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Analysis of boundary bubbling solutions for an anisotropic Emden–Fowler equation

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Abstract

We consider the following anisotropic Emden–Fowler equation

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and a is a positive smooth function. We study here the phenomenon of boundary bubbling solutions which *do not exist* for the isotropic case $a \equiv$ *constant*. We determine the localization and asymptotic behavior of the boundary bubbles, and construct some boundary bubbling solutions. In particular, we prove that if $\bar{x} \in \partial\Omega$ is a strict local minimum point of a, there exists a family of solutions such that $\varepsilon^2 a(x)e^u\,dx$ tends to $8\pi a(\bar{x})\delta_{\bar{x}}$ in $\mathcal{D}'(\mathbb{R}^2)$ as $\varepsilon \to 0$. This result will enable us to get a new family of solutions for the isotropic problem $\Delta u + \varepsilon^2 e^u = 0$ in rotational torus of dimension $N \geqslant 3$.

Keywords: Boundary bubble; Blow-up analysis; Localized energy method

1. Introduction

The classical Emden–Fowler equation, or Gelfand equation

$$\Delta u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

has motivated a lot of studies, because it has both geometrical and physical background. When $N = 2$, (1) or more generally Eq. (2) below relates to the geometric problem of Riemannian surfaces with prescribed Gaussian curvature (see [7] and references therein). For $N \geqslant 3$, it arises in the theory of thermionic emission, isothermal gas sphere, gas combustion. It is also considered in relation with Onsager's formulation in statistical mechanics, the Keller–Segel system of chemotaxis, Chern–Simon–Higgs gauge theory and many other physical applications (see [4–6,11,13,19, 17,25] and the references therein).

It is well known that there exists a critical value $\varepsilon^* > 0$ such that when $\varepsilon > \varepsilon^*$, no solution of (1) exists while for $\varepsilon \in (0, \varepsilon^*)$, we have a family of minimal solutions which tend uniformly to zero as $\varepsilon \to 0$. When $N = 2$, for any

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$\varepsilon \in (0, \varepsilon^*)$, we have also a second solution which is nonstable and blows up as $\varepsilon \to 0$. The asymptotic behavior of nonstable solutions to (1), or to a more general equation

$$\Delta u + \varepsilon^2 k(x) e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where $k(x)$ is a positive smooth function has been studied in [3,14,15,18,21,27]. Let G_D denote the standard Green's function of $-\Delta$ with Dirichlet boundary condition and H_D denote the regular part of G_D, i.e.

$$H_D(x, y) = G_D(x, y) + \frac{1}{2\pi} \log|x - y|. \tag{3}$$

If u_ε is a family of solutions to (2) satisfying

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon}\, dx \to \ell,$$

as $\varepsilon \to 0$ and $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega)} = \infty$, then up to a subsequence, there holds either $\ell = \infty$, $u_\varepsilon \to \infty$ for all $x \in \Omega$; or $\ell = 8\pi m$, $m \in \mathbb{N}^*$ and u_ε makes m points *simple* blow-up on $\mathcal{S} = \{x_1, \ldots, x_m\} \subset \Omega$ such that

$$\varepsilon^2 k(x) e^{u_\varepsilon}\, dx \to 8\pi \sum_{j=1}^m \delta_{x_j}, \quad u_\varepsilon \to 8\pi \sum_{j=1}^m G_D(\cdot, x_j) \quad \text{in } C^k_{\text{loc}}(\overline{\Omega} \setminus \mathcal{S}),\ \forall k \in \mathbb{N},$$

where $(x_1, \ldots, x_m)$ is a critical point of Ψ defined by

$$\Psi(x) = \sum_{j=1}^m H_D(x_j, x_j) + \sum_{i \neq j} G_D(x_i, x_j) + 2 \sum_{j=1}^m \log k(x_j).$$

Conversely, many authors have constructed blow-up solutions, see for example [2,9,10,20]. So the solutions of Eq. (1) or (2) are now well understood in dimension two.

Here we consider the following generalized Emden–Fowler equation

$$\Delta_a u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \partial\Omega \tag{4}$$

where Ω is a smooth bounded domain, Δ_a is the operator

$$\Delta_a u = \frac{1}{a(x)} \nabla\big[a(x)\nabla u\big] = \Delta u + \nabla \log a \nabla u$$

and $a(x)$ is a smooth function over Ω satisfying

$$0 < a_1 \leqslant a(x) \leqslant a_2 < +\infty. \tag{5}$$

Our motivation is due to the fact that few is known for Eq. (1) in dimension $N \geqslant 3$. As far as we know, the only explicit results in higher dimensions concern the radial solutions in spheres (see [11,13]) or in annuli (see [22]). It is worth to mention that Pacard proved in [23] (see also [16]) that for annuli, i.e. $\Omega = A_{r_0} = \{x \in \mathbb{R}^N, r_0 < \|x\| < 1\}$, there exists $X \subset (0, 1)$ of measure equal to 1 such that for all $r_0 \in X$, there are infinitely many symmetry breaking points with bifurcation from the branch of radial solutions. Unfortunately, we do not have no more precise information about the behavior of these nonradial solutions. However, through these results, we observe already a quite different situation with the case in dimension two. Our idea here is to consider axially symmetric solutions of (1) in a torus, and try to give some precise descriptions of them. In fact, let $\mathbb{T}$ be a standard N-dimensional torus ($N \geqslant 3$), i.e.

$$\mathbb{T} = \Big\{x = (x_i) \in \mathbb{R}^N;\ \Big(\sqrt{x_1^2 + \cdots + x_{N-1}^2} - 1\Big)^2 + x_N^2 \leqslant r_0^2\Big\} \tag{6}$$

with $0 < r_0 < 1$. If we look for solutions of (1) in the form of $u(x) = u(r, s)$ where

$$r = \sqrt{x_1^2 + \cdots + x_{N-1}^2} \quad \text{and} \quad s = x_N,$$

a direct calculus yields that the problem (1) is transformed to

$$\nabla(r^{N-2}\nabla u) + \varepsilon^2 r^{N-2} e^u = 0 \quad \text{in } \Omega_{\mathbb{T}}, \qquad u = 0 \quad \text{on } \partial\Omega_{\mathbb{T}}, \tag{7}$$

where $\Omega_{\mathbb{T}} = \{(r,s) \in \mathbb{R}^2; (r-1)^2 + s^2 < r_0^2\}$. This is just Eq. (4) with $a(r,s) = r^{N-2}$, so problem (4) represents a special case of (1) in higher dimension.

Eq. (4) seems to be similar to (2) or (1). We can show the existence of critical value $\varepsilon^* > 0$ (depending on a and Ω), the existence of minimal solution and nonstable solution for all $\varepsilon \in (0, \varepsilon^*)$. But the structure of nonstable solutions is quite different. In [28], the authors studied the asymptotic behavior of bubbling solutions to (4), they proved that if $\mathcal{T}_\varepsilon = \mathrm{O}(1)$, then either $u_\varepsilon \to 0$ uniformly on any compact subset of Ω, or there exists a finite set $\mathcal{S} = \{x_i\} \subset \Omega$ and $m_i \in \mathbb{N}^*$ such that $u_\varepsilon \to u^*$ weakly in $W^{1,p}(\Omega)$ for any $p \in (1,2)$, where u^* verifies

$$\Delta_a u^* + 8\pi \sum_i m_i \delta_{x_i} = 0 \quad \text{in } \Omega, \qquad u^* = 0 \quad \text{on } \partial\Omega. \tag{8}$$

Moreover, $m_i \in \mathbb{N}^*$ and each x_i must be a *critical point* of a. Recently, we have constructed in [26] bubbling solutions near any *topologically nontrivial critical point* of a in Ω. In particular, near any *interior strict local maximum* of a, we have solutions with arbitrary given number of bubbles, which illustrates again the contrast with the isotropic situation.

Nevertheless, if we look at Eq. (7), the function $a(r,s) = r^{N-2}$ has no critical point in $\Omega_{\mathbb{T}}$. Consequently the blow-up cannot occur in the interior of the domain, it must appear near the boundary. A natural question is to understand these boundary bubbling solutions, which is just the aim of this paper.

First, we show the localization and asymptotic behavior of boundary blow-up when a has no critical point in Ω.

Theorem 1.1. *Suppose that the anisotropic coefficient a has no critical point in Ω. Let u_ε be a family of solutions to problem* (4) *satisfying*

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega e^{u_\varepsilon}\, dx \to \ell < \infty \quad \textit{and} \quad \max_{\overline{\Omega}} u_\varepsilon \to \infty$$

as ε tends to 0. *Then $\ell = 8\pi m$ with $m \in \mathbb{N}^*$ and up to a subsequence, there exists a finite set $\mathcal{S} = \{x_1, \ldots, x_q\} \subset \partial\Omega$ and $m_1, \ldots, m_q \in \mathbb{N}^*$ such that*

$$\varepsilon^2 e^{u_\varepsilon} \chi_\Omega\, dx \to \sum_{1 \leqslant j \leqslant q} 8\pi m_j \delta_{x_j} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Moreover, the tangential derivative $\partial_\tau a(x_j) = 0$ for any $1 \leqslant j \leqslant p$ and

$$u_\varepsilon \to 0 \quad \text{in } C^k_{\mathrm{loc}}(\overline{\Omega} \setminus \mathcal{S}), \quad \forall k \in \mathbb{N}.$$

Remark 1.2. We should emphasize that the boundary blow-up phenomenon does not exist for the isotropic case, or more generally when a is constant in a neighborhood of the boundary. In that case, solutions of Eq. (4) are decreasing with respect to $d(x, \partial\Omega)$ in a fixed neighborhood of $\partial\Omega$, by moving plane argument as showed in [21] (see also [18]).

Remark 1.3. We can combine Theorem 1.1 with the result in [28] to get a more general conclusion for a satisfying just (5), see Proposition 3.4.

The second part of the paper concerns the existence of boundary bubbling solutions. Let us introduce some notations. Let $G(x,y)$ be the Green's function associated to $-\Delta_a$, that is, for any $y \in \Omega$,

$$\Delta_a G(x,y) + 8\pi \delta_y = 0 \quad \text{in } \Omega \quad \text{and} \quad G(x,y) = 0 \quad \text{if } x \in \partial\Omega. \tag{9}$$

Define H to be the regular part of $G(x,y)$ as

$$H(x,y) = G(x,y) + 4 \log|x-y|. \tag{10}$$

Then our main results for this part can be stated as follows.

Theorem 1.4. *Let $\bar{x} \in \partial\Omega$ be a local minimum point of a on $\partial\Omega$, i.e.*

$$\exists \delta > 0 \quad \textit{such that} \quad a(\bar{x}) < a(y), \quad \forall y \in B_\delta(\bar{x}) \cap \overline{\Omega},\ y \neq \bar{x}.$$

We assume also $\partial_\nu a(\bar{x}) < 0$*. Then for* $\varepsilon > 0$ *sufficiently small, problem* (4) *has a family of solutions* u_ε *such that* $\varepsilon^2 e^{u_\varepsilon}\chi_\Omega\,dx \to 8\pi\delta_{\bar{x}}$ *in* $\mathcal{D}'(\mathbb{R}^2)$*. More precisely, we have*

$$u_\varepsilon(x) = \log\frac{1}{(\varepsilon^2\mu_\varepsilon^2 + |x-\xi_\varepsilon|^2)^2} + H(x,\xi_\varepsilon) + \mathrm{o}(1) \quad \text{in } \overline{\Omega} \tag{11}$$

where ξ_ε, μ_ε *satisfy*

$$\xi_\varepsilon \to \bar{x}, \quad (\xi_\varepsilon, \partial\Omega) \sim \frac{1}{|\log\varepsilon|} \quad \text{and} \quad \mu_\varepsilon \sim \frac{1}{|\log\varepsilon|^2} \quad \text{as } \varepsilon\to 0. \tag{12}$$

Here we use the symbol $f \sim g$ to mean the existence of $C > 0$ such that

$$\frac{1}{C} \leqslant \liminf_{\varepsilon\to 0}\frac{f}{g} \leqslant \limsup_{\varepsilon\to 0}\frac{f}{g} \leqslant C.$$

Throughout the work, the symbol C denotes always a positive constant independent of ε, it could be changed from one line to another.

The following result shows the reason why we construct the bubbling solutions near a local minimum point of a on the boundary, but not near a maximum point.

Theorem 1.5. *Assume that* $\mathcal{T}_\varepsilon = \mathrm{O}(1)$ *and* $\bar{x} \in \partial\Omega$ *is a nondegenerate local maximum point of* a*, then* $\bar{x} \notin \mathcal{S}$*.*

If we return to the original equation (1) over $\mathbb{T}$, we get then a family of solutions which blows up at a $(N-2)$-dimensional submanifold on $\partial\mathbb{T}$.

