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Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials ☆

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Abstract

We establish *a priori* upper bounds for solutions to the spatially inhomogeneous Landau equation in the case of moderately soft potentials, with arbitrary initial data, under the assumption that mass, energy and entropy densities stay under control. Our pointwise estimates decay polynomially in the velocity variable. We also show that if the initial data satisfies a Gaussian upper bound, this bound is propagated for all positive times.

Résumé

Nous établissons des estimations *a priori* pour les solutions de l'équation de Landau non homogène en espace, dans le cas de potentiels faiblement mous, pour toute donnée initiale, sous l'hypothèse que la masse, l'énergie et la densité d'entropie restent contrôlées. Nos estimations ponctuelles ont une décroissance polynomiale par rapport à la variable de vitesse. Nous démontrons également que si la donnée initiale est bornée par une gaussienne, alors cette borne est propagée pour tous les temps positifs.

Keywords: Landau equation; A priori estimates

1. Introduction

We consider the spatially inhomogeneous Landau equation, a kinetic model from plasma physics that describes the evolution of a particle density $f(t,x,v) \geq 0$ in phase space (see, for example, [4,13]). It is written in divergence form as

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left[\bar{a}(t,x,v)\nabla_v f\right] + \bar{b}(t,x,v) \cdot \nabla_v f + \bar{c}(t,x,v) f, \tag{1.1}$$

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where $t \in [0, T_0]$, $x \in \mathbb{R}^d$, and $v \in \mathbb{R}^d$. The coefficients $\bar{a}(t,x,v) \in \mathbb{R}^{d\times d}$, $\bar{b}(t,x,v) \in \mathbb{R}^d$, and $\bar{c}(t,x,v) \in \mathbb{R}$ are given by

$$\bar{a}(t,x,v) := a_{d,\gamma} \int_{\mathbb{R}^d} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(t,x,v-w)\,\mathrm{d}w, \tag{1.2}$$

$$\bar{b}(t,x,v) := b_{d,\gamma} \int_{\mathbb{R}^d} |w|^{\gamma} w f(t,x,v-w)\,\mathrm{d}w, \tag{1.3}$$

$$\bar{c}(t,x,v) := c_{d,\gamma} \int_{\mathbb{R}^d} |w|^{\gamma} f(t,x,v-w)\,\mathrm{d}w, \tag{1.4}$$

where γ is a parameter in $[-d, \infty)$, and $a_{d,\gamma}$, $b_{d,\gamma}$, and $c_{d,\gamma}$ are constants. When $\gamma = -d$, the formula for $\bar{c}$ must be replaced by $\bar{c} = c_{d,\gamma} f$. Equation (1.1) arises as the limit of the Boltzmann equation as grazing collisions predominate, i.e. as the angular singularity approaches 2 (see the discussion in [2]). The case $d = 3$, $\gamma = -3$, corresponds to particles interacting by Coulomb potentials in small scales. The case $\gamma \in [-d, 0)$ is known as *soft potentials*, $\gamma = 0$ is known as *Maxwell molecules*, and $\gamma > 0$ *hard potentials*. In this paper, we focus on *moderately soft potentials*, which is the case $\gamma \in (-2, 0)$.

We assume that the mass density, energy density, and entropy density are bounded above, and the mass density is bounded below, uniformly in t and x:

$$0 < m_0 \le \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \le M_0, \tag{1.5}$$

$$\int_{\mathbb{R}^d} |v|^2 f(t,x,v)\,\mathrm{d}v \le E_0, \tag{1.6}$$

$$\int_{\mathbb{R}^d} f(t,x,v) \log f(t,x,v)\,\mathrm{d}v \le H_0. \tag{1.7}$$

In the space homogeneous case, because of the conservation of mass and energy, and the monotonicity of the entropy, it is not necessary to make the assumptions (1.5), (1.6) and (1.7). It would suffice to require the initial data to have finite mass, energy and entropy. It is currently unclear whether these hydrodynamic quantities will stay under control for large times and away from equilibrium in the space inhomogeneous case. Thus, at this point, it is simply an assumption we make.

We now state our main results. Our first theorem makes no further assumption on the initial data $f_{in} : \mathbb{R}^{2d} \to \mathbb{R}$ beyond what is required for a weak solution to exist in $[0, T_0]$.

Theorem 1.1. *Let $\gamma \in (-2, 0]$. If $f : [0, T_0] \times \mathbb{R}^{2d} \to \mathbb{R}$ is a bounded weak solution of (1.1) satisfying (1.5), (1.6), and (1.7), then there exists $K_0 > 0$ such that f satisfies*

$$f(t,x,v) \le K_0 \left(1 + t^{-d/2}\right) (1 + |v|)^{-1}, \tag{1.8}$$

for all $(t,x,v) \in [0, T_0] \times \mathbb{R}^{2d}$. The constant K_0 depends on d, γ, m_0, M_0, E_0, and H_0.

Note that even though we work with a bounded weak solution f, none of the constants in our estimates depend on $\|f\|_{L^\infty}$. Note also that our estimate does not depend on T_0. We use a definition of weak solution for which the estimates in [8] apply, since that is the main tool in our proofs.

We will show in Theorem 4.3 that an estimate of the form (1.8) cannot hold with a power of $(1 + |v|)$ less than $-(d + 2)$, which also implies there is no *a priori* exponential decay. On the other hand, if f_{in} satisfies a Gaussian upper bound in the velocity variable, this bound is propagated:

Theorem 1.2. *Let $f : [0, T_0] \times \mathbb{R}^{2d} \to \mathbb{R}$ be a bounded weak solution of the Landau equation (1.1) such that $f_{in}(x,v) \le C_0 e^{-\alpha|v|^2}$, for some $C_0 > 0$ and a sufficiently small $\alpha > 0$. Then*

$$f(t,x,v) \le C_1 e^{-\alpha|v|^2},$$

where C_1 depends on C_0, α, d, γ, m_0, M_0, E_0 and H_0. The value of α must be smaller than some $\alpha_0 > 0$ that depends on γ, d, m_0, M_0, E_0 and H_0.

This estimate is also independent of T_0. As a consequence of Theorem 1.2, we will show in Theorem 5.4 that in this regime, f is uniformly Hölder continuous on $[t_0, T_0] \times \mathbb{R}^{2d}$ for any $t_0 \in (0, T_0)$.

Note that under some formal asymptotic regime, the hydrodynamic quantities of the inhomogeneous Landau equations converge to solutions of the compressible Euler equation [3], which is known to develop singularities in finite time. Should we expect singularities to develop in finite time for the inhomogeneous Landau equation as well? That question seems to be out of reach with current techniques. A more realistic project is to prove that the solutions stay smooth for as long as the hydrodynamic quantities stay under control (as in (1.5), (1.6) and (1.7)). The results in this paper are an important step forward in that program.

1.1. Related work

It was established in [14] that solutions to (1.1) become C^∞ smooth in all three variables conditionally to the solution being away from vacuum, bounded in H^8 (in the $d = 3$ case) and having infinitely many finite moments. It would be convenient to extend this conditional regularity result to have less stringent assumptions. In particular, the assumptions (1.5), (1.6) and (1.7) are a much weaker assumption, which is also in terms of physically relevant hydrodynamic quantities. In [8], the authors show how their local Hölder continuity result for linear kinetic equations with rough coefficients can be applied to solutions of the Landau equation provided that (1.5), (1.6) and (1.7) hold and in addition the solution f is assumed to be bounded. While we also assume boundedness of f, our results do not quantitatively rely on this and in addition tell us some information about the decay for large velocities.

The local estimates for parabolic kinetic equations with rough coefficients play an important role in this work. Local L^∞ estimates were obtained in [16] using Moser iteration, and local Hölder estimates were proven in [21,22] using a weak Poincaré inequality. A new proof was given in [8] using a version of De Giorgi's method.

Classical solutions for (1.1) have so far only been constructed in a close-to-equilibrium setting: see the work of Guo [10] and Mouhot–Neumann [15]. A suitable notion of weak solution, for general initial data, was constructed by Alexandre–Villani [2,19].

The global L^∞ estimate we prove in Theorem 1.1 is similar to an estimate in [17] for the Boltzmann equation. The techniques in the proof are completely different. The propagation of Gaussian bounds that we give in Theorem 1.2 is reminiscent of the result in [7]. That result is for the space-homogeneous Boltzmann equation with cut-off, which is in some sense the opposite of the Landau equation in terms of the angular singularity in the cross section.

In order to keep track of the constants for parabolic regularization estimates (as in [8]) for large velocities, we describe a change of variables in Lemma 4.1. This change of variables may be useful in other contexts. It is related to one mentioned in the appendix of [12] for the Boltzmann equation.

For the homogeneous Landau equation, which arises when f is assumed to be independent of x in (1.1), the theory is more developed. The C^∞ smoothing is established for hard potentials in [6] and for Maxwell molecules in [20], under the assumption that the initial data has finite mass and energy. Propagation of L^p estimates in the case of moderately soft potentials was shown in [23] and [1]. Global upper bounds in a weighted $L^1_t(L^3_v)$ space were established in [5], even for $\gamma = -3$, as a consequence of entropy dissipation. Global L^∞ bounds that do not depend on f_{in} and that do not degenerate as $t \to \infty$ were derived in [18] for moderately soft potentials, and this result also implies C^2 smoothing by standard parabolic regularity theory.

Note that in the space homogeneous case our assumptions (1.5), (1.6) and (1.7) hold for all $t > 0$ provided that the initial data has finite mass, energy and entropy. Both Theorems 1.1 and 1.2 are new results even in the space homogeneous case. The previous results for soft potentials do not address the decay of the solution for large velocities.

