N+\alpha}}$. In this case for any $\delta > 0$,

$$\begin{aligned}
\mathcal{I}^{1,\delta}[x,\phi] &= \int_{|z|\leqslant\delta} \big(\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z\mathbf{1}_B(z)\big)\mu(dz),\\
&= \int_{|z|\leqslant\delta} \big(\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z\big)\mu(dz),\\
\mathcal{I}^{2,\delta}[x,p,\phi] &= \int_{|z|\geqslant\delta} \big(\phi(x+z) - \phi(x) - p\cdot z\mathbf{1}_B(z)\big)\mu(dz)\\
&= \int_{|z|\geqslant\delta} \big(\phi(x+z) - \phi(x)\big)\mu(dz)
\end{aligned}$$

(in addition we have used here the fact that μ is odd). If $\alpha > 1$, sublinear functions are integrable away from the origin and one can look for solutions with sublinear growth at infinity.

Finally, we assume that F is a continuous function satisfying the *ellipticity assumption*:

Assumption (E). For any $x \in \mathbb{R}^d$, $u \in \mathbb{R}$, $p \in \mathbb{R}^d$, $M, N \in \mathbb{S}_d$, $l_1, l_2 \in \mathbb{R}$

$$F(x,u,p,M,l_1) \leqslant F(x,u,p,N,l_2) \quad \text{if } M \geqslant N,\ l_1 \geqslant l_2.$$

As we point it out by making such an assumption, the fact that $F(x,u,p,M,l)$ is nonincreasing in l is indeed part of the ellipticity assumption on F.

Remark 2. As far as (3) is concerned, F is assumed to be nondecreasing with respect to all nonlocal operators.

1.2. Different definitions for viscosity solutions of PIDE

In this subsection, we follow [17] by giving several definitions of viscosity solutions for Eq. (2). We will next prove that they are equivalent. Let us now give a first definition of viscosity solution for (2).

Definition 1 *(Viscosity sub and supersolutions).* An usc function $u \in \mathcal{C}$ is a *viscosity subsolution* of (2) if, for any test-function $\phi \in \mathcal{C} \cap C^2(\mathbb{R}^d)$, if x is a global maximum point of $u - \phi$, then

$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}[x,\phi]\big) \leqslant 0.$$

A lsc semicontinuous function $v \in \mathcal{C}$ is a *viscosity supersolution* of (2) if, for any test-function $\phi \in \mathcal{C} \cap C^2(\mathbb{R}^d)$, if x is a global minimum point of $u - \phi$, then

$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}[x,\phi]\big) \geqslant 0.$$

Remarks 1.

- It is also worth pointing out that, in Definition 1, we can as well assume that ϕ is C^2 in a small neighborhood of 0 and is only continuous outside this neighborhood. This a consequence of the fourth and fifth points in Assumption (NLT).
- One can develop a theory with subsolutions (resp. supersolutions) merely bounded from above (resp. from below) by a function of $\mathcal{C}$. Such an idea is somehow used in Proposition 2.

In the examples we present below, it will be clear (at least we hope so) that nonlocal terms $\mathcal{I}[x, w]$ are, in general, only defined for smooth functions w because, typically, of the singularity of the Lévy measure at 0 and for functions with a suitable growth at infinity; this definition takes care of these two difficulties by using a test-function $\phi \in \mathcal{C} \cap C^2(\mathbb{R}^d)$ where we recall that $\mathcal{C}$ encodes the growth information.

Assumption (NLT) is made in order that the following definition is equivalent to the previous one.

Definition 2 *(Viscosity sub and supersolutions).* An usc function $u \in \mathcal{C}$ is a *viscosity subsolution* of (2) iff, for any test-function $\phi \in C^2(\mathbb{R}^d)$, if x is a maximum of $u - \phi$ on $B(x, R_\delta)$, then

$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}^{1,\delta}[x, \phi] + \mathcal{I}^{2,\delta}\big[x, \nabla\phi(x), u\big]\big) \leqslant 0.$$

A lsc function $v \in \mathcal{C}$ is a *viscosity supersolution* of (2) iff, for any test-function $\phi \in C^2(\mathbb{R}^d)$, if x is a minimum of $u - \phi$ on $B(x, R_\delta)$, then

$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}^{1,\delta}[x, \phi] + \mathcal{I}^{2,\delta}\big[x, \nabla\phi(x), v\big]\big) \geqslant 0.$$

We now turn to a third, less classical definition, where we mix test-functions and sub-superjets. We first recall the definition of sub and superjets.

Definition 3 *(Subjets and superjets).* Let $u : \mathbb{R}^d \to \mathbb{R}$ be an usc function and $v : \mathbb{R}^d \to \mathbb{R}$ be a lsc function.

(i) A couple $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$ is a *superjet* of u at $x \in \mathbb{R}^d$ if

$$u(x+z) \leqslant u(x) + p \cdot z + \frac{1}{2} Xz \cdot z + \mathrm{o}\big(|z|^2\big).$$

(ii) A couple $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$ is a *subjet* of v at $x \in \mathbb{R}^d$ if

$$u(x+z) \geqslant u(x) + p \cdot z + \frac{1}{2} Xz \cdot z + \mathrm{o}\big(|z|^2\big).$$

(iii) A couple $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$ is a *limiting superjet* of u at x if there exists $(x_n, p_n, X_n) \to (x, p, X)$ such that (p_n, X_n) is a superjet of u at x_n and $u(x_n) \to u(x)$.

(iv) A couple $(p, X) \in \mathbb{R}^d \times \mathbb{S}_d$ is a *limiting subjet* of v at x if there exists $(x_n, p_n, X_n) \to (x, p, X)$ such that (p_n, X_n) is a subjet of v at x_n and $u(x_n) \to u(x)$.

We will denote respectively by $J^+u(x)$, $\bar{J}^+u(x)$ the set of superjets and limiting superjets of u at x and by $J^-v(x)$, $\bar{J}^-v(x)$ the set of subjets and limiting subjets of u at x.

Definition 4 *(Sub-supersolutions and super-subjets).* A usc function $u \in \mathcal{C}$ is a *viscosity subsolution* of (2) if, for any test-function $\phi \in C^2(\mathbb{R}^d)$, if x is a maximum point of $u - \phi$ on $B(x, R_\delta)$ and if $(p, X) \in J^+u(x)$ with $p = D\phi(x)$ and $X \leqslant D^2\phi(x)$, then

$$F\big(x, u(x), p, X, \mathcal{I}^{1,\delta}[x, \phi] + \mathcal{I}^{2,\delta}\big[x, \nabla\phi(x), u\big]\big) \leqslant 0.$$

A lsc function $v \in \mathcal{C}$ is a *viscosity supersolution* of (2) if, for any test-function $\phi \in C^2(\mathbb{R}^d)$, if x is a minimum point of $u - \phi$ on $B(x, R_\delta)$ and if $(q, Y) \in J^-v(x)$ with $q = D\phi(x)$ and $Y \geqslant D^2\phi(x)$, then

$$F\big(x, u(x), q, Y, \mathcal{I}^{1,\delta}[x, \phi] + \mathcal{I}^{2,\delta}\big[x, \nabla\phi(x), v\big]\big) \geqslant 0.$$

At first glance, this definition seems useless or at least rather strange; as Section 4.3 will show it, this is however the type of situation we face when applying the nonlocal Jensen–Ishii's lemma of Section 4.

Proposition 1. *Definitions* 1*,* 2 *and* 4 *are equivalent.*

Proof. We already justified that the first two definitions are equivalent. Let us prove that so are Definitions 2 and 4. We do it only for the subsolution case, the supersolution one being treated analogously. Changing ϕ in $\phi+\chi$ where χ is a C^∞, positive function with compact support and such that $\nabla\chi(x)=0$, $D^2\chi(x)=\alpha I$ with $\alpha>0$, we can assume that $X\leqslant D^2\phi(x)-\alpha I$. Translating also ϕ, we can also assume that $\phi(x)=u(x)$.

