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Energy concentration for the Landau–Lifshitz equation

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Abstract

For the Landau–Lifshitz equation on a domain with three space dimensions, we consider energy concentration phenomena arising in the context of weakly convergent sequences of solutions. The concentration measure can be interpreted as a family of generalized curves. We establish a connection to a geometric flow.

Résumé

Pour l'équation de Landau–Lifshitz sur un domaine en trois dimensions d'espace, nous considérons des phénomènes de concentration d'énergie survenant dans le contexte des suites faiblement convergentes de solutions. La mesure de concentration peut être interprétée comme famille de courbes généralisées. Nous établissons une connexion avec un flot géométrique.

Keywords: Landau–Lifshitz equation; Energy concentration; Geometric flow; Curvature

1. Introduction

For an open set $\Omega \subset \mathbb{R}^3$ and $T > 0$, define $\Omega_T = \Omega \times (0, T)$. We consider the Landau–Lifshitz equation

$$\frac{\partial u}{\partial t} + au \wedge (u \wedge \Delta u) + bu \wedge \Delta u = 0 \quad \text{in } \Omega_T \tag{1}$$

for a map $u : \Omega_T \to \mathbb{S}^2$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit 2-sphere. Here $\wedge$ denotes the vector product in $\mathbb{R}^3$ and $a, b \in \mathbb{R}$ are fixed constants. We assume that $a > 0$, which makes the equation parabolic.

For a map $u \in C^\infty(\Omega, \mathbb{S}^2)$, the Laplacian Δu has the orthogonal decomposition

$$-\Delta u = |\nabla u|^2 u + u \wedge (u \wedge \Delta u). \tag{2}$$

The expression $\Delta u + |\nabla u|^2 u$ is called the tension field of u; it is minus the L^2-gradient of the Dirichlet energy

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$$

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under the constraint that u takes values in $\mathbb{S}^2$. Using the decomposition (2), we find that (1) is equivalent to

$$\tilde{a}\frac{\partial u}{\partial t} + \tilde{b}u \wedge \frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u \tag{3}$$

as well as

$$\tilde{a}u \wedge \frac{\partial u}{\partial t} - \tilde{b}\frac{\partial u}{\partial t} = \operatorname{div}(u \wedge \nabla u), \tag{4}$$

where

$$\tilde{a} = \frac{a}{a^2+b^2} \quad \text{and} \quad \tilde{b} = \frac{b}{a^2+b^2}.$$

Both (3) and (4) can also be interpreted in the weak sense for a map in the Sobolev space

$$H^1_{\mathrm{loc}}(\Omega_T, \mathbb{S}^2) = \big\{u \in H^1_{\mathrm{loc}}(\Omega_T, \mathbb{R}^3)\colon\ |u| = 1 \text{ almost everywhere}\big\}.$$

Eq. (1) stems from a model in micromagnetics which describes the dynamics of the magnetization vector field of a ferromagnetic body. The equation is also of geometrical interest, however. It can be thought of as a hybrid of two other geometric evolution equations: the so-called harmonic map heat flow

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u \tag{5}$$

(which is the negative L^2-gradient flow for the functional E) on the one hand and the equation

$$\frac{\partial u}{\partial t} = u \wedge \Delta u \tag{6}$$

on the other hand. Solutions of (6) are often called Schrödinger maps, because the equation is of the type of a non-linear Schrödinger equation (which is most obvious when u is composed with the stereographic projection).

Suppose that we have a sequence of solutions $u_k \in C^\infty(\overline{\Omega} \times [0,T), \mathbb{S}^2)$ of (1) such that

$$\sup_{k\in\mathbb{N}} E\big(u_k(\cdot,0)\big) < \infty.$$

Also assume for the moment that Ω is bounded and has a smooth boundary with outer normal vector ν, and that $\nu \cdot \nabla u_k = 0$ on $\partial\Omega \times [0,T)$. Taking the scalar product with Δu_k on both sides of (1), integrating over $\Omega \times \{t\}$, and performing an integration by parts, we see that

$$\frac{d}{dt}E\big(u_k(\cdot,t)\big) + a \int_{\Omega\times\{t\}} |u_k \wedge \Delta u_k|^2\,dx = 0. \tag{7}$$

Hence

$$E(u_k(\cdot,t_0)) + a \int_0^{t_0}\int_\Omega |u_k \wedge \Delta u_k|^2\,dx\,dt = E\big(u_k(\cdot,0)\big)$$

for every $t_0 \in [0,T)$. In particular the sequence $\{u_k\}$ is bounded in $H^1(\Omega_T, \mathbb{R}^3)$ and there exists a weakly convergent subsequence $\{u_{k_i}\}$. Using the representation (4) for the Landau–Lifshitz equation, it is readily verified that the limit map is a weak solution. The observed convergence, however, is not strong in $H^1(\Omega_T, \mathbb{S}^2)$ in general, since the energy density $\frac{1}{2}|\nabla u_{k_i}|^2$ may concentrate near a certain subset of Ω_T. Suppose that $\mathcal{L}^4$ is the Lebesgue measure on Ω_T. Then a way to describe this energy concentration is to consider the Radon measures

$$m_k = \frac{1}{2}\mathcal{L}^4 \llcorner |\nabla u_k|^2.$$

For a suitable choice of the above subsequence, we have convergence of m_{k_i} to a Radon measure m on Ω_T, which need not be absolutely continuous with respect to $\mathcal{L}^4$. The singular part of m measures the energy concentration, therefore we call it the concentration measure. In the situation studied here, it turns out that the concentration measure can be interpreted as a geometric object. Namely, for almost every $t \in [0,T)$, it gives rise to a type of generalized curve in Ω. Our aim is to study the evolution of these generalized curves.

Without the above boundary conditions, the energy identity (7) is no longer valid, but it can be replaced by a localized version. For any $\eta \in C_0^1(\Omega_T)$, a smooth solution of the Landau–Lifshitz equation satisfies

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\times\{t\}} \eta|\nabla u|^2\,dx = \int_{\Omega\times\{t\}} \left(\frac{1}{2}\frac{\partial\eta}{\partial t}|\nabla u|^2 - a\eta|u\wedge\Delta u|^2 - a\nabla\eta\cdot\langle\Delta u,\nabla u\rangle + b\nabla\eta\cdot\langle u\wedge\Delta u,\nabla u\rangle\right)dx. \tag{8}$$

Here and throughout the paper, we denote scalar products in $T\Omega$ by a dot and in $T\mathbb{S}^2$ by $\langle\cdot,\cdot\rangle$ for clarity. Using (8) instead of (7), we now obtain local estimates for the energy under suitable conditions, and the questions we study remain the same.

For the harmonic map heat flow (5), the corresponding problem has been studied by Li and Tian [12] and by Lin and Wang [18] (also for higher-dimensional domains and other target manifolds). These papers give a connection between the energy concentration measure and the mean curvature flow. We will show a similar connection for the Landau–Lifshitz equation, but the mean curvature flow has to be replaced by another geometric flow.

We briefly recall the definition of the mean curvature flow, before we modify it in order to obtain the equation that is relevant in the context of this paper. For simplicity, we restrict our attention to the flow for closed curves in Ω here. In this situation, the mean curvature flow is also known as the curve shortening flow. Suppose that $F:\mathbb{S}^1\times[0,T)\to\Omega$ is a smooth map such that $\Sigma_t = F(\mathbb{S}^1\times\{t\})$ is an embedded curve for every $t\in[0,T)$. Let $H_t:\Sigma_t\to\mathbb{R}^3$ be the curvature vector for Σ_t. We say that F is a solution of the mean curvature flow (curve shortening flow) if it satisfies the equation

$$\frac{\partial F}{\partial t}(s,t) = H_t\big(F(s,t)\big) \quad \text{for } s\in\mathbb{S}^1,\ t\in[0,T),$$

or in shorter notation,

$$\frac{\partial F}{\partial t} = H_t. \tag{9}$$

This is the negative L^2-gradient flow for the length functional. We write $\mathcal{H}^1$ for the 1-dimensional Hausdorff measure. If we have a smooth solution of (9), then for any $\eta\in C_0^1(\Omega_T)$, we can compute

$$\frac{d}{dt}\int_{\Sigma_t}\eta\,d\mathcal{H}^1 = \int_{\Sigma_t}\left(\frac{\partial\eta}{\partial t} - \eta|H_t|^2 + \nabla\eta\cdot H_t\right)d\mathcal{H}^1. \tag{10}$$

A relaxed version of this identity (with the equality replaced by an inequality for test functions with non-negative values) was used by Brakke [3] to define a generalization of the mean curvature flow. This generalization is also the formulation that is used in [18], and with a modification in [12], to describe the evolution of the energy concentration set for the harmonic map heat flow.

Next we regard Σ_t as oriented curves, and we choose a unit tangent vector $\tau_t:\Sigma_t\to\mathbb{S}^2$ that is continuous (also with respect to t). We replace (9) by the equation

$$\frac{\partial F}{\partial t} = aH_t - b\tau_t\wedge H_t, \tag{11}$$

or, equivalently,

$$\tilde{a}\frac{\partial F}{\partial t} + \tilde{b}\tau_t\wedge\frac{\partial F}{\partial t} = H_t.$$

Then instead of (10) we obtain

$$\frac{d}{dt}\int_{\Sigma_t}\eta\,d\mathcal{H}^1 = \int_{\Sigma_t}\left(\frac{\partial\eta}{\partial t} - a\eta|H_t|^2 + a\nabla\eta\cdot H_t - b\nabla\eta\cdot\tau_t\wedge H_t\right)d\mathcal{H}^1. \tag{12}$$

A certain formal similarity between (8) and (12) can immediately be seen. We will establish a rigorous connection in this paper and show that (a generalized version of) the flow (11) describes the behaviour of the concentration measure for the Landau–Lifshitz equation in a certain sense.

Before we can state our main result, we need some notation. We assume that the reader is familiar with the notion of countable rectifiability. If not, we refer to Federer [7] or Simon [24]. Suppose $\Sigma\subset\Omega$ is a countably 1-rectifiable set. Then we write $T_x\Sigma$ for the approximate tangent space at every point $x \in \Sigma$ where it exists. If $\phi \in C^1(\Omega, \mathbb{R}^3)$ and L is a linear subspace of $\mathbb{R}^3$, then $\operatorname{div}_L \phi(x)$ is the divergence of ϕ at x with respect to L. That is, if Π_L denotes the matrix of the orthogonal projection onto L, then

$$\operatorname{div}_L \phi(x) = \operatorname{tr}\big(\Pi_L \nabla\phi(x)\big).$$

Similarly, we write

$$\operatorname{div}_\Sigma \phi(x) = \operatorname{div}_{T_x\Sigma} \phi(x).$$

Suppose that $p \in \mathbb{S}^2$ and $X = (X_1, X_2, X_3)$, $Y = (Y_1, Y_2, Y_3) \in \mathbb{R}^3 \otimes T_p\mathbb{S}^2$; then we write $X \otimes Y$ for the (3×3)-matrix with (α, β)-th component $\langle X_\alpha, Y_\beta\rangle$.

Theorem 1.1. *Suppose $u_k \in C^\infty(\Omega_T, \mathbb{S}^2)$ are solutions of the Landau–Lifshitz equation* (1) *such that*

$$\sup_{k\in\mathbb{N}} \int_{\Omega_T} |\nabla u_k|^2\, dz < \infty.$$

Then there exists a subsequence $\{u_{k_i}\}$ such that the following is true.

