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Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights ☆

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Abstract

By the Mountain Pass Theorem and the constrained minimization method existence of positive or compactly supported radial ground states for quasilinear singular elliptic equations with weights are established. The paper also includes the discussion of regularity and the validity of useful qualitative properties of the solutions, which seems of independent interest. Finally, a Pohozaev type identity is produced to deduce some non-existence results.

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1. Introduction

Existence and non-existence, as well as qualitative properties, of non-trivial non-negative solutions for elliptic equations with singular coefficients were recently studied by several authors, e.g., in bounded domains and for $p = 2$ cf. [2,10,12,18,23,42], and for general $p > 1$ see [17,20,24,50]; while in unbounded domains and for $p = 2$ cf. [27, 31,42,47], and for general $p > 1$ see [1,11,14,21,25,36,45]. There is a large literature on p-Laplacian equations in the entire $\mathbb{R}^n$, but the nonlinear structure, objectives and methods differ somehow from those presented here.

More precisely, we consider the following quasilinear singular elliptic equation

$$\begin{aligned} &\Delta_p u - \lambda |u|^{p-2}u + \mu |x|^{-\alpha}|u|^{q-2}u + h\big(|x|\big)f(u) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ &\lambda, \mu \in \mathbb{R}, \quad 1 < p < n, \end{aligned} \tag{1.1}$$

where $\Delta_p = \operatorname{div}(|Du|^{p-2}Du)$, $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ and either $0 \leqslant \alpha < p \leqslant q < p_\alpha^* = p(n-\alpha)/(n-p)$ or $\alpha = q = p\ (= p_\alpha^*)$; while $h : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Homogeneous Dirichlet problems associated to equations of type (1.1) are studied by Ekeland and Ghoussoub in [17] and by Ghoussoub and Yuan in [24] in smooth bounded domains of $\mathbb{R}^n$ containing zero, when $\lambda = 0$, $h \equiv 1$,

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$f(u) = c|u|^{s-2}u$ with $c > 0$ and $p \leqslant s < p^* = pn/(n-p)$. They give existence and multiplicity results of non-trivial non-negative solutions by means of variational methods and of the Hardy–Sobolev inequality (see, e.g., [2,12,20,34]), when either $0 \leqslant \alpha < p \leqslant q < p^*_\alpha$ or $\alpha = q = p\ (= p^*_\alpha)$.

In this paper we extend the existence results of [17,24] to the entire $\mathbb{R}^n$, and to the case in which $\lambda > 0$, h is a general non-trivial weight such that $h(|x|) = \mathrm{o}(|x|^{-\beta})$ as $|x| \to 0$, with $\beta \in [0, p)$, bounded at infinity, see Section 2, while f is possibly different from a pure power. In particular, in Theorem 3.1 we prove the existence of a radial ground state of (1.1), i.e. a non-trivial non-negative weak solution which tends to zero at infinity, by the celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz and the Hardy–Sobolev inequality, when f is of the canonical form required in [3] and with $uf(u) > 0$ for $u \neq 0$. Then, using the regularity results of [41], and an application of the strong maximum principle due to Vázquez [48] (see also [38,39]), we show that the constructed ground state u is of class $C^1(\mathbb{R}^n \setminus \{0\})$ and by Proposition 3.2 also of class $C(\mathbb{R}^n)$, provided that $0 \leqslant \max\{\alpha, \beta\} < p$ and so *positive everywhere* in $\mathbb{R}^n$. As a consequence of Theorems 2.1 and 2.2 of [41], we also show that the solution u given by Theorem 3.1 is in $L^m_{\mathrm{loc}}(\mathbb{R}^n)$ for any $m \in [1, \infty)$ if $0 \leqslant \alpha < p \leqslant q < p^*_\alpha$, while, if $0 \leqslant \max\{\alpha, \beta\} < p$, then $u \in L^\infty(\mathbb{R}^n)$. Moreover, as an application of [41, Theorem 2.5], we prove that $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ when $1 < p \leqslant 2$ and $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ if furthermore $0 \leqslant \max\{\alpha, \beta\} \leqslant p - 1$.

Since in the degenerate case $p > 2$ the uniform ellipticity of Δ_p is lost at zeros of Du, the best we can expect, even in the standard no weighted case of (1.1), is to have solutions of class $C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ (see [16]). Of course, for (1.1) much less could be expected and regularity was an open problem. A partial result is given in Proposition 3.2 for radial ground states of (1.1), provided they are assumed a priori *bounded*. However Proposition 3.2 applies to the solution constructed in Theorem 3.1 when $0 \leqslant \max\{\alpha, \beta\} < p$. Also this result seems to the authors new.

In Section 4 we prove a Pohozaev type identity which yields some non-existence statements for (1.1). The main non-existence Theorems 4.10 and 4.11 for (1.1) extend several previous results, see e.g. [20, Lemma 3.7] established in bounded star-shaped domains, when $\lambda = \beta = 0$ and [24, Theorem 2.1] stated for $\lambda = 0$, $q = p^*_\alpha$, $\alpha \in [0, p]$, $h \equiv 1$ and $f(u) = c|u|^{p^*-2}u$, $c > 0$. In particular, as a consequence of Theorems 4.10 and 4.11 we obtain a non-existence result for the *doubly critical* equation

$$\begin{aligned}&\Delta_p u - \lambda|u|^{p-2}u + \mu|x|^{-p}|u|^{p-2}u + \gamma|x|^{-\beta}|u|^{p^*_\beta-2}u = 0 \quad \text{a.e. in } \mathbb{R}^n,\\ &\quad\lambda \neq 0, \quad \gamma, \mu \in \mathbb{R}, \quad \beta < p, \quad p^*_\beta = p(n-\beta)/(n-p),\end{aligned} \tag{1.2}$$

which extends, e.g., [33, Theorem 1.3] given for $p = 2$ and $\lambda = \beta = 0$.

Moreover, we show that

$$\Delta_p u + \gamma|x|^{-\beta}|u|^{s-2}u = 0 \quad \text{in } \Omega, \qquad \gamma \in \mathbb{R}, \quad s > 1, \tag{1.3}$$

where $\Omega = \mathbb{R}^n$ when $\beta \leqslant 0$, while $\Omega = \mathbb{R}^n \setminus \{0\}$ when $\beta \in (0, p)$, admits only the trivial solution whenever either $\gamma \leqslant 0$, or $\gamma > 0$ and $s \neq p^*_\beta$. This result, when the regular periodic function of [47] is zero, extends [47, Theorem 0.1(ii)] proved by Terracini in the case $p = 2$ and $\beta = 0$. Therefore, in Section 5 we give the explicit positive radial ground state for (1.3) only in the remaining possible case, that is when $s = p^*_\beta$, $\beta < p$ and $\gamma > 0$, defined by

$$u(x) = c\big(1 + |x|^{(p-\beta)/(p-1)}\big)^{-(n-p)/(p-\beta)}, \quad c = \left[\frac{n-\beta}{\gamma}\left(\frac{n-p}{p-1}\right)^{p-1}\right]^{(n-p)/p(p-\beta)}. \tag{1.4}$$

This includes the standard solution for the no weighted case $\beta = 0$, $p = 2$ and $s = 2^*$ (see, e.g., [28,44,49] and references therein), as well as the ground state found for (1.3) when $0 \leqslant \beta < p$ and $s = p^*_\beta$ in [17,24] for all $p > 1$, and in [10,27] for $p = 2$. Clearly the solution u given in (1.4) is of class $D^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$, while is of class $H^{1,p}(\mathbb{R}^n)$ if and only if $n > p^2$. The regularity of (1.4) at $x = 0$ is presented in Table 2 below.

In Section 6 we use a different approach for studying (1.1) when $\lambda = \mu = 0$, $h(r) = r^{-\beta}$, but $\beta < p$ is possibly negative. More precisely, we consider

$$\Delta_p u + |x|^{-\beta} f(u) = 0 \quad \text{in } \Omega, \quad \beta < p, \tag{1.5}$$

where Ω is defined as above, in the so called *normal case* (see [35]), that is when f is a general continuous function which is negative near the origin and positive at infinity. Roughly speaking, $f(u) \cong -c|u|^{q-1}$ as $u \to 0^+$, with $q > 1$, $c > 0$, and $\lim_{u\to\infty} u^{1-p^*_\beta} f(u) = 0$, with $p^*_\beta = p(n-\beta)/(n-p)$. After the papers of [4,5] related to elliptic problems with the Laplace operator in the normal case, equations involving the p-Laplacian operator in $\mathbb{R}^n$, were

Table 1

	$1 < p \leqslant 2$	$p > 2$
$\beta < 1$	$C^2(\mathbb{R}^n)$	$C^1(\mathbb{R}^n)$
$1 \leqslant \beta < p$	$C^{0,(p-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$	

Table 2

	$1 < p \leqslant 2$	$p > 2$
$\beta < 2 - p$	$C^2(\mathbb{R}^n)$	
$\beta = 2 - p$	$C^2(\mathbb{R}^n)$	$C^{1,1}_{\text{loc}}(\mathbb{R}^n)$
$2 - p < \beta < 1$	$C^2(\mathbb{R}^n)$	$C^{1,(1-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$
$1 \leqslant \beta < p$	$C^{0,(p-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$	

treated largely in literature (see e.g. [13,19,22] for the no weighted case, that is when $\beta = 0$ in (1.5), and [11] for general weighted equations).

In Propositions 6.1 and 6.2, we give some qualitative properties of *bounded* radial ground states of class $C^1(\mathbb{R}^n \setminus \{0\})$ of (1.5) and discuss their regularity, using the theory of singular elliptic problems with general weights developed in [36].

In Theorem 6.7, adapting the constrained minimization method of Coleman, Glaser and Martin [15], we prove the existence of a *bounded* continuous radial ground state u of (1.5) in the entire $\mathbb{R}^n$. It is an *open problem* if any radial ground state of (1.5) is bounded in $\mathbb{R}^n$, that is near $x = 0$. Theorem 6.7 extends to the weighted case $\beta < p$ the existence results obtained for $\beta = 0$ by Berestycki and Lions in [4] when $p = 2$, and by Citti in [13] for general $p > 1$; and also Theorems 7 and 10 of [11] given for $\operatorname{div}(g(|x|)|Du|^{p-2}Du) + h(|x|)f(u) = 0$ in the special case in which $g \equiv 1$, $h(r) = r^{-\beta}$, $\beta < p$, and f is continuous also at $u = 0$. Finally, we observe that in this paper no locally Lipschitz continuity is assumed on f as required in [11] and in several previous papers in the no weighted case (see e.g. [14,19,22,36] and the papers quoted there).

Moreover, when $1 < p \leqslant 2$, as a consequence of [41, Theorem 2.5], we show that the ground state u given in Theorem 6.7 is of class $H^{2,p}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and $u \in H^{2,p}_{\text{loc}}(\mathbb{R}^n)$, if furthermore $\beta < n/p'$.

While the ground states of (1.1) and (1.3) are necessarily positive in $\mathbb{R}^n$, for (1.5) they are compactly supported when $1 < q < p$ and positive when $q \geqslant p$, where q represents as above the growth of f near zero, see condition $(F5)$. Moreover, when $1 < q < p$, the ground state u constructed in Theorem 6.7 has the further regularity described in Table 1 (see Proposition 6.1). On the other hand, when $q \geqslant p$, then u has the further regularity described in Table 2 (see Proposition 6.2).

Table 2 extends to the general nonlinear weighted equation (1.5) *in the normal case the regularity established for* (1.3) *when* $s = p^*_\beta$, $\gamma > 0$ *and the explicit ground state* (1.4) *is known*. Furthermore both tables improve the regularity results given by Citti in Remarks 1.2 and 1.3 of [13] for the no weighted case $\beta = 0$. It remains an *open problem* if any solution of (1.5) is radially symmetric with respect to $x = 0$.

In Theorem 6.9 we present an application of Theorem 6.7 and Lemma 4.6 to the case when the nonlinearity in (1.5) is of polynomial type. Theorem 6.9 extends to the weighted p-Laplacian case the existence and non-existence results given by Berestycki and Lions in [4, Example 2] in the no weighted Laplacian case $\beta = 0$ and $p = 2$.

The main results of this paper and of [41] were presented in a unified way, but without proofs, in the survey [40].

2. Preliminaries

Here some preliminary results are presented as well as embedding theorems of the Sobolev spaces $H^{1,p}(\mathbb{R}^n)$, $H^{1,p}_{\text{rad}}(\mathbb{R}^n)$ and $D^{1,p}(\mathbb{R}^n)$ into weighted Lebesgue spaces related to (1.1), proved by means of the Hardy–Sobolev inequality (see, e.g., [2,12,20,34]) and the Caffarelli–Kohn–Nirenberg inequality [9]. For other results of this type see, e.g., [2,12,50] and references therein.

Throughout the paper we assume that $1 < p < n$. In the sequel $H^{1,p}(\mathbb{R}^n)$ denotes the usual Sobolev space, endowed with the norm $\|u\| = (\|u\|_p^p + \|Du\|_p^p)^{1/p}$, and $D^{1,p}(\mathbb{R}^n)$ is the closure of $C_0^1(\mathbb{R}^n)$ with respect to the norm $\|Du\|_p$. Let $\alpha \in \mathbb{R}$ and consider also the weighted Lebesgue space

$$L^p_\alpha(\mathbb{R}^n) = L^p(\mathbb{R}^n, |x|^{-\alpha}\,dx) = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\colon \int\limits_{\mathbb{R}^n} |u(x)|^p |x|^{-\alpha}\,dx < \infty\right\},$$

endowed with the norm $\|u\|_{p,\alpha} = (\int_{\mathbb{R}^n} |u(x)|^p |x|^{-\alpha}\,dx)^{1/p}$. Note that $L^p_\alpha(\mathbb{R}^n)$ is a reflexive Banach space and $L^{p'}_\alpha(\mathbb{R}^n)$ is its dual space (see Theorem 5.3 in [30]), where $1/p + 1/p' = 1$.

By the Hardy–Sobolev inequality (cf. Lemma 2.1 of [20]) we have that

$$\|u\|^p_{p,p} = \int\limits_{\mathbb{R}^n} |u(x)|^p |x|^{-p}\,dx \leqslant C^p_{n,p} \int\limits_{\mathbb{R}^n} |Du(x)|^p\,dx = C^p_{n,p}\|Du\|^p_p \tag{2.1}$$

for any $u \in D^{1,p}(\mathbb{R}^n)$, and so in particular

$$\|u\|_{p,p} \leqslant C_{n,p}\|u\| \quad \text{for all } u \in H^{1,p}(\mathbb{R}^n).$$

Actually, the best Hardy–Sobolev constant (see [20]) is

$$\inf_{D^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|Du\|_p}{\|u\|_{p,p}} = C^{-1}_{n,p}, \qquad C_{n,p} = \frac{p}{n-p}.$$

In the following we adopt the notation $B_R = \{x \in \mathbb{R}^n\colon |x| \leqslant R\}$, $\Omega_R = \mathbb{R}^n \setminus B_R$, $R > 0$.

For $\beta \in \mathbb{R}$, let us introduce, for general weights w, the functional space

$$\mathcal{W}_\beta = \left\{w \in L^\infty(\Omega_R) \text{ for any } R > 0\colon\ w \not\equiv 0,\ w \geqslant 0 \text{ a.e. in } \mathbb{R}^n \text{ and } \lim_{|x|\to 0} |x|^\beta w(x) = 0\right\},$$

and $L^s_w(\mathbb{R}^n) = L^s(\mathbb{R}^n, w(x)\,dx) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} |u(x)|^s w(x)\,dx < \infty\}$, endowed with the norm $\|u\|_{s,w} = (\int_{\mathbb{R}^n} |u(x)|^s w(x)\,dx)^{1/s}$, when $1 \leqslant s < \infty$.

Let c_T denote the best constant of the Sobolev embedding $D^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$, that is

$$c_T = c_T(n,p) = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1/p'} \left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{1/n},$$

see [46], and put for $w \in \mathcal{W}_\beta$ and $R > 0$

$$C_{\mathrm{HS}} = C_{\mathrm{HS}}(n,p,\beta,s,w) = c_T^{n(s-p)/ps} \max\{C^{\beta/s}_{n,p}, \|w\|^{1/s}_{L^\infty(\Omega_R)}\}. \tag{2.2}$$

Lemma 2.1.

(i) *Let $0 \leqslant \beta < p$ and let $w \in \mathcal{W}_\beta$. Then, the embedding $H^{1,p}(\mathbb{R}^n) \hookrightarrow L^s_w(\mathbb{R}^n)$ is continuous for all $s \in [p, p^*_\beta]$, where*

$$p^*_\beta = p\,\frac{n-\beta}{n-p}. \tag{2.3}$$

(ii) *If $\beta = p$, then the embedding $H^{1,p}(\mathbb{R}^n) \hookrightarrow L^p_w(\mathbb{R}^n)$ is continuous for all $w \in \mathcal{W}_p$.*
Moreover in both cases (i) *and* (ii) *for all $u \in H^{1,p}(\mathbb{R}^n)$*

$$\|u\|_{s,w} \leqslant C_{\mathrm{HS}}\|u\|, \tag{2.4}$$

where C_{HS} is given in (2.2)*, with $R = R(w)$ so small that $0 \leqslant w(x) \leqslant |x|^{-\beta}$ for all $x \in B_R \setminus \{0\}$.*

(iii) *For the standard weight $w(x) = |x|^{-\beta}$, $\beta \in [0,p]$, cases* (i) *and* (ii) *trivially apply, with*

$$C_{\mathrm{HS}} = C_{\mathrm{HS}}(n,p,\beta,s) = c_T^{n(s-p)/ps} C^{\beta/s}_{n,p}. \tag{2.5}$$

Proof. (i) Since $w \in \mathcal{W}_\beta$, there exists $R > 0$ depending on w such that $w(x) \leqslant |x|^{-\beta}$ for $0 < |x| \leqslant R$. For any $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, by the properties of w and Hőlder's inequality,

$$\begin{aligned}
\|u\|_{s,w}^s &= \int_{B_R} |u(x)|^s w(x)\,dx + \int_{\Omega_R} |u(x)|^s w(x)\,dx \\
&\leqslant \int_{B_R} |u(x)|^\beta |x|^{-\beta} |u(x)|^{s-\beta}\,dx + \|w\|_{L^\infty(\Omega_R)} \int_{\Omega_R} |u(x)|^s\,dx \\
&\leqslant \Big(\int_{B_R} |u(x)|^p |x|^{-p}\,dx\Big)^{\beta/p} \Big(\int_{B_R} |u(x)|^q\,dx\Big)^{(p-\beta)/p} + \|w\|_{L^\infty(\Omega_R)} \|u\|_s^s \\
&\leqslant \|u\|_{p,p}^\beta \|u\|_q^{s-\beta} + \|w\|_{L^\infty(\Omega_R)} \|u\|_s^s,
\end{aligned} \tag{2.6}$$

where $q = p(s-\beta)/(p-\beta)$. Now, for any $t \in [p, p^*]$, setting

$$\varepsilon_t = \frac{p(p^* - t)}{t(p^* - p)} \in [0, 1],$$

we have for all $u \in H^{1,p}(\mathbb{R}^n)$

$$\|u\|_t \leqslant \|u\|_p^{\varepsilon_t} \|u\|_{p^*}^{1-\varepsilon_t} \leqslant c_T^{1-\varepsilon_t} \|u\|^{\varepsilon_t} \|u\|^{1-\varepsilon_t} = c_T^{1-\varepsilon_t} \|u\|, \tag{2.7}$$

by the interpolation inequality and the Sobolev embedding theorem. Since $s \in [p, p_\beta^*] \subset [p, p^*]$ and so also $q \in [p, p^*]$, by (2.6), (2.7), and the Hardy–Sobolev inequality (2.1), it follows that

$$\begin{aligned}
\|u\|_{s,w}^s &\leqslant c_T^{(1-\varepsilon_q)(s-\beta)} C_{n,p}^\beta \|Du\|_p^\beta \|u\|^{s-\beta} + c_T^{(1-\varepsilon_s)s} \|w\|_{L^\infty(\Omega_R)} \|u\|^s \\
&\leqslant c_T^{n(s-p)/p} \big[C_{n,p}^\beta + \|w\|_{L^\infty(\Omega_R)}\big] \|u\|^s \leqslant \big(C_{\mathrm{HS}} \|u\|\big)^s,
\end{aligned}$$

that is (2.4) holds. Hence, $H^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^s_w(\mathbb{R}^n)$ for all $s \in [p, p_\beta^*]$.

