Centro-affine normal flows on curves: Harnack estimates and ancient solutions

Mohammad N. Ivaki

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8–10, 1040 Wien, Austria

Abstract

We prove that the only compact, origin-symmetric, strictly convex ancient solutions of the planar p centro-affine normal flows are contracting origin-centered ellipses.

© 2014 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

1. Introduction

The setting of this paper is the two-dimensional Euclidean space, $\mathbb{R}^2$. A compact convex subset of $\mathbb{R}^2$ with non-empty interior is called a *convex body*. The set of smooth, strictly convex bodies in $\mathbb{R}^2$ is denoted by $\mathcal{K}$. Write $\mathcal{K}_0$ for the set of smooth, strictly convex bodies whose interiors contain the origin of the plane.

Let K be a smooth, strictly convex body $\mathbb{R}^2$ and let $X_K : \partial K \to \mathbb{R}^2$ be a smooth embedding of ∂K, the boundary of K. Write $\mathbb{S}^1$ for the unit circle and write $\nu : \partial K \to \mathbb{S}^1$ for the Gauss map of ∂K. That is, at each point $x \in \partial K$, $\nu(x)$ is the unit outwards normal at x. The support function of $K \in \mathcal{K}_0$ as a function on the unit circle is defined by $s(z) := \langle X(\nu^{-1}(z)), z \rangle$, for each $z \in \mathbb{S}^1$. We denote the curvature of ∂K by κ which as a function on ∂K is related to the support function by

$$\frac{1}{\kappa(\nu^{-1}(z))} := \mathfrak{r}(z) = \frac{\partial^2}{\partial \theta^2} s(z) + s(z).$$

Here and afterwards, we identify $z = (\cos\theta, \sin\theta)$ with θ. The function $\mathfrak{r}$ is called the radius of curvature. The affine support function of K is defined by $\sigma : \partial K \to \mathbb{R}$ and $\sigma(x) := s(\nu(x))\mathfrak{r}^{1/3}(\nu(x))$. The affine support function is invariant under the group of special linear transformations, $SL(2)$, and it plays a basic role in our argument.

Let $K \in \mathcal{K}_0$. A family of convex bodies $\{K_t\}_t \subset \mathcal{K}_0$ given by the smooth map $X : \partial K \times [0, T) \to \mathbb{R}^2$ is said to be a solution to the p centro-affine normal flow, in short p-flow, with the initial data X_K, if the following evolution equation is satisfied:

$$\partial_t X(x, t) = -\left(\frac{\kappa(x, t)}{\langle X(x, t), \nu(x, t) \rangle^3} \right)^{\frac{p}{p+2} - \frac{1}{3}} \kappa^{\frac{1}{3}}(x, t) \nu(x, t), \qquad X(\cdot, 0) = X_K, \tag{1.1}$$

for a fixed $1 < p < \infty$. In this equation, $0 < T < \infty$ is the maximal time that the solution exists, and $\nu(x, t)$ is the unit normal to the curve $X(\partial K, t) = \partial K_t$ at $X(x, t)$. This family of flows for $p > 1$ was defined by Stancu [12]. The case $p = 1$ is the well-known affine normal flow whose asymptotic behavior was investigated by Sapiro and Tannenbaum [11], and by Andrews in a more general setting [2,4]: Any convex solution to the affine normal flow, after appropriate rescaling converges to an ellipse in the $\mathcal{C}^\infty$ norm. For $p > 1$, similar result was obtained with smooth, origin-symmetric, strictly convex initial data by the author and Stancu [7,8]. Moreover, ancient solutions of the affine normal flow have been also classified: the only compact, convex ancient solutions of the affine normal flow are contracting ellipsoids. This result in $\mathbb{R}^n$, for $n \geq 3$, was proved by Loftin and Tsui [9] and in dimension two by S. Chen [5], and also by the author with a different method. We recall that a solution of flow is called an ancient solution if it exists on $(-\infty, T)$. Here we classify compact, origin-symmetric, strictly convex ancient solutions of the planar p centro-affine normal flows:

Theorem. *The only compact, origin-symmetric, strictly convex ancient solutions of the p-flows are contracting origin-centered ellipses.*

Throughout this paper, we consider origin-symmetric solutions.

