Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 1173–1188

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Isospectral periodic Torii in dimension 2

Alden Waters

CNRS Ecole Normale Superieure, 45 Rue d'Ulm, Paris 75005, France

Received 21 January 2014; accepted 3 June 2014

Available online 28 July 2014

Abstract

We consider two dimensional real-valued analytic potentials for the Schrödinger equation which are periodic over a lattice $\mathbb{L}$. Under certain assumptions on the form of the potential and the lattice $\mathbb{L}$, we can show there is a large class of analytic potentials which are Floquet rigid and dense in the set of $C^\infty(\mathbb{R}^2/\mathbb{L})$ potentials. The result extends the work of Eskin et al., in "On isospectral periodic potentials in $\mathbb{R}^n$, II."

Résumé

Nous considérons les potentiels analytiques à valeurs réelles en dimension deux pour l'equation de Schrödinger qui sont périodiques sur un réseau $\mathbb{L}$. Sous certaines hypothèses sur la forme du potentiel et du réseau $\mathbb{L}$, nous montrons qu' il y a une grande classe de potentiels analytiques Floquet rigides et denses dans l'ensemble de $C^\infty(\mathbb{R}^2/\mathbb{L})$ potentiels. Ce résultat prolonge le travail de Eskin et al., dans "Les potentiels périodiques isospectraux dans $\mathbb{R}^n$, II."

MSC: 35J10; 35P05; 65M32

Keywords: Inverse problems; Spectral theory; Schrödinger equations

1. Introduction

The subject of multi-dimensional inverse spectral theory has seen a small amount of growth in the past few decades after the work of Eskin et al., in [3] and [4] in the context of Floquet rigidity. The reason for this is that it is difficult to calculate exactly the structure of spectral invariants for multi-dimensional periodic Schrödinger operators. The authors of [3] and [4] essentially are only able to consider perturbations of the zero potential in their work. The goal of this paper is to show that a larger class of analytic periodic potentials can be considered by use of the abelian functionals. Its and Matveev [1] have shown that the abelian functionals categorize all finite gap potentials.

The focus of this paper is the class of Schrödinger operators

$$P : u(x) \mapsto \big(-\Delta + q(x)\big)u(x),$$

where

http://dx.doi.org/10.1016/j.anihpc.2014.06.001

$$\Delta = \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2},$$

and

$$q(x) : \mathbb{R}^2 \to \mathbb{R}$$

is a real-valued periodic potential over a lattice, $\mathbb{L} \subset \mathbb{R}^2$. In other words we have

$$q(x+d) = q(x) \quad \forall d \in \mathbb{L}.$$

We will study the question of spectral rigidity for the operator P and derive results which could extend to $\mathbb{R}^n$ for $n \geq 3$. We consider the set of λ in $\mathbb{R}$ for which the self-adjoint eigenvalue problem

$$Pu(x) = \lambda u(x) \qquad u(x+d) = \exp(2\pi i k \cdot d) u(x) \tag{1.1}$$

has a solution for k in $\mathbb{R}^2$ and d in $\mathbb{L}$. When there is a nonzero solution to (1.1) we say that λ is in $\mathrm{Spec}_k(-\Delta + q)$. We refer to

$$\bigcup_{k \in \mathbb{R}} \mathrm{Spec}_k(-\Delta + q)$$

as the Floquet spectrum. However, when $k = 0$, we simply say 'spectrum' which we denote by $\mathrm{Spec}(-\Delta + q)$. Two potentials q and $\tilde{q}$ are Floquet isospectral if

$$\mathrm{Spec}_k(-\Delta + q) = \mathrm{Spec}_k(-\Delta + \tilde{q}) \quad \forall k \in \mathbb{R}^2$$

and isospectral if $\mathrm{Spec}(-\Delta + q) = \mathrm{Spec}(-\Delta + \tilde{q})$. Following the convention in [4], we consider a potential to be Floquet (spectrally) rigid if there are only a finite number of potentials modulo translations which are Floquet isospectral (resp. isospectral) to it.

In [3], Eskin et al. showed that under the assumptions

1. q is real analytic
2. $\mathbb{L}$ has the property $|d| = |d'| \Rightarrow d = \pm d'$ for all d, d' in $\mathbb{L}$

then $\mathrm{Spec}(-\Delta + q)$ determines $\mathrm{Spec}_k(-\Delta + q)$ for all k in $\mathbb{R}^n$.

It is important to note that we are considering only lattices which satisfy a type of non-orthogonality condition. The results in [3] and [4] for lattices of the form $\mathbb{Z} \times \mathbb{Z}$ were examined by Gordon and Kappeler in [6] and [7]. When the lattice satisfies the type of non-orthogonality condition, the analysis is a bit different. We only consider potentials which break down into a finite number of one dimensional finite gap potentials. It was the author's original goal to derive spectral rigidity results when the decomposition into one dimensional potentials contained a one dimensional potential with infinitely many gaps. The analysis here implies it would be difficult to derive spectral rigidity for such a class of potentials with the current machinery available. We use the invariants coming from spectral asymptotics of the heat trace in any dimension. We review the one dimensional spectral theory first. The standard references for the one dimensional theory are given by [12] and [14]. For a more modern reference reviewing the notation we refer the reader to Kappeler [9]. Koroteyv has also proved stronger characterizations of the one dimensional potentials in terms of the gap lengths of the spectra in [11], and [10], than the ones presented here. It would be interesting if explicitly calculatable invariants two dimensional operators which did not involve decomposition to one dimensional operators existed.

In the sequel to [3,4], Eskin, et al., show that there is a set of analytic potentials satisfying the conditions (1) and (2) which are dense in $C^\infty(\mathbb{R}^2/\mathbb{L})$ such that if $q(x)$ is in this set, then $q(x)$ is Floquet rigid. Furthermore, there is a smaller, but still dense set of analytic potentials in $C^\infty(\mathbb{R}^2/\mathbb{L})$ such that if $q(x)$ is in this set and $\tilde{q}(x)$ is Floquet isospectral to $q(x)$ then, $\tilde{q}(x) = q(\pm x + a)$ where a is an arbitrary constant. Under the assumptions (1) and (2), if a potential in $\mathbb{R}^2$ is spectrally rigid (resp. unique) then it is Floquet rigid (resp unique), so their results are also true with the words "Floquet rigid" (resp. unique) replacing "isospectrally rigid" (resp unique). The main result of this paper is to show that there is a more general class of potentials which satisfy the conditions for Floquet rigidity than in [4].

2. The isospectral manifold in $\mathbb{R}^1$

In $\mathbb{R}^1$ the structure of the isospectral sets of periodic potentials has been well studied and contains many results which are useful in higher dimensions. In $\mathbb{R}^1$ the Schrödinger operator becomes Hill's operator.

$$-\frac{d^2}{ds^2}+q(s)$$

where $q(s)$ has period 1 and is real-valued. We start by assuming that q is at least three times differentiable, so that we can use many of the standard results which may be found in Magnus and Winkler [12]. For the rest of this paper, we will also assume that $q(x)$ has mean zero. We look at the set of λ where there is a solution to

$$\begin{aligned} &-\frac{d^2\phi(s)}{ds^2}+q(s)\phi(s)=\lambda\phi(s)\\ &\phi(s+1)=(-1)^m\phi(s). \end{aligned} \tag{2.1}$$

The scalars λ are known as the periodic and anti-periodic eigenvalues. Through curious use of notation, the scalar, $\lambda_m^{\pm}$, denotes the eigenvalue corresponding to the eigenfunction $\phi_m^{\pm}(s+1)=(-1)^m\phi_m^{\pm}(s)$ so that

$$\lambda_0<\lambda_1^-\leq\lambda_1^+<\lambda_2^-\leq\lambda_2^+... \tag{2.2}$$

Hence the periodic spectrum consists of $\{\lambda_m^{\pm}, m \text{ even}\}$ and the antiperiodic spectrum is $\{\lambda_m^{\pm}, m \text{ odd}\}$.

If we change the problem (2.1) so that $\phi(s)$ obeys the boundary condition

$$\phi(0)=\phi(1)=0,$$

then the associated spectrum is called the Dirichlet spectrum. The Dirichlet spectrum are denoted $\mu_m(q)$ and they interlace the periodic and anti-periodic spectra. We will often use the fact

$$\left|\lambda_m^+-\lambda_n^+\right|=\mathcal{O}\left(\left|m^2-n^2\right|\right), \tag{2.3}$$

and find it worthwhile to mention it here. Although $\lambda_m^+<\lambda_{m+1}^-$, it is possible to have $\lambda_m^-=\lambda_m^+$. The spectrum of

$$-\frac{d^2}{ds^2}+q(s)$$

as an operator in $L^2(\mathbb{R})$ is

$$\bigcup_{m=0}^{\infty}\left[\lambda_m^+,\lambda_{m+1}^-\right]$$

Each of the intervals $[\lambda_m^+,\lambda_{m+1}^-]$ in the union above is called a "band", or interval of stability. The complement of the set of bands is union of the intervals $(\lambda_m^-,\lambda_m^+)$ which are called "gaps" or intervals of stability. In each gap, the operator $-\frac{d^2}{ds^2}+q(s)$ does not have a bounded eigenfunction. A gap is referred to as open whenever $\lambda_m^-<\lambda_m^+$ and closed if $\lambda_m^-=\lambda_m^+$. The length of a gap is denoted as γ_m.

