

Nonlinear Schrödinger equation with a point defect

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Abstract

We study the nonlinear Schrödinger equation with a delta-function impurity in one space dimension. Local well-posedness is verified for the Cauchy problem in $H^1(\mathbb{R})$. In case of attractive delta-function, orbital stability and instability of the ground state is proved in $H^1(\mathbb{R})$.

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Résumé

On étudie l'équation de Schrödinger non linéaire avec une impureté delta de Dirac en dimension 1. On vérifie que le problème de Cauchy est localement bien posé dans $H^1(\mathbb{R})$. Dans le cas de delta de Dirac attractif, la stabilité et l'instabilité orbitale des ondes stationnaires sont vérifiées dans $H^1(\mathbb{R})$.

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1. Introduction

In this paper we study nonlinear Schrödinger equations of the form

$$i\partial_t u + \frac{1}{2}D^2 u + Z\delta u = -|u|^{p-1}u, \quad (1.1)$$

where u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $D = \partial/\partial x$, δ is the Dirac measure at the origin, $Z \in \mathbb{R}$, and $1 \leq p < \infty$. For $Z \neq 0$, the equations of the form (1.1) arise in a wide variety of physical models with a point defect on the line [13] and references therein. In spite of a large literature on (1.1) with $Z = 0$ [25], there seems only a few mathematical studies in (1.1) with $Z \neq 0$ [13,17] available so far.

To be more specific, it was shown in [13] that the Cauchy problem is globally well-posed in $H^1(\mathbb{R}) = (1 - D^2)^{-1/2}L^2(\mathbb{R})$ in the case where $Z > 0$ and $p = 3$. Moreover, the authors in [13] studied the stability of nonlinear bound states u_{Def} given by

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$$u_{\text{Def}}(x) = \sqrt{2\omega} e^{i\omega t} \operatorname{sech}\left(\sqrt{2\omega}|x| + \tanh^{-1} \frac{Z}{\sqrt{2\omega}}\right) \tag{1.2}$$

with $\sqrt{2\omega} > Z$ and $\omega > 0$ in the orbitally Lyapunov sense in H^1 in the case where $Z > 0$ and $p = 3$.

The purpose in this paper is to generalize those results and to compare our results with the available results with $Z = 0$. We note that the main topic addressed in [13] is an interaction of solitons and defect. In this perspective, we also refer to [17].

The following proposition is concerned with the well-posedness of Eq. (1.1) in $H^1(\mathbb{R})$.

Proposition 1. *For any $u_0 \in H^1$, there exists $T > 0$ and a unique solution $u \in C([0, T]; H^1) \cap C^1([0, T]; H^{-1})$ of (1.1) with $u(0) = u_0$ such that either $T = \infty$ or $T < \infty$ and $\lim_{t \rightarrow T} \|Du\|_{L^2} = \infty$. Moreover, u satisfies the conservation of the energy and the charge:*

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \tag{1.3}$$

for all $t \in [0, T)$, where

$$E(v) = \frac{1}{4} \|Dv\|_{L^2}^2 - \frac{Z}{2} \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}, \quad Q(v) = \frac{1}{2} \|v\|_{L^2}^2.$$

This proposition follows from Theorem 3.7.1 in [4]. We will note briefly how to check it later on. The global well-posedness of the Cauchy problem holds in $H^1(\mathbb{R})$ for any p with $1 < p < 5$ by Gagliardo Nirenberg inequality and the conservation laws.

Remark 1.1. We recall the definition of the self-adjoint operator H as the precise formulation of a formal expression $-(1/2)D^2 - Z\delta$.

$$Hu = -\frac{1}{2}D^2u, \quad u \in \text{Dom}(H),$$

where

$$\begin{aligned} \text{Dom}(H) &= \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}): Du(0+) - Du(0-) = -2Zu(0)\}, \\ H^m(I) &= \{u \in L^2(I): D^j u \in L^2 \text{ for all with } 0 \leq j \leq m\}, \quad I \subset \mathbb{R}. \end{aligned}$$

All self-adjoint extensions of $\dot{H} \equiv -(1/2)D^2$ with domain

$$\text{Dom}(\dot{H}) = \{u \in H^2(\mathbb{R}): u(0) = 0\}$$

are parametrized by H with $Z \in [-\infty, +\infty)$ (see [1]).