(i) *There exists a weak solution $u \in H^1_{\mathrm{loc}}(\Omega_T, \mathbb{S}^2)$ of (4) such that $u_{k_i} \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\Omega_T, \mathbb{R}^3)$ and $\Delta u + |\nabla u|^2 u \in L^2_{\mathrm{loc}}(\Omega_T, \mathbb{R}^3)$.*

(ii) *There exists a function $\theta : \Omega_T \to \mathbb{N} \cup \{0\}$ such that for almost every $t \in [0, T)$,*
- *$\theta_t = \theta(\cdot, t)$ is locally $\mathcal{H}^1$-integrable, and*
- *$\Sigma_t = \theta_t^{-1}(\mathbb{N})$ is closed relative to Ω and countably* 1*-rectifiable,*

and for every $\eta \in C^0_0(\Omega_T)$,

$$\lim_{i\to\infty} \int_0^T\int_\Omega \eta|\nabla u_{k_i}|^2\, dx\, dt = \int_0^T\int_\Omega \eta|\nabla u|^2\, dx\, dt + 8\pi \int_0^T\int_\Omega \eta\theta\, d\mathcal{H}^1\, dt. \tag{13}$$

(iii) *There exists a vector field $H : \Omega_T \to \mathbb{R}^3$ such that for almost every $t \in [0, T)$,*
- *$H_t = H(\cdot, t) \in L^2_{\mathrm{loc}}(\mathcal{H}^1 \llcorner \theta_t, \mathbb{R}^3)$, and*
- *$H_t(x) \perp T_x\Sigma_t$ at $\mathcal{H}^1$-almost every $x \in \Sigma_t$,*

and for every $\phi \in C^1_0(\Omega_T, \mathbb{R}^3)$,

$$\int_0^T\int_\Omega \left(\frac{1}{2}|\nabla u|^2 \operatorname{div}\phi - \operatorname{tr}\big((\nabla u \otimes \nabla u)\nabla\phi\big) - \langle \Delta u, \nabla u\rangle \cdot \phi\right) dx\, dt$$
$$+ 4\pi \int_0^T\int_\Omega (\operatorname{div}_{\Sigma_t}\phi + H\cdot\phi)\theta\, d\mathcal{H}^1\, dt = 0. \tag{14}$$

(iv) *There exist three vector fields $\tau : \Omega_T \to \mathbb{S}^2$ and $H^+, H^- : \Omega_T \to \mathbb{R}^3$ with $H^+ + H^- = H$, such that for almost every $t \in [0, T)$,*
- *$\tau_t = \tau(\cdot, t)$ is $\mathcal{H}^1$-measurable,*
- *$H_t^\pm = H^\pm(\cdot, t) \in L^2_{\mathrm{loc}}(\mathcal{H}^1 \llcorner \theta_t, \mathbb{R}^3)$, and*
- *$\tau_t(x) \in T_x\Sigma_t$ and $H_t^\pm(x) \perp T_x\Sigma_t$ for $\mathcal{H}^1$-almost every $x \in \Sigma_t$,*

and for every $\eta \in C^1_0(\Omega_T, [0, \infty))$,

$$\int_0^T\int_\Omega \left(\frac{1}{2}|\nabla u|^2 \frac{\partial\eta}{\partial t} - a\eta|u \wedge \Delta u|^2 - a\nabla\eta \cdot \langle \Delta u, \nabla u\rangle + b\nabla\eta \cdot \langle u \wedge \Delta u, \nabla u\rangle\right) dx\, dt$$
$$+ 4\pi \int_0^T\int_\Omega \left(\frac{\partial\eta}{\partial t} - a\eta|H|^2 + a\nabla\eta \cdot H - b\nabla\eta \cdot \big(\tau \wedge (H^+ - H^-)\big)\right)\theta\, d\mathcal{H}^1\, dt \geqslant 0. \tag{15}$$

Next we give a few remarks on how the quantities in this theorem can be interpreted. First note that the functions θ_t, which are supported on the countably 1-rectifiable sets Σ_t, may be regarded as multiplicity functions for a generalized type of curves in Ω. By (13), these generalized curves describe the energy concentration for the subsequence $\{u_{k_i}\}$. If u happens to be sufficiently smooth, then it is readily checked that the first integral in (14) vanishes. Hence in this case we have

$$\int_{\Sigma_t} (\operatorname{div}_{\Sigma_t} \phi + H_t \cdot \phi)\theta_t \, d\mathcal{H}^1 = 0$$

for every $\phi \in C_0^1(\Omega, \mathbb{R}^3)$ and almost every $t \in [0, T)$. In other words, the vector field H_t is the curvature of the generalized curve given by θ_t. In general, however, the above identity may be false. The pair consisting of the map $u(\cdot, t)$ and θ_t should then be thought of as a geometrical object which has a "curvature" given jointly by $\Delta u + |\nabla u|^2 u$ and H_t.

Inequality (15), finally, describes (partially) the evolution of the generalized curves given by θ_t. Again, if u is smooth, then the first integral vanishes and we have

$$\int_0^T \int_{\Sigma_t} \left(\frac{\partial \eta}{\partial t} - a\eta |H_t|^2 + a\nabla\eta \cdot H_t - b\nabla\eta \cdot \big(\tau_t \wedge (H_t^+ - H_t^-)\big) \right) \theta_t \, d\mathcal{H}^1 \, dt \geqslant 0$$

for every $\eta \in C_0^1(\Omega_T)$ with $\eta \geqslant 0$. This is a relaxed version of (12), integrated over $[0, T)$—up to one discrepancy in the last term of the integrand. The fact that the quantity $\tau_t \wedge H_t$ in (12) splits up into two parts is not surprising, since the multiplicity function θ_t may encode a piece of a curve several times with opposite orientations. If this is the reason for the appearance of H_t^+ and H_t^-, one would suspect that they must be parallel to H_t and the ratios of the lengths must be fractions with the denominator θ_t almost everywhere. We are unable, however, to prove this, owing to the lack of a sufficiently strong convergence of the relevant quantities.

For the proof of Theorem 1.1, we use similar arguments as in [12] and [18] up to a certain point. In contrast to the harmonic map heat flow, however, the Landau–Lifshitz equation gives rise to an additional difficulty. The terms

$$b \int_{\Omega \times \{t\}} \nabla\eta \cdot \langle u \wedge \Delta u, \nabla u \rangle \, dx \quad \text{and} \quad b \int_{\Sigma_t} \nabla\eta \cdot \tau_t \wedge H_t \, d\mathcal{H}^1$$

in (8) and (12), respectively, are harder to control than the other terms. Unlike the terms with the coefficient a, they cannot be simplified by a mere integration by parts. This is the reason for most of the new concepts and arguments that we use.

Several generalizations of the theorem are conceivable. On the one hand, we may consider a sequence of weak solutions of the Landau–Lifshitz equation. In order to apply the tools that we need for the proof, we have to impose certain additional conditions (such as the stability hypothesis introduced by Feldman [8] for the harmonic map heat flow; cf. [20] for a version for the Landau–Lifshitz equation). But then a similar result follows with the same methods. On the other hand, we may change the dimension of Ω. The expected energy concentration set is always of codimension 2. Thus in dimensions 1 and 2, it is clearly not described by a curvature driven flow. No result of the type of Theorem 1.1 can then be expected. In dimension 4, the same arguments that we use in this paper still work with only minor modifications. In order to keep the presentation simple, we leave it to the reader to verify this (as well as the results for weak solutions). In dimension 5 and higher, however, several of the tools that we use are no longer available. The most important obstacle to proving results similar to Theorem 1.1 in higher dimensions is the lack of a so-called monotonicity formula for the Landau–Lifshitz equation. Such a formula exists, e.g., for the harmonic map heat flow (cf. Struwe [26]) and is used extensively in [12] and [18].

We close this section with the introduction of some more notation. We write $B_r(x_0)$ for an open ball in $\mathbb{R}^3$ with centre x_0 and radius r. Sometimes we work with two-dimensional balls, and then we normally use the notation $B_r^2(x_0')$ to avoid confusion (where $x_0' \in \mathbb{R}^2$). At one point, however, we work exclusively in $\mathbb{R}^2$, and then we drop the superscript. Furthermore, we use the abbreviations $B_r = B_r(0)$ and $B_r^2 = B_r^2(0)$. We denote the j-dimensional Lebesgue measure by $\mathcal{L}^j$ and the j-dimensional Hausdorff measure by $\mathcal{H}^j$ (as we have already done for certain dimensions).

We write $z = (x,t)$ for a generic point in $\mathbb{R}^3 \times \mathbb{R}$. In space-time, it is natural to use the parabolic metric

$$d\big((x,t),(y,s)\big) = \max\big\{|x-y|, \sqrt{|t-s|}\big\}$$

rather than the Euclidean metric in the context of a parabolic problem such as (1). Therefore, we also consider "balls" in this metric, for which we use the notation

$$P_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2).$$

Moreover, $P_r = P_r(0)$. The j-dimensional Hausdorff measure with respect to d is denoted by $\mathcal{H}_d^j$.

We often work with Radon measures, and we use the fact that a Radon measure (on Ω, say) can be identified with a functional in the dual space of $C_0^0(\Omega)$. When we indicate a convergence of a sequence of Radon measures, we always mean weak* convergence in $(C_0^0(\Omega))^*$.

2. A few tools from geometric measure theory

To study a blow-up measure as in Theorem 1.1, the notion of generalized varifolds that has been introduced by Ambrosio and Soner [2], and independently by Lin [14,15], is very useful. For the problem that we study in this paper, we have to modify the theory somewhat, in order to make sense of the notion of an orientation in the generalized setting. For convenience, we also use a different representation for the generalized varifolds. Before we give the details, we recall the basic definitions for (ordinary) varifolds. For further details, see Allard [1] or Simon [24]. A good source for other information on geometric measure theory is the book by Federer [7].

Let $G(3,1)$ be the Grassmann manifold of all 1-dimensional linear subspaces of $\mathbb{R}^3$ (which can of course be identified with the real projective plane). Moreover, let $G_1(\Omega) = \Omega \times G(3,1)$. A 1-varifold on Ω is a Radon measure on $G_1(\Omega)$. Important examples are the so-called integral varifolds. Suppose $\theta : \Omega \to \mathbb{N} \cup \{0\}$ is a locally $\mathcal{H}^1$-integrable function such that $\Sigma = \theta^{-1}(\mathbb{N})$ is countably 1-rectifiable. Then the varifold V on Ω such that

$$\int_{G_1(\Omega)} \psi \, dV = \int_{\Sigma} \psi(x, T_x\Sigma)\theta(x)\, d\mathcal{H}^1(x)$$

for every $\psi \in C_0^0(G_1(\Omega))$, is called an integral 1-varifold. (Thus in particular the functions θ_t in Theorem 1.1 give rise to integral varifolds.)

We also consider the Grassmann manifold $G^0(3,1)$ consisting of all oriented 1-dimensional subspaces of $\mathbb{R}^3$ (in other words, the 2-sphere). We set $G_1^0(\Omega) = \Omega \times G^0(3,1)$. An oriented 1-varifold on Ω is a Radon measure on $G_1^0(\Omega)$. If V is an oriented 1-varifold on Ω, then the projection $P : G^0(3,1) \to G(3,1)$ induces naturally a 1-varifold on Ω by the push-forward $(\mathrm{id}_\Omega \times P)_\# V$.

Suppose θ and Σ are as above and $\tau : \Omega \to G^0(3,1)$ is an $\mathcal{H}^1$-measurable function such that $P(\tau(x)) = T_x\Sigma$ for $\mathcal{H}^1$-almost every $x \in \Sigma$. Furthermore, let $\theta_1, \theta_2 : \Omega \to \mathbb{N} \cup \{0\}$ be $\mathcal{H}^1$-measurable functions such that $\theta = \theta_1 + \theta_2$. Then the oriented 1-varifold W on Ω defined by the condition that

$$\int_{G_1^0(\Omega)} \psi \, dW = \int_{\Sigma} \big(\psi\big(x, \tau(x)\big)\theta_1(x) + \psi\big(x, -\tau(x)\big)\theta_2(x)\big)\, d\mathcal{H}^1(x)$$

for every $\psi \in C_0^0(G_1(\Omega))$, is called an oriented integral 1-varifold.

For a 1-varifold V on Ω, the first variation δV is the functional on $C_0^1(\Omega, \mathbb{R}^3)$ given by

$$\delta V(\phi) = \int_{G_1(\Omega)} \operatorname{div}_L \phi(x)\, dV(x,L).$$

For an oriented 1-varifold W, we set $\delta W = \delta(\mathrm{id}_\Omega \times P)_\# W$.

We now give another representation of the same concepts, before we finally generalize them. Note that any $L \in G(3,1)$ can be identified with the (3×3)-matrix belonging to the orthogonal projection onto the orthogonal complement of L. We denote this matrix by $\Pi_L^\perp$, and similarly we write Π_L for the matrix of the orthogonal projection onto L. Any point $\tau \in \mathbb{S}^2 = G^0(3,1)$ can be identified with the pair $(\Pi_{P(\tau)}^\perp, \Lambda_\tau)$, where $\Lambda_\tau \in \mathbb{R}^{3\times 3}$ is the matrix such that $\Lambda_\tau \xi = \tau \wedge \xi$ for all $\xi \in \mathbb{R}^3$. Let $\tilde{F}$ be the space of all pairs $(A,B) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ such that

(a) $A^T = A$,
(b) $A^2 = A$,
(c) $\operatorname{tr} A = 2$,
(d) $B^T = -B$,
(e) $B^2 = -A$.

Then we have a natural diffeomorphism $\Phi : G^0(3,1) \to \tilde{F}$. If V is an oriented 1-varifold on Ω, then $(\mathrm{id}_\Omega \times \Phi)_\# V$ is a Radon measure on $\Omega \times \tilde{F}$, and this mapping provides a natural identification of the space of oriented 1-varifolds on Ω with the space of Radon measures on $\Omega \times \tilde{F}$. If $\mu = (\mathrm{id}_\Omega \times \Phi)_\# V$, then we have

$$\delta V(\phi) = \int_{\Omega \times \tilde{F}} \big(\operatorname{div} \phi(x) - \operatorname{tr}\big(A \nabla \phi(x)\big)\big)\, d\mu(x, A, B). \tag{16}$$

Next we relax the conditions (a)–(e) as follows. We replace (b) and (e) by

(b′) A is positive semidefinite and $|A|^2 \leqslant 4$,
(e′) $|B|^2 \leqslant 2$.