In [5] Garnett and Trubowitz gave a compete characterization of the gaps for q in $L^2_{\mathbb{R}}[0,1]$.

Theorem 1. *(See [5].) Let γ_n, $n\geq 1$, be any sequence of nonnegative numbers satisfying*

$$\sum_{n\geq 1}\gamma_n^2<\infty$$

Then there is a way of placing the sequence of open tiles of lengths γ_n, $n\geq 1$ in order on the positive axis $(0,\infty)$ so that the complement is the set of bands for a function q in $L^2_{\mathbb{R}}[0,1]$. In other words, the map

$$q\to\gamma(q)=\left\{\gamma_n(q)\right\}_{n\geq 1}, \tag{2.4}$$

from $L^2_{\mathbb{R}}[0,1]$ to $(l^2)^+$, is onto.

Furthermore if we multiply the gap lengths γ_m by ϵ where ϵ is in $[0, 1]$ then the map (2.4) is still onto. The fundamental result in $\mathbb{R}^1$ is that the set of analytic periodic potentials $M(\epsilon)$ with the same periodic and anti-periodic spectra is equivalent to a torus with dimension equal to I [13]. Here I is the number of m for which $\lambda_m^- < \lambda_m^+$. The coordinates $\alpha_m(q)$, on this manifold with m referring to the mth gap on $q(s)$, are related to the Dirichlet spectra and the gap lengths. They are defined as follows

$$\sin^2 \alpha_m(q) = \frac{\mu_m(q) - \lambda_m^-}{\lambda_m^+ - \lambda_m^-} \quad -\frac{\pi}{2} < \alpha_m \leq \frac{\pi}{2} \tag{2.5}$$

where $\mu_m(q)$ is the Dirichlet eigenvalue for q such that $\lambda_m^- \leq \mu_m(q) \leq \lambda_m^+$. These coordinates are further discussed in Section 4.

Finally we will need the fact that all the gap lengths are exponentially decreasing if and only if $q(s)$ is real analytic. Whenever q has only a finite number of open gaps, then q must be real analytic, [16]. The analyticity of $q(s)$ with finitely many gaps is crucial in many of the proofs of the theorems in this paper.

3. Review of necessary results in $\mathbb{R}^n$

We outline some necessary results and definitions from [3] and [4] which will be used in the rest of this paper. Let $\mathbb{L}$ be an n-dimensional lattice generated by n vectors $v_1, v_2, ..., v_n$. We can then consider it's dual $\mathbb{L}^*$ where

$$\mathbb{L}^* = \left\{\delta \in \mathbb{R}^n : \delta \cdot v \in \mathbb{Z}, \forall v \in \mathbb{L}\right\},$$

to be generated by some basis $\delta_1, \delta_2, ..., \delta_n$. A function is periodic over the lattice $\mathbb{L}$ if $q(x + d) = q(x)$ for all d in $\mathbb{L}$. For any arbitrary lattice $\mathbb{L}$ satisfying condition (2) and basis fixed as above, let $\mathbb{S}^*$ be the set of fundamental directions for $\mathbb{L}$, that is

$$\mathbb{S}^* = \left\{\delta \in \mathbb{L}^* : \delta \cdot d = 1 \text{ for some } d \in \mathbb{L}\right\}.$$

It is clear that whenever δ is in $\mathbb{S}^*$ then $-\delta$ is also in this set, so we reduce the set to $\mathbb{S}$ by only picking δ in $\mathbb{S}^*$. Therefore any element of $\mathbb{L}^*/\{0\}$ has a unique representation as $m\delta$ with δ in $\mathbb{S}$ and m in $\mathbb{Z}$.

If q is a function which is periodic over $\mathbb{L}$, then it has the following Fourier series representation

$$q(x) = \sum_{\delta \in \mathbb{L}^*} a_\delta \exp(2\pi i \delta \cdot x)$$

with

$$a_\delta = \frac{1}{\text{Vol}(\Gamma)} \int_\Gamma q(x) \exp(-2\pi i \delta \cdot x)\, dx$$

where Γ the fundamental domain of the lattice $\mathbb{L}$ as given by

$$\Gamma = \{s_1 v_1 + ... + s_n v_n : 0 \leq s_i \leq 1\}.$$

If we write

$$|\delta|^2 q_\delta(s) = \sum_{n \in \mathbb{Z}} a_{n\delta} \exp(2\pi i n s)$$

then we have that

$$q(x) = \sum_{\delta \in \mathbb{S}} \sum_{n \in \mathbb{Z}} a_{n\delta} \exp(2\pi i n \delta \cdot x) = \sum_{\delta \in \mathbb{S}} |\delta|^2 q_\delta(\delta \cdot x)$$

where each $q_\delta(s)$ is a periodic potential on $\mathbb{R}^1$. These one-dimensional potentials $q_\delta(s)'s$ are called directional potentials. The assumption that $q(x)$ has mean zero is equivalent to setting $a_0 = 0$ for all the directional potentials.

Theorem 2 in [3,4] states that

Theorem 2. $\mathrm{Spec}(-\Delta + q)$ *determines*

$$\mathrm{Spec}_k\left(-\frac{d^2}{ds^2} + q_\delta(s)\right) \quad \forall \delta \in \mathbb{S},\ k \in \mathbb{R}$$

The theorems in $\mathbb{R}^1$ we mentioned will help reduce the study of periodic potentials in $\mathbb{R}^n$ to the study of $\mathbb{R}^1$ potentials, about which much more is known.

4. Potentials in $\mathbb{R}^2$

Following [4], for the rest of this paper we assume that the elements of the lattice $\mathbb{L}$ satisfy condition (2) as stated in the introduction, and we consider analytic periodic potentials $q(x)$ such that $q(x+d) = q(x)$ for all d in $\mathbb{L}$. We also only consider potentials with a finite number of directional potentials. For this section, we make the additional assumptions that the number of gaps in each direction δ_j is finite, and that there are at least 3 directions. This setup differs from [4] where two of the directional potentials were fixed translates of the one gap potentials and the other directions were viewed as perturbations of the zero potential.

Under these assumptions we can simplify the form of $q(x)$ as follows

$$q(x) = \sum_{j=1}^{S} |\delta_j|^2 q_j(\delta_j \cdot x). \tag{4.1}$$

Each one dimensional directional potential $q_j(\delta_j \cdot x)$ corresponds to a one dimensional operator with corresponding eigenvalue and eigenfunction pair $(\lambda, \phi(s))$ satisfying

$$-\frac{d^2}{ds^2}\phi(s) + q_j(s)\phi(s) = \lambda\phi(s). \tag{4.2}$$

In order to simplify the computations needed in this paper we make the following assumptions $(*)$

1. $\delta_3 = \delta_1 + \delta_2$
2. q_1, q_2 and q_3 have the same number of open gaps

We will discuss how, given sufficient time and energy, using spectral invariants and the standard perturbation techniques that one could remove the assumptions $(*)$. The invariants are derived from the trace theorems. If we let the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u - qu \qquad u(0,x) = f(x) \tag{4.3}$$

on $\mathbb{R}^n$ be $G(x,y,t)$ then

$$\sum_{\lambda \in \mathrm{Spec}_k} \exp(-\lambda t) = \sum_{d \in \mathbb{L}} \exp(-2\pi i k \cdot d) \int_\Gamma G(x+d,x,t)\,dx \tag{4.4}$$

Therefore if one knows $\mathrm{Spec}_k(-\Delta + q)$ for all k, then one knows

$$\int_\Gamma G(x+d,x,t)\,dx \quad \forall t > 0,\ d \in \mathbb{L} \tag{4.5}$$

In [3] and [4], they derive Theorem 2 from the asymptotics of

$$\int_\Gamma G(x+Nd+e,x,t)\,dx \quad \forall t > 0,\ d \in \mathbb{L} \tag{4.6}$$

as $N \to \infty$.