Nonlinear bound states mean the solutions to (1.1) having the form $u_\omega(t, x) = e^{i\omega t} \phi_\omega(x)$, where $\omega > 0$ is the frequency and ϕ_ω should satisfy the following semilinear elliptic equations:

$$-\frac{1}{2}D^2\phi + \omega\phi - Z\delta\phi = |\phi|^{p-1}\phi, \quad x \in \mathbb{R}, \quad Z \in \mathbb{R}. \tag{1.4}$$

There exists a unique positive symmetric solution of (1.4) which is explicitly described as:

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{\sqrt{2}}|x| + \tanh^{-1}\left(\frac{Z}{\sqrt{2\omega}}\right)\right) \right\}^{\frac{1}{p-1}} \tag{1.5}$$

if $\sqrt{2\omega} > |Z|$. Precisely, this solution is constructed from the solution with $Z = 0$ on each side of the defect pasted together at $x = 0$ to satisfy the conditions of continuity and the jump condition in the first derivative at $x = 0$, $Du(0+) - Du(0-) = -2Zu(0)$.

In case of $Z = 0$ it is unique for $\omega > 0$ up to translations, which, we denote by $\psi_\omega(x)$. Orbital stability for the case of $Z = 0$ has been well studied (see [2,4,5,7,14,15,26,27]). Cazenave and Lions [5] proved that $e^{i\omega t} \psi_\omega(x)$ is stable for any $\omega > 0$ if $p < 5$. On the other hand, it was shown that $e^{i\omega t} \psi_\omega(x)$ is unstable for any $\omega > 0$ if $p \geq 5$ (see Berestycki and Cazenave [2] for $p > 5$, and Weinstein [26] for $p = 5$).

As we have mentioned, Goodman, Holmes and Weinstein [13] claimed in the case where $Z > 0$ and $p = 3$ that (1.2) are nonlinearly orbitally Lyapunov stable by the same method as that of Rose and Weinstein [22], Weinstein [26]. More exactly, the authors established a variational characterization for (1.2) and proved the stability using the bifurcation from the linear mode, i.e., the corresponding eigenfunction to the eigenvalue $\lambda = -Z^2/2$. They also remark that as $\omega \rightarrow \infty$, (1.2) looks more and more like the solitary wave of (1.1) with $Z = 0$ and $\omega = 1$. However, we address in this article a different point from the case $Z = 0$.

The notion of the stability and instability in this paper is formulated as follows.

Definition 1. For $\eta > 0$, we put

$$U_\eta(\phi_\omega) := \left\{ v \in H^1(\mathbb{R}) : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta} \phi_\omega\|_{H^1} < \eta \right\}.$$

We say that a standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (1.1) is stable in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $u_0 \in U_\eta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t} \phi_\omega(x)$ is said to be unstable in $H^1(\mathbb{R})$.

Before we mention our result, we should remark a variational characterization of ϕ_ω for the discussion below. From now on, we will consider only the case of $Z > 0$.

Definition 2. For $Z > 0$ and $\omega > Z^2/2$, we define two C^1 functionals on $H^1(\mathbb{R})$:

$$\begin{aligned} S_\omega(v) &:= E(v) + \omega Q(v), \\ I_\omega(v) &:= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx - \|v\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z |v(0)|^2 - \|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Let \mathcal{G}_ω be the set of all nonnegative minimizers for the minimization problem

$$d(\omega) = \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) = 0 \}. \tag{1.6}$$

The existence of nonnegative minimizers for (1.6) is proved by the standard variational argument. We will briefly show the following proposition in Section 3 for the sake of completeness.

Proposition 2. Let $Z > 0$. For any $\omega > Z^2/2$, the minimization problem (1.6) is attained by a symmetric nonincreasing function vanishing at infinity.

