Constructing center-stable tori

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Abstract

We show that certain derived-from-Anosov diffeomorphisms on the 2-torus may be realized as the dynamics on a center-stable or center-unstable torus of a 3-dimensional strongly partially hyperbolic system. We also construct examples of center-stable and center-unstable tori in higher dimensions.
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1. Introduction

Partially hyperbolic dynamical systems have received a large amount of attention in recent years. These systems display a wide variety of highly chaotic behavior [2], but they have enough structure to allow, in some cases, for the dynamics to be understood and classified [5,13].

A diffeomorphism f is *strongly partially hyperbolic* if there is a splitting of the tangent bundle into three invariant subbundles $TM = E^u \oplus E^c \oplus E^s$ such that the derivative Df expands vectors in the unstable bundle E^u, contracts vectors in stable bundle E^s, and these dominate any expansion or contraction in the center direction E^c. (See section 2 for a precise definition.) The global properties of these systems are often determined by analyzing invariant foliations tangent to the subbundles of the splitting.

The bundles E^u and E^s are uniquely integrable [14]. That is, there are foliations W^u and W^s such that any curve tangent to E^u or to E^s lies in a single leaf of the respective foliation. For the center bundle E^c, however, the situation is more complicated. There may not be a foliation tangent to E^c. Even if such a foliation exists, the bundle may not be uniquely integrable since, in general, the center bundle is only Hölder continuous and not C^1 regular. The first discovered examples of partially hyperbolic systems without center foliations were algebraic in nature. In these examples, both f and the splitting can be taken as smooth, and the center bundle is not integrable because it does not satisfy Frobenius' condition of involutivity [18,20]. Such non-involutive examples are only possible if the dimension of the center bundle is at least two, and for a long time it was an open question if a one-dimensional center bundle was necessarily integrable.

Rodriguez Hertz, Rodriguez Hertz, and Ures recently answered this question by constructing a counterexample [17]. They defined a partially hyperbolic system on the 3-torus with a one-dimensional center bundle which does not integrate to a center foliation. In fact, the center bundle is uniquely integrable everywhere except for an invariant embedded 2-torus tangent to $E^c \oplus E^u$ where the center curves approaching from either side of the torus meet in cusps. This discovery has shifted our view on the possible dynamics a partially hyperbolic system can possess, and leads to questions of how commonly invariant submanifolds of this type occur in general. In this paper, we build further examples of partially hyperbolic systems having compact submanifolds tangent either to $E^c \oplus E^u$ or $E^c \oplus E^s$, both in dimension 3 and in higher dimensions.

In the construction in [17], the dynamics on the 2-torus tangent to $E^c \oplus E^u$ is Anosov. In fact, it is given by a hyperbolic linear map on $\mathbb{T}^2$, the cat map. It has long been known that a *weakly partially hyperbolic* system, that is, a diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ with a splitting of the form $E^c \oplus E^u$ or $E^c \oplus E^s$, need not be Anosov. Therefore, one can ask if a weakly partially hyperbolic system which is not Anosov may be realized as the dynamics on an invariant 2-torus sitting inside a 3-dimensional strongly partially hyperbolic system. We show, in fact, that derived-from-Anosov dynamics with sinks or sources may be realized on these tori.

To state the results, we say that diffeomorphisms f_0 and f_1 of the 2-torus are *dom-isotopic* if there an isotopy $\{f_t\}_{t \in [0,1]}$ such that every f_t has a dominated splitting.

Theorem 1.1. *Let $g : \mathbb{T}^2 \to \mathbb{T}^2$ be a weakly partially hyperbolic diffeomorphism which is dom-isotopic to a linear toral automorphism. Then, there is an embedding $i : \mathbb{T}^2 \to \mathbb{T}^3$ and a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that $i(\mathbb{T}^2)$ is a center-stable or center-unstable torus and $i^{-1} \circ f \circ i = g$.*

To be precise, a *center-stable torus* is an embedded copy of $\mathbb{T}^D$ with $D \geq 2$ which is tangent to $E_f^{cs} := E_f^c \oplus E_f^s$. Similarly, a *center-unstable torus* is tangent to $E_f^{cu} := E_f^c \oplus E_f^u$. We also use the terms *cs*-torus and *cu*-torus as shorthand.

If the diffeomorphism g has a weakly partially hyperbolic of splitting of the form $E_g^s \oplus E_g^c$, then $i(\mathbb{T}^2)$ will be a *cs*-torus. If the splitting is of the form $E_g^c \oplus E_g^u$, then $i(\mathbb{T}^2)$ will be a *cu*-torus. In the case where the derivative of g preserves the orientation of the center bundle, E_g^c, we may be more specific about the construction.

Theorem 1.2. *Let $g_0 : \mathbb{T}^2 \to \mathbb{T}^2$ be a weakly partially hyperbolic diffeomorphism which preserves the orientation of its center bundle and is dom-isotopic to a linear Anosov diffeomorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$ and let $0 < \epsilon < \frac{1}{2}$. Then there is a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that*

(1) *$f(x, t) = (A(x), t)$ for all $(x, t) \in \mathbb{T}^3$ with $|t| > \epsilon$,*
(2) *$f(x, t) = (g_0(x), t)$ for all $(x, t) \in \mathbb{T}^3$ with $|t| < \frac{\epsilon}{2}$, and*
(3) *$\mathbb{T}^2 \times 0$ is either a center-stable or center-unstable torus.*

Since the construction is local in nature, different weakly partially hyperbolic diffeomorphisms may be inserted into the system at different places.

Corollary 1.3. *Suppose that $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic linear automorphism and for each $i \in \{1, \ldots, n\}$ that $g_i : \mathbb{T}^2 \to \mathbb{T}^2$ is a weakly partially hyperbolic diffeomorphism which preserves the orientation of its center bundle and is dom-isotopic to A. Let $\{t_1, \ldots, t_n\}$ be a finite subset of the circle, S^1. Then there is a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that*

(1) *$f(x, t_i) = (g_i(x), t_i)$ for each t_i and all $x \in \mathbb{T}^2$, and*
(2) *each $\mathbb{T}^2 \times t_i$ is a center-stable or center-unstable torus.*

In the above theorems, the examples may be constructed in such a way that the resulting diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ is not dynamically coherent. See section 4 for details.

We also note that the presence of a *cs* or *cu*-torus affects the dynamics only in a neighborhood of that torus and does not place global restrictions on the dynamics on $\mathbb{T}^3$. For instance, one could easily construct a system which has a *cs* or *cu*-torus $\mathbb{T}^2 \times 0$ and has a robustly transitive blender elsewhere on $\mathbb{T}^3$ [1].

The technical assumption of dom-isotopy can likely be relaxed. Gourmelon and Potrie and have announced a result showing that in the C^1-open set of diffeomorphisms of $\mathbb{T}^2$ with dominated splittings, the subset of diffeomorphisms isotopic to a given hyperbolic toral automorphism is connected. Should this hold, Theorem 1.1 would imply that any weakly partially hyperbolic diffeomorphism on $\mathbb{T}^2$ may be realized as the dynamics on a center-stable or center-unstable torus.

The original construction of Rodriguez Hertz, Rodriguez Hertz, and Ures on the 3-torus may be viewed as a skew product with Anosov dynamics in the fibers. In fact, the example can be given as a map of the form

$$F(x, v) = (f(x), Av + h(x))$$

where f is a Morse–Smale diffeomorphism of the circle, A is the cat map on $\mathbb{T}^2$, and $h : S^1 \rightarrow \mathbb{T}^2$ is smooth. The diffeomorphism f has a sink at a point x_0 and the fiber $x_0 \times \mathbb{T}^2$ over this sink gives the embedded 2-torus tangent to E_F^{cu}.

This description of F naturally suggests a way to construct higher-dimensional examples of the same form. We will show that, starting from any diffeomorphism f of any closed manifold M, one may construct a strongly partially hyperbolic diffeomorphism F of $M \times \mathbb{T}^D$ using sinks of f to construct center-unstable tori for F and sources to construct center-stable tori.

Theorem 1.4. *Let $f_0 : M \rightarrow M$ be a diffeomorphism and $X \subset M$ a finite invariant set such that every $x \in X$ is either a periodic source or sink. Then there is a strongly partially hyperbolic diffeomorphism $F : M \times \mathbb{T}^D \rightarrow M \times \mathbb{T}^D$ of the form*

$$F(x, v) = (f(x), Av + h(x))$$

such that f is isotopic to f_0 and, for each $x \in X$, the submanifold $x \times \mathbb{T}^D$ is tangent either to E^{cs} or E^{cu}.