Here $|\cdot|$ denotes the Hilbert–Schmidt norm. Let F be the space of all pairs $(A, B) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ such that (a), (b′), (c), (d), and (e′) are satisfied. We now consider Radon measures on $\Omega \times F$.

We have the inclusion map $\iota : \tilde{F} \to F$, thus any oriented 1-varifold V on Ω induces a Radon measure $(\mathrm{id}_\Omega \times (\iota \circ \Phi))_\# V$ on $\Omega \times F$. Moreover, we will see that as a result of the relaxation, a map $u \in C^\infty(\Omega, \mathbb{S}^2)$ also induces a Radon measure on $\Omega \times F$ in a natural way. This is the reason why we consider this space.

For any $u \in C^\infty(\Omega, \mathbb{S}^2)$, we define the functions $A_u, B_u : \Omega \to \mathbb{R}^{3\times 3}$ with

$$A_u(x) = 2\frac{\nabla u(x) \otimes \nabla u(x)}{|\nabla u(x)|^2} \quad \text{and} \quad B_u(x) = 2\frac{(u(x) \wedge \nabla u(x)) \otimes \nabla u(x)}{|\nabla u(x)|^2} \tag{17}$$

for every $x \in \Omega$ with $\nabla u(x) \neq 0$, and $A_u(x) = A_0$, $B_u(x) = B_0$ if $\nabla u(x) = 0$ for a fixed (but arbitrary) point $(A_0, B_0) \in F$. It is readily checked that $(A_u(x), B_u(x)) \in F$ at every $x \in \Omega$. Now consider the Radon measure μ on Ω such that for every $\psi \in C_0^0(\Omega \times F)$,

$$\int_{\Omega \times F} \psi\, d\mu = \frac{1}{2} \int_\Omega \psi\big(x, A_u(x), B_u(x)\big) \big|\nabla u(x)\big|^2\, dx. \tag{18}$$

Measures of this type will play an important role in the proof of Theorem 1.1.

Motivated by (16), we define

$$\delta\mu(\phi) = \int_{\Omega \times F} \big(\operatorname{div} \phi(x) - \operatorname{tr}\big(A \nabla \phi(x)\big)\big)\, d\mu(x, A, B)$$

for $\phi \in C_0^1(\Omega, \mathbb{R}^3)$ whenever μ is a Radon measure on $\Omega \times F$. In order to simplify the notation, we often write such an integral in the form

$$\int_{\Omega \times F} \big(\operatorname{div} \phi - \operatorname{tr}(A \nabla \phi)\big)\, d\mu.$$

That is, the symbols A and B represent the standard coordinate functions on F. Moreover, whenever it is convenient, we identify a function on Ω with a function on $\Omega \times F$ that depends only on the first argument. For a measure of the form (18), we then compute

$$\delta\mu(\phi) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 \operatorname{div} \phi - \operatorname{tr}\big((\nabla u \otimes \nabla u) \nabla \phi\big)\right) dx = \int_\Omega \langle \Delta u, \nabla u\rangle \cdot \phi\, dx \tag{19}$$

by an integration by parts.

For any Radon measure μ on $\Omega \times F$, we define

$$W(\mu) = \sup\left\{\left(\delta\mu(\phi)\right)^2 \colon \phi \in C_0^1(\Omega, \mathbb{R}^3) \text{ with } \int\limits_{\Omega \times F} \phi \cdot A\phi \, d\mu \leqslant 1\right\}.$$

Suppose $\mu = (\mathrm{id}_\Omega \times (\iota \circ \Phi))_\# V$ for an integral 1-varifold V given by the function $\theta : \Omega \to \mathbb{N} \cup \{0\}$. Let $\Sigma = \theta^{-1}(\mathbb{N})$, as before. If there exists a vector field $H : \Omega \to \mathbb{R}^3$ such that

$$\delta V(\mu) + \int\limits_{\Sigma} \phi \cdot H\theta \, d\mathcal{H}^1 = 0$$

for every $\phi \in C_0^1(\Omega, \mathbb{R}^3)$ and $H(x) \perp T_x\Sigma$ for $\mathcal{H}^1$-almost every $x \in \Sigma$, then we easily calculate

$$W(\mu) = \int\limits_{\Sigma} |H|^2 \theta \, d\mathcal{H}^1.$$

If no such H exists, then we have $W(\mu) = \infty$. Thus the functional W can be regarded as a generalization of the Willmore functional. (See Willmore [28] for a definition and basic properties of the Willmore functional.)

If μ is defined by (18) for a map $u \in C^\infty(\Omega, \mathbb{S}^2)$, then we obtain

$$W(\mu) \leqslant \int\limits_{\Omega} |u \wedge \Delta u|^2 \, dx$$

from (19). If the rank of ∇u is either 2 or 0 at almost every point of Ω, then we have even equality here.

Consider again a general Radon measure μ on $\Omega \times F$. We use the projection $Q : \Omega \times F \to \Omega$ to define the weight measure $\|\mu\| = Q_\# \mu$, which is a Radon measure on Ω. The fibre measure $\mu^{(x_0)}$ on F, which is defined by

$$\int\limits_F \zeta \, d\mu^{(x_0)} = \lim_{r \searrow 0} \frac{\int_{B_r(x_0) \times F} \zeta(A, B) \, d\mu(x, A, B)}{\|\mu\|(B_r(x_0))} \quad \text{for } \zeta \in C^0(F),$$

then exists for $\|\mu\|$-almost every $x_0 \in \Omega$. Moreover, we have

$$\int\limits_{\Omega \times F} \psi \, d\mu = \int\limits_{\Omega}\int\limits_{F} \psi(x, A, B) \, d\mu^{(x)}(A, B) d\|\mu\|(x)$$

for every $\psi \in C_0^0(\Omega \times F)$ (cf. Allard [1], Section 3.3). Finally, we define the functions $A^{(\mu)}, B^{(\mu)} : \Omega \to \mathbb{R}^{3 \times 3}$ by

$$A^{(\mu)}(x) = \int\limits_F A \, d\mu^{(x)},$$

$$B^{(\mu)}(x) = \int\limits_F B \, d\mu^{(x)}$$

at every point where this exists, and $A^{(\mu)} = A_0$, $B^{(\mu)} = B_0$ elsewhere. Since F is convex, we have $(A^{(\mu)}(x), B^{(\mu)}(x)) \in F$ for every $x \in \Omega$.

If $W(\mu) < \infty$, then Proposition 2.1 in [19], applied to the measure

$$\|\mu\| \llcorner A^{(\mu)},$$

implies that there exists a unique $\|\mu\|$-measurable function $H : \Omega \to \mathbb{R}^3$ with

$$H(x) \perp \ker A^{(\mu)}(x) \tag{20}$$

for $\|\mu\|$-almost every $x \in \Omega$, such that

$$\delta\mu(\phi) + \int\limits_{\Omega \times F} \phi \cdot AH \, d\mu = 0 \tag{21}$$

for every $\phi \in C_0^1(\Omega, \mathbb{R}^3)$ and

$$W(\mu) = \int_{\Omega \times F} H \cdot AH \, d\mu.$$

If μ belongs to an integral varifold, then this H coincides of course with the vector field considered earlier.

If μ belongs to a map $u \in C^\infty(\Omega, \mathbb{S}^2)$—that is, if it is given by (18)—then we have

$$\|\mu\| = \frac{1}{2}\mathcal{L}^3 \llcorner |\nabla u|^2.$$

We denote the Dirac measure centred at a point p (in Ω, F, or $\Omega \times F$) by δ_p. Then we have

$$\mu^{(x)} = \delta_{A_u(x)} \times \delta_{B_u(x)}$$

and

$$A^{(\mu)}(x) = A_u(x), \qquad B^{(\mu)}(x) = B_u(x)$$

for every $x \in \Omega$ where $\nabla u(x) \neq 0$. According to (19) and (21), we have

$$\int_{\Omega \times F} \phi \cdot AH \, d\mu = -\int_{\Omega} \phi \cdot \langle \Delta u, \nabla u \rangle \, dx$$

for every $\phi \in C_0^1(\Omega, \mathbb{R}^3)$. Hence

$$(\nabla u \otimes \nabla u)H = -\langle \Delta u, \nabla u \rangle$$

almost everywhere in Ω. We set

$$Z = \{x \in \Omega:\ \dim \ker \nabla u(x) \neq 2\}.$$

We claim that

$$\big((u \wedge \nabla u) \otimes \nabla u\big)H = -\langle u \wedge \Delta u, \nabla u \rangle$$

almost everywhere in Z. That is,

$$\int_{Z \times F} \phi \cdot BH \, d\mu = -\int_{Z} \phi \cdot \langle u \wedge \Delta u, \nabla u \rangle \, dx \tag{22}$$

for every $\phi \in C_0^0(\Omega, \mathbb{R}^3)$.

This is in fact quite easy to verify if the right coordinates are used. Suppose $x \in Z$ with $\nabla u(x) \neq 0$. Since $\nabla u(x) \otimes \nabla u(x)$ is positive semidefinite and of rank 2, there exists some $R \in \mathrm{SO}(3)$ such that

$$R^T\big(\nabla u(x) \otimes \nabla u(x)\big)R = \begin{pmatrix} c_1^2 & 0 & 0 \\ 0 & c_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

for certain numbers $c_1, c_2 > 0$. Replace u by the map $\tilde{u}(y) = u(Ry + x)$, then $\nabla \tilde{u}(0) \otimes \nabla \tilde{u}(0)$ is the matrix on the right hand side of (23). We work with $\tilde{u}$ instead of u now (and we drop the tilde again). We also choose coordinates in the target space such that $u(x) = (0, 0, 1)$ and

$$\frac{\partial u}{\partial x^1}(x) = (c_1, 0, 0) \quad \text{and} \quad \frac{\partial u}{\partial x^2}(x) = (0, \pm c_2, 0).$$

Then

$$(u(x) \wedge \nabla u(x)) \otimes \nabla u(x) = \pm \begin{pmatrix} 0 & c_1c_2 & 0 \\ -c_1c_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let

$$\Delta u(x) + \big|\nabla u(x)\big|^2 u(x) = (d_1, d_2, 0).$$

Then

$$\langle \Delta u, \nabla u \rangle = \begin{pmatrix} c_1 d_1 \\ \pm c_2 d_2 \\ 0 \end{pmatrix},$$

hence

$$H(x) = \begin{pmatrix} d_1/c_1 \\ \pm d_2/c_2 \\ 0 \end{pmatrix}.$$

Thus we have

$$\big(\big(u(x) \wedge \nabla u(x)\big) \otimes \nabla u(x)\big) H(x) = \begin{pmatrix} c_1 d_2 \\ \mp c_2 d_1 \\ 0 \end{pmatrix} = -\big\langle u(x) \wedge \Delta u(x), \nabla u(x)\big\rangle,$$

which proves (22).

If we assume that $Z = \Omega$, then the energy identities (8) and (12) become formally the same in the framework of generalized varifolds. This is one of the reasons why this is a useful tool for our problem.

When we work with a map $u \in C^\infty(\Omega_T, \mathbb{S}^2)$, then we define the functions $A_u, B_u : \Omega_T \to \mathbb{R}^{3\times 3}$ similarly as above. They give rise to a Radon measure μ on $\Omega_T \times F$. The quantities $\|\mu\|$, $\delta\mu$, $\mu^{(z)}$, $A^{(\mu)}$, $B^{(\mu)}$, etc., are then defined similarly as on Ω.

We now briefly discuss another tool from geometric measure theory, namely the measure-function pairs introduced by Hutchinson [10].

Let M be a manifold, with or without boundary, which is embedded in $\mathbb{R}^m$. If μ is a Radon measure on M and $f : M \to \mathbb{R}^n$ a locally μ-integrable function on M, then we say that (μ, f) is a measure-function pair over M (with values in $\mathbb{R}^n$). We are mainly interested in the case $f \in L^2(\mu, \mathbb{R}^n)$.