Remark 1.2.

(i) For $Z > 0$, let

$$\lambda = \inf \left\{ \frac{1}{2} \|Dv\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx : \|v\|_{L^2} = 1, v \in H^1(\mathbb{R}) \right\}.$$

Then we have $\lambda = -Z^2/2$ and the corresponding eigenfunction is $\Phi(x) = Ze^{-Z|x|}$.

(ii) We note that

$$I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v \rangle$$

for $\lambda > 0$.

(iii) Let $v_\omega \in \mathcal{G}_\omega$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(v_\omega) = \Lambda I'_\omega(v_\omega)$. Thus, we have $\langle S'_\omega(v_\omega), v_\omega \rangle = \Lambda \langle I'_\omega(v_\omega), v_\omega \rangle$. Since $\langle S'_\omega(v_\omega), v_\omega \rangle = I_\omega(v_\omega) = 0$ and $\langle I'_\omega(v_\omega), v_\omega \rangle = -(p-1) \|v_\omega\|_{L^{p+1}}^{p+1} < 0$, we have $\Lambda = 0$. Namely, v_ω satisfies (1.4). Moreover, for any $v \in H^1(\mathbb{R}) \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of \mathcal{G}_ω , we have $S_\omega(v_\omega) \leq S_\omega(v)$. Namely, $v_\omega \in \mathcal{G}_\omega$ is a ground state (minimal action solution) of (1.4) in $H^1(\mathbb{R})$. It is easy to see that a ground state of (1.4) in $H^1(\mathbb{R})$ is a minimizer of (1.6).

(iv) The minimizer $v \in \mathcal{G}_\omega$ obtained above, which is nonnegative, symmetric and nonincreasing, satisfies the boundary value problem (see Lemma 3.2 below):

$$\begin{aligned} v &\in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad v(x) > 0, \quad x \in \mathbb{R}, \\ -\frac{1}{2}D^2v + \omega v - v^p &= 0, \quad x \neq 0, \\ Dv(0+) - Dv(0-) &= -2Zv(0), \\ Dv(x), v(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

(v) We note that there is no nontrivial solution in $H^1(\mathbb{R})$ when $\omega \leq Z^2/2$.

Remark 1.3. The minimizer obtained in Proposition 2 is precisely the same as ϕ_ω defined by (1.5) since the positive solution with the boundary value problem in Remark 1.2(iv) is uniquely determined.

To prove stability and instability, we use the following sufficient condition originally obtained by Shatah [23] and Shatah and Strauss [24] (see also [9,10] for the proof).

Proposition 3. Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Let $v_\omega \in \mathcal{G}_\omega$. Assume that $\omega \mapsto v_\omega$ is a C^1 mapping.

- (i) If $\partial_\omega \|v_\omega\|_{L^2}^2 > 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t} v_{\omega_0}(x)$ is stable in $H^1(\mathbb{R})$.
- (ii) If $\partial_\omega \|v_\omega\|_{L^2}^2 < 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t} v_{\omega_0}(x)$ is unstable in $H^1(\mathbb{R})$.

Remark 1.4. In case of $Z = 0$, it is easy to verify this condition since we have $\|\psi_\omega\|_{L^2}^2 = \omega^{2/(p-1)-1/2} \|\psi_1\|_{L^2}^2$ by the scaling invariance even in the higher dimensional case. Due to the potential term we lost the scaling invariance in general (see [6,10–12,18,20,28,29] for example). However, in the present one-dimensional case where the potential is a Dirac-delta, we can compute exactly the increase and decrease of L^2 norm of (1.5).

Theorem 1. Let $Z > 0$ and $\omega > Z^2/2$.