Theorem 2 has the consequence that the set of real-analytic $\tilde{q}(x)$ isospectral to $q(x)$ can be identified with a subset of a real analytic manifold

$$M = T_1 \times T_2 \times \ldots \times T_S.$$

Here each torus T_j has dimension equal to the number of open gaps associated to each directional potential $q(\delta_j \cdot x)$; we call this set I_j. This manifold M has dimension $\sum_j |I_j| = N$. Again, the coordinates on the manifold $\alpha_{j,m}(q)$ are given for each j by (2.5).

In our case, we would like our set of potentials which we will call $M(\epsilon)$ to have open gap lengths which are parametrized as follows. Let E_0 denote the set

$$\big\{(j,m) : (j,m) = (1,1), (2,1)\big\},$$

and E_1 denote the set

$$\big\{(j,m) : j \leq 2, m > 1\big\}.$$

Now we let ϵ be the vector with four components $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ so we can parametrize the new gap lengths so they depend on $\vec{\epsilon}$ and γ as follows

$$\begin{aligned}
&\gamma_{j,m}(\epsilon,\gamma) = \epsilon_j \gamma_{j,m} \quad \text{for } (j,m) \in E_0\\
&\gamma_{j,m}(\epsilon,\gamma) = \epsilon_4 \gamma_{j,m} \quad \text{for } (j,m) \in E_1\\
&\gamma_{3,m}(\epsilon,\gamma) = \epsilon_3 \gamma_{3,m} \quad \text{for } m \in I_3\\
&\gamma_{j,m}(\epsilon,\gamma) = \epsilon_4 \gamma_{j,m} \quad \text{for } j > 3,\ m \in I_j
\end{aligned}$$

and are associated with the potential $q(\epsilon, x, \alpha)$. Here, suppressing the "q", we have $\alpha = \{\alpha_{j,m}\}$ is the rescaled vector of coordinates, where for each directional potential, the coordinates are given by (2.5). Notice that we have also written our gap lengths in terms of finitely many parameters and this does not destroy the fact the mapping (2.4) is onto and in this case analytic.

The following spectral invariants are derived from higher order terms in the asymptotics of (4.6) in [4] which we will use in our computations:

Theorem 3. *The periodic and anti-periodic spectra for the one dimensional potentials $q_\delta(x)$ which form $q(x)$ and the invariants*

$$\begin{aligned}
&\Phi_{\delta_j,m}(\epsilon,\alpha)\\
&\quad = \Phi_{j,m}(\epsilon,\alpha) = \int_\Gamma \big|h(\epsilon,x,\alpha)\big|^2 \big(\phi^{\pm}_{j,m}(\epsilon,\delta_j\cdot x,\alpha)\big)^2\,dx
\end{aligned} \tag{4.7}$$

when $\lambda^+_{j,m} > \lambda^-_{j,m}$ and

$$\begin{aligned}
&\Phi_{\delta_j,m}(\epsilon,\alpha) = \Phi_{j,m}(\epsilon,\alpha)\\
&\qquad = \int_\Gamma \big|h(\epsilon,x,\alpha)\big|^2 \Big(\big(\phi^+_{j,m}(\epsilon,\delta_j\cdot x,\alpha)\big)^2 + \big(\phi^-_{j,m}(\epsilon,\delta_j\cdot x,\alpha)\big)^2\Big)\,dx
\end{aligned} \tag{4.8}$$

when $\lambda^+_{j,m} = \lambda^-_{j,m}$ maybe recovered from the spectra of $q(x)$. Here $\alpha = \{\alpha_{j,m}\}$ is the collection of coordinates associated to each gap length and we have set

$$h(\epsilon,x,\alpha) = \sum_{\substack{e\in\mathbb{S}\\ e\cdot d_j\neq 0}} \frac{e}{e\cdot d_j} q_e(\epsilon, e\cdot x, \alpha)$$

with $\delta_j \cdot d_j = 0$, and d_j of minimal length.

Setting $\Phi^{+}_{\delta_j,m}(\epsilon,\alpha) = \Phi_{j,m}(\epsilon,\alpha)$, then the number of invariants with $\lambda^{+}_m > \lambda^{-}_m$ has dimension equal to the manifold $M(\epsilon)$. We would like to show that the Jacobian determinant of the invariants with respect to the coordinates α is nonzero so that we may apply the implicit function theorem.

We will primarily be calculating the spectral invariants for potentials at a specific parameter $\epsilon = \epsilon_0$. We let ϵ_0 be the vector with $(\epsilon_1, \epsilon_2, 0, 0)$ where ϵ_1 and ϵ_2 are in $(0,1)$. When $\epsilon = \epsilon_0$ the potential $q(\epsilon_0, x, \alpha)$ has

$$\gamma_{j,m}(\epsilon_0,\gamma) = \epsilon_j \gamma_{j,m} \quad \text{for } (j,m) \in E_0$$
$$\gamma_{j,m}(\epsilon_0,\gamma) = 0 \quad \text{for } (j,m) \in E_0^c$$

for gap lengths. The potential $q(\epsilon_0, x, \alpha)$ is therefore the sum of 2 potentials with only one gap, one in each direction δ_j, $j = 1, 2$. The rest of the directional potentials are zero. While the limit $q(\epsilon_0, x, \alpha)$ coincides with the form of the potential as calculated in [4], one specific difference remains – the first two directional have finitely many gaps, they are not just translates of the $\wp$ function. We Taylor expand the entries of the Jacobian determinant with respect to ϵ_3 around $\epsilon \neq \epsilon_0$ and use these computations to show that the Jacobian determinant for certain fixed α is not identically zero.

For the rest of this paper, we let $\wp(s + \frac{i\tau}{2}, \tau)$ denote a general normalized Weierstrass $\wp$ function. Whenever the parameter τ is real and greater than zero, then $\wp(s + \frac{i\tau}{2}, \tau)$ is real-valued with periods 1 and τ [15]. The real-valued $\wp$-function is always even about $\frac{1}{2}$, and by a theorem of Hochstadt [8], all one gap potentials are translates of the $\wp$-function. The directional potential, in the limit, $q_j(\epsilon_0, s, \alpha) = \wp(s + \frac{i\tau_j}{2} + \nu_j, \tau_j)$ has eigenfunctions which satisfy the following equation:

$$-\frac{d^2}{ds^2}\phi(\epsilon_0, s, \alpha) + q_j(s)\phi(\epsilon_0, s, \alpha) = \lambda\phi(\epsilon_0, s, \alpha),$$

where $q_j(\epsilon_0, 0, \alpha) = \wp(\frac{i\tau_j}{2} + \nu_j, \tau_j)$ has bands given by

$$\left[-\wp\left(\frac{1}{2}\right), -\wp\left(\frac{i\tau_j + 1}{2}\right)\right] \cup \left[-\wp\left(\frac{i\tau_j}{2}\right), +\infty\right). \tag{4.9}$$

Aligning the classical elliptic function theory with spectral theory [2] we have that,

$$-\wp\left(\frac{1}{2}\right) = \lambda_0 \qquad -\wp\left(\frac{i\tau_j + 1}{2}\right) = \lambda_1^{-} \qquad -\wp\left(\frac{i\tau_j}{2}\right) = \lambda_1^{+}. \tag{4.10}$$

We will need the parameters τ_j later in the computation of the Fourier coefficients of the $\wp$ function and the perturbation calculations for the eigenfunctions. From Eq. (4.9) we know that they are related to the ϵ_j as follows

$$\wp\left(\frac{i\tau_j + 1}{2}\right) - \wp\left(\frac{i\tau_j}{2}\right) = \epsilon_j \gamma_{j,1} \tag{4.11}$$

for $j = 1, 2$. Therefore if we pick ϵ_j, we pick τ_j and vice versa.

Since any potential $q(x, \epsilon, \alpha)$ is always Floquet isospectral to $q(\pm x + a, \epsilon, \alpha)$ where a is arbitrary, we cannot hope to remove the sign or translation degeneracy. We know that when $\epsilon = \epsilon_0$ that $\delta_1 \cdot a = \nu_1$ and $\delta_2 \cdot a = \nu_2$, so for simplicity we fix a so when $\epsilon = \epsilon_0$ then $a = 0$. As a result we have that

$$q_j(s, \alpha, \epsilon_0) = \wp_j\left(s + \frac{i\tau_j}{2}, \tau_j\right) = \sum_{n\in\mathbb{N}} a_n^j \cos(2\pi n s)$$

for $j = 1, 2$, where the coefficients a_n^j are given by Appendix A. We consider our manifold $M(\epsilon)$ of potentials which have translation fixed as above.