(ii) Fix $\varepsilon \in (0, 1]$ and let $\delta = \delta(\varepsilon)$ such that $w(x) \leqslant \varepsilon |x|^{-p}$ for all x, with $0 < |x| \leqslant \delta$. By the Hardy–Sobolev inequality we easily obtain, as in the proof of part (i) ,

$$\|u\|_{p,w}^p \leqslant \varepsilon \int_{B_\delta} |u(x)|^p |x|^{-p}\,dx + \|w\|_{L^\infty(\Omega_\delta)} \int_{\Omega_\delta} |u(x)|^p\,dx \leqslant \varepsilon C_{n,p}^p \|Du\|_p^p + \|w\|_{L^\infty(\Omega_\delta)} \|u\|_p^p,$$

which yields (2.4) when $\varepsilon = 1$ and $R = \delta(1)$. Hence, $H^{1,p}(\mathbb{R}^n)$ is continuously embedded in $L^p_w(\mathbb{R}^n)$.

(iii) If $\beta = 0$, the assertion is the standard Sobolev embedding $H^{1,p}(\mathbb{R}^n) \hookrightarrow L^s(\mathbb{R}^n)$ for all $s \in [p, p^*]$. For $\beta \in (0, p)$ we can repeat the proof of part (i) with $R = (n-p)/p = 1/C_{n,p}$. When $\beta = p$, case (iii) is a trivial consequence of (2.1). □

The next result is a particular case of the Caffarelli–Kohn–Nirenberg inequality (see [9]).

Lemma 2.2. *Let $\alpha \in [0, p]$. Then, the embedding $D^{1,p}(\mathbb{R}^n) \hookrightarrow L_\alpha^{p_\alpha^*}(\mathbb{R}^n)$ is continuous, where $p_\alpha^* = p(n-\alpha)/(n-p)$ and in turn also $H^{1,p}(\mathbb{R}^n) \hookrightarrow L_\alpha^{p_\alpha^*}(\mathbb{R}^n)$ is continuous.*

When $\alpha = p$, Lemma 2.2 reduces to the Hardy–Sobolev inequality. Of course in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, i.e. the space of functions $u \in H^{1,p}(\mathbb{R}^n)$ which are radial, much more can be said.

Lemma 2.3.

(i) *Let $0 \leqslant \beta < p$ and let $w \in \mathcal{W}_\beta$. Then, the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^s_w(\mathbb{R}^n)$ is continuous for all $s \in [p, p_\beta^*]$ and compact for all $s \in [p, p_\beta^*)$, where p_β^* is given in* (2.3)*.*

(ii) *If $\beta = p$, then the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^p_w(\mathbb{R}^n)$ is compact for all $w \in \mathcal{W}_p$.*

Proof. (i) The continuity of the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^s_w(\mathbb{R}^n)$ follows by Lemma 2.1.

Now, let $(u_k)_k$ be a bounded sequence in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Thus, there exists $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ as $k \to \infty$. Since $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ is compactly embedded in $L^s(\mathbb{R}^n)$ for all $s \in [p, p^*)$, we have that

$$Du_k \rightharpoonup Du \quad \text{in } L^p\big(\mathbb{R}^n\big) \quad \text{and} \quad u_k \to u \quad \text{in } L^s\big(\mathbb{R}^n\big) \tag{2.8}$$

as $k \to \infty$. Arguing as in (2.6), for $s \in [p, p^*_\beta)$ we have

$$\begin{aligned}\|u_k - u\|^s_{s,w} &\leqslant C^\beta_{n,p}\|Du_k - Du\|^\beta_p \|u_k - u\|^{s-\beta}_q + \|w\|_{L^\infty(\Omega_R)}\|u_k - u\|^s_s \\ &\leqslant KC^\beta_{n,p}\|u_k - u\|^{s-\beta}_q + \|w\|_{L^\infty(\Omega_R)}\|u_k - u\|^s_s,\end{aligned} \tag{2.9}$$

where $q = p(s-\beta)/(p-\beta)$ and for some positive constant K by virtue of (2.8). Since $q < p^*$ because $s \in [p, p^*_\beta)$, the assertion follows from (2.8) and (2.9).

(ii) The continuity of the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^p_w(\mathbb{R}^n)$ follows by Lemma 2.1. Let $(u_k)_k$ be a bounded sequence in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Up to a subsequence, still denoted by $(u_k)_k$, we have that $u_k \rightharpoonup u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ and $u_k \to u$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. As above

$$\|u_k - u\|^p_{p,w} \leqslant \varepsilon C^p_{n,p}\|Du_k - Du\|^p_p + \|w\|_{L^\infty(\Omega_\delta)}\|u_k - u\|^p_p. \tag{2.10}$$

Since $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ is compactly embedded in $L^p(\mathbb{R}^n)$, then (2.10) implies that

$$\limsup_k \|u_k - u\|^p_{p,w} \leqslant \varepsilon C^p_{n,p} \sup_k \|Du_k - Du\|^p_p = \mathrm{Const.}\varepsilon,$$

and the assertion follows at once since $\varepsilon > 0$ is arbitrary. □

Remark 2.4. If $w(x) = |x|^{-\alpha}$, with $\alpha \in [0, p)$, then the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^p_\alpha(\mathbb{R}^n)$ is compact. Indeed, for $\alpha = 0$, the assertion is trivial, while for $\alpha \in (0, p)$ it follows from Lemma 2.3(i) . For $\alpha = p$ the embedding $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^p_p(\mathbb{R}^n)$ is continuous by Hardy–Sobolev inequality (2.1).

Lemma 2.1 can be refined when $H^{1,p}(\mathbb{R}^n)$ is replaced by $H^{1,p}_0(\Omega)$, where Ω is any bounded open set of $\mathbb{R}^n$. Indeed in this case the embedding $H^{1,p}_0(\Omega) \hookrightarrow L^s_w(\Omega)$ is continuous for all $s \in [1 + \beta/p', p^*_\beta]$ and compact for all $s \in [1 + \beta/p', p^*_\beta)$, $\beta \in [0, p)$. The proof of the continuity is similar to that of Lemma 2.1 since in the bounded case (2.7) holds for all $t \in [1, p^*]$, and $q \in [1, p^*]$ if $s \in [1 + \beta/p', p^*_\beta]$. For the compactness we can argue as in Lemma 2.3, since $H^{1,p}_0(\Omega)$ is compactly embedded in $L^s(\Omega)$ for all $s \in [1, p^*)$. When $\beta = 0$ the result reduces to the usual Sobolev theorem (see [7]); however $1 + \beta/p' < p$ for all $\beta \in [0, p)$.

Now we present some preliminary results which will be useful in the sequel. The first lemma, stated here for the weighted spaces $L^s_w(\mathbb{R}^n)$, is well-known in the usual Lebesgue spaces (see, for instance, Theorem IV.9 of [7]). The proof is left to the reader, since it is standard.

Lemma 2.5. *Let $1 \leqslant s < \infty$ and let $w \in \mathcal{W}_\beta$. If $(u_k)_k$ is a sequence in $L^s_w(\mathbb{R}^n)$ and $u \in L^s_w(\mathbb{R}^n)$ such that $u_k \to u$ in $L^s_w(\mathbb{R}^n)$, then there exist a subsequence $(u_{k_j})_j$ of $(u_k)_k$ and a function $\varphi \in L^s_w(\mathbb{R}^n)$ such that a.e. in $\mathbb{R}^n$*

(i) $u_{k_j} \to u$ *as* $j \to \infty$; (ii) $\big|u_{k_j}(x)\big| \leqslant \varphi(x)$ *for all* $j \in \mathbb{N}$.

In this paper we need also the following lemma, which is a corollary of Theorem 1 of Brézis and Lieb [8].

Lemma 2.6. *Let $1 \leqslant s < \infty$, let $(u_k)_k$ be a sequence in $L^s(\mathbb{R}^n)$ and let $u \in L^s(\mathbb{R}^n)$. If $u_k \rightharpoonup u$ in $L^s(\mathbb{R}^n)$ and $u_k \to u$ a.e. on $\mathbb{R}^n$ as $k \to \infty$, then*

$$\lim_k \big(\|u_k\|^s_s - \|u_k - u\|^s_s\big) = \|u\|^s_s.$$

We recall that when $s > 1$ and $(u_k)_k$ converges a.e. in $\mathbb{R}^n$ to u, then $u_k \rightharpoonup u$ in $L^s(\mathbb{R}^n)$ if and only if $\sup_k \|u_k\|_s < \infty$ (see Theorem 13.44 of [26]).

A result similar to Lemma 2.6 continues to hold, essentially with the same proof, in the weighted Lebesgue space $L^s_s(\mathbb{R}^n) = L^s(\mathbb{R}^n, |x|^{-s}\,dx)$.

Lemma 2.7. *Let $1 \leqslant s < \infty$, let $(u_k)_k$ be a sequence in $L^s_s(\mathbb{R}^n)$ and let $u \in L^s_s(\mathbb{R}^n)$. If $u_k \rightharpoonup u$ in $L^s_s(\mathbb{R}^n)$ and $u_k \to u$ a.e. in $\mathbb{R}^n$ as $k \to \infty$, then*

$$\lim_k \big(\|u_k\|^s_{s,s} - \|u_k - u\|^s_{s,s}\big) = \|u\|^s_{s,s}.$$

Even if Boccardo and Murat in [6] treat only the case of bounded domains Ω of $\mathbb{R}^n$, the almost everywhere convergence of the gradients together with Remarks 2.1 and 2.2 of [6] continues to hold when Ω is replaced by $\mathbb{R}^n$. In particular, as a consequence of Theorem 2.1 of [6], we have the following lemma.

Lemma 2.8. *Let $(u_k)_k$ be a sequence in $H^{1,p}(\mathbb{R}^n)$ and $u \in H^{1,p}(\mathbb{R}^n)$ such that $u_k \rightharpoonup u$ in $H^{1,p}(\mathbb{R}^n)$, $u_k \to u$ in $L^p(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$ as $k \to \infty$. Let $(g_k)_k$ be a bounded sequence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume finally that each u_k is a weak solution of*

$$\Delta_p u_k = g_k \quad \text{in } \mathbb{R}^n.$$

Then $Du_k \to Du$ a.e. in $\mathbb{R}^n$ as $k \to \infty$.

The next lemma is essentially due to Strauss in [43] and is useful to prove the main Theorems 3.1 and 6.7. Let $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ denote the space of radial functions $u \in D^{1,p}(\mathbb{R}^n)$.

Lemma 2.9.

(i) *Let $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Then for all $R > 0$ we have, a.e. in $\mathbb{R}^n$,*

$$|u(x)| \leqslant \begin{cases} [(p-1)/(n-p)]^{1/p'} \omega_n^{-1/p} \|Du\|_p |x|^{-(n-p)/p}, & \text{if } 0 < |x| \leqslant R, \\ [\max\{p-1, 1\}]^{1/p} \omega_n^{-1/p} \|u\| \cdot |x|^{-(n-1)/p}, & \text{if } |x| > R, \end{cases}$$

where ω_n is the measure of the unit sphere of $\mathbb{R}^n$. Moreover, $|x|^{(n-1)/p} u(x) \to 0$ as $|x| \to \infty$ and $u \in C^{0,1/p'}(\mathbb{R}^n \setminus B_R)$.

(ii) *If $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, then*

$$|u(x)| \leqslant \big[(p-1)/(n-p)\big]^{1/p'} \omega_n^{-1/p} \|Du\|_p |x|^{-(n-p)/p} \quad \text{a.e. in } \mathbb{R}^n,$$

$|x|^{(n-p)/p} u(x) \to 0$ as $|x| \to \infty$ and again $u \in C^{0,1/p'}(\mathbb{R}^n \setminus B_R)$ for any $R > 0$.

Proof. (i) It is enough to prove this part for $u \in C^1_{0,\mathrm{rad}}(\mathbb{R}^n)$, since $C^1_{0,\mathrm{rad}}(\mathbb{R}^n)$ is dense in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Let $\rho > r > 0$. By using Hőlder's inequality we have that

$$\begin{aligned} |u(\rho) - u(r)| &\leqslant \int_r^\rho |u'(t)|\,dt \leqslant \left(\int_r^\rho |u'(t)|^p t^{n-1}\,dt\right)^{1/p} \left(\int_r^\rho t^{-(n-1)/(p-1)}\,dt\right)^{1/p'} \\ &\leqslant \left(\frac{p-1}{n-p}\right)^{1/p'} \omega_n^{-1/p} \|Du\|_p \big|\rho^{-(n-p)/(p-1)} - r^{-(n-p)/(p-1)}\big|^{1/p'}. \end{aligned}$$

Passing to the limit as $\rho \to \infty$ and taking into account that $1 < p < n$, it follows that $|u(x)| \leqslant [(p-1)/(n-p)]^{1/p'} \omega_n^{-1/p} \|Du\|_p |x|^{-(n-p)/p}$ a.e. in $\mathbb{R}^n$. In particular, the first inequality of part (i) holds.

Now, let $r > 0$. By the Young inequality we get

$$
\begin{aligned}
\left|u(r)\right|^p &\leqslant p\int_r^\infty \left|u(t)\right|^{p-1}\left|u'(t)\right|dt \leqslant pr^{-(n-1)}\int_r^\infty \left|u(t)\right|^{p-1}\left|u'(t)\right|t^{n-1}\,dt \\
&\leqslant r^{-(n-1)}\left[(p-1)\int_r^\infty \left|u(t)\right|^p t^{n-1}\,dt + \int_r^\infty \left|u'(t)\right|^p t^{n-1}\,dt\right] \\
&\leqslant \max\{p-1,1\} r^{-(n-1)}\left(\int_r^\infty \left|u(t)\right|^p t^{n-1}\,dt + \int_r^\infty \left|u'(t)\right|^p t^{n-1}\,dt\right),
\end{aligned}
$$

and in turn $|x|^{(n-1)/p}u(x) \to 0$ as $|x| \to \infty$. Moreover $|u(x)| \leqslant [\max\{p-1,1\}]^{1/p}\omega_n^{-1/p}\|u\| \cdot |x|^{-(n-1)/p}$ a.e. in $\mathbb{R}^n$.

Finally, for $\rho > r \geqslant R > 0$, again by Hőlder's inequality

$$
\begin{aligned}
\left|u(\rho)-u(r)\right| &\leqslant \int_r^\rho \left|u'(t)\right|dt \leqslant (\rho-r)^{1/p'}\left(\int_r^\rho \left|u'(t)\right|^p dt\right)^{1/p} \\
&\leqslant r^{-(n-1)/p}\left(\int_r^\rho \left|u'(t)\right|^p t^{n-1}\,dt\right)^{1/p}(\rho-r)^{1/p'} \\
&\leqslant \omega_n^{-1/p} R^{-(n-1)/p}\|Du\|_p(\rho-r)^{1/p'},
\end{aligned}
$$

which completes the proof of (i) .

(ii) For the second part of the lemma we can argue as above. □

Throughout the paper by a *ground state* of (1.1) we mean a non-trivial non-negative weak solution of (1.1) which tends to zero as $|x| \to \infty$. While by a *fast decay* solution of (1.1) we mean a non-trivial weak solution u of (1.1) such that

$$\lim_{|x|\to\infty} |x|^{(n-p)/(p-1)}u(x) \quad \text{exists and is finite.}$$

3. An existence theorem

In this section we prove the existence of a positive radial ground state of (1.1), when the function $f : \mathbb{R}_0^+ \to \mathbb{R}$ satisfies the following conditions

(F1) *f is continuous in $\mathbb{R}_0^+$;*
(F2) *there exist $a \geqslant 0$, $b > 0$ and $p < s$ such that $|f(u)| \leqslant au^{p-1} + bu^{s-1}$ in $\mathbb{R}_0^+$;*
(F3) $\lim_{u\to 0^+} u^{-p}F(u) = 0$, *where* $F(u) = \int_0^u f(v)\,dv$ for all $u \in \mathbb{R}_0^+$;
(F4) $0 < sF(u) \leqslant uf(u)$ *for all* $u \in \mathbb{R}^+$.

Clearly $f(0) = 0$ by (F1) and (F2). The standard prototype for f, verifying (F1), (F2), with $uf(u) > 0$ for $u \neq 0$, as required in (F4), is given by $f(u) = a|u|^{p-2}u + b|u|^{s-2}u$, with $a \geqslant 0$, $b > 0$. Of course condition (F4) fails when $s > p$. Indeed, for $u \neq 0$ we have exactly the reverse inequality $sF(u) - uf(u) = a(s-p)|u|^p/p \geqslant 0$. Therefore these nonlinearities do not verify the structure (F1)–(F4).

The power function $f(u) = b|u|^{s-2}u$ or $f(u) = b|u|^{s-1}\arctan u$, both with $b > 0$, satisfy (F1)–(F4) when $s > p$.

By (F1)–(F3) it follows that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|F(u)\right| \leqslant \varepsilon|u|^p + C_\varepsilon|u|^s \tag{3.1}$$

for any $u \in \mathbb{R}_0^+$. Clearly, when $a = 0$ in (F1), then (3.1) holds also for $\varepsilon = 0$, with $C_\varepsilon = b/s$. Condition (F4) easily yields that there exist $U > 0$ and a positive constant d such that

$$F(u) \geqslant d|u|^s \quad \text{for all } u \text{ with } u \geqslant U. \tag{3.2}$$

For the weight function $h : \mathbb{R}^+ \to \mathbb{R}^+$ in (1.1) we assume condition

(H1) $h = h(|x|) \in \mathcal{W}_\beta$ for some $\beta \in [0, p)$.

Now we give an existence result for (1.1) by means of the Mountain Pass Theorem of [3].

Theorem 3.1. *Assume* (F1)–(F4) *and* (H1). *Consider* (1.1)*, with*

$$\begin{aligned} &0 \leqslant \beta < p < s < p^*_\beta, \quad \lambda > 0, \quad 0 \leqslant \mu p C^q_{\mathrm{HS}} < q \min\{1, \lambda\}, \\ &\text{and either} \quad 0 \leqslant \alpha < p \leqslant q < p^*_\alpha \quad \text{or} \quad \alpha = q = p (= p^*_\alpha), \end{aligned} \tag{3.3}$$

where $C_{\mathrm{HS}} = C_{\mathrm{HS}}(n, p, \alpha, q)$ *is the constant of the embedding* $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^q_\alpha(\mathbb{R}^n)$ *given in* (2.5)*. Then* (1.1), (3.3) *admits a radial ground state* $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$*. Moreover,*

(i) $u \in C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *for some* $\theta \in (0, 1)$;
(ii) $|Du|^{p-2} Du \in C^1(\mathbb{R}^n \setminus \{0\})$;
(iii) *u is positive, solves Eq.* (1.1), (3.3) *pointwise in* $\mathbb{R}^n \setminus \{0\}$, $\langle x, Du(x)\rangle < 0$ *for all x with* $|x|$ *sufficiently large and* $|Du(x)| \to 0$ *as* $|x| \to \infty$;
(iv) *u is a fast decay solution of* (1.1), (3.3);
(v) *if* $0 \leqslant \alpha < p \leqslant q < p^*_\alpha$*, then* $u \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ *for any* $m \in [1, \infty)$;
(vi) *if* $0 \leqslant \max\{\alpha, \beta\} < p$*, then* $u \in L^\infty(\mathbb{R}^n)$;
(vii) *if* $1 < p \leqslant 2$*, then* $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$; *if furthermore* $0 \leqslant \max\{\alpha, \beta\} \leqslant p - 1$*, then* $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Since we are interested in positive solutions of (1.1), we extend f in the entire $\mathbb{R}$ as $f(u) = 0$ for $u < 0$.