- (i) Let $1 < p \leq 5$. Then $e^{i\omega t} \phi_\omega(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/2, \infty)$.
- (ii) Let $p > 5$. Then there exists a unique $\omega_1 > Z^2/2$ such that $e^{i\omega t} \phi_\omega(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/2, \omega_1)$, and that it is unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, \infty)$, where ω_1 is exactly defined as follows:

$$\begin{aligned} \frac{p-5}{p-1} J(\omega_1) &= \frac{Z}{\sqrt{2\omega_1}} \left(1 - \frac{Z^2}{2\omega_1} \right), \\ J(\omega_1) &= \int_{A(\omega_1)}^{\infty} \operatorname{sech}^{4/(p-1)} y \, dy, \quad A(\omega_1) = \tanh^{-1} \left(\frac{Z}{\sqrt{2\omega_1}} \right). \end{aligned}$$

Remark 1.5. Concerning the critical case $\partial_\omega \|\phi_\omega\|_{L^2}^2 = 0$, we conjecture that $e^{i\omega_1 t} \phi_{\omega_1}(x)$ would be unstable in view of the result of Comech and Pelinovsky [7]. For that purpose, the above variational characterizations of ϕ_ω would be useful to investigate the number of nonpositive eigenvalues of the linearized operators around $e^{i\omega t} \phi_\omega(x)$ (see [16,8]).

Remark 1.6. A similar result is known for one dimensional nonlinear Schrödinger equations with $Z = 0$ and with double power nonlinearity (see Ohta [21]).

In Section 2, we give an idea of Proposition 1. In Section 3, we prove Proposition 2 and we complete the proof of Theorem 1 by checking the increase and the decrease of L^2 norm of ϕ_ω as a function of ω . Also, we give the outline of the proof of Proposition 3.

2. Remarks on Proposition 1

In this section, we give some remarks about the verification of Proposition 1.

We apply Theorem 3.7.1 of [4] to our problem. We check the assumptions of Theorem 3.7.1.

First, we remark that H defined in Remark 1.1 satisfies $H \geq -m$, where $m = Z^2/2$, if $Z > 0$ and $m = 0$ if $Z < 0$. Thus, $A = -H - m$ is a self-adjoint operator on $X = L^2(\mathbb{R})$ with domain $\text{Dom}(A) = \text{Dom}(H)$, and $A \leq 0$. Moreover, in our case, we may take $X_A = H^1(\mathbb{R})$ whose norm is equivalent to $H^1(\mathbb{R})$ norm, namely,

$$\|u\|_{X_A}^2 = \frac{1}{2} \|Du\|_{L^2}^2 + (m + 1) \|u\|_{L^2}^2 - Z|u(0)|^2.$$

It is easy to see that the uniqueness of solutions and the conditions (3.7.1), (3.7.3)–(3.7.6) hold choosing $r = \rho = 2$, since we are in one dimensional case.

Lastly, the condition (3.7.2) with $p = 2$ is satisfied for the reason that A is a self-adjoint operator on $L^2(\mathbb{R})$. Here, we note that only the case $p = 2$ in (3.7.2) is needed for our case since we can take $r = \rho = 2$ in (3.7.5).

3. Existence of ground states and proof of Theorem 1

To show Proposition 2, we remark that the following variational problem is equivalent to $d(\omega)$:

$$d_1(\omega) = \inf \left\{ \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) \leq 0 \right\}. \tag{3.1}$$

We will make use of the following lemma by Brézis and Lieb [3].

Lemma 3.1. *Let $2 \leq q < \infty$ and $\{w_j\}$ be a bounded sequence in $L^q(\mathbb{R})$. Assume that $w_j(x) \rightarrow w_0(x)$ a.e. $x \in \mathbb{R}$ as $j \rightarrow \infty$. Then we have*

$$\|w_j\|_{L^q}^q - \|w_j - w_0\|_{L^q}^q - \|w_0\|_{L^q}^q \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