To a measure-function pair (μ, f) we can assign the graph measure $[\mu, f]$ on $M \times \mathbb{R}^n$, which is a Radon measure defined by the condition

$$\int_{M\times\mathbb{R}^n} \xi \, d[\mu, f] = \int_M \xi\big(x, f(x)\big)\, d\mu(x) \quad \text{for } \xi \in C_0^0(M \times \mathbb{R}^n).$$

Obviously we can represent the generalized varifold belonging to a map in $C^\infty(\Omega, S^2)$ as a graph measure of this type. More important for our purpose, however, are measure-function pairs of the form (μ, H), where μ is a generalized varifold with $W(\mu) < \infty$ and H is the function characterized by (20) and (21), interpreted as a function on $\Omega \times F$ which depends only on the first argument. In general, this function is not necessarily in $L^1_{\mathrm{loc}}(\mu, \mathbb{R}^3)$. However, if μ belongs to a map $u \in C^\infty(\Omega, \mathbb{S}^2)$ such that ∇u is of rank 2 in Ω, then we have even $H \in L^\infty_{\mathrm{loc}}(\mu, \mathbb{R}^3)$.

Now suppose that we have a sequence of measure-function pairs (μ_k, f_k) over M with values in $\mathbb{R}^n$ such that $f_k \in L^2(\mu_k, \mathbb{R}^n)$. Furthermore, we assume that (μ, f) is another measure-function pair of this type, such that $\mu_k \to \mu$. We say that the sequence $\{(\mu_k, f_k)\}$ converges weakly to (μ, f) if we have $\mu_k \llcorner f_k \to \mu \llcorner f$. We say that the convergence is strong if $[\mu_k, f_k] \to [\mu, f]$ and

$$\int_{\{(x,y)\in M\times\mathbb{R}^n:\ |x|^2+|y|^2 \geqslant R\}} y^2\, d[\mu_k, f_k](x, y) \to 0 \quad \text{as } R \to \infty$$

uniformly in k. Other characterizations of these (and similar) notions of convergence are given in [10]. The proof of the following result is also to be found there.

Proposition 2.1.

(i) *Suppose (μ_k, f_k) are measure-function pairs over M such that*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^2(\mu_k)} < \infty.$$

If μ is a Radon measure on M with $\mu_k \to \mu$, then there exists a function $f \in L^2(\mu)$ such that a subsequence of $\{(\mu_k, f_k)\}$ converges weakly to (μ, f).

(ii) *Suppose that the sequence* $\{(\mu_k, f_k)\}$ *converges weakly to* (μ, f). *Then*

$$\|f\|_{L^2(\mu)} \leqslant \liminf_{k\to\infty} \|f_k\|_{L^2(\mu_k)}.$$

The convergence is strong if and only if

$$\|f\|_{L^2(\mu)} = \lim_{k\to\infty} \|f_k\|_{L^2(\mu_k)}.$$

3. Other tools

In this section we collect a few facts that are either proved elsewhere or that follow from known arguments. First we have an estimate for the derivatives of a solution of the Landau–Lifshitz equation with small energy. The arguments that prove this result can be found in Sections 5.2 and 5.5 of [20] (although the result is not stated in the same form in that work).

Lemma 3.1. *There exist two numbers* $C, \epsilon_0 > 0$ *such that every solution* $u \in C^\infty(P_1, \mathbb{S}^2)$ *of* (1) *with*

$$\frac{1}{2}\int_{P_1} |\nabla u|^2\, dz = \epsilon \leqslant \epsilon_0,$$

satisfies

$$\sup_{P_{1/2}} \bigl(|\nabla u|^2 + |\nabla^2 u|^2\bigr) \leqslant C\epsilon.$$

The next statement gives a version of the so-called monotonicity formula for harmonic maps. A proof for this formula is given in [20], Section 4.1.

Lemma 3.2. *Suppose* $u \in C^\infty(B_1, \mathbb{S}^2)$. *For* $0 < \rho \leqslant 1$, *set*

$$\Phi(\rho) = \frac{1}{\rho}\int_{B_\rho} \left(\frac{1}{2}|\nabla u|^2 - x \cdot \langle \Delta u, \nabla u\rangle\right) dx.$$

Then

$$\Phi(r) - \Phi(s) = \int_{B_r\setminus B_s} \left(\frac{|x\cdot\nabla u|^2}{|x|^3} - \frac{x\cdot\langle\Delta u, \nabla u\rangle}{|x|}\right) dx$$

for $0 < s \leqslant r \leqslant 1$.

We will also need the following estimates. Here we write $x = (x', x^3)$ for a point in $\mathbb{R}^3$, where $x' = (x^1, x^2)$.

Lemma 3.3. *Suppose* $\xi \in C_0^1(-\frac{1}{2}, \frac{1}{2})$ *and* $\zeta \in C_0^1(-1, 1)$ *satisfy*

$$\int_{-1/2}^{1/2} \xi(s)\, ds = \int_{-1}^{1} \zeta(t)\, dt = 1.$$

Then there exists a constant C with the following properties.

(i) *For every* $u \in C^\infty(B_1, \mathbb{S}^2)$ *and every* $\eta \in C_0^1(B_{1/2}^2)$, *the inequality*

$$\left| \int_{B_1} \eta(x')\xi(x^3)\bigl|\nabla u(x', x^3)\bigr|^2\, dx - \int_{B_{1/2}^2} \eta(x')\bigl|\nabla u(x', 0)\bigr|^2\, dx' \right|$$
$$\leqslant C\|\eta\|_{C^1(B_{1/2}^2)} \|\nabla u\|_{L^2(B_1)} \left(\left\|\frac{\partial u}{\partial x^3}\right\|_{L^2(B_1)} + \|u\wedge\Delta u\|_{L^2(B_1)} \right) + C\|\eta\|_{C^0(B_{1/2}^2)} \left\|\frac{\partial u}{\partial x^3}(\cdot, 0)\right\|_{L^2(B_1^2)}^2$$

holds.

(ii) *For every solution $u \in C^\infty(P_1, \mathbb{S}^2)$ of (1) and every $\eta \in C_0^1(B_1)$, the inequality*

$$\left| \int_{P_1} \eta(x)\zeta(t)\left|\nabla u(x,t)\right|^2 dz - \int_{B_1} \eta(x)\left|\nabla u(x,0)\right|^2 dx \right| \leqslant C\|\eta\|_{C^1(B_1)} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(P_1)} \left(\|\nabla u\|_{L^2(P_1)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(P_1)} \right)$$

holds.

(iii) *For every solution $u \in C^\infty(P_1, \mathbb{S}^2)$ of (1) and every $\eta \in C_0^1(B_{1/2})$, the inequality*

$$\left| \int_{P_1} \eta(x)\zeta(t)\left|\nabla u(x,t)\right|^2 dz - \int_{B_1} \eta(x)\left|\nabla u(x,0)\right|^2 dx \right| \leqslant C\|\eta\|_{C^1(B_{1/2})} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(P_1)} \left(\left\|\nabla u(\cdot,0)\right\|_{L^2(B_1)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(P_1)} \right)$$

holds.

Proof. For part (i) of the lemma, we define $\tilde{\eta}(x', x^3) = \eta(x')$ and $\tilde{\xi}(x', x^3) = \xi(x^3)$. We consider the vector field

$$A = \tilde{\eta}\left(\frac{1}{2}|\nabla u|^2 e_3 - \left\langle \frac{\partial u}{\partial x^3}, \nabla u \right\rangle\right),$$

where $e_3 = (0, 0, 1)$. We compute

$$\operatorname{div} A = -\tilde{\eta}\left\langle \frac{\partial u}{\partial x^3}, \Delta u \right\rangle - \nabla\tilde{\eta} \cdot \left\langle \frac{\partial u}{\partial x^3}, \nabla u \right\rangle.$$

Thus we find

$$\int_{B^2_{1/2}\times\{s\}} \tilde{\eta}\left(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial u}{\partial x^3}\right|^2\right) dx' - \int_{B^2_{1/2}\times\{0\}} \tilde{\eta}\left(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial u}{\partial x^3}\right|^2\right) dx'$$
$$= -\int_{B^2_{1/2}\times(0,s)} \left(\tilde{\eta}\left\langle \frac{\partial u}{\partial x^3}, \Delta u \right\rangle + \nabla\tilde{\eta} \cdot \left\langle \frac{\partial u}{\partial x^3}, \nabla u \right\rangle\right) dx$$

for $s \in (0, \frac{1}{2})$ and a similar formula for $s \in (-\frac{1}{2}, 0)$. Multiplying both sides with $\xi(s)$, integrating over s, and using Fubini's theorem, we obtain

$$\int_{B^2_{1/2}\times(-\frac{1}{2},\frac{1}{2})} \tilde{\eta}\tilde{\xi}\left(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial u}{\partial x^3}\right|^2\right) dx - \int_{B^2_{1/2}\times\{0\}} \tilde{\eta}\left(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial u}{\partial x^3}\right|\right) dx'$$
$$= -\int_{B^2_{1/2}\times(-\frac{1}{2},\frac{1}{2})} \tilde{\sigma}\left(\tilde{\eta}\left\langle \frac{\partial u}{\partial x^3}, \Delta u \right\rangle + \nabla\tilde{\eta} \cdot \left\langle \frac{\partial u}{\partial x^3}, \nabla u \right\rangle\right) dx,$$

where

$$\tilde{\sigma}(x', x^3) = \begin{cases} \int_{x^3}^{1/2} \xi(s)\,ds & x^3 > 0, \\ -\int_{-1/2}^{x^3} \xi(s)\,ds & x^3 < 0. \end{cases}$$

The estimate in (i) follows with Young's inequality.

Part (ii) is an easy consequence of the energy identity (8).

For part (iii), it suffices to show that there exists a constant C_1 that depends only on a and b, such that

$$\int_{B_{1/2}\times(-1,1)} |\nabla u|^2\,dz \leqslant C_1\left(\int_{B_1} \left|\nabla u(x,0)\right|^2 dx + \int_{P_1} \left|\frac{\partial u}{\partial t}\right|^2 dz\right). \tag{24}$$

Once this is verified, also the inequality in (iii) follows directly from (8). To prove (24), choose a cut-off function $\omega \in C_0^\infty(B_1)$ with $\omega \equiv 1$ in $B_{1/2}$. As a consequence of (8), we have a constant $C_2 = C_2(a, b, \omega)$ such that

$$\left|\frac{d}{dt}\int_{B_1} \omega^2(x)\left|\nabla u(x,t)\right|^2 dx\right| \leqslant \int_{B_1} \omega^2(x)\left|\nabla u(x,t)\right|^2 dx + C_2\int_{B_1}\left|\frac{\partial u}{\partial t}\right|^2 dx.$$

Now (24) follows easily from this inequality. □

Next we consider a sequence of maps $u_k \in C^\infty(\Omega, \mathbb{S}^2)$ such that

$$\sup_{k\in\mathbb{N}}\left(E(u_k) + \int_\Omega |u_k \wedge \Delta u_k|^2\,dx\right) < \infty.$$

For such a sequence, a similar energy concentration phenomenon as described in Theorem 1.1 can be observed. This behaviour and the corresponding energy concentration measure have been studied in [19]. We repeat here the main results of that paper (in the case of a three-dimensional domain and the target manifold $\mathbb{S}^2$).

Theorem 3.1. *Under the above conditions, there exists a subsequence $\{u_{k_i}\}$ with the following properties.*

(i) *There exists a map $u \in H^1(\Omega, \mathbb{S}^2)$ with*

$$\Delta u + |\nabla u|^2 u \in L^2(\Omega, \mathbb{R}^3),$$

such that $u_{k_i} \rightharpoonup u$ weakly in $H^1(\Omega, \mathbb{R}^3)$ and

$$\Delta u_{k_i} + |\nabla u_{k_i}|^2 u_{k_i} \rightharpoonup \Delta u + |\nabla u|^2 u \quad \text{weakly in } L^2(\Omega, \mathbb{R}^3).$$

(ii) *There exists an $\mathcal{H}^1$-integrable function $\theta : \Omega \to [0, \infty)$, such that for any $\eta \in C_0^0(\Omega)$,*

$$\lim_{i\to\infty}\int_\Omega \eta|\nabla u_{k_i}|^2\,dx = \int_\Omega \eta|\nabla u|^2\,dx + 8\pi\int_\Omega \eta\theta\,d\mathcal{H}^1,$$

and the set $\Sigma = \theta^{-1}((0,\infty))$ is closed relative to Ω and countably 1-rectifiable.

(iii) *There exists an $\mathcal{H}^1$-measurable vector field $H : \Omega \to \mathbb{R}^3$ with $H(x) \perp T_x\Sigma$ for $\mathcal{H}^1$-almost every $x \in \Sigma$, such that*

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 \operatorname{div}\phi - \operatorname{tr}\big((\nabla u \otimes \nabla u)\nabla\phi\big) - \langle \Delta u, \nabla u\rangle \cdot \phi\right) dx + 4\pi\int_\Omega (\operatorname{div}_\Sigma \phi + H\cdot\phi)\theta\,d\mathcal{H}^1 = 0$$

for every $\phi \in C_0^1(\Omega, \mathbb{R}^3)$.