By classical results, there exists a smooth function $\psi:\mathbb{R}^d\to\mathbb{R}$ such that $\psi(x)=u(x)$, $\nabla\psi(x)=p$, $D^2\psi(x)=X$ and $\psi\geqslant u$ in $\mathbb{R}^d$. We deduce from these properties that x is a maximum of $u-\min(\psi,\phi)$ on $B(x,R_\delta)$ but, since $\psi(x)=\phi(x)$, $\nabla\psi(x)=\nabla\phi(x)$, $D^2\psi(x)\leqslant D^2\phi(x)-\alpha I$, we are sure that $\min(\psi,\phi)=\psi$ in a neighborhood of 0. By Assumption (NLT) (in particular the fact that the test-function needs to be smooth only in a neighborhood of 0), we can use Definition 1 to obtain

$$F\big(x,u(x),\nabla\psi(x),D^2\psi(x),\mathcal{I}^{1,\delta}\big[x,\min(\psi,\phi)\big]+\mathcal{I}^{2,\delta}\big[x,\nabla\phi(x),u\big]\big)\leqslant 0.$$

By using the monotonicity of F in l, we can as well replace $\min(\psi,\phi)$ by ϕ, and the proof is complete. □

1.3. An additional proposition

In [10], general test-functions are replaced with functions $\phi(t,x)=\alpha t+p\cdot x\pm\sigma|x|^2+o(|t|+|x|^2)$. In particular, it is noticed that one can choose $\delta=0$ in Definition 2 for such test-functions. Let us explain this in a general setting.

For simplicity, consider a bounded viscosity subsolution u of a special case of (2)

$$F\big(x,u,\nabla u,D^2u,\mathcal{I}[u]\big)=0\quad\text{in }\mathbb{R}^d \tag{4}$$

where, if μ denotes a singular odd measure,

$$\mathcal{I}[u](x)=\int\big(u(x+z)-u(x)-\nabla u(x)\cdot z\mathbf{1}_B(z)\big)\mu(dz).$$

For such a nonlocal operator, we choose $R_\delta=\delta\leqslant 1$ and

$$\mathcal{I}^{1,\delta}[x,\phi]=\int_{|z|\leqslant\delta}\big(\phi(x+z)-\phi(x)-\nabla\phi(x)\cdot z\big)\mu(dz),$$

$$\mathcal{I}^{2,\delta}[x,p,u]=\int_{|z|\geqslant\delta}\big(u(x+z)-u(x)-p\cdot z\mathbf{1}_B(z)\big)\mu(dz).$$

If $\phi\in C^2(\mathbb{R}^d)$ is a test-function such that $u-\phi$ attains a global maximum at x and $p=\nabla\phi(x)$, we have

$$\forall z\in B_\delta,\quad u(x+z)-u(x)-p\cdot z\leqslant\phi(x+z)-\phi(x)-p\cdot z,$$

and therefore $\psi(z):=[u(x+z)-u(x)-p\cdot z]-[\phi(x+z)-\phi(x)-p\cdot z]$ is nonpositive. This implies that $\int_{B_\delta}\psi(z)\mu(dz)$ is well-defined and since $z\mapsto\phi(x+z)-\phi(x)-p\cdot z\in L^1(B_\delta,d\mu)$, the integral $\int_{B_\delta}(u(x+z)-u(x)-p\cdot z)\mu(dz)$ is well-defined too. Moreover, by the monotone convergence theorem, we can pass to the limit in the equality

$$\int_{r\leqslant|z|\leqslant\delta}\big(u(x+z)-u(x)-p\cdot z\big)\mu(dz)=\int_{r\leqslant|z|\leqslant\delta}\big(\phi(x+z)-\phi(x)-p\cdot z\big)\mu(dz)-\int_{r\leqslant|z|\leqslant\delta}\psi(z)\mu(dz)$$

and, consequently, one can define

$$\mathcal{I}^{2,0}[x,p,u]=\lim_{\delta\to 0+}\mathcal{I}^{2,\delta}[x,p,u]\in\{-\infty\}\cup\mathbb{R}.$$

The (nonincreasing) function $F(\dots,l)$ has a limit $F_\infty\in\mathbb{R}\cup\{+\infty\}$ as l goes to $-\infty$. But since

$$F\big(x,u(x),\nabla\phi(x),D^2\phi(x),\mathcal{I}^{1,r}[x,\phi]+\mathcal{I}^{2,\delta}[x,p,u]\big)\leqslant 0, \tag{5}$$

we conclude that

$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}^{2,0}[x, p, u]\big) \leqslant 0.$$

Indeed, if F_∞ is infinite, then (5) implies that $\mathcal{I}^{2,0}[x, p, u]$ is finite and passing to the limit as $\delta \to 0$ gives the result. Hence if $l^{2,0}[x, p, u]$ is infinite, F_∞ is finite and $F(x, u(x), \nabla\phi(x), D^2\phi(x), l) \leqslant 0$ for any $l \in \mathbb{R} \cup \{-\infty\}$. It is also clear that such a discussion can be adapted to the case of supersolutions. Let us sum up this discussion in the following proposition.

Proposition 2. *For any subsolution u of (2) and any test-function $\phi \in C^2(\mathbb{R}^d)$ such that $u - \phi$ attains a global maximum at x,*

$$\mathcal{I}^{2,0}\big[x, \nabla\phi(x), u\big] \in \{-\infty\} \cup \mathbb{R},$$
$$F\big(x, u(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}^{2,0}\big[x, \nabla\phi(x), u\big]\big) \leqslant 0.$$

Moreover, if $F \to +\infty$ as $l \to -\infty$, then $\mathcal{I}^{2,0}[x, \nabla\phi(x), u]$ is finite.

Remarks 2.

- For clarity, we only treated a special case but the attentive reader can check that such a proposition holds true for all the examples we will give below.
- Remark that one can use this proposition to give an alternative proof of the fact that Definition 4 is equivalent to Definitions 1 and 2.

Following this idea, Arisawa [3] considered quadratic test-functions

$$\alpha t + p \cdot x + \frac{1}{2} A x \cdot x + \mathrm{o}\big(|t| + |x|^2\big)$$

and kept $r > 0$. Since this definition involves some technicalities when proving comparison principles, we will not use it.

2. Examples

PIDE's involve nonlocal operators that are (partially) characterized by an (eventually singular) measure. In this section, we give examples of such measures, operators and equations. Let us start with measures.

2.1. Singular measures

In applications, typically in stochastic control with jump processes, positive singular measures appear in nonlocal operators. For instance, the pure jump part of a Lévy process is characterized by such a measure that is referred to as the Lévy measure. It consequently appears in the associated infinitesimal operator one has to deal with in the associated Bellman equation.

Let us next give two examples of Lévy measures

$$\mu_1(dz) = g\left(\frac{z}{|z|}\right) \frac{1}{|z|^{N+\alpha}}\, dz \quad \text{with } \alpha \in (0, 2), \text{ in } \mathbb{R}^d, \tag{6}$$

$$\mu_2(dz) = \big(\mathbf{1}_{(0,+\infty)}(z) \mathrm{e}^{-\gamma^+ z} + \mathbf{1}_{(-\infty,0)}(z) \mathrm{e}^{\gamma^- z}\big) \frac{1}{|z|}\, dz, \quad \text{in } \mathbb{R}. \tag{7}$$

The measure μ_1 (resp. μ_2) is an anisotropic (resp. tempered) α-stable Lévy measure on $\mathbb{R}^d$ (resp. on $\mathbb{R}$).

We would like next to discuss how singular the measure is around the origin and how it decreases at infinity. If one considers the measures μ_1 and μ_2 introduced previously

$$\int_B |z|^{\alpha'} \mu_1(dz) < +\infty \quad \text{for } \alpha' > \alpha,$$

$$\int_{-1}^{1} |z|^{\alpha'} \mu_2(dz) < +\infty \quad \text{for } \alpha' > 0,$$

$$\int_{\mathbb{R}^d \setminus B} |z|^{\alpha'} \mu_1(dz) < +\infty \quad \text{for } \alpha' < \alpha,$$

$$\int_{\mathbb{R} \setminus [-1,1]} |z|^{p} \mu_2(dz) < +\infty \quad \text{for any } p \in \mathbb{N}.$$

Let us mention that a Lévy measure μ always satisfy

$$\int \min\big(|z|^2, 1\big)\mu(dz) < +\infty. \tag{8}$$

2.2. Nonlocal operators

Now we examine typical examples of nonlocal operators appearing in the applications. For instance, if φ denotes a smooth function satisfying suitable growth conditions, the following two nonlocal operators appear in [5]

$$K\varphi(x) = \int_{\mathbb{R}^d} \big(\varphi\big(x + \beta(x,z)\big) - \varphi(x) - \nabla\varphi(x) \cdot \beta(x,z)\big)\lambda(dz), \tag{9}$$

$$B\varphi(x) = \int_{\mathbb{R}^d} \big(\varphi\big(x + \beta(x,z)\big) - \varphi(x)\big)\gamma(x,z)\lambda(dz); \tag{10}$$

as a matter of fact, we redefine the measure λ appearing in the compensator at the origin by setting: $\lambda(\{0\}) = 0$ so that the domain of integration is the whole space $\mathbb{R}^d$ and not $E := \mathbb{R}^d \setminus \{0\}$. The functions β, γ have to satisfy suitable assumptions.

