Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 25 (2008) 969–985
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Breaking of resonance and regularizing effect of a first order quasi-linear term in some elliptic equations ☆

Boumediene Abdellaoui [a], Ireneo Peral [b,*], Ana Primo [b]

[a] *Département de Mathématiques, Université Aboubekr Belkaïd, Tlemcen, Tlemcen 13000, Algeria*
[b] *Departamento de Matemáticas, U. Autonoma de Madrid, 28049 Madrid, Spain*

Received 9 November 2006; received in revised form 14 June 2007; accepted 14 June 2007

Available online 17 October 2007

Abstract

In this article we study the problem

$$
\text{(P)} \quad \begin{cases} -\Delta u + |\nabla u|^q = \lambda g(x)u + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

with $1 \leqslant q \leqslant 2$ and f, g are positive measurable functions. We give assumptions on g with respect to q for which for all $\lambda > 0$ and all $f \in L^1$, $f \geqslant 0$, problem (P) has a positive solution. In particular we focus our attention on $g(x) = 1/|x|^2$ to prove that the assumptions on g are optimal.

Résumé

Dans cet article nous étudions le problème

$$
\text{(P)} \quad \begin{cases} -\Delta u + |\nabla u|^q = \lambda g(x)u + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

où $1 \leqslant q \leqslant 2$ et f, g sont des fonctions mesurables positives. Nous donnons des hypothèses sur g dépendant de q telles que pour tout $\lambda > 0$ et pour tout $f \in L^1$, $f \geqslant 0$, le problème (P) a une solution positive. Nous portons une attention particulière au cas $g(x) = 1/|x|^2$ pour montrer que les hypothèses sur g sont optimales.

MSC: 35D05; 35D10; 35J20; 35J25; 35J70; 46E30; 46E35

Keywords: Quasilinear elliptic equations; Existence and nonexistence; Regularization; Resonance

☆ Work partially supported by projects MTM2004-02223, MEC, Spain and CCG06-UAM/ESP-0340, CAM-UAM.
* Corresponding author.
E-mail addresses: boumediene.abdellaoui@uam.es (B. Abdellaoui), ireneo.peral@uam.es (I. Peral), ana.primo@uam.es (A. Primo).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.06.003

1. Introduction

This paper is devoted to obtain existence and nonexistence results for nonlinear elliptic equations of the form

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda g(x)u + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $1 \leqslant q \leqslant 2$ and $\lambda \in \mathbb{R}$. In the whole of this work, we suppose that f and g are positive measurable functions with some summability assumptions that we will specify in each case. For $\lambda \equiv 0$, equations of the form (1.1) have been widely studied in the literature. We refer to [1,4,5,9,7,8,17,19] and the references therein. The case $q > 2$ and f Lipschitz has been studied in [22] with quite different methods. The case $\lambda > 0$ and the presence of the term $|\nabla u|^q$ in the right-hand side of the equation has been recently studied in [2]. In [10] it is proved that for all $\lambda > 0$ the equation

$$-\Delta u = \lambda \frac{u}{|x|^2} + f(x) \quad \text{in } \Omega \subset \mathbb{R}^N, N \geqslant 3 \text{ and } 0 \in \Omega, \tag{1.2}$$

has in general no solution for a positive $f \in L^1(\Omega)$. However, in [3], by adding the gradient term $|\nabla u|^2$ on the left-hand side of Eq. (1.2), we find a positive solution for all $\lambda > 0$ and for all positive $f \in L^1(\Omega)$. This means that if $q = 2$ and $g(x) = |x|^{-2}$ in problem (1.1), the term $|\nabla u|^2$ produces a strong regularizing effect. The new feature in this paper is to find the assumptions on g in terms of q in order to solve problem (1.1) for all $\lambda > 0$ and for all $f \in L^1(\Omega)$, $f \geqslant 0$. We can reformulate the main objective of the paper as follows: fixed g find the *optimal* q for which for all $\lambda > 0$ and for all positive function $f \in L^1(\Omega)$, problem (1.1) has a positive solution. Precisely we prove the existence of solution *for all* $\lambda > 0$ *and for all* $f \in L^1(\Omega)$, $f \geqslant 0$, if $g \geqslant 0$ is an *admissible weight* in the sense of (2.1) below. This condition, in general, also becomes optimal. The previous result also proves that the absorption term $|\nabla u|^q$ is sufficient to break down any resonant effect of the linear zero order term. For $f \in L^1(\Omega)$, we say that u is a weak solution to problem (1.1) if $u \in W_0^{1,q}(\Omega)$, $g(x)u \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta\phi)\,dx + \int_\Omega |\nabla u|^q \phi\,dx = \lambda \int_\Omega g(x)u\phi\,dx + \int_\Omega f(x)\phi\,dx, \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega).$$

Since we consider solutions to (1.1) with data in $L^1(\Omega)$, we refer to [16] and [15] for a complete discussion about this framework. The paper is organized as follows. Section 2 is devoted to prove existence results. In Subsection 2.1, fixed $1 < q \leqslant 2$ we prove the main result, that is, if g is an admissible weight then for all $\lambda > 0$ and for all $f \in L^1(\Omega)$, $f \geqslant 0$, there exists $u \in W_0^{1,q}(\Omega)$ positive solution to problem (1.1). This is the result of Theorem 2.3.

In Subsection 2.2, we obtain conditions on g and λ such that for all $1 < q \leqslant 2$, there exists solution to problem (1.1) provided that f is in a suitable class associated explicitly to g. Due to the presence of the Laplace operator and the linear term $\lambda g(x)u$, then the natural condition is to assume (2.12). In Subsection 2.3, we give some results on uniqueness of solution. In Section 3, we consider the Hardy potential to prove nonexistence results that show the optimality of the condition required for existence in Theorems 2.3 and 2.9. These nonexistence results are related to the classical Hardy inequality,

$$\int_\Omega |\nabla u|^2\,dx \geqslant \Lambda_N \int_\Omega \frac{u^2}{|x|^2}\,dx, \quad \text{for all } u \in \mathcal{C}_0^\infty(\Omega) \text{ where } \Lambda_N = \left(\frac{N-2}{2}\right)^2$$

(see for instance [18]). More precisely by setting $g(x) = |x|^{-2}$, we prove nonexistence results for a very weak solution if $1 \leqslant q \leqslant \frac{N}{N-1}$ and $\lambda > \Lambda_N$, the Hardy constant. Next we prove the optimality of the summability assumption on f in Theorem 2.9. Finally in Subsection 3.1 we study the potential $g(x) = |x|^{-\alpha}$, $1 < \alpha < \frac{N+2}{2}$. Given u a measurable function we consider the k-truncation of u defined by

$$T_k(u) = \begin{cases} u, & |u| \leqslant k, \\ k\frac{u}{|u|}, & |u| > k. \end{cases}$$

2. Existence of weak positive solutions

In this section we prove the existence of positive solution to problem (1.1) according to a relation between q, λ, g and f. In the first part, we prove that for all $\lambda > 0$ and for all $f \in L^1(\Omega)$, $f \geqslant 0$, there exists a positive solution to (1.1), provided that g satisfies some assumptions related to q. In the second part, we study the class of data f for which, for all q in $[1, 2]$ and under some conditions on g, there exists positive solution for small values of λ.

2.1. Admissible weights: Global existence result

Let consider in (1.1) a fixed q, $1 < q \leqslant 2$, then we say that g is an *admissible weight* in the sense of (2.1) below,

$$g \gneqq 0, g \in L^1(\Omega) \quad \text{and} \quad C(g,q) = \inf_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{(\int_\Omega |\nabla\phi|^q\,dx)^{\frac{1}{q}}}{\int_\Omega g|\phi|\,dx} > 0. \tag{2.1}$$

Remark 2.1. It is clear that if g satisfies (2.1), then $g \in W^{-1,q'}(\Omega) \cap L^1(\Omega)$, $q' = \frac{q}{q-1}$. Moreover (2.1) implies that

(a) $\int_\Omega g|u|\,dx < \infty$ for all $u \in W_0^{1,q}(\Omega)$.
(b) Defining $T : W_0^{1,q}(\Omega) \to \mathbb{R}$ by

$$\langle T, u\rangle \equiv \int_\Omega gu\,dx, \tag{2.2}$$

then T is a linear continuous form on $W_0^{1,q}(\Omega)$. That means that there exists $\vec{F} = (f_1, f_2, \ldots, f_N) \in (L^{q'}(\Omega))^N$ such that $g = -\operatorname{div}(\vec{F})$ and then

$$\langle T, u\rangle \equiv \int_\Omega gu\,dx = \int_\Omega \langle \vec{F}, \nabla u\rangle\,dx.$$

As a consequence, the duality product is equivalent to the first integral and

$$\|T\|_{W^{-1,q'}(\Omega)} = \|\vec{F}\|_{(L^{q'}(\Omega))^N}.$$

Proposition 2.2. *Assume that g satisfies* (2.1) *and let T be defined by* (2.2)*. Consider $g_n(x) = \min\{g(x), n\}$ and the corresponding linear continuous form*

$$T_n : W_0^{1,q}(\Omega) \to \mathbb{R},$$
$$u \to T_n(u) = \int_\Omega g_n u\,dx.$$

The following statements hold.