First of all suppose $0 \leqslant \alpha < p \leqslant q < p^*_\alpha$. By Lemma 2.3 and (F2) the functional

$$\mathcal{J}(u) = \frac{1}{p}\left(\|Du\|^p_p + \lambda \|u\|^p_p\right) - \frac{\mu}{q}\|u\|^q_{q,\alpha} - \int_{\mathbb{R}^n} F\big(u(x)\big) h\big(|x|\big)\, dx$$

is well-defined in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. By Lemma 2.1(iii) we have

$$q\left(\|Du\|^p_p + \lambda\|u\|^p_p\right) - \mu p \|u\|^q_{q,\alpha} \geqslant \left(q \min\{1, \lambda\} - \mu p C^q_{\mathrm{HS}} \|u\|^{q-p}\right)\|u\|^p. \tag{3.4}$$

Let $\delta \in (0, 1]$. Then for any $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ with $\|u\| = \delta$ we get

$$q \min\{1, \lambda\} - \mu p C^q_{\mathrm{HS}} \|u\|^{q-p} \geqslant q \min\{1, \lambda\} - \mu p C^q_{\mathrm{HS}} := \gamma p q > 0$$

by assumption. Hence, from (3.4),

$$\mathcal{J}(u) \geqslant \gamma \|u\|^p - \int_{\mathbb{R}^n} F\big(u(x)\big) h\big(|x|\big)\, dx.$$

By (3.1) for every $\varepsilon > 0$ there is a positive constant C_ε such that

$$\mathcal{J}(u) \geqslant \gamma\|u\|^p - \varepsilon \|u\|^p_{p,h} - C_\varepsilon \|u\|^s_{s,h} \geqslant \big(\gamma - \varepsilon C_1\big)\|u\|^p - C_\varepsilon C_2 \|u\|^s, \tag{3.5}$$

where $C_1 = \max\{C^\beta_{n,p}, \|h\|_{L^\infty(\Omega_R)}\}$ and $C_2 = c_T^{n(s-p)/p} C_1$ and $R = R(h)$ is so small that $0 \leqslant h(|x|) \leqslant |x|^{-\beta}$ for $0 < |x| \leqslant R$, cf. Lemma 2.1 and the proof of part (i) .

Now we fix $\varepsilon > 0$ so small that $\gamma - \varepsilon C_1 > 0$ (say, e.g., $\varepsilon = \gamma/(2C_1)$), then we take $\delta \in (0, 1]$ so small that $C_\varepsilon C_2 \delta^{s-p} < \gamma - \varepsilon C_1$. The latter can be done since $s > p$. Thus for any $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ with $\|u\| = \delta$ by (3.5) we have

$$\mathcal{J}(u) \geqslant \left[(\gamma - \varepsilon C_1) - C_\varepsilon C_2 \delta^{s-p}\right]\delta^p := \varrho > 0.$$

Clearly there is $u \in C^1_{\mathrm{rad},0}(\mathbb{R}^n)$, compactly supported in $\mathbb{R}^n$, with $\|u\| = 1$ and $|u|h > 0$ on some measurable subset E of $\mathrm{supp}(u)$, with $|E| > 0$. For otherwise $h = 0$ a.e. in $\mathbb{R}^n$, contradicting the definition of $\mathcal{W}_\beta$. Hence $\|u\|_{s,h} > 0$. Take

$$\tau > \max\left\{\delta, [\max\{1, \lambda\}/pd\|u\|^s_{s,h}]^{1/(s-p)}\right\}$$

so large that $\tau u(x) \geqslant U$ for all $x \in \mathrm{supp}(u)$, with d and U given in (3.2). By (3.2) and the fact that $p < s$ we obtain that

$$\begin{aligned}\mathcal{J}(\tau u) &\leqslant \frac{\tau^p}{p}\max\{1,\lambda\}\|u\|^p - \int\limits_{\mathrm{supp}(u)} F\big(\tau u(x)\big)h\big(|x|\big)\,dx\\ &\leqslant \frac{\tau^p}{p}\max\{1,\lambda\} - d\int\limits_{\mathbb{R}^n}\big|\tau u(x)\big|^s h\big(|x|\big)\,dx\\ &= \tau^p\big(\max\{1,\lambda\}/p - d\tau^{s-p}\|u\|_{s,h}^s\big) < 0.\end{aligned}$$

Therefore $v = \tau u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, $\|v\| = \tau > \delta$ and $\mathcal{J}(v) < 0$. Consequently, $\mathcal{J}$ has the geometric structure required by the Mountain Pass Theorem.

Let $(u_k)_k$ be a Palais–Smale sequence in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ at some level c, that is for every $v \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$

$$\big\langle \mathcal{J}'(u_k), v\big\rangle \to 0 \quad \text{and} \quad \mathcal{J}(u_k) \to c \quad \text{as } k \to \infty. \tag{3.6}$$

We claim that $(u_k)_k$ is bounded in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Indeed, by (3.6) there are two positive constants K_1 and K_2 such that

$$\big|\mathcal{J}(u_k)\big| \leqslant K_1 \quad \text{and} \quad \big|\big\langle \mathcal{J}'(u_k), u_k\big\rangle\big| \leqslant K_2\|u_k\| \tag{3.7}$$

for any $k \in \mathbb{N}$.

Let us consider two cases. If $q = p$, let $\sigma \in [1/s, 1/p)$. By Lemma 2.1, (F4) and the choice of σ we have that

$$\begin{aligned}\mathcal{J}(u_k) - \sigma\big\langle \mathcal{J}'(u_k), u_k\big\rangle &\geqslant (1/p - \sigma)\big(\min\{1,\lambda\}\|u_k\|^p - \mu\|u_k\|_{p,\alpha}^p\big)\\ &\quad + \int\limits_{\mathbb{R}^n}\big[\sigma u_k(x) f\big(u_k(x)\big) - F\big(u_k(x)\big)\big]h\big(|x|\big)\,dx\\ &\geqslant (1/p - \sigma)\big(\min\{1,\lambda\} - \mu C_{\mathrm{HS}}^p\big)\|u_k\|^p + (\sigma - 1/s)\int\limits_{\mathbb{R}^n} F\big(u_k(x)\big)h\big(|x|\big)\,dx\\ &\geqslant (1/p - \sigma)\big(\min\{1,\lambda\} - \mu C_{\mathrm{HS}}^p\big)\|u_k\|^p\end{aligned}$$

for any $k \in \mathbb{N}$. Therefore by (3.7) b we obtain

$$0 \leqslant (1/p - \sigma)\big(\min\{1,\lambda\} - \mu C_{\mathrm{HS}}^p\big)\|u_k\|^p \leqslant K_1 + \sigma K_2\|u_k\|$$

for any $k \in \mathbb{N}$, and the claim is proved since $p > 1$.

If $q > p$ let $\sigma \in (\max\{1/q, 1/s\}, 1/p]$. By Lemma 2.1, (F4) and the choice of σ

$$\begin{aligned}\mathcal{J}(u_k) - \sigma\big\langle \mathcal{J}'(u_k), u_k\big\rangle &\geqslant (1/p - \sigma)\min\{1,\lambda\}\|u_k\|^p + \mu C_{\mathrm{HS}}^q(\sigma - 1/q)\|u_k\|^q\\ &\quad + \int\limits_{\mathbb{R}^n}\big[\sigma u_k(x) f\big(u_k(x)\big) - F\big(u_k(x)\big)\big]h\big(|x|\big)\,dx\\ &\geqslant \mu C_{\mathrm{HS}}^q(\sigma - 1/q)\|u_k\|^q + (\sigma - 1/s)\int\limits_{\mathbb{R}^n} F\big(u_k(x)\big)h\big(|x|\big)\,dx\\ &> \mu C_{\mathrm{HS}}^q(\sigma - 1/q)\|u_k\|^q\end{aligned}$$

for any $k \in \mathbb{N}$. Hence by (3.7)

$$0 \leqslant \mu C_{\mathrm{HS}}^q(\sigma - 1/q)\|u_k\|^q \leqslant K_1 + \sigma K_2\|u_k\|$$

for any $k \in \mathbb{N}$, and the claim is proved since $q > 1$.

Since $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ is compactly embedded in $L^p(\mathbb{R}^n)$ and by Lemma 2.3, up to a subsequence, still denoted by $(u_k)_k$, we have $u_k \rightharpoonup u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ as $k \to \infty$, and

$$u_k \to u \quad \text{in } L^p\big(\mathbb{R}^n\big), \qquad Du_k \rightharpoonup Du \quad \text{in } L^p\big(\mathbb{R}^n\big), \tag{3.8}$$

$$u_k \to u \quad \text{a.e. in } \mathbb{R}^n, \qquad u_k \to u \quad \text{in } L^q_\alpha\big(\mathbb{R}^n\big), \tag{3.9}$$

$$u_k \to u \quad \text{in } L^p_h\big(\mathbb{R}^n\big), \qquad u_k \to u \quad \text{in } L^s_h\big(\mathbb{R}^n\big). \tag{3.10}$$

Using Lemma 2.9, the fact that $\alpha < p < n$ and $\beta < p < n$ we have $g_k = \lambda|u_k|^{p-2}u_k - \mu|x|^{-\alpha}|u_k|^{q-2}u_k - h(|x|)f(u_k) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$. Moreover, by (3.8)–(3.10) and Lemma 2.5, $(g_k)_k$ is bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence Lemma 2.8 yields

$$Du_k \to Du \quad \text{a.e. in } \mathbb{R}^n \text{ as } k \to \infty,$$

and so, by (3.8) and Lemma 2.6 applied to the sequence $(Du_k)_k$, we get

$$\|Du_k\|_p^p - \|Du_k - Du\|_p^p \to \|Du\|_p^p \tag{3.11}$$

as $k \to \infty$. By means of Lemma 2.5, (F2), (3.10) and by the dominated convergence theorem we have

$$\int_{\mathbb{R}^n} F\big(u_k(x)\big)h\big(|x|\big)\,dx \to \int_{\mathbb{R}^n} F\big(u(x)\big)h\big(|x|\big)\,dx \tag{3.12}$$

$$\int_{\mathbb{R}^n} u_k(x)f\big(u_k(x)\big)h\big(|x|\big)\,dx \to \int_{\mathbb{R}^n} u(x)f\big(u(x)\big)h\big(|x|\big)\,dx \tag{3.13}$$

as $k \to \infty$. By (3.8), (3.9), (3.11) and (3.12)

$$c = \lim_k \mathcal{J}(u_k) = \mathcal{J}(u) + \frac{1}{p}\lim_k \|Du_k - Du\|_p^p. \tag{3.14}$$

Arguing as in (3.13) we also get

$$\int_{\mathbb{R}^n} f\big(u_k(x)\big)u(x)h\big(|x|\big)\,dx \to \int_{\mathbb{R}^n} f\big(u(x)\big)u(x)h\big(|x|\big)\,dx \tag{3.15}$$

as $k \to \infty$. Moreover, by (3.9)

$$\int_{\mathbb{R}^n} \big|u_k(x)\big|^{q-2}u_k(x)u(x)|x|^{-\alpha}\,dx \to \int_{\mathbb{R}^n} \big|u(x)\big|^q|x|^{-\alpha}dx = \|u\|_{q,\alpha}^q \tag{3.16}$$

as $k \to \infty$. By (3.8), (3.15) and (3.16) we deduce that $\langle \mathcal{J}'(u_k), u\rangle \to \langle \mathcal{J}'(u), u\rangle$ as $k \to \infty$. On the other hand, $\langle \mathcal{J}'(u_k), u\rangle \to 0$ as $k \to \infty$ by (3.6). Thus

$$\big\langle \mathcal{J}'(u), u\big\rangle = 0. \tag{3.17}$$

Since $(u_k)_k$ is bounded in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ and (3.6) holds, $\langle \mathcal{J}'(u_k), u_k\rangle \to 0$ as $k \to \infty$. Hence, by (3.6), (3.9), (3.12), (3.13) and (3.17) we get

$$\begin{aligned}
pc &= \lim_k\big(p\mathcal{J}(u_k) - \big\langle \mathcal{J}'(u_k), u_k\big\rangle\big)\\
&= \lim_k\bigg(\int_{\mathbb{R}^n}\big[f\big(u_k(x)\big)u_k(x) - pF\big(u_k(x)\big)\big]h\big(|x|\big)\,dx + \mu(1-p/q)\|u_k\|_{q,\alpha}^q\bigg)\\
&= \int_{\mathbb{R}^n}\big[f\big(u(x)\big)u(x) - pF\big(u(x)\big)\big]h\big(|x|\big)\,dx + \mu(1-p/q)\|u\|_{q,\alpha}^q\\
&= p\mathcal{J}(u) - \big\langle \mathcal{J}'(u), u\big\rangle = p\mathcal{J}(u).
\end{aligned}$$

In other words $c = \mathcal{J}(u)$, which combined with (3.14) yields $u_k \to u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ as $k \to \infty$. Hence, the functional $\mathcal{J}$ satisfies the Palais–Smale condition.

The existence of a non-trivial solution u for problem (1.1), (3.3) follows from the Mountain Pass Theorem.

Now, we consider the case $\alpha = q = p(= p_\alpha^*)$. The functional

$$\mathcal{J}(u) = \frac{1}{p}\big(\|Du\|_p^p + \lambda\|u\|_p^p - \mu\|u\|_{p,\alpha}^p\big) - \int_{\mathbb{R}^n} F\big(u(x)\big)h\big(|x|\big)\,dx,$$

is well-defined in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, by Lemma 2.3 and (F2), and again has the geometric structure of the Mountain Pass Theorem.

Let $(u_k)_k$ be a Palais–Smale sequence for $\mathcal{J}$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ at some level c. Arguing as in the other case, $(u_k)_k$ is bounded in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Hence, up to a subsequence, we have $u_k \rightharpoonup u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ and so, by Remark 2.4, we get again the validity of (3.8) and

$$u_k \to u \quad \text{a.e. in } \mathbb{R}^n, \qquad u_k \rightharpoonup u \quad \text{in } L^p_p(\mathbb{R}^n), \tag{3.18}$$

as $k \to \infty$. Remark 2.4 says that $(u_k)_k$ is bounded in $L^p_p(\mathbb{R}^n)$. By (3.18) and Lemma 2.7 applied to $(u_k)_k$ we have

$$\|u_k\|^p_{p,p} - \|u_k - u\|^p_{p,p} \to \|u\|^p_{p,p} \tag{3.19}$$

as $k \to \infty$. Using (3.11), (3.12), (3.8), (3.19) and Lemma 2.1(iii) we get

$$\begin{aligned} c = \lim_k \mathcal{J}(u_k) &= \mathcal{J}(u) + \frac{1}{p}\lim_k\big(\|Du_k - Du\|^p_p + \lambda\|u_k - u\|^p_p - \mu\|u_k - u\|^p_{p,p}\big) \\ &\geqslant \mathcal{J}(u) + \gamma \limsup_k \|u_k - u\|^p \geqslant \mathcal{J}(u) + \gamma \lim_k \|Du_k - Du\|^p_p, \end{aligned} \tag{3.20}$$

where $p\gamma = \min\{1 - \mu C^p_{\mathrm{HS}}, \lambda - \mu C^p_{\mathrm{HS}}\} > 0$. Also in this case $c = \mathcal{J}(u)$. Indeed, by (3.6), (3.12), (3.13) and (3.17)

$$\begin{aligned} pc &= \lim_k\big(p\mathcal{J}(u_k) - \langle \mathcal{J}'(u_k), u_k\rangle\big) = \lim_k \int_{\mathbb{R}^n} \big[f\big(u_k(x)\big)u_k(x) - pF\big(u_k(x)\big)\big]h\big(|x|\big)\,dx \\ &= \int_{\mathbb{R}^n} \big[f\big(u(x)\big)u(x) - pF\big(u(x)\big)\big]h\big(|x|\big)\,dx = p\mathcal{J}(u) - \langle \mathcal{J}'(u), u\rangle = p\mathcal{J}(u). \end{aligned}$$

Then, by (3.20) we deduce that $Du_k \to Du$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Thus, $u_k \to u$ in $H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$ by (3.8), and the Palais–Smale condition is proved. The Mountain Pass Theorem gives a non-trivial solution u of problem (1.1) under (3.3).

Now we prove that u is non-negative. Since u is a solution of (1.1), then $\langle \mathcal{J}'(u), \varphi\rangle = 0$ for any $\varphi \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$. Taking $\varphi = u^- = \max\{-u, 0\}$, we obtain

$$\begin{aligned} 0 &= \big\|D(u^-)\big\|^p_p + \lambda\|u^-\|^p_p - \mu\|u^-\|^q_{q,\alpha} - \int_{\mathbb{R}^n} f\big(u(x)\big)u^-(x)h\big(|x|\big)\,dx \\ &\geqslant \big\|D(u^-)\big\|^p_p + \lambda\|u^-\|^p_p - \mu\|u^-\|^q_{q,\alpha} \geqslant \gamma\|u^-\|^p, \end{aligned}$$

where $\gamma > 0$ is given in (3.5) when $q > p$ and in (3.20) when $q = p$. Hence $u^- \equiv 0$, that is u is non-negative.

By Lemma 2.9 we know $|x|^{(n-1)/p}u(x) \to 0$ as $|x| \to \infty$, from which we deduce that $u(x) \to 0$ as $|x| \to \infty$, that is u is a ground state.

Let $g(r,u) = \lambda u^{p-1} - \mu r^{-\alpha}u^{q-1} - h(r)f(u)$ in $\mathbb{R}^+ \times \mathbb{R}^+_0$. Since $u \in H^{1,p}_{\mathrm{rad}}(\mathbb{R}^n)$, the choice of α and β, Lemmas 2.9 and 2.3 and (F2) imply that $g(\cdot, u(\cdot)) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, the regularity of u and the fact that u solves Eq. (1.1) pointwise in $\mathbb{R}^n \setminus \{0\}$ follow from [41, Theorem 3.2]. Moreover, $g(\cdot, 0) = 0$ in $\mathbb{R}^+$ and $g(r,u) \geqslant [\lambda - r^{-\beta} - \mu r^{-\alpha}u^{q-p} - r^{-\beta}u^{s-p}]u^{p-1} > 0$ in $(R,\infty) \times (0,\delta)$, for some $R, \delta > 0$, by (H1) and (F2). Hence, $\langle x, Du(x)\rangle < 0$ for $|x|$ sufficiently large and $|Du(x)| \to 0$ as $|x| \to \infty$ by [41, Theorem 3.3].

Now we show that $u > 0$ in $\mathbb{R}^n \setminus \{0\}$. Indeed, since $u \geqslant 0$ in $\mathbb{R}^n$, $\mu \geqslant 0$, $h \geqslant 0$ and (F4) holds, we have $g(|x|, u(x)) \leqslant \lambda u(x)^{p-1}$ in $\mathbb{R}^n \setminus \{0\}$. Then, u is a C^1 weak solution also of $\Delta_p u - \lambda u^{p-1} \leqslant 0$ in $\mathbb{R}^n \setminus \{0\}$. Hence $u > 0$ in $\mathbb{R}^n \setminus \{0\}$ by the famous strong maximum principle due to Vázquez [48] (see also [38,39]).

By (H1), (F2), the fact that u is positive and $u'(r) < 0$, we have $[r^{n-1}|u'(r)|^{p-1}]' = -r^{n-1}g(r, (u(r)) < 0$ for all r sufficiently large. Hence, $r^{n-1}|u'(r)|^{p-1}$ is decreasing in $[R,\infty)$ for R sufficiently large and so admits a finite limit $\ell' \geqslant 0$ as $r \to \infty$. Since u is a positive ground state, by L'Hospital's rule, $r^{(n-p)/(p-1)}u(r) \cong -(p-1)r^{(n-1)/(p-1)}u'(r)/(n-p)$. Thus $r^{(n-p)/(p-1)}u(r)$ decreases to the limit $\ell = \ell'(p-1)/(n-p) \geqslant 0$ as $r \to \infty$ and in turn u is a fast decay solution of (1.1).