The natural assumptions coming from the probabilistic formulation are (8) and

$$\big|\beta(x,z)\big|, \big|\gamma(x,z)\big| \leqslant K|z| \quad \text{if } |z| \leqslant 1.$$

It is easy to check that, under these assumptions, the $\mathcal{I}^{1,\delta}[x,\phi]$ terms, namely

$$\int_{\{|z|\leqslant\delta\}} \big(\phi\big(x + \beta(x,z)\big) - \phi(x) - \nabla\phi(x)\cdot\beta(x,z)\big)\lambda(dz) \quad \text{or} \quad \int_{\{|z|\leqslant\delta\}} \big(\phi\big(x + \beta(x,z)\big) - \phi(x)\big)\gamma(x,z)\lambda(dz)$$

are well-defined for any $x \in \mathbb{R}^d$ and any smooth function ϕ, and they satisfy (NLT).

On the other hand, the $\mathcal{I}^{2,\delta}[x,p,\phi]$ terms, namely

$$\int_{\{|z|\geqslant\delta\}} \big(\phi\big(x + \beta(x,z)\big) - \phi(x) - p\cdot\beta(x,z)\big)\lambda(dz) \quad \text{or} \quad \int_{\{|z|\geqslant\delta\}} \big(\phi\big(x + \beta(x,z)\big) - \phi(x)\big)\gamma(x,z)\lambda(dz)$$

are well-defined and satisfy (NLT) if β, γ have suitable integrability properties w.r.t the measure $\lambda(dz)$; these integrability properties determine also the space $\mathcal{C}$ on which the operators $\mathcal{I}^{2,\delta}[x,p,\phi]$ are defined. In [5], β is assumed to be bounded and therefore (NLT) is readily satisfied for $\mathcal{C} = C(\mathbb{R}^d)$.

In stochastic control with jump processes (see for instance [15]), one can consider the so-called Lévy–Ito diffusions whose infinitesimal generators are of the form:

$$\mathcal{I}_{LI}[u](x) = \int \big(u\big(x + j(x,z)\big) - u(x) - \nabla u(x)\cdot j(x,z)\mathbf{1}_B(z)\big)\mu(dz) \tag{11}$$

where $\mathbf{1}_B$ denotes the indicator function of the ball B, μ is a Lévy measure (hence it satisfies (8)) and $j(x,z)$ is the size of the jumps at x. In order that the operator is well-defined, one assumes:

$$\left|j(x,z)\right| \leqslant \bar{c}|z|. \tag{12}$$

The simplest example of j function is $j(x,z) = z$ and in this case, operators are the infinitesimal generators of any pure jump Lévy process; hence, it is referred to as Lévy operators (see [7] for details). One can check that (NLT) is also satisfied with $\mathcal{C} = C_b(\mathbb{R}^d)$ and:

$$\mathcal{I}^{1,\delta}_{LI}[x,\phi] = \int\limits_{|z|\leqslant\delta} \big(\phi\big(x + j(x,z)\big) - \phi(x) - \nabla\phi(x)\cdot j(x,z)\mathbf{1}_B(z)\big)\mu(dz),$$

$$\mathcal{I}^{2,\delta}_{LI}[x,p,\phi] = \int\limits_{|z|\geqslant\delta} \big(\phi\big(x + j(x,z)\big) - \phi(x) - p\cdot j(x,z)\mathbf{1}_B(z)\big)\mu(dz).$$

Let us conclude this subsection by giving other examples one can find in the (huge) literature concerning nonlocal operators:

$$\mathcal{I}_{Si}[x,u](x) = \int \big(u(x+z) - u(x) - \nabla u(x)\cdot z\mathbf{1}_B(z)\big)K(x,z)\,dz \tag{13}$$

$$\mathcal{I}_{Sa}[x,u](x) = \int \big(u(x+z) - u(x) - \nabla u(x)\cdot z\mathbf{1}_B(z)\big)\mu(x,dz) \tag{14}$$

$$\mathcal{I}[x,u](x) = \int \big(u\big(x + j(x,z)\big) - u(x) - \nabla u(x)\cdot j(x,z)\mathbf{1}_B(z)\big)\mu(x,dz) \tag{15}$$

where $\mu(x,dz)$ are measures that are bounded away from the origin possibly singular at the origin. Operators appearing in (13) are considered in [18]; see this paper for further details. Sayah [17] considered operators of the form (14). To finish with, operators of the form (15) are probably the most general ones and they appear for instance in [9].

2.3. Examples of PIDE

A first example of (semilinear) PIDE is one appearing in the context of growing interfaces [20]

$$\partial_t u + \frac{1}{2}|\nabla u|^2 - \mathcal{I}_L[u] = 0$$

where $\mathcal{I}_L$ is a Lévy operator.

A second simple example of nonlinear PIDE is for instance the nonlinear diffusion arising in the context of the homogenization for dislocation dynamics [11]

$$\partial_t u = \overline{H}\big(\nabla u, \mathcal{I}_L[u]\big)$$

where $\mathcal{I}_L$ is an anisotropic Lévy operator of order 1.

An important example of a nonlinear elliptic PIDE is the Bellmann equation associated with a stochastic control problem

$$\lambda u + \sup_{\alpha\in\mathcal{A}}\left\{-\mathcal{I}^\alpha[u] - \frac{1}{2}D^2u(x)\sigma_\alpha(x)\cdot\sigma_\alpha(x) - b^\alpha(x)\cdot\nabla u(x) - f^\alpha(x)\right\} = 0 \tag{16}$$

where

$$\mathcal{I}^\alpha[u] = \int \big(u\big(x + j_\alpha(x,z)\big) - u(x) - \nabla u(x)\cdot j_\alpha(x,z)\mathbf{1}_B(z)\big)\mu(dz).$$

3. Stability results

In this section, we state and prove general stability results for viscosity solutions of PIDE. These results are a slight generalization of all previous analogous results [17,2,6] since they cover the case of general unbounded solutions. To do so, we use the framework introduced in Section 1.1.

For $\varepsilon > 0$, we consider sub or supersolutions u^ε of

$$F_\varepsilon\big(x, u^\varepsilon, \nabla u^\varepsilon, D^2 u^\varepsilon, \mathcal{I}[x, u^\varepsilon]\big) = 0 \quad \text{in } \mathbb{R}^d. \tag{17}$$

The main stability result is the following one.

Theorem 1. *Let* $(F_\varepsilon)_\varepsilon$ *be a sequence of locally uniformly bounded nonlinearities satisfying the ellipticity condition* (E) *and let* $(u^\varepsilon)_\varepsilon$ *be a sequence of subsolutions* (*resp. supersolutions*) *of* (17) *such that there exists* $\bar{c} > 0$ *such that*

$$\big|u^\varepsilon(x)\big| \leqslant \bar{c}\big(1 + R(x)\big) \quad \text{in } \mathbb{R}^d.$$

Then $\bar{u} := \limsup^* u^\varepsilon$ (*resp.* $\bar{u} := \liminf_* u^\varepsilon$) *is a subsolution* (*resp. supersolution*) *of* (2) *with* $\underline{F} := \liminf_* F_\varepsilon$ (*resp.* $\overline{F} := \limsup^* F_\varepsilon$).

The second stability result (which is partly a consequence of the first one) concerns the supremum/infimum of a family of sub/supersolutions. It is the cornerstone of Perron's method when proving the existence of a solution.