In dimension 3, the presence of a compact submanifold tangent to E^{cs} or E^{cu} has strong consequences on the global topology of the manifold. In fact, Rodriguez Hertz, Rodriguez Hertz, and Ures showed that the 3-manifold can only be one of a few possibilities [16]. The proof of Theorem 1.4 uses a local argument and the global topology of M has no impact on the construction. This suggests that in higher dimensions, compact submanifolds tangent to E^{cs} and E^{cu} may arise naturally in many examples of partially hyperbolic systems.

Theorem 1.4 is stated for the trivial fiber bundle $M \times \mathbb{T}^D$ only for the sake of simplicity. As the proof is entirely local in nature, the same technique may be used to introduce center-stable and center-unstable tori in a system defined on a non-trivial fiber bundle, so long as the dynamics in the fibers is given by a linear Anosov map. By adapting the examples in [9], it might be possible to define a system with a center-stable torus so that the total space is simply connected. See also [8] for further constructions, and [6] for conditions which imply that the fiber bundle must be trivial. We suspect that, just as in the case of dimension 3, the future study of compact center-stable submanifolds in higher dimensions will be full of surprises.

Section 2 gives the definitions of domination and partial hyperbolic and establishes several preliminary results. Section 3 gives the proof of Theorem 1.2. Section 4 generalizes this construction and proves Theorem 1.1. Finally, section 5 handles higher-dimensional examples and proves Theorem 1.4.

2. Preliminaries

Let f be a diffeomorphism of a Riemannian manifold M and let $\Lambda \subset M$ be a compact invariant subset. A splitting of $T_\Lambda M$ into two non-zero bundles $T_\Lambda M = E \oplus \hat{E}$ is *dominated* if it is invariant under the derivative of f and there is $k \geq 1$ such that $\|Df^k v\| < \|Df^k \hat{v}\|$ for all $x \in \Lambda$ and unit vectors $v \in E(x)$ and $\hat{v} \in \hat{E}(x)$.

An invariant bundle $E \subset TM$ is *expanding* if there is $k \geq 1$ such that $\|D^k v\| > 2$ for all unit vectors $v \in E$. The bundle is contracting if there is $k \geq 1$ such that $\|D^k v\| < \frac{1}{2}$ for all unit vectors $v \in E$. A diffeomorphism f of a closed manifold M is *weakly partially hyperbolic* if it has a dominated splitting either of the form $TM = E^s \oplus E^c$ with E^s contracting or $TM = E^s \oplus E^u$ with E^u expanding.

A diffeomorphism f of a closed manifold M is *strongly partially hyperbolic* if it has a splitting $TM = E^s \oplus E^c \oplus E^u$ where E^s is contracting, E^u is expanding and both $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated splittings.

Notation. For a non-zero vector $v \in TM$ and $n \in \mathbb{Z}$, let v^n denote the unit vector

$$v^n = \frac{Df^n v}{\|Df^n v\|}.$$

Of course, v^n depends on the diffeomorphism $f : M \to M$ being studied, so this notation is used only when the f under study is clear.

To show that the constructions given in this paper are strongly partially hyperbolic, we use the following two results.

Proposition 2.1. *Suppose f is a diffeomorphism of a manifold M, and Y and Z are compact invariant subsets such that*

(1) *all chain recurrent points of $f|_Y$ lie in Z,*
(2) *Z has a dominated splitting $T_Z M = E \oplus \hat{E}$ with $d = \dim E$, and*
(3) *for every $x \in Y \setminus Z$, there is a point y in the orbit of x and a subspace V_y of dimension d such that for any non-zero $v \in V_y$, each of the sequences v^n and v^{-n} accumulates on a vector in $T_Z M \setminus E$ as $n \to +\infty$.*

Then the dominated splitting on Z extends to a dominated splitting on $Y \cup Z$.

For a diffeomorphism f, let $NW(f)$ denote the non-wandering set.

Proposition 2.2. *Suppose f is a diffeomorphism of a compact manifold M, $TM = E^s \oplus E^c \oplus E^u$ is an invariant splitting, and there is $k \geq 1$ such that*

$$\|Tf^k v^s\| < \|Tf^k v^c\| < \|Tf^k v^u\| \quad and \quad \|Tf^k v^s\| < 1 < \|Tf^k v^u\|$$

for all $x \in NW(f)$ and unit vectors $v^u \in E^u(x)$, $v^c \in E^c(x)$, and $v^s \in E^s(x)$. Then, f is partially hyperbolic.

The techniques used to prove the above results are similar in form to results developed by Mañé to study quasi-Anosov systems [15, Lemma 1.9], by Hirsch, Pugh, Shub in regards to normally hyperbolicity [14, Theorem 2.17], and by Franks and Williams in constructing non-transitive Anosov flows [7, Theorem 1.2]. For further details and the proofs of Propositions 2.1 and 2.2, see [10].

Proposition 2.3. *If g_0 and g_1 are dom-isotopic diffeomorphisms defined on $\mathbb{T}^2$, then there is a C^1 function $g : \mathbb{T}^2 \times [0, 1] \to \mathbb{T}^2$ such that $g_0 = g(\cdot, 0)$, $g_1 = g(\cdot, 1)$ and each $g(\cdot, t)$ has a dominated splitting.*

Proof. As diffeomorphisms with a dominated splitting comprise an open subset of all C^1 diffeomorphisms, we may assume there is a piecewise linear dom-isotopy $G : \mathbb{T}^2 \times [0, 1] \to \mathbb{T}^2$ such that $G(\cdot, 0) = g_0$ and $G(\cdot, 1) = g_1$. Define a smooth bijection $\alpha : [0, 1] \to [0, 1]$ such that the derivative of α equals zero at every time t_i where G changes from one linear function to the next. One may then show that $g(x, t) = G(x, \alpha(t))$ is the desired C^1 function. $\square$

In the C^1 topology, the dominated splitting depends continuously on the diffeomorphism. By adapting results on dominated splittings and cone families (see, for instance, [4, Section 2]), one can further show the following.

Corollary 2.4. *For a function g as in Proposition 2.3, there is a constant $\eta > 0$ and a family $\mathscr{C}$ of convex cones such that if, at a point $(x, t) \in \mathbb{T}^2 \times [0, 1]$, the dominated splitting is given by*

$$T_x \mathbb{T}^2 = E(x, t) \oplus \hat{E}(x, t),$$

then the cone $\mathscr{C}(x, t) \subset T_x \mathbb{T}^2$ satisfies the properties

(1) *$\hat{E}(x, t) \subset \mathscr{C}(x, t)$,*
(2) *$E(x, t) \cap \mathscr{C}(x, t) = 0$, and*

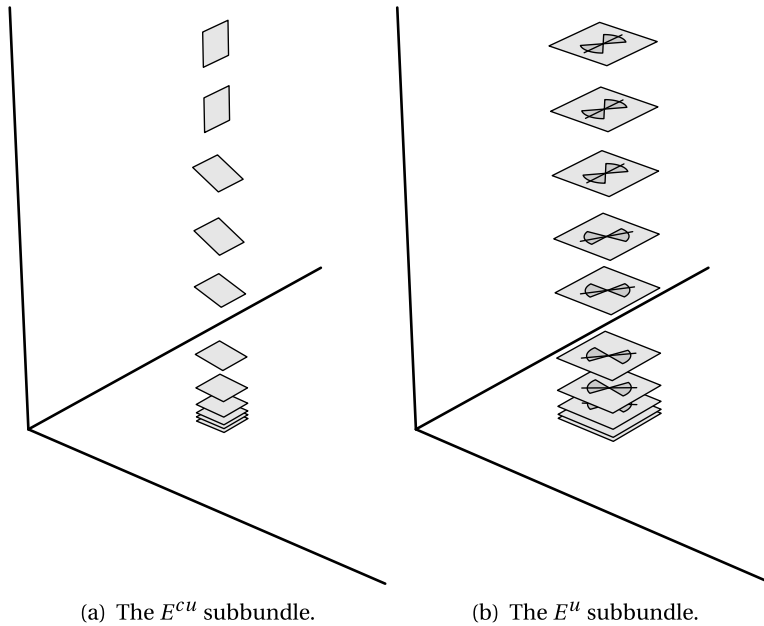(a) The E^{cu} subbundle.   (b) The E^u subbundle.