2. Harnack estimate

In this section, we follow [1] to obtain the Harnack estimates for p-flows.

Proposition. *Under the flow (1.1) we have* $\partial_t(s^{1 - \frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}} t^{\frac{p}{2p+2}}) \geq 0$.

Proof. For simplicity we set $\alpha = -\frac{p}{p+2}$. To prove the proposition, using the parabolic maximum principle we prove that the quantity defined by

$$\mathcal{R} := t\mathcal{P} - \frac{\alpha}{\alpha - 1} s^{1+3\alpha} \mathfrak{r}^\alpha \tag{2.1}$$

remains negative as long as the flow exists. Here $\mathcal{P}$ is defined as follows

$$\mathcal{P} := \partial_t \left(-s^{1+3\alpha} \mathfrak{r}^\alpha \right).$$

Lemma 2.1. *(See [7].)*

- $\partial_t s = -s^{1+3\alpha} \mathfrak{r}^\alpha,$
- $\partial_t \mathfrak{r} = -[(s^{1+3\alpha} \mathfrak{r}^\alpha)_{\theta\theta} + s^{1+3\alpha} \mathfrak{r}^\alpha].$

Using the evolution equations of s and $\mathfrak{r}$ we find

$$\mathcal{P} = (1 + 3\alpha) s^{1+6\alpha} \mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \left[\left(s^{1+3\alpha} \mathfrak{r}^\alpha \right)_{\theta\theta} + s^{1+3\alpha} \mathfrak{r}^\alpha \right]$$

$$:= (1 + 3\alpha) s^{1+6\alpha} \mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \mathcal{Q}. \tag{2.2}$$

Lemma 2.2. *We have the following evolution equation for $\mathcal{P}$ as long as the flow exists:*

$$\partial_t \mathcal{P} = -\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta} + \mathcal{P}] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha} \mathfrak{r}^{3\alpha}$$

$$+ \left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right] s^{3\alpha} \mathfrak{r}^{\alpha} \mathcal{P} - \frac{\alpha-1}{\alpha} \frac{\mathcal{P}^2}{s^{1+3\alpha} \mathfrak{r}^{\alpha}}.$$

Proof. We repeatedly use the evolution equation of s and $\mathfrak{r}$ given in Lemma 2.1.

$$\partial_t \mathcal{P}$$
$$= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha} \mathfrak{r}^{3\alpha} - 2\alpha(1+3\alpha)s^{1+6\alpha} \mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha} \mathfrak{r}^{\alpha}\right]$$
$$- \alpha(1+3\alpha)s^{1+6\alpha} \mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha} \mathfrak{r}^{\alpha}\right]$$
$$- \alpha(\alpha-1)s^{1+3\alpha} \mathfrak{r}^{\alpha-2}\left[\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha} \mathfrak{r}^{\alpha}\right]^2 - \alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta} + \mathcal{P}]$$
$$= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha} \mathfrak{r}^{3\alpha} - 3\alpha(1+3\alpha)s^{1+6\alpha} \mathfrak{r}^{2\alpha-1} \mathcal{Q}$$
$$- \alpha(\alpha-1)s^{1+3\alpha} \mathfrak{r}^{\alpha-2} \mathcal{Q}^2 - \alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta} + \mathcal{P}].$$

By the definition of $\mathcal{Q}$, (2.2), we have

$$\mathcal{Q}^2 = \frac{\mathcal{P}^2}{\alpha^2 s^{2+6\alpha} \mathfrak{r}^{2\alpha-2}} - \frac{2(3\alpha+1)}{\alpha^2} \frac{\mathcal{P} \mathfrak{r}^2}{s} + \frac{(3\alpha+1)^2}{\alpha^2} s^{6\alpha} \mathfrak{r}^{2\alpha+2}$$

and

$$\mathcal{Q} = \frac{\mathcal{P} - (1+3\alpha)s^{1+6\alpha} \mathfrak{r}^{2\alpha}}{\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}}.$$