We recall that $\psi_\omega(x)$ is a unique positive symmetric solution of (1.4) with $Z = 0$. It is known that $\psi_\omega(x)$ is a minimizer of

$$\begin{aligned} d^0_\omega &= \inf \{ S^0_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, I^0_\omega(v) = 0 \} \\ &= \inf \left\{ \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} : v \in H^1(\mathbb{R}) \setminus \{0\}, I^0_\omega(v) \leq 0 \right\}, \end{aligned}$$

where

$$\begin{aligned} S^0_\omega(v) &= \frac{1}{4} \|Dv\|_{L^2}^2 + \frac{\omega}{2} \|v\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}, \\ I^0_\omega(v) &= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Proof of Proposition 2. Let $\{v_j\}$ be a minimizing sequence for $d_1(\omega)$, then we have that $\|v_j\|_{H^1}^2$ is bounded. Indeed, v_j satisfies

$$\frac{1}{2} \|Dv_j\|_{L^2}^2 + \omega \|v_j\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v_j(x)|^2 dx \leq C.$$

By (i) of Remark 1.2, $(\omega + \lambda) \|v_j\|_{L^2}^2 \leq C$. Also, by Sobolev embedding,

$$\begin{aligned} \frac{1}{2} \|Dv_j\|_{L^2}^2 &\leq C + Z|v_j(0)|^2 \\ &\leq C + C \|v_j\|_{L^2} \|Dv_j\|_{L^2} \\ &\leq C + C \left(\frac{1}{2\varepsilon} \|v_j\|_{L^2}^2 + \frac{\varepsilon}{2} \|Dv_j\|_{L^2}^2 \right). \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small, we have that $\|Dv_j\|_{L^2}^2$ is bounded and so that $\|v_j\|_{H^1}^2$ is bounded. Thus, there exists $v_0 \in H^1(\mathbb{R})$ such that a subsequence of $\{v_j\}$, which we will denote by the same letter, converges to v_0 weakly in $H^1(\mathbb{R})$. Therefore, $v_j(x)$ converges to $v_0(x)$ a.e. $x \in \mathbb{R}$. Now suppose that $v_0 \equiv 0$. Then we have, $I_\omega^0(v_j) \rightarrow 0$, as $j \rightarrow \infty$. Since $Z > 0$, we have $I_\omega(\psi_\omega) < 0$, and hence we obtain,

$$d_1(\omega) < \frac{p-1}{2(p+1)} \|\psi_\omega\|_{L^{p+1}}^{p+1} = d^0(\omega). \tag{3.2}$$

We set

$$\lambda_j = \left(\frac{\|Dv_j\|_{L^2}^2 + \omega \|v_j\|_{L^2}^2}{\|v_j\|_{L^{p+1}}^{p+1}} \right)^{\frac{1}{p-1}}.$$

We here remark that $\liminf_{j \rightarrow \infty} \|v_j\|_{L^{p+1}}^{p+1} = d_1(\omega) > 0$. It then follows that

$$\lambda_j^{p-1} - 1 = \frac{I_\omega^0(v_j)}{\|v_j\|_{L^{p+1}}^{p+1}} \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

and we see that $I_\omega^0(\lambda_j v_j) = 0$ and $\lambda_j v_j \not\equiv 0$. By the definition of $d^0(\omega)$, we obtain,

$$d^0(\omega) \leq \frac{p-1}{2(p+1)} \lambda_j^{p+1} \|v_j\|_{L^{p+1}}^{p+1} \rightarrow d_1(\omega), \quad \text{as } j \rightarrow \infty,$$

which is a contradiction to (3.2) and we conclude that $v_0 \not\equiv 0$. By Lemma 3.1, we have, as $j \rightarrow \infty$,

$$\|v_j\|_{L^{p+1}}^{p+1} - \|v_j - v_0\|_{L^{p+1}}^{p+1} - \|v_0\|_{L^{p+1}}^{p+1} \rightarrow 0, \tag{3.3}$$

$$I_\omega(v_j) - I_\omega(v_j - v_0) - I_\omega(v_0) \rightarrow 0. \tag{3.4}$$

Now we suppose that $I_\omega(v_0) > 0$. Then, it follows from (3.4) and the fact $I_\omega(v_j) \leq 0$ that $I_\omega(v_j - v_0) < 0$ for sufficiently large j . Accordingly, from the definition of $d_1(\omega)$, we obtain $\frac{p-1}{2(p+1)} \|v_j - v_0\|_{L^{p+1}}^{p+1} \geq d_1(\omega)$ for large j . On the other hand, by (3.3), we have