Fig. 1. A depiction of the E^{cu} and E^u subbundles in the construction given in section 3. Consider a point $(x, s) \in \mathbb{T}^2 \times (e, \epsilon)$ and its forward orbit $(x_n, s_n) := f^n(x, s)$. For simplicity, we assume the sequence $\{x_n\}$ is constant. The construction of f ensures that $\{s_n\}$ decreases towards 0. Subfigure (a) shows, for each $n \geq 0$, the two-dimensional subbundle $E^{cu}(x_n, s_n)$. When $s_n > d$, the E^{cu} subbundle is vertical. When $c < s_n < d$, a shearing effect in the dynamics pushes the E^{cu} planes to be closer to horizontal. When n is large and therefore $0 < s_n < a$, the strong vertical contraction means the slopes of these planes tend to zero as n tends to $+\infty$. Subfigure (b) shows, for each $n \geq 0$, the one-dimensional subbundle $E^u(x_n, s_n)$ lying inside the horizontal plane $T_{x_n}\mathbb{T}^2 \times 0$. It also depicts the cone field $\mathscr{C}(x_n, s_n)$ determined by Corollary 2.4. Both the horizontal planes and E^u are unaffected by the shearing. In the region $\mathbb{T}^2 \times [a, b]$, the E^u direction moves around as different horizontal maps $g(\cdot, t)$ are applied. However, the E^u direction always stays within the cone field $\mathscr{C}$.

(3) $Dg(\mathscr{C}(x, s))$ lies in the interior of $\mathscr{C}(g(x, t), t)$
for all $s \in [0, 1]$ with $|s - t| < \eta$.

The next lemma is used to determine the effect of shearing in the proof of Theorem 1.2.

Lemma 2.5. *Suppose $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a hyperbolic toral automorphism which preserves the orientation of its stable bundle. Lift A to a linear map on $\mathbb{R}^2$ and let $E^u_A(0)$ denote the lifted unstable manifold through the origin. For any $C > 1$, there is $z \in \mathbb{Z}^2$ such that $\mathrm{dist}\big(\zeta \cdot A(z) + \xi \cdot z, \, E^u_A(0)\big) \geq C(\zeta + \xi)$ for all $\zeta, \xi \geq 0$.*

The proof is left to the reader. Note how the condition on orientation is necessary.

3. Proof of Theorem 1.2

We now construct the diffeomorphism in the conclusion of Theorem 1.2 and show that it is strongly partially hyperbolic. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$, $g_0 : \mathbb{T}^2 \to \mathbb{T}^2$, and $\epsilon > 0$ be as in the theorem. Choose constants $\epsilon/2 < a < b < c < d < e < \epsilon$.

We give an intuitive description of the construction before diving into the details. The diffeomorphism f will contract the region $\mathbb{T}^2 \times (-e, e)$ down towards $\mathbb{T}^2 \times 0$. In the region $\mathbb{T}^2 \times [c, d]$, a strong shear pushes the vertical center direction to be almost horizontal. Then in the region $\mathbb{T}^2 \times [a, b]$, the dynamics in the horizontal direction is changed from A to g_0. Finally in $\mathbb{T}^2 \times [0, a)$, the vertical contraction is dialed up so that $\mathbb{T}^2 \times 0$ is a normally attracting submanifold. The effect of the dynamics on the E^{cu} and E^u subbundles is shown in Fig. 1.

Let $g : \mathbb{T}^2 \times [0, 1] \to \mathbb{T}^2$ be a C^1 function as in Proposition 2.3. By a reparameterization of the $[0, 1]$ coordinate, assume without loss of generality that $g(\cdot, t) = g_0$ for all $t \leq a$ and $g(\cdot, t) = A$ for all $t \geq b$. With g now determined, let $\eta > 0$ be as in Corollary 2.4.

Fix a value $\lambda \in (0, 1)$ such that $\|Dg_0 v\| > 2\lambda$ for all unit vectors v in the tangent bundle of $\mathbb{T}^2$. Define a smooth diffeomorphism $h : [0, \epsilon] \to [0, \epsilon]$ with the following properties.

(1) $h(s) = \lambda s$ for all $s \in [0, \epsilon/2]$,
(2) $h(s) < s$ for all $s \in (0, e)$,
(3) $|h(s) - s| < \eta$ for all $s \in [a, b]$,
(4) $h^2(d) < c < h(d)$, and
(5) $h(s) = s$ for all $s \in [e, \epsilon]$.

Define a smooth bump function $\rho : [0, \epsilon] \to [0, 1]$ such that

(1) $\rho(s) = 0$ for all $s \in [0, c]$,
(2) $\rho'(s) > 0$ for all $s \in (c, d)$, and
(3) $\rho(s) = 1$ for all $s \in [d, \epsilon]$.

Let z be a non-zero element of $\mathbb{Z}^2$. The precise conditions for choosing z will be given later in this section.

With these objects in place, define f for $(x, s) \in \mathbb{T}^2 \times [0, \epsilon]$ by

$$f(x, s) = (g(x, s) + \rho(s) \cdot z, h(s)).$$

Extend f to all of $\mathbb{T}^2 \times [-\epsilon, \epsilon]$ by the requirement that $(y, t) = f(x, s)$ if and only if $(y, -t) = f(x, -s)$. Finally, set $f(x, s) = (A(x), s)$ for all $(x, s) \notin \mathbb{T}^2 \times [-\epsilon, \epsilon]$.

We now consider the effect of Df on vectors of the tangent bundle. The identity $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ means that, for a point $p = (x, s)$, a tangent vector $u \in T_p \mathbb{T}^3$ may be decomposed as $u = (v, w)$ with $v \in T_x \mathbb{T}^2$ and $w \in T_s S^1$. During the proof, we will routinely write vectors this way and refer to v and w as the horizontal and vertical components of u. The linear toral automorphism A gives a linear splitting $T_x \mathbb{T}^2 = E_A^u(x) \oplus E_A^s(x)$ which further defines subspaces $E_A^u(x) \times 0$ and $E_A^s(x) \times 0$ of $T_p \mathbb{T}^3$. Also, if $\mathscr{C}(p) = \mathscr{C}(x, s) \subset T_x \mathbb{T}^2$ is the cone given by Corollary 2.4, then $\mathscr{C}(p) \times 0$ may be considered as a subset of $T_p \mathbb{T}^3$.

Lemma 3.1. *If $p = (x, s) \in \mathbb{T}^2 \times [c, \epsilon]$ and $y \in \mathbb{T}^2$ is such that $Df(p) = (y, h(s))$, then $Df_p(E_A^u(x) \times 0) = E_A^u(y) \times 0$.*

Proof. In this region, f is given by $f(x, s) = (A(x) + \rho(s)z, h(s))$ and both A and the translation $x \mapsto x + \rho(s)z$ leave the linear unstable foliation of A invariant. $\square$

Lemma 3.2. *If $p \in \mathbb{T}^2 \times [0, c]$, then $Df(\mathscr{C}(p) \times 0) \subset \mathscr{C}(f(p)) \times 0$.*

Proof. This follows directly from the use of $\eta > 0$ in the definition of f. $\square$

Lemma 3.3. *f has a dominated splitting of the form $E^u \oplus_> E^{cs}$ with $\dim E^u = 1$.*

Proof. We will apply Proposition 2.1 with $Y = \mathbb{T}^2 \times [0, e]$ and $Z = \mathbb{T}^2 \times \{0, e\}$. Note that Z has a well-defined partially hyperbolic splitting. If $p = (x, e) \in \mathbb{T}^2 \times e$, then $E_f^u(p) = E_A^u(x) \times 0$. If $p = (x, 0) \in \mathbb{T}^2 \times 0$, then $E_f^u(p) = E_{g_0}^u(x) \times 0$.

Consider an orbit $\{f^n(p)\}_{n \in \mathbb{Z}}$ where $p \in \mathbb{T}^2 \times (0, e)$. Up to shifting along the orbit, one may assume $p = (x, s)$ with $s \in [h(c), c]$. Define $V_p \subset T_p \mathbb{T}^3$ by $V_p = E_A^u(x) \times 0$ and let u be a non-zero vector in V_p. Write $p_n = (x_s, s_n) = f^n(p)$ for all $n \in \mathbb{Z}$. First, consider the backwards orbit of u. By Lemma 3.1, $u^{-m} \in E_A^u(x_{-m}) \times 0$ for all $m > 0$. For a subsequence $\{m_j\}$, if p_{-m_j} converges to a point $p_- = (x_-, e)$, then u^{-m_j} converges to a vector in $E_A^u(x_-) \times 0 = E_f^u(p_-)$.