Theorem 1.6. *Let* $\mathbb{T}$ *be the torus defined by* (6)*, then there exists a family of solutions* u_ε *for* (1) *such that as* $\varepsilon\to 0$*,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}} \varepsilon^2 e^{u_\varepsilon}\,dx = 0 \tag{13}$$

and u_ε *blows up exactly on*

$$\mathcal{S}_{\mathbb{T}} = \left\{(x_i)\in\mathbb{T};\ \sqrt{x_1^2+\cdots+x_{N-1}^2} = 1-r_0,\ x_N = 0\right\}.$$

The paper is organized as follows. First we consider the behavior of boundary bubbles, we prove Theorems 1.1 and 1.5 by potential analysis, Pohozaev identity, combined with blow-up arguments used in [15] and [28]. Then we prove Theorem 1.4 via the *localized energy method*, a combination of Liapunov–Schmidt reduction method and variational techniques similar to our previous paper [26]. The difficulties for proving all these results come from the fact that the distance between the bubbles and the boundary will tend to zero, therefore some refinements, in particular some precise informations for the Green's function and the corresponding Robin function need to be developed to make our approach successful. Theorem 1.6 can be shown as a direct consequence of Theorems 1.5 and 1.4.

2. Behavior of the Green's function $G(x, y)$ near $\partial\Omega$

In order to study the bubbles of (4) which tend to the boundary, we need to have a good understanding of the Green's function $G(x,y)$ associated to $-\Delta_a$, when y is near $\partial\Omega$, and its regular part H, especially the corresponding Robin function $x \mapsto H_R(x) = H(x,x)$.

Lemma 2.1. *There exist positive constants* d_0 *and* C *such that for any* $x\in\Omega$, $d(x,\partial\Omega)\leqslant d_0$*, there exists a unique point* $x_\nu\in\partial\Omega$ *satisfying* $d(x,\partial\Omega) = |x-x_\nu|$ *and if* $x^* = 2x_\nu - x$ *denotes the reflection point of* x *with respect to* $\partial\Omega$*, then*

$$\left|G(y,x) + 4\log|x-y| - 4\log|x^*-y|\right| \leqslant C, \quad \forall y\in\overline{\Omega}. \tag{14}$$

Moreover,

$$\lim_{d(x,\partial\Omega)\to 0}\left\|G(y,x) - 4\log\frac{|x^*-y|}{|x-y|}\right\|_{L^\infty(\Omega)} = 0. \tag{15}$$

Proof. Consider $L(y,x) = G(y,x) + 4\log|x-y| - 4\log|x^*-y|$. We have

$$-\Delta_{a(y)}L(y,x) = -4\nabla\log a(y)\cdot\left(\frac{y-x}{|y-x|^2} - \frac{y-x^*}{|y-x^*|^2}\right) \quad \text{in } \Omega.$$

It is clear that the right-hand side is uniformly bounded in $L^p(\Omega)$ for any $p\in[1,2)$. Furthermore, by the regularity of the domain Ω, we know that $\|L(\cdot,x)\|_{L^\infty(\partial\Omega)} = \mathrm{O}(d(x,\partial\Omega))$ when $d(x,\partial\Omega)\leqslant d_0$. The standard elliptic theory implies then $\|L(\cdot,x)\|_{L^\infty(\Omega)} = \mathrm{O}(1)$.

More precisely, remark that for $1\leqslant p<2$, $\|\Delta_{a(y)}L(y,x)\|_{L^p(\Omega)}$ tends to 0 as $|x-x^*|\to 0$. So $\|L(\cdot,x)\|_{L^\infty(\Omega)}\to 0$ as $d(x,\partial\Omega)$ tends to 0. □

Recall the following expansion of $x\mapsto H(x,y)$ proved in [26].

Lemma 2.2. *Let $H_y(x) = H(x,y)$ for any $y\in\Omega$. Then $y\mapsto H_y$ is a continuous map from Ω into $C^{0,\gamma}(\overline{\Omega})$, $\forall\gamma\in(0,1)$. Let H_D be the regular part of the standard Green's function defined by* (3)*, we have*

$$H(x,y) = 8\pi H_D(x,y) + \nabla\log a(y)\cdot\nabla\big(|x-y|^2\log|x-y|\big) + H_1(x,y), \tag{16}$$

where $y\mapsto H_1(\cdot,y)$ is a continuous map from Ω into $C^{1,\gamma}(\overline{\Omega})$ for all $\gamma\in(0,1)$. Furthermore, the function $(x,y)\mapsto H_1(x,y)\in C^1(\Omega\times\Omega)$, in particular $x\mapsto H(x,x)\in C^1(\Omega)$.

Using the equation satisfied by H_y, we can get, for any $\gamma\in(0,1)$,

$$\|H_y\|_{C^{0,\gamma}(\overline{\Omega})} = \mathrm{O}\left(\frac{1}{d(y,\partial\Omega)}\right) \quad \text{uniformly in } \Omega. \tag{17}$$

Now we show the behavior of the Robin function $x\mapsto H(x,x)$ near the boundary.

Lemma 2.3. *Let H_R denote the Robin function $x\mapsto H(x,x)$, then*

$$H_R(x) = 4\log d(x,\partial\Omega) + \mathrm{O}(1), \quad \nabla H_R(x) = \mathrm{O}\left(\frac{1}{d(x,\partial\Omega)}\right) \quad \text{uniformly in } \Omega. \tag{18}$$

Sketch of Proof. Using the equation of $x\mapsto H(x,y)$, clearly $H(x,y) = 8\pi H_D(x,y) + \mathrm{O}(1)$ in $\Omega\times\Omega$. By the behavior of H_D (see for example [1]), we have $H(x,x) = 4\log d(x,\partial\Omega) + \mathrm{O}(1)$ in Ω. For the estimate of ∇H_R, using the equation satisfied by $H_D(\cdot,y)$, we obtain

$$\big\|H_D(\cdot,y)\big\|_{C^1(\overline{\Omega})} = \mathrm{O}\left(\frac{1}{d(y,\partial\Omega)}\right) \quad \text{for } y\in\Omega.$$

Consider the equation of $x\mapsto H_1(x,y)$,

$$\begin{aligned}\Delta_{a(x)}H_1(x,y) = {}& 4\big[\nabla\log a(x) - \nabla\log a(y)\big]\cdot\frac{x-y}{|x-y|^2} - 8\pi\nabla\log a(x)\nabla_x H_D(x,y)\\ &- \nabla_x^2\big(|x-y|^2\log|x-y|\big)\cdot\big(\nabla\log a(x),\nabla\log a(y)\big)\end{aligned}$$

in Ω and $H_1(x,y) = -\nabla\log a(y)\cdot\nabla_x(|x-y|^2\log|x-y|)$ if $x\in\partial\Omega$. We get then

$$\big\|H_1(\cdot,y)\big\|_{C^1(\overline{\Omega})} = \mathrm{O}\left(\frac{1}{d(y,\partial\Omega)}\right) \quad \text{uniformly in } \Omega. \tag{19}$$

Moreover, for $1\leqslant p<2$, we can prove, by direct calculus,

$$\left\|\frac{x-y}{|x-y|^2}\right\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \mathrm{O}\left(\frac{1}{d(y,\partial\Omega)}\right) \quad \text{uniformly for } y\in\Omega,$$

hence $\|\nabla_y H_D(\cdot,y)\|_{W^{1,p}(\Omega)} = \mathrm{O}(d(y,\partial\Omega)^{-1})$ in Ω. Checking carefully the equation satisfied by $\nabla_y H_1(\cdot,y)$, we obtain

$$\big\|\nabla_y H_1(\cdot,y)\big\|_{C^0(\overline{\Omega})} = \mathrm{O}\left(\frac{1}{d(y,\partial\Omega)}\right) \quad \text{uniformly in } \Omega. \tag{20}$$

As $H_D(x,y)=H_D(y,x)$ in $\Omega\times\Omega$ and $H_R(x)=8\pi H_D(x,x)+H_1(x,x)$, we are done. □

Remark 2.4. Here we have $a(x)G(x,y)=a(y)G(y,x)$ in $\Omega\times\Omega$, but not the usual symmetry for the standard Green's function G_D and thanks to the expansion (16), we see that H_y is not in $C^1(\Omega)$ in general (except if $\nabla a(y)=0$). These facts make our estimate for the Robin function H_R more involved.

3. Boundary blow-up analysis

Now we are in position to prove Theorems 1.1 and 1.5. If $\mathcal{T}_\varepsilon$ is bounded, we know that u_ε is bounded in $W^{1,p}(\Omega)$ for any $p\in[1,2)$. We get first a Brezis–Merle type result.

Lemma 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^2$. There exists $\alpha>0$ (depending on Ω and a) such that if a solution of (4) u_ε satisfies, for $x\in\overline{\Omega}$ and $\delta>0$*

$$\int\limits_{B_\delta(x)\cap\overline{\Omega}}\varepsilon^2 e^{u_\varepsilon}\,dy\leqslant\alpha,$$

then $\|u_\varepsilon\|_{L^\infty(B_{\delta/2}(x)\cap\overline{\Omega})}\leqslant C$.

By the results in [3], we need just to consider the situation near the boundary, i.e. when $x\in\partial\Omega$. Indeed, we can use conformal transformation to change $\overline{\Omega}$ locally as a part of $\mathbb{R}\times\mathbb{R}_+$ (see the proof of Lemma 3.2 below), then we use a reflection argument in order to apply methods in [3] (see also [28]). We leave the detail for interested readers.

We define the blow up set for u_ε as follows:

$$\mathcal{S}\stackrel{\text{def}}{=}\left\{x\in\overline{\Omega};\ \exists\varepsilon_k\to0\text{ and }x_{\varepsilon_k}\to x\text{ such that }u_{\varepsilon_k}(x_{\varepsilon_k})\to\infty\right\}.$$

Thanks to Lemma 3.1, we obtain

$$\mathcal{S}=\Sigma\stackrel{\text{def}}{=}\left\{x\in\overline{\Omega};\ \forall\delta>0,\ \liminf_{\varepsilon\to0}\int\limits_{B_\delta(x)\cap\overline{\Omega}}\varepsilon^2e^{u_\varepsilon}\,dy>\alpha\right\}.$$

Consequently, up to a subsequence, we have that $\varepsilon^2e^{u_\varepsilon}\chi_\Omega\,dx$ tends to $\sum_{x_j\in\mathcal{S}}\eta_j\delta_{x_j}$ in $\mathcal{D}'(\mathbb{R}^2)$ and $\#(\mathcal{S})<\infty$, since $\mathcal{T}_\varepsilon=\mathrm{O}(1)$. By the result in [28], if we assume that a has no critical point in Ω, then $\mathcal{S}\subset\partial\Omega$.

Lemma 3.2. *For any $k\in\mathbb{N}$, $u_\varepsilon\to0$ in $C^k_{\mathrm{loc}}(\overline{\Omega}\setminus\mathcal{S})$. Moreover, for any $x_0\in\mathcal{S}$, we have $\partial_\tau a(x_0)=0$ and $\eta_0\geqslant8\pi$.*

Proof. First, by the definition of the Green's function G,

$$u_\varepsilon(x)=\frac{1}{8\pi a(x)}\int\limits_\Omega a(z)G(z,x)\varepsilon^2e^{u_\varepsilon(z)}\,dz,\quad\forall x\in\Omega.\tag{21}$$

Fix any subset $K\subset\Omega$ such that $\overline{K}\cap\mathcal{S}=\emptyset$. Given $\lambda>0$, since $\mathcal{S}\subset\partial\Omega$, applying (15) and Remark 2.4, there exists $\delta>0$ such that $\|G(y,x)\|_{L^\infty(\overline{K})}\leqslant\lambda$, if $d(y,\mathcal{S})\leqslant\delta$. Therefore, if we decompose Ω as $\Omega_1=\{y\in\Omega;\,d(y,\mathcal{S})\leqslant\delta\}$ and $\Omega_2=\{y\in\Omega;\,d(y,\mathcal{S})>\delta\}$, for any $x\in K$,

$$\int\limits_{\Omega_1}a(y)G(y,x)\varepsilon^2e^{u_\varepsilon(y)}\,dy\leqslant a_2\lambda\times\int\limits_{\Omega_1}\varepsilon^2e^{u_\varepsilon(y)}\,dy\leqslant a_2\mathcal{T}_\varepsilon\lambda=\mathrm{O}(\lambda).$$

On the other hand, the definition of $\mathcal{S}$ implies that u is uniformly bounded in any compact set away from $\mathcal{S}$. Hence for fixed $\delta>0$, there exists $C>0$ such that

$$\int\limits_{\Omega_2}a(y)G(y,x)\varepsilon^2e^{u_\varepsilon(y)}\,dy\leqslant C\varepsilon^2\int\limits_{\Omega_2}G(y,x)\,dy=\mathrm{O}(\varepsilon^2),$$

because $\|G(\cdot,x)\|_{L^1(\Omega)}$ is uniformly bounded for $x\in\Omega$. By (21), we get $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(K)}=0$. The usual elliptic theory shows then $u_\varepsilon\to 0$ in $C^k(K)$ for all $k\in\mathbb{N}$.