(i) *$T_n \to T$ strongly in $W^{-1,q'}(\Omega)$.*
(ii) *Assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$, $u_n \geqslant 0$ and $u_n \to u$ a.e., then $g_n u_n \to gu$ strongly in $L^1(\Omega)$.*

Proof. (i) As in Remark 2.1 we also have that

$$\langle T_n, u\rangle = \int_\Omega g_n u\,dx,$$

that is, the duality is realized by the integral and moreover

$$C(g_n, q) = \inf_{\phi \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{(\int_\Omega |\nabla\phi|^q\,dx)^{\frac{1}{q}}}{\int_\Omega g_n|\phi|\,dx} > 0.$$

Notice that

$$\|T_n\|_{W^{-1,q'}} = \sup_{\|u\|_{W_0^{1,q}(\Omega)}\leqslant 1} \left|\langle T_n, u\rangle\right| \leqslant \sup_{\|u\|_{W_0^{1,q}(\Omega)}\leqslant 1} \left|\int_\Omega g_n u\right| \leqslant \sup_{\|u\|_{W_0^{1,q}(\Omega)}\leqslant 1} \int_\Omega g|u| \leqslant \|T\|_{W^{-1,q'}}.$$

Then $\{T_n\}_{n\in\mathbb{N}}$, up to a subsequence, converges weakly in $W^{-1,q'}(\Omega)$. Since for all $u \in W_0^{1,q}(\Omega)$, the Lebesgue theorem proves that

$$g_n|u| \to g|u| \quad \text{strongly in } L^1(\Omega), \text{ then } \{T_n\}_{n\in\mathbb{N}} \rightharpoonup T$$

therefore,

$$\|T\|_{W^{-1,q'}} \leqslant \liminf_{n\to\infty} \|T_n\|_{W^{-1,q'}} \leqslant \limsup_{n\to\infty} \|T_n\|_{W^{-1,q'}} \leqslant \|T\|_{W^{-1,q'}},$$

hence,

$$T_n \to T, \quad \text{strongly in } W^{-1,q'}(\Omega).$$

(ii) According to the strong convergence proved in (i), we have that if $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$, then,

$$\langle T_n, u_n\rangle = \int_\Omega g_n u_n\,dx \to \int_\Omega gu\,dx = \langle T, u\rangle \quad \text{as } n \to \infty.$$

If we assume that, moreover, $u_n \geqslant 0$ and $u_n \to u$ a.e., then by using a well-known result in [12] we obtain that $g_n u_n \to gu$ strongly in $L^1(\Omega)$. (See also [21], Theorem 1.9, page 21.) □

The main result of this section is the following one.

Theorem 2.3. *Assume that* $1 < q \leqslant 2$ *and let* g *be a positive function such that* (2.1) *holds, then for all* $\lambda \geqslant 0$ *and for all* $f \in L^1(\Omega)$, $f \geqslant 0$, *there exists* $u \in W_0^{1,q}(\Omega)$, $u \geqslant 0$, *that solves problem* (1.1) *in the sense of distributions.*

To prove Theorem 2.3 we start by proving the result in some particular cases and then we proceed by approximation of g and f. Notice that since $1 < q \leqslant 2$, then $\frac{N}{2} \leqslant \frac{N}{q}$, therefore the following first approach is quite natural.

Theorem 2.4. *Assume that* $f, g \in L^r(\Omega)$ *are positive functions with* $r > \frac{N}{q}$, *then for all* $\lambda \geqslant 0$, *there exists* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *a weak positive solution to problem* (1.1).

Proof. *Step 1*: For every fixed $k > 0$, consider $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $-\Delta v = \lambda k g(x) + f(x)$ in Ω and denote $M_k = \|v\|_{L^\infty}$. Notice that v is bounded by the assumptions on g and f and by standard elliptic estimates (see [24]). It follows that zero is a subsolution and v is a supersolution to problems

$$\begin{cases} -\Delta w_n + \frac{|\nabla w_n|^q}{1+\frac{1}{n}|\nabla w_n|^q} = \lambda g(x) T_k w_n + f(x), \\ w_n \in W_0^{1,2}(\Omega), \quad w_n \geqslant 0 \end{cases} \tag{2.3}$$

for all $n \in \mathbb{N}$. By a simple variation of the arguments used in [8] and [4], there exists a sequence of nonnegative minimal solutions $\{w_n\}$ to problems (2.3). It follows that $-\Delta w_n \leqslant \lambda k g(x) + f(x) = -\Delta v$, hence by the weak comparison principle we conclude that $0 \leqslant w_n \leqslant v \leqslant M$, uniformly in n, in particular, $w_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Taking w_n as test function in (2.3) we obtain,

$$\int_\Omega |\nabla w_n|^2\,dx + \int_\Omega H_n(\nabla w_n) w_n\,dx = \lambda \int_\Omega g(x) T_k w_n w_n\,dx + \int_\Omega f(x) w_n\,dx,$$

where $H_n(s) = |s|^q/(1 + \frac{1}{n}|s|^q)$. It is easy to check that there exists a positive constant C such that

$$\int_\Omega |\nabla w_n|^2\,dx \leqslant C(f, g, \Omega, k) \quad \text{uniformly in } n,$$

therefore, up to subsequences $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$. By weak-* convergence in $L^\infty(\Omega)$ we also have that $u_k \in W_0^{1,2} \cap L^\infty(\Omega)$ and $u_k \leqslant M_k$. Next we investigate the equation satisfied by u_k. To do that we prove the following claim.

Convergence claim. *$w_n \to u_k$ strongly in $W_0^{1,2}(\Omega)$.*

Proof of the convergence claim. We follow closely the argument used in [5]. Consider $\phi(s) = s\mathrm{e}^{\frac{1}{4}s^2}$, which satisfies $\phi'(s) - |\phi(s)| \geqslant \frac{1}{2}$. Taking $\phi(w_n - u_k)$ as test function in (2.3),

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)\,dx + \int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)\,dx = \lambda \int_\Omega g(x)T_k w_n \phi(w_n - u_k)\,dx + \int_\Omega f(x)\phi(w_n - u_k)\,dx. \tag{2.4}$$

As $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$, then a direct computation shows that

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)\,dx = \int_\Omega \big|\nabla(w_n - u_k)\big|^2 \phi'(w_n - u_k)\,dx + \mathrm{o}(1).$$

Since $q \leqslant 2$, it is well known that $\forall\varepsilon > 0$ there exists a nonnegative constant C_ε such that

$$s^q \leqslant \varepsilon s^2 + C_\varepsilon, \quad s \geqslant 0. \tag{2.5}$$

Hence the second term in the left-hand side can be estimated in the following way,

$$\begin{aligned}\int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)\,dx &\leqslant \varepsilon \int_\Omega |\nabla w_n|^2 \big|\phi(w_n - u_k)\big|\,dx + C(\varepsilon)\int_\Omega \big|\phi(w_n - u_k)\big|\,dx \\ &= \varepsilon \int_\Omega |\nabla w_n - \nabla u_k|^2 \big|\phi(w_n - u_k)\big|\,dx - \varepsilon \int_\Omega |\nabla u_k|^2 \big|\phi(w_n - u_k)\big|\,dx \\ &\quad + 2\varepsilon \int_\Omega \nabla w_n \nabla u_k \big|\phi(w_n - u_k)\big|\,dx + C(\varepsilon)\int_\Omega \big|\phi(w_n - u_k)\big|\,dx.\end{aligned}$$

Since $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$ and by the fact that $|\phi(w_n - u_k)| \to 0$ almost everywhere (and in $L^2(\Omega)$), then it follows

(i) $\int_\Omega |\nabla u_k|^2 |\phi(w_n - u_k)|\,dx \to 0$ as $n \to \infty$,
(ii) $\int_\Omega \nabla w_n \nabla u_k \phi(w_n - u_k)\,dx \to 0$ as $n \to \infty$.

Therefore, passing to the limit as n tends to ∞, we have

$$\int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)\,dx \leqslant \varepsilon \int_\Omega |\nabla w_n - \nabla u_k|^2 \big|\phi(w_n - u_k)\big|\,dx + \mathrm{o}(1).$$

Moreover, it is clear that the right-hand side in (2.4) goes to zero as $n \to \infty$. Since $\phi'(s) - |\phi(s)| > \frac{1}{2}$, choosing $\varepsilon \leqslant 1$ we conclude that

$$\frac{1}{2}\int_\Omega |\nabla w_n - \nabla u_k|^2\,dx \leqslant \int_\Omega \big(\phi'(w_n - u_k) - \varepsilon\big|\phi(w_n - u_k)\big|\big)|\nabla w_n - \nabla u_k|^2\,dx \leqslant \mathrm{o}(1),$$

whence $w_n \to u_k$ in $W_0^{1,2}(\Omega)$ and the claim is proved. Moreover, from (2.5) it follows that

$$H_n(\nabla w_n) \leqslant c_1|\nabla w_n|^2 + c_2.$$

By the claim, we have in particular the almost everywhere convergence of the gradients and therefore we conclude that

$$H_n(\nabla w_n) \to |\nabla u_k|^q \quad \text{in } L^1(\Omega).$$

Hence we find that u_k solves problem

$$-\Delta u_k + |\nabla u_k|^q = \lambda g(x) T_k u_k + f(x) \quad \text{in } \Omega, \quad u_k \in W_0^{1,2}(\Omega). \tag{2.6}$$

Step 2. We claim that there exists $M > 0$ such that $\|u_k\|_{L^\infty(\Omega)} \leqslant M$ for all $k > 0$.