When $0 \leqslant \alpha < p \leqslant q < p_\alpha^*$, it remains to show that $u \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ for any $m \in [1, \infty)$. By [41, Theorem 2.1] we have only to prove that $a \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$, where $a(x) = g(|x|, u(x))/(1 + |u(x)|^{p-1})$, $x \in \mathbb{R}^n \setminus \{0\}$. By (F2), the definition of g, the fact that $h \in \mathcal{W}_\beta$, for any $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$\left|a(x)\right| \leqslant c_1 + \mu|x|^{-\alpha}\left|u(x)\right|^{q-p} + a|x|^{-\beta} + b|x|^{-\beta}\left|u(x)\right|^{s-p} + c_2\left|u(x)\right|^{s-p}, \tag{3.21}$$

where $c_1 = \lambda + \|h\|_{L^\infty(\Omega_R)}$ and $c_2 = b\|h\|_{L^\infty(\Omega_R)}$ for some positive R. Clearly c_1 and $|x|^{-\beta} \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$, since $\beta \in [0, p)$, and also $|u|^{s-p} \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$, since $u \in C(\mathbb{R}^n \setminus \{0\})$, Lemma 2.9 holds and $s < p_\beta^* \leqslant p^*$. When $q = p$ we get $|x|^{-\alpha}|u(x)|^{q-p} = |x|^{-\alpha} \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$, being $\alpha < p$; while if $q > p$ by Lemma 2.9, a.e. in $\mathbb{R}^n$,

$$|x|^{-\alpha n/p}\left|u(x)\right|^{(q-p)n/p} \leqslant \begin{cases} C|x|^{-\alpha n/p-(n-p)(q-p)n/p^2} & \text{if } |x| < 1, \\ |u(x)|^{(q-p)n/p} & \text{if } |x| \geqslant 1, \end{cases}$$

where $C^{p/(q-p)n} = [(p-1)/(n-p)]^{1/p'}\omega_n^{-1/p}\|Du\|_p$. Since $0 \leqslant \alpha < p < q < p_\alpha^*$, then $|x|^{-\alpha n/p-(n-p)(q-p)n/p^2} \in L^1(B_1)$, where B_1 is the unit ball centered at $x = 0$; while $|u|^{(q-p)n/p} \in C(\mathbb{R}^n \setminus \{0\})$, being $u \in C(\mathbb{R}^n \setminus \{0\})$ by (i). Hence $|x|^{-\alpha n/p}|u|^{(q-p)n/p}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Arguing in the same way we get that $|x|^{-\beta}|u|^{s-p} \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$, since $\beta \in [0, p)$ and $p < s < p_\beta^*$. Then, $a \in L^{n/p}_{\mathrm{loc}}(\mathbb{R}^n)$ by (3.21) and so $u \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ for all $m \in [1, \infty)$ by [41, Theorem 2.1].

Now, using again (3.21), we show (vi). We claim that $a \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$ small enough. Since $\beta < p$ by assumption, taking ε so small that $\beta < p(1-\varepsilon)$ we get $|x|^{-\beta} \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$. By Lemma 2.9 we have $|u(x)|^{s-p} \leqslant |x|^{-(n-p)(s-p)/p}$ in B_1. Hence $|u|^{s-p} \in L^{n/p(1-\varepsilon)}(B_1)$ if ε is even smaller so that $s < p^* - p^2\varepsilon/(n-p)$. This is possible since $s < p_\beta^* \leqslant p^*$. Thus $|u|^{s-p} \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$, being $u \in C(\mathbb{R}^n \setminus \{0\})$. Of course, when $q = p$, we get $|x|^{-\alpha}|u(x)|^{q-p} = |x|^{-\alpha} \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$ for ε small enough, so that $\alpha < p(1-\varepsilon)$. This choice is possible being $\alpha < p$ by assumption. While if $q > p$ by Lemma 2.9, a.e. in $\mathbb{R}^n$,

$$|x|^{-\alpha}\left|u(x)\right|^{q-p} \leqslant \begin{cases} C|x|^{-\alpha-(n-p)(q-p)/p} & \text{if } |x| < 1, \\ |u(x)|^{q-p} & \text{if } |x| \geqslant 1, \end{cases}$$

where $C^{1/(q-p)} = [(p-1)/(n-p)]^{1/p'}\omega_n^{-1/p}\|Du\|_p$. Since $0 \leqslant \alpha < p < q < p_\alpha^*$, taking ε so small that $q < p_\alpha^* - p^2\varepsilon/(n-p)$, we have $|x|^{-\alpha-(n-p)(q-p)/p} \in L^{n/p(1-\varepsilon)}(B_1)$, while $|u|^{q-p} \in C(\mathbb{R}^n \setminus \{0\})$. Hence $|x|^{-\alpha}|u|^{q-p} \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$ for ε small enough. The same argument shows that $|x|^{-\beta}|u|^{s-p} \in L^{n/p(1-\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$ when ε is sufficiently small, since $\beta < p < s < p_\beta^*$. By (3.21) the claim is proved and so, as an application of [41, Theorem 2.4], we get $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Hence $u \in L^\infty(\mathbb{R}^n)$, since u is a ground state which is continuous in $\mathbb{R}^n \setminus \{0\}$.

Finally, $g(\cdot, u(\cdot)) \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, since $u \in C(\mathbb{R}^n \setminus \{0\})$. Thus, an application of [41, Theorem 2.5] yields $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. Furthermore suppose that $0 \leqslant \max\{\alpha, \beta\} \leqslant p - 1$. For a suitable constant $C > 0$ we have

$$\left|g(r, u)\right| \leqslant Cr^{-\max\{\alpha,\beta\}}\left(u^{p-1} + u^{q-1} + u^{s-1}\right)$$

for $r \in (0, 1]$ and $u \in \mathbb{R}_0^+$, by (F2) and the fact that $h \in \mathcal{W}_\beta$. If $\max\{\alpha, \beta\} = 0$, then $g(\cdot, u(\cdot)) \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ for all $m \geqslant 1$ by (v). Otherwise, if $0 < \max\{\alpha, \beta\} \leqslant p - 1$, take $t \in (1, n(p-1)/p\max\{\alpha, \beta\})$. This choice is possible by the assumptions on α and β. By Hőlder's inequality we have

$$\left\|g\left(\cdot, u(\cdot)\right)\right\|^{p'}_{L^{p'}(B_1)} \leqslant C\left(\|u\|^{p-1}_{(p-1)t'} + \|u\|^{q-1}_{(q-1)t'} + \|u\|^{s-1}_{(s-1)t'}\right)\left(\int\limits_{B_1} |x|^{-\max\{\alpha,\beta\}pt/(p-1)}\,dx\right)^{1/t'}$$

and so $g(\cdot, u(\cdot)) \in L^{p'}(B_1)$ by (v). Then, $g(\cdot, u(\cdot)) \in L^{p'}_{\mathrm{loc}}(\mathbb{R}^n)$, being $u \in C(\mathbb{R}^n \setminus \{0\})$. Hence, [41, Theorem 2.5] applies when $1 < p \leqslant 2$ and so $u \in H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$. □

Now we give a regularity result for *bounded* radial ground states of class $C^1(\mathbb{R}^n \setminus \{0\})$ of Eq. (1.1).

Proposition 3.2. *Assume* (F1)–(F4) *and* (H1). *Consider* (1.1) *with* $\lambda > 0$, $\mu \geqslant 0$, $q > 1$ *and* $0 \leqslant \max\{\alpha, \beta\} < p$. *Let* $u \in C^1(\mathbb{R}^n \setminus \{0\})$, *with* $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$, *be a bounded radial ground state which solves* (1.1) *also pointwise in* $\mathbb{R}^n \setminus \{0\}$. *Then u is positive in* $\mathbb{R}^n \setminus \{0\}$. *Moreover*

(i) *if* $\alpha, \beta \in [0,1)$, *then* $u \in C^1(\mathbb{R}^n)$, *with* $u(0) > 0$ *and* $Du(0) = \mathbf{0}$;
(ii) *if* $\alpha = \beta = 1$, *then* $u \in C^{0,1}_{\text{loc}}(\mathbb{R}^n)$;
(iii) *if* $1 < \max\{\alpha, \beta\} < p$, *then* $u \in C^{0,\theta}_{\text{loc}}(\mathbb{R}^n)$, *with* $\theta = (p - \max\{\alpha,\beta\})/(p-1)$.

Therefore u is continuous at $x = 0$ *in all the cases* (i)–(iii).

In particular this proposition applies to the bounded radial ground state constructed in Theorem 3.1 *when* $0 \leqslant \max\{\alpha,\beta\} < p$.

Proof. First note that u is positive in $\mathbb{R}^n \setminus \{0\}$ by the strong maximum principle. From the regularity assumptions it is obvious that we have to discuss the smoothness of u only locally at $x = 0$. Using the same notation as in the proof of [41, Theorem 3.2], it is clear that $u(x) = u(r)$, $r = |x|$, solves

$$r^{n-1}\left|u'(r)\right|^{p-2}u'(r) = \int_0^r \rho^{n-1} g\big(\rho, u(\rho)\big)\,d\rho,$$

for corresponding g noted in the proof of Theorem 3.1. Hence, since u is bounded and $h \in \mathcal{W}_\beta$, by (F2) for all $r > 0$ we have

$$\left|u'(r)\right|^{p-1} \leqslant c r^{1-n} \int_0^r \rho^{n-1}\left[1 + \rho^{-\alpha} + \rho^{-\beta}\right] d\rho \leqslant C r^{1-\max\{\alpha,\beta\}},$$

where $c, C > 0$ are suitable constants depending on h and u.

If $\max\{\alpha,\beta\} < 1$, then $u'(0) = 0$ and $u \in C^1(\mathbb{R}^n)$. Since u is non-negative and $u'(0) = 0$, then u solves $\Delta_p u - \lambda u^{p-1} \leqslant 0$ in $\mathbb{R}^n$ and the strong maximum principle can be applied in the entire $\mathbb{R}^n$. Hence $u(0) > 0$ and so case (i) is proved.

When $\alpha = \beta = 1$, for any $r, r' > 0$ we get $|u(r) - u(r')| \leqslant C^{1/(p-1)}|r - r'|$, and so (ii) is proved.

If $1 < \max\{\alpha,\beta\} < p$, then $r^{(1-\max\{\alpha,\beta\})/(p-1)} \in L^1(0,\delta)$, $\delta > 0$. Therefore for any $r, r' > 0$

$$\left|u(r) - u(r')\right| \leqslant L|r - r'|^{(p-\max\{\alpha,\beta\})/(p-1)},$$

with $L = (p-1)C^{1/(p-1)}/(p - \max\{\alpha,\beta\})$. Hence (iii) holds.

It is now clear that u can be extended by continuity at $x = 0$ also in all the cases (i)–(iii) . The final part of the proposition is an obvious consequence of Theorem 3.1(vi). □

Proposition 3.2 does not cover the case $\alpha = q = p$ in (1.1) which remains still open.

4. Non-existence results

In this section we give some non-existence results for (1.1) by a Pohozaev–Pucci–Serrin type identity. Throughout the section, without further mentioning, we assume that $1 < p < n$, and either

$$\alpha = q = p(= p_\alpha^*) \quad \text{or}$$
$$\alpha \in [0,p), \quad \text{if } q \in [p, p_\alpha^*],\ \ p_\alpha^* = p(n-\alpha)/(n-p) > p.$$

Moreover we suppose that condition (F1) holds and $h : \mathbb{R}^+ \to \mathbb{R}$ is continuous.

Lemma 4.1. *Let* $u \in H^{1,p}(\mathbb{R}^n)$ *be a weak solution of* (1.1). *Then the following identity holds*

$$\|Du\|_p^p + \lambda\|u\|_p^p = \mu\|u\|_{q,\alpha}^q + \int_{\mathbb{R}^n} f\big(u(x)\big)u(x)h\big(|x|\big)\,dx$$

for any $\lambda, \mu \in \mathbb{R}$.

Furthermore, if $\lambda = \mu = 0$*, any weak solution* $u \in D^{1,p}(\mathbb{R}^n)$ *of* (1.1) *verifies the following identity*

$$\|Du\|_p^p = \int_{\mathbb{R}^n} f\big(u(x)\big)u(x)h\big(|x|\big)\,dx.$$

Hence, if $uf(u)h(|x|) \leqslant 0$ *a.e. in* $\mathbb{R}^n$*, then* u *is the trivial solution.*

Proof. Since $u \in H^{1,p}(\mathbb{R}^n)$ is a weak solution of (1.1), then for any $\varphi \in H^{1,p}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \big|Du(x)\big|^{p-2}\big\langle Du(x), D\varphi(x)\big\rangle\,dx + \lambda \int_{\mathbb{R}^n} \big|u(x)\big|^{p-2}u(x)\varphi(x)\,dx$$
$$= \mu \int_{\mathbb{R}^n} \big|u(x)\big|^{q-2}u(x)\varphi(x)|x|^{-\alpha}\,dx + \int_{\mathbb{R}^n} f\big(u(x)\big)\varphi(x)h\big(|x|\big)\,dx.$$

Also the second integral on the right–hand side must converge, since all the other integrals are convergent by Lemma 2.1(iii), and the choice of α and q. The assertion follows at once taking $\varphi = u$.

The second identity of the lemma can be proved in a similar way. When $uf(u)h(|x|) \leqslant 0$ a.e. in $\mathbb{R}^n$ the corresponding non-existence result for (1.1) follows as a consequence of this identity. □

From now on in the section we suppose that the weight function h satisfies also the further condition

(H2) *h is differentiable a.e. in* $\mathbb{R}^+$.

Lemma 4.2. *Let* $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *satisfy* (1.1) *a.e. in* $\mathbb{R}^n$ *and assume that* $F \circ u \in L^1_h(\mathbb{R}^n)$*. Then, the following identity holds*

$$\frac{n-p}{p}\|Du\|_p^p + \frac{\lambda n}{p}\|u\|_p^p - \frac{\mu(n-\alpha)}{q}\|u\|_{q,\alpha}^q = \int_{\mathbb{R}^n} \big[nh\big(|x|\big) + |x|h'\big(|x|\big)\big]F\big(u(x)\big)\,dx$$

for any $\lambda, \mu \in \mathbb{R}$.

Furthermore, if $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *satisfies*

$$\Delta_p u + h\big(|x|\big)f(u) = 0 \quad \text{a.e. in } \mathbb{R}^n, \tag{4.1}$$

and $F \circ u \in L^1_h(\mathbb{R}^n)$*, then*

$$(n-p)\|Du\|_p^p = p\int_{\mathbb{R}^n} \big[nh\big(|x|\big) + |x|h'\big(|x|\big)\big]F\big(u(x)\big)\,dx.$$

Proof. The regularity of u yields that the function $\langle x, Du\rangle \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. The idea consists of multiplying Eq. (1.1) by $\langle x, Du\rangle$ and of integrating on $B_R \setminus B_\varepsilon$, where $0 < \varepsilon < R$. We get

$$\int_{B_R\setminus B_\varepsilon} \operatorname{div}\big(|Du|^{p-2}Du\big)\langle x, Du\rangle\,dx - \lambda \int_{B_R\setminus B_\varepsilon} |u|^{p-2}u\langle x, Du\rangle\,dx$$
$$+ \mu \int_{B_R\setminus B_\varepsilon} |x|^{-\alpha}|u|^{q-2}u\langle x, Du\rangle\,dx + \int_{B_R\setminus B_\varepsilon} h\big(|x|\big)f(u)\langle x, Du\rangle\,dx = 0. \tag{4.2}$$

The last integral must converge, since all the other integrals are convergent, thanks to the choice of α and q and since $\langle x, Du\rangle \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$.

The first term in (4.2) becomes

$$\begin{aligned}\int_{B_R\setminus B_\varepsilon} \operatorname{div}\big(|Du|^{p-2}Du\big)\langle x, Du\rangle\,dx &= -\int_{B_R\setminus B_\varepsilon} |Du|^{p-2}\big\langle Du, D\big(\langle x, Du\rangle\big)\big\rangle\,dx \\ &\quad + R\int_{\partial B_R} |Du|^{p-2}\big|\langle Du, \nu\rangle\big|^2\,dS + \varepsilon\int_{\partial B_\varepsilon} |Du|^{p-2}\big|\langle Du, \nu\rangle\big|^2\,dS \\ &= -\int_{B_R\setminus B_\varepsilon} |Du|^p dx - \frac{1}{p}\int_{B_R\setminus B_\varepsilon} \big\langle x, D\big(|Du|^p\big)\big\rangle\,dx \\ &\quad + R\int_{\partial B_R} |Du|^p\,dS + \varepsilon\int_{\partial B_\varepsilon} |Du|^p\,dS \\ &= \frac{n-p}{p}\int_{B_R\setminus B_\varepsilon} |Du|^p\,dx + R\frac{p-1}{p}\int_{\partial B_R} |Du|^p\,dS + \varepsilon\frac{p-1}{p}\int_{\partial B_\varepsilon} |Du|^p\,dS.\end{aligned}$$

Taking into account that $p|u|^{p-2}u\langle x, Du\rangle = \langle x, D(|u|^p)\rangle$ and integrating by parts we get

$$\int_{B_R\setminus B_\varepsilon} |u|^{p-2}u\langle x, Du\rangle\,dx = -\frac{n}{p}\int_{B_R\setminus B_\varepsilon} |u|^p\,dx + \frac{R}{p}\int_{\partial B_R} |u|^p\,dS + \frac{\varepsilon}{p}\int_{\partial B_\varepsilon} |u|^p\,dS.$$

Arguing in the same way we also have

$$\int_{B_R\setminus B_\varepsilon} |u|^{q-2}u\langle x, Du\rangle|x|^{-\alpha}\,dx = -\frac{(n-\alpha)}{q}\int_{B_R\setminus B_\varepsilon} |u|^q|x|^{-\alpha}\,dx + \frac{R}{p}\int_{\partial B_R} |u|^q|x|^{-\alpha}\,dS + \frac{\varepsilon}{p}\int_{\partial B_\varepsilon} |u|^q|x|^{-\alpha}\,dS.$$

Moreover

$$\begin{aligned}\int_{B_R\setminus B_\varepsilon} f(u)\langle x, Du\rangle h\big(|x|\big)\,dx &= \int_{B_R\setminus B_\varepsilon} \big\langle h\big(|x|\big)x, DF(u)\big\rangle\,dx \\ &= -\int_{B_R\setminus B_\varepsilon} \big[nh\big(|x|\big) + |x|h'\big(|x|\big)\big]F\big(u(x)\big)\,dx + R\int_{\partial B_R} F(u)h\big(|x|\big)\,dS \\ &\quad + \varepsilon\int_{\partial B_\varepsilon} F(u)h\big(|x|\big)\,dS.\end{aligned}$$

Hence, by (4.2) we get

$$\begin{aligned}&\frac{n-p}{p}\int_{B_R\setminus B_\varepsilon} |Du|^p\,dx + \lambda\frac{n}{p}\int_{B_R\setminus B_\varepsilon} |u|^p\,dx - \mu\frac{n-\alpha}{q}\int_{B_R\setminus B_\varepsilon} |u|^q|x|^{-\alpha}\,dx \\ &\qquad - \int_{B_R\setminus B_\varepsilon} \big[nh\big(|x|\big) + |x|h'\big(|x|\big)\big]F\big(u(x)\big)\,dx \\ &\quad = -R\int_{\partial B_R}\left[\frac{p-1}{p}|Du|^p - \frac{\lambda}{p}|u|^p + \frac{\mu}{q}|u|^q|x|^{-\alpha} + F(u)h\big(|x|\big)\right]dS \\ &\qquad - \varepsilon\int_{\partial B_\varepsilon}\left[\frac{p-1}{p}|Du|^p - \frac{\lambda}{p}|u|^p + \frac{\mu}{q}|u|^q|x|^{-\alpha} + F(u)h\big(|x|\big)\right]dS. \qquad (4.3)\end{aligned}$$

Let us define $\Phi(x) = |Du(x)|^p + |u(x)|^p + |x|^{-\alpha}|u(x)|^q + |h(|x|)| \cdot |F(u(x))|$ for any $x \in \mathbb{R}^n$ and $\Psi(r) = \int_{\partial B_r} \Phi(x)\, dS$ for any $r \in \mathbb{R}^+$. We have

$$\int_{\mathbb{R}^n} \Phi(x)\, dx = \int_0^\infty \Psi(r)\, dr.$$

Since $u \in H^{1,p}(\mathbb{R}^n)$, by Lemma 2.1, the choice of α and q and the assumption $F \circ u \in L^1_h(\mathbb{R}^n)$, we have $\Phi \in L^1(\mathbb{R}^n)$, that is

$$\Psi \in L^1(\mathbb{R}^+). \tag{4.4}$$

We claim that there exists a sequence $(R_k)_k$ tending to infinity as $k \to \infty$ such that $R_k \Psi(R_k) \to 0$, i.e.

$$R_k \int_{\partial B_{R_k}} \Phi(x)\, dS \to 0 \tag{4.5}$$

as $k \to \infty$. Suppose, by contradiction, that $\liminf_{r\to\infty} r\Psi(r) = \ell > 0$. Then, for r sufficiently large, say $r > M > 0$, we have $r\Psi(r) \geqslant \ell/2$, which yields

$$\int_0^\infty \Psi(r)\, dr \geqslant \int_M^\infty \Psi(r)\, dr \geqslant \frac{\ell}{2} \int_M^\infty r^{-1}\, dr = \infty.$$

This contradicts (4.4), and so the claim is proved.