In order to prove that $M(\epsilon)$ actually is an analytic manifold with coordinates $\alpha = \{\alpha_{j,m}(q)\}$ we must first remind the reader of a few definitions involved in the selection of the coordinates $\{\alpha_{j,m}\}$ defined by (2.5) as they are related to the Dirichlet spectra $\mu_{j,m}(q)$ of the operator. We define the discriminant $\Delta(\lambda)$ as follows

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda)\prod_{n=1}^{\infty} \frac{(\lambda_n^{+} - \lambda)(\lambda_n^{-} - \lambda)}{n^4\pi^4}. \tag{4.12}$$

Let $\mu_m(s, q_j) = \mu_{j,m}(\epsilon, s, \alpha)$ be the solution to the system (where here we are suppressing the j)

$$\frac{d\mu_m(\epsilon, s, \alpha)}{ds} = m^2\pi^2 \frac{\sqrt{\Delta^2(\mu_m) - 4}}{\prod_{\substack{n\in I,\\ n\neq m}} (\mu_n(\epsilon, s, \alpha) - \mu_m(\epsilon, s, \alpha))/n^2\pi^2} \tag{4.13}$$

with $\mu_{j,m}(\epsilon, 0, \alpha) = \mu_m(0, q_j)$, $k \in I$, where the choice of signs is initially by the sign of numerator, and changes whenever $\mu_{j,m}(\epsilon, s, \alpha)$ hits $\lambda^{\pm}_{j,m}$. The proof of analyticity of μ by examining (4.13) remains almost exactly the same as in [4] and is omitted here. Since there are a finite number of coordinates, it is easy to see that analyticity in each coordinate is preserved, and hence $M(\epsilon)$ is still an analytic manifold.

By McKean–Van Moerbeke [13], the initial value the sum of the initial values, $\mu_{j,m}(\epsilon, 0, \alpha)$, is related to each directional potential $q_j(\epsilon, 0, \alpha)$ in the following way

$$q_j(\epsilon, 0, \alpha) = \lambda_0 + \sum_{m\in I_j} \left(\lambda^+_{j,m} + \lambda^-_{j,m} - 2\mu_{j,m}(\epsilon, 0, \alpha)\right)$$

and this relationship remains true when the parameter s is varied

$$q_j(\epsilon, s, \alpha) = \lambda_0 + \sum_{m\in I_j} \left(\lambda^+_{j,m} + \lambda^-_{j,m} - 2\mu_{j,m}(\epsilon, s, \alpha)\right). \tag{4.14}$$

Using a combination of formulas on pp. 325 and 329, in [16], the eigenfunctions for each directional potential corresponding to $\lambda^+_{j,m}$ for all j can be written as

$$\left(\phi^+_m(\epsilon, s, \alpha)\right)^2 = \prod_{n\in I_j} \left(\frac{\lambda^+_m - \mu_n(\epsilon, s, \alpha)}{\lambda^+_m - \dot\lambda_n}\right) \tag{4.15}$$

where $\dot\lambda_m$ is the zero of $\frac{\partial\Delta}{\partial\lambda}$ lying between λ^-_m and λ^+_m. It is important to note here that the formula in [4] is a misprint. We will also need the derivatives of the eigenfunctions which from Eq. (4.15) are

$$2\phi^+_m(\epsilon, s, \alpha)\frac{d\phi^+_m(\epsilon, s, \alpha)}{ds} = \sum_{n\in I_j} \frac{-1}{\lambda^+_n - \dot\lambda_k} \left(\frac{d\mu_n(\epsilon, s, \alpha)}{ds}\right) \prod_{k\neq n} \frac{\lambda^+_m - \mu_k(\epsilon, s, \alpha)}{\lambda^+_m - \dot\lambda_k} \tag{4.16}$$

with the derivative for $\phi^-(\epsilon_0, s, \alpha)$ computed similarly. Let us start by considering the eigenfunctions for those directional potentials with $j > 3$. Because we are looking for the root between $\lambda^+_{j,m}$ and $\lambda^-_{j,m}$ when $\epsilon = \epsilon_0$, we make the substitution $\lambda = \lambda^-_{j,m} + \epsilon_4\gamma_{j,m}\tilde\lambda$ into (4.12) to find that

$$\Delta^2(\tilde\lambda) - 4 = \epsilon_4^2\tilde\lambda(1 - \tilde\lambda) f(\epsilon_4\tilde\lambda, \epsilon_4)$$

where $f(z, \epsilon_4)$ is analytic and $f(0, 0) = \gamma_m^2 \neq 0$. Therefore for ϵ_4 sufficiently small, $\dot\lambda_m$ corresponds to the root of

$$0 = (1 - 2\tilde\lambda) f(\epsilon_4\tilde\lambda, \epsilon_4) + \epsilon_4\tilde\lambda(1 - \tilde\lambda)\frac{\partial f}{\partial z}(\epsilon_4\tilde\lambda, \epsilon_4)$$

near $\tilde\lambda = \frac{1}{2}$. As a result, the following estimate holds

$$\frac{\lambda^+_m(\epsilon) - \lambda^-_m(\epsilon)}{\lambda^+_m(\epsilon) - \dot\lambda_m(\epsilon)} = 2 + \mathcal{O}(\epsilon_4), \tag{4.17}$$

giving that

$$\frac{\lambda^+_m(\epsilon_0) - \mu_m(\epsilon_0, \alpha, s)}{\lambda^+_m(\epsilon_0) - \dot\lambda_m(\epsilon_0)} = 2\cos^2\left(\tilde\alpha_m(s, \alpha)\right). \tag{4.18}$$

The variable $\tilde\alpha_m(s, \alpha)$ denotes the solution to the system (4.13) where $\epsilon = \epsilon_0$ with initial condition α under the change of variables (2.5). The same estimates above are true for the eigenfunctions $\phi^+_{3,m}(\epsilon_0, s, \alpha)$, m in I_3 when expanded with respect to ϵ_3. We can conclude for all $j \geq 3$

$$\phi^+_{j,m}(\epsilon_0, s, \alpha) = \sqrt{2}\cos\tilde\alpha_{j,m}(s, \alpha) \tag{4.19}$$

where we know we have picked the right sign by verifying the derivative (4.16) in the limit.

Now we consider the case when $j \leq 2$. When $\epsilon = \epsilon_0$, we have for all $n > 1$ that $\lambda^+_{j,n} = \lambda^-_n = \mu_n = \dot{\lambda}_n$ so that terms in the product (4.15) where $n \neq m$ and $n > 1$ become

$$\frac{\lambda^+_{j,m}(\epsilon_0) - \mu_{j,n}(\epsilon_0, s, \alpha)}{\lambda^+_{j,m}(\epsilon_0) - \dot{\lambda}_{j,n}(\epsilon_0)} = 1, \tag{4.20}$$

and for $n = 1$ we have

$$\frac{\lambda^+_{j,m}(\epsilon_0) - \mu_{j,1}(\epsilon_0)}{\lambda^+_{j,m}(\epsilon_0) - \dot{\lambda}_{j,1}(\epsilon_0)} = \frac{\lambda^+_{j,m}(\epsilon_j) - \lambda^-_{j,1}(\epsilon_j) - \epsilon_j \gamma_{j,1} \sin^2(\tilde{\alpha}_{j,1}(s,\alpha))}{\lambda^+_{j,m} - \dot{\lambda}_{j,1}(\epsilon_j)}. \tag{4.21}$$

Combining Eq. (4.18) (which is still true for $j \leq 2$) and (4.20), we see that for $\epsilon = \epsilon_0$, and (j,m) in E_1,

$$\left(\phi^+_{j,m}(\epsilon_0, \alpha, s)\right)^2 = 2\cos^2\left(\tilde{\alpha}_m(s,\alpha)\right)\left(\frac{\lambda^+_{j,m}(\epsilon_j) - \lambda^-_{j,1}(\epsilon_j) - \epsilon_j \gamma_{j,1} \sin^2(\tilde{\alpha}_{j,1}(s,\alpha))}{\lambda^+_{j,m}(\epsilon_j) - \dot{\lambda}_{j,1}(\epsilon_j)}\right). \tag{4.22}$$

Comparing with the derivative computed in (4.16) we know that the correct choice of sign is

$$\left(\phi^+_{j,m}(\epsilon_0, \alpha, s)\right) = \sqrt{2}\cos\left(\tilde{\alpha}_m(s,\alpha)\right)\sqrt{\frac{\lambda^+_{j,m}(\epsilon_j) - \lambda^-_{j,1}(\epsilon_j) - \epsilon_j \gamma_{j,1} \sin^2(\tilde{\alpha}_{j,1}(s,\alpha))}{\lambda^+_{j,m}(\epsilon_j) - \dot{\lambda}_{j,1}(\epsilon_j)}}. \tag{4.23}$$

The introduction of this setup provides the necessary background to introduce the following theorem:

Theorem 4. *For all but an analytic set of (ϵ_3, ϵ_4) in $[0,1]^2$, there is an open set of potentials satisfying the hypotheses (1), (2) and (∗) in $M(\epsilon)$ which are isospectral to only a finite number of other analytic potentials.*