First of all we prove that $\{u_k\}$ is uniformly bounded in $L^a(\Omega)$ for all $a > 1$.

Consider

$$\lambda_1(\theta, g) = \inf_{\phi \in \mathcal{C}_0^\infty(\Omega),\, \phi \neq 0} \frac{\int_\Omega |\nabla \phi|^\theta \, dx}{\int_\Omega g|\phi|^\theta \, dx}.$$

Since $g \in L^r(\Omega)$ with $r > \frac{N}{q}$, it follows that $\lambda_1(\theta) > 0$ for all $\theta \geqslant q$.

Since $u_k \in L^\infty(\Omega)$, then using u_k^a, with $a \gg 2$, as a test function in (2.6), we get

$$a \int_\Omega u_k^{a-1} |\nabla u_k|^2 \, dx + \int_\Omega u_k^a |\nabla u_k|^q \, dx \leqslant \lambda \int_\Omega g u_k^{a+1} \, dx + \int_\Omega f u^a \, dx.$$

Thus

$$\frac{4a}{(2+a)^2} \int_\Omega |\nabla u_k^{\frac{a}{2}+1}|^2 \, dx + \left(\frac{q}{a+q}\right)^q \int_\Omega |\nabla u_k^{\frac{a}{q}+1}|^q \, dx \leqslant \lambda \int_\Omega g u_k^{a+1} \, dx + \int_\Omega f u^a \, dx.$$

Using Hölder, Young and Poincaré inequalities there results that

$$\begin{aligned}
\int_\Omega g u^{a+1} \, dx &= \int_\Omega g^{\frac{a}{2(a+q)}} u^{\frac{a}{2}} g^{\frac{1}{2}} u^{\frac{a}{2}+1} g^{\frac{q}{2(a+q)}} \, dx \\
&\leqslant \left(\int_\Omega g u^{a+q} \, dx\right)^{\frac{a}{2(a+q)}} \left(\int_\Omega g u^{a+2} \, dx\right)^{\frac{1}{2}} \left(\int_\Omega g \, dx\right)^{\frac{q}{2(a+q)}} \\
&\leqslant \varepsilon \int_\Omega g u^{a+q} \, dx + \varepsilon \int_\Omega g u^{a+2} \, dx + C(\varepsilon) \int_\Omega g \, dx \\
&\leqslant \frac{\varepsilon}{\lambda_1(q, g)} \int_\Omega |\nabla u_k^{\frac{a}{q}+1}|^q \, dx + \frac{\varepsilon}{\lambda_1(2, g)} \int_\Omega |\nabla u_k^{\frac{a}{2}+1}|^2 \, dx + C(\varepsilon, g),
\end{aligned}$$

where ε is a positive constant that will be chosen later. On the other hand we have

$$\begin{aligned}
\int_\Omega f u^a \, dx &\leqslant \left(\int_\Omega u^{2^*(\frac{a}{2}+1)} dx\right)^{\frac{a}{2^*(a/2+1)}} \left(\int_\Omega f^{\frac{N(a+2)}{2(N+a)}} \, dx\right)^{\frac{2(N+a)}{N(a+2)}} \\
&\leqslant C(f, S) \left(\int_\Omega |\nabla u^{\frac{a}{2}+1)}|^2 \, dx\right)^{\frac{a}{a+2}} \leqslant \varepsilon \int_\Omega |\nabla u^{\frac{a}{2}+1)}|^2 \, dx + C(f, S, \varepsilon),
\end{aligned}$$

where S is the Sobolev constant and $2^* = \frac{2N}{N-2}$. Therefore, putting together the above inequalities it follows that

$$\left(\frac{4a}{(1+a)^2} - \frac{\varepsilon\lambda}{\lambda_1(2, g)} - \varepsilon\right) \int_\Omega |\nabla u_k^{\frac{a}{2}+1}|^2 \, dx + \left(\left(\frac{q}{a+q}\right)^q - \frac{\varepsilon\lambda}{\lambda_1(q, g)}\right) \int_\Omega |\nabla u_k^{\frac{a}{q}+1}|^q \, dx \leqslant C(f, g, S, \varepsilon).$$

Choosing ε small enough we conclude that

$$\int_\Omega |\nabla u_k^{\frac{a}{2}+1}|^2 \, dx + \int_\Omega |\nabla u_k^{\frac{a}{q}+1}|^q \, dx \leqslant C,$$

where C is independent of k. Hence using Sobolev inequality we obtain that $\|u_k\|_{L^a(\Omega)} \leqslant C(a, \lambda, f, g)$ for all $a > 1$. Whence $\{gu_k + f\}_{L^r(\Omega)}$ is uniformly bounded in k, for some $r > \frac{N}{2}$. The uniform boundedness is a consequence of classical results about elliptic regularity, see [24].

Therefore if $k > M$, $u \equiv u_k$ and is a solution to problem (1.1). □

Remark 2.5. Notice that if $q \leqslant 2$, then the passage to the limit in the *convergence claim* above can be performed in a different way; indeed using a compactness result by Boccardo–Murat in [6], the gradients converge almost everywhere, therefore using Vitali theorem we get the strong convergence of the gradient.

However, we prove the *convergence claim* with arguments valid in a more general setting, which are needed to obtain strong convergence in the next theorems.

In the following result, we continue considering a weight with extra summability, but a general $f \in L^1(\Omega)$.

Theorem 2.6. *Assume that f, g are positive functions, $f \in L^1(\Omega)$, $f \geqslant 0$ and $g \in L^r(\Omega)$ with $r > \frac{N}{q}$, then for all $\lambda \geqslant 0$, problem* (1.1) *has a positive solution $u \in W_0^{1,q}(\Omega)$.*

Proof. Consider a sequence $f_n \in L^\infty(\Omega)$ such that $f_n \uparrow f$ in $L^1(\Omega)$. Thanks to Theorem 2.4, there exists a sequence of positive bounded functions $\{u_n\}$, solutions to problems

$$\begin{cases} -\Delta u_n + |\nabla u_n|^q = \lambda g(x) u_n + f_n(x) & \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega, \\ u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{2.7}$$

Notice that, in particular u_n solves a problem of the form $-\Delta u_n = F_n \in L^1(\Omega)$, then u_n is the unique entropy solution to this problem. As a consequence we can use $T_k(u_n)$ as a test function. See for instance [24] or [16].

Taking $T_k u_n$ as test function in (2.7), it follows that

$$\int_\Omega |\nabla T_k u_n|^2\, dx + \int_\Omega |\nabla u_n|^q T_k u_n\, dx = \lambda \int_\Omega g(x) u_n T_k u_n\, dx + \int_\Omega f_n(x) T_k u_n\, dx \quad \text{in } \Omega.$$

Define $\Psi_k(s) = \int_0^s T_k(t)^{\frac{1}{q}}\, dt$, that explicitly is,

$$\Psi_k(s) = \begin{cases} \frac{q}{q+1} s^{\frac{q+1}{q}} & \text{if } s < k, \\ \frac{q}{q+1} k^{\frac{q+1}{q}} + (s-k) k^{\frac{1}{q}} & \text{if } s > k, \end{cases}$$

then by using the assumptions on g and f, it follows that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla T_k u_n|^2\, dx + \int_\Omega |\nabla \Psi_k u_n|^q\, dx &\leqslant k\varepsilon\lambda \left(\int_\Omega g(x) u_n\, dx \right)^q + \lambda k C(\varepsilon) + k\|f_n\|_{L^1} \\ &\leqslant \frac{\varepsilon k \lambda}{C(q,g)} \int_\Omega |\nabla u_n|^q\, dx + \lambda k C(\varepsilon) + k\|f_n\|_{L^1}. \end{aligned} \tag{2.8}$$