Substituting these expressions into the evolution equation of $\mathcal{P}$ we find that

$$\partial_t \mathcal{P} = -\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta} + \mathcal{P}] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha} \mathfrak{r}^{3\alpha}$$

$$- 3(3\alpha+1)s^{3\alpha} \mathfrak{r}^{\alpha} \mathcal{P} - \frac{\alpha-1}{\alpha} \frac{\mathcal{P}^2}{s^{1+3\alpha} \mathfrak{r}^{\alpha}} + \frac{2(\alpha-1)(3\alpha+1)}{\alpha} s^{3\alpha} \mathfrak{r}^{\alpha} \mathcal{P}.$$

This completes the proof of Lemma 2.2. $\square$

We now proceed to find the evolution equation of $\mathcal{R}$ which is defined by (2.1). First notice that

$$-\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \mathcal{R}_{\theta\theta} = -t\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \mathcal{P}_{\theta\theta} + \frac{\alpha^2}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)_{\theta\theta}.$$

Therefore, by Lemma 2.2 and identity (2.2) we get

$$\partial_t \mathcal{R}$$
$$= -t\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1}[\mathcal{P}_{\theta\theta} + \mathcal{P}] + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha} \mathfrak{r}^{3\alpha}$$

$$+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right] s^{3\alpha} \mathfrak{r}^{\alpha} \mathcal{P} - t\frac{\alpha-1}{\alpha} \frac{\mathcal{P}^2}{s^{1+3\alpha} \mathfrak{r}^{\alpha}} + \mathcal{P} + \frac{\alpha}{\alpha-1} \mathcal{P}$$

$$- \alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \mathcal{R}_{\theta\theta} + t\alpha s^{1+3\alpha} \mathfrak{r}^{\alpha-1} \mathcal{P}_{\theta\theta} - \frac{\alpha^2}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)_{\theta\theta}$$

$$+ \frac{\alpha^2}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right) - \frac{\alpha^2}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha} \mathfrak{r}^{\alpha}\right)$$

$$+ \frac{\alpha(3\alpha+1)}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha} \mathfrak{r}^{2\alpha}\right) - \frac{\alpha(3\alpha+1)}{\alpha-1} s^{1+3\alpha} \mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha} \mathfrak{r}^{2\alpha}\right)$$

$$= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}$$

$$+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P}$$

$$+ \frac{\alpha}{\alpha-1}\mathcal{P} - \frac{\alpha}{\alpha-1}\mathcal{P} - t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}$$

$$= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}$$

$$+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P}$$

$$- t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}.$$

In the last expression, using the definition of $\mathcal{R}$, identity (2.1), we replace $t\mathcal{P}$ by $\mathcal{R} + \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha}$. Therefore, at the point where the maximum of $\mathcal{R}$ is achieved we obtain

$$\partial_t\mathcal{R}$$
$$\leq \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right]$$
$$+ \frac{\alpha}{\alpha-1}\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{2+6\alpha}\mathfrak{r}^{2\alpha} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}$$
$$+ t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha}$$
$$\leq \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right]s^{3\alpha}\mathfrak{r}^{\alpha}\right].$$

To get the last inequality, we used the fact that the terms on the second and third line are negative for $p \geq 1$. Hence, by the parabolic maximum principle and the fact that at the time zero we have $\mathcal{R} \leq 0$, we conclude $\mathcal{R} = t\mathcal{P} - \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha} \leq 0$. Negativity of $\mathcal{R}$ is equivalent to $\partial_t\ln(s^{1+3\alpha}\mathfrak{r}^{\alpha}) \geq \frac{\alpha}{1-\alpha}\frac{1}{t}$ for $t > 0$. From this we infer that $\partial_t(s^{1+3\alpha}\mathfrak{r}^{\alpha}t^{\frac{\alpha}{\alpha-1}}) \geq 0$ for $t > 0$. $\square$

Proposition 2.3. *Ancient solutions of the flow (1.1) satisfy* $\partial_t(s(\frac{1}{\mathfrak{r}s^3})^{\frac{p}{p+2}}) \geq 0$.