Now consider the forward orbit of u. By Lemma 3.2, $u^n \in \mathscr{C}(x_n, s_n) \times 0$ for all $n > 0$. If a subsequence $\{p_{n_j}\}$ converges to a point $p_+ = (x_+, 0)$ and v^{n_j} converges to a vector $v_+ \in T_x \mathbb{T}^2 \times 0$, then $v_+ \in \mathscr{C}(p_+) \times 0$. In particular, v_+ does not lie in $E_A^s(x_+) \times 0$.

This shows that the conditions of Proposition 2.1 are satisfied and a dominated splitting exists on all of $\mathbb{T}^2 \times [0, e]$. By symmetry, a dominated splitting exists on $\mathbb{T}^2 \times [-e, 0]$. Since f is linear outside of $\mathbb{T}^2 \times [-e, e]$, there is a global dominated splitting on all of $\mathbb{T}^3$. $\quad\square$

For a non-zero vector $u \in T\mathbb{T}^3$ with horizontal component $v \in T\mathbb{T}^2$ and vertical component $w \in TS^1$, define the *slope* of u by

$$\text{slope}(u) = \frac{\|w\|}{\|v\|} \in [0, \infty].$$

Note that f maps a horizontal torus $\mathbb{T}^2 \times s$ to a horizontal torus $\mathbb{T}^2 \times h(s)$ and therefore $\text{slope}(u) = 0$ implies that slope $Df(u) = 0$.

Lemma 3.4. *If $p \in \mathbb{T}^2 \times [0, \frac{\epsilon}{2}]$ and $u \in T_p\mathbb{T}^3$ with* $\text{slope}(u) < \infty$*, then*

$$\text{slope } Df(u) < \tfrac{1}{2} \text{slope}(u).$$

Proof. This follows from the choice of λ at the start of the section. $\quad\square$

Lemma 3.5. *There is $k \geq 1$ and $\delta > 0$ such that if $p \in \mathbb{T}^2 \times [h^3(d), h^2(d)]$ and $u \in T_p\mathbb{T}^3$ with* $\text{slope}(u) < \delta$*, then $f^k(p) \in \mathbb{T}^2 \times [0, \frac{\epsilon}{2}]$ and slope $Df^k(u) < 1$.*

Proof. Since $h(s) < s$ for all $s \in (0, e)$, there is $k \geq 1$ so that $s < h^2(d)$ implies $h^k(s) < \epsilon/2$. Let K be the compact set of all unit vectors based at points in $\mathbb{T}^2 \times [h^3(d), h^2(d)]$, and let $K_0 \subset K$ be those vectors with slope zero. Define

$$\gamma : K \to [0, \infty], \quad v \mapsto \text{slope } Df^k(v).$$

Since $\gamma(K_0) = \{0\}$ and γ is uniformly continuous, one may find $\delta > 0$ as desired. $\quad\square$

Since $h^2(d) < c$, the choice of $z \in \mathbb{Z}^2$ does not affect the definition of f in the region $\mathbb{T}^2 \times [0, h^2(d)]$. Hence, the values k and δ may be determined before specifying z. The next lemma, however, does rely on this choice and the conditions on z are given in the lemma's proof.

Lemma 3.6. *For any $\delta > 0$, the $z \in \mathbb{Z}^2$ used in the definition of f may be chosen such that the following property holds:*
If $p = (x, s) \in \mathbb{T}^2 \times [h(d), d]$ and $u \in E_A^u(x) \times T_sS^1 \subset T_p\mathbb{T}^3$, then slope $Df^2(u) < \delta$.

Proof. Write $u = (v, w)$ as before. If $w = 0$, then slope $Df^2(u) = 0$. Therefore, one need only consider the case where w is non-zero. Up to rescaling the vector u, assume w is a unit vector pointing in the "up" direction of S^1. That is, pointing in the direction of increasing s. By calculating the derivative of

$$f^2(x, s) = \big(A^2(x) + \rho(s) \cdot A(z) + (\rho \circ h)(s) \cdot z, \; h^2(s)\big)$$

one can show that

$$Df^2(v, w) = \big(A^2(v) + \rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z, \; Dh^2(w)\big).$$

Define

$$\alpha := \min\big\{\rho'(s) + (\rho \circ h)'(s) : s \in [h(d), d]\big\}$$

and

$$\beta := \max\big\{(h^2)'(s) : s \in [h(d), d]\big\}$$

and note that $\alpha > 0$. For some $C > 1$, if z is given by Lemma 2.5, then

$$\|A^2(v) + \rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z\| \geq$$
$$\text{dist}\big(\rho'(s) \cdot A(z) + (\rho \circ h)'(s) \cdot z, \; E_A^u(0)\big) > C\alpha$$

and therefore slope $Df^2(u) < \beta/C\alpha$. Take C large enough that $\beta/C\alpha < \delta$. $\quad\square$

For the remainder of the proof, assume z was chosen so that Lemma 3.6 holds with $\delta > 0$ given by Lemma 3.5. The last three lemmas then combine to show the following.

Corollary 3.7. *If $p = (x, s) \in \mathbb{T}^2 \times [h(d), d]$ and $u \in E_A^u(x) \times 0 \subset T_p \mathbb{T}^3$, then*

$$\lim_{n \to +\infty} \text{slope } Df^n(u) = 0.$$

Lemma 3.8. *f has a dominated splitting of the form $E^{cu} \oplus_> E^s$ with $\dim E^{cu} = 2$.*

Proof. This proof follows the same general outline as the proof of Lemma 3.3. Let Y and Z be as in that proof. If $p = (x, e) \in \mathbb{T}^2 \times e$, then $E_f^{cu}(p) = E_A^u(x) \times T_e S^1$. If $p = (x, 0) \in \mathbb{T}^2 \times 0$, then $E_f^u(p) = T_x \mathbb{T}^2 \times 0$.

Now, consider an orbit $\{f^n(p)\}_{n \in \mathbb{Z}}$ where $p \in \mathbb{T}^2 \times (0, e)$. Up to shifting along the orbit, one may assume $p = (x, s)$ with $s \in [h(d), d]$. Define $V_p \subset T_p \mathbb{T}^3$ by $V_p = E_A^u(x) \times T_s S^1$ and let u be a non-zero vector in V_p. Write $p_n = (x_n, s_n) = f^n(p)$ for all $n \in \mathbb{Z}$. First, consider the backwards orbit of u. Note that

$$Df^n(V_p) = E_A^u(x_n) \times T_{s_n} S^1$$

for all $n < 0$. Hence, if $\{u^{-m_j}\}$ is a convergent subsequence, then p_{-m_j} converges to a point $p_- \in \mathbb{T}^2 \times e$, and u^{-m_j} converges to a vector in $E_f^{cu}(p_-)$. In the other direction, Corollary 3.7 implies that slope(u^n) tends to 0 as $n \to \infty$. If $\{u^{n_j}\}$ is a convergent subsequence, then p_{n_j} converges to a point $p_+ \in \mathbb{T}^2 \times 0$, and u_j^n converges to a vector in $E_f^{cu}(p_+)$. One may then use Proposition 2.1 to show that the dominated splitting extends to all of $\mathbb{T}^3$.   □

Now that the global invariant dominated splittings $E^u \oplus E^{cs}$ and $E^{cu} \oplus E^s$ are known to exist, Proposition 2.2 implies that f is strongly partially hyperbolic on all of $\mathbb{T}^3$.

4. Further constructions

Rodriguez Hertz, Rodriguez Hertz and Ures gave two different constructions of a system on the 3-torus with an invariant center-unstable 2-torus [17]. In the first of these constructions, the system is not dynamically coherent as there is no invariant foliation tangent to E^c. In the second of their constructions, the center bundle E^c is integrable, but not uniquely integrable. The construction we gave in section 3 corresponds to the first of these cases.

Proposition 4.1. *The construction of f given in section 3 is not dynamically coherent.*

Proof. The diffeomorphism f leaves the foliation of horizontal planes invariant. Therefore, if a vector u in the tangent bundle $T\mathbb{T}^3$ has a non-zero vertical component, then $Df(u)$ also has a non-zero vertical component. If $p \in \mathbb{T}^2 \times [h(d), e]$, and u is a unit vector in $E_f^c(p)$, then u has a non-zero vertical component. By iterating forward, one sees that the same property holds for any $p \in \mathbb{T}^2 \times (0, e]$. Hence, one may choose an orientation for the line bundle E_f^c on $\mathbb{T}^2 \times (0, e]$ so that the center direction always points in the direction of decreasing s. That is, the orientation always points towards $\mathbb{T}^2 \times 0$.