Arguing in the same way we get that there exists a sequence $(\varepsilon_k)_k$ tending to zero as $k \to \infty$ such that

$$\varepsilon_k \int_{\partial B_{\varepsilon_k}} \Phi(x)\, dS \to 0$$

as $k \to \infty$. Hence, using also (4.5) we deduce

$$\begin{aligned} &R_k \int_{\partial B_{R_k}} \left[\frac{p-1}{p}|Du|^p - \frac{\lambda}{p}|u|^p + \frac{\mu}{q}|u|^q|x|^{-\alpha} + F(u)h(|x|)\right] dS \to 0,\\ &\varepsilon_k \int_{\partial B_{\varepsilon_k}} \left[\frac{p-1}{p}|Du|^p - \frac{\lambda}{p}|u|^p + \frac{\mu}{q}|u|^q|x|^{-\alpha} + F(u)h(|x|)\right] dS \to 0 \end{aligned} \tag{4.6}$$

as $k \to \infty$. We also have

$$\begin{cases} \int_{B_{R_k}\setminus B_{\varepsilon_k}} |Du|^p\, dx \to \int_{\mathbb{R}^n} |Du|^p\, dx, \quad \int_{B_{R_k}\setminus B_{\varepsilon_k}} |u|^p\, dx \to \int_{\mathbb{R}^n} |u|^p\, dx,\\ \int_{B_{R_k}\setminus B_{\varepsilon_k}} |u|^q|x|^{-\alpha} dx \to \int_{\mathbb{R}^n} |u|^q|x|^{-\alpha} dx,\\ \int_{B_{R_k}\setminus B_{\varepsilon_k}} [nh(|x|) + |x|h'(|x|)]F(u(x))\, dx \to \int_{\mathbb{R}^n} [nh(|x|) + |x|h'(|x|)]F(u(x))\, dx \end{cases} \tag{4.7}$$

as $k \to \infty$. Choosing $R = R_k$ and $\varepsilon = \varepsilon_k$ in (4.3) and taking into account (4.6) and (4.7) the first part of the lemma is proved.

The latter part of the lemma can be proved in the same way. □

Remark 4.3. If $u \in H^{1,p}(\mathbb{R}^n)$ satisfies Eq. (1.1) a.e. in $\mathbb{R}^n$, then u is a weak solution of (1.1). Indeed, multiplying (1.1) by $\varphi \in H^{1,p}(\mathbb{R}^n)$ and integrating by parts we get

$$\int_{\mathbb{R}^n} |Du(x)|^{p-2}\langle Du(x), D\varphi(x)\rangle\,dx + \lambda \int_{\mathbb{R}^n} |u(x)|^{p-2}u(x)\varphi(x)\,dx$$
$$= \mu \int_{\mathbb{R}^n} |u(x)|^{q-2}u(x)\varphi(x)|x|^{-\alpha}\,dx + \int_{\mathbb{R}^n} f\big(u(x)\big)\varphi(x)h\big(|x|\big)\,dx.$$

Here the last integral is convergent because all the others converge, since $u \in H^{1,p}(\mathbb{R}^n)$ and Lemma 2.1(iii) holds. Clearly arguing in the same way we can prove that if $u \in D^{1,p}(\mathbb{R}^n)$ satisfies (4.1) a.e. in $\mathbb{R}^n$, then u is a weak solution of (4.1).

On the other hand, if $u \in H^{1,p}(\mathbb{R}^n)$ or $u \in D^{1,p}(\mathbb{R}^n)$ is a weak solution of (1.1) or (4.1), respectively, then $F \circ u \in L^1_h(\mathbb{R}^n)$ whenever $0 \leqslant F(u) \leqslant uf(u)$, $u > 0$, holds. Clearly this occurs if f is non-negative and non-decreasing in $\mathbb{R}^+$, as well as when f verifies (F4).

As a consequence of Lemma 4.2 we have

Proposition 4.4. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy Eq.* (1.1) *a.e. in $\mathbb{R}^n$ and $F \circ u \in L^1_h(\mathbb{R}^n)$. Assume $\mu \leqslant 0 \leqslant \lambda$ and that along the solution u*

$$\big[nh\big(|x|\big) + |x|h'\big(|x|\big)\big]F\big(u(x)\big) \leqslant 0 \quad \text{for a.a. } x \in \mathbb{R}^n. \tag{4.8}$$

Then, $u \equiv 0$.

If $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies (4.1) *and* (4.8) *a.e. in $\mathbb{R}^n$, and $F \circ u \in L^1_h(\mathbb{R}^n)$, then $u \equiv 0$.*

Proof. By Lemma 4.2 we get

$$(n-p)\|Du\|_p^p + \lambda n\|u\|_p^p - \mu\, p(n-\alpha)\|u\|_{q,\alpha}^q/q \leqslant 0.$$

Since $\mu \leqslant 0 \leqslant \lambda$ and $\alpha \leqslant p < n$ the first assertion is proved.

When $\lambda = \mu = 0$, again by Lemma 4.2 and (4.8) we have $(n-p)\|Du\|_p^p \leqslant 0$. Hence the second assertion of the lemma follows. □

In particular, if $h(|x|) = |x|^{-\beta}$, $\beta < p$, then (4.8) holds when $F(u) \leqslant 0$. But in this special case much more can be said.

Proposition 4.5. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy*

$$\Delta_p u - \lambda|x|^{-\beta}|u|^{p-2}u + \mu|x|^{-\alpha}|u|^{q-2}u + |x|^{-\beta}f(u) = 0 \quad \text{a.e. in } \mathbb{R}^n, \tag{4.9}$$

and assume $F \circ u \in L^1_\beta(\mathbb{R}^n)$ and $\mu \leqslant 0$. If $F(u) \leqslant C|u|^p$ for all $u \in \mathbb{R}$, where $C > 0$ is an appropriate constant, then $u \equiv 0$ provided that $\lambda \geqslant p(n-\beta)C/n$.

Proof. Indeed, by Lemma 4.2

$$0 \leqslant (n-p)\|Du\|_p^p - \mu\, p(n-\alpha)\|u\|_{q,\alpha}^q/q = \int_{\mathbb{R}^n} \big[p(n-\beta)F\big(u(x)\big) - \lambda n|u|^p\big]|x|^{-\beta}\,dx \leqslant 0,$$

and the assertion follows at once. □

Lemma 4.6. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy* (1.1) *a.e. in $\mathbb{R}^n$ and assume $F \circ u \in L^1_h(\mathbb{R}^n)$. Then, the following identity*

$$\lambda pq\|u\|_p^p + q\int_{\mathbb{R}^n} \big[(n-p)u(x)f\big(u(x)\big) - npF\big(u(x)\big)\big]h\big(|x|\big)\,dx - pq\int_{\mathbb{R}^n} F\big(u(x)\big)|x|h'\big(|x|\big)\,dx$$
$$= \mu(n-p)(p^*_\alpha - q)\|u\|_{q,\alpha}^q$$

holds for any $\lambda, \mu \in \mathbb{R}$. In the particular case when $q = p$ and $\alpha \in [0, p]$ in (1.1) *the identity becomes*

$$\lambda p\|u\|_p^p + \int_{\mathbb{R}^n} \big[(n-p)u(x)f\big(u(x)\big) - npF\big(u(x)\big)\big]h\big(|x|\big)\,dx - p\int_{\mathbb{R}^n} F\big(u(x)\big)|x|h'\big(|x|\big)\,dx = \mu(p-\alpha)\|u\|_{p,\alpha}^p.$$

Analogously, if $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfies (4.1) *a.e. in $\mathbb{R}^n$ and $F \circ u \in L^1_h(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \big[(n-p)u(x)f\big(u(x)\big) - npF\big(u(x)\big)\big]h\big(|x|\big)\,dx = p\int_{\mathbb{R}^n} F\big(u(x)\big)|x|h'\big(|x|\big)\,dx.$$

Proof. By Remark 4.3 the function u is a weak solution of (1.1). Thus the assertion follows from Lemmas 4.1 and 4.2. □

Now, by Lemma 4.6 we can easily deduce the following non-existence results for (1.1).

Proposition 4.7. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy* (1.1) *a.e. in $\mathbb{R}^n$. Assume $F \circ u \in L^1_h(\mathbb{R}^n)$, $\lambda > 0$, h is non-negative and non-increasing in $\mathbb{R}^+$ and* (F4) *holds along u for some $s \geqslant p^* = np/(n-p)$. Then $u \equiv 0$, whenever either $\mu \leqslant 0$ or $q = p^*_\alpha$.*

Remark 4.8. The case $q = p = p^*_\alpha$ occurs only when $\alpha = p$.

Of course $p^*_\alpha < p^*$ when $\alpha \in (0, p]$, while they coincide in the more standard case $\alpha = 0$. Hence Proposition 4.7 is not completely satisfactory for (1.1) in the general case $\alpha \in (0, p]$.

Proposition 4.9. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy* (1.1) *a.e. in $\mathbb{R}^n$ and assume $F \circ u \in L^1_h(\mathbb{R}^n)$.*

If $\lambda = 0$ and either

$$\big[(n-p)uf(u) - npF(u)\big]h(r) \geqslant pF(u)rh'(r) \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \text{ or} \tag{4.10}$$

$$\big[(n-p)uf(u) - npF(u)\big]h(r) \leqslant pF(u)rh'(r) \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{4.11}$$

holds, then $u \equiv 0$, whenever $q \neq p^*_\alpha$ and either $\mu < 0$ under the validity of (4.10) *or $\mu > 0$ under* (4.11).

If $\lambda > 0$ and (4.10) *is valid, then* (1.1) *admits in $H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ only the trivial solution $u \equiv 0$, whenever either $\mu \leqslant 0$ or $q = p^*_\alpha$; while if $\lambda < 0$ and* (4.11) *holds, then* (1.1) *admits in $H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ only the trivial solution $u \equiv 0$, whenever either $\mu \geqslant 0$ or $q = p^*_\alpha$.*

Proof. It follows from Lemma 4.6. □

When $f(u) = c|u|^{s-2}u$, $s > 1$, $c > 0$, condition (4.10) becomes

$$(n-p)(s-p^*)h(r) \geqslant prh'(r) \quad \text{in } \mathbb{R}^+,$$

i.e. it gives a link between the growth exponent s of f and the weight h.

Theorem 4.10. *Let $u \in H^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy*

$$\Delta_p u - \lambda|u|^{p-2}u + \mu|x|^{-\alpha}|u|^{q-2}u + |x|^{-\beta}f(u) = 0 \quad \text{a.e. in } \mathbb{R}^n, \tag{4.12}$$

and assume $F \circ u \in L^1_\beta(\mathbb{R}^n)$.

If $\lambda > 0$ and (F4) *holds along u for some $s \geqslant p^*_\beta = p(n-\beta)/(n-p)$, $\beta < p$, then $u \equiv 0$, whenever either $\mu \leqslant 0$ or $q = p^*_\alpha$.*

If $\lambda > 0$ and along u

$$(n-p)uf(u) - p(n-\beta)F(u) \geqslant 0, \tag{4.13}$$

then $u \equiv 0$, whenever either $\mu \leqslant 0$ or $q = p^*_\alpha$; while if $\lambda < 0$ and along u

$$(n-p)uf(u) - p(n-\beta)F(u) \leqslant 0, \tag{4.14}$$

then $u \equiv 0$, whenever either $\mu \geqslant 0$ or $q = p^*_\alpha$. Finally, if $\lambda = 0$ and along u either (4.13) *or* (4.14) *holds, then $u \equiv 0$, whenever $q \neq p^*_\alpha$ and either $\mu < 0$ or $\mu > 0$.*

Proof. When $h(r)=r^{-\beta}$, $\beta\in\mathbb{R}$, along solutions u of class $H^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ of (4.12), the first identity in Lemma 4.6 reduces to

$$\lambda pq\|u\|_p^p+q\int_{\mathbb{R}^n}\big[(n-p)u(x)f\big(u(x)\big)-p(n-\beta)F\big(u(x)\big)\big]|x|^{-\beta}\,dx=\mu(n-p)(p^*_\alpha-q)\|u\|^q_{q,\alpha} \tag{4.15}$$

for any $\lambda,\mu\in\mathbb{R}$, and the results follows at once. □

Theorem 4.10 extends several previous results, see, for instance, [20, Lemma 3.7] established in bounded star-shaped domains, when $\lambda=\beta=0$ and [24, Theorem 2.1] stated for $\lambda=0$, $q=p^*_\alpha=p(n-\alpha)/(n-p)$, $\alpha\in[0,p]$, $h\equiv 1$ and $f(u)=c|u|^{p^*-2}u$, $c>0$.

From Theorem 4.10 we deduce that Eq. (4.12) with $f\equiv 0$, i.e.

$$\Delta_p u-\lambda|u|^{p-2}u+\mu|x|^{-\alpha}|u|^{q-2}u=0\quad \text{a.e. in } \mathbb{R}^n,\ \lambda\neq 0,$$

admits in $H^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ only the trivial solution $u\equiv 0$, whenever either $q=p^*_\alpha$ or $\lambda\mu\leqslant 0$. Hence, in particular, if $\alpha=q=p=p^*_\alpha$, then $u\equiv 0$ for all $\mu\in\mathbb{R}$.

In the special case when also f is a pure power, much more can be deduced.

Theorem 4.11. *Assume $\beta<p$ and $s>1$. Then the equation*

$$\Delta_p u-\lambda|u|^{p-2}u+\mu|x|^{-\alpha}|u|^{q-2}u+\gamma|x|^{-\beta}|u|^{s-2}u=0\quad \text{a.e. in } \mathbb{R}^n \tag{4.16}$$

admits in $H^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ only the trivial solution $u\equiv 0$, whenever either

$$\lambda\neq 0,\quad \lambda\gamma(s-p^*_\beta)\geqslant 0\quad\text{and}\quad \lambda\mu\leqslant 0;\quad\text{or}$$

$\lambda=0$, $[\gamma(s-p^*_\beta)]^2+[\mu(p^*_\alpha-q)]^2>0$ and either $\gamma\mu(p^*_\alpha-q)(s-p^*_\beta)=0$ or

$$\gamma\mu(p^*_\alpha-q)(s-p^*_\beta)\neq 0\quad\text{and}\quad \operatorname{sign}\mu=\operatorname{sign}\big[\gamma(p^*_\beta-s)\big].$$

In particular when $\lambda=0$ and $\alpha=q=p=p^*_\alpha$, then (4.16) admits only the trivial solution $u\equiv 0$ in $H^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ whenever $\gamma(s-p^*_\beta)\neq 0$ and $\mu\in\mathbb{R}$.

Proof. Of course $F(u)=\gamma|u|^s/s=\gamma uf(u)/s$, so that $F\circ u\in L^1_\beta(\mathbb{R}^n)$ by Remark 4.3. Moreover, the identity (4.15) reduces to

$$\lambda pqs\|u\|_p^p+\gamma q(n-p)(s-p^*_\beta)\|u\|^s_{s,\beta}=\mu s(n-p)(p^*_\alpha-q)\|u\|^q_{q,\alpha}$$

for any $\lambda,\mu,\gamma\in\mathbb{R}$ and $\beta<p$, and the result follows at once. □

Next we consider the *doubly critical* equation (1.2), that is (4.12) with $\alpha=q=p$ and $f(u)=|u|^{p^*_\beta-2}u$.

Corollary 4.12. *Assume $\beta<p$. If $u\in H^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ solves (1.2) a.e in $\mathbb{R}^n$, then $u\equiv 0$ for any $\lambda\neq 0$ and $\mu,\gamma\in\mathbb{R}$.*

Proof. In this case $q=p$. Hence, taking into account Remark 4.8, the assertion follows from Theorem 4.11. □

Corollary 4.12 extends to the case $\lambda\neq 0$, $p>1$ and $\beta<p$ a result obtained in [33, Theorem 1.3] for $p=2$ and $\lambda=\beta=0$.

Finally, combining the last part of Lemma 4.1 and Theorem 4.11 we have the following non-existence result.

Corollary 4.13. *Let $u\in D^{1,p}(\mathbb{R}^n)\cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$ satisfy Eq. (1.3) a.e. in $\mathbb{R}^n$ and let $\beta<p$. Then $u\equiv 0$, whenever either $\gamma\leqslant 0$ or $\gamma>0$ and $s\neq p^*_\beta$.*

When the regular periodic function in [47] is zero, Corollary 4.13 extends [47, Theorem 0.1(ii)] due to Terracini in the case $p = 2$ and $\beta = 0$.

In the next section we shall give an existence result for Eq. (1.3) in the critical case, i.e. $s = p^*_\beta$, $\beta < p$, and $\gamma > 0$, in order to complete the discussion for (1.3).

5. A weighted critical equation

We continue to assume that $1 < p < n$. Consider the following weighted critical equation

$$\Delta_p u + \gamma |x|^{-\beta} |u|^{p^*_\beta - 2} u = 0 \quad \text{in } \Omega, \qquad \beta < p,\ p^*_\beta = p\frac{n-\beta}{n-p}, \tag{5.1}$$

where $\gamma > 0$ and $\Omega = \mathbb{R}^n$ if $\beta \leqslant 0$, while $\Omega = \mathbb{R}^n \setminus \{0\}$ if $\beta \in (0, p)$.

When $\beta = 0$, then (5.1) reduces to the classical critical equation

$$\Delta_p u + \gamma |u|^{p^* - 2} u = 0 \quad \text{in } \mathbb{R}^n. \tag{5.2}$$

The existence of a non-trivial solution for (5.2) was considered by many authors which have also given an explicit form of such solution (for the case $p = 2$ see, for instance, [28,44,49] and references therein). When $\gamma > 0$ and $\beta \in [0, p)$ problem (5.1) was studied in several papers for general p (see, for instance, [17,24]) and for $p = 2$ (see, for example, [10,27]).

With respect to the weighted critical equation (5.1) our result is the following:

Theorem 5.1. *Let u be the function defined in* (1.4). *Then*

(i) $u \in D^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *is a positive radial fast decay ground state of* (5.1);
(ii) $u \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$ *solves* (5.1) *pointwise in* $\mathbb{R}^n \setminus \{0\}$;
(iii) $u \in H^{1,p}(\mathbb{R}^n)$ *if and only if* $n > p^2$.

Proof. Clearly $u \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^\infty(\mathbb{R}^n)$ is a radial positive function, since $\gamma > 0$.