Proof. By the Harnack estimate every solution of the flow (1.1) satisfies

$$\partial_t\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) + \frac{p}{2t(p+1)}\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) \geq 0. \tag{2.3}$$

We let the flow starts from a fixed time $t_0 < 0$. So the inequality (2.3) becomes

$$\partial_t\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) + \frac{p}{2(t-t_0)(p+1)}\left(s\left(\frac{1}{\mathfrak{r}s^3}\right)^{\frac{p}{p+2}}\right) \geq 0.$$

Now letting t_0 goes to $-\infty$ proves the claim. $\square$

Corollary 2.4. *Every ancient solution of the flow (1.1) satisfies* $\partial_t(s\mathfrak{r}^{\frac{1}{3}}) \leq 0$.

Proof. The $s(\cdot, t)$ is decreasing on the time interval $(-\infty, 0]$. The claim now follows from the previous proposition. $\square$

3. Affine differential setting

We will recall several definitions from affine differential geometry. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedded strictly convex curve with the curve parameter θ. Define $\mathfrak{g}(\theta) := [\gamma_\theta, \gamma_{\theta\theta}]^{1/3}$, where for two vectors u, v in $\mathbb{R}^2$, $[u, v]$ denotes the determinant of the matrix with rows u and v. The affine arc-length is defined as

$$\mathfrak{s}(\theta) := \int_0^\theta \mathfrak{g}(\alpha) d\alpha.$$

Furthermore, the affine normal vector $\mathfrak{n}$ is given by $\mathfrak{n} := \gamma_{\mathfrak{s}\mathfrak{s}}$. In the affine coordinate $\mathfrak{s}$, there hold $[\gamma_\mathfrak{s}, \gamma_{\mathfrak{s}\mathfrak{s}}] = 1$, $\sigma = [\gamma, \gamma_\mathfrak{s}]$, and $\sigma_{\mathfrak{s}\mathfrak{s}} + \sigma\mu = 1$, where $\mu = [\gamma_{\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}]$ is the affine curvature.

We can express the area of $K \in \mathcal{K}$, denoted by $A(K)$, in terms of affine invariant quantities:

$$A(K) = \frac{1}{2} \int_{\partial K} \sigma \, d\mathfrak{s}.$$

The p-affine perimeter of $K \in \mathcal{K}_0$ (for $p = 1$ the assumption $K \in \mathcal{K}_0$ is not necessary and we may take $K \in \mathcal{K}$), denoted by $\Omega_p(K)$, is defined as

$$\Omega_p(K) := \int_{\partial K} \sigma^{1 - \frac{3p}{p+2}} d\mathfrak{s},$$

[10]. We call the quantity $\Omega_p^{2+p}(K)/A^{2-p}(K)$, the p-affine isoperimetric ratio and mention that it is invariant under $GL(2)$. Moreover, for $p > 1$ the p-affine isoperimetric inequality states that if K has its centroid at the origin, then

$$\frac{\Omega_p^{2+p}(K)}{A^{2-p}(K)} \leq 2^{p+2}\pi^{2p} \tag{3.1}$$

and equality cases are obtained only for origin-centered ellipses. In the final section, we will use the 2-affine isoperimetric inequality.

Let $K \in \mathcal{K}_0$. The polar body of K, denoted by K^*, is a convex body in $\mathcal{K}_0$ defined by

$$K^* = \left\{ y \in \mathbb{R}^2 \mid \langle x, y \rangle \leq 1, \ \forall x \in K \right\}.$$

The area of K^*, denoted by $A^* = A(K^*)$, can be represented in terms of affine invariant quantities:

$$A^* = \frac{1}{2} \int_{\partial K} \frac{1}{\sigma^2} d\mathfrak{s} = \frac{1}{2} \int_{\mathbb{S}^1} \frac{1}{s^2} d\theta.$$

Let $K \in \mathcal{K}_0$. We consider a family of convex bodies $\{K_t\}_t \subset \mathcal{K}$, given by the smooth embeddings $X : \partial K \times [0, T) \to \mathbb{R}^2$, which are evolving according to (1.1). Then up to a time-dependant diffeomorphism, $\{K_t\}_t$ evolves according to

$$\frac{\partial}{\partial t} X := \sigma^{1 - \frac{3p}{p+2}} \mathfrak{n}, \qquad X(\cdot, 0) = X_K(\cdot). \tag{3.2}$$

Therefore, classification of compact, origin-symmetric ancient solutions to (1.1) is equivalent to the classification of compact, origin-symmetric ancient solutions to (3.2). In what follows our reference flow is the evolution equation (3.2).