In order to find the Jacobian corresponding to the invariants as given by Eq. (4.7), we must first figure out what it means to calculate their derivatives with respect to $\{\alpha_{j,m}\}$ with (j,m) in E^c_0. We start with the following lemma

Lemma 1. *For (j,m) in E^c_0, we have*

$$\frac{\partial \tilde{\alpha}_{j,m}(s,\alpha)}{\partial \alpha_{j,m}} = 1, \quad \textit{and} \quad \frac{\partial \tilde{\alpha}_{j,m}(s,\alpha)}{\partial \alpha_{r,k}} = 0 \quad \textit{when } (r,k) \neq (j,k)$$

Proof. Examining (4.13) under the change of variables given by (2.5) for (j,m) in E_1 and $\epsilon = \epsilon_0$

$$\frac{d\tilde{\alpha}_{j,m}(s,\alpha)}{ds} = \frac{\sqrt{(\lambda^+_{j,m} - \lambda_0)(\lambda^+_{j,m} - \lambda^+_{j,1})(\lambda^+_{j,m} - \lambda^-_{j,1})}}{\lambda^+_{j,m} - \lambda^-_{j,1} - \epsilon_j \gamma_{j,1} \sin^2 \tilde{\alpha}_{j,1}(s,\alpha)} \tag{4.24}$$

Therefore $\tilde{\alpha}_{j,m}(s,\alpha)$ depends only on $\alpha_{j,1}$ and the initial data for $\tilde{\alpha}_{j,m}(0,\alpha) = \alpha_{j,m}$ so the result follows.

The case whenever $j \geq 3$ and $\epsilon = \epsilon_0$, is much easier to compute. We have for all such corresponding m

$$\frac{d\tilde{\alpha}_{j,m}(s)}{ds} = m\pi \tag{4.25}$$

so again the result follows by the same reasoning above. □

For the computations done in the appendix, we need to know that when $\epsilon_j = 0$, (4.23) agrees with the limit one would expect. In other words for (j,m) in E_1, we have

$$\phi^+_{j,m}(\epsilon_0, \alpha, s) = \sqrt{2}\cos(\pi m s + \alpha_{j,m}) + \mathcal{O}(\epsilon_j) \tag{4.26}$$

which is easily verifiable by Lemma 1, and the estimates (4.17) and (4.20). We have computed the eigenfunctions in (4.23) to illustrate that they are expressed in terms of elliptic functions, and therefore the invariants will not be explicitly computable.

We can now prove the main lemma. If we consider a potential $q(\epsilon, x, \alpha)$ in $M(\epsilon)$ then it is associated to a fixed set of coordinates α. Let $\det(J)(\epsilon, \alpha)$ be the Jacobian determinant of the invariants $\Phi_{j,m}(\epsilon, \alpha)$ with respect to the coordinates $\{\alpha_{j,m}\}$ with j, m in E_0^c, and $\det(J)(\epsilon, \alpha)$ is an $(N-2) \times (N-2)$ determinant.

The proof of Theorem 4 will be based on the following lemma:

Lemma 2. *There is a choice of ϵ_1, ϵ_2 in $[0, 1]$ such that on a dense open set of α,*

$$\det(J)(\epsilon, \alpha) \neq 0 \tag{4.27}$$

Proof. We will proceed by showing that for all $k = 1$ to $n-1$

$$\frac{\partial^k \det(J)}{\partial \epsilon_3^k}(\epsilon_0, \alpha) = 0$$

while

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0, \alpha) \neq 0$$

where $n = |I_1| + |I_2| - 2 = |E_1|$. The desired result will follow since we notice that if for some n

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0, \alpha) \neq 0 \quad \text{and} \quad \det(J)(\epsilon, \alpha) \equiv 0$$

then this is a contradiction since all of the derivatives of $\det(J)(\epsilon, \alpha)$ evaluated at any ϵ should be identically zero as well, since $\det(J)(\epsilon, \alpha)$ is an analytic function of ϵ.

Now we proceed to calculate the derivatives of $\det(J)(\epsilon, \alpha)$. Let the columns $v_i(\epsilon, \alpha)$ of $\det(J)(\epsilon, \alpha)$ be indexed by i where i ranges from 1 to $N-2$. Each i corresponds to a pair of indices (j, m) such that

$$v_i(\epsilon, \alpha) = \nabla_\alpha \Phi_{j,m}(\epsilon, \alpha)$$

where we are considering the pairs (j, m) ordered first by the j and then by the m. The perturbation calculations to find the derivatives of the invariants are located in Appendices. In order to examine the Jacobian further, we need the following key observations:

1. $\frac{\partial q_j}{\partial \alpha_{l,k}}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \ \forall (l,k) \in E_o^c$, and $\forall j$
2. $\frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{l,k}}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \ \forall (l,k), (j,m) \in E_0^c$ unless $(j,k) = (l,m)$
3. $\frac{\partial q_j}{\partial \epsilon_3}(\epsilon_0, \delta_j \cdot x, \alpha) = \frac{\partial (\phi^+_{j,m})^2}{\partial \epsilon_3}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \ \forall j \neq 3$

The first two observations follow from Lemma 1 and formulae (4.14) and (4.15), respectively. The last observation follows from the parametrization of the open gaps since only $q_3(\epsilon, \delta_3 \cdot x, \alpha)$ and $\phi_{3,m}(\epsilon, \delta_3 \cdot x, \alpha)$ for m in I_3 depend on ϵ_3.

Going back to Eq. (4.7), each invariant has the form as follows

$$\Phi_{j,m}(\epsilon, \alpha) = \int_\Gamma \left| \sum_{\substack{l \in N \\ l \neq j}} \frac{\delta_l}{\delta_l \cdot d_j} q_l(\epsilon, \delta_l \cdot x, \alpha) \right|^2 \left(\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha)\right)^2 dx \tag{4.28}$$

Now we let D denote a generic constant independent of the coordinates. When $\epsilon = \epsilon_0$ the form of the invariants (4.28) for $j \geq 3$ coincides with that of [4]. Since δ_1 and δ_2 form a basis for $\mathbb{S}$, we know that there exists a nonzero pair of integers (p_l, r_l) such that for any third vector $\delta_l \neq \delta_1, \delta_2$ we have $\delta_l = p_l \delta_1 + r_l \delta_2$. Therefore when $j \geq 3$

$$\Phi_{j,m}(\epsilon_0, \alpha) = D \int_0^1 \int_0^1 \left(\wp_2\left(t + \frac{i\tau_2}{2}, \tau_2\right)\right)\left(\wp_1\left(s + \frac{i\tau_1}{2}, \tau_1\right)\right) \cos^2\left(\pi m(p_j s + r_j t) + \alpha_{j,m}\right) ds\, dt + D \tag{4.29}$$

Exactly as in [4], we have that when (j,m) is such that $j \geq 3$

$$\Phi_{j,m}(\epsilon_0,\alpha) = c_{1,2,j}a^1_{mp_j}a^2_{mr_j}\cos 2\alpha_{j,m} + D$$

The coefficients $c_{1,2,j}a^1_{mp_j}a^2_{mr_j}$ are independent of the coordinates and nonzero. They can be found in Appendix A. However for j in $\{1,2\}$, we come across the degeneracy that

$$\frac{\partial \Phi_{j,m}}{\partial \alpha_{l,k}}(\epsilon_0,\alpha) = 0 \tag{4.30}$$

for all (l,k) in E_0^c. We know from our observations (1) and (2) that (4.30) holds except for possibly when $(l,k) = (j,m)$. In this case since again δ_1 and δ_2 form a basis for $\mathbb{S}$ we can write

$$\begin{aligned}\frac{\partial \Phi_{j,m}}{\partial \alpha_{j,m}}(\epsilon_0,\alpha) &= \int_\Gamma \left|\frac{\delta_l}{\delta_l\cdot d_j}q_l(\epsilon_0,\delta_l\cdot x,\alpha)\right|^2 \frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0,\delta_j\cdot x,\alpha)\,dx \\ &= D\int_0^1 \wp_l^2\left(s+\frac{i\tau_l}{2},\tau_l\right)ds \int_0^1 \frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0,t,\alpha)\,dt \end{aligned}\tag{4.31}$$

where $l \neq j$ and l is in $\{1,2\}$. But since we consider our eigenfunctions as normalized for all (j,m), e.g. $\|\phi^+_{j,m}(\epsilon,\delta_j\cdot x,\alpha)\|_{L^2(\mathbb{R})} = 1$, the right hand side of (4.31) is just zero.