The radial form of (5.1) is

$$|u'(r)|^{p-2}\left[(p-1)u''(r) + \frac{n-1}{r}u'(r)\right] + \gamma r^{-\beta} |u(r)|^{p^*_\beta - 2} u(r) = 0 \quad \text{in } \mathbb{R}^+, \tag{5.3}$$

where $r = |x|$. From the definition of u it follows that for any $r \in \mathbb{R}^+$

$$u'(r) = -\frac{n-p}{p-1} u(r) \left[1 + r^{(p-\beta)/(p-1)}\right]^{-1} r^{(1-\beta)/(p-1)},$$
$$u''(r) = \frac{n-p}{(p-1)^2} u(r) \left[1 + r^{(p-\beta)/(p-1)}\right]^{-2} r^{(2-2\beta)/(p-1)} \left\{n - \beta - (n-p)\left[1 + r^{(p-\beta)/(p-1)}\right] r^{-(p-\beta)/(p-1)}\right\},$$

so that, taking into account the value of the constant c, we have that u satisfies (5.3) in $\mathbb{R}^+$.

Of course $|Du| = |u'| \in L^p(\mathbb{R}^n)$ since $1 < p < n$, so that $u \in L^{p^*}(\mathbb{R}^n)$. Hence $u \in D^{1,p}(\mathbb{R}^n)$ is a weak solution of (5.3) by Remark 4.3. Clearly $u(r) \cong c\, r^{-(n-p)/(p-1)}$, and so u is a fast decay ground state. Finally $u \in H^{1,p}(\mathbb{R}^n)$ if and only if $n > p^2$. $\square$

When $\gamma = 1$ and $\beta \in [0, p)$, the explicit solution u in Theorem 5.1 was first given in Theorem 3.1 of [24] by a different argument and approach.

The regularity at $x = 0$ of the solution u given in Theorem 5.1 is expressed in terms of the parameters p and β, as summarized in Table 2 of Section 1.

By means of the main change of variable of [36] (see also [37]), given here simply by $t(r) = \int_0^r s^{-\beta/p} ds = p\, r^{(p-\beta)/p}/(p-\beta)$, Eq. (5.1) is transformed into the equivalent form

$$\left[t^{N-1} |v_t(t)|^{p-2} v_t(t)\right]_t + \gamma t^{N-1} |v(t)|^{p^*_N - 2} v(t) = 0 \quad \text{in } \mathbb{R}^+, \tag{5.4}$$

where $N = p(n-\beta)/(p-\beta)$ and $v(t) = u(r(t))$, being $r(t)$ the inverse of t. When N is an integer, (5.4) is the radial version of

$$\Delta_p v + \gamma |v|^{p_N^*-2} v = 0 \quad \text{in } \mathbb{R}^N.$$

6. The case $\lambda = \mu = 0$

In this section we take $1 < p < n$ and consider (1.1) when $\lambda = \mu = 0$ and the weight function h is a power, that is we treat Eq. (1.5), where $\beta < p$, and $\Omega = \mathbb{R}^n$ if $\beta \leqslant 0$, while $\Omega = \mathbb{R}^n \setminus \{0\}$ if $\beta \in (0, p)$; finally $f : \mathbb{R}_0^+ \to \mathbb{R}$ satisfies condition (F1) of Section 4.

When f is a *pure power*, non-existence for (1.5) is proved in Corollary 4.13, while existence in Theorem 5.1. When f is negative in all $\mathbb{R}_0^+$ then equation (1.5) admits only the trivial solution $u \equiv 0$ by Lemma 4.2 with $h(|x|) = |x|^{-\beta}$.

In this section we shall prove existence of positive radial ground states of (1.5) by means of the constrained minimization method (see [4,15]), when f is not modelled by a pure power, but actually f is negative near the origin and positive at infinity. This is usually called the *normal case* (see [35]).

After the papers of [4,5] related to elliptic problems with the Laplace operator, equations with no weights, that is when $\beta = 0$ in (1.5), involving the p-Laplacian operator in $\mathbb{R}^n$, were treated largely in literature when f is negative near the origin and positive at infinity, see e.g. [13,19,22] for the no weighted case and [11] for general weighted equations.

In this section we introduce the following further condition on f

(F5) *there exist* $a > 0$ *and* $q > 1$ *such that* $\lim_{u \to 0^+} u^{1-q} f(u) = -a$.

Clearly $f(0) = 0$ by (F1) and (F5). Now we give some qualitative properties of *bounded* radial ground states of class $C^1(\mathbb{R}^n \setminus \{0\})$ of Eq. (1.5).

Proposition 6.1. *Assume* (F1)*, with* $f(0) = 0$*. Let* $u \in C^1(\mathbb{R}^n \setminus \{0\})$*, with* $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$*, be a radial ground state which solves* (1.5) *pointwise in* $\mathbb{R}^n \setminus \{0\}$*. Then* $|Du(x)| \to 0$ *as* $|x| \to \infty$*.*

Moreover, if u is locally bounded at $x = 0$, then u is continuous at $x = 0$, $u(0) > 0$ *and* $\langle x, Du(x)\rangle \leqslant 0$ *in* $\mathbb{R}^n \setminus \{0\}$*.*

If in addition (F5) *holds, then u has compact support in* $\mathbb{R}^n$ *if* $1 < q < p$*; while* $u > 0$ *in* $\mathbb{R}^n$ *and* $\langle x, Du(x)\rangle < 0$ *for* $|x|$ *sufficiently large if* $q \geqslant p$*. Furthermore, for any* $q > 1$*, the solution* $u \in C^2(\mathbb{R}^n \setminus B_R)$*, for some* $R > 0$*, and u has the regularity in* $\mathbb{R}^n$*, as described in Table* 1 *of Section* 1*.*

Proof. The radial version of (1.5) is

$$\left[r^{n-1}\left|u'(r)\right|^{p-2}u'(r)\right]' + r^{n-1-\beta} f\left(u(r)\right) = 0 \quad \text{in } \mathbb{R}^+. \tag{6.1}$$

Using the main change of variable of [36] (see also [37]) given here simply by

$$t(r) = \int_0^r s^{-\beta/p}\, ds = \frac{p r^{(p-\beta)/p}}{p-\beta},$$

Eq. (6.1) is transformed into the equivalent form

$$\left[t^{N-1}\left|v_t(t)\right|^{p-2}v_t(t)\right]_t + t^{N-1} f\left(v(t)\right) = 0 \quad \text{in } \mathbb{R}^+, \tag{6.2}$$

where $N = p(n-\beta)/(p-\beta) > 1$, since $\beta < p$, and $v(t) = u(r(t))$, being $r(t)$ the inverse of the transformation t. Clearly $N > p$, since $n > p > \beta$, and when N is an integer, then (6.2) is the radial version of

$$\Delta_p v + f(v) = 0 \quad \text{in } \mathbb{R}^N, \tag{6.3}$$

that is N is the underline dimension of (1.5). Note that $p_N^* = p_\beta^*$, where $1/p_N^* = 1/p - 1/N$ (see Section 4 of [36]).

Define the energy associated to v in this way

$$E(t) = \left|v_t(t)\right|^p/p' + F\left(v(t)\right) \tag{6.4}$$

for any $t \in \mathbb{R}^+$. By Lemma 5.3 of [36], using (6.2), we get

$$E'(t) = -(N-1)\left|v_t(t)\right|^p/t \leqslant 0. \tag{6.5}$$

Hence E is non-increasing in $\mathbb{R}^+$ and so there exists finite $\lim_{t\to\infty} E(t) = \ell \geqslant 0$. Suppose, by contradiction, that $\ell > 0$. Since u is a ground state and $F(0) = 0$, from $|v_t(t)|^p \to p'\ell > 0$, we get an immediate contradiction. Hence $\ell = 0$, i.e.

$$E(t) \to 0, \quad v_t(t) \to 0 \quad \text{as } t \to \infty, \tag{6.6}$$

and so $|Du(x)| \to 0$ as $|x| \to \infty$.

Moreover by (6.5) for $0 < t < s$ we have $E(t) - E(s) = (N-1)\int_t^s (|v_t(\tau)|^p/\tau)\,d\tau$, and so passing to the limit as $s \to \infty$, by (6.6) we get

$$E(t) = (N-1)\int_t^\infty \left(\left|v_t(\tau)\right|^p/\tau\right)d\tau \tag{6.7}$$

for any $t \in \mathbb{R}^+$.

Assume now that u is locally bounded also at $x = 0$. Then by (F1) and the fact that v is ground state of (6.2) locally bounded at zero, we have that $|v_t(t)| \leqslant Ct^{1/(p-1)}$ for any $t \in \mathbb{R}^+$, where C is a positive constant. Hence, $v_t(0) = 0$ and $v \in C^1(\mathbb{R}_0^+)$. Thus by the main change of variable $u \in C^1(\mathbb{R}_0^+)$, with $u'(0) = 0$, when $\beta \leqslant 0$, while $u \in C(\mathbb{R}_0^+)$ in the remaining case $\beta \in (0, p)$.

Suppose, by contradiction, that $u(0) = 0$. Then, by (6.4), (6.7) and the fact that $v(0) = v_t(0) = 0$, we have $0 = F(v(0)) = E(0) = (N-1)\int_0^\infty (|v_t(\tau)|^p/\tau)\,d\tau$, which yields $v_t \equiv 0$ in $\mathbb{R}^+$. This is a contradiction since u is a non-trivial solution. Thus $u(0) = v(0) > 0$ and $F(u(0)) > 0$.

Moreover, by Proposition 5.6 of [36] we have $v_t(t) \leqslant 0$ in $\mathbb{R}^+$, and so $u' \leqslant 0$ in $\mathbb{R}^+$, being $u'(r) = v_t(t(r))t'(r)$, that is $\langle x, Du(x)\rangle \leqslant 0$ in Ω.

Assume now that $f(z) < 0$ for $z > 0$ sufficiently small. Then by Corollary 5.8 of [36], the solution v has compact support in $\mathbb{R}^+$ if and only if $1 < q < p$ and $v > 0$ in $\mathbb{R}^+$ if and only if $q \geqslant p$. Thus u has compact support in $\mathbb{R}^n$ if and only if $1 < q < p$, while $u > 0$ in $\mathbb{R}^n$ if and only if $q \geqslant p$.

Let $q \geqslant p$. We prove that $\langle x, Du(x)\rangle < 0$ for $|x|$ sufficiently large. By assumption there exists the maximal $\delta > 0$ such that $F(u) \leqslant 0$ in $[0, \delta]$. Since $v_t \leqslant 0$ in $\mathbb{R}^+$, v is a ground state of (6.2) and $F(v(0)) > 0$, there exists $t_\delta > 0$ such that $v(t_\delta) = \delta$ and $0 < v(t) \leqslant \delta$ in $[t_\delta, \infty)$. Suppose, by contradiction, that $v_t(t_0) = 0$ for some $t_0 \in [t_\delta, \infty)$. By (6.4) and (6.7) we get

$$0 \leqslant (N-1)\int_{t_0}^\infty \left(\left|v_t(\tau)\right|^p/\tau\right)d\tau = E(t_0) = F\left(v(t_0)\right) \leqslant 0, \tag{6.8}$$

that is $v_t(t) \equiv 0$ in $[t_0, \infty)$. Hence, $v(t) = v(t_0) > 0$ in $[t_0, \infty)$ which contradicts the fact that v is a ground state. Thus, $v_t < 0$ in $[t_\delta, \infty)$ and so $\langle x, Du(x)\rangle < 0$ for $|x| \geqslant r_\delta = t_\delta^{p/(p-\beta)}$. By Corollary 5.2 of [36] we get $v \in C^2([t_\delta, \infty))$, that is $u \in C^2(\Omega_{r_\delta})$.

Of course, also in the case $1 < q < p$ when u is compactly supported in $\mathbb{R}^n$, then $u \in C^2(\Omega_R)$, $\operatorname{supp}(u) \subset B_R$.

Now we prove the regularity of u in the entire $\mathbb{R}^n$. Since u is a ground state of (1.5) locally bounded at zero, using the same notations and arguments as in the proof of Proposition 3.2 it is clear that for all $r > 0$

$$\left|u'(r)\right| \leqslant Cr^{(1-\beta)/(p-1)}, \tag{6.9}$$

for a suitable constant $C > 0$. If $\beta < 1$ then $u \in C^1(\mathbb{R}^n)$ with $Du(0) = \mathbf{0}$. If furthermore $1 < p \leqslant 2$, since $u'(r) \leqslant 0$ in $\mathbb{R}^+$, $u'(0) = 0$ and $|u'|^{p-2}u' \in C^1(\mathbb{R}_0^+)$, we have $u' = -(|u'|^{p-1})^{1/(p-1)} \in C^1(\mathbb{R}_0^+)$, with value 0 at $r = 0$. In other words $u \in C^2(\mathbb{R}^n)$.

Finally, if $1 \leqslant \beta < p$, then $r^{(1-\beta)/(p-1)} \in L^1(0, \delta)$, $\delta > 0$, so that $u \in C^{0,(p-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$ by (6.9). □

Proposition 6.2. *Assume* (F1)*, with* $f(0) = 0$*, and that* $F(z) \leqslant 0$ *whenever* $f(z) = 0$*. Let* $u \in C^1(\mathbb{R}^n \setminus \{0\})$*, with* $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$*, be a bounded radial ground state which solves* (1.5) *pointwise in* $\mathbb{R}^n \setminus \{0\}$*. Then*

$|Du(x)| \to 0$ as $|x| \to \infty$, u is continuous at $x = 0$, with $u(0) > 0$, and with the property that $x \neq 0$ and $u(x) > 0$ implies $\langle x, Du(x)\rangle < 0$, while $x \neq 0$ and $u(x) = 0$ implies that u is compactly supported in $\mathbb{R}^n$ and $Du(x) = \mathbf{0}$.

Finally, if in addition either $f \geqslant 0$ in $[0, u(0)]$ or $f < 0$ in some open interval $(0, \delta)$, $\delta > 0$, and

$$\int_{0^+} \frac{du}{|F(u)|^{1/p}} = \infty, \tag{6.10}$$

then u is positive in $\mathbb{R}^n$, $u \in C^2(\mathbb{R}^n \setminus \{0\})$ and has the regularity in $\mathbb{R}^n$, as described in Table 2 of Section 1.

Proof. The proof of the first part of the proposition is the same of the Proposition 6.1, and so we already know that the corresponding solution $v(t) = u(r(t))$ of (6.2) is such that $v(t) \geqslant 0$ and $v_t(t) \leqslant 0$ in the entire $\mathbb{R}_0^+$, with $v_t(0) = 0$, for all $\beta < p$. For the first part it remains to show that $t_0 > 0$ and $v(t_0) > 0$ imply $v_t(t_0) < 0$. Assume by contradiction that $v_t(t_0) = 0$. Clearly $w = |v_t|^{p-2}v_t$ is zero at t_0 and we claim that $w_t(t_0) = 0$. For otherwise, v_t will change sign at $t = t_0$ contradicting the fact that $v_t(t) \leqslant 0$ in the entire $\mathbb{R}_0^+$. Hence by (6.2)

$$w_t + \frac{N-1}{t} w + f(v) = 0 \quad \text{in } \mathbb{R}^+,$$

so that $f(v(t_0)) = 0$ and by assumption $F(v(t_0)) \leqslant 0$. Now by (6.4) and (6.7) we get again (6.8), that is $v_t \equiv 0$ in $[t_0, \infty)$. In other words $v(t) = v(t_0) > 0$ for all $t \geqslant t_0$, contradicting the fact the $v(t)$ approaches zero as $t \to \infty$. Finally, by Theorem 5.4 of [36] if $v(t_0) = 0$ for some $t_0 > 0$, then $v \equiv 0$ in $[t_0, \infty)$ and so u has compact support in $\mathbb{R}^n$.

In the last part of the proposition we can apply Corollary 5.8 of [36] so that $v > 0$ in $\mathbb{R}_0^+$. Hence, from the argument above, $v_t < 0$ in $\mathbb{R}^+$, and so by Corollary 5.2 of [36] we have $v \in C^2(\mathbb{R}^+)$. Thus u' is negative in $\mathbb{R}^+$, $u \in C^2(\mathbb{R}^+)$ and u solves pointwise

$$\left|u'(r)\right|^{p-2}\left[(p-1)u''(r) + \frac{n-1}{r}u'(r)\right] + r^{-\beta} f\big(u(r)\big) = 0 \quad \text{in } \mathbb{R}^+. \tag{6.11}$$

To prove the regularity of u in the entire $\mathbb{R}^n$ we can argue as in the final part of the proof of Proposition 6.1 (see (6.9)). In particular, if $\beta < 2 - p$ by (6.11) and (6.9) we get $u''(0) = 0$ and so $u \in C^2(\mathbb{R}^n)$. While if $2 - p < \beta < 1$ and $p > 2$, it is easily seen that $u'' \in L^1(0, \delta)$, for some $\delta > 0$ sufficiently small, and in turn $u \in C^{1,(1-\beta)/(p-1)}_{\text{loc}}(\mathbb{R}^n)$. □

The critical power nonlinearity of equation (5.1) verifies all the hypotheses of Proposition 6.2, since in particular both $F(z) \leqslant 0$ whenever $f(z) = 0$ and (6.10) hold. Indeed, the regularity of the explicit solution of the critical equation (5.1) is exactly that described by Table 2 of Proposition 6.2 given in Section 1. Proposition 6.2 extends completely to the general weighted equation (1.5) the regularity established for the critical problem (5.1).

Corollary 6.3. *Assume* (F1) *and* (F5) *with $q \geqslant p$. Then every bounded radial ground state u of class $C^1(\mathbb{R}^n \setminus \{0\})$, with $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$, which solves* (1.5) *pointwise in $\mathbb{R}^n \setminus \{0\}$, is continuous in $\mathbb{R}^n$, with $u(0) > 0$, positive in $\mathbb{R}^n$, $u \in C^2(\mathbb{R}^n \setminus \{0\})$ and has the regularity in $\mathbb{R}^n$ as described in Table 2 of Proposition 6.2 given in Section 1.*

Proof. It is a consequence of Propositions 6.1 and 6.2. □

Proposition 6.4. *Assume* (F1), (F5), *with $q \geqslant p$, and that $f(\bar{u}) \geqslant 0$ for some $\bar{u} > 0$. If*

$$f(u) \geqslant 0 \quad \text{for all } u \geqslant \bar{u} \tag{6.12}$$

holds, then every $C^1(\mathbb{R}^n \setminus \{0\})$ non-trivial non-negative weak solution of (1.5) *is positive in $\mathbb{R}^n \setminus \{0\}$.*

If (6.12) *does not hold, then every $C^1(\mathbb{R}^n \setminus \{0\})$ non-trivial non-negative weak solution of* (1.5), *which is bounded above by $u^* = \min\{u \geqslant \bar{u}\colon f(u) = 0\}$, is positive in $\mathbb{R}^n \setminus \{0\}$.*

Proof. Let (6.12) hold. Then

$$f^-(u) \leqslant Cu^{q-1} \qquad \text{in } \mathbb{R}_0^+, \tag{6.13}$$

for some positive constant C, where $f^-(u) = \max\{-f(u), 0\}$. Indeed, $f^-(u) \leqslant 2au^{q-1}$ in $[0,\delta]$, for some $\delta > 0$, by (F5). Thus by (6.12) condition (6.13) holds with $C = \max\{2a, M\}$, where $M = \max_{u\in[\delta,\bar u]} f^-(u)/u^{q-1} \geqslant 0$. Clearly for $x \in \Omega_r$, $r > 0$, and $u \geqslant 0$ we have

$$|x|^{-\beta}\big[f^+(u) - f^-(u)\big] \geqslant -|x|^{-\beta} f^-(u) \geqslant -C_r u^{q-1},$$

where $C_r = Cr^{-\beta} > 0$. Let u be a $C^1(\mathbb{R}^n \setminus \{0\})$ non-trivial non-negative weak solution of (1.5). Then u is a $C^1(\Omega_r)$ weak solution of $\Delta_p u - C_r u^{q-1} \leqslant 0$. Thus $u > 0$ in Ω_r for all $r > 0$ by the strong maximum principle (see [39,48]). In conclusion, $u > 0$ in $\mathbb{R}^n \setminus \{0\}$.