Notice that as a family of convex bodies evolve according to the evolution equation (3.2), in the Gauss parametrization their support functions and radii of curvature evolve according to Lemma 2.1. Assume Q and $\bar{Q}$ are two smooth functions $Q : \partial K \times [0, T) \to \mathbb{R}$, $\bar{Q} : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ that are related by $Q(x, t) = \bar{Q}(\nu(x, t), t)$. It can be easily verified that

$$\partial_t \bar{Q} = \partial_t Q - Q_\mathfrak{s}\left(\sigma^{1 - \frac{3p}{p+2}}\right)_\mathfrak{s}.$$

In particular, for ancient solutions of (3.2), in views of Corollary 2.4, $Q = \sigma$ must satisfy $0 \geq \partial_t \sigma - \sigma_\mathfrak{s}(\sigma^{1 - \frac{3p}{p+2}})_\mathfrak{s}$. The preceding argument proves the next proposition.

Proposition 3.1. *Every ancient solution satisfies* $\partial_t \sigma \leq -(\frac{3p}{p+2} - 1)\sigma_{\mathfrak{s}}^2 \sigma^{-\frac{3p}{p+2}}$.

The next two lemmas were proved in [7].

Lemma 3.2. *(See [7, Lemma 3.1].) The following evolution equations hold:*

(1) $\frac{\partial}{\partial t}\sigma = \sigma^{1-\frac{3p}{p+2}}(-\frac{4}{3} + (\frac{p}{p+2} + 1)(1 - \frac{3p}{p+2})\frac{\sigma_{\mathfrak{s}}^2}{\sigma} + \frac{p}{p+2}\sigma_{\mathfrak{s}\mathfrak{s}})$,

(2) $\frac{d}{dt}A = -\Omega_p$.

Lemma 3.3. *(See [7, Section 6].) The following evolution equation for Ω_l holds for every $l \geq 2$ and $p \geq 1$:*

$$\frac{d}{dt}\Omega_l(t) = \frac{2(l-2)}{l+2}\int_{\gamma_t}\sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}}d\mathfrak{s} + \frac{18pl}{(l+2)^2(p+2)}\int_{\gamma_t}\sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}, \tag{3.3}$$

where $\gamma_t := \partial K_t$ is the boundary of K_t.

Lemma 3.4. *(See [12].) The area product, $A(t)A^*(t)$, and the p-affine isoperimetric ratio are both non-decreasing along (3.2).*

Write respectively $\max_{\gamma_t}\sigma$ and $\min_{\gamma_t}\sigma$ for σ_M and σ_m.

Lemma 3.5. *There is a constant $0 < c < \infty$ such that $\frac{\sigma_M}{\sigma_m} \leq c$ on $(-\infty, 0]$.*

Proof. By Corollary 3.1 and part (1) of Lemma 3.2 we have

$$-\left(\frac{3p}{p+2} - 1\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3} \geq \frac{\partial_t \sigma}{\sigma^{3-\frac{3p}{p+2}}}$$

$$= -\frac{4}{3\sigma^2} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3} + \frac{p}{p+2}\frac{\sigma_{\mathfrak{s}\mathfrak{s}}}{\sigma^2}. \tag{3.4}$$

Integrating the inequality (3.4) against $d\mathfrak{s}$ we obtain

$$\frac{4}{3}\int_{\gamma_t}\frac{1}{\sigma^2}d\mathfrak{s} \geq \frac{p}{p+2}\left(2 - \frac{3p}{p+2}\right)\int_{\gamma_t}\frac{\sigma_{\mathfrak{s}}^2}{\sigma^3}d\mathfrak{s}$$

$$= \frac{p}{p+2}\left(3 - \frac{3p}{p+2}\right)\int_{\gamma_t}\frac{(\ln\sigma)_{\mathfrak{s}}^2}{\sigma}d\mathfrak{s}$$

$$\geq \frac{p}{p+2}\left(3 - \frac{3p}{p+2}\right)\frac{(\int_{\gamma_t}|(\ln\sigma)_{\mathfrak{s}}|d\mathfrak{s})^2}{\int_{\gamma_t}\sigma d\mathfrak{s}}. \tag{3.5}$$