To state it in a general way, we consider sub and supersolutions u_α, for $\alpha \in \mathcal{A}$, of

$$F_\alpha\big(x, u_\alpha, \nabla u_\alpha, D^2 u_\alpha, \mathcal{I}[x, u_\alpha]\big) = 0 \quad \text{in } \mathbb{R}^d \tag{18}$$

where $\mathcal{A}$ can be any set.

Theorem 2. *Let* $(F_\alpha)_{\alpha\in\mathcal{A}}$ *be a family of uniformly locally bounded from above* (*resp. from below*) *nonlinearities satisfying the ellipticity condition* (E) *and let* $(u_\alpha)_{\alpha\in\mathcal{A}}$ *be a family of subsolutions* (*resp. supersolutions*) *of* (18) *such that there exists* $\bar{c} > 0$ *such that, for any* $\alpha \in \mathcal{A}$ *and* $x \in \mathbb{R}^d$

$$u_\alpha(x) \leqslant \bar{c}\big(1 + R(x)\big) \quad \big(\text{resp. } u_\alpha(x) \geqslant -\bar{c}\big(1 + R(x)\big)\big). \tag{19}$$

We set $u = \sup_{\alpha\in\mathcal{A}} u_\alpha$ (*resp.* $v = \inf_{\alpha\in\mathcal{A}} u_\alpha$). *Then* u^* (*resp.* v_*) *is a subsolution* (*resp. supersolution*) *of* (2) *where* $\underline{F} = (\inf_{\alpha\in\mathcal{A}} F_\alpha)_*$ *(resp.* $\overline{F} = (\sup_{\alpha\in\mathcal{A}} F_\alpha)^*$*).*

Remarks 3.

- The reader can check that both stability results can be easily adapted to the general case of several nonlocal operators such as (3).
- The condition (19) is not necessary for local equations but it cannot be avoided for nonlocal ones. A special case of it appears in [6].

The proofs of both theorems are easy adaptations of classical ones. For the sake of completeness, we give a proof of the first one and we let the reader check that the classical proof for the first one can be also adapted.

Proof. We only provide the proof for $\bar{u}$ since the other case follows along the same lines. In order to prove that $\bar{u}$ is a subsolution of the limit equation, we consider a test-function ϕ and a maximum point x of $\bar{u} - \phi$ on $B(x, R_\delta)$ (see Assumption (NLT)).

First we can assume without loss of generality that x is a strict maximum point of $\bar{u} - \phi$ on $B(x, R_\delta)$: indeed we may replace ϕ with $\tilde{\phi} = \phi + \alpha\chi$ where χ is a C^∞ function whose support is $B(x, 2R_\delta)$ and such that $\chi > 0$ on $B(x, R_\delta) \setminus \{x\}$, $\chi(x) = 0$, $\nabla\chi(x) = 0$, $D^2\chi(x) = 0$.

Next we consider a subsequence such that

$$\bar{u}(x) = \lim_{\varepsilon'} u_{\varepsilon'}(x_{\varepsilon'}).$$

Since x is a strict maximum point of $\bar{u} - \phi$ on $B(x, R_\delta)$, classical arguments show that $u_{\varepsilon'} - \tilde{\phi}$ attains a maximum on $B(x, R_\delta)$ at $y_{\varepsilon'} \in B(x, R_\delta)$; moreover

$$x = \lim_{\varepsilon'} y_{\varepsilon'} \quad \text{and} \quad \bar{u}(x) = \lim u_{\varepsilon'}(y_{\varepsilon'}).$$

Since $u_{\varepsilon'}$ is a subsolution of (17), we have

$$F_{\varepsilon'}\big(x_{\varepsilon'}, u_{\varepsilon'}(x_{\varepsilon'}), \nabla\tilde{\phi}(x_{\varepsilon'}), D^2\tilde{\phi}(x_{\varepsilon'}), l_{\varepsilon'}\big) \leqslant 0$$

with

$$l_{\varepsilon'} = \mathcal{I}^{1,\delta}[x_{\varepsilon'}, \tilde{\phi}] + \mathcal{I}^{2,\delta}[x_{\varepsilon'}, \nabla\tilde{\phi}(x_{\varepsilon'}), u_{\varepsilon'}].$$

By (NLT) and (19), we can conclude that

$$\limsup_{\varepsilon'} l_{\varepsilon'} \leqslant \mathcal{I}^{1,\delta}[x, \tilde{\phi}] + \mathcal{I}^{2,\delta}[x, \nabla\phi(x), \bar{u}]$$

and the definition of $\underline{F}$ together with the nonlocal ellipticity permits to get

$$\underline{F}\big(x, \bar{u}(x), \nabla\phi(x), D^2\phi(x), \mathcal{I}^{1,\delta}[x, \tilde{\phi}] + \mathcal{I}^{2,\delta}\big[x, \nabla\phi(x), \bar{u}\big]\big) \leqslant 0.$$

Finally we let α tend to 0 and the proof is complete. □

4. A nonlocal version of Jensen–Ishii's lemma

In order to state our result, we first introduce the inf and sup-convolution operations we are going to use.

4.1. Modified inf/sup-convolution procedures

For any usc function $U : \mathbb{R}^m \to \mathbb{R}$ and any lsc function $V : \mathbb{R}^m \to \mathbb{R}$, we set

$$R^\alpha[U](z, r) = \sup_{|Z-z|\leqslant 1} \left\{ U(Z) - r \cdot (Z - z) - \frac{|Z - z|^2}{2\alpha} \right\},$$

$$R_\alpha[V](z, r) = \inf_{|Z-z|\leqslant 1} \left\{ V(Z) + r \cdot (Z - z) + \frac{|Z - z|^2}{2\alpha} \right\}.$$

Notice that $R_\alpha[V] = -R^\alpha[-V]$.

Proposition 3. *For any usc function $U : \mathbb{R}^m \to \mathbb{R}$ and any lsc function $V : \mathbb{R}^m \to \mathbb{R}$, the function $R^\alpha[U]$, $R_\alpha[V]$ satisfy the following properties*

1. *For any $x, r \in \mathbb{R}^m$, $R^\alpha[U](x, r) \geqslant U$ and $R_\alpha[V](x, r) \leqslant V$.*
2. *For any $x \in \mathbb{R}^m$ and $\bar{K} > 0$, there exists $\bar{\alpha} = \bar{\alpha}(x, \bar{K})$ such that, for $0 < \alpha \leqslant \bar{\alpha}$, $R^\alpha[U](\cdot, r)$ is semi-convex in $B(x, \bar{K})$ (resp. $R_\alpha[V](\cdot, r)$ is semi-concave in $B(x, \bar{K})$).*
3. *Assume that $U \in C^2(\mathbb{R}^m)$ (resp. $V \in C^2(\mathbb{R}^m)$). For any $x \in \mathbb{R}^m$ and $\bar{K} > 0$, there exists $\bar{\alpha} = \bar{\alpha}(x, \bar{K})$ such that, for $0 < \alpha \leqslant \bar{\alpha}$, then $R^\alpha[U]$ (resp. $R_\alpha[V]$) is C^2 in $B(0, \bar{K})$.*
4. *If $R^\alpha[U](z, r) = U(\bar{z}) - r \cdot (\bar{z} - z) - |\bar{z} - z|^2/(2\alpha)$ and if $|\bar{z} - z| < 1$, then*

$$(s, A) \in J^+ R^\alpha[U](z, r) \Rightarrow (s, A) \in J^+ U(\bar{z}) \text{ and } s = r - \frac{z - \bar{z}}{\alpha}, \tag{20}$$

$$(r, A) \in \overline{D}^{2,+} R^\alpha[U](z, r) \Rightarrow (s, A) \in \overline{D}^{2,+} U(z). \tag{21}$$

Proof. The first two points are clear. The third point is a consequence of the analogous classical result: indeed, for α small enough, the supremum and infimum are achieved for $|Z - z| < 1$ and one can apply the same proof as for the classical sup and inf-convolutions (maximizing or minimizing w.r.t. $Z \in \mathbb{R}^m$ as if U and V were bounded). Let us focus on the fourth one. Eq. (20) is a simple adaptation of the classical result about sup-convolution. Eq. (21) is a consequence of it. Indeed, by definition of limiting superjets, there exists $(r_n, A_n) \in J^+ R^\alpha[U](z_n, r)$ such that $(r_n, A_n, z_n) \to (r, A, z)$. Moreover, by (20), we have: $(r_n, A_n) \in J^+ U(z_n + \alpha(r - r_n))$. The proof is now complete. □

4.2. Statement and proof of the lemma

We can now state our nonlocal version of Jensen–Ishii's lemma.