(iv) *The inequality*

$$\int_\Omega |u \wedge \Delta u|^2\,dx + 4\pi\int_\Omega |H|^2\theta\,d\mathcal{H}^1 \leqslant \liminf_{k\to\infty}\int_\Omega |u_k \wedge \Delta u_k|^2\,dx$$

holds.

If we have a sequence of solutions $u_k \in C^\infty(\Omega, \mathbb{S}^2)$ of (1) that satisfy the hypotheses of Theorem 1.1, then the energy identity (8) implies that for any precompact set $K \Subset \Omega$ and for every $t_0 \in (0, T)$, we have

$$\sup_{k\in\mathbb{N}}\left(\int_K |\nabla u_k(x,t_0)|^2\,dx + \int_{t_0}^T\int_K |u_k \wedge \Delta u_k|^2\,dx\,dt\right) < \infty.$$

Thus, the above result can be applied to restrictions of u_k to $K \times \{t_0\}$ for almost every $t_0 \in [0, T)$, at least after the choice of a subsequence (which may depend on t_0). Parts (i)–(iii) of Theorem 1.1 then follow with relatively little extra work, except for the statement that θ is integer-valued. Most of the rest of the paper is therefore dedicated to this quantization property and part (iv) of Theorem 1.1.

4. Two-dimensional blow-up analysis

In this section we work in two-dimensional domains. The results will later be applied to cross-sections of a higher-dimensional domain. We now write $B_r(x_0)$ for an open ball in $\mathbb{R}^2$ (or B_r if $x_0 = 0$). We also use the notations $\hat{F} = \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Remember that a Dirac measure centred at a point p is denoted by δ_p.

We study a sequence of maps $u_k \in C^\infty(B_1, \mathbb{S}^2)$. We consider the corresponding measures μ_k on $B_1 \times \hat{F}$ given by

$$\int_{B_1 \times \hat{F}} \psi \, d\mu_k = \frac{1}{2} \int_{B_1} \psi\left(x, 2\frac{\nabla u_k \otimes \nabla u_k}{|\nabla u_k|^2}, 2\frac{(u_k \wedge \nabla u_k) \otimes \nabla u_k}{|\nabla u_k|^2}\right) |\nabla u_k|^2 \, dx \tag{25}$$

for $\psi \in C_0^0(B_1 \times K)$; in other words, the equivalent of (18) for the maps u_k. The weight measures $\|\mu_k\|$ on B_1 are then defined similarly as in Section 2.

We examine the blow-up behaviour of this sequence under the assumption that there exists a number ϵ_0 such that the following conditions are satisfied.

(I) There exists a number $\theta > 0$ such that $\|\mu_k\| \to 4\pi\theta\delta_0$.

(II) $\sup_{k\in\mathbb{N}} \|\nabla^2 u_k\|_{L^1(B_1)} < \infty$.

(III) Suppose $x_k \in B_1$ and $r_k > 0$ are such that $x_k \to 0$ and $r_k \to 0$. If

$$0 < \limsup_{k\to\infty} \int_{B_{r_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant 2\epsilon_0,$$

then the sequence of rescaled maps $v_k(x) = u_k(r_k x + x_k)$ subconverges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$. The limit is a non-constant harmonic map $v \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ (i.e., it satisfies $v \wedge \Delta v = 0$) with

$$\int_{\mathbb{R}^2} |\nabla v|^2 \, dx < \infty.$$

(IV) There exists a constant C_0 such that for any $\epsilon \in (0, \epsilon_0]$ and for any sequence of balls $B_{r_k}(x_k) \subset B_1$ with

$$\limsup_{k\to\infty} \int_{B_{r_k}(x_k)} |\nabla u_k|^2 \, dx \leqslant 2\epsilon,$$

the inequality

$$\limsup_{k\to\infty} \sup_{x \in B_{r_k/2}(x_k)} \left(r_k^2 \left|\nabla u_k(x)\right|^2 + r_k^4 |\nabla^2 u_k|^2\right) \leqslant C_0 \epsilon$$

holds.

We want to determine the possible limit measures of the sequence $\{\mu_k\}$ under these assumptions. We achieve this by a blow-up analysis similar to what has been done for harmonic maps by Jost [11] and for the harmonic map heat flow and Palais–Smale sequences for the Dirichlet energy by Qing [21], Ding and Tian [5], Qing and Tian [22], and Lin and Wang [17]. Typical for this method is that the limit measures are described in terms of so-called "harmonic bubbles". These bubbles are harmonic maps which are obtained from rescaled sequences as in (III). It turns out that they contain all the information about the limit measure. Since harmonic maps of this type are well understood, we obtain a good description of any possible limit of $\{\mu_k\}$.

Proposition 4.1. *Under the hypotheses* (I)–(IV)*, the number θ is an integer. If $\mu_k \to \mu$ for a Radon measure μ on $B_1 \times \hat{F}$, then there exists a number $\sigma \in [0,1]$, such that*

$$\mu = 4\pi\theta\delta_0 \times \delta_I \times \big(\sigma\delta_J + (1-\sigma)\delta_{-J}\big).$$

For the proof of this proposition, we use ideas from the papers mentioned earlier and also from Lin and Rivière [16]. One of the tools we need is the following lemma.

Lemma 4.1. *There exists a constant C with the following property. Suppose $r \in (0,1)$ and $f \in W^{1,1}(B_1)$ satisfy*

$$\sup_{x \in B_1 \setminus B_r} \big|xf(x)\big| \leqslant c. \tag{26}$$

Then

$$\int\limits_{B_1 \setminus B_r} f^2\,dx \leqslant Cc\big(\|f\|_{W^{1,1}(B_1)} + c\big). \tag{27}$$

Proof. It suffices to consider $r \in (0, \frac{1}{8}]$. Choose a cut-off function $\xi \in C_0^1(B_1 \setminus B_r)$ with $0 \leqslant \xi \leqslant 1$ and $\xi \equiv 1$ in $B_{1/2} \setminus B_{2r}$, such that $|\nabla\xi| \leqslant 4$ in $B_1 \setminus B_{1/2}$ and $|\nabla\xi| \leqslant 2/r$ in $B_{2r} \setminus B_r$. Define $g = \xi f$. Then

$$\|g\|_{W^{1,1}(\mathbb{R}^2)} \leqslant \|f\|_{W^{1,1}(B_1)} + 16\pi c.$$

For every $s \geqslant c$, we have

$$\big|\big\{x \in \mathbb{R}^2:\ \big|g(x)\big| > s\big\}\big| \leqslant \pi\frac{c^2}{s^2}$$

by (26), thus we have the estimate

$$\|g\|_{L^{(2,\infty)}(\mathbb{R}^2)} \leqslant \sqrt{\pi}\,c$$

in the Lorentz space $L^{(2,\infty)}(\mathbb{R}^2)$. The Sobolev space $W^{1,1}(\mathbb{R}^2)$ is continuously embedded in $L^{(2,1)}(\mathbb{R}^2)$ (cf. Tartar [27]). Thus it follows that

$$\int\limits_{\mathbb{R}^2} g^2\,dx \leqslant C_1\|g\|_{L^{(2,1)}(\mathbb{R}^2)}\|g\|_{L^{(2,\infty)}(\mathbb{R}^2)} \leqslant C_2 c\big(\|f\|_{W^{1,1}(B_1)} + c\big)$$

for certain universal constants C_1 and C_2. To estimate the remaining part of the integral in (27), we use (26) again. □

Proof of Proposition 4.1. We may assume that a limit μ of the sequence $\{\mu_k\}$ exists. Let $m \in \mathbb{N}$ be the minimal integer such that $\theta \leqslant m$. We prove the proposition by induction on m.

Suppose first that $m = 1$. Then we choose a sequence of radii $r_k \to 0$ such that

$$\frac{1}{2}\int\limits_{B_{r_k}} |\nabla u_k|^2\,dx = \min\{\epsilon_0, 2\pi\theta\}.$$

By (III), there exists a subsequence of the sequence defined by $v_k(x) = u_k(r_k x)$ which converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ to a non-constant harmonic map $v \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ with finite energy. Such a harmonic map must be conformal and must satisfy

$$\frac{1}{2}\int\limits_{\mathbb{R}^2} |\nabla v|^2\,dx = 4\pi M$$

for some $M \in \mathbb{N}$ (cf. Sacks and Uhlenbeck [23] and Section 11.5 in Eells and Lemaire [6]). It follows immediately that $M = \theta = 1$. Moreover, the above convergence is strong in $H^1(K, \mathbb{R}^3)$ for every bounded set $K \subset \mathbb{R}^2$.

Fix $x_0 \in \mathbb{R}^2$ and choose $r > 0$ such that

$$\int_{B_r(x_0)} |\nabla v|^2\,dx \leqslant \epsilon_0.$$

Then it follows from (IV) that the above subsequence of $\{v_k\}$ converges to v even in the topology of $C^1(B_{r/2}(x_0), \mathbb{S}^2)$. Since v is conformal, we have

$$2\frac{\nabla v \otimes \nabla v}{|\nabla v|^2} = I \quad \text{and} \quad 2\frac{(v \wedge \nabla v) \otimes \nabla v}{|\nabla v|^2} = \pm J$$

on $\mathbb{R}^2$. The claim of the proposition then follows immediately (with $\sigma = 0$ or $\sigma = 1$).

Next we assume that the statement of the proposition is true whenever θ does not exceed $m - 1$. We want to show that it still holds for $\theta \leqslant m$.

We fix $\epsilon \in (0, \epsilon_0]$. We choose a sequence $r_k \to 0$ such that

$$\frac{1}{2}\int_{B_{2r_k}\setminus B_{r_k}} |\nabla u_k|^2\,dx = \epsilon,$$

but

$$\frac{1}{2}\int_{B_{2\rho}\setminus B_\rho} |\nabla u_k|^2\,dx \leqslant \epsilon$$

for every $\rho \in (r_k, \frac{1}{2}]$. Then by (IV), there exists a constant C_1, such that

$$\limsup_{k\to\infty} \sup_{x\in B_{1/4}\setminus B_{4r_k}} |x|\bigl|\nabla u_k(x)\bigr| \leqslant C_1\sqrt{\epsilon}.$$

We apply Lemma 4.1 to $f = |\nabla u_k|$. Using also (II), we conclude that

$$\limsup_{k\to\infty} \int_{B_1\setminus B_{r_k}} |\nabla u_k|^2\,dx \leqslant C_2\sqrt{\epsilon}, \tag{28}$$

where C_2 is a constant that depends only ϵ_0, C_0, and the supremum in (II).

Consider the rescaled maps

$$v_k(x) = u_k(r_k x)$$

and the corresponding measures $\tilde{\mu}_k$ which are defined similarly as in (25), but with v_k instead of u_k. Using (III), we see that there exists a subsequence which converges weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^3)$ to a non-constant harmonic map $v \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ with finite energy. We assume for simplicity that this convergence holds for the full sequence. We may also assume that $\tilde{\mu}_k \to \tilde{\mu}$ for a Radon measure $\tilde{\mu}$ on $\mathbb{R}^2 \times \hat{F}$. If we can show that there exist $\hat{m} \in \mathbb{N}$ and $\sigma \in [0, 1]$ such that

$$\int_{\mathbb{R}^2\times\hat{F}} \psi(A, B)\,d\tilde{\mu}(x, A, B) = 4\pi\hat{m}\bigl(\sigma\psi(I, J) + (1 - \sigma)\psi(I, -J)\bigr) \tag{29}$$

for every $\psi \in C^0_0(\hat{F})$, then the claim of the proposition follows, because we have (28) for an arbitrarily small number $\epsilon \in (0, \epsilon_0]$.