1.2. Organization of the paper

In Section 2, we establish precise bounds on the coefficients $\bar{a}$, $\bar{b}$, and $\bar{c}$ in (1.1). In Section 3, we derive the local estimates we will use to prove Theorem 1.1, starting from the Harnack estimate of [8]. Section 4 contains the proof of

Theorem 1.1 and a propagating lower bound that implies the exponent of $(1+|v|)$ in (1.8) cannot be arbitrarily high. In Section 5, we prove Theorem 1.2 and the Hölder estimate, Theorem 5.4. In Appendix A, we derive a convenient maximum principle for kinetic Fokker–Planck equations.

1.3. Notation

We say a constant is universal if it depends only on d, γ, m_0, M_0, E_0, and H_0. The notation $A \lesssim B$ means that $A \le CB$ for a universal constant C, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We will let $z=(t,x,v)$ denote a point in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$. For any $z_0=(t_0,x_0,v_0)$, define the Galilean transformation

$$\mathcal{S}_{z_0}(t,x,v) := (t_0+t, x_0+x+tv_0, v_0+v).$$

We also have

$$\mathcal{S}_{z_0}^{-1}(t,x,v) := (t-t_0, x-x_0-(t-t_0)v_0, v-v_0).$$

For any $r>0$ and $z_0=(t_0,x_0,v_0)$, let

$$Q_r(z_0) := (t_0-r^2, t_0] \times \{x : |x-x_0-(t-t_0)v_0| < r^3\} \times B_r(v_0),$$

and $Q_r = Q_r(0,0,0)$. The shift $\mathcal{S}_{z_0}$ and the scaling of Q_r correspond to the symmetries of the left-hand side of (1.1). We will sometimes write ∂_i or ∂_{ij}, and these will always refer to differentiation in v.

2. The coefficients of the Landau equation

In this section we review various estimates of the coefficients $\bar{a}$, $\bar{b}$ and $\bar{c}$ in (1.1). In calculating these upper and lower bounds, the dependence of f on t and x is irrelevant, so in this section we will write $f(v)$ and $\bar{a}(v)$, etc.

Lemma 2.1. *Let $\gamma \in [-2,0)$, and assume f satisfies (1.5), (1.6), and (1.7). Then there exist constants c and C depending on d, γ, m_0, M_0, E_0, and H_0, such that for unit vectors $e \in \mathbb{R}^d$,*

$$\bar{a}_{ij}(v)e_ie_j \ge c \begin{cases} (1+|v|)^{\gamma}, & e \in \mathbb{S}^{d-1}, \\ (1+|v|)^{\gamma+2}, & e\cdot v = 0, \end{cases} \tag{2.1}$$

and

$$\bar{a}_{ij}(v)e_ie_j \le C \begin{cases} (1+|v|)^{\gamma+2}, & e \in \mathbb{S}^{d-1}, \\ (1+|v|)^{\gamma}, & e\cdot v = |v|, \end{cases} \tag{2.2}$$

where $\bar{a}_{ij}(v)$ is defined by (1.2).

Proof. The lower bounds (2.1) are proven in [18, Lemma 3.1]. For the upper bounds, the formula (1.2) implies

$$\begin{aligned}
\bar{a}_{ij}(v)e_ie_j &= a_{d,\gamma} \int_{\mathbb{R}^d} \left(1 - \left(\frac{w\cdot e}{|w|}\right)^2\right) |w|^{\gamma+2} f(v-w)\,\mathrm{d}w \\
&\lesssim \int_{\mathbb{R}^d} |w|^{\gamma+2} f(v-w)\,\mathrm{d}w \\
&= \int_{\mathbb{R}^d} |v-z|^{\gamma+2} f(z)\,\mathrm{d}z \\
&\lesssim \int_{\mathbb{R}^d} (|v|^{\gamma+2} + |z|^{\gamma+2}) f(z)\,\mathrm{d}z \\
&\lesssim M_0(1+|v|^{\gamma+2}) + E_0,
\end{aligned}$$

since $0 \le \gamma+2 \le 2$.

The above bound is valid for all $e \in \mathbb{S}^{d-1}$. If e is parallel to v, then

$$\begin{aligned}\int_{\mathbb{R}^d}\left(1-\left(\frac{w\cdot e}{|w|}\right)^2\right)|w|^{\gamma+2}f(v-w)\,\mathrm{d}w &= \int_{\mathbb{R}^d}\left(1-\left(\frac{(v-z)\cdot e}{|v-z|}\right)^2\right)|v-z|^{\gamma+2}f(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^d}\left(|v-z|^2-(|v|-z\cdot e)^2\right)|v-z|^{\gamma}f(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^d}\left(|z|^2-(z\cdot e)^2\right)|v-z|^{\gamma}f(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^d}|z|^2\sin^2\theta|v-z|^{\gamma}f(z)\,\mathrm{d}z,\end{aligned}$$

where θ is the angle between v and z. Let $R = |v|/2$. If $z \in B_R(v)$, then $|\sin\theta| \le |v-z|/|v|$, and

$$\begin{aligned}\int_{B_R(v)}|z|^2\sin^2\theta|v-z|^{\gamma}f(z)\,\mathrm{d}z &\le \int_{B_R(v)}|z|^2|v|^{-2}|v-z|^{\gamma+2}f(z)\,\mathrm{d}z\\ &\le \frac{|v|^{\gamma}}{2^{\gamma+2}}\int_{B_R(v)}|z|^2f(z)\,\mathrm{d}z \lesssim E_0|v|^{\gamma}.\end{aligned}$$

If $|v-z| \ge R = |v|/2$, then $|v-z|^{\gamma} \lesssim |v|^{\gamma}$, and we have

$$\int_{\mathbb{R}^d\setminus B_R(v)}|z|^2\sin^2\theta|v-z|^{\gamma}f(z)\,\mathrm{d}z \lesssim |v|^{\gamma}\int_{\mathbb{R}^d\setminus B_R(v)}|z|^2f(z)\,\mathrm{d}z \lesssim E_0|v|^{\gamma}. \quad \square$$

In the proof of Theorem 1.1, we will need to keep track of how the bounds on $\bar{b}$ and $\bar{c}$ in the next two lemmas depend on the local L^∞ norm of f. In Lemma 2.2 and Lemma 2.3, $\|f\|_{L^\infty(A)}$ means $\|f(t,x,\cdot)\|_{L^\infty(A)}$ for any set $A \subseteq \mathbb{R}^d$.

Lemma 2.2. *Let f satisfy (1.5), (1.6), and (1.7). Then $\bar{c}(v)$ defined by (1.4) satisfies*

$$\bar{c}(v) \lesssim \begin{cases}(1+|v|)^{\gamma}(1+\|f\|_{L^\infty(B_\rho(v))})^{-\gamma/d}, & \dfrac{-2d}{d+2} \le \gamma < 0,\\ (1+|v|)^{-2-2\gamma/d}\left(1+\|f\|_{L^\infty(B_\rho(v))}\right)^{-\gamma/d}, & -d < \gamma < \dfrac{-2d}{d+2},\end{cases}$$

where the constants depend on d, γ, M_0, and E_0, and

$$\rho = \begin{cases}1, & |v| < 2,\\ |v|^{-2/d}, & |v| \ge 2.\end{cases}$$

Proof. Assume first $|v| \ge 2$. Let $r := |v|^{-2/d}(1+\|f\|_{L^\infty(B_\rho(v))})^{-1/d} < \rho$. Consider

$$I_1 = \int_{B_r}|w|^{\gamma}f(v-w)\,\mathrm{d}w, \quad I_2 = \int_{B_{|v|/2}\setminus B_r}|w|^{\gamma}f(v-w)\,\mathrm{d}w,$$

$$I_3 = \int_{\mathbb{R}^d\setminus B_{|v|/2}}|w|^{\gamma}f(v-w)\,\mathrm{d}w.$$

We have

$$I_1 \lesssim \|f\|_{L^\infty(B_\rho(v))}r^{d+\gamma} \lesssim |v|^{-2-2\gamma/d}\|f\|^{-\gamma/d}_{L^\infty(B_\rho(v))}.$$

$$I_2 \lesssim r^\gamma |v|^{-2} \int_{B_{|v|/2}} |v-w|^2 f(v-w)\,\mathrm{d}w \lesssim E_0 |v|^{-2-2\gamma/d}(1+\|f\|_{L^\infty(B_\rho(v))})^{-\gamma/d}.$$

Finally, for $|w| \geq |v|/2$, we have $|w|^\gamma \lesssim |v|^\gamma$, and

$$I_3 \lesssim |v|^\gamma \int_{\mathbb{R}^d \setminus B_{|v|/2}} f(v-w)\,\mathrm{d}w \leq M_0 |v|^\gamma.$$

Thus $\bar{c}(v) \lesssim (1+\|f\|_{L^\infty(B_\rho(v))})^{-\gamma/d}|v|^{-2-2\gamma/d} + |v|^\gamma$ for $|v| > 2$.

When $\gamma \in \left(-d, \frac{-2d}{d+2}\right)$, $-2-2\gamma/d > \gamma$ and we get

$$\bar{c}(v) \lesssim (1+\|f\|_{L^\infty(B_\rho(v))})^{-\gamma/d}|v|^{-2-2\gamma/d}.$$

When $\gamma \in \left[\frac{-2d}{d+2}, 0\right)$, $\gamma > -2-2\gamma/d$ and we get

$$\bar{c}(v) \lesssim (1+\|f\|_{L^\infty(B_\rho(v))})^{-\gamma/d}|v|^{\gamma}.$$

This completes the proof in the case $|v| > 2$.