Hence, as in the second step of the proof of Theorem 2.4,

$$\begin{aligned} \int_\Omega |\nabla u_n|^q\, dx &\leqslant \int_\Omega |\nabla T_k u_n|^2\, dx + k \int_{\{u_n \geqslant k\}} |\nabla u_n|^q dx + C_q|\Omega| \\ &\leqslant \frac{\varepsilon k \lambda}{C(q,g)} \int_\Omega |\nabla u_n|^q\, dx + \lambda k C(\varepsilon) + C_q|\Omega| + k\|f\|_{L^1}, \end{aligned}$$

then $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$ and $T_k u_n \rightharpoonup T_k u$ weakly in $W_0^{1,2}(\Omega)$. It is clear by the assumption on g that $gu_n \to gu$ strongly in $L^1(\Omega)$. Let $G_k(s) = s - T_k(s)$ and define $\psi_{k-1}(s) = T_1(G_{k-1}(s))$, then $\psi_{k-1}(u_n)|\nabla u_n|^q \geqslant |\nabla u_n|^q \chi_{\{u_n \geqslant k\}}$. As in [5], using $\psi_{k-1}(u_n)$ as a test function in (2.7) there results that

$$\int_\Omega \left|\nabla \psi_{k-1}(u_n)\right|^2 dx + \int_\Omega \psi_{k-1}(u_n)|\nabla u_n|^q\, dx = \int_\Omega \big(\lambda g(x)u_n + f_n(x)\big)\psi_{k-1}(u_n)\, dx.$$

Since $\{u_n\}$ is uniformly bounded in $L^p(\Omega), \forall p \leqslant q^*$, it follows that

$$\left|\left\{x \in \Omega,\ \text{such that } k-1 < u_n(x) < k\right\}\right| \to 0, \quad \left|\left\{x \in \Omega,\ \text{such that } u_n(x) > k\right\}\right| \to 0 \quad \text{as } k \to \infty,$$

uniformly in n. Thus we conclude

$$\lim_{k\to\infty} \int_{\{u_n \geqslant k\}} |\nabla u_n|^q\, dx = 0, \quad \text{uniformly in } n. \tag{2.9}$$

We follow the same arguments as in the proof of the *convergence claim* in Theorem 2.4. Take $\phi(T_k(u_n) - T_k(u))$ as a test function in (2.7), then

$$T_k u_n \to T_k u \quad \text{strongly in } W_0^{1,2}(\Omega). \tag{2.10}$$

To finish the proof, it is sufficient to show that

$$|\nabla u_n|^q \to |\nabla u|^q \quad \text{strongly in } L^1(\Omega).$$

Since the sequence of gradients converges a.e. in Ω, we only have to show that is equi-integrable and to apply Vitali's theorem. Let $E \subset \Omega$ be a measurable set. Then,

$$\int_E |\nabla u_n|^q\, dx \leqslant \int_E |\nabla T_k u_n|^q\, dx + \int_{\{u_n \geqslant k\}\cap E} |\nabla u_n|^q\, dx.$$

From (2.10) it follows that for all $k > 0$, $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(\Omega)$ for all $p \leqslant 2$. In particular we obtain the strong convergence for $p = q$. Hence the integral $\int_E |\nabla T_k(u_n)|^q\, dx$ is uniformly small if $|E|$ is small enough. On the other hand, thanks to (2.9) we obtain that

$$\int_{\{u_n \geqslant k\}\cap E} |\nabla u_n|^q\, dx \leqslant \int_{\{u_n \geqslant k\}} |\nabla u_n|^q\, dx \to 0 \quad \text{as } k \to \infty \text{ uniformly in } n.$$

The equi-integrability of $|\nabla u_n|^q$ follows immediately, and the proof is completed. □

Proof of Theorem 2.3. We consider the truncation $g_n(x) = \min\{g(x), n\} \in L^\infty(\Omega)$. Due to Theorem 2.6, there exists a sequence of positive functions $\{u_n\}$ such that u_n solves

$$\begin{cases} -\Delta u_n + |\nabla u_n|^q = \lambda g_n(x)u_n + f(x) & \text{in } \Omega, \\ u_n > 0 \quad \text{in } \Omega, \quad u_n = 0 & \text{on } \partial\Omega, \\ u_n \in W_0^{1,q}(\Omega). \end{cases} \tag{2.11}$$

Since $T_k u_n \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, then we can use $T_k u_n$ as a test function in (2.11), it follows that

$$\begin{aligned} \int_\Omega |\nabla T_k u_n|^2\, dx + \int_\Omega |\nabla \Psi_k u_n|^q\, dx &\leqslant \lambda \int_\Omega g_n(x) T_k u_n u_n\, dx + \int_\Omega f(x) T_k u_n\, dx \\ &\leqslant k\lambda \int_\Omega g_n(x) u_n\, dx + k \int_\Omega f(x)\, dx. \end{aligned}$$

Since

$$\int_\Omega |\nabla \Psi_k u_n|^q\, dx \geqslant \int_{\{u_n \geqslant k\}} |\nabla \Psi_k u_n|^q\, dx \geqslant k \int_{\{u_n \geqslant k\}} |\nabla u|^q\, dx,$$

then as above

$$\int_{\Omega} \left|\nabla T_k(u_n)\right|^2 dx + k \int_{\{u_n \geqslant k\}} |\nabla u_n|^q \, dx \leqslant k\varepsilon\lambda \left(\int_{\Omega} g_n(x) u_n \, dx \right)^q + k \int_{\Omega} f(x)\, dx + \lambda k C(\varepsilon, \Omega).$$

Therefore by (2.1) we have that

$$\int_{\Omega} |\nabla u_n|^q \, dx \leqslant \frac{k\varepsilon\lambda}{C(g,q)} \int_{\Omega} |\nabla u_n|^q \, dx + k \int_{\Omega} f(x)\, dx + \lambda k C(\varepsilon, q, \Omega),$$

hence $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$. We have that $u_n \geqslant 0$ and by Sobolev and Rellich theorems, up to a subsequence, we obtain that $u_n \to u$ a.e. Then we apply Proposition 2.2 to obtain that

$$g_n u_n \to g u \quad \text{strongly in } L^1(\Omega).$$

Therefore to finish the proof it is sufficient to show that $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$. This fact follows by proving that $T_k u_n \to T_k u$ strongly in $W_0^{1,2}(\Omega)$ and using Vitali's theorem as in the previous steps. □

Remarks 2.7.

(1) The solution of problem (1.1) obtained in Theorem 2.3 is also an entropy solution in the sense that we can take in problem (1.1) test functions of the form $T_k(u - v)$ with $v \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$.

(2) The same existence result holds if f is a bounded positive Radon measure such that $f \in L^1(\Omega) + W^{-1,2}(\Omega)$ (f is absolutely continuous with respect to the classical capacity). These results are obtained with some minor technical changes, i.e. the result follows using the same approximation arguments. See [15] to find the precise meaning of solution in this framework and equivalent definitions. By the regularity theory of renormalized solutions we easily get that if u is a positive solution to problem (1.1), then $u \in W_0^{1,q}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $1 \leqslant p < \frac{N}{N-1}$.

(3) The existence result obtained in Theorem 2.3 in particular means that resonance phenomenon doesn't occur if we add $|\nabla u|^q$ as an absorption term. Without the gradient term, positive solution exists just by assuming that λ is less than the infimum of the spectrum of the operator $-\Delta$ with the corresponding weight g and under a suitable condition on f.

As a direct application of Theorem 2.3 we obtain the following well known result.

Corollary 2.8. *Assume that $\lambda = 0$, then problem* (1.1) *has an entropy positive solution for all $0 \leqslant f \in L^1(\Omega)$ and for all $1 \leqslant q \leqslant 2$.*

In fact we can consider $g \in L^{\infty}(\Omega)$ in Theorem 2.3 and then pass to the limit using a priori estimates as λ goes to 0. A direct proof of this particular existence result can be obtained using truncation argument. See [5]. We will analyze in particular the Hardy potential in section 3 in order to prove the optimality of the assumptions in Theorem 2.3.

2.2. Existence of solution for all $q \in [1, 2]$ and small λ

In this section we will find general conditions on g and f that assure the existence of solution for all $q \leqslant 2$. The presence of the linear term $\lambda g(x) u$ motivates the following assumption on g,

$$g \geqslant 0,\ g \not\equiv 0 \text{ and } g \in L^1(\Omega) \text{ with } \lambda_1(g,2) = \inf_{\phi \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_{\Omega} |\nabla \phi|^2 \, dx}{\int_{\Omega} g|\phi|^2 \, dx} > 0. \tag{2.12}$$

It is easy to check that under assumption (2.12), for every $\bar{\lambda} < \lambda_1(g,2)$ there exists a unique $\varphi \in W_0^{1,2}(\Omega)$, positive weak solution to the problem

$$-\Delta\varphi = \bar{\lambda} g(x)\varphi + g(x) \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega. \tag{2.13}$$

The main result in this section is the following one.