Let $x_0\in\mathcal{S}$. Choosing a smooth open neighborhood U of x_0 such that $\overline{U}\cap\mathcal{S}=\{x_0\}$, we can assume that there exists a conformal diffeomorphism Φ from $\overline{B}_\delta$ into $\overline{U}$ satisfying $\Phi(0)=x_0$,

$$\Phi^{-1}(\partial\Omega\cap\overline{U})=[-\delta,\delta]\times\{0\}\quad\text{and}\quad \Phi^{-1}(\overline{\Omega}\cap\overline{U})=(\mathbb{R}\times\mathbb{R}_+)\cap\overline{B}_\delta\stackrel{\text{def}}{=}\overline{\Omega}_0.$$

It is not difficult to check that $v_\varepsilon=u_\varepsilon\circ\Phi$ is a solution of

$$-\Delta_b v_\varepsilon=\varepsilon^2|\nabla\Phi|^2e^{v_\varepsilon}\quad\text{in }\Omega_0,\quad\text{with } b=a\circ\Phi \tag{22}$$

and $v_\varepsilon=0$ on $\Gamma_0=\partial\Omega_0\cap(\mathbb{R}\times\{0\})$. Taking $b(x)\partial_1 v_\varepsilon$ as a test function, since $\nu_1=0$ and $v_\varepsilon=0$ on Γ_0, we get

$$\int_{\Omega_0}\frac{\partial b(x)}{\partial x_1}\frac{|\nabla v_\varepsilon|^2}{2}\,dx=\int_{\Omega_0}\varepsilon^2\frac{\partial(b(x)|\nabla\Phi|^2)}{\partial x_1}(e^{v_\varepsilon}-1)\,dx-\varepsilon^2\int_\Gamma b(x)|\nabla\Phi|^2(e^{v_\varepsilon}-1)\nu_1\,d\sigma$$
$$+\int_\Gamma\frac{b(x)}{2}|\nabla v_\varepsilon|^2\nu_1\,d\sigma-\int_\Gamma b(x)\frac{\partial v_\varepsilon}{\partial x_1}\frac{\partial v_\varepsilon}{\partial\nu}\,d\sigma,$$

where $\Gamma=\partial\Omega_0\setminus\Gamma_0$. Thus all the right-hand side terms are uniformly bounded. If $\partial_\tau a(x_0)\neq 0$, by taking δ small enough, we can assume without loss of generality that $\partial_1 b(x)>0$ in Ω_0, because $|\partial_{x_1}b(0)|=|\nabla\Phi(0)|\times|\partial_\tau a(x_0)|\neq 0$. Therefore

$$\int_{\Omega_0}|\nabla v_\varepsilon|^2\,dx=\mathrm{O}(1),$$

this deduces that u_ε is bounded in $H^1(U\cap\Omega)$. The Moser–Trudinger inequality shows then the boundedness of e^{u_ε} in $L^p(U\cap\Omega)$ for all $p\geqslant 1$, which yields $\|u_\varepsilon\|_{L^\infty(U\cap\Omega)}=\mathrm{O}(1)$ by the equation, hence contradicts with $x_0\in\mathcal{S}=\Sigma$.

Furthermore, let $x_\varepsilon\in\overline{U}$ realize $\max_{\overline{U}\cap\overline{\Omega}}u_\varepsilon(x)$. So $\lim_{\varepsilon\to 0}x_\varepsilon=x_0$. Let $d_\varepsilon=d(x_\varepsilon,\partial\Omega)$ and $-2\log\lambda_\varepsilon=u_\varepsilon(x_\varepsilon)+2\log\varepsilon$. We claim

$$\lim_{\varepsilon\to 0}\frac{d_\varepsilon}{\lambda_\varepsilon}=\infty. \tag{23}$$

First, we have $\lim_{\varepsilon\to 0}\lambda_\varepsilon=0$, since otherwise $\varepsilon^2e^{u_\varepsilon}$ is uniformly bounded in $\overline{U}\cap\overline{\Omega}$ and no blow-up will occur. Suppose now (23) is not true, up to a subsequence, we may assume that $d_\varepsilon/\lambda_\varepsilon\leqslant C$. Define

$$w_\varepsilon(y)=u_\varepsilon(x_\varepsilon+\lambda_\varepsilon y)+2\log\varepsilon+2\log\lambda_\varepsilon\quad\text{in }\Omega_\varepsilon=\{y\in\mathbb{R}^2;\,x_\varepsilon+\lambda_\varepsilon y\in U\cap\Omega\}.$$

Clearly $w_\varepsilon(y)\leqslant 0$ in Ω_ε,

$$-\Delta_{a(x_\varepsilon+\lambda_\varepsilon y)}w_\varepsilon=e^{w_\varepsilon}\quad\text{in }\Omega_\varepsilon\quad\text{and}\quad\int_{\Omega_\varepsilon}e^{w_\varepsilon(y)}\,dy=\varepsilon^2\int_{U\cap\Omega}e^{u_\varepsilon(x)}\,dx=\mathrm{O}(1).$$

Taking $x=x_\varepsilon$ in (21) and applying Lemma 2.1 (as $d(x_\varepsilon,\partial\Omega)\to 0$), we obtain

$$\begin{aligned}0&=\frac{1}{8\pi a(x_\varepsilon)}\int_\Omega a(z)G(z,x_\varepsilon)\varepsilon^2e^{u_\varepsilon(z)}\,dz-u_\varepsilon(x_\varepsilon)\\&=\frac{1}{2\pi a(x_\varepsilon)}\int_\Omega\left[\log\frac{|x_\varepsilon^*-z|}{|x_\varepsilon-z|}+\mathrm{O}(1)\right]a(z)\varepsilon^2e^{u_\varepsilon(z)}\,dz-u_\varepsilon(x_\varepsilon)\\&=\frac{1}{2\pi a(x_\varepsilon)}\int_{U\cap\Omega}a(z)\varepsilon^2e^{u_\varepsilon(z)}\log\frac{|x_\varepsilon^*-z|}{|x_\varepsilon-z|}\,dz-u_\varepsilon(x_\varepsilon)+\mathrm{O}(1).\end{aligned}\tag{24}$$

Here, we have used the fact that $|x_\varepsilon^*-z|/|x_\varepsilon-z|\to 1$ uniformly in $\Omega\setminus U$, as $x_\varepsilon\to x_0$. Let $z=x_\varepsilon+\lambda_\varepsilon y$ and $y_\varepsilon^*=(x_\varepsilon^*-x_\varepsilon)/\lambda_\varepsilon$, the equality (24) leads to

$$0=\frac{1}{2\pi a(x_\varepsilon)}\int_{\Omega_\varepsilon}a(x_\varepsilon+\lambda_\varepsilon y)e^{w_\varepsilon(y)}\log\frac{|y_\varepsilon^*-y|}{|y|}\,dy-u_\varepsilon(x_\varepsilon)+\mathrm{O}(1). \tag{25}$$

Notice that $|y_\varepsilon^*| \leqslant 2C$ and $\|e^{w_\varepsilon}\|_{L^1\cap L^\infty(\Omega_\varepsilon)} = \mathrm{O}(1)$. By decomposing the last integral over domains $\{|y| \leqslant C\} \cap \Omega_\varepsilon$ and $\{|y| \geqslant C\} \cap \Omega_\varepsilon$, we obtain clearly

$$\int_{\Omega_\varepsilon} a(x_\varepsilon + \lambda_\varepsilon y) e^{w_\varepsilon(y)} \log \frac{|y_\varepsilon^* - y|}{|y|}\, dy \leqslant a_2 \times \int_{\Omega_\varepsilon} e^{w_\varepsilon(y)} \log\left(1 + \frac{2C}{|y|}\right) dy = \mathrm{O}(1).$$

But $\lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = \infty$, we reach a contradiction with (25), so the claim (23) holds.

Thus, Ω_ε tends to the whole space $\mathbb{R}^2$ and standard blow-up analysis implies that up to a subsequence, w_ε converges to w in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ where

$$-\Delta w = e^w \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \int_{\mathbb{R}^2} e^w\, dy < \infty.$$

It is well known from [8] that

$$\int_{\mathbb{R}^2} e^w\, dy = 8\pi.$$

Fatou's lemma yields then $\eta_0 \geqslant 8\pi$. □

More precisely, we will quantify η_0.

Proposition 3.3. *For any $x_0 \in \mathcal{S}$, we have $\eta_0 \in 8\pi\mathbb{N}^*$.*

Proof. Let $x_0 \in \mathcal{S}$, under conformal transformation, we can assume that $x_0 = 0$ and there exists $\delta > 0$ such that $\overline{B}_\delta \cap \mathcal{S} = \{0\}$,

$$\Omega_0 = B_\delta(0) \cap \Omega = B_\delta \cap (\mathbb{R} \times \mathbb{R}_+^*) \quad \text{and} \quad \overline{B}_\delta(0) \cap \partial\Omega = [-\delta, \delta] \times \{0\} = \Gamma_0.$$

It suffices to prove that (recall that $v_\varepsilon = u_\varepsilon \circ \Phi$ verifies (22)),

$$\lim_{\varepsilon\to 0} \int_{\Omega_0} \varepsilon^2 |\nabla\Phi|^2 e^{v_\varepsilon}\, dx \in 8\pi\mathbb{N}^*. \tag{26}$$

Define ζ_ε by

$$-\Delta\zeta_\varepsilon = \nabla \log b \cdot \nabla v_\varepsilon \quad \text{in } \Omega_0, \qquad \zeta_\varepsilon = v_\varepsilon \quad \text{on } \partial\Omega_0.$$

Then $g_\varepsilon = v_\varepsilon - \zeta_\varepsilon$ satisfies

$$-\Delta g_\varepsilon = \varepsilon^2 V_\varepsilon e^{g_\varepsilon} \quad \text{in } \Omega_0, \qquad g_\varepsilon = 0 \quad \text{on } \partial\Omega_0,$$

where $V_\varepsilon = |\nabla\Phi|^2 e^{\zeta_\varepsilon}$. Since $\|v_\varepsilon\|_{C(\partial\Omega_0)} \to 0$ and v_ε converges to 0 weakly in $W^{1,p}(\Omega_0)$ for $1 < p < 2$ by Lemma 3.2, so ζ_ε is a family of continuous function satisfying $\|\zeta_\varepsilon\|_{C(\overline{\Omega}_0)} \to 0$, thus V_ε is a family of continuous functions which converges uniformly to the positive function $V = |\nabla\Phi|^2$.

We will prove a Li–Shafrir type result as in [15]. Indeed, we can get a first bubble by considering $\max_{\overline{\Omega}_0} g_\varepsilon = g_\varepsilon(x_\varepsilon)$ as in the proof of Lemma 3.2. We claim:

$$\text{If } \max_{\overline{\Omega}_0}\bigl[g_\varepsilon(x) + 2\log\varepsilon + 2\log|x - x_\varepsilon|\bigr] = \mathrm{O}(1), \text{ then } \eta_0 = \lim_{\varepsilon\to 0} \int_{\Omega_0} \varepsilon^2 V_\varepsilon e^{g_\varepsilon}\, dx = 8\pi. \tag{27}$$

Recall that $d_\varepsilon = d(x_\varepsilon, \partial\Omega)$, let

$$\overline{w}_\varepsilon(y) = v_\varepsilon(x_\varepsilon + d_\varepsilon y) + 2\log\varepsilon + 2\log d_\varepsilon$$

be defined in B_1. Then $-\Delta\overline{w}_\varepsilon = V_\varepsilon(x_\varepsilon + d_\varepsilon y) e^{\overline{w}_\varepsilon}$. Since all the assumptions of Proposition 2 in [15] are verified by $\overline{w}_\varepsilon$, we conclude then

$$\lim_{\varepsilon\to 0} \int_{B_{d_\varepsilon}(x_\varepsilon)} \varepsilon^2 V_\varepsilon e^{g_\varepsilon}\, dx = \lim_{\varepsilon\to 0} \int_{B_1} V_\varepsilon(x_\varepsilon + d_\varepsilon y) e^{\overline{w}_\varepsilon}\, dy = 8\pi. \tag{28}$$

Let $D_\varepsilon = \Omega_0 \setminus B_{d_\varepsilon}(x_\varepsilon)$, we will prove $\|g_\varepsilon\|_{L^\infty(D_\varepsilon)} = \mathrm{O}(1)$. Unfortunately we do not have the sup + inf inequality as in [15], so we need some new arguments. In fact, we will use the potential analysis. Write

$$g_\varepsilon(x) = \int_{\Omega_0} G_0(y,x)\varepsilon^2 V_\varepsilon(y) e^{g_\varepsilon(y)}\,dy,$$

where G_0 is the standard Green's function associate to $-\Delta$ over the domain Ω_0. Let $x_\varepsilon = (x_\varepsilon^1, x_\varepsilon^2)$, thanks to Lemma 3.2, we can assume that $x_\varepsilon^1 = 0$ by translation. By Lemma 2.1, we can also fix $\delta_0 \in (0,\delta)$ small enough such that for any $|x| < \delta_0$,

$$\big|2\pi G_0(y,x) + \log|x-y| - \log|x^*-y|\big| \leqslant C, \quad \forall y \in \overline{\Omega}_0.$$

Let $x \in B_{\delta_0} \cap D_\varepsilon$. If $|x|/4 \leqslant |x-y| \leqslant 3|x|$, as $|x-x^*| \leqslant 2|x|$, $|y-x^*| \leqslant |y-x| + |x^*-x| \leqslant 5|x|$ and $2\pi G_0(y,x) \leqslant \log 20 + \mathrm{O}(1)$; if $|x-y| \geqslant 3|x|$, then $|y| \geqslant |x-y| - |x| \geqslant 2|x|$ hence $|x-y| \geqslant |y| - |x| \geqslant |y|/2$. Now $|y-x^*| \leqslant |y| + |x| \leqslant 3|y|/2$, we get $2\pi G_0(y,x) \leqslant \log 3 + \mathrm{O}(1)$. So $G_0(\cdot, x)$ is uniformly bounded in the domain $\Omega_0 \setminus B_{|x|/4}(x)$.