Lemma 1 *(Nonlocal Jensen–Ishii's lemma). Let u and v be respectively a usc and a lsc function defined on $\mathbb{R}^d$ and let ϕ be a C^2 function defined on $\mathbb{R}^{2d}$. If $(\bar{x},\bar{y})\in\mathbb{R}^{2d}$ is a zero global maximum point of $u(x)-v(y)-\phi(x,y)$ and if $p:=D_x\phi(\bar{x},\bar{y})$, $q:=D_y\phi(\bar{x},\bar{y})$, then the following holds*

$$u(x)-v(y)\leqslant R^\alpha[u](x,p)-R_\alpha[v](y,q),$$
$$\leqslant R^\alpha[\phi]\big((x,y),(p,q)\big), \tag{22}$$
$$u(\bar{x})=R^\alpha[u](\bar{x},p), \tag{23}$$
$$v(\bar{y})=R_\alpha[v](\bar{y},q), \tag{24}$$
$$R^\alpha[\phi]\big((\bar{x},\bar{y}),(p,q)\big)=\phi(\bar{x},\bar{y}). \tag{25}$$

Moreover, for any $\bar{K}>0$, there exists $\bar{\alpha}(\bar{K})>0$ such that, for any $0<\alpha\leqslant\bar{\alpha}(\bar{K})$, we have: there exist sequences $x_k\to x$, $y_k\to y$, $p_k\to p$, $q_k\to q$, matrices X_k,Y_k and a sequence of functions ϕ_k, converging to the function $\phi_\alpha:=R^\alpha[\phi]((x,y),(p,q))$ uniformly in $\mathbb{R}^m$ and in $C^2(B((\bar{x},\bar{y}),\bar{K}))$, such that

$$(x_k,y_k)\ \text{is a global maximum point of}\ u-v-\phi_k, \tag{26}$$
$$u(x_k)\to u(\bar{x}),\ v(y_k)\to v(\bar{y}), \tag{27}$$
$$(p_k,X_k)\in J^+u(x_k), \tag{28}$$
$$(-q_k,Y_k)\in J^-v(y_k), \tag{29}$$
$$-\frac{1}{\alpha}I\leqslant\begin{bmatrix}X_k & 0\\ 0 & -Y_k\end{bmatrix}\leqslant D^2\phi_k(x_k,y_k). \tag{30}$$

Remark 3. The nonlocal Jensen–Ishii's lemma is stated for functions u and v which are defined in $\mathbb{R}^d$ but the same result holds if u and v are defined only on a (closed) subset of $\mathbb{R}^d$. Indeed, following the "User's guide" (cf. [8], p. 57), it suffices to extend u and v in a suitable way outside this subset (and typically u by $-\infty$ and v by $+\infty$). This remark is important when one wants to deal with problems set in a domain of $\mathbb{R}^d$ with boundary conditions.

Proof. Eq. (22) is a simple consequence of Proposition 3 of the fact that $u-v-\phi$ attains a zero global maximum. Eq. (25) is a consequence of the regularity of ϕ and more precisely of the property

$$\phi(x,y)\leqslant\phi(\bar{x},\bar{y})+p\cdot(x-\bar{x})+q\cdot(y-\bar{y})+K\big|(x,y)-(\bar{x},\bar{y})\big|^2$$

for some constant $K>0$ and for $|(x,y)-(\bar{x},\bar{y})|\leqslant 1$. Eq. (25) implies (23) and (24).

The function $(x,y)\mapsto R^\alpha[u](x,p)-R_\alpha[v](y,-q)-R^\alpha[\phi]((x,y),(p,q))$ is semi-convex and achieves a global maximum point at $(\bar{x},\bar{y})$. In order to apply Lemma A.3 of [8], we have to transform $(\bar{x},\bar{y})$ into a strict maximum point. To do so, we consider a smooth bounded function χ such that $\chi>0$ in $B((\bar{x},\bar{y}),1)\setminus\{(\bar{x},\bar{y})\}$ and $\chi=0$ outside; $(\bar{x},\bar{y})$ is a strict maximum point of

$$(x,y)\mapsto R^\alpha[u](x,p)-R_\alpha[v](y,-q)-R^\alpha[\phi]\big((x,y),(p,q)\big)-\delta\chi(x,y),$$

for any $\delta>0$.

Next we consider a smooth function $\psi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, with compact support and such that $\psi(x,y)=1$ if $|(x,y)-(\bar{x},\bar{y})|\leqslant 1$. We are going to apply Lemma A.3 of [8] to the function

$$(x,y)\mapsto R^\alpha[u](x,p)-R_\alpha[v](y,-q)-R^\alpha[\phi]\big((x,y),(p,q)\big)-\delta\chi(x,y)+\psi(x,y)(r\cdot x+s\cdot y),$$

for $r,s\in\mathbb{R}^d$ close to 0. On one hand, the fact that $(\bar{x},\bar{y})$ is a strict global maximum of $R^\alpha[u](x,p)-R_\alpha[v](y,-q)-R^\alpha[\phi]((x,y),(p,q))-\delta\chi(x,y)$ implies that this function has global maximum points near $(\bar{x},\bar{y})$ for r,s close enough to 0, and, on the other hand, since ψ is 1 in a neighborhood of $(\bar{x},\bar{y})$, we can readily apply Lemma A.3 of [8] in this neighborhood where the function is nothing but

$$(x,y)\mapsto R^\alpha[u](x,p)-R_\alpha[v](y,-q)-R^\alpha[\phi]\big((x,y),(p,q)\big)-\delta\chi(x,y)+r\cdot x+s\cdot y.$$

Combining it with Theorem A.2 of [8], we deduce that, for any $\delta > 0$, there exists sequences $(r_m^\delta)_m$, $(s_m^\delta)_m$ and global maximum points $(x_m^\delta, y_m^\delta)_m$ of the above function such that $R^\alpha[u]$ is twice differentiable at x_m^δ and $R_\alpha[v]$ is twice differentiable at y_m^δ. Choosing $\delta = k^{-1}$, $(\tilde{x}_k, \tilde{y}_k) = (x_m^\delta, y_m^\delta)$ for m large enough and

$$\tilde{\phi}_k(x, y) = R^\alpha[\phi]\big((x, y), (p, q)\big) - \delta\chi(x, y) + r_m^\delta \cdot x + s_m^\delta \cdot y,$$

we have a sequence $(\tilde{x}_k, \tilde{y}_k)$ of maximum points of $R^\alpha[u](x, p) - R_\alpha[v](y, -q) - \phi_k((x, y), (p, q))$, where ϕ_k are small, smooth perturbations of $R^\alpha[\phi]((x, y), (p, q))$. By Proposition 3, there then exists x_k and y_k such that:

$$R^\alpha[u](\tilde{x}_k, p) = u(x_k) - p \cdot (x_k - \tilde{x}_k) - \frac{|x_k - \tilde{x}_k|^2}{2\alpha} \quad \text{and}$$
$$R_\alpha[v](\tilde{y}_k, -q) = u(x_k) - q \cdot (y_k - \tilde{y}_k) + \frac{|y_k - \tilde{y}_k|^2}{2\alpha}.$$

Hence, using (20), we get two sequences of points $(x_k)_k$ and $(y_k)_k$ with the following properties:

1. (x_k, y_k) is a maximum point of $(x, y) \mapsto u(x) - v(y) - \phi_k(x, y)$,
2. $x_k \to x$, $y_k \to y$, $p_k \to p$, $q_k \to q$ as $k \to +\infty$,
3. $R^\alpha[u]$ and $R_\alpha[v]$ are twice differentiable at x_k and y_k respectively,

where $\phi_k(x, y) = \tilde{\phi}_k(x + \tilde{x}_k - x_k, y + \tilde{y}_k - y_k)$.