Theorem 2.9. *Assume that $0 < \lambda < \lambda_1(g, 2)$ and let φ be the solution to problem* (2.13) *with $\lambda < \bar{\lambda} < \lambda_1(g, 2)$. Suppose that f is a positive function such that $\int_\Omega f\varphi\,dx < \infty$, then there exists $u \in W_0^{1,q}(\Omega)$ positive solution to problem* (1.1) *and moreover, $\int_\Omega |\nabla u|^p\,dx < \infty, \forall p \leqslant q$ if $q \geqslant \frac{N}{N-1}$ and $p < \frac{N}{N-1}$, on the contrary case.*

Proof. By using Theorem 2.4, we find a sequence $\{u_n\}$ of positive solutions to the approximated problems

$$\begin{cases} -\Delta u_n + |\nabla u_n|^q = \lambda g_n(x) u_n + f_n(x) & \text{in } \Omega, \\ u_n \in W_0^{1,q}(\Omega), \end{cases} \tag{2.14}$$

with $g_n(x) = \min\{g(x), n\} \in L^\infty(\Omega)$ and $f_n(x) = \min\{f(x), n\} \in L^\infty(\Omega)$. It is clear that $\int_\Omega f_n(x)\varphi\,dx < C$ uniformly in n. By a duality argument we obtain

$$\int_\Omega u_n(-\Delta\varphi)\,dx + \int_\Omega |\nabla u_n|^q\varphi\,dx = \lambda\int_\Omega g_n(x)u_n\varphi\,dx + \int_\Omega f_n(x)\varphi\,dx.$$

Therefore,

$$(\bar{\lambda} - \lambda)\int_\Omega g(x)u_n\varphi\,dx + \int_\Omega g(x)u_n\,dx + \int_\Omega |\nabla u_n|^q\varphi\,dx \leqslant \int_\Omega f_n(x)\varphi\,dx \leqslant \int_\Omega f(x)\varphi\,dx \leqslant C,$$

and hence we conclude that

$$\int_\Omega g(x)u_n\varphi\,dx \leqslant \frac{C}{\bar{\lambda} - \lambda} \quad \text{uniformly in } n,$$

$$\int_\Omega g(x)u_n\,dx \leqslant C \quad \text{uniformly in } n,$$

$$\int_\Omega |\nabla u_n|^q\varphi\,dx \leqslant C \quad \text{uniformly in } n.$$

Taking $T_k u_n$ as a test function in (2.14), we have that

$$\int_\Omega |\nabla T_k u_n|^2\,dx + \int_\Omega |\nabla u_n|^q T_k u_n\,dx \leqslant \lambda k\int_\Omega g_n(x)u_n\,dx + k\int_\Omega f(x)\,dx.$$

Thus $\frac{1}{k}\int_\Omega |\nabla T_k u_n|^2 dx \leqslant C$ and $T_k u_n \rightharpoonup T_k u$ weakly in $W_0^{1,2}(\Omega)$. In particular for $k = 1$ we obtain that

$$\int_\Omega |\nabla u_n|^q \leqslant \int_\Omega |\nabla T_1 u_n|^2\,dx + C(q, \Omega) + \int_\Omega |\nabla u_n|^q T_1 u_n\,dx \leqslant C,$$

therefore $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$. If $q > \frac{N}{N-1}$ and $g \in L^m(\Omega)$ with $m > \frac{N}{2}$ or $g(x) = |x|^{-2}$ the existence result is a consequence of the previous theorem. On the contrary, if $q \leqslant \frac{N}{N-1}$, then using the regularity result for entropy solutions obtained in [16] (see also [15] for the case of Radon measures), we obtain an extra regularity, that is, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for all $p < \frac{N}{N-1}$. We claim that $g_n(x)u_n \to g(x)u$ strongly in $L^1(\Omega)$. To prove the claim we consider the sequence $\{w_n\} \in W_0^{1,2}(\Omega)$ of solutions to the problem,

$$\begin{cases} -\Delta w_n = \lambda g_n(x) w_n + f_n(x) & \text{in } \Omega, \\ w_n = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Since $\lambda < \lambda_1(g, 2)$, then using the assumption on f we obtain that $w_n \nearrow w$ everywhere and $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$, for all $p < \frac{N}{N-1}$, with w the unique entropy solution to problem $-\Delta w = \lambda g(x)w + f$. Notice that solutions to (2.14) are subsolutions to (2.15), hence $g_n u_n \leqslant g_n w_n \leqslant g w$. Thanks to the dominated convergence theorem, $g_n(x)u_n \to g(x)u$ in $L^1(\Omega)$ and we conclude. To finish we have just to prove the strong convergence of $|\nabla u_n|^q$ to $|\nabla u|^q$ in $L^1(\Omega)$. The proof is done in two steps. First we start by proving the strong convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,2}(\Omega)$, which follows by applying to $T_k(u_n)$ and $T_k(u)$ the techniques used in the proof of *convergence claim* in

Theorem 2.4. Then to get the main convergence we use Vitali's Lemma and a suitable test function as in the last step in the proof of Theorem 2.6. □

Remarks 2.10.

(1) Notice that if $g(x) = |x|^{-2}$, then the regularity required on f in Theorem 2.9, depends on λ. See [10].
(2) In Section 3 we will show that the condition on the integrability of f in Theorem 2.9 is, in general, optimal.

2.3. Some remarks on the uniqueness

We consider the case $q = 2$ for which a change of variables allows us to find a comparison principle and uniqueness.

Lemma 2.11. *Let $u \in W_0^{1,2}(\Omega)$ be such that $u \neq 0$ and $-\Delta u + |\nabla u|^2 \geqslant 0$, then there exist constants $C, R > 0$ such that $u \geqslant C$ in $B_R(0) \subset \Omega$.*

Proof. The change of variables $v = 1 - \mathrm{e}^{-u}$ transforms $-\Delta u + |\nabla u|^2 \geqslant 0$ into $-\Delta v \geqslant 0$ with $v \neq 0$. It is sufficient to apply the classical maximum principle of the Laplacian operator and we conclude. □

Lemma 2.12. *Assume that $f \in L^1(\Omega)$ is a positive function and g satisfies* (2.1). *Then for all $\lambda > 0$ the problem*

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda g(x)u + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

has at most a positive solution $u \in W_0^{1,2}(\Omega)$.

Proof. If $u \in W_0^{1,2}(\Omega)$ is a solution to (2.16) then $v = 1 - \mathrm{e}^{-u}$ satisfies $0 \leqslant v \leqslant 1$ in Ω and it is a solution to

$$\begin{cases} -\Delta v = \lambda g(x)(1-v)\log(\frac{1}{1-v}) + (1-v)f(x) & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

Define

$$H(x, v) = \begin{cases} \lambda g(x)(1-v)\log(\frac{1}{1-v}) + (1-v)f(x), & \text{if } 0 \leqslant v < 1, \\ 0, & \text{if } v \geqslant 1. \end{cases}$$

By a direct computation we find that $\frac{H(v,x)}{v}$ is a nonincreasing function in v for $v \geqslant 0$, then by similar arguments as in [11] we conclude that v is the unique solution to (2.17). Therefore, u is the unique solution to (2.16). □

Remark 2.13. Using the same ideas of Lemma 2.12 we get a comparison principle for a sub- and super-solution to problem (2.16), namely if u_1 is a nonnegative subsolution and u_2 is a nonnegative supersolution such that $u_1, u_2 \in W^{1,2}(\Omega)$ and $u_2 \geqslant u_1$ on $\partial\Omega$, then $u_2 \geqslant u_1$ in Ω.

We will prove the following comparison principle, that we will use below, without change of variables.

Theorem 2.14. *Assume that $1 \leqslant q < \frac{N}{N-1}$ and $0 \leqslant f \in L^1(\Omega)$. Let v, u be two nonnegative functions such that $u, v \in W^{1,q}(\Omega)$, $\Delta u, \Delta v \in L^1(\Omega)$ and*

$$\begin{cases} -\Delta u + |\nabla u|^q \geqslant f(x) & \text{in } \Omega, \\ -\Delta v + |\nabla v|^q \leqslant f(x) & \text{in } \Omega, \\ v \leqslant u & \text{on } \partial\Omega, \end{cases} \tag{2.18}$$

then $v \leqslant u$ in Ω.

Proof. Consider $w = v - u$, then it is clear that $w \in W^{1,q}(\Omega)$, $w \leqslant 0$ on $\partial\Omega$ and $\Delta w \in L^1(\Omega)$. In order to conclude, it is sufficient to prove that $w_+ = 0$. It follows that

$$-\Delta w + |\nabla v|^q - |\nabla u|^q \leqslant 0.$$

Since $1 \leqslant q \leqslant 2$, we obtain

$$\begin{cases} -\Delta w \leqslant a(q,x)|\nabla w|, \\ w \leqslant 0 \quad \text{on } \partial\Omega, \\ w \in W^{1,q}(\Omega),\ \Delta w \in L^1(\Omega), \end{cases}$$

with $a(q,x) \leqslant q|\nabla u|^{q-1}$ if $q > 1$ and $a(q,x) = 1$ if $q = 1$. Therefore, applying Kato's inequality (see [20] and [13]) it follows that

$$\begin{cases} -\Delta w_+ \leqslant a(q,x)|\nabla w_+|, \\ w_+ = 0 \quad \text{on } \partial\Omega, \\ w_+ \in W_0^{1,q}(\Omega). \end{cases}$$

Since $q < \frac{N}{N-1}$, then $a(q,x) \in L^r(\Omega)$ with $r > N$, therefore we can apply the results of [4], thus we conclude that $w_+ = 0$. $\square$

Corollary 2.15.