This choice extends continuously to $\mathbb{T}^2 \times [0, e]$. Further, by the symmetry of the construction, the center orientation may be extended to $\mathbb{T}^2 \times [-e, e]$, and on both sides, the center orientation points towards $\mathbb{T}^2 \times 0$. This means that any parameterized curve $\gamma : [0, +\infty) \to \mathbb{T}^3$ that starts in $\mathbb{T}^2 \times 0$, stays tangent to E^c, and agrees with the orientation of E^c, must remain for all time inside of $\mathbb{T}^2 \times 0$.

The constructed f is homotopic to A times the identity map on S^1. If f were dynamically coherent, then by the leaf conjugacy given in [11, Theorem 1.3], there would be a circle tangent to E_f^c though every point in $\mathbb{T}^3$. In particular, there would be an invariant foliation of center circles lying in $\mathbb{T}^2 \times 0$. As the dynamics g_0 on $\mathbb{T}^2 \times 0$ is homotopic to a hyperbolic toral automorphism, this is not possible and gives a contradiction.   □

We now look at ways in which the construction in the previous section may be modified. The definition of f may be stated piecewise as
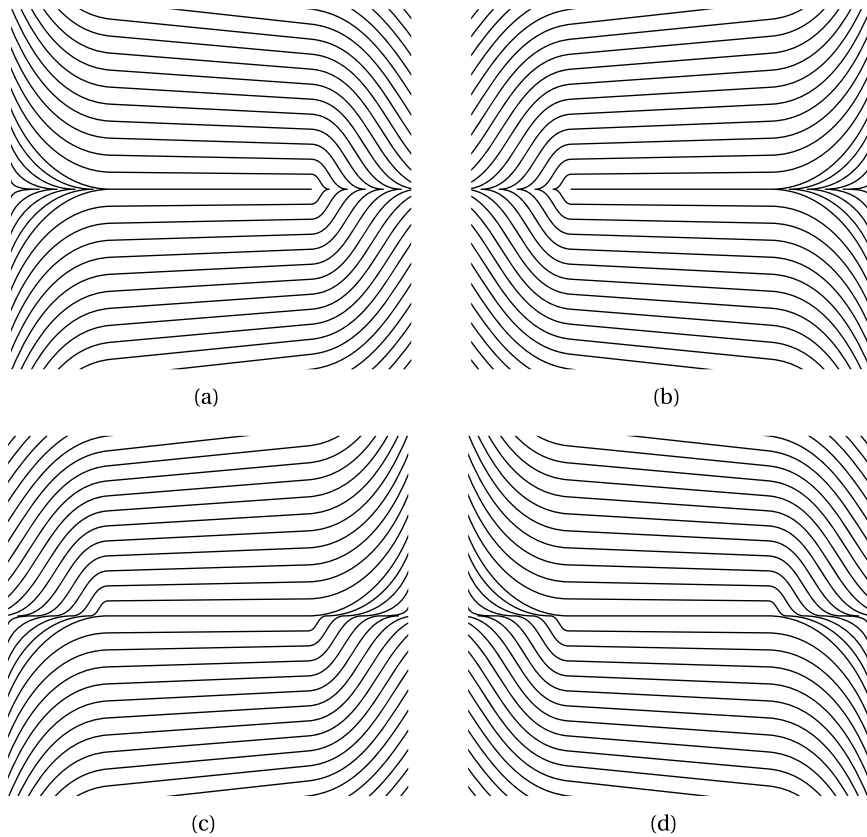
Fig. 2. Four possible ways in which the center bundle may behave near a center-unstable torus with derived-from-Anosov dynamics. Shown here are lines tangent to the E^c direction inside a cs-leaf. In each subfigure, the cs-leaf intersects the cu-torus in a horizontal line passing through the middle of the subfigure. In this example, the middle of this line intersects the basin of repulsion of a repelling fixed point inside the cu-torus so that there are no cusps here.

$$f(x,s) = \begin{cases} \big(g(x,s) + \rho(s) \cdot z,\ h(s)\big), & \text{if } s \in [0, \epsilon] \\ \big(g(x,-s) + \rho(-s) \cdot z,\ -h(-s)\big), & \text{if } s \in [-\epsilon, 0] \\ (Ax,\ s), & \text{if } s \notin [-\epsilon, \epsilon]. \end{cases}$$

Recall that $z \in \mathbb{Z}^2$ was chosen to satisfy the conclusions of Lemma 2.5. If k is any non-zero integer, then the product $k \cdot z \in \mathbb{Z}^2$ also satisfies those same conclusions. Thus, for any choice of non-zero integers k_1 and k_2, one may show that the function defined by

$$(x,s) \mapsto \begin{cases} \big(g(x,s) + k_1\rho(s) \cdot z,\ h(s)\big), & \text{if } s \in [0, \epsilon] \\ \big(g(x,-s) + k_2\rho(-s) \cdot z,\ -h(-s)\big), & \text{if } s \in [-\epsilon, 0] \\ (Ax,\ s), & \text{if } s \notin [-\epsilon, \epsilon] \end{cases}$$

is strongly partially hyperbolic with a cu-torus at $\mathbb{T}^2 \times 0$.

The choices of sign for k_1 and k_2 give four different ways to realize g_0 as the dynamics on an invariant cu-torus. These correspond to the two different ways the center bundle can approach a horizontal direction on either side of $\mathbb{T}^2 \times 0$ and are depicted in Fig. 2. The cases (a) and (b) in the figure are not dynamically coherent, as may be shown by the argument in the proof of Proposition 4.1. From the figure, it appears that the dynamics depicted in each of cases (c) and (d) has an invariant center foliation with leaves which topologically cross the torus. Rigorously proving the existence of this center foliation will require a sophisticated analysis of the Franks semiconjugacy of the system and its relation to the branching foliations of Brin, Burago and Ivanov. This work is left to a future paper.

The above modifications to the construction suggest a way to prove Theorem 1.1 in the case where g_0 reverses the orientation of E^c.

Proof of Theorem 1.1. Let g_0 be weakly partially hyperbolic with a splitting of the form $E^c \oplus E^u$. The case where g_0 preserves the orientation of E^c was already handled in section 3, so assume here that g_0 reverses the center orientation. Then g_0 is homotopic to a hyperbolic toral automorphism A which reverses the orientation of its stable bundle E_A^s. Analogously to Lemma 2.5, for any $C > 1$, there is $z \in \mathbb{Z}^2$ such that $\mathrm{dist}(\zeta \cdot A(z) - \xi \cdot z, \ E_A^u(0)) \geq C(\zeta + \xi)$ for all $\zeta, \xi \geq 0$. (Note now the minus sign before $\xi \cdot z$.)

Our constructed diffeomorphism on $\mathbb{T}^3$ will be the result of modifying the linear map $A \times (-\mathrm{id})$ defined on $\mathbb{T}^2 \times S^1$. Fix a small $\epsilon > 0$ and define $h : [0, \epsilon] \to [0, \epsilon]$ and $\rho : [0, \epsilon] \to [0, 1]$ with the properties as listed in section 3. Define f by

$$f(x, s) = \begin{cases} \big(g(x, s) + \rho(s) \cdot z, \ -h(s)\big), & \text{if } s \in [0, \epsilon] \\ \big(g(x, -s) - \rho(-s) \cdot z, \ h(-s)\big), & \text{if } s \in [-\epsilon, 0] \\ (Ax, \ -s), & \text{if } s \notin [-\epsilon, \epsilon]. \end{cases}$$

If $s \in [h(d), d]$, then

$$f^2(x, s) = (A^2(x) + \rho(s) \cdot A(z) - (\rho \circ h)(s) \cdot z, h^2(s)).$$

The above analogue of Lemma 2.5 then establishes an analogue of Lemma 3.6 in this context. The other parts of the proof in section 3 are also easily adapted and one may show that f is strongly partially hyperbolic. □

For simplicity, the previous section constructed a diffeomorphism on $\mathbb{T}^3$. It is a simple matter to apply the same techniques to a 3-manifold defined by the suspension of either an Anosov map or "minus the identity" on $\mathbb{T}^2$. The important condition in each case is that there is a strongly partially hyperbolic map and an invariant subset of the manifold homeomorphic to $\mathbb{T}^2 \times [-\epsilon, \epsilon]$ where the dynamics is given by $A \times \mathrm{id}$. As shown in [16], these are the only orientable 3-manifolds which allow a torus tangent to E^{cu} or E^{cs}.