$$\frac{p-1}{2(p+1)} \|v_0\|_{L^{p+1}}^{p+1} = d_1(\omega) - \frac{p-1}{2(p+1)} \lim_{j \rightarrow \infty} \|v_j - v_0\|_{L^{p+1}}^{p+1},$$

which is a contradiction since $\|v_0\|_{L^{p+1}}^{p+1} > 0$. Therefore, we get $I_\omega(v_0) \leq 0$ and then,

$$d_1(\omega) \leq \frac{p-1}{2(p+1)} \|v_0\|_{L^{p+1}}^{p+1} \leq \frac{p-1}{2(p+1)} \liminf_{j \rightarrow \infty} \|v_j\|_{L^{p+1}}^{p+1} = d_1(\omega).$$

This concludes that v_0 is a minimizer of $d_1(\omega)$. We may verify that the minimizer is nonnegative, symmetric and nonincreasing using a rearrangement inequality of [19, Theorem 3.4]. \square

To assure that the minimizer satisfies the boundary condition and decays at infinity, we prove the following lemma.

Lemma 3.2. *Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Assume that $v \in \mathcal{G}_\omega$. Then v is symmetric, positive and satisfies the following.*

$$v \in C^j(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2, \tag{3.5}$$

$$-\frac{1}{2} D^2 v + \omega v - v^p = 0, \quad x \neq 0, \tag{3.6}$$

$$Dv(0+) - Dv(0-) = -2Zv(0), \tag{3.7}$$

$$Dv(x), v(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \tag{3.8}$$

Proof. Since v satisfies $S'_\omega(v) = 0$, v satisfies (1.4). To check (3.5) and (3.8), we take an appropriate test function $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Then ξv satisfies

$$-\frac{1}{2}D^2(\xi v) + \omega \xi v = -\frac{1}{2}(D^2\xi)v - (D\xi)(Dv) + \xi v^p,$$

in the sense of distributions. We employ the standard bootstrap argument for this equation (see Section 8 of [4] for details). The right-hand side is in $L^2(\mathbb{R})$ and so $\xi v \in H^2(\mathbb{R})$, that is, $v \in H^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R} \setminus \{0\})$. The case of $j = 2$ is similar. Eq. (3.6) follows from the fact that $C_0^\infty(\mathbb{R} \setminus \{0\})$ is dense in $L^2(\mathbb{R})$. Concerning (3.7), we integrate $S'_\omega(v) = 0$ from $-\varepsilon$ to ε .

$$-\frac{1}{2} \int_{-\varepsilon}^{\varepsilon} D^2v \, dx + \omega \int_{-\varepsilon}^{\varepsilon} v \, dx - Z \int_{-\varepsilon}^{\varepsilon} \delta(x)v \, dx = \int_{-\varepsilon}^{\varepsilon} v^p \, dx.$$

Then we have the initial boundary condition

$$Dv(0+) - Dv(0-) = -2Zv(0)$$

letting $\varepsilon \rightarrow 0$. Multiplying Eq. (3.6) by Dv and integrating resulting terms in $x > 0$ and in $x < 0$, we have

$$-\frac{1}{4}(Dv)^2 = F(v(x)), \quad x \neq 0. \tag{3.9}$$

We note that $v(x) > 0$ for $x \in \mathbb{R}$. If not, there exists x_0 such that $v(x_0) = 0$. From (3.9), we have $Dv(x_0) = 0$. It implies $v \equiv 0$, which is impossible. \square

To show Theorem 1, we check the sufficient condition for stability and instability in Proposition 3.