Set $d_p = \frac{p}{p+2}(3 - \frac{3p}{p+2})$. Applying the Hölder inequality to the left-hand side and the right-hand side of inequality (3.5) yields

$$\left(\int_{\gamma_t}|(\ln\sigma)_{\mathfrak{s}}|d\mathfrak{s}\right)^2 \leq d_p' A^*(t)A(t),$$

for a new positive constant d_p'. Here we used the identities $\int_{\gamma_t}\frac{1}{\sigma^2}d\mathfrak{s} = 2A^*(t)$ and $\int_{\gamma_t}\sigma d\mathfrak{s} = 2A(t)$. Now by Lemma 3.4 we have $A(t)A^*(t) \leq A(0)A^*(0)$. This implies that

$$\left(\ln\frac{\sigma_M}{\sigma_m}\right)^2 \leq d_p'',$$

for a new positive constant d_p''. Therefore, on $(-\infty, 0]$ we find that

$$\frac{\sigma_M}{\sigma_m} \leq c \tag{3.6}$$

for some positive constant c. $\square$

Let $\{K_t\}_t$ be a solution of (3.2). Then the family of convex bodies, $\{\tilde{K}_t\}_t$, defined by

$$\tilde{K}_t := \sqrt{\frac{\pi}{A(K_t)}} K_t$$

is called a normalized solution to the p-flow, equivalently a solution that the area is fixed and is equal to π.

Furnish every quantity associated with the normalized solution with an over-tilde. For example, the support function, curvature, and the affine support function of $\tilde{K}$ are denoted by $\tilde{s}$, $\tilde{\kappa}$, and $\tilde{\sigma}$, respectively.

Lemma 3.6. *There is a constant $0 < c < \infty$ such that on the time interval $(-\infty, 0]$ we have*

$$\frac{\tilde{\sigma}_M}{\tilde{\sigma}_m} \leq c. \tag{3.7}$$

Proof. The estimate (3.6) is scaling invariant, so the same estimate holds for the normalized solution. $\square$

Lemma 3.7. $\Omega_2(t)$ *is non-decreasing along the p-flow. Moreover, we have*

$$\frac{d}{dt}\Omega_2(t) \geq \frac{9p}{4(p+2)} \int_{\gamma_t} \sigma^{-\frac{3p}{p+2}-\frac{3}{2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

Proof. Use the evolution equation (3.3) for $l = 2$. $\square$

Corollary 3.8. *There exists a constant $0 < b_p < \infty$ such that*

$$\frac{1}{\Omega_2^4(t)} < b_p$$

on $(-\infty, 0]$.

Proof. Notice that $\Omega_2(t) = (\int_{\partial \gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s})$ is a $GL(2)$ invariant quantity. Therefore, we need only to prove the claim after applying appropriate $SL(2)$ transformations to the normalized solution of the flow. By the estimate (3.7) and the facts that $\Omega_2(\tilde{K}_t)$ is non-decreasing and $A(\tilde{K}_t) = \pi$ we have

$$\frac{c^{\frac{3}{2}}}{2}\tilde{\sigma}_m^{\frac{3}{2}}(t)\tilde{\Omega}_2(0) \geq \frac{1}{2}\tilde{\sigma}_M^{\frac{3}{2}}(t)\tilde{\Omega}_2(0) \geq \frac{1}{2}\tilde{\sigma}_M^{\frac{3}{2}}(t)\tilde{\Omega}_2(t) \geq \tilde{A}(t) = \pi.$$