For $|v| \leq 2$, $\gamma \in (-d, 0]$, and any $R \in (0,1]$ we have that

$$\begin{aligned}\int_{\mathbb{R}^d} |w|^\gamma f(v-w)\,\mathrm{d}w &= \int_{B_R} |w|^\gamma f(v-w)\,\mathrm{d}w + \int_{\mathbb{R}^d\setminus B_R} |w|^\gamma f(v-w)\,\mathrm{d}w,\\ &\lesssim R^{d+\gamma}\|f\|_{L^\infty(B_1(v))} + R^\gamma M_0.\end{aligned}$$

Choosing $R = (1+\|f\|_{L^\infty(B_1(v))})^{-1/d}$, we then have

$$\bar{c}(v) \lesssim (R^{d+\gamma}\|f\|_{L^\infty(B_1(v))} + R^\gamma M_0) \lesssim (1+\|f\|_{L^\infty(B_1(v))})^{-\gamma/d},$$

for $|v| \leq 2$, completing the proof. □

Lemma 2.3. *Let f satisfy (1.5), (1.6), and (1.7). Then $\bar{b}(v)$ defined by (1.3) satisfies the estimate*

$$|\bar{b}(v)| \lesssim \begin{cases} (1+|v|)^{\gamma+1}(1+\|f\|_{L^\infty(B_\rho(v))})^{-(\gamma+1)/d}, & \gamma \in [-2,-1), \\ (1+|v|)^{\gamma+1}, & \gamma \in [-1, 0] \end{cases} \tag{2.3}$$

where the constants depend on d, γ, M_0, and E_0, and

$$\rho = \begin{cases} 1, & |v| < 2, \\ |v|^{-2/d}, & |v| \geq 2. \end{cases}$$

Proof. Taking norms, we have

$$|\bar{b}(v)| \lesssim \int_{\mathbb{R}^d} |w|^{1+\gamma} f(v-w)\,\mathrm{d}w.$$

If $\gamma \in [-2,-1)$, then $0 > 1+\gamma \geq -1 \geq \frac{-2d}{d+2}$, and the conclusion follows from Lemma 2.2. If $\gamma \in [-1, 0]$, we have

$$\begin{aligned}|\bar{b}(v)| &\lesssim \int_{\mathbb{R}^d} (|v|^{\gamma+1} + |v-w|^{\gamma+1}) f(v-w)\,\mathrm{d}w \\ &\lesssim |v|^{\gamma+1} M_0 + E_0^{(1+\gamma)/2} M_0^{(1-\gamma)/2} \lesssim (1+|v|)^{\gamma+1}. \quad \square\end{aligned}$$

3. Local estimates

In this section we refine the local estimates in [16] and [8] for linear kinetic equations with rough coefficients. Essentially, we start from their results and apply scaling techniques to improve the local L^∞ estimates.

We will need the following technical lemma. See [11, Lemma 4.3] for the proof.

Lemma 3.1. *Let $\eta(r) \geq 0$ be bounded in $[r_0, r_1]$ with $r_0 \geq 0$. Suppose for $r_0 \leq r < R \leq r_1$, we have*

$$\eta(r) \leq \theta\eta(R) + \frac{A}{(R-r)^\alpha} + B$$

for some $\theta \in [0,1)$ and $A, B, \alpha \geq 0$. Then there exists $c(\alpha,\theta) > 0$ such that for any $r_0 \leq r < R \leq r_1$, there holds

$$\eta(r) \leq c(\alpha,\theta)\left(\frac{A}{(R-r)^\alpha} + B\right).$$

Proposition 3.2. *If $g(t,x,v) \geq 0$ is a weak solution of*

$$\partial_t g + v \cdot \nabla_x g = \nabla_v \cdot (A\nabla_v g) + B \cdot \nabla_v g + s \tag{3.1}$$

in Q_1, with

$$\begin{aligned} 0 < \lambda I \leq A(t,x,v) &\leq \Lambda I, \qquad (t,x,v) \in Q_1,\\ |B(t,x,v)| &\leq \Lambda, \qquad (t,x,v) \in Q_1,\\ s &\in L^\infty(Q_1), \end{aligned}$$

then

$$\sup_{Q_{1/2}} g \leq C\left(\|g\|_{L^\infty_{t,x}L^1_v(Q_1)} + \|s\|_{L^\infty(Q_1)}\right), \tag{3.2}$$

with C depending only on d, λ, and Λ.

Proof. It is proven in [8] that if $g(t,x,v)$ solves (3.1) weakly with A, B, and s as in the statement of the proposition, then

$$\|g\|_{L^\infty(Q_{1/2})} \leq C\left(\|g\|_{L^2(Q_1)} + \|s\|_{L^\infty(Q_1)}\right),$$

with C depending on d, λ, and Λ. Since $\|g\|_{L^2(Q_1)} \leq \sqrt{\omega_d}\|g\|_{L^\infty_{t,x}L^2_v(Q_1)}$, where $\omega_d = \mathcal{L}^d(B_1)$, we also have

$$\|g\|_{L^\infty(Q_{1/2})} \leq C\left(\|g\|_{L^\infty_{t,x}L^2_v(Q_1)} + \|s\|_{L^\infty(Q_1)}\right). \tag{3.3}$$

To replace $\|g\|_{L^\infty_{t,x}L^2_v(Q_1)}$ with $\|g\|_{L^\infty_{t,x}L^1_v(Q_1)}$, we use an interpolation argument. For $0 < r \leq 1$, define

$$\begin{aligned} &g_r(t,x,v) := g(r^2t, r^3x, rv), \quad s_r(t,x,v) := s(r^2t, r^3x, rv),\\ &A_r(t,x,v) := A(r^2t, r^3x, rv), \;\; B_r(t,x,v) := B(r^2t, r^3x, rv), \end{aligned} \tag{3.4}$$

and note that g_r satisfies

$$\partial_t g_r + v \cdot \nabla_x g_r = \nabla_v \cdot (A_r \nabla_v g_r) + rB_r \cdot \nabla_v g_r + r^2 s_r \tag{3.5}$$

in Q_1. Since $r \leq 1$, we may apply (3.3) to g_r, which gives

$$\|g\|_{L^\infty(Q_{r/2})} \leq C\left(\frac{1}{r^{d/2}}\|g\|_{L^\infty_{t,x}L^2_v(Q_r)} + r^2\|s\|_{L^\infty(Q_r)}\right), \tag{3.6}$$

for any $r \in (0,1]$. Now, for $\theta, R \in (0,1)$, apply (3.6) in $Q_{(1-\theta)R}(z)$ for each $z \in Q_{\theta R}$ to obtain

$$\|g\|_{L^\infty(Q_{\theta R})} \le C\left(\frac{1}{[(1-\theta)R]^{d/2}}\|g\|_{L^\infty_{t,x}L^2_v(Q_R)} + R^2\|s\|_{L^\infty(Q_R)}\right)$$
$$\le C\left(\frac{1}{[(1-\theta)R]^{d/2}}\|g\|_{L^\infty_{t,x}L^2_v(Q_R)} + \|s\|_{L^\infty(Q_1)}\right).$$

By the Hölder and Young inequalities, we have

$$\|g\|_{L^\infty(Q_{\theta R})} \le C\left(\frac{1}{[(1-\theta)R]^{d/2}}\|g\|^{1/2}_{L^\infty(Q_R)}\|g\|^{1/2}_{L^\infty_{t,x}L^1_v(Q_R)} + \|s\|_{L^\infty(Q_1)}\right)$$
$$\le \frac{1}{2}\|g\|_{L^\infty(Q_R)} + C\left(\frac{1}{[(1-\theta)R]^{d}}\|g\|_{L^\infty_{t,x}L^1_v(Q_R)} + \|s\|_{L^\infty(Q_1)}\right).$$

Define $\eta(\rho) = \|g\|_{L^\infty(Q_\rho)}$ for $\rho \in (0,1]$. Then for any $0 < r < R \le 1$, we have

$$\eta(r) \le \frac{1}{2}\eta(R) + \frac{C}{(R-r)^d}\|g\|_{L^\infty_{t,x}L^1_v(Q_1)} + C\|s\|_{L^\infty(Q_1)}.$$

Applying Lemma 3.1, we obtain

$$\eta(r) \le \frac{C}{(R-r)^d}\|g\|_{L^\infty_{t,x}L^1_v(Q_1)} + C\|s\|_{L^\infty(Q_1)}.$$

Let $R \to 1-$ and set $r = \frac{1}{2}$ to conclude (3.2). □

Lemma 3.3. *Let $g(t,x,v)$ solve (3.1) weakly in $Q_R(z_0)$ for some $z_0 \in \mathbb{R}^{2d+1}$ and $R > 0$, with*

$$0 < \lambda I \le A(t,x,v) \le \Lambda I, \qquad (t,x,v) \in Q_R,$$
$$|B(t,x,v)| \le \Lambda/R, \qquad (t,x,v) \in Q_R,$$
$$s \in L^\infty(Q_R).$$

Then the improved estimate

$$g(t_0,x_0,v_0) \le C\left(\|g\|^{2/(d+2)}_{L^\infty_{t,x}L^1_v(Q_R)}\|s\|^{d/(d+2)}_{L^\infty(Q_R)} + R^{-d}\|g\|_{L^\infty_{t,x}L^1_v(Q_R)}\right) \tag{3.7}$$

holds, with C depending only on d, λ, and Λ.

Proof. By applying the change of variables

$$(t,x,v) \mapsto \left(\frac{t-t_0}{R^2}, \frac{x-x_0-(t-t_0)v_0}{R^3}, \frac{v-v_0}{R}\right)$$

to g and s, we may suppose $(t_0,x_0,v_0) = (0,0,0)$ and $R = 1$.