Therefore for all i from 1 to n we have

$$v_i(\epsilon_0,\alpha) = 0,$$

while for all i from $n+1$ to $(N-2)$ we see that

$$\big(v_i(\epsilon_0,\alpha)\big)^t_l = \begin{cases} 0 & l = 1,..,i-1 \\ c_{1,2,j}a^1_{mp_j}a^2_{mr_j}\sin 2\alpha_{j,m} & l = i \\ 0 & l > i \end{cases}. \tag{4.32}$$

Because the determinant is a multi-linear function of its rows, we may write

$$\det(J)(\epsilon_0,\alpha) = \det(v_1, v_2, ..., v_n, v_{n+1}, .., v_{N-2})$$

It is now clear that for all $k = 1$ to $n-1$

$$\frac{\partial^k \det(J)}{\partial \epsilon_3^k}(\epsilon_0,\alpha) = 0$$

however for $k = n$ we have

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0,\alpha) = C(n)\det\left(\frac{\partial v_1}{\partial\epsilon_3}, \frac{\partial v_2}{\partial\epsilon_3}, ..., \frac{\partial v_n}{\partial\epsilon_3}, v_{n+1}, ..., v_{N-2}\right), \tag{4.33}$$

where $C(n)$ is a constant depending on n only.

From observations (1–3) we know for j in $\{1,2\}$

$$\frac{\partial^2 \Phi_{j,m}}{\partial\epsilon_3\partial\alpha_{l,k}}(\epsilon_0,\alpha) = 0$$

except for possibly when $l = 3$ or $(l,k) = (j,m)$. We then note that corresponding rows with $1 \leq i \leq n$ in (4.34) take the form

$$\left(\frac{\partial v_i}{\partial\epsilon_3}\right)^t_l = \begin{cases} 0 & l = 1,..,i-1 \\ \frac{\partial^2\Phi_{j,m}}{\partial\alpha_{j,m}\partial\epsilon_3}(\epsilon_0,\alpha) & l = i \\ 0 & r > l > i \\ \frac{\partial^2\Phi_{j,m}}{\partial\alpha_{3,j}\partial\epsilon_3}(\epsilon_0,\alpha) & i = r...k \\ 0 & l > r \end{cases} \tag{4.34}$$

Here the index r corresponds to $(3,1)$ and $k-r=|I_3|$. We can conclude from (4.32) and (4.34) the determinant (4.33) is an upper triangular one. The determinant (4.33) looks like

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix}$$

where A is an $n\times n$ block diagonal matrix, and C is an $(N-n-2)\times(N-n-2)$ block diagonal matrix. If the diagonal entries in the upper triangular determinant (4.34) are nonzero, then we will arrive at the desired result that

$$\frac{\partial^n \det(J)}{\partial^n \epsilon_3}(\epsilon_0,\alpha)\neq 0 \tag{4.35}$$

The collection of diagonal entries for (j,m) in E_1 corresponding the block A, for $1\leq i\leq n$ are $\frac{\partial^2 \Phi_{j,m}}{\partial\epsilon_3\partial\alpha_{j,m}}(\epsilon_0,\alpha)$. From Appendix B, we know that there is a choice of ϵ_1 and ϵ_2 so that these invariants are nonzero except on an analytic set of $\alpha_{j,m}$. Also from Appendix B and Eq. (B.23), whenever $i>n$ we have diagonal entries corresponding to (j,m) with $j\geq 3$, corresponding to the block C are

$$\frac{\partial \Phi_{j,m}}{\partial\alpha_{j,m}}(\epsilon_0,\alpha)=-2c_{1,2,j}a^1_{mp_j}a^2_{mr_j}\sin 2\alpha_{j,m} \tag{4.36}$$

These entries are only zero whenever $\alpha_{j,m}\equiv 0 \mod \pi/2$ for $j\geq 3$. The lemma is finished.

Remark. It should be possible to remove the assumption $(*)$ by using the standard perturbation series to calculate $(\phi^+_{j,m}(\epsilon_j,s,\alpha))^2$ around $\epsilon_j=0$. If δ_3 were generically of the from $p_3\delta_1+r_3\delta_2$, then we conjecture that (B.8) is nonzero provided we expanded the eigenfunctions to order n with n satisfying the relation $m\pm l=np_3$ or $m\pm l=nr_3$ for some l in $\mathbb{N}$. The calculations required to do so are difficult. This conjecture is discussed further in Appendix B. □

Proof of Theorem 4. This proof is very similar to the one in [4] and is again included for completeness. Let us start by assuming the matrix J is invertible on $M(\epsilon)$ except for on an analytic set, say U, of (ϵ_3,ϵ_4) Recall that on the manifold ϵ_j and the corresponding $\alpha_{j,1}$ for $j=1,2$ are fixed. Then given some $\tilde{\epsilon}$ with variable components (ϵ_3,ϵ_4) in $[0,1]^2/U$, we let

$$F=\left\{\alpha : \frac{\partial\Phi}{\partial\alpha}(\tilde{\epsilon},\alpha)=0\right\}.$$

Since

$$\Phi(\tilde{\epsilon},\alpha): M(\tilde{\epsilon})\to\mathbb{R}^{N-2},$$

the corollary follows if we can show that the set $\Phi^{-1}(\Phi(F)^c)$ is open and dense. We know the set is open since Φ^{-1} is open, and F is compact. If we assume that it is not dense, then the set contains an open set O which also contains a point α_0 which is not in F. Because the Jacobian is nonzero, Φ is a homeomorphism on a neighborhood of α_0, which implies $\Phi(F)$ contains an open set. The last statement contradicts Sard's theorem. Now we assume that $\Phi(\alpha_1)$ is not in $\Phi(F)$ and $\Phi^{-1}(\Phi(F))$ is infinite. Let α_2 be an accumulation point of $\Phi^{-1}(\Phi(\alpha_1))$. Because Φ is continuous, $\Phi(\alpha_2)=\Phi(\alpha_1)$ and $\frac{\partial\Phi}{\partial\alpha_2}\neq 0$. It follows that there is a neighborhood, N, of α_2 such that α is in N and $\Phi(\alpha)=\Phi(\alpha_2)$ implies $\alpha=\alpha_2$. This is a contradiction to our assumption so we know $\Phi^{-1}(\Phi(\alpha_1))$ is finite. Because Φ is a spectral invariant, then $\Phi^{-1}(\Phi(F)^c)$ is a subset of the manifold which satisfies the conditions of Theorem 4. □

This theorem has a nice corollary if we make the following observations:

1. Any two directions δ_1 and δ_2 form a basis for the lattice $\mathbb{L}$, so our choice of basis and translate is arbitrary.
2. The potentials on $M(\epsilon)$ satisfying the conditions of the theorem are dense in the set of all analytic potentials in the C^∞ topology.
3. The set of smooth periodic potentials which are a sum of only a finite number of directional potentials each with a finite number of gaps in each direction are dense in the set of finite gap periodic potentials in the $C^\infty(\mathbb{R}^2/\mathbb{L})$ topology.
4. The set of finite gap potentials is dense in the set of all $C^6(\mathbb{R}^2/\mathbb{L})$ potentials in the C^∞ topology.

Corollary 1. *The set of analytically rigid potentials is dense in the set of smooth potentials on* $\mathbb{R}^2/\mathbb{L}$ *in the* $C^\infty(\mathbb{R}^2/\mathbb{L})$ *topology.*

Proof. Observations 1, 3, and 4 along with Theorem 4 imply the existence of a set of potentials which are analytically rigid. Combined with observation 2, we have proved Corollary 1. □

Conflict of interest statement

The author declares that there is no conflict of interest.

Appendix A. Fourier coefficients of the $\wp$ function

As detailed in Section 1, the $\wp$-function depends on a parameter $\tau_j > 0$. The complex valued function $\wp(z,\tau)$ is given by

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2/0} \left(\frac{1}{(z-n-im\tau)^2} - \frac{1}{(n+im\tau)^2} \right)$$

which as before is real on the line $x + \frac{i\tau_j}{2}$ and setting,

$$a = e^{-2\pi\tau_j} \qquad b = e^{2\pi i(x+\frac{i\tau_j}{2})}$$

gives

$$\frac{1}{(2\pi i)^2}\wp(x,\tau) = \frac{1}{12} + \sum_{n=-\infty}^{\infty} \frac{ab}{(1-a^m b)^2} - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Because

$$\frac{a^m b}{(1-a^m b)^2} = \sum_{n=1}^{\infty} n\left(a^m b\right)^n \quad m \geq 0$$

and

$$\frac{a^m b}{(1-a^m b)^2} = \sum_{n=1}^{\infty} n\left(a^{-m} b^{-1}\right)^n \quad m < 0$$

the representation

$$\frac{1}{(2\pi i)^2}\wp(x,\tau)$$
$$= \frac{1}{12} + \sum_{n=1}^{\infty} n a^{\frac{n}{2}} e^{2\pi i n x} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n\left(a^{n(m+\frac{1}{2})} e^{2\pi i n x} + a^{n(m-\frac{1}{2})} e^{-2\pi i x}\right) - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Changing the order of summation we get

$$\frac{-1}{4\pi^2}\wp(x,\tau) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2na^{\frac{n}{2}}}{1-a^n}\cos(2\pi n x) - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Therefore the Fourier coefficients for the $\wp$ functions in the first three directions are given by:

$$a_n^j = \frac{-8\pi^2 n \exp(-\pi n\tau_j)}{1-\exp(-2\pi n\tau_j)} \quad \text{for } n \geq 1 \tag{A.1}$$

$$a_0 = -\frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \frac{n\exp(-\pi n\tau_j)}{1-\exp(-\pi n\tau_j)} \tag{A.2}$$

where $j = 1, 2$. The appropriate τ_j will depend on the choice of ϵ_j as given in Section 1.