It remains to consider $|y-x| \leqslant |x|/4$. In this case, since $|x| \leqslant |x-x_\varepsilon| + d_\varepsilon$ and $x \in D_\varepsilon$,

$$|y-x_\varepsilon| \geqslant |x-x_\varepsilon| - |y-x| \geqslant |x-x_\varepsilon| - \frac{|x|}{4} \geqslant \frac{3|x-x_\varepsilon| - d_\varepsilon}{4} \geqslant \frac{d_\varepsilon}{2},$$

so $|y| \leqslant |y-x_\varepsilon| + |x_\varepsilon| \leqslant 3|y-x_\varepsilon|$. By the hypothesis in (27), we get $g(y) + 2\log\varepsilon + 2\log|y| \leqslant C$, if $|y-x| \leqslant |x|/4$, $y \in \Omega_0$ and $x \in D_\varepsilon$. Hence

$$I = \int_{B_{|x|/4}(x)\cap\Omega_0} G_0(y,x)\varepsilon^2 V_\varepsilon e^{g_\varepsilon(y)}\,dy \leqslant \int_{B_{|x|/4}(x)} \left(\left|\log\frac{|y-x^*|}{|y-x|}\right| + \mathrm{O}(1)\right)\varepsilon^2 \times \frac{C}{\varepsilon^2|y|^2}\,dy.$$

Using polar coordinates $y = x + |x| r e^{i\theta}$, as $|y-x^*| \leqslant 9|x|/4$ and $|y| \geqslant 3|x|/4$ in $B_{|x|/4}(x)$,

$$I \leqslant C\int_0^{1/4}\left[\log\left(\frac{9}{4r}\right) + 1\right] r\,dr < \infty.$$

Finally, for $x \in B_{\delta_0} \cap D_\varepsilon$,

$$\begin{aligned}
g_\varepsilon(x) &= \int_{\Omega_0} G_0(y,x)\varepsilon^2 V_\varepsilon e^{g_\varepsilon}\,dy \\
&= \int_{\Omega_0\setminus B_{|x|/4}(x)} G_0(y,x)\varepsilon^2 V_\varepsilon e^{g_\varepsilon}\,dy + \int_{\Omega_0\cap B_{|x|/4}(x)} G_0(y,x)\varepsilon^2 V_\varepsilon e^{g_\varepsilon}\,dy \\
&\leqslant \mathrm{O}(1) \times \int_{\Omega_0\setminus B_{|x|/4}(x)} \varepsilon^2 V_\varepsilon e^{g_\varepsilon}\,dy + I \\
&= \mathrm{O}(1).
\end{aligned}$$

Applying Lemma 3.2, we conclude then $\|g_\varepsilon\|_{L^\infty(D_\varepsilon)} = \mathrm{O}(1)$, which implies $\|\varepsilon^2 V_\varepsilon e^{g_\varepsilon}\|_{L^1(D_\varepsilon)} = \mathrm{O}(\varepsilon^2)$. Combining with (28), the claim (27) is proved.

If now

$$\max_{\Omega_0}\big[g_\varepsilon(x) + 2\log\varepsilon + 2\log|x-x_\varepsilon|\big] \to \infty, \tag{29}$$

we prove similar results as Lemmas 4 and 5 in [15] by induction, that is, up to a subsequence, we can get a family $\{x_{\varepsilon,k}\}_{1\leqslant k\leqslant m}$ such that

$$\max_{\Omega_0}\left[\frac{g_\varepsilon(x)}{2} + \log\varepsilon + \min_{1\leqslant k\leqslant m}\log|x-x_{\varepsilon,k}|\right] = \mathrm{O}(1) \quad \text{and} \quad \eta_0 = 8\pi m. \tag{30}$$

The main idea is to compare the distance between the different bubbles and their distances to the boundary. By suitable gauge transformation, we can either transform some of them into interior bubbles and then use the result in [15]; or split them into boundary bubbles which concentrate in different places of $\partial\Omega$ and then use the induction hypothesis. We show here just the situation with $m=2$ and leave the complete proof for interested readers.

Assume that (29) holds. Then by considering the point y_ε which realizes the maximum of $g_\varepsilon(x)+2\log\varepsilon+2\log|x-x_\varepsilon|$ over Ω_0, we get a second bubble by repeating the argument in [15] and the proof of Lemma 3.2. Suppose that the first estimate in (30) holds with $x_{\varepsilon,1}=x_\varepsilon$ and $x_{\varepsilon,2}=y_\varepsilon$. Define

$$\tilde{d}_\varepsilon=\min\bigl[|x_\varepsilon-y_\varepsilon|,d(x_\varepsilon,\partial\Omega),d(y_\varepsilon,\partial\Omega)\bigr].$$

Up to a subsequence and a permutation of index, we have the following possibilities:

(i) $\tilde{d}_\varepsilon=d(x_\varepsilon,\partial\Omega)$ and $|y_\varepsilon-x_\varepsilon|\leqslant C_0\tilde{d}_\varepsilon$. Under translation, we can assume $\tilde{d}_\varepsilon=|x_\varepsilon|$. Take $M_\varepsilon=(C_0+1)\tilde{d}_\varepsilon$ and consider $w_\varepsilon(x)=g_\varepsilon(M_\varepsilon x)+2\log\varepsilon+2\log M_\varepsilon$ in $B_4\cap(\mathbb{R}\times\mathbb{R}_+)$. Up to a subsequence, we have then two interior bubbles where we may use the result in [15] to claim that

$$\lim_{\varepsilon\to 0}\int\limits_{B_{4M_\varepsilon}\cap\Omega_0}\varepsilon^2V_\varepsilon e^{g_\varepsilon}\,dx=16\pi. \tag{31}$$

(ii) $\tilde{d}_\varepsilon=d(x_\varepsilon,\partial\Omega)$ and $|y_\varepsilon-x_\varepsilon|/\tilde{d}_\varepsilon\to\infty$. So $|y_\varepsilon|/|x_\varepsilon|$ tends to infinity. Take $M_\varepsilon=|y_\varepsilon|$ and consider $w_\varepsilon(x)=g_\varepsilon(M_\varepsilon x)+2\log\varepsilon+2\log M_\varepsilon$ in $B_4\cap(\mathbb{R}\times\mathbb{R}_+)$. The two bubbles are now split away, either we have one interior bubble and one boundary bubble, or we have two separated boundary bubbles. So we turn back to the simple boundary bubble case and again (31) holds.

(iii) $\tilde{d}_\varepsilon=|y_\varepsilon-x_\varepsilon|$, it suffices to take $M_\varepsilon=|x_\varepsilon|$ and w_ε as above, we transform then the bubbles to two interior bubbles as in case (i), so the result in [15] yields also (31).

Meanwhile, for all three cases, since $|x-x_\varepsilon|\geqslant|x-y_\varepsilon|/2$ in $\Omega_0\setminus B_{4M_\varepsilon}$ (for corresponding M_ε), the first estimate in (30) implies $g_\varepsilon(x)+2\log\varepsilon+2\log|x-y_\varepsilon|=\mathrm{O}(1)$ in $\Omega_0\setminus B_{4M_\varepsilon}$. We can show always $\|g_\varepsilon\|_{L^\infty(\Omega_0\setminus B_{4M_\varepsilon})}=\mathrm{O}(1)$ by using the Green's function as for (27). Now it is easy to conclude that $\eta_0=16\pi$. □

Combining with the result in [28], we get

Proposition 3.4. *Let u_ε be a family of solutions to problem* (4) *satisfying*

$$\mathcal{T}_\varepsilon=\varepsilon^2\int\limits_\Omega e^{u_\varepsilon}\,dx=\mathrm{O}(1)\quad\text{and}\quad\lim_{\varepsilon\to 0}\max_{\overline{\Omega}}u_\varepsilon=\infty.$$

Then up to a subsequence, u_ε tends to u^* in $\mathcal{D}'(\Omega)$, where u^* is the solution given by (8) *with a finite set $\mathcal{S}'=\{x_i\}\subset\Omega$, composed by critical points of a. On the other hand, there exists a finite set $\mathcal{S}=\{z_j\}\subset\partial\Omega$ such that in $\mathcal{D}'(\mathbb{R}^2)$,*

$$\varepsilon^2e^{u_\varepsilon}\chi_\Omega\,dx\to\sum_{x_i\in\mathcal{S}'}8\pi m_i\delta_{x_i}+\sum_{z_j\in\mathcal{S}}8\pi\lambda_j\delta_{z_j}.$$

Moreover $m_i,\lambda_j\in\mathbb{N}^*$, $\partial_\tau a(z_j)=0$ for any $z_j\in\mathcal{S}$ and

$$u_\varepsilon\to u^*\quad\text{in } C^k_{\mathrm{loc}}\bigl(\overline{\Omega}\setminus(\mathcal{S}'\cup\mathcal{S})\bigr),\quad\forall k\in\mathbb{N}.$$

Remark 3.5. As we have mentioned earlier, when a is constant in a neighborhood of $\partial\Omega$, $\mathcal{S}=\emptyset$.

Proof of Theorem 1.5. If it is not true, we have $\bar{x}\in\mathcal{S}$ which is a nondegenerate local maximum point of a. As in the proof of Lemma 3.2, we will take a conformal transformation to change a small neighborhood of $\bar{x}$ in Ω into $\Omega_0=B_\delta\cap(\mathbb{R}\times\mathbb{R}^*_+)$ with a replaced by $b=a\circ\Phi$ and u_ε replaced by $v_\varepsilon=u_\varepsilon\circ\Phi$. Moreover, $0=\Phi(\bar{x})$ becomes a nondegenerate local maximum point of b. Choosing $\delta>0$ small enough, we may assume that $\overline{\Omega}_0\cap\mathcal{S}=\{0\}$ and $x\cdot\nabla b(x)\leqslant 0$ in $\overline{\Omega}_0$. Using the Pohozaev identity (see [24]) obtained by multiplying $b(x)x\cdot\nabla v_\varepsilon$ to Eq. (22), we get ($\Gamma=\partial\Omega_0\setminus(\mathbb{R}\times\{0\})$)

$$\int_{\Omega_0} \operatorname{div}\left[b(x)|\nabla\Phi|^2 x\right]\varepsilon^2 e^{v_\varepsilon}\,dx = \int_\Gamma \varepsilon^2 b(x)|\nabla\Phi|^2(x\cdot\nu)e^{v_\varepsilon}\,d\sigma - \frac{1}{2}\int_\Gamma b(x)(x\cdot\nu)|\nabla v_\varepsilon|^2\,d\sigma + \int_\Gamma b(x)\frac{\partial v_\varepsilon}{\partial\nu}(x\cdot\nabla v_\varepsilon)\,dx + \frac{1}{2}\int_{\Omega_0}(x\cdot\nabla b)|\nabla v_\varepsilon|^2\,dx,$$

since $x\cdot\nu = x\cdot\nabla v_\varepsilon = 0$ over $[-\delta,\delta]\times\{0\}$. Thanks to Lemma 3.2, we see that the first three terms in the right-hand side tend to zero as $\varepsilon\to 0$ while the last term is nonpositive. However, the left-hand side tends to $16a(\bar{x})\pi m$ with $m\in\mathbb{N}^*$, this is just impossible. □

Remark 3.6. The nondegeneracy condition can be erased if we know that $x\cdot\nabla b(x)\leqslant 0$ near 0 in Ω_0. It is worth to mention that Theorem 1.5 cannot be proved by the usual moving plane method, however we state here a special case which is a direct consequence of this method.

Proposition 3.7. *Assume that $a(x)\equiv a(r)$ for $x=(r,s)\in\overline{\Omega}$ and a is nondecreasing with respect to r. Assume also $\mathcal{T}_\varepsilon = \mathrm{O}(1)$. Then $\mathcal{S}\cap\{r=r_{\max}\}$ is empty, where $r_{\max}=\max_{x\in\overline{\Omega}} r$.*

Proof. In this case, we see that Eq. (4) is in the form $\Delta u + c_1(x)\partial_1 u + \varepsilon^2 e^u = 0$ with $c_1\geqslant 0$ in Ω, thus the moving plane argument (see Theorem 2.1 in [12]) implies that for any $x_0=(r_{\max},s)\in\partial\Omega$, there exists a neighborhood U of x_0, independent of ε, such that u_ε is decreasing with respect to r in U. Hence no blow up can occur at x_0 because $\mathcal{T}_\varepsilon=\mathrm{O}(1)$. □

4. Existence of boundary bubbling solution

From now on, we will construct a family of blow up solutions stated in Theorem 1.4. We should define some approximate solutions, and choose an appropriate configuration space for the parameters. Then we need to understand the behavior of linearized operator around these approximate solutions, to get suitable functional settings and the inverse of these linearized operators. Finally, we solve the nonlinear problem and conclude by variational techniques. This procedure has been used successfully in constructing interior bubbles for Eq. (2) in [9,10] and for Eq. (4) in [26]. Here we construct boundary bubbles. Our difficulty is to estimate precisely the distance between the bubble and the boundary.