Define

$$\Sigma_0 = \bigl\{x \in \mathbb{R}^2\colon\ \|\tilde{\mu}\|\bigl(\{x\}\bigr) \geqslant \epsilon_0/2\bigr\}.$$

Because of (28), we have $\Sigma_0 \subset \overline{B_1}$ whenever ϵ is chosen sufficiently small. Moreover, this is a finite set. At every point $x_0 \in \Sigma_0$, it is easy to find a sequence $s_k \to 0$, such that the maps $w_k(x) = v_k(s_k x + x_0)$ satisfy

$$\frac{1}{2}\mathcal{L}^2 \llcorner |\nabla w_k|^2 \to \|\tilde{\mu}\|\bigl(\{x_0\}\bigr)\delta_0.$$

Such a sequence also satisfies the conditions (I)–(IV), but with the number

$$\theta_{x_0} = \frac{\|\tilde{\mu}\|(\{x_0\})}{4\pi}$$

in (I) instead of θ, which satisfies $\theta_{x_0} \leqslant \theta - 1$. By our assumptions, this means that we have

$$\int_{\{x_0\}\times\hat{F}} \psi\, d\tilde{\mu} = 4\pi m_{x_0}\big(\sigma_{x_0}\psi(I,J) + (1-\sigma_{x_0})\psi(I,-J)\big)$$

for certain numbers $m_{x_0} \in \mathbb{N}$ and $\sigma_{x_0} \in [0,1]$. Similarly as in the first part of the proof we obtain

$$\int_{(\mathbb{R}^2\setminus\Sigma_0)\times\hat{F}} \psi\, d\tilde{\mu} = 4\pi\tilde{m}\big(\tilde{\sigma}\psi(I,J) + (1-\tilde{\sigma})\psi(I,-J)\big)$$

for some $\tilde{m} \in \mathbb{N}$ and $\tilde{\sigma} \in [0,1]$. Combining these identities, we obtain (29) for

$$\hat{m} = \tilde{m} + \sum_{x_0\in\Sigma_0} m_{x_0}$$

and

$$\sigma = \frac{1}{\hat{m}}\bigg(\tilde{m}\tilde{\sigma} + \sum_{x_0\in\Sigma_0} m_{x_0}\sigma_{x_0}\bigg),$$

and the proof is complete. □

5. Three-dimensional blow-up analysis

We now examine a sequence of solutions $u_k \in C^\infty(P_1,\mathbb{S}^2)$ of the Landau–Lifshitz equation (1) that arises from rescaling a sequence as in Theorem 1.1 around a typical blow-up point. (We will see later what "typical" means here.)

We define the measures μ_k on $P_1 \times F$ which belong to u_k and are given by

$$\int_{P_1\times F} \psi\, d\mu_k = \frac{1}{2}\int_{P_1} \psi\big(z, A_{u_k}(z), B_{u_k}(z)\big)\big|\nabla u_k(z)\big|^2\, dz \quad \text{for } \psi \in C_0^0(P_1\times F).$$

Here A_{u_k} and B_{u_k} are the functions defined similarly as in (17). We assume that $\mu_k \to \mu$ for a Radon measure μ on $P_1 \times F$. We also assume that this limit measure has a special structure: namely, that there exist a one-dimensional linear subspace $L \subset \mathbb{R}^3$ and a number $\theta > 0$ such that

$$\|\mu\| = 4\pi\theta\mathcal{H}^2 \llcorner \big((L\times\mathbb{R})\cap P_1\big). \tag{30}$$

We then fix a unit vector $\tau \in L\cap\mathbb{S}^2$. Finally, we assume that

$$\lim_{k\to 0}\int_{P_1}\left|\frac{\partial u_k}{\partial t}\right|^2 dz = 0. \tag{31}$$

Proposition 5.1. *Under the above hypotheses, there exist $m \in \mathbb{N}$ and $\sigma \in [0,1]$, such that for every $\psi \in C^0(F)$,*

$$\int_{P_{1/4}\times F} \psi(A,B)\, d\mu(z,A,B) = \frac{\pi m}{4}\big(\sigma\psi(\Pi_L^\perp,\Lambda_\tau) + (1-\sigma)\psi(\Pi_L^\perp,-\Lambda_\tau)\big).$$

Proof. We consider the functional on $C^0(F)$ that assigns to ψ the number

$$\int_{P_{1/4}\times F} \psi(A,B)\, d\mu(z,A,B).$$

Clearly this is represented by a Radon measure on F with total mass $\pi\theta/4$. Thus it suffices to show that

(i) $\theta \in \mathbb{N}$ and
(ii) for any $\psi \in C^0(F)$ with $\psi \geqslant 0$ and $\psi(\Pi_L^\perp, \Lambda_\tau) = \psi(\Pi_L^\perp, -\Lambda_\tau) = 0$, we have

$$\int_{P_{1/4}\times F} \psi(A,B)\,d\mu(z,A,B) = 0.$$

For the proofs of both statements we use ideas of Lin [13] and of Lin and Rivière [16].

We first exploit the fact that the Landau–Lifshitz equation can be represented in the form (3) or (4). We note that

$$|\nabla u_k|^2 u_k = \sum_{\alpha=1}^{3} \frac{\partial u_k}{\partial x^\alpha} \wedge \left(u_k \wedge \frac{\partial u_k}{\partial x^\alpha}\right)$$

and

$$\sqrt{a^2+b^2}\,\bigl|\operatorname{div}(u_k \wedge \nabla u_k)\bigr| = \left|\frac{\partial u_k}{\partial t}\right|.$$

With the compensated compactness method of Coifman, Lions, Meyer, and Semmes [4], we can prove the estimate

$$\bigl\| |\nabla u_k(\cdot,t)|^2 u_k(\cdot,t)\bigr\|_{\mathcal{H}^1(B_{3/4})} \leqslant C_1 \|\nabla u_k(\cdot,t)\|_{L^2(B_1)}^2 + C_1 \|\nabla u_k(\cdot,t)\|_{L^2(B_1)} \left\|\frac{\partial u_k}{\partial t}(\cdot,t)\right\|_{L^2(B_1)}$$

in the Hardy space $\mathcal{H}^1(B_{3/4})$ for every $t \in (-1,1)$, where C_1 is a constant that depends only on a and b. We also have

$$\sqrt{a^2+b^2}\,\bigl|\Delta u_k + |\nabla u_k|^2 u_k\bigr| = \left|\frac{\partial u_k}{\partial t}\right|,$$

thus

$$\|\Delta u_k(\cdot,t)\|_{\mathcal{H}^1(B_{3/4})} \leqslant C_2 \|\nabla u_k(\cdot,t)\|_{L^2(B_1)}^2 + C_2\bigl(\|\nabla u_k(\cdot,t)\|_{L^2(B_1)} + 1\bigr) \left\|\frac{\partial u_k}{\partial t}(\cdot,t)\right\|_{L^2(B_1)}$$

for another constant C_2 that depends only on a and b. Standard estimates for singular integrals involving Hardy spaces (see, e.g., Theorem 3 in Section III.3.1 of Stein [25]) now imply that there exists a constant $C_3 = C_3(a,b)$, such that

$$\|\nabla^2 u_k(\cdot,t)\|_{L^1(B_{1/2})} \leqslant C_3\bigl(\|\nabla u_k(\cdot,t)\|_{L^2(B_1)} + 1\bigr)\left(\|\nabla u_k(\cdot,t)\|_{L^2(B_1)} + \left\|\frac{\partial u_k}{\partial t}(\cdot,t)\right\|_{L^2(B_1)}\right).$$

In particular we have

$$\limsup_{k\to\infty} \|\nabla^2 u_k\|_{L^1(P_{1/2})} < \infty. \tag{32}$$

We now assume that $L = \{(0,0)\} \times \mathbb{R}$ for simplicity. Because of (30) and (31), the monotonicity formula of Lemma 3.2 (applied to $u_k(\cdot,t)$ for every $k \in \mathbb{N}$ and $t \in (-1,1)$) implies

$$\lim_{k\to\infty} \int_{-1}^{1} \int_{B_1\setminus B_r} \frac{|x\cdot\nabla u_k|^2}{|x|^3}\,dx\,dt = 0$$

for every $r > 0$. Since $\|\mu\|$ is supported on $\{(0,0)\} \times \mathbb{R}^2$, this means that

$$\lim_{k\to\infty} \int_{P_1} \left|\frac{\partial u_k}{\partial x^3}\right|^2 dz = 0.$$

For $r > 0$, let $P_r^2 = (-r,r) \times (-r^2, r^2)$. On $P_{1/2}^2$, define the functions

$$F_k(s,t) = \int_{B_{1/2}^2\times\{(s,t)\}} \left|\frac{\partial u_k}{\partial x^3}\right|^2 dx'$$

and

$$G_k(s,t) = \int\limits_{B^2_{1/2}\times\{(s,t)\}} \left|\frac{\partial u_k}{\partial t}\right|^2 dx'.$$

Moreover, for $(s_0,t_0)\in P^2_{1/4}$, let

$$F^*_k(s_0,t_0) = \sup_{0<r\leqslant 1/4}\left(r^{-3}\int\limits_{t_0-r^2}^{t_0+r^2}\int\limits_{s_0-r}^{s_0+r} F_k\,ds\,dt + r^{-1}\int\limits_{s_0-r}^{s_0+r} F_k(s,t_0)\,ds\right)$$

and

$$G^*_k(s_0,t_0) = \sup_{0<r\leqslant 1/4}\left(r^{-1}\int\limits_{t_0-r^2}^{t_0+r^2}\int\limits_{s_0-r}^{s_0+r} G_k\,ds\,dt + r\int\limits_{s_0-r}^{s_0+r} G_k(s,t_0)\,ds\right).$$

We have

$$\lim_{k\to\infty}\|F_k\|_{L^1(P^2_{1/2})} = \lim_{k\to\infty}\|G_k\|_{L^1(P^2_{1/2})} = 0,$$

hence for any $c>0$,

$$\lim_{k\to\infty}\mathcal{L}^2\left(\left\{(s,t)\in P^2_{1/4}\colon\ F^*_k(s,t)\geqslant c\right\}\right) = 0 \tag{33}$$

and

$$\lim_{k\to\infty}\mathcal{L}^2\left(\left\{(s,t)\in P^2_{1/4}\colon\ G^*_k(s,t)\geqslant c\right\}\right) = 0. \tag{34}$$

Using also (32), we can find a sequence of points $(s_k,t_k)\in P^2_{1/4}$, such that

$$\lim_{k\to\infty} F^*(s_k,t_k) = \lim_{k\to\infty} G^*(s_k,t_k) = 0 \tag{35}$$

and

$$\limsup_{k\to\infty}\left\|\nabla^2 u_k(\cdot,s_k,t_k)\right\|_{L^1(B^2_{1/4})} < \infty. \tag{36}$$

Define

$$v_k(x',x^3,t) = u_k\left(\frac{x'}{4},\frac{x^3}{4}+s_k,\frac{t}{16}+t_k\right).$$

These are again solutions of (1). Because of (35), there exists a sequence $\epsilon_k\to 0$ such that

$$\int\limits_{B^2_1\times P^2_r}\left(r^{-3}\left|\frac{\partial v_k}{\partial x^3}\right|^2 + r^{-1}\left|\frac{\partial v_k}{\partial t}\right|^2\right)dz + \int\limits_{B^2_1\times(-r,r)}\left(r^{-1}\left|\frac{\partial v_k}{\partial x^3}(x,0)\right|^2 + r\left|\frac{\partial v_k}{\partial t}(x,0)\right|^2\right)dx \leqslant \epsilon_k \tag{37}$$

for every $k\in\mathbb{N}$ and every $r\in(0,1]$. Moreover, for any $\eta\in C^0_0(P_1)$, we have

$$\lim_{k\to\infty}\int\limits_{P_1}\eta|\nabla v_k|^2\,dz = 8\pi\theta\int\limits_{-1}^{1}\int\limits_{-1}^{1}\eta(0,0,s,t)\,ds\,dt \tag{38}$$

by (30). Using part (ii) of Lemma 3.3, we see that

$$\lim_{k\to\infty}\int\limits_{B_1}\left|\nabla v_k(x,0)\right|^2 dx = 16\pi\theta.$$

Since we have (37), Lemma 3.2 then implies

$$\frac{1}{r}\limsup_{k\to\infty}\int_{B_r(x_0)}\left|\nabla v_k(x,0)\right|^2 dx \leqslant 32\pi\theta \tag{39}$$

uniformly in $x_0 \in B_{1/2}$ and $r \in (0, \frac{1}{2}]$.

Choose two cut-off functions $\xi \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$ and $\zeta \in C_0^\infty(-1, 1)$ such that

$$\int_{-1/2}^{1/2} \xi(s)\,ds = \int_{-1}^{1} \zeta(t)\,dt = 1.$$

For $r \in (0, \frac{1}{2}]$, set

$$\xi_r(s) = r^{-1}\xi(s/r), \quad \zeta_r(t) = r^{-2}\zeta(t/r^2).$$

Fix also a function $\eta \in C_0^\infty(B_{1/2}^2)$, and for $x_0' \in B_{1/2}$ and $r \in (0, \frac{1}{2}]$ set

$$\eta_{x_0',r}(x') = \eta\left(\frac{x'-x_0'}{r}\right).$$

Then (37), (39), and Lemma 3.3 imply

$$\left|\int_{P_1} \eta_{x_0',r}(x')\xi_r(x^3)\zeta_r(t)\left|\nabla v_k(x',x^3,t)\right|^2 dz - \int_{B_1^2} \eta_{x_0',r}(x')\left|\nabla v_k(x',0,0)\right|^2 dx'\right| \to 0 \quad \text{as } k\to\infty \tag{40}$$

uniformly in x_0' and r.