(1) *Assume that $q < \frac{N}{N-1}$ and let $f \in L^1(\Omega)$ be a positive function, then problem*

$$\begin{cases} -\Delta u + |\nabla u|^q = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

has a unique positive solution.

(2) *If $q < \frac{N}{N-1}$, then problem* (1.1) *has a minimal solution.*

3. The Hardy potential: optimal results

Consider the problem,

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda \frac{u}{|x|^2} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with $N \geqslant 3$, $0 \in \Omega$. We have that $|x|^{-2} \in L^{(q^*)'}$ if and only if $q > \frac{N}{N-1}$. According to the definition in (2.1) and Hölder inequality, it follows that

$$C\left(\frac{1}{|x|^2}, q\right) > 0 \quad \text{if and only if} \quad q > \frac{N}{N-1}.$$

We have the following strong nonexistence result.

Theorem 3.1. *Assume that $q < \frac{N}{N-1}$. If $\lambda > \Lambda_N = \frac{(N-2)^2}{4}$, then problem* (3.1) *has no positive very weak positive supersolution in the sense that u, $\frac{u}{|x|^2}$, $|\nabla u|^q \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$\int \big(u(-\Delta\phi) + |\nabla u|^q \phi\big)\,dx \geqslant \lambda \int \frac{u\phi}{|x|^2}\,dx + \int f(x)\phi\,dx, \quad \text{for all } \phi \in \mathcal{C}_0^\infty(\Omega).$$

Proof. We argue by contradiction. Suppose that problem (3.1) has for some $\lambda > \Lambda_N$ a nonnegative very weak supersolution u in the sense defined above , then $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. Without loss of generality we can assume that $f \in L^\infty(\Omega)$. We claim that problem (3.1) has an entropy solution. To prove the claim we consider the sequence $\{u_n\}$ defined by

$$-\Delta u_1 + |\nabla u_1|^q = f(x), \quad u_1 \in W_0^{1,q}\big(B_r(0)\big),$$

with $B_r(0) \Subset \Omega$. The existence of u_1 follows using the result of the previous sections. Therefore by Theorem 2.14 it follows that $0 \leqslant u_1 \leqslant u$. Let us define now u_n by setting

$$-\Delta u_n + |\nabla u_n|^q = \lambda \frac{u_{n-1}}{|x|^2} + f(x), \quad u_n \in W_0^{1,q}\big(B_r(0)\big). \tag{3.2}$$

According with the comparison result of Theorem 2.14 we prove by recurrence that

$$0 \leqslant u_{n-1} \leqslant u_n \leqslant u.$$

Using $T_k(u_n)$ as a test function in (3.2) and taking into account that $u_n \leqslant u$ we obtain that $\|u_n\|_{W_0^{1,q}(B_r(0))} \leqslant C$ independently of n. Hence there exists $w \geqslant 0$ such that $u_n \rightharpoonup w$ weakly in $W_0^{1,q}(B_r(0))$, $u_n \uparrow w$ in $L^\alpha(B_r(0))$, $\alpha < q^*$ and $w \leqslant u$. By the monotone convergence theorem,

$$\lambda \frac{u_{n-1}}{|x|^2} + f(x) \to \lambda \frac{w}{|x|^2} + f(x) \quad \text{strongly in } L^1\big(B_r(0)\big),$$

therefore with the same arguments as in the proof of Theorem 2.3 we obtain that $w \in W_0^{1,q}(B_r(0))$ solves problem (3.1) in $B_r(0)$ in the sense of distributions. Since $|\nabla w|^q, \frac{w}{|x|^2} \in L^1(B_r(0))$ we conclude that w is an entropy solution to problem (3.1) in the sense defined in [16]. Hence the claim follows. Thus by [14] it follows that $w \in W_0^{1,p}(B_r(0))$ for all $p < \frac{N}{N-1}$, in particular $w \in L^m(B_r(0))$ for all $m < \frac{N}{N-2}$. Since $q \geqslant 1$ then using the strong maximum principle proved in [23] there results that $w > 0$. As a consequence, we can consider $\frac{\phi^2}{w}$ as a test function in (3.1), with $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$ and $\eta << r$ a small positive number that we will chose later. Hence,

$$-\int\limits_{B_\eta(0)} \frac{|\nabla w|^2\phi^2}{w^2}\,dx + 2\int\limits_{B_\eta(0)} \frac{\phi\nabla\phi}{w}\nabla w\,dx + \int\limits_{B_\eta(0)} \frac{|\nabla w|^q\phi^2}{w}\,dx \geqslant \lambda \int\limits_{B_\eta(0)} \frac{\phi^2}{|x|^2}\,dx. \tag{3.3}$$

Let us analyze the left-hand side of previous inequality (3.3) term by term

$$\begin{aligned}
\int\limits_{B_\eta(0)} \frac{|\nabla w|^q\phi^2}{w}\,dx &= \int\limits_{B_\eta(0)} \frac{|\nabla w|^q}{w^q} w^{q-1}\phi^2\,dx \\
&\leqslant \Bigg(\int\limits_{B_\eta(0)} \frac{|\nabla w|^2}{w^2}\phi^2\,dx\Bigg)^{\frac{q}{2}} \Bigg(\int\limits_{B_\eta(0)} w^{\frac{2(q-1)}{2-q}}\phi^2\,dx\Bigg)^{\frac{2-q}{2}} \\
&\leqslant \frac{q}{2}\varepsilon_0^{\frac{2}{q}} \int\limits_{B_\eta(0)} \frac{|\nabla w|^2}{w^2}\phi^2\,dx + \frac{2-q}{2}\varepsilon_0^{-\frac{2}{2-q}} \int\limits_{B_\eta(0)} w^{\frac{2(q-1)}{2-q}}\phi^2\,dx,
\end{aligned}$$

where ε_0 is a positive number that we will choose later. On the other hand we have

$$2\int\limits_{B_\eta(0)} \frac{\phi\nabla\phi}{w}\nabla w\,dx \leqslant \varepsilon_1^2 \int\limits_{B_\eta(0)} \frac{\phi^2|\nabla w|^2}{w^2}\,dx + \varepsilon_1^{-2}\int\limits_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

Hence it follows that

$$\lambda \int\limits_{B_\eta(0)} \frac{\phi^2}{|x|^2}\,dx \leqslant -\bigg(1 - \varepsilon_1^2 - \frac{q}{2}\varepsilon_0^{\frac{2}{q}}\bigg) \int\limits_{B_\eta(0)} \left|\frac{\nabla w}{w}\right|^2 \phi^2\,dx + \frac{2-q}{2}\varepsilon_0^{-\frac{2}{2-q}} \int\limits_{B_\eta(0)} w^{\frac{2(q-1)}{2-q}}\phi^2\,dx + \varepsilon_1^{-2}\int\limits_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

Fixed $\varepsilon_1 > 0$ such that $\varepsilon_1^2\lambda > \Lambda_N$, then we can fix ε_0 small enough such that $(1 - \varepsilon_1^2 - \frac{q}{2}\varepsilon_0^{\frac{2}{q}}) \geqslant 0$, thus we conclude that

$$\varepsilon_1^2\lambda \int\limits_{B_\eta(0)} \frac{\phi^2}{|x|^2}\,dx \leqslant \varepsilon_1^2 \frac{2-q}{2}\varepsilon_0^{-\frac{2}{2-q}} \int\limits_{B_\eta(0)} w^{\frac{2(q-1)}{2-q}}\phi^2\,dx + \int\limits_{B_\eta(0)} |\nabla\phi|^2\,dx.$$

We deal now with the mixed term,

$$\int\limits_{B_\eta(0)} w^{\frac{2(q-1)}{2-q}}\phi^2\,dx \leqslant \Bigg(\int\limits_{B_\eta(0)} |\phi|^{2^*}\,dx\Bigg)^{\frac{2}{2^*}} \Bigg(\int\limits_{B_\eta(0)} w^{\frac{N(q-1)}{2-q}}\,dx\Bigg)^{\frac{2}{N}}$$

$$\leqslant S^{-1}\left(\int_{B_\eta(0)} w^{\frac{N(q-1)}{2-q}}\,dx\right)^{\frac{2}{N}}\int_{B_\eta(0)}|\nabla\phi|^2\,dx,$$

where S is the classical Sobolev constant. Since $q<\frac{N}{N-1}$ we get $\frac{N(q-1)}{2-q}<\frac{N}{N-2}$, thus we conclude that

$$\int_{B_\eta(0)} w^{\frac{N(q-1)}{2-q}}\,dx\to 0\quad\text{as }\eta\to 0.$$

Hence we can choose ε_0 and ε_1 such that

$$\varepsilon_1^2\frac{2-q}{2}\varepsilon_0^{-\frac{2}{2-q}}S^{-1}\left(\int_{B_\eta(0)} w^{\frac{N(q-1)}{2-q}}\,dx\right)^{\frac{2}{N}}\to 0\quad\text{as }\eta\to 0.$$