As explored in [3, Section 4] and [12, Appendix A], it is possible to define partially hyperbolic diffeomorphisms on non-orientable manifolds which are double covered by the 3-torus. A similar construction works in the current setting to define one-sided center-stable and center-unstable tori.

Proposition 4.2. *For any weakly partially hyperbolic diffeomorphism $g_0 : \mathbb{T}^2 \to \mathbb{T}^2$ which preserves its center orientation, there is a non-orientable 3-manifold M, an embedding $i : \mathbb{T}^2 \to M$ and a strongly partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that the one-sided torus $i(\mathbb{T}^2)$ is tangent either to E_f^{cs} or E_f^{cu} and $i^{-1} \circ f \circ i = g_0$.*

Proof. Assume g_0 has a splitting of the form $E^u \oplus E^c$ and construct $f : \mathbb{T}^3 \to \mathbb{T}^3$ as in section 3. Assume $\mathbb{T}^3$ is defined as $\mathbb{R}^3/\mathbb{Z}^3$ and lift f to a map $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\tilde{f}(\mathbb{R}^2 \times 0) = \mathbb{R}^2 \times 0$. Construct a new closed 3-manifold by quotienting $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ by the group generated by the translations $(v, s) \mapsto (v, s + 1)$ and $(v, s) \mapsto (v + (0, 1), s)$ and the isometry $(v, s) \mapsto (v + (1, 0), -s)$. □

This concludes our construction of examples in dimension 3. The rest of paper handles constructions in higher dimension.

5. Compact center-stable manifolds of higher dimension

This section proves Theorem 1.4. In fact, we will prove the following restatement of the theorem which gives more technical details about the nature of the constructed diffeomorphism F.

Proposition 5.1. *Let $f_0 : M \to M$ be a diffeomorphism, let $X \subset M$ be a finite invariant set such that every $x \in X$ is either a periodic source or sink, and let U be a neighborhood of X. Then, there are a diffeomorphism $f : M \to M$, a toral automorphism $A : \mathbb{T}^D \to \mathbb{T}^D$, a smooth map $h : M \to \mathbb{T}^D$, and a diffeomorphism $F : M \times \mathbb{T}^D \to M \times \mathbb{T}^D$ defined by*

$$F(x, v) = (f(x), Av + h(x))$$

such that:

(1) *F is strongly partially hyperbolic;*
(2) *A is a linear Anosov diffeomorphism with eigenvalues $\lambda < 1$ and $\lambda^{-1} > 1$, each of multiplicity $d = \dim M$;*
(3) *$f(x) = f_0(x)$ and $h(x) = 0$ for all $x \in M \setminus U$;*
(4) *if $x \in NW(f) \setminus X$ and $v \in \mathbb{T}^D$, then*

$$E_F^s(x, v) = 0 \oplus E_A^s(v), \quad E_F^c(x, v) = T_x M \oplus 0, \quad and \quad E_F^u(x, v) = 0 \oplus E_A^u(v);$$

(5) *if $x \in X$ is a sink and $v \in \mathbb{T}^D$, then*

$$E_F^c(x, v) = 0 \oplus E_A^s(v) \quad and \quad E_F^u(x, v) = 0 \oplus E_A^u(v);$$

(6) *if $x \in X$ is a source and $v \in \mathbb{T}^D$, then*

$$E_F^s(x, v) = 0 \oplus E_A^s(v) \quad E_F^c(x, v) = 0 \oplus E_A^u(v).$$

Note that the notation and, in particular, the functions f, g, and h play very different roles here than in previous sections.

The basic idea of the construction is to replace the possibly non-linear behavior of f_0 in a neighborhood of a point $x \in X$ with a simple linear contraction or expansion. Then, both f and A are linear maps and there are exactly three rates of contraction or expansion given by f and the stable and unstable directions of A. This allows us to restrict our consideration to the case of a linear map

$$F(w, x, y) = (\lambda^{-1} w, bx, \lambda y)$$

defined on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ and where $0 < \lambda < b < 1$. We deform this map so that a d-dimensional subspace which lies roughly in the direction of $0 \times \mathbb{R}^d \times 0$ converges to the subspace $0 \times 0 \times \mathbb{R}^d$ under application of the derivative DF^n as $n \to +\infty$. This provides the effect of pushing the center direction into the stable direction of A.

The first step is to establish the following.

Lemma 5.2. *For $0 < \lambda < b < 1$ and $C > 1$, there is a diffeomorphism f of $\mathbb{R}^d$ and a smooth map $h : \mathbb{R}^d \to \mathbb{R}^d$ such that the diffeomorphism F of $\mathbb{R}^d \times \mathbb{R}^d$ defined by*

$$F(x, y) = (f(x), \lambda y + h(x))$$

has the following properties. If $p = (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $b \le \|x\| \le 1$, then

(1) *$f(x) = bx$ and $h(x) = 0$; and*
(2) *if $V \subset \mathbb{R}^d \times \mathbb{R}^d$ is the graph of a linear map $L : \mathbb{R}^d \to \mathbb{R}^d$ with $\|L\| < C$, then $DF_p^n(V)$ tends to $0 \times \mathbb{R}^d$ as n tends to $+\infty$.*

As an aid in proving Lemma 5.2, we first introduce a notion of the "quality" of a square matrix. This is closely related to the idea of a row diagonally dominated matrix, however we use different wording here in order to avoid potential confusion between different notions of domination.

Let A be a $d \times d$ matrix with entries a_{ij}. Define the *quality* of the matrix as

$$q(A) := \frac{\min\{a_{ii} : 1 \le i \le d\}}{\sum\{|a_{ij}| : 1 \le i, j \le d, \ i \ne j\}}.$$

To have positive quality, a matrix must have positive diagonal entries. We allow $q(A) = +\infty$ which occurs if and only if A is diagonal and positive definite.

Lemma 5.3. *If $q(A) > 2$, then A is invertible and the operator norm of the inverse satisfies*

$$\left\| A^{-1} \right\| \le \max \left\{ \frac{2d}{a_{ii}} : 1 \le i \le d \right\}.$$

Proof. This is a variation on the Gershgorin circle theorem. Suppose $v \in \mathbb{R}^d$ is non-zero and let i be an index such that $|v_i| \geq |v_j|$ for all j. Then,

$$\left| \sum_{j=1}^{d} a_{ij} v_j \right| \geq \left(a_{ii} - \sum_{j \neq i} |a_{ij}| \right) |v_i| \geq \tfrac{1}{2} a_{ii} |v_i|$$

which implies that $\|Av\| \geq \frac{1}{2d} a_{ii} \|v\|$. $\quad \square$

Lemma 5.4. *If A is a $d \times d$ matrix with $q(A) > 0$ and B is a positive definite diagonal matrix with entries b_{ij}, then*

$$q(AB) \geq q(A) \min \left\{ \frac{b_{ii}}{b_{jj}} : 1 \leq i, j \leq d \right\}.$$

Proof. Multiply A and B and check. $\quad \square$

Proof of Lemma 5.2. We prove Lemma 5.2 in the specific case where

$$\frac{b - \lambda}{b - 1} < \lambda.$$

Showing that the general case of $\lambda < b < 1$ may be proved from this special case is left to the reader. With this assumption added, there is a constant $0 < a < \lambda$ such that

$$\frac{b - a}{b - 1} < a.$$

Define a function $g_0 : [0, \infty) \to [a, b]$ such that

(1) $g_0(t) = a$ for $t \leq b$,
(2) $g_0(t) = b$ for $t \geq 1$, and
(3) $0 \leq t g_0'(t) < a$ for all $t \geq 0$.

Define a smooth bump function $\rho : [0, \infty) \to [0, 1]$ with $\rho(t) = 0$ for $t \geq b$, and $\rho(t) = 1$ for $t \leq b^2$. Define $h : \mathbb{R}^d \to \mathbb{R}^d$ by $h(x) = \rho(\|x\|)x$.