Appendix B. Calculation of the invariants

In order to prove Lemma 2 we need to show that there exist ϵ_1 and ϵ_2 in $[0,1]$ such that

$$\frac{\partial^2 \Phi_{1,m}}{\partial \epsilon_3 \partial \alpha_{1,m}}(\epsilon_0,\alpha) \quad \text{and} \quad \frac{\partial^2 \Phi_{2,n}}{\partial \epsilon_3 \partial \alpha_{2,n}}(\epsilon_0,\alpha) \tag{B.1}$$

are nonzero except perhaps on an analytic set of α.

We know by (4.7)

$$\Phi_{j,m}(\epsilon,\alpha) = \int_\Gamma \left| \sum_{j\neq k} \frac{\delta_k}{\delta_k \cdot d_j} q_k(\epsilon, \delta_k \cdot x, \alpha) \right|^2 \left(\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha)\right)^2 dx \tag{B.2}$$

Each $q_j(\epsilon, \delta_j \cdot x, \alpha)$ is independent of ϵ_3 when $j \neq 3$. Furthermore since $q_k(\epsilon, \delta_k \cdot x, \alpha)$ and $(\phi^+_{k,m}(\epsilon, \delta_j \cdot x, \alpha))^2$ are independent of $\mu_{j,m}(\epsilon, \delta_j \cdot x, \alpha)$ for all $j \neq k$, so only the function $(\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2$ depends on $\alpha_{j,m}$ in the above integral. As a result we can write

$$\frac{\partial^2 \Phi_{j,m}}{\partial \epsilon_3 \partial \alpha_{j,m}}(\epsilon,\alpha) = \int_\Gamma \frac{\partial}{\partial \epsilon_3} \left| \sum_{j\neq k} \frac{\delta_k}{\delta_k \cdot d_j} q_k(\epsilon, \delta_k \cdot x, \alpha) \right|^2 \frac{\partial}{\partial \alpha_{j,m}} \left(\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha)\right)^2 dx \tag{B.3}$$

Whenever $\epsilon = \epsilon_0$, then $q_3(\epsilon_0, \delta_3 \cdot x, \alpha) = 0$ and the derivative $\partial_{\epsilon_3} q_3(\epsilon_0, \delta_3 \cdot x, \alpha)$ can be calculated using the Fredholm alternative as in [4]. Following Appendix I of [4], we may write

$$\frac{\partial q_3}{\partial \epsilon_3}(\epsilon_0, \delta_3 \cdot x, \alpha) = \sum_{n \in I_3} \gamma_{3,n} \cos(2\pi \delta_3 \cdot x + 2\alpha_{3,n}). \tag{B.4}$$

Also from the derivation of Eq. (4.19), we can conclude that

$$\left(\phi^+_{j,m}(\epsilon_0, s, \alpha)\right)^2 = 2\cos^2\left(\pi m(\delta_j \cdot x) + \alpha_{j,m}\right) + \mathcal{O}(\epsilon_j) \tag{B.5}$$

where by Lemma 1 the order terms are bounded by $\epsilon_j C$ where C depends only on $\alpha_{j,m}$. Hence from analytic perturbation theory and the derivation of (4.19) we can use (B.5) to conclude that

$$\frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{j,m}}(\epsilon_0, \delta_j \cdot x, \alpha) = -2\sin\left(2\pi(\delta_j \cdot x)m + 2\alpha_{j,m}\right) + \mathcal{O}(\epsilon_j) \tag{B.6}$$

where the $\mathcal{O}(\epsilon_j)$ terms are bounded by $\epsilon_j C$ with C a constant depending only on the coordinate $\alpha_{j,m}$. Because any two directions δ_1 and δ_2 in $\mathbb{S}$ form a basis, we know that there exists a nonzero pair of integers (p_l, r_l) such that for any third vector $\delta_l \neq \delta_1, \delta_2$ we have $\delta_l = p_l \delta_1 + r_l \delta_2$. For easier computations we make the initial variable change $\delta_1 \cdot x = s$ and $\delta_2 \cdot x = t$, with the associated Jacobian, $\mathrm{Vol}(\Gamma)$, and rewrite the invariants. We also let D denote a generic constant which is independent of the coordinates, and we let

$$c_{l,k,j} = \frac{\delta_l \cdot \delta_j}{2(\delta_l \cdot d_j)(\delta_k \cdot d_j)} \left(\mathrm{Vol}(\Gamma)\right). \tag{B.7}$$

From statements (1–3) in Section 3, (B.4), (B.5) and (B.3), when $\epsilon = \epsilon_0$, we have

$$\left(c_{3,l,j}\mathrm{Vol}(\Gamma)\right)^{-1} \frac{\partial^2 \Phi_{j,m}}{\partial \epsilon_3 \partial \alpha_{j,m}}(\epsilon_0,\alpha) \tag{B.8}$$

$$= 4\int_0^1 \int_0^1 \left(\sum_{n\in I_3} \gamma_{3,n} \cos\left(2\pi n(s+t) + 2\alpha_{3,n}\right) \right) \wp_l\left(t + i\frac{\tau_l}{2}, \tau_l\right) \frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{j,m}}(\epsilon_0, s, \alpha)^2 \, ds\, dt$$

$$= 2\sum_{n\in I_3} \gamma_{3,n} a^l_n \int_0^1 \cos(2\pi n s + 2\alpha_{3,n}) \frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{j,m}}(\epsilon_0, s, \alpha)\, ds \tag{B.9}$$

where $0 \leq j, l \leq 2$, $j \neq l$.

When $j = 1$, by the hypothesis $(*)$ on the number of open gaps that q_3 has, the right hand side of (B.8) is just

$$2a_m^2\gamma_{3,m}\sin(2\alpha_{3,m} - 2\alpha_{1,m}) + \mathcal{O}(\epsilon_1) \tag{B.10}$$

Here the $\mathcal{O}(\epsilon_1)$ terms are bounded by $\epsilon_1 C$ where the constant depends only on $\alpha_{1,m}$ and $\alpha_{3,n}$ for all $n \in I_3$. We recall that $a_n^l \to 0$ as $\epsilon_l \to 0$ for all n in $\mathbb{N}$ and $l = 1, 2$ since a_n^l is related to ϵ_l by Eq. (4.11) and (A.1) However, we can make the constant uniform in ϵ_2. If we let

$$\sup_{s\in[0,1]}\left|\frac{\partial(\phi_{1,m}^+)^2}{\partial\alpha_{1,m}}(\epsilon_0, s, \alpha)\right| = M_m < \infty \tag{B.11}$$

then this follows from the rough estimate

$$\begin{aligned}&\left|\sum_{n\in I_3}\int_0^1\int_0^1\left(\sum_{n\in I_3}\gamma_{3,n}\cos\big(2\pi n(s+t)+2\alpha_{3,n}\big)\right)p_2\left(t+i\frac{\tau_2}{2},\tau_2\right)\right.\\ &\qquad\left.\times\left(\frac{\partial(\phi_{1,m}^+)^2}{\partial\alpha_{1,m}}(\epsilon_0,s,\alpha)\right)^2 - \sin(2\pi ms + 2\alpha_{1,m})\,ds\,dt\right|\\ &\quad\le\sum_{n\in I_3}\gamma_{3,n}a_n^2\cos(2\alpha_{3,n})(M_m+2) \le 2n(M_m+2)\end{aligned} \tag{B.12}$$

since the gap lengths $\gamma_{3,n}$ and the Fourier coefficients a_n^2 are exponentially decreasing. Now let β in $(0, 1)$ be a small fixed parameter. We consider the set of α such that

$$|2\alpha_{3,m} - 2\alpha_{1,m} - k\pi| \ge \beta \quad \forall k \in \mathbb{Z},\ m \in I_1 \tag{B.13}$$

We let this set be denoted as A_1, and note that its complement is an analytic set. Therefore provided we chose ϵ_1 and ϵ_2 which satisfy the inequality

$$(M_m + 2)\epsilon_1 < \frac{|a_m^2|\gamma_{3,m}}{2n}\sin(\beta) \tag{B.14}$$

for all m in I_1 and α in A_1 then (B.8) is nonzero for $j = 1$ and all m in I_1. The tricky step is to prove that we can pick ϵ_1, ϵ_2 in $(0, 1)$ such that (B.14) holds for all m in I_1 but also so

$$\frac{\partial^2\Phi_{2,n}}{\partial\epsilon_3\partial\alpha_{2,n}}(\epsilon_0,\alpha) \ne 0 \tag{B.15}$$

for all n in I_2 except on an analytic set of α.