4.1. Approximate solution

Given $\xi\in\Omega$ and $\mu>0$, we define

$$u(x)=\log\frac{8\mu^2}{(\varepsilon^2\mu^2+|x-\xi|^2)^2}.$$

The configuration space for ξ is chosen as the following

$$\Lambda \stackrel{\text{def}}{=} \left\{\xi\in\overline{B}_\delta(\bar{x})\cap\Omega;\ \frac{C_1}{|\log\varepsilon|}\leqslant(\xi,\partial\Omega)\leqslant\frac{C_2}{|\log\varepsilon|}\right\} \tag{32}$$

where $C_1, C_2>0$ will be determined later on and μ is chosen as

$$\log(8\mu^2)=H_R(\xi)=H(\xi,\xi). \tag{33}$$

Using the choice of Λ and the estimate (18), there exists $C>0$ such that for $\varepsilon>0$ small,

$$\frac{1}{C|\log\varepsilon|^2}\leqslant\mu\leqslant\frac{C}{|\log\varepsilon|^2}. \tag{34}$$

We get also $u(x)=\log(8\mu^2)-4\log|x-\xi|+\mathrm{O}(\varepsilon^2\mu)$ on $\partial\Omega$. The ansatz is then $U(x)=u(x)+H^\varepsilon(x)$ where H^ε is a correction term defined as the solution of

$$\Delta_a H^\varepsilon + \nabla\log a\nabla u = 0 \quad \text{in } \Omega, \qquad H^\varepsilon = -u \quad \text{on } \partial\Omega. \tag{35}$$

By the same proof as for Lemma 2.4 in [26], we obtain that for any $0 < \alpha < 1$,

$$H^\varepsilon(x) = H(x,\xi) - \log(8\mu^2) + \mathrm{O}(\varepsilon^\alpha)$$

uniformly for $x \in \overline{\Omega}$ and $\xi \in \Lambda$. Moreover, it will be convenient to work with the scaling of u given by

$$v(y) = u(\varepsilon y) + 4\log\varepsilon = \log\frac{8\mu^2}{(\mu^2 + |y-\xi'|^2)^2}$$

where $\xi' = \xi/\varepsilon$ and $\Omega_\varepsilon = \Omega/\varepsilon$. To resolve (4), it suffices to obtain w such that

$$\Delta_{a(\varepsilon y)} w + e^w = 0 \quad \text{in } \Omega_\varepsilon, \qquad w = 4\log\varepsilon \quad \text{on } \partial\Omega_\varepsilon. \tag{36}$$

We will seek a solution w in the form $w = V + \phi$ where $V(y) = U(\varepsilon y) + 4\log\varepsilon = v(y) + H^\varepsilon(\varepsilon y)$. Problem (36) can be then stated as to finding ϕ, a solution to

$$\Delta_{a(\varepsilon y)}\phi + e^V\phi + N(\phi) + E = 0 \quad \text{in } \Omega_\varepsilon \quad \text{and} \quad \phi = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

where the nonlinear term is $N(\phi) = e^V(e^\phi - 1 - \phi)$ and the error term is $E = \Delta_{a(\varepsilon y)}V + e^V$. Note that V satisfies $\Delta_{a(\varepsilon y)}V + e^v = 0$ in Ω_ε, we claim the following estimate for E:

Lemma 4.1. *For any $\alpha \in (0,1)$, there exists C independent of $\varepsilon > 0$ small enough and $\xi \in \Lambda$ such that*

$$\big|E(y)\big| \leqslant C\varepsilon^\alpha\left[\varepsilon^2 + \frac{1}{\mu^2(1+\mu^{-3}|y-\xi'|^3)}\right], \quad \forall y \in \Omega_\varepsilon. \tag{37}$$

Proof. By definition, we have

$$E = \frac{8\mu^2}{(\mu^2+|y-\xi'|^2)^2}\big[e^{H^\varepsilon(\varepsilon y)} - 1\big].$$

Using the choice of μ and (17)

$$\begin{aligned} H^\varepsilon(\varepsilon y) &= H(\varepsilon y,\xi) - \log(8\mu^2) + \mathrm{O}(\varepsilon^\alpha) = H(\varepsilon y,\xi) - H(\xi,\xi) + \mathrm{O}(\varepsilon^\alpha) \\ &= \mathrm{O}\big(\varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha\big) + \mathrm{O}(\varepsilon^\alpha). \end{aligned}$$

Fix $1/2 < \beta < \alpha \leqslant 1$. For $|y-\xi'| \leqslant \mu\varepsilon^{-1}$, then

$$\begin{aligned} E &= \frac{8\mu^2}{(\mu^2+|y-\xi'|^2)^2}\big[e^{\mathrm{O}(\varepsilon^\alpha+\varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha)} - 1\big] = \frac{8\mu^2 \times \mathrm{O}(\varepsilon^\alpha + \varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha)}{(\mu^2+|y-\xi'|^2)^2} \\ &= \frac{\mathrm{O}(\varepsilon^\beta)}{\mu^2(1+\mu^{-3}|y-\xi'|^3)}. \end{aligned} \tag{38}$$

Here we have used (34), $\varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha = \mathrm{O}(1)$ and $\sup_{\mathbb{R}_+} t^\alpha/(1+t) \leqslant 1$ for all $\alpha < 1$.

Notice that H is uniformly upper bounded over $\Omega \times \Omega$, which can be seen by its equation and the maximum principle. Therefore, for $|y-\xi'| > \mu\varepsilon^{-1}$, using again (34), we get

$$E = \frac{8\mu^2}{(\mu^2+|y-\xi'|^2)^2}\big[e^{H(\varepsilon y,\xi)-\log(8\mu^2)+\mathrm{O}(\varepsilon^\alpha)} - 1\big] = \frac{8}{(\mu^2+|y-\xi'|^2)^2} \times \mathrm{O}(1) = \mathrm{O}(\varepsilon^{4-\alpha}),$$

for any $\alpha \in (0,1)$. Combining the two parts of estimate, we get immediately (37). $\square$

The same arguments deduce also the estimate for $W = e^V$ as follows:

$$W(y) = \begin{cases} \mathrm{O}\big(\mu^{-2}\big(1+\mu^{-1}|y-\xi'|\big)^{-4}\big) & \text{if } |y-\xi'| \leqslant \mu\varepsilon^{-1}, \\ \mathrm{O}(\varepsilon^{4-\alpha}) & \text{if } |y-\xi'| \geqslant \mu\varepsilon^{-1}. \end{cases} \tag{39}$$

4.2. Linearized equations and nonlinear problem

Consider now the following linear problem associated to the approximate solution V: Given $h \in L^\infty(\Omega_\varepsilon)$, find ϕ, c_1 and c_2 such that

$$\begin{cases} -\Delta_{a(\varepsilon y)}\phi = W\phi + h + \frac{1}{a(\varepsilon y)}\sum_{i=1}^{2} c_i Z_i \chi & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi Z_i \chi \, dy = 0, & \text{for } i = 1, 2 \end{cases} \tag{40}$$

where W is a function satisfying (39), Z_i, χ are defined as follows. Denote

$$Z_0 = 1 - \frac{2\mu^2}{\mu^2 + |y - \xi'|^2} \quad \text{and} \quad Z_i = \frac{(y - \xi')_i}{\mu^2 + |y - \xi'|^2} \quad \text{for } i = 1, 2.$$

We choose a large but fixed number R_0 and a nonnegative smooth function χ_0: $\mathbb{R} \to \mathbb{R}$ so that $\chi_0(r) = 1$ for $r \leqslant R_0$ and $\chi_0(r) = 0$ for $r \geqslant R_0 + 1$, $0 \leqslant \chi_0 \leqslant 1$. The cut-off we use is just given by $\chi(y) = \chi_0(|y - \xi'|/\mu)$. The functions Z_i and χ depend on μ, ε and ξ, but we omit this dependence in the notation for simplicity. Eq. (40) will be solved for $h \in L^\infty(\Omega_\varepsilon)$, and we will estimate the size of the solution in terms of the following norm

$$\|h\|_* = \sup_{y \in \Omega_\varepsilon} \frac{|h(y)|}{\varepsilon^2 + \mu^{-2}(1 + |y - \xi'|/\mu)^{-3}}. \tag{41}$$

Proposition 4.2. *There exist $\varepsilon_0 > 0$, $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $\xi \in \Lambda$ and $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, $c_i \in \mathbb{R}$ to* (40). *Moreover*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C|\log\varepsilon|\|h\|_* \quad \textit{and} \quad |c_i| \leqslant C|\log\varepsilon|^2\|h\|_*.$$

To prove this result, a crucial argument is to get the following *a priori* estimates of solutions, with respectively orthogonality conditions to all Z_i, $0 \leqslant i \leqslant 2$; or just to Z_1, Z_2.

Lemma 4.3. *There are $R_0 > 0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and ψ, ϕ solutions respectively to*

$$\begin{cases} -\Delta_{a(\varepsilon y)}\psi = W\psi + h & \text{in } \Omega_\varepsilon, \\ \psi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi Z_i \psi \, dy = 0 & \forall i = 0, 1, 2, \end{cases} \tag{42}$$

and

$$\begin{cases} -\Delta_{a(\varepsilon y)}\phi = W\phi + h & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi Z_i \phi \, dy = 0 & \forall i = 1, 2, \end{cases} \tag{43}$$

we have

$$\|\psi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\|h\|_* \quad \textit{and} \quad \|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C|\log\varepsilon|\|h\|_* \tag{44}$$

where C is independent of $\varepsilon \in (0, \varepsilon_0)$.

Proof. The proof of estimate for ψ is totally similar to the proof of Lemma 3.2 in [26], we need just to remark that since $\mu \sim |\log\varepsilon|^{-2}$, then for any fixed $R > 0$, we have $B_{\mu R}(\xi') \subset \Omega_\varepsilon$ for ε small enough and $\xi \in \Lambda$ since $d(\xi', \Omega_\varepsilon) \sim \varepsilon^{-1}|\log\varepsilon|^{-1}$.

Let ϕ satisfy (43). We will modify ϕ to satisfy all the orthogonality relations as for ψ (see (47) below). For this purpose we consider modifications with compact support of the function Z_0. Let $R > R_0 + 1$ be large enough which value will be determined later on. Let

$$a_0 = \frac{1}{-4\log(\varepsilon\mu R) + H(\xi, \xi)}.$$

Note that we have $H(\xi, \xi) = O(\log|\log\varepsilon|)$, since $\xi \in \Lambda$. So it is easy to see that

$$\lim_{\varepsilon \to 0} 4|\log\varepsilon| a_0 = 1, \quad \text{uniformly in } \Lambda. \tag{45}$$

Let η_0 be a radial smooth cut-off function on $\mathbb{R}^2$ so that

$$0 \leqslant \eta_0 \leqslant 1, \quad \eta_0 \equiv 1 \quad \text{in } B_R(0) \quad \text{and} \quad \eta_0 \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R+1}(0).$$

Denote

$$\eta(y) = \eta_0\left(\frac{|y-\xi'|}{\mu}\right)$$

and also

$$\widehat{Z}_0(y) = Z_0(y) - 1 + a_0 G(\varepsilon y, \xi) + T(\varepsilon y),$$

where T is a correction term, solution of

$$-\Delta_a T = 0 \quad \text{in } \Omega, \qquad T(x) = 1 - Z_0(x/\varepsilon) \quad \text{on } \partial\Omega. \tag{46}$$

Take now $\widetilde{Z}_0 = \eta Z_0 + (1-\eta)\widehat{Z}_0$ and $\tilde{\phi} = \phi + \lambda \widetilde{Z}_0$, we adjust $\tilde{\phi}$ to satisfy the orthogonality condition:

$$\int_{\Omega_\varepsilon} \tilde{\phi} Z_i \chi \, dy = 0, \quad \text{for all } 0 \leqslant i \leqslant 2. \tag{47}$$

Let $L = -\Delta_{a(\varepsilon y)} - W$, we claim then

$$\|\widetilde{Z}_0\|_\infty \leqslant C, \quad \|L\widetilde{Z}_0\|_* \leqslant \frac{C}{|\log\varepsilon|} \quad \text{and} \quad |\lambda| \leqslant C|\log\varepsilon|\|h\|_*. \tag{48}$$

Estimate (44) for ϕ is now a direct consequence of (48). Indeed, as $L\tilde{\phi} = h + \lambda L\widetilde{Z}_0$, using conclusion for ψ, we have

$$\|\tilde{\phi}\|_\infty \leqslant C\big(\|h\|_* + |\lambda|\big\|L(\widetilde{Z}_0)\big\|_*\big) \leqslant C\|h\|_*. \tag{49}$$