Therefore we can take $p_k = \nabla R^\alpha[u](\tilde{x}_k)$, $X_k = D^2 R^\alpha[u](\tilde{x}_k)$, $q_k := \nabla R_\alpha[v](\tilde{y}_k)$, $Y_k := D^2 R_\alpha[v](y_k)$ and all the claims of Lemma 1 are easy consequences of either the semi-convexity or semi-concavity of $R^\alpha[u]$ and $R_\alpha[v]$ or of classical properties. □

4.3. How to apply the lemma?

Now we address the question: how to apply the nonlocal Jensen–Ishii's lemma? The (partial) answer is given by the

Corollary 1. *Let u be an usc viscosity subsolution of* (2)*, let v be a lsc viscosity supersolution of* (2) *and let $\phi \in C^2(\mathbb{R}^d)$. If $(\bar{x}, \bar{y}) \in \mathbb{R}^{2d}$ is a global maximum point of $u(x) - v(y) - \phi(x, y)$, then, for any $\delta > 0$, there exists $\bar{\alpha}$ such that, for $0 < \alpha < \bar{\alpha}$, we have*

$$F\big(\bar{x}, u(\bar{x}), p, X, \mathcal{I}^{1,\delta}\big[\bar{x}, \phi_\alpha(\cdot, \bar{y})\big] + \mathcal{I}^{2,\delta}[\bar{x}, p, u]\big) \leqslant 0,$$
$$F\big(\bar{y}, v(\bar{y}), q, Y, \mathcal{I}^{1,\delta}\big[\bar{y}, -\phi_\alpha(\bar{x}, \cdot)\big] + \mathcal{I}^{2,\delta}[\bar{y}, q, v]\big) \geqslant 0,$$

where $p = \nabla_x \phi(\bar{x}, \bar{y}) = \nabla_x \phi_\alpha(\bar{x}, \bar{y})$, $q = -\nabla_y \phi(\bar{x}, \bar{y}) = \nabla_y \phi_\alpha(\bar{x}, \bar{y})$ and with also

$$-\frac{1}{\alpha} I \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant D^2 \phi_\alpha(\bar{x}, \bar{y}) = D^2 \phi(\bar{x}, \bar{y}) + o_\alpha(1). \tag{31}$$

This situation is exactly the one which is needed in the uniqueness proofs, including the ones which consist in proving first that $u - v$ is a subsolution of an auxiliary PIDE (see [5] for details).

It is worth pointing out that, in the nonlocal term, one has a priori to use the function ϕ_α instead of ϕ: this is consistent with the fact that the second derivatives are also estimated by $D^2\phi_\alpha(\bar{x}, \bar{y})$. However, because of our assumption on the $\mathcal{I}^{1,\delta}$-term, the difference between $\mathcal{I}^{1,\delta}[\bar{x}, \phi_\alpha]$ and $\mathcal{I}^{1,\delta}[\bar{x}, \phi]$ is a small error term and we will see in the next section that under mild additional assumptions on F, one can use $\mathcal{I}^{1,\delta}[\bar{x}, \phi] + o_\alpha(1)$ as well.

Proof. Without loss of generality, changing ϕ in $\phi - M$ for some well-chosen constant M, we can assume that $(\bar{x}, \bar{y}) \in \mathbb{R}^{2d}$ is a zero maximum point of $u(x) - v(y) - \phi(x, y)$.

Applying the lemma together with Definition 4, we get

$$F\big(x_k, u(x_k), p_k, X_k, \mathcal{I}^{1,\delta}\big[x_k, \phi_k(\cdot, y_k)\big] + \mathcal{I}^{2,\delta}\big[x_k, \nabla_x \phi_k(x_k, y_k), u\big]\big) \leqslant 0,$$
$$F\big(y_k, v(y_k), q_k, Y_k, \mathcal{I}^{1,\delta}\big[y_k, -\phi_k(x_k, \cdot)\big] + \mathcal{I}^{2,\delta}\big[y_k, \nabla_y \phi_k(x_k, y_k), v\big]\big) \geqslant 0.$$

Choosing α small enough in order that $\phi_k \to \phi_\alpha := R^\alpha[\phi]((x,y),(p,q))$ in $C^2(B(x,R_\delta))$, we can pass to the limit and obtain the result. □

We conclude this section by an easy extension of the nonlocal Jensen–Ishii's lemma. It concerns the case of time-dependent equations

$$u_t + F(x,u,Du,D^2u,\mathcal{I}[x,u]) = 0 \quad \text{in } \mathbb{R}^d \times (0,T), \tag{32}$$

where $T > 0$.

We formulate without proof the analogue of Corollary 1 where J^+, J^- denotes the "parabolic" super and subjets which take only into account the second-order derivatives in space (and not in time).

Corollary 2. *Let u be an usc viscosity subsolution of* (32)*, let v be a lsc viscosity supersolution of* (32) *and let $\phi \in C^2(\mathbb{R}^d \times (0,T))$. If $(\bar{x},\bar{y},\bar{t}) \in \mathbb{R}^{2d} \times (0,T)$ is a global maximum point of $u(x,t) - v(y,t) - \phi(x,y,t)$, then, for any $\delta > 0$, there exists $\bar{\alpha}$ such that, for $0 < \alpha < \bar{\alpha}$, there exists $(a,p,X) \in J^+u(\bar{x},\bar{t})$, $(b,q,Y) \in J^-v(\bar{y},\bar{t})$ such that we have*

$$a + F(\bar{x},u(\bar{x}),p,X,\mathcal{I}^{1,\delta}[\bar{x},\phi_\alpha] + \mathcal{I}^{2,\delta}[\bar{x},p,u]) \leqslant 0,$$
$$b + F(\bar{y},v(\bar{y}),q,Y,\mathcal{I}^{1,\delta}[\bar{y},\phi_\alpha] + \mathcal{I}^{2,\delta}[\bar{y},q,v]) \geqslant 0,$$

with, in addition, $p = \nabla_x\phi(\bar{x},\bar{y})$, $q = -\nabla_y\phi(\bar{x},\bar{y})$ and

$$a - b = \phi_t(\bar{x},\bar{y}),$$
$$-\frac{1}{\alpha}I \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant D^2\phi_\alpha(\bar{x},\bar{y}) = D^2\phi(\bar{x},\bar{y}) + o_\alpha(1).$$

The proof of Corollary 2 follows readily the classical ideas to prove the local Jensen–Ishii lemma and the above arguments to treat the nonlocal part.

5. Application to comparison principles

In this section, we consider the equation

$$F(x,u,\nabla u,D^2u,\mathcal{I}_{LI}[u](x)) = 0 \quad \text{in } \mathbb{R}^d \tag{33}$$

where $\mathcal{I}_{LI}[\phi]$ is given by (11). We state and prove a comparison principle in the class of bounded (sub and super) solutions. We treat a case where there is no strong interaction between the nonlocal term and the x-dependence of F; we discuss this assumption in Subsection 5.2.

5.1. Statement of the comparison principle

We first need to strengthen the condition (12) on μ and j.

- (A1) The measure $\mu(dz)$ and the function $j(x,z)$ satisfy: there exists a constant $C > 0$ such that

$$\int_B |j(x,z)|^2 \mu(dz) < +\infty, \qquad \int_{\mathbb{R}^d \setminus B} \mu(dz) < +\infty, \tag{34}$$

$$\int_{\mathbb{R}^d} |j(x,z) - j(y,z)|^2 \mu(dz) \leqslant \bar{c}|x-y|^2 \quad \text{and} \quad \int_{\mathbb{R}^d \setminus B} |j(x,z) - j(y,z)| \mu(dz) \leqslant \bar{c}|x-y|. \tag{35}$$

For the nonlinearity F, we first introduce the classical assumption on the dependence of F in u.

- (A2) There exists $\gamma > 0$ such that for any $x \in \mathbb{R}^d$, $u, v \in \mathbb{R}$, $p \in \mathbb{R}^d$, $X \in \mathbb{S}_d$ and $l \in \mathbb{R}$

$$F(x,u,p,X,l) - F(x,v,p,X,l) \geqslant \gamma(u-v) \quad \text{when } u \geqslant v.$$

Next we have to impose assumptions on the dependence of F in x and we can do it in two ways.