Before defining f, we first consider the behavior of $\hat{F}(x, y) := (bx, \lambda y + h(x))$ under iteration. Let $p = (x, y)$, V, and L be as in item (2) of the statement of the lemma being proved. In particular, $b \leq \|x\| \leq 1$. For $n \geq 0$, define $\hat{V}_n := D\hat{F}_p^n(V)$ and let $\hat{L}_n : \mathbb{R}^d \to \mathbb{R}^d$ be the linear map such that $\mathrm{graph}(\hat{L}_n) = \hat{V}_n$. The definition of $\hat{F}$ implies that

$$\hat{L}_{n+1} = \tfrac{\lambda}{b} \hat{L}_n + \tfrac{1}{b} Dh$$

where the derivative Dh is evaluated at $b^n x$. If $n > 2$, then Dh is the identity map, I, and

$$\hat{L}_{n+1} = \tfrac{\lambda}{b} \hat{L}_n + \tfrac{1}{b} I.$$

It follows that $\hat{L}_n$ converges exponentially fast to $(b - \lambda)^{-1} I$. When viewed as a matrix, $(b - \lambda)^{-1} I$ is diagonal and positive definite and so its "quality," as defined above, is $q((b - \lambda)^{-1} I) = +\infty$. Therefore, there is $N > 2$ such that $q(\hat{L}_n) > 4$ for all $n \geq N$. By compactness, one may find a uniform value of N such that this lower bound on $q(\hat{L}_n)$ holds for any starting $p = (x, y)$, V, and L with $\|L\| < C$.

With N now fixed, define $g : [0, \infty) \to [a, b]$ by $g(t) := g_0(b^{-N} t)$ and observe that

(1) $g(t) = a$ for $t \leq b^{N+1}$,
(2) $g(t) = b$ for $t \geq b^N$, and
(3) $0 \leq t g'(t) < a$ for all $t \geq 0$.

Define f by $f(x) = g(\|x\|)x$. With f and h now defined, we show that $F(x, y) = (f(x), \lambda y + h(x))$ satisfies the conclusions of the lemma.

This definition of F has a form of radial symmetry: if R is a rigid rotation about the origin in $\mathbb{R}^d$, then $f \circ R = R \circ f$, $h \circ R = R \circ h$, and $F \circ (R \times R) = (R \times R) \circ F$. Further, any one-dimensional subspace in $\mathbb{R}^d$ is invariant under f. Because of this symmetry, when analyzing orbits of F, we need only consider points of the form $p = (x, y)$ where $x \in \mathbb{R} \times 0$. That is, if x is written in coordinates as $x = (x_1, x_2, \ldots, x_d)$, then $x_2 = x_3 = \cdots = x_d = 0$.

The partial derivatives of $f : \mathbb{R}^d \to \mathbb{R}^d$ are given by

$$\frac{\partial f_i}{\partial x_j} = g(\|x\|)\delta_{ij} + \frac{x_i x_j}{\|x\|} g'(\|x\|).$$

Since we are assuming $x \in \mathbb{R} \times 0$, the terms $x_i x_j$ all evaluate to 0 except for the term $x_1 x_1$. Therefore

$$\frac{\partial f_1}{\partial x_1} = g(\|x\|) + \|x\| g'(\|x\|)$$

$$\frac{\partial f_i}{\partial x_i} = g(\|x\|) \qquad\qquad \text{if } i > 1, \text{ and}$$

$$\frac{\partial f_i}{\partial x_j} = 0 \qquad\qquad \text{if } i \neq j.$$

Further $g'(\|x\|)$ is non-zero only when $b^{N+1} < \|x\| < b^N$ and one may show that

$$g(\|x\|) \leq g(\|x\|) + \|x\| g'(\|x\|) \leq 2g(\|x\|).$$

In other words, the Jacobian of f is a diagonal matrix where no entry is more than twice as large as any other.

Let $p = (x, y)$ with $x \in \mathbb{R} \times 0$ and $b \leq \|x\| \leq 1$. Let V and L be as in item (2) of the statement of the lemma. For $n \geq 0$, define $V_n := DF_p^n(V)$ and $L_n : \mathbb{R}^d \to \mathbb{R}^d$ such that $\operatorname{graph}(L_n) = V_n$. We now analyze L_n as n tends to $+\infty$. First, if $n < N$, then $\|f^n(x)\| \geq b^N$ and the functions F^n and $\hat{F}^n$ are equal in a neighborhood of p. Therefore $L_N = \hat{L}_N$ and in particular $q(L_N) > 4$.

For the case $n = N$, the equality $\operatorname{graph}(L_{N+1}) = DF(\operatorname{graph}(L_N))$ may be written as

$$\left\{ (u, L_{N+1}(u)) : u \in \mathbb{R}^d \right\} = \left\{ (Df(v), \lambda L_N(v) + v) : v \in \mathbb{R}^d \right\}$$

showing that $L_{N+1} = (\lambda L_N + I) \circ Df^{-1}$ where Df is evaluated at $f^N(x)$. Lemma 5.4, along with the above remark about the Jacobian of f, shows that

$$q\left((\lambda L_N + I) \circ Df^{-1} \right) \geq \tfrac{1}{2} q\left(\lambda L_N + I \right)$$

and this implies that $q(L_{N+1}) \geq \tfrac{1}{2} q(L_N) > 2$.

Finally, for $n > N$, the point $f^n(x)$ satisfies $\|f^n(x)\| \leq b^{N+1}$. For points in this region, $Df = aI$ and so $L_{N+1} = \frac{\lambda}{a} L_n + \frac{1}{a} I$ which implies that $q(L_{n+1}) > q(L_n) > 2$ for all large n. Since $\frac{\lambda}{a} > 1$, the linear map L_n when viewed as a matrix has positive entries on its diagonal and these entries tend to $+\infty$ as n tends to $+\infty$. Lemma 5.3 implies that $\|L_n^{-1}\|$ tends to zero as $n \to +\infty$ and therefore the sequence of subspaces V_n tends to $0 \times \mathbb{R}^d$. $\quad\square$

The next result simply adds an expanding direction to Lemma 5.2.

Corollary 5.5. *For $0 < \lambda < b < 1$ and $C > 1$, there is a diffeomorphism f of $\mathbb{R}^d$ and a smooth map $h : \mathbb{R}^d \to \mathbb{R}^d$ such that the diffeomorphism F of $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ defined by*

$$F(w, x, y) = (\lambda^{-1} w, f(x), \lambda y + h(x))$$

has the following properties. If $p = (w, x, y) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ with $b \leq \|x\| \leq 1$, then

(1) *$f(x) = bx$ and $h(x) = 0$;*
(2) *if $V \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ is the graph of a linear map $L : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ with $\|L\| < C$, then $DF_p^n(V)$ tends to $\mathbb{R}^d \times 0 \times 0$ as n tends to $+\infty$; and*
(3) *if $V \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ is the graph of a linear map $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with $\|L\| < C$, then $DF_p^n(V)$ tends to $\mathbb{R}^d \times 0 \times \mathbb{R}^d$ as n tends to $+\infty$.*

Proof. Use the same f and h as in Lemma 5.2. □

With this established, we now consider diffeomorphisms defined on closed manifolds. For a closed manifold M and a hyperbolic toral automorphism $A : \mathbb{T}^D \to \mathbb{T}^D$, an *A-map* is a map $F : M \times \mathbb{T}^D \to M \times \mathbb{T}^D$ of the form

$$F(x, v) = (f(x), Av + h(x)).$$

See [9] for a more general definition and further details. If F is also a (strongly) partially hyperbolic diffeomorphism, we call it a *partially hyperbolic A-map*. Note that we do not a priori assume that the partially hyperbolic splitting has any relation to the fibers of the torus bundle.

There is a small subtlety in proving Proposition 5.1 in the case where the basin of a sink overlaps the basin of a source. To handle this, we will prove Proposition 5.1 by induction and keep track of a property we call being "graph like" for the splitting at a point.

For a partially hyperbolic A-map and a point $x \in M$, the subbundle E^u is *graph like* at x if, for all $v \in \mathbb{T}^D$, $E^u(x, v)$ is the graph of a linear function from $E^u_A(v)$ to $E^s_A(v) \oplus T_x M$. Similarly, E^{cu}, E^{cs}, and E^s are graph like at x if they are graphs of linear functions

$$T_x M \oplus E^u_A(v) \to E^s_A(v), \quad T_x M \oplus E^s_A(v) \to E^u_A(v), \quad \text{and} \quad E^s_A(v) \to E^u_A(v) \oplus T_x M$$

respectively. If all of E^u, E^{cs}, E^{cu}, and E^s are graph like at x, we say the splitting is graph like at x.

Since the bundles in the splitting are continuous and DF-invariant the following is easily verified.

Lemma 5.6. *Let F be a partially hyperbolic A-map with base map $f : M \to M$. For a bundle $E \in \{E^u, E^{cu}, E^{cs}, E^s\}$, the set of graph-like points is open and f-invariant.*

Next, we consider a normally attracting fiber.