Therefore $\|\phi\|_\infty \leqslant \|\tilde{\phi}\|_\infty + |\lambda|\|\widetilde{Z}_0\|_\infty \leqslant C|\log\varepsilon|\|h\|_*$. For getting (48), we show first the estimate of functions T and Z_0. Since

$$1 - Z_0\left(\frac{x}{\varepsilon}\right) = \frac{2\varepsilon^2\mu^2}{\varepsilon^2\mu^2 + |x-\xi|^2} \quad \text{on } \partial\Omega,$$

we get readily for $\xi \in \Lambda$,

$$\left\|1 - Z_0\left(\frac{x}{\varepsilon}\right)\right\|_{C^k(\partial\Omega)} = \mathrm{O}(\varepsilon^{2-\alpha}), \quad \forall \alpha > 0,\ k \in \mathbb{N}. \tag{50}$$

Thus, the elliptic theory implies that

$$\|T\|_{C^k(\overline{\Omega})} = \mathrm{O}(\varepsilon^{2-\alpha}), \quad \text{for any } \alpha > 0,\ k \in \mathbb{N}. \tag{51}$$

On the other hand, we have $\|\Delta_{a(\varepsilon y)} Z_0 + e^v Z_0\|_* = \mathrm{O}(\varepsilon\mu)$, because

$$\Delta_{a(\varepsilon y)} Z_0 + e^v Z_0 = \varepsilon \nabla \log a(\varepsilon y) \nabla Z_0 = \mathrm{O}\left(\varepsilon\mu^{-1}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right). \tag{52}$$

To estimate $\|\widetilde{Z}_0\|_\infty$, as $\|Z_0\|_\infty \leqslant 1$, we need only to consider the term $(1-\eta)a_0 G(\varepsilon y, \xi)$. When $1-\eta \neq 0$, as $G(\varepsilon y, \xi) = -4\log(\varepsilon|y-\xi'|) + H(\varepsilon y, \xi)$ and $\varepsilon|y-\xi'| \in (\varepsilon\mu R, \mathrm{diam}_\Omega)$, combined with the estimates (17) and (45), we obtain

$$\|\widetilde{Z}_0\|_\infty \leqslant C, \quad \text{for } \varepsilon > 0 \text{ small enough}. \tag{53}$$

For the estimate of $\|L\widetilde{Z}_0\|_*$, we decompose Ω_ε into three regions: $\Omega_1 = \{|y-\xi'| \leqslant \mu R\}$, $\Omega_2 = \{\mu R < |y-\xi'| \leqslant \mu(R+1)\}$ and $\Omega_3 = \{|y-\xi'| \geqslant \mu(R+1)\}$.

On Ω_1, we get by (52),

$$L\widetilde{Z}_0 = LZ_0 = \mathrm{O}\left(\varepsilon\mu^{-1}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right) + (e^v - W)Z_0. \tag{54}$$

According to (38),

$$(e^v - W)Z_0 = -EZ_0 = \mathrm{O}\left(\varepsilon^\alpha \mu^{-2}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right), \quad \text{for any } \alpha \in (0,1).$$

Therefore

$$\left|L\widetilde{Z}_0(y)\right| = \mathrm{O}\left(\varepsilon^\alpha \mu^{-2}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right), \quad \forall y \in \Omega_1. \tag{55}$$

On Ω_2, we have

$$\begin{aligned} L\widetilde{Z}_0 &= \eta L Z_0 + (1-\eta)L\widehat{Z}_0 + 2\nabla\eta\nabla(\widehat{Z}_0 - Z_0) + (\widehat{Z}_0 - Z_0)\Delta_{a(\varepsilon y)}\eta \\ &= LZ_0 - (1-\eta)W(\widehat{Z}_0 - Z_0) + 2\nabla\eta\nabla(\widehat{Z}_0 - Z_0) + (\widehat{Z}_0 - Z_0)\Delta_{a(\varepsilon y)}\eta. \end{aligned}$$

The estimate of LZ_0 is the same as (54). Using (17) and (51), for $\xi \in \Lambda$,

$$\begin{aligned} \widehat{Z}_0 - Z_0 &= a_0 G(\varepsilon y, \xi) - 1 + T(\varepsilon y) = a_0\left[4\log\frac{\mu R}{|y-\xi'|} + H(\varepsilon y, \xi) - H(\xi,\xi)\right] + \mathrm{O}(\varepsilon^{2-\alpha}) \\ &= a_0\left[\mathrm{O}(1) + \mathrm{O}\left(\varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha\right)\right] + \mathrm{O}\left(\varepsilon^{2-\alpha}\right). \end{aligned}$$

We have also

$$\nabla(\widehat{Z}_0 - Z_0) = a_0\left[\mathrm{O}\left(\frac{1}{|y-\xi'|}\right) + \varepsilon\nabla_x H(\varepsilon y, \xi)\right] + \mathrm{O}(\varepsilon^{3-\alpha}).$$

Applying (45), the expansion of H and (19), we derive then

$$\widehat{Z}_0 - Z_0 = \mathrm{O}\left(\frac{1}{|\log\varepsilon|}\right), \quad \nabla(\widehat{Z}_0 - Z_0) = \mathrm{O}\left(\frac{1}{\mu|\log\varepsilon|}\right) \quad \text{in } \Omega_2. \tag{56}$$

Moreover, $|\nabla\eta| = \mathrm{O}(\mu^{-1})$ and $\Delta_{a(\varepsilon y)}\eta = \mathrm{O}(\mu^{-2})$, we obtain finally by (39)

$$\left\|(1-\eta)W(Z_0 - \widehat{Z}_0) + 2\nabla\eta\nabla(\widehat{Z}_0 - Z_0) + (\widehat{Z}_0 - Z_0)\Delta_{a(\varepsilon y)}\eta\right\|_{L^\infty(\Omega_2)} = \mathrm{O}\left(\frac{1}{\mu^2|\log\varepsilon|}\right).$$

Hence

$$\|L\widetilde{Z}_0\|_{L^\infty(\Omega_2)} = \mathrm{O}\left(\frac{1}{\mu^2|\log\varepsilon|}\right). \tag{57}$$

On Ω_3, since $\widetilde{Z}_0 = \widehat{Z}_0$, so

$$L\widetilde{Z}_0 = -\Delta_{a(\varepsilon y)}Z_0 - W\widehat{Z}_0 = -(\Delta_{a(\varepsilon y)}Z_0 + e^v Z_0) + EZ_0 + W(Z_0 - \widehat{Z}_0).$$

We have always (52), it suffices to consider the last two terms. For this propose, we decompose Ω_3 to two subregions: $\Omega_{31} = \{\mu(R+1) \leqslant |y-\xi'| < \mu\varepsilon^{-1}\}$ and $\Omega_{32} = \{|y-\xi'| \geqslant \mu\varepsilon^{-1}\}$. For $y \in \Omega_{31}$ and any $\alpha \in (0,1)$,

$$\begin{aligned} Z_0 - \widehat{Z}_0 &= a_0\left[4\log\frac{\mu R}{|y-\xi'|} + H(\varepsilon y, \xi) - H(\xi,\xi)\right] + \mathrm{O}(\varepsilon^{2-\alpha}) \\ &= \mathrm{O}\left(\frac{1}{|\log\varepsilon|}\right) \times \left[\log\frac{\mu R}{|y-\xi'|} + \mathrm{O}\left(\varepsilon^\alpha|\log\varepsilon||y-\xi'|^\alpha\right)\right] + \mathrm{O}(\varepsilon^{2-\alpha}) \\ &= \mathrm{O}\left(\frac{1}{|\log\varepsilon|}\right) \times \mathrm{O}\left(1 + \frac{|y-\xi'|}{\mu}\right). \end{aligned}$$

Making use of (38) and (39),

$$EZ_0 + W(Z_0 - \widehat{Z}_0) = \mathrm{O}\left(\frac{1}{\mu^2|\log\varepsilon|}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right) \quad \text{in } \Omega_{31}. \tag{58}$$

For $y \in \Omega_{32}$, we have $EZ_0 + W(Z_0 - \widehat{Z}_0) = e^v Z_0 - W\widetilde{Z}_0$. Thus $W = \mathrm{O}(\varepsilon^{4-\alpha})$, $e^v = \mathrm{O}(\varepsilon^{4-\alpha})$ and Z_0, $\widetilde{Z}_0$ are uniformly bounded,

$$\left|L\widetilde{Z}_0(y)\right| = \mathrm{O}\left(\varepsilon^\alpha \mu^{-2}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right) \quad \text{in } \Omega_{32}. \tag{59}$$

Combining the estimates (55), (57)–(59), we conclude finally

$$\|L\widetilde{Z}_0\|_* \leqslant \frac{C}{|\log\varepsilon|}, \quad \forall \xi \in \Lambda. \tag{60}$$

Now we prove the estimate for λ. Multiplying the equation $L\tilde{\phi} = h + \lambda L\widetilde{Z}_0$ by $a(\varepsilon y)\widetilde{Z}_0$, integrating by parts (use $\widetilde{Z}_0 = 0$ on $\partial\Omega_\varepsilon$) and the first inequality in (49), we find

$$\begin{aligned}\left|\lambda \int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_0 L\widetilde{Z}_0\, dy\right| &= \left|-\int_{\Omega_\varepsilon} a(\varepsilon y) h\widetilde{Z}_0\, dy + \int_{\Omega_\varepsilon} a(\varepsilon y)\tilde{\phi} L\widetilde{Z}_0\, dy\right| \\ &\leqslant C\|h\|_*\left(1 + \|L\widetilde{Z}_0\|_*\right) + C|\lambda|\|L\widetilde{Z}_0\|_*^2.\end{aligned} \tag{61}$$

We need just to show a convenient lower bound of the left-hand side. Decompose the domain Ω_ε as before, by (55), we have

$$\int_{\Omega_1} a(\varepsilon y)\widetilde{Z}_0 L\widetilde{Z}_0\, dy = \mathrm{O}(\varepsilon^\alpha), \quad \forall \alpha \in (0,1).$$

From (60) and (53), we derive that

$$\int_{\Omega_3} a(\varepsilon y)\widetilde{Z}_0 L\widetilde{Z}_0\, dy \leqslant \frac{C}{|\log\varepsilon|} \int_{R+1}^{\infty} \frac{r\, dr}{1+r^3} = \mathrm{O}\left(\frac{1}{R|\log\varepsilon|}\right).$$

It remains to estimate the integrate over Ω_2. Let

$$\begin{aligned} I &= \int_{\Omega_2} a(\varepsilon y)\widetilde{Z}_0 L\widetilde{Z}_0\, dy \\ &= \int_{\Omega_2} a(\varepsilon y)\widetilde{Z}_0\left[\eta L Z_0 + (1-\eta)L\widehat{Z}_0\right] dy + \int_{\Omega_2} 2a(\varepsilon y)\widetilde{Z}_0 \nabla\eta\nabla(\widehat{Z}_0 - Z_0)\, dy \\ &\quad + \int_{\Omega_2} \widetilde{Z}_0(\widehat{Z}_0 - Z_0)\nabla\cdot\left[a(\varepsilon y)\nabla\eta\right] dy.\end{aligned}$$

The integration by parts for the last term will deduce

$$\begin{aligned} I &= \int_{\Omega_2} a(\varepsilon y)\widetilde{Z}_0\left[L Z_0 - (1-\eta)W(\widehat{Z}_0 - Z_0)\right] dy + \int_{\Omega_2} a(\varepsilon y)(\widehat{Z}_0 - Z_0)^2|\nabla\eta|^2\, dy \\ &\quad + \int_{\Omega_2} a(\varepsilon y)\widehat{Z}_0\nabla\eta\nabla(\widehat{Z}_0 - Z_0)\, dy - \int_{\Omega_2} a(\varepsilon y)(\widehat{Z}_0 - Z_0)\nabla\eta\nabla\widehat{Z}_0\, dy \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Applying (52), (56) and (39),

$$L Z_0 - (1-\eta)W(\widehat{Z}_0 - Z_0) = \mathrm{O}\left(\frac{1}{\mu^2|\log\varepsilon|(1+r^3)}\right)$$

where $r = |y-\xi'|/\mu$. Thus

$$I_1 = \mathrm{O}\left(\frac{1}{R^2|\log\varepsilon|}\right).$$

Moreover, $|\nabla\eta| = \mathrm{O}(\mu^{-1})$ and $|\nabla\widehat{Z}_0| = \mathrm{O}(\mu^{-1}r^{-3})$ in Ω_2. Using again (56), we get

$$I_2 = \mathrm{O}\left(\frac{R}{|\log\varepsilon|^2}\right) \quad \text{and} \quad I_4 = \mathrm{O}\left(\frac{1}{|\log\varepsilon| R^2}\right).$$

As $\widehat{Z}_0 - Z_0 = \mathrm{o}(Z_0)$ in Ω_2, using (45),

$$I_3 = \int_{\Omega_2} a(\varepsilon y)\widehat{Z}_0 \nabla\eta\nabla(\widehat{Z}_0 - Z_0)\,dy = a_0 \int_R^{R+1} a(\xi)\eta_1'(r)\frac{1-r^2}{1+r^2}\big[4 + \mathrm{o}(1)\big]\,dr$$
$$= \frac{a(\xi)}{|\log\varepsilon|}\big[1 + \mathrm{o}(1) + \mathrm{O}(R^{-2})\big].$$