For $r \in (0,1]$ to be determined, we make the transformation (3.4) as in the proof of Proposition 3.2 and get a function g_r satisfying (3.5) in Q_1. Then Proposition 3.2 implies

$$g(0,0,0) \le C\left(\|g_r\|_{L^\infty_{t,x}L^1_v(Q_1)} + \|r^2 s_r\|_{L^\infty(Q_1)}\right)$$
$$= C\left(r^{-d}\|g\|_{L^\infty_{t,x}L^1_v(Q_r)} + r^2\|s\|_{L^\infty(Q_r)}\right)$$
$$\le C\left(r^{-d}\|g\|_{L^\infty_{t,x}L^1_v(Q_1)} + r^2\|s\|_{L^\infty(Q_1)}\right).$$

If $\|g\|_{L^\infty_{t,x}L^1_v(Q_1)} \le \|s\|_{L^\infty(Q_1)}$, then the choice $r = (\|g\|_{L^\infty_{t,x}L^1_v(Q_1)}/\|s\|_{L^\infty(Q_1)})^{1/(d+2)}$ implies

$$g(0,0,0) \le C\|g\|^{2/(d+2)}_{L^\infty_{t,x}L^1_v(Q_1)}\|s\|^{d/(d+2)}_{L^\infty(Q_1)}.$$

On the other hand, if $\|s\|_{L^\infty(Q_1)} \le \|g\|_{L^\infty_{t,x}L^1_v(Q_1)}$, the choice $r = 1$ implies $g(0,0,0) \le C\|g\|_{L^\infty_{t,x}L^1_v(Q_1)}$, so we have

$$g(0,0,0) \le C\left(\|g\|^{2/(d+2)}_{L^\infty_{t,x}L^1_v(Q_1)}\|s\|^{d/(d+2)}_{L^\infty(Q_1)} + \|g\|_{L^\infty_{t,x}L^1_v(Q_1)}\right)$$

in both cases. □

4. Global estimates

In this section, we prove global upper bounds for solutions f of (1.1). Our bounds depend only on the estimates on the hydrodynamic quantities (1.5), (1.6) and (1.7). Our bound does not depend on an upper bound of the initial data. We also get that the solution will have certain polynomial decay in v for $t > 0$.

From Lemma 2.1, we see that the bounds on $\bar{a}_{ij}(t,x,v)$ degenerate as $|v| \to \infty$. In the first lemma, we show how to change variables to obtain an equation with uniform ellipticity constants independent of $|v|$.

Lemma 4.1. *Let $z_0 = (t_0, x_0, v_0) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$ be such that $|v_0| \geq 2$, and let T be the linear transformation such that*

$$Te = \begin{cases} |v_0|^{1+\gamma/2} e, & e \cdot v_0 = 0 \\ |v_0|^{\gamma/2} e, & e \cdot v_0 = |v_0|. \end{cases}$$

Let $\tilde{T}(t,x,v) = (t, Tx, Tv)$, and define

$$\begin{aligned} \mathcal{T}_{z_0}(t,x,v) &:= \mathcal{S}_{z_0} \circ \tilde{T}(t,x,v) \\ &= (t_0 + t, x_0 + Tx + tv_0, v_0 + Tv). \end{aligned}$$

Then,

(a) *There exists a constant $C > 0$ independent of $v_0 \in \mathbb{R}^d \setminus B_2$ such that for all $v \in B_1$,*

$$C^{-1}|v_0| \leq |v_0 + Tv| \leq C|v_0|.$$

(b) *If $f_T(t,x,v) := f(\mathcal{T}_{z_0}(t,x,v))$, then f_T satisfies*

$$\partial_t f_T + v \cdot \nabla_x f_T = \nabla_v \left[A(z) \nabla_v f_T \right] + B(z) \cdot \nabla_v f_T + C(z) f_T \tag{4.1}$$

in Q_R for any $0 < R < \min\{\sqrt{t_0}, c_1 |v_0|^{-1-\gamma/2}\}$, where c_1 is a universal constant, and

$$\lambda I \leq A(z) \leq \Lambda I,$$

$$|B(z)| \lesssim \begin{cases} |v_0|^{1+\gamma/2} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\rho(v))}\right)^{-(\gamma+1)/d}, & \gamma \in [-2,-1), \\ |v_0|^{1+\gamma/2}, & \gamma \in [-1,0], \end{cases}$$

$$|C(v)| \lesssim \begin{cases} |v_0|^{\gamma} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\rho(v))}\right)^{-\gamma/d}, & \dfrac{-2d}{d+2} \leq \gamma < 0, \\ |v_0|^{-2-2\gamma/d} \left(1 + \|f(t,x,\cdot)\|_{L^\infty(B_\rho(v))}\right)^{-\gamma/d}, & -2 < \gamma < \dfrac{-2d}{d+2}, \end{cases}$$

with λ and Λ universal, and $\rho \lesssim 1 + |v_0|^{-2/d}$.

Proof. Since $|v| \leq 1$ and $|v_0| > 2$,

$$|v_0| - |v_0|^{1+\gamma/2} \leq |v_0| - |Tv| \leq |v_0 + Tv| \leq |v_0| + |Tv| \leq |v_0| + |v_0|^{1+\gamma/2}.$$

Thus, (a) follows since $\gamma \in (-2, 0)$.

For (b), by direct computation, f_T satisfies (4.1) with

$$A(z) = T^{-1}\bar{a}(\mathcal{T}_{z_0}(z))T^{-1}, \quad B(z) = T^{-1}\bar{b}(\mathcal{T}_{z_0}(z)), \quad C(z) = \bar{c}(\mathcal{T}_{z_0}(z)).$$

In order to keep the proof clean, let us write $\bar{a}_{ij}$ and A_{ij} instead of $\bar{a}_{ij}(\mathcal{T}_{z_0}(z))$ and $A_{ij}(z)$ for the rest of the proof.

Fix $z = (t,x,v) \in Q_R$, and let $\tilde{v} = v_0 + Tv$. From part (a), we know that $|\tilde{v}| \approx |v_0|$. Applying Lemma 2.1, we have that for any unit vector e,

$$\bar{a}_{ij} e_i e_j \lesssim \begin{cases} (1 + |v_0|)^{\gamma}, & e = \tilde{v}/|\tilde{v}|, \\ (1 + |v_0|)^{\gamma+2}, & e \in S^{d-1}, \end{cases} \tag{4.2}$$

and,

$$\bar{a}_{ij}e_ie_j \gtrsim \begin{cases}(1+|v_0|)^\gamma, & e \in S^{d-1},\\ (1+|v_0|)^{\gamma+2}, & e\cdot\tilde{v}=0.\end{cases} \tag{4.3}$$

Our first step is to verify that we can switch $\tilde{v}$ for v_0 in (4.2) and (4.3).

Let us start with (4.2). This is where the assumption $|v| < R \le C_1|v_0|^{-1-\gamma/2}$ plays a role. We can choose c_1 so as to ensure that $|Tv| \le 1$. Since $v_0 = \tilde{v} - Tv$ and using the fact that $\bar{a}_{ij}$ is positive definite,

$$\bar{a}_{ij}(v_0)_i(v_0)_j \le 2\bar{a}_{ij}\tilde{v}_i\tilde{v}_j + 2\bar{a}_{ij}(Tv)_i(Tv)_j \le C|v_0|^{2+\gamma}.$$

Let $e_0 = v_0/|v_0|$. The computation above tells us that $\bar{a}_{ij}(e_0)_i(e_0)_j \lesssim |v_0|^\gamma$.

Let us now turn to (4.3). We will show that

$$\bar{a}_{ij}w_iw_j \gtrsim (1+|v_0|)^{\gamma+2}|w|^2 \qquad \text{if } w\cdot v_0 = 0. \tag{4.4}$$

Note that $(1+|v_0|)^{2+\gamma}$ and $(1+|v_0|)^\gamma$ are comparable when $|v_0|$ is small, so we only need to verify (4.4) for $w\cdot v_0 = 0$ and $|v_0|$ arbitrarily large. For such vector w, we write $w = \eta\tilde{v} + w'$ with $w'\cdot\tilde{v} = 0$. Since $|\tilde{v} - v_0| = |Tv| \le 1$, we have $|\eta| = |w\cdot\tilde{v}|/|\tilde{v}|^2 = |w\cdot(\tilde{v}-v_0)|/|\tilde{v}|^2 \le |w||\tilde{v}|^{-2}$. Moreover, $|w'| \approx |w|$.

Since $\bar{a}_{ij}$ is positive definite,

$$\bar{a}_{ij}(\sqrt{2}\eta\tilde{v} - w'/\sqrt{2})_i(\sqrt{2}\eta\tilde{v} - w'/\sqrt{2})_j \ge 0,$$

then we have

$$\begin{aligned}\bar{a}_{ij}w_iw_j &\ge \frac{1}{2}\bar{a}_{ij}w'_iw'_j - \eta^2\bar{a}_{ij}\tilde{v}_i\tilde{v}_j\\ &\ge \left(c(1+|v_0|)^{\gamma+2} - (1+|v_0|)^\gamma\right)|w|^2 \gtrsim (1+|v_0|)^{\gamma+2}|w|^2,\end{aligned}$$

as desired.

Let $w\in\mathbb{R}^d$ be arbitrary. We will estimate $A_{ij}w_iw_j$ from above. Writing $w = \mu e_0 + \tilde{w}$, with $\tilde{w}\cdot e = 0$.

$$A_{ij}w_iw_j = |v_0|^{-\gamma}\left(\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + 2\mu|v_0|^{-1}\bar{a}_{ij}(e_0)_i\tilde{w}_j + |v_0|^{-2}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j\right),$$

and using that $\bar{a}_{ij}$ is positive definite,

$$\begin{aligned}A_{ij}w_iw_j &\le 2|v_0|^{-\gamma}\left(\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + |v_0|^{-2}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j\right),\\ &\le C\left(\mu^2 + |\tilde{w}|^2\right) =: \Lambda|w|^2.\end{aligned}$$

This establishes upper bound $\{A_{ij}\} \le \Lambda I$ for some $\Lambda > 0$.