Define now

$$w_k(x') = v_k(x',0,0).$$

From (38) and (40), we obtain

$$\lim_{k\to\infty}\int_{B_1^2} \eta|\nabla w_k|^2\,dx' = 8\pi\theta\eta(0)$$

for every $\eta \in C_0^0(B_1^2)$. That is, the sequence $\{w_k\}$ satisfies the hypothesis (I) from Section 4. Hypothesis (II) follows from (36). Combining (40) with Lemma 3.1, we find that (IV) is true. Using also (37) and the regularity results of Hélein [9], we easily show (III). Thus we can apply Proposition 4.1, and we immediately obtain (i).

To show (ii), we argue by contradiction. Suppose there exists a function $\psi \in C^0(F)$ with $\psi \geqslant 0$ and $\psi(\Pi_L^\perp, \Lambda_\tau) = \psi(\Pi_L^\perp, -\Lambda_\tau) = 0$, such that

$$\int_{P_{1/4}\times F} \psi(A,B)\,d\mu(z,A,B) > 0. \tag{41}$$

For $(s,t) \in P_{1/4}^2$, define

$$f_k(s,t) = \frac{1}{2}\int_{B_{1/4}^2} \psi\big(A_{u_k}(x',s,t), B_{u_k}(x',s,t)\big)\left|\nabla u_k(x',s,t)\right|^2 dx'.$$

Then we have

$$\liminf_{k\to\infty}\int_{-1/16}^{1/16}\int_{-1/4}^{1/4} f_k(s,t)\,ds\,dt > 0. \tag{42}$$

For $c > 0$, define

$$Z_k^c = \{(s,t) \in P_{1/4}^2 \colon f_k(s,t) \geqslant c\}.$$

We claim that there exists a number $c > 0$ such that

$$\liminf_{k\to\infty} \mathcal{L}^2(Z_k^c) > 0. \tag{43}$$

To see this, suppose $c > 0$ is such that

$$\liminf_{k\to\infty} \mathcal{L}^2(Z_k^c) = 0.$$

We may assume that $\mathcal{L}^2(Z_k^c) \leqslant 2^{-k}$ for every $k \in \mathbb{N}$; otherwise we replace our sequence by a subsequence. Define

$$Y_\ell^c = \bigcup_{\ell=k}^{\infty} Z_k^c.$$

Then $\mathcal{L}^2(Y_\ell^c) \leqslant 2^{1-\ell}$, and $f_k < c$ outside of Y_ℓ^c for any $k \geqslant \ell$. Thus

$$\begin{aligned}\liminf_{k\to\infty} \int_{-1/16}^{1/16}\int_{-1/4}^{1/4} f_k(s,t)\,ds\,dt &\leqslant \frac{c}{16} + \frac{1}{2}\|\psi\|_{C^0(F)} \liminf_{k\to\infty} \int_{B_{1/4}^2\times Y_\ell^c} |\nabla u_k|^2\,dz \\ &\leqslant \frac{c}{16} + 2^{2-\ell}\pi\theta\|\psi\|_{C^0(F)}\end{aligned}$$

for any $\ell \in \mathbb{N}$. By (42), this is only possible for sufficiently large values of c.

Now because of (32)–(34) and (43), we can find a sequence of points $(s_k, t_k) \in P_{1/4}^2$ such that (35) and (36) hold true, and in addition,

$$\liminf_{k\to\infty} f_k(s_k, t_k) > 0. \tag{44}$$

Similarly as in the first part of the proof, we see that a subsequence of the maps

$$w_k(x') = u_k\left(\frac{x'}{4}, s_k, t_k\right)$$

satisfies the conditions (I)–(IV) in section 4. But then Proposition 4.1 contradicts (44). This concludes the proof. □

6. Proof of Theorem 1.1

Suppose now that $u_k \in C^\infty(\Omega \times [0,T), \mathbb{S}^2)$ are solutions of the Landau–Lifshitz equation (1) that satisfy the hypotheses of Theorem 1.1. Then it follows from the energy identity (8) that for every precompact set $K \Subset \Omega$ and every $t_0 \in (0,T)$, we have

$$\sup_{k\in\mathbb{N}}\left(\int_K |\nabla u_k(x,t_0)|^2\,dx + \int_{t_0}^{T}\int_K |u_k \wedge \Delta u_k|^2\,dx\,dt\right) < \infty. \tag{45}$$

We consider the measures μ_k on $\Omega_T \times F$ given by

$$\int_{\Omega_T\times F} \psi\,d\mu_k = \frac{1}{2}\int_{\Omega_T} \psi(z, A_{u_k}, B_{u_k})|\nabla u_k|^2\,dz \quad \text{for } \psi \in C_0^0(\Omega_T \times F).$$

Because of (45), we may assume that there exist a function $u \in H^1_{\mathrm{loc}}(\Omega_T, \mathbb{S}^2)$ and a Radon measure μ on $\Omega_T \times F$ such that $u_k \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\Omega_T, \mathbb{R}^3)$ and pointwise almost everywhere, and $\mu_k \to \mu$ (possibly after the choice of a subsequence). Using the representation (4) of the Landau–Lifshitz equation and passing to the limit, we see immediately that u is a weak solution.

It also follows from (45) that there exist Radon measures μ^t on $\Omega \times F$ for almost every $t \in [0, T)$, such that

$$\int_{\Omega_T \times F} \psi \, d\mu = \int_0^T \int_{\Omega \times F} \psi \, d\mu^t \, dt$$

for every $\psi \in C_0^0(\Omega_T \times F)$. We define $m^t = \|\mu^t\|$. We also consider the measures

$$m_k^t = \frac{1}{2} \mathcal{L}^3 \llcorner \left|\nabla u_k(\cdot, t)\right|^2.$$

Fix a function $\xi \in C_0^1(\Omega)$ and define

$$f_{k\xi}(t) = \int_{\Omega} \xi \, dm_k^t$$

and

$$f_\xi(t) = \int_{\Omega} \xi \, dm^t.$$

For any $\zeta \in C_0^0(0, T)$, we have

$$\lim_{k \to \infty} \int_0^T f_{k\xi}(t)\zeta(t)\, dt = \int_0^T f_\xi(t)\zeta(t)\, dt.$$

By (8), we have

$$\frac{d}{dt} f_{k\xi}(t) = -\int_{\Omega \times \{t\}} \left(a\xi \left|\frac{\partial u_k}{\partial t}\right|^2 + a\nabla\xi \cdot \langle \nabla u_k, \Delta u_k\rangle - b\nabla\xi \cdot \langle \nabla u_k, u_k \wedge \Delta u_k\rangle \right) dx.$$

Using (45), we find that

$$\sup_{k \in \mathbb{N}} \int_{t_0}^T \left| \frac{d}{dt} f_{k\xi}(t)\right| dt < \infty$$

for every $t_0 > 0$. That is, the sequence $\{f_{k\xi}\}$ is bounded in $\mathrm{BV}(t_0, T)$. Hence we have $f_{k\xi}(t) \to f_\xi(t)$ for almost every $t \in [0, T)$. Since $C_0^1(\Omega)$ is dense in $C_0^0(\Omega)$, this means that

$$m_k^t \to m^t \tag{46}$$

for almost every $t \in [0, T)$.

For almost every $t \in [0, T)$, we also have $u_k(x, t) \to u(x, t)$ at almost every $x \in \Omega$. By (45) again, the sequence $\{u_k(\cdot, t)\}$ is bounded in $H^1(K, \mathbb{R}^3)$ for every $K \Subset \Omega$. We conclude that

$$u_k(\cdot, t) \rightharpoonup u(\cdot, t) \quad \text{weakly in } H^1_{\mathrm{loc}}(\Omega, \mathbb{R}^3) \tag{47}$$

for almost every $t \in [0, T)$. Using (45) and Fatou's lemma, we also find that

$$\liminf_{k \to \infty} \int_{\Omega} \left| \frac{\partial u_k}{\partial t}(x, t)\right|^2 dx < \infty \tag{48}$$

for almost every $t \in [0, T)$.

Fix $t \in [0, T)$, such that (46), (47), and (48) hold. Then Theorem 3.1 can be applied locally to a subsequence of $\{u_k(\cdot, t)\}$. Hence there exists a locally $\mathcal{H}^1$-integrable function $\theta_t : \Omega \to [0, \infty)$ such that $\Sigma_t = \theta_t^{-1}((0, \infty))$ is closed and countably 1-rectifiable, and

$$m^t = \frac{1}{2} \mathcal{L}^3 \llcorner \left|\nabla u(\cdot, t)\right|^2 + 4\pi \mathcal{H}^1 \llcorner \theta_t. \tag{49}$$

Moreover, we have

$$\int_\Omega \left|u(x,t)\wedge \Delta u(x,t)\right|^2 dx \leqslant \liminf_{k\to\infty} \int_\Omega \left|u_k(x,t)\wedge \Delta u_k(x,t)\right|^2 dx.$$

Fatou's lemma then implies that $\Delta u + |\nabla u|^2 u \in L^2_{\mathrm{loc}}(\Omega_T, \mathbb{R}^3)$.

Consider the measures

$$\lambda_k = \mathcal{L}^4 \llcorner \left|\frac{\partial u_k}{\partial t}\right|^2$$

on Ω_T. We may assume that there exists a Radon measure λ on Ω_T such that $\lambda_k \to \lambda$. Define

$$S = \left\{z_0 \in \Omega_T\colon\ \liminf_{r\searrow 0} r^{-1}\lambda\big(P_r(z_0)\big) > 0\right\}.$$

We claim that the parabolic Hausdorff dimension of S is at most 1. To see this, define

$$S_{ij} = \left\{z_0 \in \Omega_T\colon\ r^{-1}\lambda\big(P_r(z_0)\big) \geqslant i^{-1} \text{ for every } r \in (0, j^{-1})\right\}.$$

These are closed sets, and therefore a standard covering argument involving Vitali's covering lemma shows that S_{ij} has locally finite $\mathcal{H}^1_d$-measure. Consequently, the set

$$S = \bigcup_{i=1}^\infty \bigcup_{j=1}^\infty S_{ij}$$

satisfies $\mathcal{H}^\gamma_d(S) = 0$ for every $\gamma > 1$. Hence for almost every $t_0 \in [0, T)$, we have

$$S \cap \big(\Omega \times \{t_0\}\big) = \emptyset. \tag{50}$$

Fix a t_0 such that (49) and (50) hold. Choose a point $x_0 \in \Sigma_{t_0}$ such that θ_{t_0} is approximately continuous at x_0 and the approximate tangent space $T_{x_0}\Sigma_{t_0}$ exists. (This is true m^{t_0}-almost everywhere on Σ_{t_0}.) Then there exists a sequence $r_k \searrow 0$ such that

$$\lim_{k\to\infty} \frac{1}{r_k} \int_{P_{r_k}(x_0,t_0)} \left|\frac{\partial u_k}{\partial t}\right|^2 dz = 0,$$

and such that the rescaled maps

$$v_k(x,t) = u_k(r_k x + x_0, r_k^2 t + t_0)$$

satisfy

$$\lim_{k\to\infty} \int_{B_1} \xi(x)\left|\nabla v_k(x,0)\right|^2 dx = 8\pi\theta_{t_0}(x_0) \int_{T_{x_0}\Sigma_{t_0}} \xi \, d\mathcal{H}^1$$

for every $\xi \in C^1_0(B_2)$. With the help of part (iii) of Lemma 3.3, we conclude that

$$\lim_{k\to\infty} \int_{P_1} \eta|\nabla v_k|^2 dz = 8\pi\theta_{t_0}(x_0) \int_{-1}^{1} \int_{T_{x_0}\Sigma_{t_0}} \eta \, d\mathcal{H}^1 \, dt$$

for every $\eta \in C^0_0(P_1)$. That is, a subsequence of $\{v_k\}$ satisfies the hypotheses of Proposition 5.1. Therefore, the measures $\tilde{\mu}_k$ on $P_1 \times F$ that belong to v_k subconverge to a measure $\tilde{\mu}$ on $P_1 \times F$ which satisfies

$$\int_{P_{1/4}} \psi(A,B)\, d\tilde{\mu}(z,A,B) = \frac{\pi\tilde{m}}{4}\big(\tilde{\sigma}\psi(\Pi^\perp_{\tilde{L}}, \Lambda_{\tilde{\tau}}) + (1-\tilde{\sigma})\psi(\Pi^\perp_{\tilde{L}}, -\Lambda_{\tilde{\tau}})\big)$$

for two numbers $\tilde{m} \in \mathbb{N}$ and $\tilde{\sigma} \in [0,1]$, where $\tilde{L} = T_{x_0}\Sigma_{t_0}$ and $\tilde{\tau} \in \tilde{L} \cap \mathbb{S}^2$. By the construction of $\tilde{\mu}$, this formula determines $\theta_{t_0}(x_0)$ as well as the fibre measure $\mu^{(x_0,t_0)}$ of μ at the point (x_0,t_0). Namely, we have

$$\theta_{t_0}(x_0) = \tilde{m}$$

and

$$\mu^{(x_0,t_0)} = \delta_{\Pi_{\tilde{L}}^{\perp}} \times \big(\tilde{\sigma}\delta_{\Lambda_{\tilde{\tau}}} + (1-\tilde{\sigma})\delta_{-\Lambda_{\tilde{\tau}}}\big).$$