Proof of Theorem 1. We put $\alpha = \omega^{-1/2}$, for $\omega > Z^2/2$ and then it follows from (1.5) that

$$\begin{aligned} \frac{\partial}{\partial \omega} \|\phi_\omega\|_{L^2}^2 &= -\frac{\alpha^3}{2} \frac{\partial}{\partial \alpha} \|\phi_\alpha\|_{L^2}^2 = -C_p \alpha^{-4/(p-1)+3} g(\alpha), \\ g(\alpha) &= \frac{p-5}{p-1} J(\alpha) - \alpha Z (1 - C_\alpha^2)^{-(p-3)/(p-1)}, \\ J(\alpha) &= \int_{A(\alpha)}^{\infty} \operatorname{sech}^{4/(p-1)} y \, dy, \quad A(\alpha) = \tanh^{-1}(C_\alpha), \end{aligned}$$

where $C_\alpha = Z\alpha/\sqrt{2}$ and C_p is a positive constant depending only on p . It suffices to check the sign of $g(\alpha)$. In the case where $Z > 0$ and $p \leq 5$, we have $g(\alpha) < 0$ for any $\alpha \in (0, \sqrt{2}/Z)$. In the case where $Z > 0$ and $p > 5$, we see that $g'(\alpha) > 0$ in a neighborhood of 0, $g'(\alpha) < 0$ in a neighborhood of $\sqrt{2}/Z$ and that $g''(\alpha) < 0$ for any $\alpha \in (0, \sqrt{2}/Z)$. Therefore, there exists a unique $\alpha^* \in (0, \sqrt{2}/Z)$ such that $g(\alpha^*) = 0$, $g(\alpha) > 0$ for any $\alpha \in (0, \alpha^*)$ and that $g(\alpha) < 0$ for any $\alpha \in (\alpha^*, \sqrt{2}/Z)$ since $g(0) > 0$. We put $\alpha^* = \omega_1^{-1/2}$ and then $\omega_1 > Z^2/2$. \square

For the sake of completeness, we give a remark on the proof of Proposition 3. First, we consider the stability. We explain briefly because the proof is similar to that of [10, Proposition 1] (see also Fibich and Wang [9]). We remark that $d'(\omega) = Q(\phi_\omega)$ and it follows from the explicit form of (1.5) that the mapping $\omega \mapsto \phi_\omega$ is C^1 .

We introduce the C^1 map $\omega(\cdot) : U_\eta(\phi_\omega) \rightarrow \mathbb{R}$ defined by

$$\omega(u) = d^{-1} \left(\frac{p-1}{2(p+1)} \|u\|_{L^{p+1}}^{p+1} \right). \tag{3.10}$$

Here, let us denote ϕ_{ω_0} by ϕ_0 for simplicity. The following lemma is important to have the stability. We omit the proof since it is the same as that of Lemma 4.2 in [10].

Lemma 3.3. *Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Assume $d''(\omega) > 0$ at $\omega = \omega_0$ for some $\omega_0 \in (Z^2/2, \infty)$. Then there exists $\eta = \eta(\omega_0) > 0$ such that for all $u \in U_\eta(\phi_0)$,*

$$E(u) - E(\phi_0) + \omega(u)\{Q(u) - Q(\phi_0)\} \geq \frac{1}{4}d''(\omega_0)(\omega(u) - \omega_0)^2.$$

We verify the statement of Proposition 3(i) by contradiction. Assume that $e^{i\omega_0 t}\phi_0(x)$ is unstable in $H^1(\mathbb{R})$. Then we have $\varepsilon_0 > 0$ and initial data $u_k(0) \in U_{1/k}(\phi_0)$ such that

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u_k(t) - e^{i\theta}\phi_0\|_{H^1} \geq \varepsilon_0,$$

where $u_k(t)$ is the solution of (1.1) with initial data $u_k(0)$. Let t_k be the first time at which

$$\inf_{\theta \in \mathbb{R}} \|u_k(t_k) - e^{i\theta}\phi_0\|_{H^1} = \frac{\varepsilon_0}{2}. \tag{3.11}$$