If (6.12) does not hold, there is $u_1 > \bar u$ such that $f(u_1) < 0$, so that u^* is well defined and $f(u^*) = 0$. Put

$$\tilde f(u) = \begin{cases} f(u), & \text{if } u \in [0, u^*),\\ 0, & \text{if } u \in [u^*, \infty).\end{cases}$$

The function $\tilde f$ verifies conditions (F1) and (F5), with $q \geqslant p$, $\tilde f(\bar u) = f(\bar u) \geqslant 0$, and (6.12). Therefore every $C^1(\mathbb{R}^n \setminus \{0\})$ non-trivial non-negative weak solution of

$$\Delta_p u + |x|^{-\beta}\tilde f(u) = 0 \tag{6.14}$$

is positive in $\mathbb{R}^n \setminus \{0\}$ by the first part of this proposition. The conclusion follows at once since every non-negative $C^1(\mathbb{R}^n \setminus \{0\})$ weak solution of (1.5) which is bounded above by u^* is a non-negative weak solution of (6.14). □

Proposition 6.5. *If* (F1) *and* (F5) *hold while* (6.12) *does not, then any ground state $u \in D^{1,p}(\mathbb{R}^n)$ of* (6.14) *is a ground state of* (1.5) *bounded above by u^*, where u^* is given in Proposition* 6.4.

Proof. Indeed, if $u \in D^{1,p}(\mathbb{R}^n)$ is a ground state of (6.14), then $u \leqslant u^*$ a.e. in $\mathbb{R}^n \setminus B_R$ for some $R > 0$ sufficiently large. Hence, $\operatorname{supp}(u - u^*)^+ \subset B_R$ and $(u - u^*)^+ \in D^{1,p}(\mathbb{R}^n)$. Since u is a weak solution of (6.14), taking $\varphi = (u - u^*)^+$ as a test function, by the definition of $\tilde f$, we obtain

$$\int_{\mathbb{R}^n} \big|D(u-u^*)^+\big|^p\,dx = \int_{\mathbb{R}^n} |Du|^{p-2}\big\langle Du, D(u-u^*)^+\big\rangle\,dx = \int_{\mathbb{R}^n} |x|^{-\beta}\tilde f(u)(u-u^*)^+ dx = 0.$$

That is $D(u - u^*)^+ = \mathbf{0}$, in other words $(u - u^*)^+ = 0$, since $(u - u^*)^+ \in L^{p^*}(\mathbb{R}^n)$. Thus $u \leqslant u^*$ a.e. in $\mathbb{R}^n$. Using the definition of $\tilde f$, it is easily seen that u is a ground state of (1.5). □

In the following we denote by $D^{1,p}_N(\mathbb{R}^+)$, $N > 1$, the closure, with respect to the norm

$$\|v'\|_{p,N} = \left(\int_0^\infty t^{N-1}\big|v'(t)\big|^p\,dt\right)^{1/p},$$

of the space $\mathcal{A} = \{v \in C^1(\mathbb{R}^+_0)\colon\ v = 0 \text{ in } [R_v,\infty), \text{ for some } R_v > 0, \text{ and } v'(0) = 0\}$.

Lemma 6.6. *Let $1 < p < N$ and $1/p^*_N = 1/p - 1/N$. Then the embedding $D^{1,p}_N(\mathbb{R}^+) \hookrightarrow L^{p^*_N}_N(\mathbb{R}^+)$ is continuous. Moreover, if $v \in D^{1,p}_N(\mathbb{R}^+)$, then*

$$\big|v(t)\big| \leqslant \big[(p-1)/(N-p)\big]^{1/p'}\|v'\|_{p,N}\,t^{-(N-p)/p} \quad \text{a.e. in } \mathbb{R}^+,$$

$t^{(N-p)/p}v(t) \to 0$ as $t \to \infty$ and $v \in C^{0,1/p'}([T,\infty))$ for any $T > 0$.

Proof. The continuity of the embedding of $D^{1,p}_N(\mathbb{R}^+)$ in $L^{p^*_N}_N(\mathbb{R}^+)$ is proved in [29, Theorem 4.45] for more general weights, and is also a particular case of the inequality given in [9]. The second part can be proved as for Lemma 2.9. □

Note that if (F1) is valid and F is positive at some point $\hat u > 0$, then condition

(F6) *there exists* $\bar u > 0$ *such that* $f(\bar u) \geqslant 0$ *and* $F(\bar u) > 0$

holds. Indeed, if $f(\hat u) \geqslant 0$ we are done, otherwise by (F1) we can define

$$\bar u = \inf\{v > 0:\ f(u) < 0 \text{ for any } u \in (v, \hat u]\}.$$

Clearly $\bar u > 0$, otherwise $F(\hat u) \leqslant 0$. Moreover, $f(\bar u) = 0$ and $F(\bar u) = F(\hat u) - \int_{\bar u}^{\hat u} f(t)\,dt > 0$. Letting

$$u_0 = \inf\{v > 0:\ F(v) > 0\}, \tag{6.15}$$

if (F1), (F5) and (F6) hold, then $F(u_0) = 0$ and $0 < u_0 < \bar u$.

Finally let us introduce the natural subcritical assumption

(F7) $\lim_{u\to\infty} u^{1-p_\beta^*} f(u) = 0$; $p_\beta^* = p(n-\beta)/(n-p)$, $1 < p < n$.

Theorem 6.7. *Assume that* (F1), (F5)–(F7) *hold. Then, Eq.* (1.5)*, with* $\beta < p$*, admits a radial ground state* $u \in D_{\mathrm{rad}}^{1,p}(\mathbb{R}^n) \cap L_\beta^q(\mathbb{R}^n)$ *bounded above by* $\bar u$*. Moreover,*

(i) $u \in C_{\mathrm{loc}}^{1,\theta}(\mathbb{R}^n \setminus \{0\})$ *for some* $\theta \in (0,1)$;
(ii) $|Du|^{p-2}Du \in C^1(\mathbb{R}^n \setminus \{0\})$ *and* u *solves* (1.5) *pointwise in* $\mathbb{R}^n \setminus \{0\}$;
(iii) $|Du(x)| \to 0$ *as* $|x| \to \infty$ *and* $|Du(x)| = \mathrm{O}(|x|^{-(n-1)/(p-1)})$ *as* $|x| \to 0$;
(iv) u *is continuous at* $x = 0$, $\langle x, Du(x)\rangle \leqslant 0$ *in* $\mathbb{R}^n \setminus \{0\}$*, and* $\|u\|_\infty = u(0) \in (u_0, \bar u]$;
(v) *if* $1 < p \leqslant 2$*, then* $u \in H_{\mathrm{loc}}^{2,p}(\mathbb{R}^n \setminus \{0\})$*; if furthermore* $\beta < n/p'$*, then* $u \in H_{\mathrm{loc}}^{2,p}(\mathbb{R}^n)$.

If $1 < q < p$*, then* u *is compactly supported in* $\mathbb{R}^n$*, and of course is a fast decay solution of* (1.5) *of class* $H^{1,p}(\mathbb{R}^n)$*. Furthermore,* u *has the regularity in* $\mathbb{R}^n$ *as described in Table* 1 *of Proposition* 6.1 *given in Section* 1.

Finally, if $q \geqslant p$*, then* u *is positive in* $\mathbb{R}^n$, $u \in C^2(\mathbb{R}^n \setminus \{0\})$ *and* u *has the regularity in* $\mathbb{R}^n$ *as described in Table* 2 *of Proposition* 6.2 *given in Section* 1*. Moreover,* u *is a fast decay solution of* (1.5) *and, in particular* $r^{(n-p)/(p-1)}u$ *is decreasing in* $[R,\infty)$*, for* R *sufficiently large and approaches a limit* $\ell \geqslant 0$ *as* $r \to \infty$*. When* $\ell > 0$ *then* $u \in H^{1,p}(\mathbb{R}^n)$ *if and only if* $n > p^2$*; while if* $\ell = 0$ *and* $n > p^2$ *then* $u \in H^{1,p}(\mathbb{R}^n)$.

Proof. We extend f in (1.5) in the entire $\mathbb{R}$ as an odd function.

Condition (F5) and the fact that f is odd imply that $\lim_{u\to 0^-} |u|^{1-q} f(u) = a$. Moreover, by (F1), (F5), (F7) and the definition of F, it easily follows that there exist positive constants $\delta(a)$, k_1 and k_2 such that for any $u \in \mathbb{R}$

$$|f(u)| \leqslant k_1\big(|u|^{q-1} + |u|^{p_\beta^*-1}\big), \tag{6.16}$$

$$F(u) \leqslant -\frac{a}{2q}|u|^q + \delta(a)|u|^{p_\beta^*}, \qquad |F(u)| \leqslant k_2\big(|u|^q + |u|^{p_\beta^*}\big). \tag{6.17}$$

Since we are interested in the existence of radial solutions of (1.5), arguing as in the proof of Proposition 6.1, we use the main change of variable of [36] and consider equation (6.2) where $N = p(n-\beta)/(p-\beta)$ and $v(t) = u(r(t))$, being $r(t)$ the inverse of t. Note that $N > p$, since $\beta < p < n$.

Now we shall study (6.2) by means of the constrained minimization method (see [4,13,15,19]). Consider the functionals $\mathcal{T} : D_N^{1,p}(\mathbb{R}^+) \to \mathbb{R}$ and $\mathcal{F} : D_N^{1,p}(\mathbb{R}^+) \to \mathbb{R} \cup \{\infty\}$, defined by

$$\mathcal{T}(v) = \|v_t\|_{p,N}^p/p, \qquad \mathcal{F}(v) = \int_0^\infty t^{N-1} F\big(v(t)\big)\,dt.$$

Clearly $\mathcal{T}$ is well-defined, while $\mathcal{F}$ may not be finite in $D_N^{1,p}(\mathbb{R}^+)$.

Let $\mathcal{M} = \{v \in D_N^{1,p}(\mathbb{R}^+):\ F \circ v \in L_N^1(\mathbb{R}^+),\ \mathcal{F}(v) = 1 \text{ and } |v| \leqslant \bar u \text{ a.e. in } \mathbb{R}^+\}$, where $\bar u$ is the number given in (F6). First of all we prove that $\mathcal{M}$ is not empty. Indeed, let $T > 0$ and define

$$v_T(t) = \begin{cases} \bar u & \text{if } t < T, \\ (T+1-t)\bar u & \text{if } T \leqslant t < T+1, \\ 0 & \text{if } t \geqslant T+1. \end{cases}$$

The function $v_T \in D_N^{1,p}(\mathbb{R}^+) \cap L_N^q(\mathbb{R}^+)$ being $N > 1$. Then, $F(v_T) \in L_N^1(\mathbb{R}^+)$ by (6.17) and Lemma 6.6. Furthermore,

$$\mathcal{F}(v_T) = \int_0^T t^{N-1} F(\bar{u})\,dt + \int_T^{T+1} t^{N-1} F\big(v_T(t)\big)\,dt \geqslant \big\{F(\bar{u})T^N - \overline{F}\big(T^{N+1} - T^N\big)\big\}/N,$$

where $\overline{F}$ is the maximum of $|F|$ on $[0, \bar{u}]$. Passing to the limit as $T \to \infty$, we get $\mathcal{F}(v_T) \to \infty$, since $F(\bar{u}) > 0$ by (F6), $(T+1)^N - T^N \sim NT^{N-1}$ and $N > 1$. Thus, by the continuity of $\mathcal{F}$ on its effective domain and the fact that $\mathcal{F}(0) = 0$, it is clear that $\mathcal{M}$ is not empty.

We claim that $\mathcal{M}$ is a regular manifold. Let us proceed by contradiction and suppose that there exists $v \in \mathcal{M}$ such that $\langle d\mathcal{F}(v), \varphi\rangle = \int_0^\infty t^{N-1} f(v(t))\varphi(t)\,dt = 0$ for all $\varphi \in D_N^{1,p}(\mathbb{R}^+)$. Hence $f \circ v = 0$ a.e. in $\mathbb{R}^+$. Integrating by parts on $[\varepsilon, T]$, $0 < \varepsilon < T$, we get

$$0 = -\int_\varepsilon^T t^N f\big(v(t)\big) v_t(t)\,dt = N \int_\varepsilon^T t^{N-1} F\big(v(t)\big)\,dt - F\big(v(T)\big)T^N + F\big(v(\varepsilon)\big)\varepsilon^N. \tag{6.18}$$

There exists a sequence $(T_k)_k$ going to infinity as $k \to \infty$ such that $F(v(T_k))T_k^N \to 0$ as $k \to \infty$. Indeed, otherwise $\liminf_{k\to\infty} |F(v(T_k))|T_k^N = \ell > 0$, that is $|F(v(t))|t^N \geqslant \ell/2$ for all $t > T_\ell > 0$, and so $\int_0^\infty t^{N-1}|F(v(t))|\,dt \geqslant (\ell/2)\int_{T_\ell}^\infty dt/t = \infty$, which is an obvious contradiction, since $F \circ v \in L_N^1(\mathbb{R}^+)$. Arguing in the same way we show that there exists a sequence $(\varepsilon_k)_k$ going to zero as $k \to \infty$ such that $F(v(\varepsilon_k))\varepsilon_k^N \to 0$ as $k \to \infty$. Taking $\varepsilon = \varepsilon_k$ and $T = T_k$ in (6.18) and passing to the limit as $k \to \infty$, we obtain

$$0 = -\int_0^\infty t^N f\big(v(t)\big) v_t(t)\,dt = N\mathcal{F}(v) = N > 1,$$

since $v \in \mathcal{M}$. This contradiction proves the claim.

Let $(v_k)_k$ be a minimizing sequence for $\mathcal{T}$ on $\mathcal{M}$, that is $v_k \in \mathcal{M}$ for any $k \in \mathbb{N}$ and $\mathcal{T}(v_k) \to \inf_{\mathcal{M}} \mathcal{T}(\geqslant 0)$ as $k \to \infty$. Without loss of generality, we can suppose that v_k is non-negative. Indeed, $\mathcal{T}(v_k) = \mathcal{T}(|v_k|)$ and $|v_k| \in \mathcal{M}$ whenever $v_k \in \mathcal{M}$, being F even. Clearly

$$\sup_k \|v_{k,t}\|_{p,N} < \infty, \quad v_{k,t} = dv_k/dt. \tag{6.19}$$

Thus, there exists $v \in D_N^{1,p}(\mathbb{R}^+)$ such that, up to a subsequence, $v_{k,t} \rightharpoonup v_t$ in $L_N^p(\mathbb{R}^+)$ as $k \to \infty$ and, by Corollary 8.7 of [32],

$$v_k \to v \geqslant 0 \quad \text{a.e. in } \mathbb{R}^+ \tag{6.20}$$

as $k \to \infty$. Of course $v \leqslant \bar{u}$ a.e. in $\mathbb{R}^+$.

The sequence $(v_k)_k$ is bounded in $L_N^q(\mathbb{R}^+)$. Indeed, by (6.17) and the fact that $v_k \in \mathcal{M}$, it results

$$1 = \mathcal{F}(v_k) \leqslant -\frac{a}{2q}\|v_k\|_{q,N}^q + \delta(a)\|v_k\|_{p_N^*,N}^{p_N^*}$$

for any $k \in \mathbb{N}$. Hence, by Lemma 6.6 and (6.19) we have

$$\sup_k \|v_k\|_{q,N} < \infty. \tag{6.21}$$

Furthermore, by (6.19) and Hőlder's inequality

$$\begin{aligned}\mathcal{T}(v) &= \frac{1}{p}\lim_{k\to\infty}\int_0^\infty t^{N-1}\big|v_t(t)\big|^{p-2} v_t(t) v_{k,t}(t)\,dt \leqslant \big[\mathcal{T}(v)\big]^{1/p'} \lim_{k\to\infty}\big[\mathcal{T}(v_k)\big]^{1/p} \\ &= \big[\mathcal{T}(v)\big]^{1/p'}\Big[\inf_{\mathcal{M}} \mathcal{T}\Big]^{1/p},\end{aligned} \tag{6.22}$$

that is $\mathcal{T}(v) \leqslant \inf_{\mathcal{M}} \mathcal{T}$.

We claim that $v \in \mathcal{M}$. First of all we show that

$$\lim_{k\to\infty} \int_0^\infty t^{N-1} F^+\big(v_k(t)\big)\, dt = \int_0^\infty t^{N-1} F^+\big(v(t)\big)\, dt, \tag{6.23}$$

where $g^+(u) = \max\{g(u), 0\}$. By (F7) for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$F^+(u) \leqslant \varepsilon |u|^{p^*_\beta} \quad \text{for any } |u| > M_\varepsilon. \tag{6.24}$$

Moreover, $F^+(u) = 0$ in $(0, \delta]$, for some $\delta > 0$, by (F1) and (F5). Thus, by Lemma 6.6, for some $T > 0$ we get $F^+(v_k(t)) = 0$ for all $t \geqslant T$, and also $F^+(v(t)) = 0$ a.e. in $[T, \infty)$, by (6.20). Since $t^{N-1}F^+(v_k(t)) \to t^{N-1}F^+(v(t))$ a.e. in $[0, T]$ as $k \to \infty$, by Egoroff's theorem, there exists a measurable set $E \subset [0, T]$ such that $t^{N-1}F^+(v_k(t)) \to t^{N-1}F^+(v(t))$ uniformly in E as $k \to \infty$ and $2|\widetilde{E}|T^{N-1} \max_{v\in[0,M_\varepsilon]} F^+(v) < \varepsilon$, where $\tilde{E} = [0, T] \setminus E$. Hence,

$$\lim_{k\to\infty} \int_E t^{N-1} F^+\big(v_k(t)\big)\, dt = \int_E t^{N-1} F^+\big(v(t)\big)\, dt \tag{6.25}$$

and, denoting by $V_k = \{t \in [0, T]:\ v_k(t) \leqslant M_\varepsilon\}$, $V^k = \{t \in [0, T]:\ v_k(t) > M_\varepsilon\}$, $V_\infty = \{t \in [0, T]:\ v(t) \leqslant M_\varepsilon\}$, and $V^\infty = \{t \in [0, T]:\ v(t) > M_\varepsilon\}$, we have

$$\begin{aligned}\left| \int_{\widetilde{E}} t^{N-1} F^+\big(v_k(t)\big)\, dt - \int_{\widetilde{E}} t^{N-1} F^+\big(v(t)\big)\, dt \right| &\leqslant \int_{\widetilde{E}\cap V_k} t^{N-1} F^+\big(v_k(t)\big)\, dt + \int_{\widetilde{E}\cap V^k} t^{N-1} F^+\big(v_k(t)\big)\, dt \\ &\quad + \int_{\widetilde{E}\cap V_\infty} t^{N-1} F^+\big(v(t)\big)\, dt + \int_{\widetilde{E}\cap V^\infty} t^{N-1} F^+\big(v(t)\big)\, dt \\ &\leqslant \varepsilon\big(1 + \|v_k\|^N_{p^*_N,N} + \|v\|^N_{p^*_N,N}\big) \leqslant \varepsilon c, \end{aligned} \tag{6.26}$$

where c is a positive constant independent of k by (6.19), (6.24) and Lemma 6.6. Thus (6.23) follows from (6.25) and (6.26). Now, by the Fatou lemma and (6.20)

$$\liminf_{k\to\infty} \int_0^\infty t^{N-1} F^-\big(v_k(t)\big)\, dt \geqslant \int_0^\infty t^{N-1} F^-\big(v(t)\big)\, dt \geqslant 0, \tag{6.27}$$

where $g^-(u) = \max\{-g(u), 0\}$. Therefore,

$$\begin{aligned}\int_0^\infty t^{N-1} F^-\big(v(t)\big)\, dt &\leqslant \liminf_{k\to\infty} \int_0^\infty t^{N-1} F^-\big(v_k(t)\big)\, dt \\ &= \lim_{k\to\infty} \left(\int_0^\infty t^{N-1} F^+\big(v_k(t)\big)\, dt - \mathcal{F}(v_k) \right) \\ &= \int_0^\infty t^{N-1} F^+\big(v(t)\big)\, dt - 1, \end{aligned}$$

by (6.23) and the fact that $\mathcal{F}(v_k) = 1$ for any $k \in \mathbb{N}$. Hence, by (6.27) and (6.23), we have $F \circ v \in L^1_N(\mathbb{R}^+)$ and $\mathcal{F}(v) \geqslant 1$. Suppose, by contradiction, that $\mathcal{F}(v) > 1$. Let $v_\sigma(t) = v(t/\sigma)$, with $\sigma > 0$. Taking $\sigma = [\mathcal{F}(v)]^{-1/(N-1)} < 1$ we have $\mathcal{F}(v_\sigma) = \sigma^{N-1}\mathcal{F}(v) = 1$, that is $v_\sigma \in \mathcal{M}$, but $\mathcal{T}(v_\sigma) = \sigma^{N-1}\mathcal{T}(v) < \mathcal{T}(v) \leqslant \inf_{\mathcal{M}} \mathcal{T}$, by (6.22). This is a contradiction, so that $\mathcal{F}(v) = 1$ and v is non-trivial. Hence the claim is proved, i.e. $v \in \mathcal{M}$. By (6.17)

$$\|v\|^q_{q,N} \leqslant -\mathcal{F}(v) + \|v\|^{p^*_N}_{p^*_N,N} < \infty,$$

and so $v \in L^q_N(\mathbb{R}^+)$.