Now we will prove the lower bound for A_{ij}. Again, we write $w = \mu e_0 + \tilde{w}$ with $e_0\cdot\tilde{w} = 0$. We need to analyze the quadratic form associated with the coefficients $\bar{a}_{ij}$ more closely. From (4.3), we have that for some universal constant $c > 0$,

$$c|v_0|^\gamma(\mu^2 + |\tilde{w}|^2) \le \bar{a}_{ij}w_iw_j = \mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + 2\mu\bar{a}_{ij}(e_0)_i\tilde{w}_j + \bar{a}_{ij}\tilde{w}_i\tilde{w}_j.$$

Moreover, (4.2) implies that there is a universal constant $\delta > 0$ so that

$$c|v_0|^\gamma(\mu^2 + |\tilde{w}|^2) \ge \delta\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + \delta|v_0|^{-2}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j.$$

Subtracting the two inequalities above,

$$(1-\delta)\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + 2\mu\bar{a}_{ij}(e_0)_i\tilde{w}_j + (1-\delta|v_0|^{-2})\bar{a}_{ij}\tilde{w}_i\tilde{w}_j \ge 0.$$

The same inequality holds if we replace $w = \mu e_0 + \tilde{w}$ with $w = (1-\delta/2)^{-1/2}\mu e_0 + (1-\delta/2)^{1/2}|v_0|^{-1}\tilde{w}$, therefore

$$\frac{1-\delta}{1-\delta/2}\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + 2\mu|v_0|^{-1}\bar{a}_{ij}(e_0)_i\tilde{w}_j + (1-\delta/2)(1-\delta|v_0|^{-2})|v_0|^{-2}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j \ge 0.$$

Recalling the formula above for $A_{ij}w_iw_j$, and replacing it in the left hand side, we get

$$A_{ij}w_iw_j - \left(1 - \frac{1-\delta}{1-\delta/2}\right)|v_0|^{-\gamma}\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j - \left(1-(1-\delta/2)(1-\delta|v_0|^{-2})\right)|v_0|^{-2-\gamma}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j \ge 0.$$

Therefore, using (4.3) and (4.4),

$$A_{ij}w_iw_j \geq \left(1-\frac{1-\delta}{1-\delta/2}\right)|v_0|^{-\gamma}\mu^2\bar{a}_{ij}(e_0)_i(e_0)_j + \left(1-(1-\delta/2)(1-\delta|v_0|^{-2})\right)|v_0|^{-2-\gamma}\bar{a}_{ij}\tilde{w}_i\tilde{w}_j,$$
$$\geq \lambda(\mu^2+|\tilde{w}|^2),$$

for some universal constant $\lambda > 0$. This establishes the lower bound $\{A_{ij}\} \geq \lambda I$.

To derive the bound on $B(z)$, Lemma 2.3 and conclusion (a) imply

$$|B(z)| \lesssim \|T^{-1}\||\bar{b}(\mathcal{T}_{z_0}(z))|$$
$$\lesssim \begin{cases}(1+|v_0|)^{\gamma/2+1}(1+\|f\|_{L^\infty(B_{\rho'}(\tilde{v}))})^{-(\gamma+1)/d}, & \gamma\in(-2,-1),\\ (1+|v_0|)^{\gamma/2+1}, & \gamma\in[-1,0],\end{cases}$$

where $\rho' = |\tilde{v}|^{-2/d}$. From the triangle inequality, we have that $B_{\rho'}(\tilde{v}) \subset B_\rho(v_0)$, with $\rho \lesssim (1+|v_0|)^{-2/d} + R(1+|v_0|)^{(\gamma+2)/2} \leq 1+(1+|v_0|)^{-2/d}$. The bound on $C(z)$ follows in a similar manner, using Lemma 2.2. □

The key lemma in the proof of Theorem 1.1 is the following pointwise estimate on f:

Lemma 4.2. *Let $\gamma\in(-2,0]$, $T_0>0$, and let $f:[0,T_0]\times\mathbb{R}^{2d}\to\mathbb{R}_+$ solve the Landau equation (1.1) weakly. If*

$$f(t,x,v) \leq K(1+t^{-d/2})(1+|v|)^{-\alpha}$$

in $[0,T_0]\times\mathbb{R}^{2d}$ for some $\alpha\in[0,1]$ and $K\geq 1$, then

$$f(t,x,v) \leq C\left((K(1+t^{-d/2}))^{(d-\gamma)/(d+2)}(1+|v|)^{P(d,\alpha,\gamma)} + K^{Q(\gamma)}(1+t^{-d/2})(1+|v|)^{-1}\right), \tag{4.5}$$

for some C universal and

$$P(d,\alpha,\gamma) = \begin{cases} -1-d(1+\alpha)/(d+2), & \gamma\in\left[\dfrac{-2d}{d+2},0\right],\\ -[d(4+\gamma)+2+2\gamma+\alpha d]/(d+2), & \gamma\in\left(-2,\dfrac{-2d}{d+2}\right),\end{cases}$$

$$Q(\gamma) = \begin{cases}0, & \gamma\in[-1,0],\\ -(1+\gamma), & \gamma\in(-2,-1).\end{cases}$$

Proof. *Case 1: $\gamma\in[-1,0]$.* Let $z_0=(t_0,x_0,v_0)$ be such that such that $|v_0|\geq 2$. Define $r_0=\min\{1,\sqrt{t_0}\}$, and note that $r_0^{-d}\approx(1+t_0^{-d/2})$. Letting f_T be as in Lemma 4.1, we will estimate $f_T(t,x,v)$ in Q_R, where

$$R := c_1(r_0/2)(1+|v_0|)^{-(2+\gamma)/2},$$

with c_1 as in Lemma 4.1(b). We have that f_T solves (4.1) in Q_R, and by Lemma 4.1(a) and our assumption on f,

$$f_T(t,x,v) \lesssim K r_0^{-d}(1+|v_0|)^{-\alpha} \tag{4.6}$$

in Q_R. Feeding (4.6) into Lemma 4.1(b), we have

$$0 < \lambda I \leq A(z) \leq \Lambda I,$$
$$|B(z)| \lesssim (1+|v_0|)^{(2+\gamma)/2}, \tag{4.7}$$
$$|C(z)| \lesssim \left(K r_0^{-d}\right)^{-\gamma/d}(1+|v_0|)^\gamma, \tag{4.8}$$

in Q_R.

Let $Q_{T,R}$ be the image of Q_R under $z \mapsto \mathcal{T}_{z_0}(z)$, and note that

$$\begin{aligned}\|f_T\|_{L^\infty_{t,x}L^1_v(Q_R)} &= \det(T^{-1})\|f\|_{L^\infty_{t,x}L^1_v(Q_{T,R})}\\ &= (1+|v_0|)^{-[(d-1)(2+\gamma)/2+\gamma/2]}\|f\|_{L^\infty_{t,x}L^1_v(Q_{T,R})}\\ &\le (1+|v_0|)^{-(1+d(2+\gamma)/2)}E_0,\end{aligned} \tag{4.9}$$

where the last inequality comes from the energy bound (1.6) and Lemma 4.1(a).

By (4.7) and our choice of R, we can apply Lemma 3.3 in Q_R with $g = f_T$ and $s = C(z)f_T$ to obtain

$$\begin{aligned}f(t_0,x_0,v_0) &\le C\left(\|f_T\|^{2/(d+2)}_{L^\infty_{t,x}L^1_v(Q_R)}\|C(z)f_T\|^{d/(d+2)}_{L^\infty(Q_R)} + r_0^{-d}(1+|v_0|)^{d(2+\gamma)/2}\|f_T\|_{L^\infty_{t,x}L^1_v(Q_R)}\right)\\ &\le C\left((Kr_0^{-d})^{(d-\gamma)/(d+2)}(1+|v_0|)^{-1-d(1+\alpha)/(d+2)} + r_0^{-d}(1+|v_0|)^{-1}\right),\end{aligned} \tag{4.10}$$

using (4.6), (4.8), and (4.9). Note that we derived (4.10) assuming that $|v_0| \ge 2$. When $|v_0| \le 2$, the matrix $\bar{a}_{ij}(z)$ is uniformly elliptic and we can apply Lemma 3.3 directly to f to obtain (4.10) in this case as well.

Case 2: $\gamma \in (-2,-1]$. The argument is the same as in Case 1, but the estimates are quantitatively different as a result of the different bounds on $B(z)$ and $C(z)$ in Lemma 4.1. The changes are as follows: the radius R of the cylinder Q_R is chosen to be

$$R := K^{(1+\gamma)/d}(r_0/2)(1+|v_0|)^{-(2+\gamma)/2},$$

the bound on $B(z)$ becomes

$$|B(z)| \lesssim K^{-(1+\gamma)/d}r_0^{1+\gamma}(1+|v_0|)^{(2+\gamma)/2} \le \Lambda/R, \quad z \in Q_R,$$

and for $C(z)$ we have

$$|C(z)| \lesssim \begin{cases}\left(Kr_0^{-d}\right)^{-\gamma/d}(1+|v_0|)^{\gamma}, & \gamma \in \left[\dfrac{-2d}{d+2},-1\right],\\ \left(Kr_0^{-d}\right)^{-\gamma/d}(1+|v_0|)^{-2-2\gamma/d}, & \gamma \in \left(-2,\dfrac{-2d}{d+2}\right),\end{cases}$$

for $z \in Q_R$. After applying Lemma 3.3 and (4.9), we obtain

$$f(t_0,x_0,v_0) \le C\left((Kr_0^{-d})^{(d-\gamma)/(d+2)}(1+|v_0|)^{P(d,\alpha,\gamma)} + K^{-(1+\gamma)}r_0^{-d}(1+|v_0|)^{-1}\right),$$

as desired, with $P(d,\alpha,\gamma)$ as in the statement of the lemma. □

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. Define

$$K := \sup_{(0,T_0]\times\mathbb{R}^{2d}} \min\{t^{d/2},1\}f(t,x,v).$$

First, we will show that $K \le K_*$, where K_* is universal. We can assume $K > 1$. For each $\gamma \in (-2,0]$, define $p_\gamma : (1,\infty) \to \mathbb{R}$ by

$$p_\gamma\left(\overline{K}\right) = \begin{cases} C\left(\overline{K}^{(d-\gamma)/(d+2)} + 1\right), & \gamma \in (-1,0],\\ C\left(\overline{K}^{(d-\gamma)/(d+2)} + (\overline{K})^{-(1+\gamma)}\right), & \gamma \in (-2,-1],\end{cases} \tag{4.11}$$

where C is the appropriate constant from Lemma 4.2 for each γ. Then since $-(1+\gamma) < 1$ and $\dfrac{d-\gamma}{d+2} < 1$ for $\gamma > -2$, there is a $K_* > 1$ such that

$$\begin{aligned}&K_* = p_\gamma(K_*),\\ &\overline{K} > p_\gamma(\overline{K}), \quad \text{if } \overline{K} > K_*.\end{aligned}$$

Let $\varepsilon > 0$. By the definition of K, there exists some $(t_0, x_0, v_0) \in (0, T] \times \mathbb{R}^{2d}$ such that $f(t_0, x_0, v_0) > (K - \varepsilon)\max\{t_0^{-d/2}, 1\}$. Therefore, Lemma 4.2 implies that

$$K - \varepsilon \le p_\gamma(K).$$

Since this is true for all $\varepsilon > 0$, we have that $K \le K_*$.