Then there exists $\eta>0$ small enough and $\varepsilon_0,\varepsilon_1<1$ with $\varepsilon_1\sim 1$ such that

$$\varepsilon_1^2\lambda\left\{1+\varepsilon_1^2\frac{2-q}{2}\varepsilon_0^{-\frac{2}{2-q}}S^{-1}\left(\int_{B_\eta(0)} w^{\frac{N(q-1)}{2-q}}\,dx\right)^{\frac{2}{N}}\right\}^{-1}\equiv\lambda_1>\Lambda_N.$$

Therefore we conclude that

$$\lambda_1\int_{B_\eta(0)}\frac{\phi^2}{|x|^2}\,dx\leqslant\int_{B_\eta(0)}|\nabla\phi|^2\,dx,$$

a contradiction with Hardy inequality. $\square$

We will also take $g(x)=|x|^{-2}$ to show that the summability condition on f in Theorem 2.9 is optimal. Consider again $q<\frac{N}{N-1}$, $\lambda<\Lambda_N$, and $\alpha=\frac{N-2}{2}-\sqrt{\Lambda_N-\lambda}$. Assume that $\varphi\in W_0^{1,2}(\Omega)$ is the unique positive solution to

$$\begin{cases}-\Delta\varphi=\lambda\frac{\varphi}{|x|^2}+\frac{1}{|x|^2} & \text{in }\Omega,\\ \varphi=0 & \text{on }\partial\Omega.\end{cases}\tag{3.4}$$

An easy computation shows that $\varphi\simeq C|x|^{-\alpha}$ in $B_R(0)$, for some $R>0$. Applying Theorem 2.9 with $g(x)=\frac{1}{|x|^2}$, we deduce that if f is a nonnegative function such that $\int_\Omega f|x|^{-\alpha}\,dx<\infty$, then there exists $u\in W_0^{1,q}(\Omega)$ positive solution to (3.1) such that $\int_\Omega|\nabla u|^p\,dx<\infty$, $\forall p<\max\{q,\frac{N}{N-1}\}$.

Theorem 3.2. *Assume $q<\frac{N}{N-1}$. Let φ_λ be the solution to problem* (3.4). *There exists $\lambda(q)>0$ such that for all $0<\lambda<\lambda(q)$ and for all $f\in L^1(\Omega)$ satisfying $\int_\Omega f\varphi_\lambda\,dx=\infty$, problem* (3.1) *has no solution.*

Proof. We argue by contradiction. Assume that $u\in W_0^{1,q}(\Omega)$ is a positive solution to (3.1) such that $\frac{u}{|x|^2}\in L^1(\Omega)$. Using the regularity result for entropy solutions we obtain that $u\in W_0^{1,p}(\Omega)$ for all $p<\frac{N}{N-1}$. Let $\bar q\equiv\frac{N+2}{2(N-1)}+\frac{\sqrt{\Lambda_N-\lambda}}{N-1}$, then $0<\bar q<\frac{N}{N-1}$ and $\bar q\to\frac{N}{N-1}$ as $\lambda\to 0$. Hence there exists $\lambda(q)>0$ such that if $\lambda<\lambda(q)$, then $q<\bar q$. Notice that we also can assume that $\lambda(q)<\Lambda_N$. Consider φ_n solution to

$$-\Delta\varphi_n=\lambda a_n(x)\varphi_n+a_n(x),\quad \varphi_n=0\quad\text{on }\partial\Omega,\quad a_n(x)=\min\{n,|x|^{-2}\}.$$

Take φ_n as a test function in (3.1), then

$$\int_\Omega ua_n(x)\,dx+\int_\Omega|\nabla u|^q\varphi_n\,dx=\lambda\int_\Omega u\left(\frac{1}{|x|^2}-a_n\right)\varphi_n\,dx+\int_\Omega f\varphi_n\,dx\geqslant\int_\Omega f\varphi_n\,dx.$$

Since $\lambda<\Lambda_N$, then $\varphi_n<\varphi_{n+1}$ for all n and $\varphi_n\uparrow\varphi$, solution to (3.4). Therefore,

(i) $\int_\Omega u a_n(x)\,dx \to \int_\Omega \frac{u}{|x|^2}\,dx$ as $n \to \infty$.
(ii) Since $u \in W_0^{1,p}(\Omega)$ for all $p < \frac{N}{N-1}$, by the summability properties of φ_n and taking into account that $q < \bar{q}$, then Hölder inequality gives $\int_\Omega |\nabla u|^q \varphi_n\,dx \to \int_\Omega |\nabla u|^q \varphi\,dx < \infty$.

But then we reach a contradiction with the fact that $\int_\Omega f\varphi_n\,dx \to \int_\Omega f\varphi\,dx = \infty$. □

Remark 3.3. If $\lambda = \Lambda_N$, let $H(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|\psi\|_H^2 = \int_\Omega \left(|\nabla\psi|^2 - \left(\frac{N-2}{2}\right)^2 \frac{|\psi|^2}{|x|^2}\right) dx.$$

Problem (3.4) with $\lambda = \Lambda_N$ has a solution $\varphi \in H(\Omega)$ and then problem (3.1) has a solution provided $\int_\Omega f|x|^{-\frac{N-2}{2}}\,dx < \infty$.

3.1. Further remarks: $g(x) \equiv |x|^{-\alpha}$

We will consider the problem (1.1) with $g(x) = \frac{1}{|x|^\alpha}$, $\alpha \in (-\infty, \frac{N+2}{2})$, $1 < q \leqslant 2$, and f a suitable positive function. To avoid the trivial cases, hereafter we will consider $\alpha \in (1, \frac{N+2}{2})$. Namely, we deal with the problem

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda \frac{u}{|x|^\alpha} + f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

As a direct application of the existence result proved in Theorem 2.3, we get the following consequence.

Corollary 3.4. *Assume that $\alpha \in (1, \frac{N+2}{2})$ and define $q_\alpha = \max\{1, \frac{N}{(N+1)-\alpha}\}$. If $q > q_\alpha$, then, for all $\lambda \geqslant 0$ and for all $f \in L^1(\Omega)$, $f \geqslant 0$, problem* (3.5) *has a solution.*

Proof. Thanks to Theorem 2.3, it is sufficient to prove that $\frac{1}{|x|^\alpha} \in L^r(\Omega)$, with $r > (q^*)'$. Since $q > q_\alpha$, then $\alpha(q^*)' = \frac{\alpha N q}{(N+1)q - N} < \frac{\alpha N q_\alpha}{(N+1)q_\alpha - N} \leqslant N$. Hence we conclude that $\frac{1}{|x|^\alpha} \in L^r(\Omega)$ for some $r > (q^*)'$ and the result follows. □

If $\alpha = \frac{N+2}{2}$, then $q_\alpha = 2$ and we cannot apply Theorem 2.3. However we have the following nonexistence result.

Theorem 3.5. *Assume that $\alpha = \frac{N+2}{2}$ and consider $q = 2$, then problem* (3.5) *has no positive solution for $\lambda > \frac{(N-2)(N-1)}{2}$.*

Proof. We suppose by contradiction that there exists $u \in W_0^{1,2}(\Omega)$, a positive solution to (3.5) with $\alpha = \frac{N+2}{2}$ and $q = 2$. Thanks to Lemma 2.11 there exist constants $C, R > 0$ such that $u \geqslant C$ in $B_R(0)$. Let β be a parameter we will choose later and define $w_n(x) = \frac{1}{(r+\frac{1}{n})^\beta}$ in $B_R(0)$. For the scaled function, $\overline{w}_n = \gamma w_n$, we have

$$\begin{aligned} -\Delta\overline{w}_n(x) + |\nabla\overline{w}_n|^2 &= -\gamma\Delta w_n(x) + \gamma^2|\nabla w_n|^2 \\ &= \frac{\gamma}{r(r+1/n)^{\beta+1}}\left[(N-2-\beta)\beta + \frac{\beta}{n}\frac{(\beta+1)}{(r+1/n)}\right] + \frac{\gamma^2\beta^2}{(r+1/n)^{2(\beta+1)}} \\ &\leqslant \frac{\overline{w}_n}{r(r+1/n)}\big[(N-2-\beta)\beta + \beta(\beta+1)\big] + \frac{\gamma\beta^2\overline{w}_n}{(r+1/n)^{\beta+2}} \\ &\leqslant \frac{\overline{w}_n}{r(r+1/n)^{\beta+1}}\big[(N-2-\beta)\beta + \beta(\beta+1) + \gamma\beta^2\big]. \end{aligned}$$

We set $\beta = \frac{N-2}{2}$ and then, choosing γ small enough and using the fact that $\lambda > \frac{(N-2)(N-1)}{2}$,

$$-\Delta\overline{w}_n(x) + |\nabla\overline{w}_n|^2 \leqslant \lambda\frac{\overline{w}_n}{|x|^{\frac{N+2}{2}}} \leqslant \lambda\frac{\overline{w}_n}{|x|^{\frac{N+2}{2}}} + f \quad \text{in } B_R(0),$$

namely, w_n is a subsolution to (3.5) with $\alpha = \frac{N+2}{2}$ and $q = 2$. Since $u \in W_0^{1,2}(\Omega)$ is a solution to (3.5), then in particular is a supersolution. Similar calculations as in Lemma 2.12, give that $w_n \leqslant u$ in $B_R(0)$, thus $\overline{w}_n/|x|^{\frac{N+2}{2}} \leqslant u/|x|^{\frac{N+2}{2}}$. Since $u/|x|^{\frac{N+2}{2}} \in L^1(\Omega)$, then using the Lebesgue theorem we can pass to the limit as $n \to \infty$ to obtain that $\gamma/|x|^{\frac{N+2}{2}+\beta} \in L^1(\Omega)$, a contradiction with the choice of β. □

In the case where $q = 1$ we have the following nonexistence result.