- (A3-1) For any $R > 0$, there exist moduli of continuity ω, ω_R such that, for any $|x|, |y| \leqslant R$, $|v| \leqslant R$, $l \in \mathbb{R}$ and for any $X, Y \in \mathbb{S}_d$ satisfying

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + r(\beta)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{36}$$

for some $\varepsilon > 0$ and $r(\beta) \to 0$ as $\beta \to 0$, then, if $s(\beta) \to 0$ as $\beta \to 0$, we have

$$F\big(y, v, \varepsilon^{-1}(x-y), Y, l\big) - F\big(x, v, \varepsilon^{-1}(x-y) + s(\beta), X, l\big) \leqslant \omega(\beta) + \omega_R\big(|x-y| + \varepsilon^{-1}|x-y|^2\big). \tag{37}$$

or

- (A3-2) For any $R > 0$, F is uniformly continuous on $\mathbb{R}^n \times [-R,R] \times B_R \times D_R \times \mathbb{R}$ where $D_R := \{X \in \mathbb{S}_d;\ |X| \leqslant R\}$ and there exist a modulus of continuity ω_R such that, for any $x, y \in \mathbb{R}^d$, $|v| \leqslant R$, $l \in \mathbb{R}$ and for any $X, Y \in \mathbb{S}_d$ satisfying (36) and $\varepsilon > 0$, we have

$$F\big(y, v, \varepsilon^{-1}(x-y), Y, l\big) - F\big(x, v, \varepsilon^{-1}(x-y), X, l\big) \leqslant \omega_R\big(\varepsilon^{-1}|x-y|^2 + |x-y| + r(\beta)\big). \tag{38}$$

We provide more comments on these assumptions in the next section but clearly (A3-1) allows more general dependence in x while (A3-2) allows more general dependence in p.

- (A4) $F(x,u,p,X,l)$ is Lipschitz continuous in l, uniformly with respect to all the other variables.

Theorem 3. *Assume that* (A1), (A2), (A3-1) *or* (A3-2) *and* (A4) *hold. If u is a bounded usc subsolution of* (33) *and v is a lsc bounded supersolution v of* (33)*, then $u \leqslant v$ on $\mathbb{R}^d$.*

Proof. We consider $M = \sup_{\mathbb{R}^d}(u - v)$ and argue by contradiction by assuming that $M > 0$.

We next approximate M by dedoubling the variables

$$M_{\varepsilon,\beta} = \sup_{x,y\in\mathbb{R}^d}\left\{u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \psi_\beta(x)\right\}$$

where ε, β are small parameters devoted to tend to 0 and the functions ψ_β are built in the following way: let $\psi : \mathbb{R}^d \to \mathbb{R}+$ be a smooth function such that $\psi, \nabla\psi, D^2\psi$ are bounded in $\mathbb{R}^d$ and such that $\psi(x) = 0$ if $|x| \leqslant 1$ and $\psi(x) > \mathcal{R} := (||u||_\infty + ||v||_\infty)$ for $|x| \geqslant 2$. We then set $\psi_\beta(x) = \psi(\beta x)$. The three main properties of the ψ_β we are going to use are the following

- $\psi_\beta(x) > \mathcal{R}$ when $|x| \geqslant 2/\beta$, which ensures that the supremum defining $M_{\varepsilon,\beta}$ is achieved and therefore is a maximum,
- $\nabla\psi_\beta(x),\ D^2\psi_\beta(x) \to 0$ as $\beta \to 0$ uniformly on $\mathbb{R}^d$, which allows to control the differential terms of the ψ_β,
- $\mathcal{I}_{LI}[\psi_\beta](x) \to 0$ as $\beta \to 0$ uniformly on $\mathbb{R}^d$, which allows to control the integral terms of the ψ_β.

We classically obtain for ε and β small enough,

$$0 < \frac{M}{2} \leqslant M_{\varepsilon,\beta} \leqslant u(\bar{x}) - v(\bar{y}) \quad \text{and} \quad \frac{|\bar{x}-\bar{y}|}{\varepsilon} \leqslant \frac{C}{\sqrt{\varepsilon}} \quad \text{and} \quad \psi_\beta(\bar{x}) \leqslant \mathcal{R}.$$

In particular, $|\bar{x}| \leqslant 2/\beta$. Now we consider any maximum points $(\bar{x}, \bar{y})$ of the function $u(x) - v(y) - |x-y|^2/(2\varepsilon) - \psi_\beta(x)$. By definition of $(\bar{x}, \bar{y})$, we have

$$u(\bar{x}+d') - v(\bar{y}+d) - \frac{|\bar{x}+d'-\bar{y}-d|^2}{2\varepsilon} - \psi_\beta(\bar{x}+d') \leqslant u(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} - \psi_\beta(\bar{x}).$$

By setting $q = (\bar{x} - \bar{y})/\varepsilon$ and $p = q + \nabla\psi_\beta(\bar{x})$, we deduce from the previous inequality

$$u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) - p \cdot j(\bar{x}, z) \leqslant v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y}) - q \cdot j(\bar{y}, z) + \frac{|j(\bar{x}, z) - j(\bar{y}, z)|^2}{2\varepsilon} + \big(\psi_\beta\big(\bar{x} + j(\bar{x}, z)\big) - \psi_\beta(\bar{x}) - \nabla\psi_\beta(\bar{x}) \cdot j(\bar{x}, z)\big) \tag{39}$$

and

$$u\big(\bar{x} + j(\bar{x}, z)\big) - u(\bar{x}) \leqslant v\big(\bar{y} + j(\bar{y}, z)\big) - v(\bar{y}) + q \cdot \big(j(\bar{x}, z) - j(\bar{y}, z)\big) + \big(\psi_\beta\big(\bar{x} + j(\bar{x}, z)\big) - \psi_\beta(\bar{x})\big) + \frac{|j(\bar{x}, z) - j(\bar{y}, z)|^2}{2\varepsilon}. \tag{40}$$

For δ small enough, in order, at least, that $B(0, \delta) \subset B$, we define $\mathcal{I}^{1,\delta}$ as being the same integral as $\mathcal{I}_{LI}$ but integrating only on $B(0, \delta)$ and, in the same way, $\mathcal{I}^{2,\delta}$ stands for the same integral as $\mathcal{I}_{LI}$ but integrating only on $\mathbb{R}^d \setminus B(0, \delta)$. We also denote by ϕ_x the function $x \mapsto \phi(x, \bar{y})$ and by ϕ_y the function $y \mapsto \phi(\bar{x}, y)$.

Define $\phi(x, y) = |x - y|^2/(2\varepsilon) + \psi_\beta(x)$. Then

$$\begin{aligned}\mathcal{I}^{1,\delta}[\bar{x}, \phi_x] &\leqslant \mathcal{I}^{1,\delta}[\bar{y}, -\phi_y] + \frac{1}{\varepsilon}\int_{|z|\leqslant\delta} |z|^2\mu(dz) + \mathcal{I}^{1,\delta}[\bar{x}, \psi_\beta]\\ &\leqslant \mathcal{I}^{1,\delta}[\bar{y}, -\phi_y] + \frac{1}{\varepsilon}o_\delta(1) + o_\beta(1).\end{aligned}$$

Next we consider the $\mathcal{I}^{2,\delta}$ terms which, in fact, consist in two terms, whether we integrate on $B \setminus B(0, \delta)$ or on $\mathbb{R}^d \setminus B$. The corresponding terms are denoted respectively by $\mathcal{I}_1^{2,\delta}$ and $\mathcal{I}_2^{2,\delta}$.

For the $\mathcal{I}_1^{2,\delta}$ term, we integrate inequality (39) on $B \cap \setminus B(0, \delta)$, which yields

$$\mathcal{I}_1^{2,\delta}[\bar{x}, p, u] \leqslant \mathcal{I}_1^{2,\delta}[\bar{y}, q, v] + \int_B \frac{|j(\bar{x}, z) - j(\bar{y}, z)|^2}{2\varepsilon}\mu(dz) + \mathcal{I}_1^{2,\delta}\big[\bar{x}, \nabla\psi_\beta(\bar{x}), \psi_\beta\big],$$

where for the second term of the right-hand side, we have estimated the integral on $B \cap (\mathbb{R}^d \setminus B(0, \delta))$ by the integral on the whole ball B.