If $\gamma \in \left[\frac{-2d}{d+2}, 0\right]$, we apply Lemma 4.2 with $\alpha = 0$ to conclude (1.8) with $K_0 = CK_*$. If $\gamma \in \left(-2, \frac{-2d}{d+2}\right)$, Lemma 4.2 with $\alpha = 0$ implies

$$f(t, x, v) \le CK\left(1 + t_0^{-d/2}\right)(1 + |v_0|)^{-[d(4+\gamma)+2+2\gamma]/(d+2)},$$

so we can apply Lemma 4.2 again with $\alpha = [d(4 + \gamma) + 2 + 2\gamma]/(d + 2)$. We iterate this step, and since for any $\alpha \in (0, 1]$, we have $\alpha \le 1 < d(4 + \gamma)/2 + 1 + \gamma$, the gain of decay at each step, $-P(d, \alpha, \gamma) - \alpha$, is bounded away from 0. Therefore, after finitely many steps (with the number of steps depending only on d and γ), we obtain (1.8) for some K_0. □

The next result shows that the generating decay in Theorem 1.1 cannot be improved to polynomial decay with power greater than $d + 2$, or to exponential decay. Note that since $\bar{b}_i = -\partial_j \bar{a}_{ij}$, for smooth solutions (1.1) may be written equivalently in non-divergence form as

$$\partial_t f + v \cdot \nabla_x f = \operatorname{tr}(\bar{a}(t, x, v)D_v^2 f) + \bar{c}(t, x, v)f. \tag{4.12}$$

Theorem 4.3. *Let $\gamma \in [-2, 0]$ and $p > d + 2$. Assume f solves (1.1) in $[0, T_0] \times \mathbb{R}^{2d}$ with*

$$f_{in}(x, v) \ge c_0(1 + |v|)^{-p} \tag{4.13}$$

for $v, x \in \mathbb{R}^d$, for some $c_0 > 0$. Then there exist $c_1 > 0$ and $\beta > 0$ such that

$$f(t, x, v) \ge c_1 e^{-\beta t}(1 + |v|)^{-p} \tag{4.14}$$

for all $|v| \ge 1$, $x \in \mathbb{R}^d$, and $t \in [0, T_0]$.

Proof. Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth, decreasing function such that $\eta(r) \equiv 2$ when $r \in [0, \frac{1}{2}]$ and $\eta(r) = r^{-p}$ when $r \in [1, \infty)$. Note $\eta(r) \approx (1 + r)^{-p}$. Let us define $\psi(t, x, v) = e^{-\beta t}\eta(|v|)$ with β to be chosen later. Choose an arbitrary $R_0 > 1$, and recall from Lemma 2.1 that $\bar{a}_{ij}\partial_{ij}\psi \ge -C(1 + |v|)^{\gamma+2}|D^2\psi|$. (Throughout this proof, $\bar{a}_{ij}$ and $\bar{c}$ are defined in terms of f.) From our choice of η, it is clear that $|D^2\psi|/\psi$ is uniformly bounded from above in $\mathbb{R}_+ \times \mathbb{R}^d \times \{v : |v| \le R_0 + 1\}$, so for $\beta \ge \beta_1$ sufficiently large, we have

$$-\partial_t\psi + \bar{a}_{ij}\partial_{ij}\psi + \bar{c}\psi \ge \beta\psi - C(1 + |v|)^{\gamma+2}|D^2\psi| \ge 0, \quad |v| \le R_0 + 1.$$

For $|v| \ge R_0$, we estimate $\bar{a}_{ij}\partial_{ij}\psi$ more carefully. Since $|v| \ge 1$, we have

$$\partial_{ij}\psi = \frac{\partial_{rr}\psi}{|v|^2}v_i v_j + \frac{\partial_r\psi}{|v|}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right) = \left[p(p+1)|v|^{-4}v_i v_j - p|v|^{-2}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)\right]e^{-\beta t}|v|^{-p},$$

and Lemma 2.1 implies

$$-\partial_t\psi + \bar{a}_{ij}\partial_{ij}\psi \ge \beta\psi + \left[p(p+1)C_1|v|^{-2+\gamma} - pC_2|v|^\gamma\right]\psi \ge \left(\beta - C|v|^\gamma\right)\psi.$$

For $\beta \ge \beta_2$ sufficiently large, the right-hand side is positive for all $|v| \ge R_0$. Since $\bar{c}(t, x, v) \ge 0$, this implies $\psi(t, x, v) = e^{-\beta t}\eta(|v|)$ with $\beta = \max(\beta_1, \beta_2)$ is a subsolution of $(-\partial_t + \bar{a}_{ij}\partial_{ij} + \bar{c})g = 0$ in the entire domain $\mathbb{R}_+ \times \mathbb{R}^{2d}$. By (4.13), there is some $c_1 \ge c_0$ so that $f_{in}(x, v) \ge c_1\psi(0, x, v)$ in $\mathbb{R}^{2d}$. Now we can apply the maximum principle (see Appendix A) to $c_1\psi - f$ to conclude (4.14). □

Remark. *The bound on the energy $\int_{\mathbb{R}^d} |v|^2 f(t, x, v)\,\mathrm{d}v \le E_0 < \infty$ implies that $f_{in}(x, v)$ cannot be bounded below by $c_0|v|^{-p}$ with $p \le d + 2$ as $|v| \to \infty$.*

Remark. *In particular, Theorem 4.3 tells us that there is no generation of moments when $\gamma \in [-2, 0]$.*

5. Gaussian bounds

We show the propagation of Gaussian upper bounds. The first lemma says that a sufficiently slowly decaying Gaussian is a supersolution of the linear Landau equation for large velocities. As above, the coefficients $\bar{a}_{ij}$ and $\bar{c}$ in (5.1) are defined in terms of f.

Lemma 5.1. *Let $\gamma \in (-2, 0]$. Let f be a bounded function satisfying (1.5), (1.6), and (1.7). Let $\bar{a}$ and $\bar{c}$ be given by (1.2) and (1.4) respectively. If $\alpha > 0$ is sufficiently small, then there exists $R_0 > 0$ and $C > 0$, depending on d, γ, M_0, m_0, E_0, H_0 and $\|f\|_{L^\infty}$, such that*

$$\phi(v) := e^{-\alpha|v|^2}$$

satisfies

$$\bar{a}_{ij}\partial_{ij}\phi + \bar{c}\phi \le -C|v|^{\gamma+2}\phi, \tag{5.1}$$

for $|v| \ge R_0$.

Proof. Since ϕ is radial, we have

$$\partial_{ij}\phi = \frac{\partial_{rr}\phi}{|v|^2}v_i v_j + \frac{\partial_r\phi}{|v|}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right) = \left[\frac{4\alpha^2|v|^2 - 2\alpha}{|v|^2}v_i v_j - 2\alpha\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)\right]e^{-\alpha|v|^2},$$

and the bounds (2.1) and (2.2) imply

$$\begin{aligned}\bar{a}_{ij}\partial_{ij}\phi &\le \left[(4\alpha^2|v|^2 - 2\alpha)C_1|v|^\gamma - 2\alpha C_2|v|^{\gamma+2}\right]e^{-\alpha|v|^2}\\ &= \left((4\alpha^2 C_1 - 2\alpha C_2)|v|^{\gamma+2} - 2\alpha C_1|v|^\gamma\right)e^{-\alpha|v|^2}\\ &\le -C|v|^{\gamma+2}\phi(v),\end{aligned}$$

for $|v|$ sufficiently large, provided $\alpha < C_2/(2C_1)$. With Lemma 2.2 (this is the point where $\|f\|_{L^\infty}$ plays a role), this implies

$$\bar{a}_{ij}\partial_{ij}\phi + \bar{c}\phi \le \left[-C|v|^{\gamma+2} + C|v|^{-2-2\gamma/d}\right]\phi(v).$$

For $-2 < \gamma \le 0$, the first term on the right-hand side will dominate for large $|v|$, since $\gamma + 2 > 0 > -2 - 2\gamma/d$. □

Theorem 1.1 gives us an upper bound for a solution f to the Landau equation which is useful away from $t = 0$. If the initial data $f(0, x, v)$ is a bounded function, we can improve our upper bound for small values of t using the upper bound for $f(0, x, v)$. That is the purpose of the next lemma.