So we get $(\tilde{s}\tilde{\mathfrak{r}}^{1/3})(t) \geq a > 0$ on $(-\infty, 0]$, for an a independent of t. Moreover, as the affine support function is invariant under $SL(2)$ we may further assume, after applying a length minimizing special linear transformation at each time, that $\tilde{s}(t) < a' < \infty$, for an a' independent of t. Therefore

$$\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)} = \frac{(\int_{\mathbb{S}^1} \tilde{\mathfrak{r}}^{2/3} d\theta)^3}{\pi} > a'' > 0, \tag{3.8}$$

for an a'' independent of t. Now the claim follows from the Hölder inequality:

$$\left(\int_{\gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s}\right) \Omega_1^{\frac{1}{2}}(t) A^{\frac{1}{2}}(t) \geq \int_{\gamma_t} \sigma^{-\frac{1}{2}} d\mathfrak{s} \int_{\gamma_t} \sigma^{\frac{1}{2}} d\mathfrak{s} \geq \Omega_1^2(t),$$

so

$$\tilde{\Omega}_2(t) = \Omega_2(t) \geq \left(\frac{\Omega_1^3(t)}{A(t)}\right)^{\frac{1}{2}} = \left(\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)}\right)^{\frac{1}{2}}. \qquad \square$$

Corollary 3.9. *As K_t evolve by (3.2), then the following limit holds:*

$$\liminf_{t \to -\infty} \left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right) \int_{\gamma_t} \left(\sigma^{\frac{1}{4} - \frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} = 0. \tag{3.9}$$

Proof. Suppose on the contrary that there exists an $\varepsilon > 0$ small enough, such that

$$\left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right) \int_{\gamma_t} \left(\sigma^{\frac{1}{4} - \frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \geq \varepsilon \frac{\left(\frac{1}{4} - \frac{3p}{2(p+2)}\right)^2}{\frac{9p}{p+2}}$$

on $(-\infty, -N]$ for N large enough. Then $\frac{d}{dt}\frac{1}{\Omega_2^4(t)} \leq \varepsilon \frac{d}{dt}\ln(A(t))$. So by integrating this last inequality against dt and by Corollary 3.8 we get

$$0 < \frac{1}{\Omega_2^4(-N)} \leq \frac{1}{\Omega_2^4(t)} + \varepsilon \ln(A(-N)) - \varepsilon \ln(A(t))$$

$$< b_p + \varepsilon \ln(A(-N)) - \varepsilon \ln(A(t)).$$

Letting $t \to -\infty$ we reach to a contradiction: $\lim_{t \to -\infty} A(t) = +\infty$, that is, the right-hand side becomes negative for large values of t. $\square$

Corollary 3.10. *For a sequence of times $\{t_k\}$ as t_k converge to $-\infty$ we have*

$$\lim_{t_k \to -\infty} \tilde{\sigma}(t_k) = 1.$$

Proof. Notice that the quantity $\left(\frac{A(t)}{\Omega_p(t)\Omega_2^5(t)}\right) \int_{\gamma_t} \left(\sigma^{\frac{1}{4} - \frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}$ is scaling invariant and $\frac{\tilde{A}(t)}{\tilde{\Omega}_p(t)\tilde{\Omega}_2^5(t)}$ is bounded from below (by Lemmas 3.4 and 3.7, $\tilde{\Omega}_p(t) \leq \tilde{\Omega}_p(0)$ and $\tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0)$). Thus Corollary 3.9 implies that there exists a sequence of times $\{t_k\}_{k \in \mathbb{N}}$, such that $\lim_{k \to \infty} t_k = -\infty$ and

$$\lim_{t_k \to -\infty} \int_{\tilde{\gamma}_{t_k}} \left(\tilde{\sigma}^{\frac{1}{4} - \frac{3p}{2(p+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}} = 0.$$

On the other hand, by the Hölder inequality

$$\frac{\left(\tilde{\sigma}_M^{\frac{1}{4} - \frac{3p}{2(p+2)}}(t_k) - \tilde{\sigma}_m^{\frac{1}{4} - \frac{3p}{2(p+2)}}(t_k)\right)^2}{\tilde{\Omega}_1(t_k)} \leq \int_{\tilde{\gamma}_{t_k}} \left(\tilde{\sigma}^{\frac{1}{4} - \frac{3p}{2(p+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}}.$$

Moreover, $\tilde{\Omega}_1(t)$ is bounded from above: Indeed $\left(\frac{\tilde{\Omega}_1^3(t)}{\tilde{A}(t)}\right)^{\frac{1}{2}} \leq \tilde{\Omega}_2(t) \leq \tilde{\Omega}_2(0)$. Therefore, we find that

$$\lim_{t_k \to -\infty} \left(\tilde{\sigma}_M^{\frac{1}{4} - \frac{3p}{2(p+2)}}(t_k) - \tilde{\sigma}_m^{\frac{1}{4} - \frac{3p}{2(p+2)}}(t_k)\right)^2 = 0.$$