Lemma 5.7. *For a partially hyperbolic A-map F with base map $f : M \to M$, if $x \in M$ is a periodic sink for f and $x \times \mathbb{T}^D$ is tangent to E^{cu}, then E^s and E^{cs} are graph like for every point in the basin of x.*

Proof. Since E^s_F is transverse to $x \times \mathbb{T}^D$, it is graph like at x. By the uniqueness of the dominated splitting on $x \times \mathbb{T}^D$, $E^c_F(x, v) = E^s_A(v)$ for all $v \in \mathbb{T}^D$. Therefore, E^{cs}_F is also graph like at x. By the previous lemma, being graph like at x extends to being graph like on the basin of x. □

The next lemma allows us to replace non-linear sinks with linear ones.

Lemma 5.8. *Let $f_0 : M \to M$ be a diffeomorphism with a periodic sink $x_0 = f_0^k(x_0)$ and let $\epsilon > 0$ and $0 < b < 1$. Then there is a diffeomorphism $f : M \to M$ and a coordinate chart $\varphi : [-1, 1]^d \to M$ such that*

(1) *if $dist(x, x_0) > \epsilon$, then $f(x) = f_0(x)$,*
(2) *f and f_0 have the same non-wandering set,*
(3) *$\varphi(0) = x_0$, and*
(4) *$\varphi^{-1} \circ f^k \circ \varphi(y) = by$ for all $y \in [-1, 1]^d$.*

Proof. This follows from standard methods of pasting diffeomorphisms [19]. First, one may make a C^1 small perturbation in order to assume that $\varphi^{-1} \circ f^k \circ \varphi$ is linear in a neighborhood of 0. Then, deform the linear map inside that neighborhood to get the desired homothety. □

Now we state what will be the inductive step in proving Proposition 5.1.

Proposition 5.9. *Let A be a hyperbolic toral automorphism of $\mathbb{T}^D$ with eigenvalues $\lambda < 1$ and $\lambda^{-1} > 1$, each of multiplicity $d = \frac{1}{2} D$. Suppose F_0 is a partially hyperbolic A-map having a base map $f_0 : M \to M$ with $\dim M = d$ and x_0 is a periodic sink such that the splitting is graph like at x_0. For any $\epsilon > 0$, there is a partially hyperbolic A-map F such that*

(1) *if $dist(x, x_0) > \epsilon$, then $F(x, v) = F_0(x, v)$ for all $v \in \mathbb{T}^D$;*

(2) *if the splitting for F_0 is graph like at a point x outside the orbit of x_0, then the splitting for F is also graph like at x; and*

(3) *$x_0 \times \mathbb{T}^D$ is an F-periodic submanifold tangent to E_F^{cu}.*

Proof. This proof breaks into two steps. First, we deform F_0 to produce a partially hyperbolic map F_1 which is linear in a neighborhood of $x_0 \times \mathbb{T}^D$, but which still has a graph-like splitting at x_0. Then, we paste in the dynamics given by Corollary 5.5, to produce a partially hyperbolic map F for which E_F^{cu} is tangent to $x_0 \times \mathbb{T}^D$.

Let U be a neighborhood of the orbit of x_0 such that $\overline{U}$ is contained in the basin of attraction and $f_0(\overline{U}) \subset U$. Define a smooth function $h_1 : M \to \mathbb{T}^D$ such that $h_1(x) = h_0(x)$ for all $x \in M \setminus U$ and $h_1(x) = 0$ for all $x \in f(U)$.

Fix b such that $\lambda < b < 1$ where λ is the stable eigenvalue of A. Let k denote the period of x_0. By Lemma 5.8, there is a coordinate chart $\varphi : [-1, 1]^d \to M$ and a diffeomorphism $f_1 : M \to M$ such that $\varphi^{-1} \circ f_1^k \circ \varphi(x) = bx$ for all $x \in [-1, 1]^d$. Moreover, we may freely assume that $\varphi([-1, 1]^d) \subset f_0(U)$ and that $f_1(x) = f_0(x)$ for all $x \in M \setminus f_0(U)$. By abuse of notation, we identify $[-1, 1]^d$ with its image and regard $[-1, 1]^d$ as a subset of M.

Define a diffeomorphism F_1 of $M \times \mathbb{T}^D$ by $F_1(x, v) = (f_1(x, v), \ Av + h_1(x))$. If $x \in \overline{U} \setminus f_1(U)$ and $v \in \mathbb{T}^D$, define $E_{F_1}^u(x, v) := E_{F_0}^u(x, v)$. Using Proposition 2.1, one may then establish the existence of a dominated splitting $E_{F_1}^u \oplus E_{F_1}^{cs}$ on all of $M \times \mathbb{T}^D$. Similarly, If $x \in \overline{U} \setminus f_1(U)$ and $v \in \mathbb{T}^D$, define $E_{F_1}^{cu}(x, v) := E_{F_0}^{cu}(x, v)$ and apply the same reasoning to establish a dominated splitting of the form $E_{F_1}^{cu} \oplus E_{F_1}^s$ on all of $M \times \mathbb{T}^D$. From this, one may show that F_1 is partially hyperbolic and that the splitting of F_1 is graph like at a point x if and only if the original F_0 was graph like at x.

Since $E_{F_1}^u$ is continuous and graph like on U, there is a uniform constant $C > 1$ such that if $x \in [-1, 1]^d \subset M$ with $b \le \|x\| \le 1$ and $v \in \mathbb{T}^D$ then $E_{F_1}^u(x, v)$ is the graph of a linear function $L : E_A^u(x, v) \to T_x M \oplus E_A^s(x, v)$ with $\|L\| < C$. A similar bound also holds when $E_{F_1}^{cu}(x, v)$ is expressed as the graph of a linear function. By Corollary 5.5, there are functions $f : M \to M$ and $h : M \to \mathbb{T}^D$ such that F defined by $F(x, v) = (f(x), \ Av + h(x))$ satisfies the following properties.

(1) If either $x \in M \setminus [-1, 1]^d$ or $x \in [-1, 1]^d$ with $\|x\| > 1$, then $f(x) = f_1(x)$ and $h(x) = h_1(x)$.

(2) If $x \in [-1, 1]^d$ with $b \le \|x\| \le 1$, then $f(x) = bx$. Further, if $\{n_j\} \subset \mathbb{N}$ is such that $F^{n_j}(x, v)$ converges to a point (x_0, v_0) in $x_0 \times \mathbb{T}^D$, then $DF^{n_j}(E_{F_1}^u(x, v))$ converges to $0 \times E_A^u(v_0)$ and $DF^{n_j}(E_{F_1}^{cu}(x, v))$ converges to $0 \times T_{v_0}\mathbb{T}^D$.

Then the results in section 2 show that F is partially hyperbolic with $x_0 \times \mathbb{T}^D$ tangent to E_F^{cu}. $\quad\square$

One can ask if the new map F created in Proposition 5.9 is dynamically coherent. That is, even though the center bundle is not uniquely integrable, is there an invariant foliation tangent to the bundle? In the special case that the base manifold $M = S^1$ is one-dimensional, the $h(x)$ term in the construction shears the dynamics in opposite directions on the two sides of the sink. The resulting map F on $S^1 \times \mathbb{T}^2 = \mathbb{T}^3$ therefore corresponds to the dynamically coherent example appearing in subsection 2.3 of [17]. Based on this, the author suspects that the map F in Proposition 5.9 will be dynamically coherent exactly when the original map F_0 was dynamically coherent. This question, however, is left as a topic for future research.

With Proposition 5.9 established, Proposition 5.1 easily follows.

Proof of Proposition 5.1. Given f_0, define a hyperbolic toral automorphism $A : \mathbb{T}^D \to \mathbb{T}^D$ such that $F_0 := f_0 \times A$ is partially hyperbolic. For instance, A can be the direct product of d copies of a high iterate of the cat map. Clearly, F_0 is a partially hyperbolic A-map and the splitting is graph like at all points. Let x_0 be any point in X and apply Proposition 5.9 to F_0 and x_0 to produce a map F_1 where $x_0 \times \mathbb{T}^D$ is tangent either to E^{cs} and E^{cu}. If X contains a point x_1 which is not in the orbit of x_0, then apply Proposition 5.9 to F_1 and x_1 to produce a map F_2. After a finite number of steps of this form, the desired map F in Proposition 5.1 is constructed. $\quad\square$

Conflict of interest statement

Author declares no conflicts of interest.

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