Lemma 5.2. *Let $f : [0, T_0] \times \mathbb{R}^{2d} \to \mathbb{R}$ be a solution of the Landau equation (1.1) for some $\gamma \in (-2, 0]$, and suppose that $g : [0, T_0] \times \mathbb{R}^{2d} \to \mathbb{R}$ is bounded from above and a subsolution to the equation*

$$\partial_t g(t, x, v) + v \cdot \nabla_x g(t, x, v) \le \bar{a}_{ij}(t, x, v)\partial_{ij} g(t, x, v) + \bar{c}(t, x, v)g(t, x, v), \tag{5.2}$$

where $\bar{a}_{ij}$ and $\bar{c}$ are defined in terms of f as in (1.2) and (1.4). Let $\kappa(t)$ be defined by

$$\kappa(t) = \begin{cases} \frac{\beta}{1+\gamma/2}t^{1+\gamma/2}, & 0 \le t \le 1\\ \frac{\beta}{1+\gamma/2} + \beta(t-1), & t \ge 1\end{cases}, \tag{5.3}$$

where $\beta > 0$ depends only on d, γ, m_0, M_0, E_0, and H_0. Then

$$\sup_{[0,T_0]\times\mathbb{R}^d} e^{-\kappa(t)} g_+(t, x, v) = \sup_{\mathbb{R}^{2d}} g_+(0, x, v). \tag{5.4}$$

Proof. By Theorem 1.1, we have that $f(t,x,v) \le K_0 t^{-d/2}$ for $0 < t < 1$. Hence by Lemma 2.2, we have that $\bar{c}(t,x,v) \lesssim t^{\gamma/2}$. Since $\gamma > -2$, for some universal $\beta > 0$, $\kappa(t)$ satisfies $\bar{c}(t,x,v) \le \kappa'(t)$ for all $t > 0$. Thus $\tilde{g}(t,x,v) = e^{-\kappa(t)} g(t,x,v)$ satisfies

$$\begin{aligned}\partial_t \tilde{g}(t,x,v) + v \cdot \nabla_x \tilde{g}(t,x,v) &\le \bar{a}_{ij}(t,x,v)\partial_{ij}\tilde{g}(t,x,v) + (\bar{c}(t,x,v) - \kappa'(t))\tilde{g}(t,x,v) \\ &\le \bar{a}_{ij}(t,x,v)\partial_{ij}\tilde{g}(t,x,v).\end{aligned}$$

We apply Lemma A.2 from the Appendix to $\tilde{g}(t,x,v) - \sup_{\mathbb{R}^{2d}} g(0,x,v)$ to conclude (5.4). □

Proof of Theorem 1.2. Applying Lemma 5.2 with $g = f$ and $t \in [0,1]$ and Theorem 1.1 for $t > 1$, we have that there is some constant C_2 (depending on C_0, d, γ, M_0, m_0, E_0, and H_0) so that $f(t,x,v) \le C_2$ for all $t \ge 0$, $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$.

Let $\phi(v) := e^{-\alpha|v|^2}$. From Lemma 5.1, we have that there is a C, depending on C_2, d, γ, M_0, m_0, E_0, and H_0, such that

$$\sup_{(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d} \bar{a}_{ij}\partial_{ij}\phi + \bar{c}\phi \le C\phi.$$

Thus $C_0 e^{Ct}\phi(v)$ is a supersolution of the equation and $f(t,x,v) \le C_0 e^{Ct}\phi(v)$ for all $t > 0$, $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$.

This upper bound is good for small values of t. We see that there is some time $t_0 > 0$ so that $C_0 e^{Ct_0}\phi(v) > C_2$ for $|v| < R_0$. Here C_2 is the upper bound for f mentioned above and R_0 is the radius from Lemma 5.1. Thus, the function

$$g(t,x,v) := \left[f(t_0 + t, x, v) - C_0 e^{Ct_0}\phi(v) \right]_+$$

is a supersolution of

$$g_t + v \cdot \nabla_x g \le \bar{a}_{ij}\partial_{ij} g + \bar{c} g.$$

Applying the maximum principle (Lemma A.2), we have that $g \le 0$ for all $t > 0$, so $f(t,x,v) \le C_0 e^{t_0 C}\phi(v)$ for all $t > t_0$, and we conclude the proof. □

By combining Theorem 1.2 with the local Hölder estimates proved in [8] or [22], we derive a global Hölder estimate for solutions of (1.1) under the assumption that $f_{in}(x,v) \le C_0 e^{-\alpha|v|^2}$. The following local estimate is essentially the same as Theorem 2 of [8]:

Theorem 5.3. *Let f be a weak solution of*

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s \tag{5.5}$$

in Q_1, with $\lambda I \le A \le \Lambda I$, $|B| \le \Lambda$, and $s \in L^\infty(Q_1)$. Then f is Hölder continuous with respect to (t,x,v) in $Q_{1/2}$, and

$$\frac{|f(z_1) - f(z_1)|}{|t_1 - t_2|^{\beta/2} + |x_1 - x_2|^{\beta/3} + |v_1 + v_2|^\beta} \le C(\|f\|_{L^2(Q_1)} + \|s\|_{L^\infty(Q_1)}),$$

for all $z_1, z_2 \in Q_{1/2}$, where β and C depend on d, λ, and Λ.

To state our theorem as a global Hölder estimate, we will need an appropriate notion of distance in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ which is invariant by Galilean transformations. A natural choice is the following

$$d_P(z_1, z_2) := \min\{r : \exists z \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d : z_1 \in Q_r(z) \text{ and } z_2 \in Q_r(z)\}.$$

We can easily estimate the value of $d_P(z_1, z_2)$ by the simpler formula

$$d_P(z_1, z_2) \approx |t_1 - t_2|^{1/2} + |x_1 - x_2 - (t_1 - t_2)(v_1 + v_2)/2|^{1/3} + |v_1 - v_2|.$$

It turns out that we need to deform this distance using the transformation $\mathcal{T}_z$ described in Lemma 4.1. We define

$$d_L(z_1, z_2) := \min\{|v|^{1+\gamma/2} r : z \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d : \mathcal{T}_z^{-1} z_1 \in Q_r \text{ and } \mathcal{T}_z^{-1} z_2 \in Q_r\}.$$

(Here, we make the convention that $\mathcal{T}_z = \mathcal{S}_z$ when $|v| < 2$.) An explicit expression for $d_L(z_1, z_2)$ is messy. It involves the affine transformation $\mathcal{T}$ which is anisotropic and affects both the x and v variables. In the case that we compare two points with identical values of t and x, it is straightforward to check that when $d_L((t, x, v_1), (t, x, v_2)) < 1$, then d_L is equivalent to the metric introduced by Gressman–Strain [9] in their study of the Boltzmann equation.

Theorem 5.4. *Under the assumptions of Theorem 1.2, there exist $C > 0$ and $\beta \in (0, 1)$ depending on C_0, α, d, γ, m_0, M_0, E_0, and H_0, such that for any $z_1, z_2 \in [0, T_0] \times \mathbb{R}^{2d}$, one has*

$$|f(z_1) - f(z_2)| \le C \left(e^{-\alpha|v_1|^2} + e^{-\alpha|v_2|^2}\right) \min\left\{1, \left(1 + t_1^{-\beta/2} + t_2^{-\beta/2}\right) d_L(z_1, z_2)^\beta\right\}.$$

Proof. If $|v_1| \le 2$ or $|v_2| \le 2$, the result follows by applying Theorem 5.3 directly to f, noting that $1 \lesssim e^{-\alpha|v_1|^2} + e^{-\alpha|v_2|^2}$. So, we can assume that $|v_1| > 2$ and $|v_2| > 2$.

Let $\bar{z} = (\bar{t}, \bar{x}, \bar{v})$ be the point achieving the minimum in the definition of $d_L(z_1, z_2)$. Thus $\tilde{z}_1 := \mathcal{T}_{\bar{z}}^{-1} z_1 \in Q_\delta$ and $\tilde{z}_1 := \mathcal{T}_{\bar{z}}^{-1} z_1 \in Q_\delta$, where $\delta = |\bar{v}|^{-1-\gamma/2} d_L(z_1, z_2)$.

Let $r := \min\left(t_1^{1/2}, t_2^{1/2}, (1 + |\bar{v}|)^{-1-\gamma/2}\right)$. If $\delta \ge r/2$, then we simply estimate $|f(z_1) - f(z_2)| \le C_1(e^{-\alpha|v_1|^2} + e^{-\alpha|v_2|^2})$ from Theorem 1.2. We need to concentrate on the case $\delta < r/2$.

Let us consider the function f_T as in Lemma 4.1, with base point $\bar{z}$. By our choice of r, f_T satisfies an equation of the form (5.5) in Q_r, and since Theorem 1.2 gives us a bound on $\|f_T\|_{L^\infty}$, we have that A is uniformly elliptic (with constants independent of $\bar{z}$), $|B| \lesssim |\bar{v}|^{1+\gamma/2}$, and $|s| = |C(z) f_T| \lesssim |f_T|$. Defining $\tilde{f}_T(t, x, v) := f_T(r^2 t, r^3 x, r v)$, we see that $\tilde{f}_T$ satisfies another equation of the form (5.5) with the new $|B|$ bounded independently of $|\bar{v}|$. Moreover, the points $(r^{-2}\tilde{t}_1, r^{-3}\tilde{x}_1, r^{-1}\tilde{v}_1)$ and $(r^{-2}\tilde{t}_2, r^{-3}\tilde{x}_2, r^{-1}\tilde{v}_2)$ belong to $Q_{r^{-1}\delta} \subset Q_{1/2}$. Therefore, we can apply Theorem 5.3 to $\tilde{f}_T$ in Q_1 to obtain

$$\begin{aligned}
\frac{|f(z_1) - f(z_2)|}{r^{-\beta} d_L(z_1, z_2)^\beta} |\bar{v}|^{\beta(1+\gamma/2)} &= \frac{|f_T(\tilde{z}_1) - f_T(\tilde{z}_2)|}{r^{-\beta}\delta^\beta} \\
&= \frac{|\tilde{f}_T(r^{-2}\tilde{t}_1, r^{-3}\tilde{x}_1, r^{-1}\tilde{v}_1) - \tilde{f}_T(r^{-2}\tilde{t}_2, r^{-3}\tilde{x}_2, r^{-1}\tilde{v}_2)|}{r^{-\beta}\delta^\beta}, \\
&\lesssim \|\tilde{f}_T\|_{L^1(Q_1)} + \|\tilde{f}_T\|_{L^\infty(Q_1)} \lesssim \sup_{v \in Q_{r/2}} e^{-\alpha|\bar{v} + T v|^2} \lesssim e^{-\alpha|\bar{v}|^2}.
\end{aligned}$$

We have used Theorem 1.2 to estimate the L^∞ norm of $\tilde{f}_T$ in Q_1. Rewriting this estimate, we obtain

$$\begin{aligned}
|f(z_1) - f(z_2)| &\lesssim r^{-\beta} |\bar{v}|^{-\beta(1+\gamma/2)} d_L(z_1, z_2)^\beta e^{-\alpha|\bar{v}|^2} \\
&\lesssim \left(1 + t_1^{-\beta/2} + t_2^{-\beta/2}\right) d_L(z_1, z_2)^\beta \left(e^{-\alpha|\bar{v}_1|^2} + e^{-\alpha|\bar{v}_2|^2}\right). \quad \square
\end{aligned}$$

Conflict of interest statement

There is no conflict of interest.