By (6.22) and $v \in \mathcal{M}$ we deduce that $\mathcal{T}(v) = \inf_{\mathcal{M}} \mathcal{T}$, i.e. v is a minimum for $\mathcal{T}$ on $\mathcal{M}$. Since $\mathcal{T} \in C^1(D_N^{1,p}(\mathbb{R}^+))$ and $\mathcal{F} \in C^1(\mathcal{M})$, there exists a Lagrange multiplier $\vartheta \in \mathbb{R}$ such that

$$\int_0^\infty t^{N-1}\left|v_t(t)\right|^{p-2} v_t(t)\varphi'(t)\,dt = \vartheta \int_0^\infty t^{N-1} f\big(v(t)\big)\varphi(t)\,dt \tag{6.28}$$

for any $\varphi \in D_N^{1,p}(\mathbb{R}^+)$.

Let us show that $\vartheta > 0$. If $\vartheta = 0$, taking $\varphi = v$ in (6.28), we get $v = 0$ by the fact that $v \in D_N^{1,p}(\mathbb{R}^+)$. This is impossible, since $v \in \mathcal{M}$.

Let us suppose by contradiction that $\vartheta < 0$. Since $\mathcal{M}$ is a regular manifold, there exists $w \in D_N^{1,p}(\mathbb{R}^+)$ such that $\langle d\mathcal{F}(v), w\rangle = \int_0^\infty t^{N-1} f(v(t))w(t)\,dt > 0$. By density we can take $w \in \mathcal{A}$. Let $\tilde{w} = (\bar{u} - v)w/\|w\|_\infty$. Note that $\tilde{w} \in D_N^{1,p}(\mathbb{R}^+) \cap L_N^q(\mathbb{R}^+)$, $\tilde{w} \neq 0$ a.e. in $\mathbb{R}^+$, since $v \neq \bar{u}$ a.e. in $\mathbb{R}^+$, being $v \in D_N^{1,p}(\mathbb{R}^+)$, and finally sign $\tilde{w} =$ sign w, being $v \leqslant \bar{u}$ a.e. in $\mathbb{R}^+$. Therefore

$$\big\langle d\mathcal{F}(v), \tilde{w}\big\rangle = \int_0^\infty t^{N-1} f\big(v(t)\big)\tilde{w}(t)\,dt > 0. \tag{6.29}$$

Let $z_\varepsilon = v + \varepsilon\tilde{w}$, with $\varepsilon \in (0,1]$. By the choice of $\tilde{w}$ and the fact that $v \geqslant 0$ we get $|z_\varepsilon| \leqslant \bar{u}$ a.e. in $\mathbb{R}^+$. Clearly the segment $[v, z_\varepsilon]$ is in $D_N^{1,p}(\mathbb{R}^+) \cap L_N^q(\mathbb{R}^+)$ and so $F \circ z_\varepsilon \in L_N^1(\mathbb{R}^+)$ for any $\varepsilon \in (0,1]$ by (6.17). Moreover, $\mathcal{F}(z_\varepsilon) = \mathcal{F}(v) + \varepsilon\langle d\mathcal{F}(v), \tilde{w}\rangle + \mathrm{o}(\varepsilon)$ as $\varepsilon \to 0^+$. Hence, by (6.28) with $\varphi = \tilde{w}$, we get

$$\mathcal{T}(z_\varepsilon) = \mathcal{T}(v) + \varepsilon\big\langle d\mathcal{T}(v), \tilde{w}\big\rangle + \mathrm{o}(\varepsilon) = \mathcal{T}(v) + \varepsilon\vartheta\big\langle d\mathcal{F}(v), \tilde{w}\big\rangle + \mathrm{o}(\varepsilon).$$

By (6.29) we can choose $\varepsilon > 0$ so small that

$$\mathcal{F}(z_\varepsilon) > \mathcal{F}(v) = 1 \quad \text{and} \quad \mathcal{T}(z_\varepsilon) < \mathcal{T}(v) = \inf_{\mathcal{M}} \mathcal{T}. \tag{6.30}$$

On the other hand, $\mathcal{F}$ is finite and continuous on the segment $[0, z_\varepsilon]$ of $D_N^{1,p}(\mathbb{R}^+) \cap L_N^q(\mathbb{R}^+)$, hence by (6.30) and the fact that $\mathcal{F}(0) = 0$ there exists $\tau \in (0,1)$ such that $\mathcal{F}(\tau z_\varepsilon) = 1$, and so $\tau z_\varepsilon \in \mathcal{M}$, since clearly also $|\tau z_\varepsilon| \leqslant \bar{u}$ a.e. in $\mathbb{R}^+$. Moreover, $\mathcal{T}(\tau z_\varepsilon) = \tau^p\mathcal{T}(z_\varepsilon) < \mathcal{T}(z_\varepsilon) < \mathcal{T}(v) = \inf_{\mathcal{M}} \mathcal{T}$, which is impossible being $\tau z_\varepsilon \in \mathcal{M}$. In conclusion $\vartheta > 0$, as claimed.

By (6.28) the non-trivial non-negative function $\hat{v}(t) = v(\sigma t)$, with $\sigma = \vartheta^{1/(N-1)} > 0$, verifies

$$\int_0^\infty t^{N-1}\left|\hat{v}_t(t)\right|^{p-2}\hat{v}_t(t)\varphi'(t)\,dt = \int_0^\infty t^{N-1} f\big(\hat{v}(t)\big)\varphi(t)\,dt,$$

i.e. $\hat{v}$ is a non-trivial non-negative radial weak solution of (6.2) bounded above by $\bar{u}$.

Then $u(r) = \hat{v}(t(r))$ is a solution of (6.1), that is $u \in D^{1,p}(\mathbb{R}^n) \cap L_\beta^q(\mathbb{R}^n)$ is a non-trivial non-negative bounded radial weak solution of (1.5) in $\mathbb{R}^n$. Moreover, $|x|^{(n-p)/p}u(x) \to 0$ as $|x| \to \infty$ by Lemma 2.9, and so u is a *bounded* radial ground state of (1.5).

Define $g(r,u) = -|r|^{-\beta} f(u)$ in $\mathbb{R}^+ \times \mathbb{R}$. The constructed ground state u above is in $L_\beta^q(\mathbb{R}^n) \cap L_\beta^{p_\beta^*}(\mathbb{R}^n)$ since $v \in L_N^q(\mathbb{R}^+) \cap L_N^{p_N^*}(\mathbb{R}^+)$, and so $g(\cdot, u(\cdot)) \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$ by (6.16). Thus [41, Theorem 3.2] applies and so (i) and (ii) hold. The other regularity properties of u now follow from Proposition 6.1 and Corollary 6.3. In particular by Proposition 6.1 we have that $u(0) > 0$, $F(u(0)) > 0$ (see the proof of Proposition 6.1) and $\langle x, Du(x)\rangle \leqslant 0$ in $\mathbb{R}^n \setminus \{0\}$. Hence, $u_0 < u(0) = \|u\|_\infty \leqslant \bar{u}$.

Furthermore, $g(\cdot, u(\cdot)) \in L_{\mathrm{loc}}^{p'}(\mathbb{R}^n \setminus \{0\})$ being $u \in C(\mathbb{R}^n)$, and $g(\cdot, u(\cdot)) \in L_{\mathrm{loc}}^{p'}(\mathbb{R}^n)$ if $\beta < n/p'$, since $|g(r,u)| \leqslant Cr^{-\beta}$ in $\mathbb{R}^+ \times \mathbb{R}$ for a suitable positive constant C. Hence [41, Theorem 2.5] applies when $1 < p \leqslant 2$ and so (v) is proved.

When $1 < q < p$, since u is continuous at $x = 0$ and compactly supported in $\mathbb{R}^n$ by Proposition 6.1, then clearly u is a fast decay solution of (1.5) and $u \in H^{1,p}(\mathbb{R}^n)$. If $p \leqslant q < p_\beta^*$, then $[r^{n-1}|u'(r)|^{p-1}]' = r^{n-1-\beta} f(u(r)) < 0$ for all r sufficiently large by (6.1), (F5) and the strong maximum principle. Arguing as in the proof of Theorem 3.1, we prove that $r^{(n-p)/(p-1)}u(r)$ decreases to $\ell \geqslant 0$ as $r \to \infty$ and so u is a fast decay solution of (1.5). If $\ell \neq 0$, by direct calculation $u \in H^{1,p}(\mathbb{R}^n)$ if and only if $n > p^2$. While, if $\ell = 0$, then $0 \leqslant u(r) \leqslant Cr^{-(n-p)/(p-1)}$ for all r sufficiently large and a suitable positive constant C. Therefore $u \in H^{1,p}(\mathbb{R}^n)$ if $n > p^2$. $\square$

From the proof of Theorem 6.7 it is clear that actually we work with (6.2). When N is an integer, then (6.2) is the radial version of

$$\Delta_p v + f(v) = 0 \quad \text{in } \mathbb{R}^N,$$

that is the equation studied by Citti in [13] by means of the same constrained minimization method.

Theorem 6.7 extends the results given in [22] for the no weighted version of (1.5) when locally Lipschitz continuity on f is assumed. Theorem 6.7 extends to the weighted case also the existence results obtained by Berestycki and Lions in [4] when $p = 2$, and by Citti in [13] for general $p > 1$ (see also [19]).

The regularity results given in Theorem 6.7 extend to the general nonlinear weighted equation (1.5) in the normal case the regularity established for the critical problem (5.1) when $\gamma > 0$ and the explicit ground state is known (see Theorem 5.1). We also improve the regularity proved by Citti (see Remarks 1.2 and 1.3 of [13]) in the no weighted case $\beta = 0$.

Remark 6.8. Even if in the proof of Theorem 6.7 condition (F6) is used only to show that the regular manifold $\mathcal{M}$ is not empty, (F6) is necessary for the existence of weak solutions u for (1.5) of class $D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$, with $F \circ u \in L^1_\beta(\mathbb{R}^n)$, $\beta < p$. Indeed, by Lemma 4.2 with $h(|x|) = |x|^{-\beta}$, we have

$$(n-p)\|Du\|_p^p = p(n-\beta)\|F \circ u\|_{1,\beta}.$$

Then (1.5) has only the trivial solution if $F(u) \leqslant 0$ for any $u \in \mathbb{R}$, since $1 < p < n$.

The regularity of the solution u constructed in Theorem 6.7 when $p \leqslant q < p^*_\beta$ coincides for the main parts with the regularity of the explicit solution (1.4) of (5.1), see Theorem 5.1 and the related Table 2 of Section 1.

An interesting model for f is when f is of *polynomial* type, e.g.

$$f(u) = -a|u|^{q-2}u - b|u|^{l-2}u + c|u|^{s-2}u, \quad a \geqslant 0,\ b \geqslant 0,\ c > 0,\ a + b > 0. \tag{6.31}$$

In this case (F1), (F5)–(F7) are satisfied provided $1 < q \leqslant l < s < p^*_\beta$, and so Theorem 6.7 applies. When f is as in (6.31), in order to apply Theorem 6.7 we need that its growth exponent at zero is $q < p^*_\beta$, but there are functions verifying (F1), (F5)–(F7) whose growth in zero is critical or supercritical. For example,

$$f(u) = \begin{cases} -qu^{q-1} & \text{if } u \in [0, \tilde{u}],\ \tilde{u} > 0, \\ q\tilde{u}^{q-2}(u - 2\tilde{u}) & \text{if } u \in (\tilde{u}, 2\tilde{u}), \\ s(u - 2\tilde{u})^{s-1} & \text{if } u \in [2\tilde{u}, \infty) \end{cases}$$

verifies (F1), (F5)–(F7) for all $q > 1$ and $s \in (1, p^*_\beta)$. Thus, in particular, Theorem 6.7 applies for such f also when $q \geqslant p^*_\beta$.

Clearly extending f as an odd function, then $-u$, where u is given by Theorem 6.7, is a non-trivial non-positive weak solution of (1.5) which tends to zero as $|x| \to \infty$.

By Lemma 4.6 with $h(|x|) = |x|^{-\beta}$, $\beta < p$, the solutions $u \in D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ of (1.5) such that $F \circ u \in L^1_\beta(\mathbb{R}^n)$ verifies the following identity

$$\int_{\mathbb{R}^n} \big[u(x) f\big(u(x)\big) - p^*_\beta F\big(u(x)\big)\big]|x|^{-\beta}\,dx = 0. \tag{6.32}$$

By Theorem 6.7 and (6.32) we finally have

Theorem 6.9. *Consider* (1.5)*, with f given by* (6.31)*, where q, l, $s > 1$ and $\beta < p$.*

If $1 < q \leqslant l < s < p^*_\beta$ then (1.5)*,* (6.31) *admits a radial continuous ground state u of class $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \cap L^q_\beta(\mathbb{R}^n)$, with $\|u\|_\infty = u(0) \in (u_0, \bar{u}]$, where u_0 is given in* (6.15) *and $\bar{u}$ is any number verifying*

$$\bar{u} > \begin{cases} C^{1/(s-q)} \geqslant u_0, & \text{if } 0 < C \leqslant 1, \\ C^{1/(s-l)} \geqslant u_0, & \text{if } C \geqslant 1, \end{cases} \quad \text{and } C = s\frac{al+bq}{cql} > 0. \tag{6.33}$$

Moreover, u has the regularity as stated in Theorem 6.7 *and, if* $1 < q < p$ *the solution u is compactly supported in* $\mathbb{R}^n$*, while if* $q \geqslant p$ *the solution u is positive in* $\mathbb{R}^n$.

On the other hand, (1.5), (6.31) *admits in* $D^{1,p}(\mathbb{R}^n) \cap H^{2,p}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ *only the trivial solution* $u \equiv 0$*, whenever* $(q - p^*_\beta)(l - p^*_\beta) \geqslant 0$ *and either*

$$s = p^*_\beta \quad \text{and} \quad (q - p^*_\beta) + (l - p^*_\beta) \neq 0; \quad \text{or}$$
$$s \neq p^*_\beta \quad \text{and} \quad (s - p^*_\beta)\big[(q - p^*_\beta) + (l - p^*_\beta)\big] \leqslant 0.$$

In particular, when $1 < q \leqslant l < s$*, then* (1.5), (6.31) *admits a bounded radial continuous ground state when* $p^*_\beta > s$ *and only the trivial solution* $u \equiv 0$ *when* $p^*_\beta \in [l, s]$*. The case* $p^*_\beta \in (p, l)$ *is left open.*

Furthermore, if $l = q$ *then* (1.5), (6.31) *admits a bounded radial continuous ground state u when* $1 < q = l < s < p^*_\beta$*, with* $\|u\|_\infty = u(0) \in (C^{1/(s-q)}, \bar{u}]$ *and* $C = s(a+b)/cq$*; while only the trivial solution* $u \equiv 0$ *when* $p^*_\beta \in [q, s]$*. It remains an open problem whether there are solutions of* (1.5), (6.31) *when* $p < p^*_\beta < q < s$*. On the other hand, the case* $l = q = p$ *is completely treated, that is* (1.5), (6.31) *admits a bounded radial ground state when* $p < s < p^*_\beta$ *and only the trivial solution* $u \equiv 0$ *when* $s \geqslant p^*_\beta$.

Proof. From Remark 6.8 if $1 < q \leqslant l < s < p^*_\beta$ Theorem 6.7 applies and so there exists a radial continuous ground state $u \in D^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \cap L^q_\beta(\mathbb{R}^n)$ of (1.5), (6.31) bounded above by some $\bar{u} > 0$, verifying (F6). First note that $F(u) = u^q G(u)$, $u \in \mathbb{R}^+$, where $G(u) = -a/q - bu^{l-q}/l + cu^{s-q}/s$, and in turn $F(u) > 0$ if and only if $G(u) > 0$. It is easy to see that f, F and G have a unique positive zero and $F(u) = 0$ if and only if $G(u) = 0$. Moreover $G < 0$ in $[0, u_0)$ while $G > 0$ in (u_0, ∞), with $G(u_0) = 0$, and so $\bar{u} > u_0$. Clearly $G(1) \leqslant 0$ if and only if $C \geqslant 1$, where $C > 0$ is given in (6.33). Hence, if $G(1) \leqslant 0$, then $u_0 > 1$ and for all $u \geqslant 1$ we have $G(u) \geqslant u^{l-q}[cu^{s-l}/s - (a/q + b/l)]$ so that $C^{1/(s-l)} \geqslant u_0$ since $G(C^{1/(s-l)}) \geqslant 0$ and $s > l$. In this case it is enough to take any $\bar{u} > C^{1/(s-l)}$. On the other hand, if $G(1) > 0$, then for all $u \leqslant 1$ we have $G(u) \geqslant cu^{s-q}/s - (a/q + b/l)$ and again, since $s > q$ and $G(C^{1/(s-q)}) \geqslant 0$, we have $C^{1/(s-q)} \geqslant u_0$. Therefore, it is enough to take any $\bar{u} > C^{1/(s-q)}$. The case $l = q$ is much simpler, since $u_0 = C^{1/(s-q)} = [s(a+b)/cq]^{1/(s-q)}$.

Finally, the non-existence result is a consequence of Lemma 4.6. Indeed, the identity (6.32) becomes

$$als(q - p^*_\beta)\|u\|^q_{q,\beta} + bqs(l - p^*_\beta)\|u\|^l_{l,\beta} = cql(s - p^*_\beta)\|u\|^s_{s,\beta}$$

and the assertion follows at once. □

Theorem 6.9 extends to the weighted p-Laplacian case the existence and non-existence results given by Berestycki and Lions in [4, Example 2] for the no weighted Laplacian case, i.e. $p = 2$ and $\beta = 0$.

Remark 6.10. In [11], using the theory of singular elliptic problems with weights developed in [36], the authors study a quasilinear problem with much more general weights and prove the existence of solutions by a subcritical growth condition at infinity different from (F7) (see Section 4 of [11]). By comparing this condition with (F7) some examples show that none is more powerful than the other. To see this, it is enough to produce examples of f and give the definitions only for u sufficiently large. For instance, $f(u) = u^{p^*_\beta} + 1/u$, $u \gg 1$, verifies condition (Φ) of [11], but not (F7); while $f(u) = u^{s-1}$, $u \gg 1$, with $s < -p$, satisfies (F7), but not (Φ). Finally, $f(u) = u^{s-1}$, $u \gg 1$, with $-p < s < p^*_\beta$, verifies both (Φ) and (F7). Hence Theorem 6.7 extends Theorems 7 and 10 of [11] in the special case in which $g \equiv 1$, $h(r) = r^{-\beta}$, $\beta < p$, and f is continuous also at $u = 0$. However, we do not require that f is locally Lipschitz continuous, say, in $\mathbb{R}^+$ as required in [11] in the standard case and in many previous papers in the no weighted case (see e.g. [19,22] and the papers quoted there).

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