Because for small ϵ_1, $a_{n_1}^1 > a_{n_2}^1$ whenever $n_2 > n_1$ the right hand side of (B.8) is already written in ascending order in ϵ_1 for $j = 2, l = 1$. Let

$$b_{j,m,n}(\epsilon_0,\alpha) = \int_0^1 \cos(2\pi ns + 2\alpha_{3,n})\frac{\partial(\phi_{j,m}^+)^2}{\partial\alpha_{j,m}}(\epsilon_0, s, \alpha)\,ds. \tag{B.16}$$

Since we do not know if $b_{2,m,n}(\epsilon_0, \alpha) \equiv 0$ in α for all $m \ne n$, we pick ϵ_1 as follows. Say $b_{2,m,1}(\epsilon_0, \alpha)$ is nonzero except on an analytic set of α, and then let the set where $b_{2,m,1}(\epsilon_0, \alpha) = 0$ be denoted as $A_{2,m,1}^c$. If we can prove that for $j = 2, l = 1$, (B.8) is nonzero for some α, then it will be nonzero on some open dense set of α's. The easiest α to select is the one when $b_{2,m,1}(\epsilon_0, \alpha)$ is at its maximum. Hence we then pick ϵ_1 such that

$$\max_{\alpha\in A_{2,m,1}}\left|\gamma_{3,1}a_1^1 b_{2,m,1}(\epsilon_0,\alpha)\right| \ge \left|\sum_{\substack{k\in I_3\\ k\ne 1}}\gamma_{3,k}a_k^1 b_{2,m,k}(\epsilon_0,\alpha)\right| \tag{B.17}$$

where the max is taken over the possible values of $b_{2,m,1}(\epsilon_0, \alpha)$ with α in $A_{2,m,1}$, and we consider the right hand side of (B.17) to be evaluated at this α as well. If $b_{2,m,1}(\epsilon_0, \alpha) \equiv 0$ in α, but $b_{2,m,2}(\epsilon_0, \alpha)$ is nonzero except on an analytic set of $\alpha_{2,m}$, and let the set where $b_{2,m,2}(\epsilon_0, \alpha) = 0$ be denoted as $A_{2,m,2}^c$ then pick ϵ_1 such that

$$\max_{\alpha\in A_{2,m,2}} \left|\gamma_{3,2}a_2^1 b_{2,m,2}(\epsilon_0,\alpha)\right| \geq \left|\sum_{\substack{k\in I_3\\ k>2}} \gamma_{3,k}a_k^1 b_{2,m,k}(\epsilon_0,\alpha)\right| \tag{B.18}$$

where again the max is taken over the possible values of $b_{2,m,2}(\epsilon_0,\alpha)$ with α in $A_{2,m,2}$. We continue this process inductively. As before, let β be a small parameter in $(0,1)$. We now also consider the set of α such that

$$|2\alpha_{3,m} - 2\alpha_{2,m} - k\pi| \geq \beta \quad \forall k\in\mathbb{Z},\ m\in I_2 \tag{B.19}$$

and let this set be denoted by $A_{2,m,m}$. We know

$$b_{2,m,m}(\epsilon_0,\alpha) = \sin(2\alpha_{3,m} - 2\alpha_{2,m}) + \mathcal{O}(\epsilon_2) \tag{B.20}$$

where the $\mathcal{O}(\epsilon_2)$ terms are bounded by $\epsilon_2 C$ where is a constant depending only on $\alpha_{2,m}$ and $\alpha_{3,n}$ for all n in I_3. Hence our selection process terminates because $b_{2,m,m}(\epsilon_0,\alpha)$ is not zero for α in $A_{2,m,m}$ provided we chose ϵ_2 such that

$$\epsilon_2|C| < \sin(\beta) \tag{B.21}$$

Hence we pick ϵ_1 in terms of ϵ_2 so that

$$\min_n \max_{\alpha\in A_{2,m,n}} \left(\left|\gamma_{3,n}a_n^1 b_{2,m,n}(\epsilon_0,\alpha)\right|\right) \geq \left|\sum_{\substack{l\in I_3\\ k>n}} \gamma_{3,k}a_k^1 b_{2,m,k}(\epsilon_0,\alpha)\right| \tag{B.22}$$

for all m in I_2 where the $\min_n$ is taken over those indices n for which $b_{2,m,n}(\epsilon_0,\alpha)$ is not identically zero in α. This choice of ϵ_1 and ϵ_2 is not in contradiction to our choice of ϵ_1 small compared to ϵ_2 since the right hand side of the inequality (B.22) always has a higher order function of ϵ_1 than the left hand side. Furthermore $b_{2,m,n} = 0$ for all $m \neq n$ whenever $\epsilon_2 = 0$, so the right hand side is bounded. We conjecture using a computer and the standard perturbation series for $b_{j,m,n}(\epsilon_0,\alpha)$ that the assumption q_1, q_2 and q_3 have the same number of gaps could be removed. However, this is computationally difficult since it has been verified $b_{j,m,n}(\epsilon_0,\alpha)$ is $\mathcal{O}(\epsilon_j^{|m-n|})$ for all m up to some sufficiently large values of m and n.

For the case with $j \geq 3$, the invariants are computed almost exactly the same way as in [4] because the form of the invariants coincides for these indices. In this case we have that

$$\Phi_{j,m}(\epsilon_0,\alpha) = c_{1,2,j}a^1_{mp_j}a^2_{mr_j}\cos(2\alpha_{j,m}) + D \tag{B.23}$$

References

[1] V. Matveev, A. Its, A class of solutions of the Korteweg–de Vries equation, Probl. Math. Phys. 79 (9) (1976).
[2] K. Cai, Dispersion for Schrödinger operators with one-gap periodic potentials on $\mathbb{R}^1$, Dyn. Partial Differ. Equ. 3 (1) (2006) 71–92.
[3] G. Eskin, J. Ralston, E. Trubowitz, On isospectral periodic potentials in $\mathbf{R}^n$, Commun. Pure Appl. Math. 37 (6) (1984) 715–753.
[4] G. Eskin, J. Ralston, E. Trubowitz, On isospectral periodic potentials in $\mathbf{R}^n$. ii, Commun. Pure Appl. Math. 37 (5) (1984) 647–676.
[5] J. Garnett, E. Trubowitz, Gaps and bands of one-dimensional periodic Schrödinger operators, Comment. Math. Helv. 59 (2) (1984) 258–312.
[6] C. Gordon, T. Kappeler, On isospectral potentials on tori, Duke Math. J. 63 (2) (1991) 217–233.
[7] C. Gordon, T. Kappeler, On isospectral potentials on flat tori ii, Commun. Partial Differ. Equ. 20 (3–4) (1995) 709–728.
[8] Harry Hochstadt, On the determination of a Hill's equation from its spectrum, Arch. Ration. Mech. Anal. 19 (1965) 353–362.
[9] T. Kappeler, M. Makarov, On Birkhoff coordinates for KdV, Ann. Henri Poincaré 2 (2001) 806–856.
[10] E. Korotyaev, Estimates for the Hill operator, J. Differ. Equ. 162 (1) (2000) 1–26.
[11] E. Korotyaev, Estimates for the Hill operator, ii, J. Differ. Equ. 223 (2006) 229–260.
[12] Wilhelm Magnus, Stanley Winkler, Hill's Equation, Dover Publications Inc., New York, 1979. Corrected reprint of the 1966 edition.
[13] H.P. McKean, E. Trubowitz, Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points, Commun. Pure Appl. Math. 29 (2) (1976) 143–226.
[14] J. Poschel, E. Trubowitz, Inverse Spectral Theory, Academic Press [Harcourt Brace Jovanovich Publishers], 1987.
[15] Elias M. Stein, Rami Shakarchi, Complex Analysis, Princeton Lectures in Analysis, II, Princeton University Press, Princeton, NJ, 2003.
[16] E. Trubowitz, The inverse problem for periodic potentials, Commun. Pure Appl. Math. 30 (3) (1977) 321–337.