Combining all these estimates, we conclude that for R large enough and ε small enough,

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_0 L\widetilde{Z}_0\,dy \geqslant \frac{C}{|\log\varepsilon|}. \tag{62}$$

Inserting this lower bound and (60) in (61), we obtain

$$\frac{|\lambda|}{|\log\varepsilon|} \leqslant C\left(\|h\|_* + \frac{|\lambda|}{|\log\varepsilon|^2}\right),$$

which yields readily $|\lambda| \leqslant C|\log\varepsilon|\|h\|_*$, our proof is completed. □

Proof of Proposition 4.2. First, we prove some a priori estimates for ϕ, c_i, solutions of (40). By the previous lemma,

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C|\log\varepsilon|\left(\|h\|_* + \sum_{i=1}^{2}|c_i|\|Z_i\chi\|_*\right). \tag{63}$$

It is easy to see that $\|Z_i\chi\|_* = \mathrm{O}(\mu)$ for $i = 1, 2$, so a main step is to estimate the constants c_i. To this end, we multiply (40) by $a(\varepsilon y)\widetilde{Z}_i$ where $\widetilde{Z}_i = Z_i\eta_1$ and

$$\eta_1(y) = \chi_1\left(\frac{\varepsilon(y-\xi')}{\mu}\right), \quad \chi_1 \equiv 1 \quad \text{in } B_1, \quad \chi_1 \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_2.$$

We have then

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_i L\phi\,dy = \int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_i\,dy + \sum_{k=1}^{2} c_k \int_{\Omega_\varepsilon} \widetilde{Z}_i Z_k\chi\,dy$$
$$= \int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_i\,dy + c_i \int_{\Omega_\varepsilon} \widetilde{Z}_i^2\chi\,dy. \tag{64}$$

We claim that $\|L\widetilde{Z}_i\|_* \leqslant C\varepsilon^{1/3}\mu^{-1}$ for $i = 1, 2$. Decompose the domain Ω_ε to two regions: $\Omega_1 = \{|y-\xi'| \leqslant \mu\varepsilon^{-1}\}$ and $\Omega_2 = \{|y-\xi'| \in (\mu\varepsilon^{-1}, 2\mu\varepsilon^{-1})\}$. Since $\widetilde{Z}_i = Z_i$ in Ω_1, so

$$L\widetilde{Z}_i = -\Delta_{a(\varepsilon y)}Z_i - WZ_i = -\varepsilon\nabla\log a(\varepsilon y)\nabla Z_i - (W - e^v)Z_i. \tag{65}$$

We have, by Young's inequality,

$$-\varepsilon\nabla\log a(\varepsilon y)\nabla Z_i = \mathrm{O}\left(\varepsilon\mu^{-2}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-2}\right)$$
$$= \mathrm{O}(\varepsilon^{7/3}) + \mathrm{O}\left(\varepsilon^{1/3}\mu^{-3}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right).$$

Using (38), we obtain

$$(W - e^v)Z_i = EZ_i = \mathrm{O}\left(\varepsilon^\alpha\mu^{-2}\left[1 + \frac{|y-\xi'|}{\mu}\right]^{-3}\right) \times \mathrm{O}(\mu^{-1}), \quad \forall\alpha \in (0, 1).$$

Hence

$$\left|L\widetilde{Z}_i(y)\right| = \mathrm{O}(\varepsilon^{7/3}) + \mathrm{O}\left(\varepsilon^{1/3}\mu^{-3}\left[1+\frac{|y-\xi'|}{\mu}\right]^{-3}\right) \quad \text{in } \Omega_1. \tag{66}$$

In Ω_2, we have $L\widetilde{Z}_i = (LZ_i)\eta_1 - 2\nabla\eta_1\nabla Z_i - Z_i\Delta_{a(\varepsilon y)}\eta_1$. Since

$$e^v = \mathrm{O}(\varepsilon^{4-\alpha}), \quad W = \mathrm{O}(\varepsilon^{4-\alpha}), \quad Z_i = \mathrm{O}\big(|y-\xi'|^{-1}\big), \quad \nabla Z_i = \mathrm{O}\big(|y-\xi'|^{-2}\big)$$

and also $\nabla\eta_1 = \mathrm{O}(\varepsilon\mu^{-1})$, $\Delta_{a(\varepsilon y)}\eta_1 = \mathrm{O}(\varepsilon^2\mu^{-2})$, we deduce easily $\|L\widetilde{Z}_i\|_{L^\infty(\Omega_2)} = \mathrm{O}(\varepsilon^{2+\alpha})$ for any $\alpha \in (0,1)$. Combining with (66), our claim is true. Consequently

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_i L\phi\, dy = \int_{\Omega_\varepsilon} a(\varepsilon y)\phi L\widetilde{Z}_i\, dy = \mathrm{O}\big(\varepsilon^{1/3}\mu^{-1}\|\phi\|_\infty\big). \tag{67}$$

On the other hand,

$$\int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_i\, dy = \mathrm{O}\big(\mu^{-1}\|h\|_*\big) \tag{68}$$

and by definition,

$$\int_{\Omega_\varepsilon} \widetilde{Z}_i^2\chi\, dy = C_0.$$

Substituting (68) and (67) into (64), we obtain

$$|c_i| \leqslant C\big(\mu^{-1}\|h\|_* + \varepsilon^{1/3}\mu^{-1}\|\phi\|_\infty\big), \quad i = 1, 2.$$

Combine with (63) and recall that $\mu \sim |\log\varepsilon|^{-2}$, finally

$$\|\phi\|_\infty \leqslant C|\log\varepsilon|\|h\|_*. \tag{69}$$

Consider the Hilbert space

$$H = \left\{\phi \in H_0^1(\Omega_\varepsilon);\ \int_{\Omega_\varepsilon} \chi Z_i\phi\, dy = 0 \quad \text{for } i = 1, 2\right\}$$

endowed with the norm $\|\phi\|_{H_0^1} = \|\nabla\phi\|_{L^2(\Omega_\varepsilon)}$. Eq. (40) is equivalent to find $\phi \in H$ such that

$$\int_{\Omega_\varepsilon} \big[a(\varepsilon y)\nabla\phi\nabla\psi - a(\varepsilon y)W\phi\psi\big]\, dy = \int_{\Omega_\varepsilon} a(\varepsilon y)h\psi\, dy, \qquad \forall\psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to the problem, which is guaranteed by (69). □

Proposition 4.2 implies that the unique solution to (40), $\phi = T(h)$ defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in $L^\infty(\Omega_\varepsilon)$ endowed with the norm $\|\cdot\|_*$, into $L^\infty(\Omega_\varepsilon)$. We need also the differentiability of the operator T with respect to the variable ξ'. Indeed, we can compute the derivatives of ϕ with respect to ξ' and obtain their estimates as follows

$$\big\|\partial_{\xi'}T(h)\big\|_\infty \leqslant C|\log\varepsilon|^5\|h\|_*. \tag{70}$$

Sketch of Proof. The proof is similar to that in [9] or [26], here the difficulty comes from the fact that ξ goes to the boundary as ε tends to 0. The most delicate point is to estimate $\|\partial_{\xi'}W\|_*$. Since $W = e^V$, we need just to estimate $\|\partial_{\xi'}V\|_\infty$ thanks to (39). Consider first the variation of μ. Thanks to (18) and (33), we get readily $|\partial_\xi\mu| = \mathrm{O}(\mu|\log\varepsilon|)$ over Λ. This will lead to $\|\partial_\xi u\|_\infty = \mathrm{O}(\varepsilon^{-1}\mu^{-1})$. Using then the equation for the ansatz U and the maximum principle, we obtain

$$\|\partial_\xi U\|_\infty \leqslant \|\partial_\xi u\|_\infty\|U\|_\infty = \mathrm{O}\left(\frac{|\log\varepsilon|^3}{\varepsilon}\right).$$

After the scaling, this yields $\|\partial_{\xi'} V\|_\infty = O(|\log\varepsilon|^3)$. $\square$

Now we are in position to solve the nonlinear equation associated to V.

$$\begin{cases} -\Delta_{a(\varepsilon y)}\phi - W\phi = E + N(\phi) + \frac{1}{a(\varepsilon y)}\sum_{i=1,2} c_i Z_i \chi & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi Z_i \phi\, dy = 0, \quad \forall i = 1,2 \end{cases} \tag{71}$$

where $W = e^V$, $N(\phi) = e^V(e^\phi - 1 - \phi)$ is the nonlinear term and $E = \Delta_{a(\varepsilon y)} V + e^V$ is the error term. We have the following result.

Lemma 4.4. *Let $\alpha \in (0,1)$. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $\xi \in \Lambda$ the problem* (71) *admits a unique solution ϕ, c_i such that*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\varepsilon^\alpha |\log\varepsilon|. \tag{72}$$

Furthermore, the function $\xi' \mapsto \phi(\xi')$ is in C^1 and

$$\|\nabla_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\,\varepsilon^\alpha |\log\varepsilon|^5.$$

The proof can be done along the lines of those of Lemma 4.1 of [9] by fixed point argument, so we omit the details.

4.3. Variational reduction and expansion of the energy

In view of Lemma 4.4, given $\xi \in \Lambda$, we can define $\phi(\xi')$ and $c_i(\xi')$ to be the unique solution to (71) satisfying (72). Recall the ansatz $U(\xi) = u(x) + H^\varepsilon(x)$, we set

$$\mathcal{F}_\varepsilon(\xi) = J_\varepsilon\big(U(\xi) + \tilde\phi(\xi)\big), \tag{73}$$

where J_ε is the functional associated to Eq. (4), i.e.

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega a(x)|\nabla v|^2\, dx - \varepsilon^2 \int_\Omega a(x) e^v\, dx$$

and

$$\tilde\phi(\xi)(x) = \phi\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right) \quad \text{in } \Omega. \tag{74}$$

Lemma 4.5. *If $\xi \in \Lambda$ is a critical point of $\mathcal{F}_\varepsilon$ then $u = U(\xi) + \tilde\phi(\xi)$ is a critical point of J_ε, that is, a solution to* (4).

Sketch of Proof. The proof is similar to that of Lemma 5.1 of [9]. The most delicate point is to verify the closeness of $\partial_{\xi'_i} V$ with Z_i, here again the difficulty comes from the fact $d(\xi, \partial\Omega) \to 0$. As

$$V = \log\frac{8\mu^2}{(\mu^2 + |y - \xi'|^2)^2} + H^\varepsilon(\varepsilon y)$$

and $|\partial_{\xi'}\mu| = O(\varepsilon\mu|\log\varepsilon|)$, we need just to estimate $\|\partial_\xi H^\varepsilon\|_{L^\infty(\Omega)}$. For that, we can use Eq. (35), by differentiating the second member with respect to ξ, we can prove that $\|\partial_\xi H^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^{\alpha-1})$ for any $\alpha \in (0,1)$. This leads to $\partial_{\xi'_i} V = -4Z_i + O(\varepsilon^\alpha)$ in Ω_ε. $\square$

A key argument to get critical points of $\mathcal{F}_\varepsilon$ is its expected closeness to the functional $J_\varepsilon(U)$, for which the proof is completely similar to that of Lemma 5.2 in [26], so we leave the detail for interested readers.

Lemma 4.6. *We have*

$$\mathcal{F}_\varepsilon(\xi) = J_\varepsilon\big(U(\xi)\big) + \theta_\varepsilon(\xi),$$

where $|\theta_\varepsilon| + \|\nabla\theta_\varepsilon\| \to 0$ uniformly on Λ, as ε tends to 0.

We get also the asymptotic expansion of $J_\varepsilon(U)$ where U is the ansatz.

Lemma 4.7. *Let U be the approximate solution defined as $U = u + H^\varepsilon$. Then*

$$J_\varepsilon(U) = -16\pi a(\xi)\log\varepsilon - 4\pi a(\xi)H(\xi,\xi) + 8\pi(\log 8 - 2)a(\xi) + \mathrm{o}(1) \tag{75}$$

where the term $\mathrm{o}(1)$ *tends uniformly to* 0 *in* Λ.