We put $v_k = u_k(t_k)$. Since E and Q are conserved in t , we have

$$|E(v_k) - E(\phi_0)| = |E(u_k(0)) - E(\phi_0)| \rightarrow 0, \tag{3.12}$$

$$|Q(v_k) - Q(\phi_0)| = |Q(u_k(0)) - Q(\phi_0)| \rightarrow 0 \tag{3.13}$$

as $k \rightarrow \infty$. From (3.11), we have $\|v_k\|_{H^1} \leq C$ uniformly in k . Also we note that $\omega_k = \omega(v_k)$ is uniformly bounded in k since $\omega(u)$ is a continuous map. Here, we take η small enough so that Lemma 3.3 may be applied. Then we have

$$E(v_k) - E(\phi_0) + \omega_k\{Q(v_k) - Q(\phi_0)\} \geq \frac{1}{4}d''(\omega_0)(\omega_k - \omega_0)^2. \tag{3.14}$$

Since $d''(\omega_0) > 0$, this implies that $\omega_k \rightarrow \omega_0$ as $k \rightarrow \infty$. By using $I_{\omega_0}(\phi_0) = 0$ and the fact that $d(\cdot)$ is continuous, it follows that

$$\lim_{k \rightarrow \infty} \frac{p-1}{2(p+1)} \|v_k\|_{L^{p+1}}^{p+1} = \lim_{k \rightarrow \infty} d(\omega_k) = d(\omega_0) = \frac{p-1}{2(p+1)} \|\phi_0\|_{L^{p+1}}^{p+1}. \tag{3.15}$$

From (3.12) and (3.13), we have

$$\begin{aligned} S_{\omega_0}(v_k) &= S_{\omega_0}(v_k) - S_{\omega_0}(\phi_0) + S_{\omega_0}(\phi_0) \\ &= E(v_k) - E(\phi_0) + \omega_0(Q(v_k) - Q(\phi_0)) + d(\omega_0) \\ &\rightarrow d(\omega_0), \end{aligned} \tag{3.16}$$

as $k \rightarrow \infty$. Let $w_k = (\|\phi_0\|_{L^{p+1}}/\|v_k\|_{L^{p+1}})v_k$. Then, w_k satisfies $\|w_k\|_{L^{p+1}} = \|\phi_0\|_{L^{p+1}}$ and $\|w_k - v_k\|_{H^1} \rightarrow 0$ as $k \rightarrow \infty$. Furthermore, by (3.15) and (3.16), $S_{\omega_0}(w_k) \rightarrow d(\omega_0)$ as $k \rightarrow \infty$. Therefore, $\{w_k\}$ is a minimizing sequence for $d(\omega_0)$. By a similar argument we have done in the proof of Proposition 2 and the uniqueness of minimizers of $d(\omega_0)$, there exists a sequence $\{\theta_k\} \subset \mathbb{R}$ such that $\|w_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$ as $k \rightarrow \infty$. Namely, we get

$$\|v_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$$

as $k \rightarrow \infty$, which is a contradiction to (3.11). \square

Next, concerning the sufficient condition for instability, i.e., Proposition 3(ii), we can apply a similar method by Shatah and Strauss [24] to our present case. Indeed, we modify the function $\psi(\omega)$, which was defined in the proof of Theorem 5 in [24] and used to determine an unstable direction in solving the ordinary differential equation of Lemma 11 in [24], as the following: For any fixed $\omega_0 \in (Z^2/2, \infty)$, we put for any $v_\omega \in \mathcal{G}_\omega$,

$$\psi(\omega) = \lambda(\omega)v_\omega(x), \quad \lambda(\omega) = \frac{\|v_{\omega_0}\|_{L^2}}{\|v_\omega\|_{L^2}}$$

for any $\omega \in (Z^2/2, \infty)$ which is close to ω_0 . We also remark that our variational characterization is different from theirs only in the point that $d(\omega) = \frac{p-1}{2(p+1)}\|v_\omega\|_{L^{p+1}}^{p+1}$. Accordingly, we change the norm in the statement of Lemma 11(iv) in [24] to L^{p+1} norm. In consequence, their method works and Theorem 17 in [24] holds for the present case.

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