Theorem 3.6. *Assume $q = 1$. If $\alpha > 2$, then there is no positive solution to problem* (3.5) *for any $\lambda > 0$. If $\alpha = 2$ then problem* (3.5) *has no positive solution for $\lambda > \Lambda_N$. If $\alpha < 2$, there exists $\lambda^* > 0$ such that problem* (3.5) *has no positive solution for $\lambda > \lambda^*$.*

Proof. We follow an argument by contradiction. Suppose that u is a weak solution to (3.5) with $q = 1$. Taking $\frac{\phi^2}{u}$ as test function in (3.5) with $q = 1$ and $\phi \in \mathcal{C}_0^\infty(\Omega)$, we get

$$-\int_\Omega \frac{|\nabla u|^2\phi^2}{u^2}\,dx + 2\int_\Omega \frac{\phi\nabla\phi}{u}\nabla u\,dx + \int_\Omega \frac{|\nabla u|\phi^2}{u}\,dx \geqslant \lambda\int_\Omega \frac{\phi^2}{|x|^\alpha}\,dx. \tag{3.6}$$

As above we have

$$\int_\Omega \frac{|\nabla u|\phi^2}{u}\,dx \leqslant \varepsilon\int_\Omega \left|\frac{\nabla u}{u}\right|^2\phi^2\,dx + C(\varepsilon)\int_\Omega \phi^2\,dx,$$

$$2\int_\Omega \frac{\phi\nabla\phi}{u}\nabla u\,dx \leqslant \bar\varepsilon\int_\Omega \phi^2\left|\frac{\nabla u}{u}\right|^2\,dx + \frac{1}{\bar\varepsilon}\int_\Omega |\nabla\phi|^2\,dx.$$

Therefore it follows that the right term of Eq. (3.6) satisfies

$$\lambda\int_\Omega \frac{\phi^2}{|x|^\alpha}\,dx \leqslant -(1-\varepsilon-\bar\varepsilon)\int_\Omega \left|\frac{\nabla u}{u}\right|^2\phi^2\,dx + C(\varepsilon)\int_\Omega \phi^2\,dx + \frac{1}{\bar\varepsilon}\int_\Omega |\nabla\phi|^2\,dx.$$

Choosing $\bar\varepsilon > 0$ in such a way that $\varepsilon < 1 - \bar\varepsilon$, we get

$$\lambda\int_\Omega \frac{\phi^2}{|x|^\alpha}\,dx \leqslant C(\varepsilon)\int_\Omega \phi^2\,dx + \frac{1}{\bar\varepsilon}\int_\Omega |\nabla\phi|^2\,dx.$$

(i) If $\alpha > 2$, then independently of the value of λ, using the Poincaré inequality we reach a contradiction with the classical Hardy inequality, hence there is no solution for any $\lambda > 0$.

(ii) If $\alpha = 2$, we choose $\bar\varepsilon < 1$ such that $\Lambda_N < \lambda\bar\varepsilon$, hence we get the existence of a positive number $\sigma > 0$ such that

$$(\Lambda_N + \sigma)\int_\Omega \frac{\phi^2}{|x|^2}\,dx \leqslant \lambda\bar\varepsilon\int_\Omega \frac{\phi^2}{|x|^2}\,dx \leqslant \bar\varepsilon C(\varepsilon)\int_\Omega \phi^2\,dx + \int_\Omega |\nabla\phi|^2\,dx,$$

thus

$$(\Lambda_N + \sigma)\int_\Omega \frac{\phi^2}{|x|^2}\,dx \leqslant \bar\varepsilon C(\varepsilon)\int_\Omega \phi^2 dx + \int_\Omega |\nabla\phi|^2\,dx. \tag{3.7}$$

Consider

$$A(c) = \inf_{\psi\in\mathcal{C}_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\psi|^2\,dx + c\int_\Omega \psi^2\,dx}{\int_\Omega \frac{\psi^2}{|x|^2}\,dx},$$

then using a dilatation argument we can prove that $A(c) = \Lambda_N$ for all $c \geqslant 0$. Hence we get a contradiction with (3.7). Therefore there is no solution for $\lambda > \lambda_N$.

(iii) If $\alpha < 2$, then we get easily the existence of positive constants c_1 and c_2 depending only on the data such that

$$\lambda \int_\Omega \frac{\phi^2}{|x|^\alpha}\,dx \leqslant c_1 \int_\Omega \phi^2\,dx + c_2 \int_\Omega |\nabla\phi|^2\,dx \leqslant (c_1\lambda_1 + c_2)\int_\Omega |\nabla\phi|^2\,dx,$$

where λ_1 is the first eigenvalue of the Laplacian operator in Ω. By setting $\lambda^* = (c_1\lambda_1 + c_2)\lambda_1(\frac{1}{|x|^\alpha})$, we conclude that problem (3.5) has no solution for $\lambda > \lambda^*$. □

Finally, if $\alpha > 2$ and $q \leqslant \frac{N}{N-1}$, it is no hard to prove that problem (3.5) has no positive solution for any $\lambda > 0$.

Acknowledgements

The authors would like to express their gratitude to the anonymous referee for his/her kind suggestions which have improved the final form of the manuscript.

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, Some remarks on elliptic problems with critical growth on the gradient, J. Differential Equations 222 (1) (2006) 21–62.
[2] B. Abdellaoui, I. Peral, The equation $-\Delta u - \lambda\frac{u}{|x|^2} = |\nabla u|^p + cf(x)$: The optimal power, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) VI (2007) 1–25.
[3] B. Abdellaoui, I. Peral, A. Primo, Some elliptic problems with Hardy potential and critical growth in the gradient: non-resonance and blow-up results, J. Differential Equations 239 (2007) 386–416.
[4] N. Alaa, M. Pierre, Weak solutions of some quasilinear elliptic equations with data measures, SIAM. J. Math. Anal. 24 (1) (1993) 23–35.
[5] L. Boccardo, T. Gallouët, L. Orsina, Existence and nonexistence of solutions for some nonlinear elliptic equations, J. Anal. Math. 73 (1997) 203–223.
[6] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. TMA 19 (6) (1992) 581–597.
[7] L. Boccardo, F. Murat, J.P. Puel, Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique, in: L. Lions, H. Brezis (Eds.), Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, vol. IV, in: Research Notes in Math, vol. 84, Pitman, London, 1983, pp. 19–73.
[8] L. Boccardo, F. Murat, J.P. Puel, Resultats d'existence pour certains problèmes elliptiques quasi-linéaires, Ann. Sc. Norm. Sup. Pisa 11 (2) (1984) 213–235.
[9] L. Boccardo, F. Murat, J.P. Puel, Existence des solutions non bornées pour certains équations quasi-linéaires, Portugal. Math. 41 (1982) 507–534.
[10] L. Boccardo, L. Orsina, I. Peral, A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential, Discrete and Continuous Dynamical Systems 16 (3) (2006) 513–523.
[11] H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^N$, Manuscripta Math. 74 (1992) 87–106.
[12] H. Brezis, E.H. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[13] H. Brezis, A. Ponce, Kato's inequality when Δu is a measure, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 599–604.
[14] H. Brezis, W.A. Strauss, Semilinear second order elliptic equations in L^1, J. Math. Soc. Japan 25 (1973) 565–590.
[15] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Sc. Norm. Super. Pisa Cl. Sci. 28 (4) (1999) 741–808.
[16] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. 22 (2) (1995) 241–273.
[17] V. Ferone, V.F. Murat, Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small, in: Equations aux dérivées partialles et applications, Gauthier-Villars, Ed. Sci. Méd. Elsevier, Paris, 1998, pp. 497–515.
[18] J. Garcia Azorero, I. Peral, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.
[19] K. Hansson, V.G. Maz'ya, I.E. Verbitsky, Criteria of solvability for multidimensional Riccati equations, Ark. Mat. 37 (1999) 87–120.
[20] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972) 135–148.
[21] E.H. Lieb, M. Loos, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.
[22] P.L. Lions, Résolution de problèmes elliptiques quasilinéaires, Arch. Rational Mech. Anal. 74 (4) (1980) 335–353.
[23] P. Pucci, J. Serrin, The strong maximum principle revisited, J. Differential Equations 196 (1) (2004) 1–66.
[24] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier 15 (1965) 189–258.