Appendix A. Maximum principle for weak solutions to kinetic Fokker–Planck equations

In this appendix, we give a proof of the maximum principle in a form that is convenient for our purposes.

The following proposition is perhaps a classical result. We prove it here, since we could not find any easy reference and also for completeness. The result is for equations on a bounded domain with general coefficients (not necessarily defined by integrals as above).

Proposition A.1. *Let $Q = [0, T_0] \times \Omega$, where $\Omega \subset \mathbb{R}^{2d}$ is a bounded domain, and assume that g is a subsolution of the equation*

$$\partial_t g + v \cdot \nabla_x g \le \nabla_v \cdot [a(t, x, v)\nabla_v g] + b(t, x, v) \cdot \nabla_v g + c(t, x, v) g, \tag{A.1}$$

in the weak sense in Q, where a is uniformly elliptic in Q with constants λ and Λ, and b and c are uniformly bounded in Q.

If $g \le 0$ on the parabolic boundary of Q, then $g \le 0$ in Q.

Proof. Choosing the test function $\phi = g_+$, the weak formulation of (A.1) gives

$$\int_Q g_+ (\partial_t g + v \cdot \nabla_x g)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t \le \int_Q \left(-a\nabla_v g \nabla_v g_+ - g_+ b \cdot \nabla g + c g_+^2\right) \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t,$$

or

$$\int_Q \frac{1}{2}\frac{d}{dt}(g_+)^2\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t \le \int_Q \left(-\lambda |\nabla_v g_+|^2 - bg\nabla g_+ + c g_+^2\right) \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t$$
$$\le \left(\frac{\|b\|_{L^\infty}}{4\lambda} + \|c\|_{L^\infty}\right) \int_Q g_+^2\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t,$$

by Young's inequality. We apply Gronwall's Lemma to $\int_\Omega (g_+)^2\, \mathrm{d}x\, \mathrm{d}v$ on $[0, T_0]$ to conclude $g_+ \equiv 0$ in Q. □

Next, we derive a maximum principle on the whole space for subsolutions of a Landau-type equation without a zeroth-order term:

Lemma A.2. *Let g be a bounded function on $[0, T_0] \times \mathbb{R}^{2d}$ that satisfies*

$$\partial_t g + v \cdot \nabla_x g \le \mathrm{tr}(\bar{a}(t,x,v) D_v^2 g), \tag{A.2}$$

in the weak sense. Here, $\bar{a}(t,x,v)$ is defined as in (1.2) in terms of a function f satisfying (1.5), (1.6), and (1.7). If $g(0,x,v) \le 0$ in $\mathbb{R}^{2d}$, then $g(t,x,v) \le 0$ in $[0, T_0] \times \mathbb{R}^{2d}$.

Proof. By the bounds on $\bar{a}$ given in Lemma 2.1, we have

$$\bar{a}_{ij}\partial_{ij}(1+|v|) \le C_1(1+|v|)^{1+\gamma},$$

for some constant C_1, and thus $\phi_1(t,v) := e^{C_1 t}(1+|v|)$ satisfies

$$\partial_t \phi_1(t,v) \ge \bar{a}_{ij}(t,x,v)\partial_{ij}\phi_1(t,v).$$

Let $\varepsilon_1 > 0$ be a small constant. Since g is bounded, there is $R(\varepsilon_1) > 0$ such that $g - \varepsilon_1\phi_1 < 0$ whenever $|v| \ge R(\varepsilon_1)$. Let $R_1 > R(\varepsilon_1)$, and choosing $C_2 > 0$ large enough depending on R_1, we can define $\phi_2(t,x) := (1+|x|)e^{C_2 t}$, and we have

$$\partial_t \phi_2 + v \cdot \nabla_x \phi_2 \ge 0,$$

whenever $|v| < R_1$. Finally, for $\varepsilon_2 > 0$ arbitrary, we define

$$\tilde{g}(t,x,v) := [g(t,x,v) - \varepsilon_1\phi_1(t,v) - \varepsilon_2\phi_2(t,x)]_+.$$

It is clear that $\tilde{g}$ is a subsolution as in (A.1) with $c \equiv 0$, whenever $|v| < R_1$. For $R(\varepsilon_2)$ sufficiently large, we have that $g - \varepsilon_1\phi_1 - \varepsilon_2\phi_2 < 0$ for $|x| \ge R(\varepsilon_2)$ or $|v| \ge R(\varepsilon_1)$. Then for any $R_2 > R(\varepsilon_2)$, we have that $\tilde{g} = 0$ on the parabolic boundary of $[0, T_0] \times B_{R_2} \times B_{R_1}$, so Proposition A.1 applied to $\tilde{g}$ gives

$$g - \varepsilon_1\phi_1 - \varepsilon_2\phi_2 \le 0, \quad |v| < R_1, \ |x| < R_2.$$

Take $R_2 \to \infty$ and $\varepsilon_2 \to 0$ to conclude

$$g - \varepsilon_1\phi_1 \le 0, \quad |v| < R_1.$$

Take $R_1 \to \infty$ and $\varepsilon_1 \to 0$, and the proof is complete. □

References

[1] R. Alexandre, J. Liao, C. Lin, Some a priori estimates for the homogeneous Landau equation with soft potentials, Kinet. Relat. Models 8 (4) (2015) 617–650.

[2] R. Alexandre, C. Villani, On the Landau approximation in plasma physics, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 21 (1) (2004) 61–95.

[3] C. Bardos, F. Golse, D. Levermore, Fluid dynamic limits of kinetic equations. I. Formal derivations, J. Stat. Phys. 63 (1–2) (1991) 323–344.

[4] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases, 3rd edition, Cambridge University Press, 1970.

[5] L. Desvillettes, Entropy dissipation estimates for the Landau equation in the Coulomb case and applications, J. Funct. Anal. 269 (5) (2015) 1359–1403.

[6] L. Desvillettes, C. Villani, On the spatially homogeneous Landau equation for hard potentials part I: existence, uniqueness and smoothness, Commun. Partial Differ. Equ. 25 (1–2) (2000) 179–259.

[7] I.M. Gamba, V. Panferov, C. Villani, Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation, Arch. Ration. Mech. Anal. 194 (1) (2009) 253–282.

[8] F. Golse, C. Imbert, C. Mouhot, A. Vasseur, Harnack inequality for kinetic Fokker–Planck equations with rough coefficients and application to the Landau equation, Ann. Sc. Norm. Super. Pisa (2017), in press.

[9] P.T. Gressman, R.M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off, J. Am. Math. Soc. 24 (3) (2011) 771–847.

[10] Y. Guo, The Landau equation in a periodic box, Commun. Math. Phys. 231 (3) (2002) 391–434.

[11] Q. Han, F.-H. Lin, Elliptic Partial Differential Equations, 2nd edition, Courant Lecture Notes, Courant Institute of Mathematical Sciences, New York University, 2011.

[12] C. Imbert, L. Silvestre, Weak Harnack inequality for the Boltzmann equation without cut-off, J. Eur. Math. Soc. (2017), in press.

[13] E.M. Lifshitz, L.P. Pitaevskii, Physical Kinetics, 1st edition, Course of Theoretical Physics, vol. 10, Butterworth-Heinemann, 1981.

[14] S. Liu, X. Ma, Regularizing effects for the classical solutions to the Landau equation in the whole space, J. Math. Anal. Appl. 417 (1) (2014) 123–143.

[15] C. Mouhot, L. Neumann, Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus, Nonlinearity 19 (4) (2006) 969.

[16] A. Pascucci, S. Polidoro, The Moser's iterative method for a class of ultraparabolic equations, Commun. Contemp. Math. 06 (03) (2004) 395–417.

[17] L. Silvestre, A new regularization mechanism for the Boltzmann equation without cut-off, Commun. Math. Phys. 348 (1) (2016) 69–100.

[18] L. Silvestre, Upper bounds for parabolic equations and the Landau equation, J. Differ. Equ. 262 (3) (2017) 3034–3055.

[19] C. Villani, On the Cauchy problem for Landau equation: sequential stability, global existence, Adv. Differ. Equ. 1 (5) (1996) 793–816.

[20] C. Villani, On the spatially homogeneous Landau equation for Maxwellian molecules, Math. Models Methods Appl. Sci. 08 (06) (1998) 957–983.

[21] W. Wang, L. Zhang, The C^α regularity of a class of non-homogeneous ultraparabolic equations, Sci. China Ser. A 52 (8) (2009) 1589–1606.

[22] W. Wang, L. Zhang, The C^α regularity of weak solutions of ultraparabolic equations, Discrete Contin. Dyn. Syst. 29 (3) (2011) 1261–1275.

[23] K.-C. Wu, Global in time estimates for the spatially homogeneous Landau equation with soft potentials, J. Funct. Anal. 266 (5) (2014) 3134–3155.