Next, for the $\mathcal{I}_2^{2,\delta}$ term, we integrate inequality (40) on $\mathbb{R}^d \setminus B$, which yields

$$\begin{aligned}\mathcal{I}_2^{2,\delta}[\bar{x}, u] \leqslant \mathcal{I}_2^{2,\delta}[\bar{y}, v] &+ \int_{\mathbb{R}^d\setminus B} q \cdot \big(j(\bar{x}, z) - j(\bar{y}, z)\big)\mu(dz)\\ &+ \frac{1}{2\varepsilon}\int_{\mathbb{R}^d\setminus B} \big|j(\bar{x}, z) - j(\bar{y}, z)\big|^2\mu(dz) + \mathcal{I}_2^{2,\delta}\big[\bar{x}, \nabla\psi_\beta(\bar{x}), \psi_\beta\big].\end{aligned}$$

By using (34)–(35), summing up these inequalities, we thus obtain

$$\mathcal{I}^{2,\delta}[\bar{x}, p, u] \leqslant \mathcal{I}^{2,\delta}[\bar{y}, q, v] + O\bigg(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\bigg) + o_\beta(1)$$

and finally, we get the following estimate between the integral terms

$$l := \mathcal{I}^{1,\delta}[\bar{x}, \phi_x] + \mathcal{I}^{2,\delta}[\bar{x}, u] \leqslant \mathcal{I}^{1,\delta}[\bar{y}, -\phi_y] + \mathcal{I}^{2,\delta}[\bar{y}, v] + O\bigg(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon}\bigg) + o_\beta(1) + \frac{1}{\varepsilon}o_\delta(1). \tag{41}$$

We are next going to apply Corollary 1 with ϕ. If $(p, -q)$ denotes $\nabla\phi(\bar{x}, \bar{y})$ and A denotes $D^2\phi(\bar{x}, \bar{y})$, for any $\alpha > 0$, there exists two matrices $X, Y \in \mathbb{S}_d$ such that (31) holds true and such that

$$\begin{aligned}&F\big(\bar{x}, p, u(\bar{x}), X, \mathcal{I}^{1,\delta}[\bar{x}, \phi_x] + \mathcal{I}^{2,\delta}[\bar{x}, p, u]\big) \leqslant 0,\\ &F\big(\bar{y}, q, v(\bar{y}), Y, \mathcal{I}^{1,\delta}[\bar{y}, -\phi_y] + \mathcal{I}^{2,\delta}[\bar{y}, q, v]\big) \geqslant 0.\end{aligned}$$

We next combine the two viscosity inequalities and we use first (A2) and (A4) together with Estimate (41) and the (nonlocal) ellipticity in order to get as $\delta \to 0$

$$\gamma \frac{M}{2} \leqslant F\big(\bar{x}, v(\bar{y}), q, Y, l\big) - F\big(\bar{y}, v(\bar{y}), q + \nabla\psi_\beta(\bar{x}), X, l\big) + \mathrm{O}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\right) + \mathrm{o}_\beta(1). \tag{42}$$

The inequality (31) implies in particular that

$$-\frac{1}{\alpha}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \begin{bmatrix} D^2\psi_\beta(\bar{x}) & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{o}_\alpha(1)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
$$\leqslant \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \big(\mathrm{o}_\alpha(1) + \mathrm{o}_\beta(1)\big)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

where $\mathrm{o}_\beta(1)$ and $\mathrm{o}_\alpha(1)$ are uniform in ε; we used the properties satisfied by ψ_β listed above. In the following, we will choose $\alpha = \varepsilon/K$ with K large enough so that $|X| + |Y| \leqslant \bar{c}(\varepsilon^{-1} + \mathrm{o}_\beta(1) + \mathrm{o}_\varepsilon(1))$.

We first assume that (A3-1) holds. In this case, we choose $R = R_\beta = 2/\beta$ and write

$$\gamma \frac{M}{2} \leqslant F\big(\bar{x}, v(\bar{y}), q, Y, l\big) - F\big(\bar{y}, v(\bar{y}), q + \mathrm{o}_\beta(1), X, l\big) + \mathrm{O}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\right) + \mathrm{o}_\beta(1)$$
$$\leqslant \omega(\beta) + \omega_{R_\beta}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} + |\bar{x}-\bar{y}|\right) + \mathrm{O}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\right) + \mathrm{o}_\beta(1).$$

Using (37) and letting successively ε and β go to 0, we get the desired result.

Next we assume that (A3-2) holds. We derive from (42),

$$\gamma \frac{M}{2} \leqslant F\big(\bar{x}, v(\bar{y}), q, Y, l\big) - F\big(\bar{y}, v(\bar{y}), q, X, l\big) + \mathrm{O}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\right) + \mathrm{o}_\beta(1) + \omega_{R_\varepsilon}\big(|\nabla\psi_\beta|\big)$$
$$\leqslant \omega\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} + |\bar{x}-\bar{y}| + \mathrm{o}_\beta(1)\right) + \mathrm{O}\left(\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\right) + \mathrm{o}_\beta(1) + \bar{\omega}_{R_\varepsilon}\big(|\nabla\psi_\beta|\big)$$

where $\omega = \omega_{\|v\|_\infty}$ given by (A3-2) and $\bar{\omega}_{R_\varepsilon}$ is the modulus of continuity of F on $\mathbb{R}^n \times [-R_\varepsilon, R_\varepsilon] \times B_{R_\varepsilon} \times D_{R_\varepsilon} \times R$ with $R_\varepsilon = K/\varepsilon$ for some $K > 0$. In this case, we use (38) and we let successively β and ε tend to 0. The proof is now complete. □

Remark 4. It is worth pointing out that, in the two cases we consider in the proof (letting first ε tend to 0 and then β tend to 0, or the contrary), the behavior of $M_{\varepsilon,\beta}$ are different. Indeed

$$\lim_{\beta\to 0}\lim_{\varepsilon\to 0} M_{\varepsilon,\beta} = M \quad \text{while} \quad \lim_{\varepsilon\to 0}\lim_{\beta\to 0} M_{\varepsilon,\beta} = \lim_{s\downarrow 0}\sup_{|x-y|\leqslant s}\big(u(x) - v(y)\big) \geqslant M.$$

But, in both cases, we have the key property $|\bar{x}-\bar{y}|^2/\varepsilon \to 0$.

5.2. Discussion of the assumptions

We want to discuss here the assumptions (A3-1) and (A3-2) and, in particular, the connections with the nonlocal term.

First, as long as local equations are concerned, we recall that the equation

$$-\operatorname{Tr}\big(A(x)D^2u\big) + b(x)|Du|^p + c(x)u = f(x) \quad \text{in } \mathbb{R}^n,$$

satisfies (A3-1) if $A = \sigma^T\sigma$ where σ is a bounded, matrix-valued locally Lipschitz continuous function, $0 < p \leqslant 1$ and b is also a locally Lipschitz continuous function and c, f are continuous functions. For (A3-2), in most of the cases, we have still to assume $0 < p \leqslant 1$ but σ and b have to be (globally) bounded Lipschitz continuous functions and c, f need to be uniformly continuous. But if b is a constant function then p can be any nonnegative number.

It is worth pointing out that the assumptions on the nonlocal term are rather restrictive: for example, one can add (in fact subtract) the Lévy operator $\mathcal{I}_L[u](x)$ given by (1) from the above equation and the resulting equation still satisfies (A3-1) or (A3-2). But this is not the case anymore if the measure $\mu(dz)$ has a singularity at $z = 0$ and if we try to subtract a term like $d(x)\mathcal{I}_L[u](x)$, whatever we may assume on the function d. We can treat such term only if $\mu(dz)$ is a bounded measure. Therefore the x-dependence in the nonlocal term is rather restrictive, except perhaps if it written in the Lévy–Ito form (11) where we have a well-adapted dependence in x. This is the reason why we formulate Theorem 3 with such an operator.

Curiously the type of singularity of $\mu(dz)$ does not seem to change anything: one could think that the cases where $|z|$ is integrable at 0 and the cases where only $|z|^2$ is integrable are different and the first one easier to treat. But we were unable to see any difference in the proof where we can just use inequalities (39) and (40).

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