Since $\tilde{\sigma}_m \leq 1$ and $\tilde{\sigma}_M \geq 1$ (see [3, Lemma 10]) the claim follows. $\square$

4. Proof of the main theorem

Proof. For each time $t \in (-\infty, T)$, let $T_t \in SL(2)$ be a special linear transformation that the maximal ellipse contained in $T_t \tilde{K}_t$ is a disk. So by John's ellipsoid lemma we have

$$\frac{1}{\sqrt{2}} \le s_{T_t \tilde{K}_t} \le \sqrt{2}.$$

Then by the Blaschke selection theorem, there is a subsequence of times, denoted again by $\{t_k\}$, such that $\{T_{t_k} \tilde{K}_{t_k}\}$ converges in the Hausdorff distance to an origin-symmetric convex body $\tilde{K}_{-\infty}$, as $t_k \to -\infty$. By Corollary 3.10, and by the weak convergence of the Monge–Ampère measures, the support function of $\tilde{K}_{-\infty}$ is the generalized solution of the following Monge–Ampère equation on $\mathbb{S}^1$:

$$s^3(s_{\theta\theta} + s) = 1$$

Therefore, by Lemma 8.1 of Petty [6], $\tilde{K}_{-\infty}$ is an origin-centered ellipse. This in turn implies that $\lim_{t \to -\infty} \tilde{\Omega}_2(t_k) = 2\pi$. On the other hand, by the 2-affine isoperimetric inequality, (3.1), and by Lemma 3.7, for $t \in (\infty, 0]$ we have

$$2\pi \ge \tilde{\Omega}_2(t) \ge \lim_{t_k \to -\infty} \tilde{\Omega}_2(t_k) = 2\pi.$$

Thus $\frac{d}{dt}\tilde{\Omega}_2(t) \equiv 0$ on $(-\infty, 0]$. Hence, in view of Lemma 3.7, K_t is an origin-centered ellipse for every time $t \in (-\infty, T)$. $\quad\square$

Conflict of interest statement

The author declares that there are no conflicts of interest.

References

[1] B. Andrews, Harnack inequalities for evolving hypersurfaces, Math. Z. 217 (1994) 179–197.
[2] B. Andrews, Contraction of convex hypersurfaces by their affine normal, J. Differ. Geom. 43 (1996) 207–230.
[3] B. Andrews, Affine curve-lengthening flow, J. Reine Angew. Math. 506 (1999) 43–83.
[4] B. Andrews, Motion of hypersurfaces by Gauss curvature, Pac. J. Math. 195 (1) (2000) 1–34.
[5] S. Chen, Classifying convex compact ancient solutions to the affine curve shortening flow, J. Geom. Anal. (2013), http://dx.doi.org/10.1007/s12220-013-9456-z.
[6] C.M. Petty, Affine isoperimetric problems, in: Discrete Geometry and Convexity, Ann. N.Y. Acad. Sci. 440 (1985) 113–127, http://dx.doi.org/10.1111/j.1749-6632.1985.tb14545.x.
[7] M.N. Ivaki, Centro-affine curvature flows on centrally symmetric convex curves, Trans. Am. Math. Soc. (2014), in press, arXiv:1205.6456v2.
[8] M.N. Ivaki, A. Stancu, Volume preserving centro-affine normal flows, Commun. Anal. Geom. 21 (2013) 671–685.
[9] J. Loftin, M.P. Tsui, Ancient solutions of the affine normal flow, J. Differ. Geom. 78 (2008) 113–162.
[10] E. Lutwak, The Brunn–Minkowski–Fiery theory II: affine and geominimal surface areas, Adv. Math. 118 (1996) 244–294.
[11] G. Sapiro, A. Tannenbaum, On affine plane curve evolution, J. Funct. Anal. 119 (1994) 79–120.
[12] A. Stancu, Centro-affine invariants for smooth convex bodies, Int. Math. Res. Not. 2012 (10) (2012) 2289–2320.