These arguments work for almost every $t_0 \in [0,T)$ and $\mathcal{H}^1$-almost every $x_0 \in \Sigma_{t_0}$. Hence there exists a function $\theta : \Omega_T \to \mathbb{N} \cup \{0\}$, such that $\theta(x,t) = \theta_t(x)$ for $\|\mu\|$-almost every point $(x,t) \in \Omega_T$, and thus

$$\int_{\Omega_T} \eta \, d\|\mu\| = \frac{1}{2}\int_{\Omega_T} \eta |\nabla u|^2 \, dx + 4\pi \int_0^T \int_{\Sigma_t} \eta\theta \, d\mathcal{H}^1 \, dt$$

for every $\eta \in C_0^0(\Omega_T)$. Moreover, there exist two functions $\sigma : \Omega_T \to [0,1]$ and $\tau : \Omega_T \to \mathbb{S}^2$ with $\tau(x,t) \in T_x\Sigma_t$ for almost every t and $\mathcal{H}^1$-almost every $x \in \Sigma_t$, such that for every $\psi \in C_0^0(\Omega_T \times F)$, we have

$$\begin{aligned}\int_{\Omega_T \times F} \psi \, d\mu = \frac{1}{2}\int_{\Omega_T} \psi(z, A_u, B_u)|\nabla u|^2 \, dz + 4\pi \int_0^T \int_{\Omega} \big[\sigma\psi(x,t,\Pi_{T_x\Sigma_t}, \Lambda_\tau) \\ + (1-\sigma)\psi(x,t,\Pi_{T_x\Sigma_t}, -\Lambda_\tau)\big]\theta \, d\mathcal{H}^1 \, dt.\end{aligned} \tag{51}$$

So far we have proved parts (i) and (ii) of Theorem 1.1. For part (iii), it suffices to use Theorem 3.1 again. Only part (iv) remains to be proved.

Let ϵ_0 be the constant from Lemma 3.1. Define the set

$$\Sigma = \Big\{z_0 \in \Omega_T \colon \liminf_{r \searrow 0} r^{-3}\|\mu\|\big(P_r(z_0)\big) \geqslant \epsilon_0\Big\}.$$

Using Lemma 3.1, we see that for every point $z_0 \in \Omega_T \backslash \Sigma$, there exists a radius $r > 0$ such that a subsequence of $\{u_k\}$ converges to u in $C^1(P_r(z_0), S^2)$. We conclude that Σ is closed and $\Sigma_t \times \{t\} \subset \Sigma$ for almost every $t \in [0,T)$. With a covering argument, we see that $\mathcal{H}_d^3(\Sigma \cap K) < \infty$ for every compact set $K \Subset \Omega_T$. Thus in particular $\mathcal{L}^4(\Sigma) = 0$.

For $\delta > 0$, define

$$U_\delta = \big\{z \in \Omega_T \colon \operatorname{dist}(z, \Sigma) < \delta\big\},$$

where dist is the distance function with respect to the parabolic metric d. Then we have

$$\lim_{\delta \searrow 0} \int_{U_\delta} \eta |\nabla u|^2 \, dz = 0$$

for every $\eta \in C_0^0(\Omega_T)$. Hence

$$\int_0^T \int_{\Sigma_t} \eta\theta \, d\mathcal{H}^1 \, dt = \lim_{\delta \to 0} \int_{U_\delta} \eta \, d\|\mu\|.$$

We finally consider the vector fields $H_k : \Omega_T \to \mathbb{R}^3$ that satisfy $H_k(x) \perp \ker A_{u_k}(x)$ for $\|\mu_k\|$-almost every $x \in \Omega_T$ and

$$\int_{\Omega_T \times F} \phi \cdot AH_k \, d\mu_k = -\delta\mu_k(\phi)$$

for every $\phi \in C_0^1(\Omega_T, \mathbb{R}^3)$. Also consider the function

$$g(z, A, B) = |A|^2 - 2$$

on $\Omega_T \times F$. Note that g is non-negative by condition (c) in the definition of F in Section 2. Now define

$$V_{\delta k} = \{z \in U_\delta:\ g(z, A_{u_k}(z), B_{u_k}(z)) \leqslant 1\}.$$

Then for every $z \in V_{\delta k}$, we have either $\nabla u_k = 0$ or the rank of ∇u_k is 2. By (22), the identity

$$\int_{V_{\delta k}} \phi \cdot BH_k \, d\mu_k = -\int_0^T\int_\Omega \phi \cdot \langle u_k \wedge \Delta u_k, \nabla u_k\rangle \, dx\, dt$$

holds for every $\phi \in C_0^0(\Omega_T, \mathbb{R}^3)$. Integrating both sides of (8), we thus obtain

$$0 = \int_{V_{\delta k}} \left(\frac{\partial \eta}{\partial t} - a\eta H_k \cdot AH_k + a\nabla\eta \cdot AH_k - b\nabla\eta \cdot BH_k\right) d\mu + \int_{\Omega\setminus V_{\delta k}} \left(\frac{1}{2}|\nabla u_k|^2 \frac{\partial \eta}{\partial t} - a\eta|u_k \wedge \Delta u_k|^2 - a\nabla\eta \cdot \langle \Delta u_k, \nabla u_k\rangle + b\nabla\eta \cdot \langle u_k \wedge \Delta u_k, \nabla u_k\rangle\right) dx\, dt$$

for every $\eta \in C_0^1(\Omega_T)$.

Note that

$$\int_{\Sigma\times F} g\, d\mu = 0$$

by (51). Hence

$$\lim_{\delta\searrow 0}\lim_{k\to\infty} \int_{U_\delta\times F} g\, d\mu_k = 0,$$

and we conclude that

$$\lim_{\delta\searrow 0}\lim_{k\to\infty} \|\mu_k\|(U_\delta\setminus V_{\delta k}) = 0. \tag{52}$$

Let $\chi_{\delta k}$ be the characteristic function of $V_{\delta k}$, and define the functions

$$\tilde{H}_{\delta k}(z, A, B) = \chi_{\delta k}(z) H_k(z)$$

on $\Omega_T \times F$. Observe that

$$\left|\tilde{H}_{\delta k}(z, A, B)\right|^2 \leqslant C_1 \left|u_k(z) \wedge \Delta u_k(z)\right|^2$$

for a universal constant C_1 at every point $(z, A, B) \in \Omega_T \times F$ such that $\nabla u_k(z) \neq 0$. Thus for any $K \Subset \Omega$ and $t_0 \in (0, T)$, we have

$$\limsup_{k\to\infty} \int_{K\times(t_0,T)} |\tilde{H}_{\delta k}|^2 \, d\mu_k < \infty$$

for every $\delta > 0$. By Proposition 2.1, we may assume that the measure-function pairs $(\mu_k, \tilde{H}_{\delta k})$ converge weakly to a pair $(\mu, \tilde{H}_\delta)$. Clearly $\tilde{H}_{\delta'} = \tilde{H}_\delta$ in $U_{\delta'} \times F$ whenever $\delta' \leqslant \delta$. Let now

$$\tilde{H}(z, A, B) = \begin{cases} \tilde{H}_1(z, A, B) & \text{if } z \in \Sigma, \\ 0 & \text{else.} \end{cases}$$

This is the pointwise limit of $\tilde{H}_\delta$ as $\delta \searrow 0$. We claim that

$$\delta\mu(\phi) = \int_0^T\int_\Omega \phi \cdot \langle \Delta u, \nabla u\rangle \, dx\, dt - \int_{\Omega_T\times F} \phi \cdot A\tilde{H}\, d\mu \tag{53}$$

for every $\phi \in C_0^1(\Omega_T, \mathbb{R}^3)$. To verify this, we calculate

$$\delta\mu_k(\phi) = \int_{\Omega_T\setminus U_\delta} \phi\cdot\langle\Delta u_k, \nabla u_k\rangle\,dz + \int_{U_\delta\setminus V_{\delta k}} \phi\cdot\langle\Delta u_k, \nabla u_k\rangle\,dz - \int_{V_{\delta k}} \phi\cdot A\tilde{H}_{\delta k}\,d\mu.$$

We have

$$\lim_{\delta\searrow 0}\lim_{k\to\infty}\int_{\Omega_T\setminus U_\delta} \phi\cdot\langle\Delta u_k, \nabla u_k\rangle\,dz = \int_0^T\int_\Omega \phi\cdot\langle\Delta u, \nabla u\rangle\,dx\,dt$$

and

$$\lim_{\delta\searrow 0}\lim_{k\to\infty}\int_{V_{\delta k}} \phi\cdot A\tilde{H}_{\delta k}\,d\mu = \lim_{\delta\searrow 0}\int_{U_\delta} \phi\cdot A\tilde{H}_\delta\,d\mu = \int_\Sigma \phi\cdot A\tilde{H}\,d\mu.$$

Here we use Lebesgue's convergence theorem in the last step. Finally,

$$\lim_{\delta\searrow 0}\lim_{k\to\infty}\left|\int_{U_\delta\setminus V_{\delta k}} \phi\cdot\langle\Delta u_k, \nabla u_k\rangle\,dz\right| \leqslant \lim_{\delta\searrow 0}\lim_{k\to\infty}\sup_{\Omega_T}|\phi|\left(\|\mu_k\|(U_\delta\setminus V_{\delta k})\int_{\operatorname{supp}\phi}|u_k\wedge\Delta u_k|^2\,dz\right)^{1/2} = 0$$

by Hölder's inequality and (52) and (45). This shows (53). With a similar reasoning, we find that

$$\int_0^T\int_\Omega \phi\cdot\langle u\wedge\Delta u, \nabla u\rangle\,dx\,dt - \int_{\Omega_T\times F} \phi\cdot B\tilde{H}\,d\mu = \lim_{k\to\infty}\int_0^T\int_\Omega \phi\cdot\langle u_k\wedge\Delta u_k, \nabla u_k\rangle\,dx\,dt$$

for every $\phi\in C_0^0(\Omega_T,\mathbb{R}^3)$. With the help of Proposition 2.1, we also obtain the inequality

$$\int_0^T\int_\Omega \eta|u\wedge\Delta u|^2\,dx\,dt + \int_{\Omega_T\times F}\eta|\tilde{H}|^2\,d\mu \leqslant \liminf_{k\to\infty}\int_0^T\int_\Omega \eta|u_k\wedge\Delta u_k|^2\,dx\,dt$$

for every $\eta\in C_0^1(\Omega_T)$ with $\eta\geqslant 0$. Passing to the limit in (8), we therefore find

$$\begin{aligned}0 \leqslant{}& \int_0^T\int_\Omega\left(\frac{1}{2}|\nabla u|^2\frac{\partial\eta}{\partial t} - a\eta|u\wedge\Delta u|^2 - a\nabla\eta\cdot\langle\Delta u, \nabla u\rangle + b\nabla\eta\cdot\langle u\wedge\Delta u, \nabla u\rangle\right)dx\,dt\\ &+ \int_{\Omega_T\times F}\left(\frac{\partial\eta}{\partial t} - a\eta|\tilde{H}|^2 + a\nabla\eta\cdot A\tilde{H} - b\nabla\eta\cdot B\tilde{H}\right)d\mu.\end{aligned}$$

Now we define

$$H^+(x,t) = \sigma(x,t)\tilde{H}(x,t,\Pi_{T_x\Sigma_t},\Lambda_{\tau(x,t)})$$

and

$$H^-(x,t) = (1-\sigma(x,t))\tilde{H}(x,t,\Pi_{T_x\Sigma_t},-\Lambda_{\tau(x,t)}).$$

Taking the representation (51) for μ and identity (53) into account, we obtain (15) as well as the other claims from part (iv) of Theorem 1.1.

Remark. We have mentioned earlier that we do not know whether the vector fields H^+ and H^- are parallel to H almost everywhere. In other words, our arguments do not prove that $\tilde{H}$ is a function that depends only on the variable z. Note, however, that if we had strong convergence of the measure-function pairs $(\mu_k,\tilde{H}_{\delta k})$ to $(\mu,\tilde{H}_\delta)$, then Proposition 2.1, together with Jensen's inequality (applied to fibre measures of μ) would imply that $\tilde{H}$ depends only on the first argument. Thus the statement in part (iv) of Theorem 1.1 could be improved in this case. Although strong convergence in this sense may be too much to expect in general, a further analysis of the higher order energies given by $u_k\wedge\Delta u_k$ might give a better understanding of the energy concentration.

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