Proof. By definition,

$$J_\varepsilon(U) = \frac{1}{2}\int_\Omega a(x)|\nabla U|^2\,dx - \varepsilon^2\int_\Omega a(x)e^U\,dx = J_A + J_B.$$

Using the equation $-\Delta_a U = \varepsilon^2 e^u$ in Ω, $U = 0$ on $\partial\Omega$ and the expansion of H^ε,

$$2J_A = \varepsilon^2\int_\Omega a(x)e^u U\,dx$$
$$= \varepsilon^2\int_\Omega a(x)\frac{8\mu^2}{(\varepsilon^2\mu^2 + |x-\xi|^2)^2}\left[\log\frac{1}{(\varepsilon^2\mu^2+|x-\xi|^2)^2} + H(x,\xi) + \mathrm{O}(\varepsilon^\alpha)\right]dx.$$

Make the change of variables $x = \varepsilon\mu y + \xi$ and denote $\widetilde{\Omega} = \{y\in\mathbb{R}^2;\ \xi + \varepsilon\mu y \in \Omega\}$, we obtain

$$2J_A = \int_{\widetilde{\Omega}} \frac{8a(\xi+\varepsilon\mu y)}{(1+|y|^2)^2}\left[\log\frac{1}{(1+|y|^2)^2} + H(\xi+\varepsilon\mu y,\xi) - 4\log(\varepsilon\mu)\right]dy + \mathrm{O}(\varepsilon^\alpha).$$

But $|a(\xi+\varepsilon\mu y) - a(\xi)| \leqslant C\varepsilon\mu|y|$ and $|H(\xi+\varepsilon\mu y,\xi) - H(\xi,\xi)| = \mathrm{O}(\varepsilon^\alpha\mu^\alpha|\log\varepsilon||y|^\alpha)$ for any $\alpha\in(0,1)$, thus

$$J_A = 4\pi a(\xi)H(\xi,\xi) - 16\pi a(\xi)\log(\varepsilon\mu) - 8\pi a(\xi) + \mathrm{o}(1). \tag{76}$$

On the other hand,

$$J_B = -\varepsilon^2\int_\Omega a(x)e^U\,dx = -\varepsilon^2\int_\Omega a(x)\frac{8\mu^2}{(\varepsilon^2\mu^2+|x-\xi|^2)^2}e^{H^\varepsilon(x)}\,dx$$
$$= -\int_{\widetilde{\Omega}}\frac{8a(\xi+\varepsilon\mu y)}{(1+|y|^2)^2}e^{H^\varepsilon(\xi+\varepsilon\mu y,\xi) - H(\xi,\xi) + \mathrm{O}(\varepsilon^\alpha)}\,dy.$$

We decompose the domain $\widetilde{\Omega}$ into two subregions $\Omega_1 = \{|y|\leqslant\varepsilon^{-1}\}$ and $\Omega_2 = \widetilde{\Omega}\setminus\Omega_1$. Using the fact $H(x,y)\leqslant C$ uniformly in $\Omega\times\Omega$, we get

$$\int_{\Omega_2}\frac{8a(\xi+\varepsilon\mu y)}{(1+|y|^2)^2}e^{H^\varepsilon(\xi+\varepsilon\mu y,\xi)-H(\xi,\xi)+\mathrm{O}(\varepsilon^\alpha)}\,dy \leqslant C\int_{\varepsilon^{-1}}^{\infty}\frac{e^{-H(\xi,\xi)}r\,dr}{(1+r^2)^2} = \mathrm{O}(\varepsilon^2\mu^{-2}).$$

In Ω_1, using again the regularity of a and $H(\cdot,\xi)$, we obtain

$$J_B = -8\pi a(\xi) + \mathrm{o}(1). \tag{77}$$

Thanks to (76), (77) and employing (33), the expansion (75) is proved. □

4.4. Proof of Theorem 1.4

We suppose now $\bar{x}\in\partial\Omega$ is a strict local minimum point of $a(x)$, i.e., there exists $\delta > 0$ such that for any $y \in (B_\delta(\bar{x})\setminus\{\bar{x}\})\cap\overline{\Omega}$, $a(y) < a(\bar{x})$. We suppose also that $\partial_\nu a(\bar{x}) < 0$.

Lemma 4.8. *Assume that there exists $\delta > 0$ is small enough such that $\partial_\nu a(x) < -l_0 < 0$ for any $x \in \partial\Omega \cap B_\delta(\bar{x})$. Then we have $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the minimization problem $\min_{\xi\in\Lambda} \mathcal{F}_\varepsilon(\xi)$ has a solution in the interior of Λ.*

Proof. Let $\xi_\varepsilon \in \Lambda$ be a minimizer of $\mathcal{F}_\varepsilon$. We need to prove that ξ_ε belongs to int Λ, the interior of Λ. First, let

$$\xi^0 = \bar{x} - \frac{\nu_{\bar{x}}}{|\log\varepsilon|}$$

where $\nu_{\bar{x}}$ denotes the unit outward normal vector at $\bar{x}$. It is clear that $\xi^0 \in \Lambda$ if ε is small enough and if we choose $C_1 < 1 < C_2$. From (75) and Lemma 4.6, thanks to the expansion of $H(\xi, \xi)$, we obtain an upper bound as follows.

$$\min_\Lambda \mathcal{F}_\varepsilon(\xi) \leqslant \mathcal{F}_\varepsilon(\xi^0) \leqslant -16\pi a(\bar{x})\log\varepsilon + 16\pi a(\bar{x})\log|\log\varepsilon| + \mathrm{O}(1). \tag{78}$$

Suppose by contrary that $\xi_\varepsilon \in \partial\Lambda$. There are two possibilities: either $\xi_\varepsilon \in \partial B_\delta(\bar{x}) \cap \overline{\Omega}$; or $d(\xi_\varepsilon, \partial\Omega) = C_i/|\log\varepsilon|$ for $i = 1$ or 2. If $\xi_\varepsilon \in \partial B_\delta(\bar{x}) \cap \overline{\Omega}$, we have $\delta_0 > 0$ such that $a(\xi_\varepsilon) \geqslant a(\bar{x}) + \delta_0$. Since $d(\xi_\varepsilon, \partial\Omega) \sim |\log\varepsilon|^{-1}$, applying Lemmas 4.6 and 4.7, we have

$$\min_\Lambda \mathcal{F}_\varepsilon(\xi) \geqslant -16\pi\big[a(\bar{x}) + \delta_0\big]\log\varepsilon + \mathrm{O}\big(\log|\log\varepsilon|\big)$$

which contradicts to (78). This argument shows also $a(\xi_\varepsilon) \to a(\bar{x})$ as $\varepsilon \to 0$. Hence $\xi_\varepsilon \to \bar{x}$ by the hypothesis over a.

If $d(\xi_\varepsilon, \partial\Omega) = C_i/|\log\varepsilon|$, we denote by x_ξ the orthogonal projection of ξ_ε on $\partial\Omega$. As $H(\xi_\varepsilon, \xi_\varepsilon)$ tends to $-\infty$, we have

$$\begin{aligned}\min_\Lambda \mathcal{F}_\varepsilon(\xi) &\geqslant -16\pi\left[a(x_\xi) + \frac{l_0 C_i}{|\log\varepsilon|}\right]\log\varepsilon - 4\pi a(\bar{x})\left[4\log\frac{C_i}{|\log\varepsilon|} + \mathrm{O}(1)\right] + \mathrm{O}(1)\\ &\geqslant -16\pi a(\bar{x})\log\varepsilon + 16\pi a(\bar{x})\log|\log\varepsilon| + 16\pi l_0 C_i - 16\pi a(\bar{x})\log C_i + \mathrm{O}(1)\end{aligned}$$

where the term O(1) is independent of ε small and $\xi \in \Lambda$. Notice that the function

$$g(t) = 16\pi l_0 t - 16\pi a(\bar{x})\log t$$

satisfies $\lim_{t\to 0^+} g(t) = \lim_{t\to\infty} g(t) = \infty$, so if we choose the constants $C_1 \in (0, 1)$ small enough and $C_2 > 1$ large enough, we will reach again a contradiction with (78). The lemma is proved. □

Proof of Theorem 1.4 completed. According to Lemma 4.5, the function $U(\xi) + \tilde{\phi}(\xi)$ is a solution of problem (4), if we adjust ξ so that it is a critical point of $\mathcal{F}_\varepsilon(\xi)$ defined by (73). Lemma 4.8 guarantees then the existence of a such critical point and thus a solution u_ε for (4). On the other hand, we get from the ansatz, u_ε remains uniformly bounded on $\Omega \setminus B_{d_\varepsilon}(\xi_\varepsilon)$ where $d_\varepsilon = d(\xi_\varepsilon, \partial\Omega)$. The reason is just that $\|G(\cdot, \xi_\varepsilon)\|_{L^\infty(\Omega\setminus B_{d_\varepsilon}(\xi_\varepsilon))} = \mathrm{O}(1)$ by Lemma 2.1, the properties of u_ε can be easily seen from its decomposition. □

5. Boundary blow-up solution for $\Delta u + \varepsilon^2 e^u = 0$ and further remarks

Proof of Theorem 1.6. If we look at the solutions with rotational symmetry over $\mathbb{T}$, i.e. $u_\varepsilon(x) = v_\varepsilon(r, x_N)$, we know that Eq. (1) for u_ε is transformed in (4) for v_ε with $a(r, s) = r^{N-2}$. Since $z_0 = (1 - r_0, 0)$ is a minimum point of a on $\Omega_\mathbb{T}$, Theorem 1.4 deduces then the existence of v_ε such that $\varepsilon^2 e^{v_\varepsilon}\chi_{\Omega_\mathbb{T}}\,dz \to 8\pi\delta_{z_0}$ in $\mathcal{D}'(\mathbb{R}^2)$. This yields a family of solutions u_ε which blows up exactly on $\mathcal{S}_\mathbb{T} \subset \partial\mathbb{T}$. The equality (13) comes from the rotational symmetry of our solution and the asymptotic behavior of v_ε. □

Remark 5.1. By Theorems 1.1 and 1.5 (we can show $x \cdot \nabla b \leqslant 0$ by explicit calculus) or Proposition 3.7, the solutions v_ε will blow up near z_0, the unique minimum point of a on the boundary if $\mathcal{T}_\varepsilon(v_\varepsilon) = \mathrm{O}(1)$. The limit (13) shows another contrast with the situation in dimension two, comparing with Lemma 3.1.

In particular, we get a family of solutions with a circle as blow up set in dimension three, which is a minimal geodesic for the induced Euclidean metric in $\mathbb{R}^3$. Naturally, the following questions are raised.

Question 1. Do we have another family of blow up solutions for ε near zero? Can we have solutions of (1) on $\mathbb{T}$ which breaks the rotational symmetry?

Question 2. For any smooth domain Ω with nontrivial topology in $\mathbb{R}^3$, can we have a family of blow up solutions for ε near zero? If yes, can we characterize its blow-up set by some geometrical or topological properties of the domain?

For the anisotropic equation (4) in dimension two, many problems are also remained open for the boundary blow-up phenomenon.

Question 3. Can we construct bubbling solutions near a saddle point $\bar{x}$ of a on the boundary with $\partial_\nu a(\bar{x}) < 0$?

It seems that we need to understand more about the asymptotic behavior of the blow-up phenomena near $\partial\Omega$. For the interior bubbles, as already mentioned, we proved in [26] that near any strict local maximum point in Ω, we have a family of m-bubble solutions for each $m \in \mathbb{N}^*$. We wonder if a multi-bubbles could exist on the boundary. However, we can prove that $\mathcal{T}_\varepsilon$ is not bounded in general, even when only boundary bubbles are possible.

Proposition 5.2. *There are domains Ω and anisotropic coefficients a without any critical point in Ω such that we have a family of solutions u_ε satisfying*

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega e^{u_\varepsilon}\,dx \to \infty, \quad \text{as } \varepsilon \to 0.$$

Sketch of Proof. Indeed, if we assume that there exists a finite set of disjointed strict local minimums $\{x_i\}$ for a on $\partial\Omega$ with negative outward normal derivatives, it is not difficult to construct, by the same method as for Theorem 1.4, a family of solutions such that a single bubble appears near each point x_i. The reason is that we can consider

$$\Lambda = \left\{(\xi_i) \in \Omega^m;\ \xi_i \in \overline{B}_\delta(x_i),\ \frac{C_0}{|\log\varepsilon|} \leqslant d(\xi_i, \partial\Omega) \leqslant \frac{C_1}{|\log\varepsilon|}, \forall i\right\}, \quad \delta < \frac{1}{3}\min_{i\neq j}|x_i - x_j|,$$

and take the ansatz as the sum of corresponding solution for each x_i. Here the interaction between disjointed bubbles are negligible. For example from the expansion (11), we see that u_ε, the solution given by Theorem 1.4 (see also Lemma 3.2), tends to zero uniformly in any compact set in $\overline{\Omega} \setminus \{\bar{x}\}$.

Now let $\Omega \subset \mathbb{R}^- \times \mathbb{R}$ be a smooth bounded domain such that $\{(0, y), |y| \leqslant 1\} \subset \partial\Omega$ and

$$a(r, s) = 2 - r + h(s^{-2}), \quad \text{where } h(\sigma) = e^{-\sigma}\sin\sigma.$$

It is clear that $a \geqslant 1$ in $\mathbb{R}^- \times \mathbb{R}$ and has no critical point in $\mathbb{R}^2$. Moreover, the function a has infinitely many local minimum points on $\partial\Omega$ near the origin with $\partial_\nu a$ equal to -1. Therefore, for any $m \in \mathbb{N}^*$, we can construct a family of solutions with m *simple* bubbles, and the diagonal process will give us a family of solutions satisfying $\lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon = \infty$. □

Similarly, when a admits some critical points in Ω, we may get solutions with both interior and boundary bubbles. We can also ask the following question:

Question 4. If $\Sigma = \partial\Omega = \{y \in \overline{\Omega};\ a(y) = \min_{\overline{\Omega}} a\}$, for example, consider $\Omega = B_1$ and $a = a(\|x\|)$ decreasing along the radius, can we have boundary layer solutions?

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