Two-dimensional impinging jets in hydrodynamic rotational flows ☆

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In memory of Professor Yingming Liu

Abstract

The free streamline theory in hydrodynamics is an important and difficult issue not only in fluid mechanics but also in mathematics. The major purpose in this paper is to establish the well-posedness of the impinging jets in steady incompressible, rotational, plane flows. More precisely, given a mass flux and a vorticity of the incoming flows in the inlet of the nozzle, there exists a unique smooth impinging plane jet. Moreover, there exists a smooth free streamline, which goes to infinity and initiates at the endpoint of the nozzle smoothly. In addition, asymptotic behavior in upstream and downstream, uniform direction and other properties of the impinging jet are also obtained. The main ingredients of the mathematic analysis in this paper are based on the modified variational method developed by H. W. Alt, L. A. Caffarelli and A. Friedman in the elegant works [1,17], which has been shown to be powerful to deal with the steady irrotational flows with free streamlines.

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1. Introduction and main results

Important advancement in understanding impingement of a planar jet upon an edge has resulted from the earlier theoretical investigations of A. Powell in [22] and the more investigations summarized by D. Rockwell in [23]. Therefore, it is well known that studying the jet impinging upon a wall is directly relevant to understanding of a very complex flow field and the dynamical behavior generated underneath a Vertical/Short Take-off and Landing (V/STOL) aircraft operating close to the ground. It has been long recognized that, when an aircraft is in this condition, there are a huge number of complexities associated with the flow field created underneath the aircraft. From an analytical point of view, it is also one of the most difficult problems in fluid mechanics.
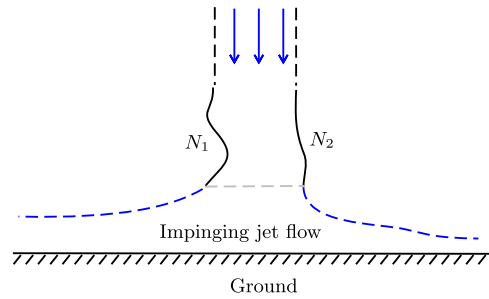
Fig. 1. Impinging jet flow.

One of the most interesting and significant aspects of impinging jet flows in mathematics was mentioned by A. Friedman in Chapter 3 in his book [17], that the existence of steady incompressible irrotational impinging planar jets had been considered by L. A. Caffarelli and himself in their unpublished paper. Fig. 1 schematically represents the planar jet impinging on the ground. Recently, the authors investigated the oblique impinging jet flow with the irrotational condition in [11], and established the systematical wellposedness of the oblique impinging planar jet with one or two asymptotic directions.

The fundamental ingredients in mathematical analysis to investigate the incompressible irrotational impinging jet are stream function method and variational method. The latter has been developed by the three mathematicians, H. W. Alt, L. A. Caffarelli, A. Friedman in [1,17] in 1980's, which has been shown to be powerful to solve the elliptic equations with free boundaries. Based on the frameworks, a series of significant works on irrotational flows with free streamlines have been solved, such as, 3D axially symmetric jet flow in [4], asymmetric jet flow in [2], jets with two fluids in [5–7], axially symmetric infinite cavities in [10], jets with gravity in [3], and so on.

In this paper, we will investigate the incompressible impinging jet in rotational flow. The incompressible, inviscid planar flows are governed by the following two-dimensional incompressible Euler system,

$$\begin{cases} \partial_{x_1} u + \partial_{x_2} v = 0, \\ \partial_{x_1}(u^2 + p) + \partial_{x_2}(uv) = 0, \\ \partial_{x_1}(uv) + \partial_{x_2}(v^2 + p) = 0, \end{cases} \tag{1.1}$$

where $(x_1, x_2) \in \mathbb{R}^2$, (u, v) and p denote the velocity field and the pressure of incompressible fluid, respectively.

Before we state the impinging jet flow problem, we give a semi-infinitely long channel as follows.

Consider a symmetric nozzle here, which is bounded by the symmetric axis $x_1 = 0$ and the nozzle wall (see Fig. 2)

$$N : x_1 = f(x_2) > 0 \quad \text{for } H \leq x_2 < +\infty, \tag{1.2}$$

where $f(x_2) \in C^{2,\alpha}$-smooth $(0 < \alpha < 1)$ function in $[H, +\infty)$, with

$$f(H) = b \quad \text{and} \quad f(x_2) \to a > 0 \quad \text{as } x_2 \to +\infty. \tag{1.3}$$

Given an impermeable wall $M^0 : \{x_1 > 0, x_2 = 0\}$, which blocks the path of the jets. H is the impingement length (the distance from the nozzle opening to the ground) and $2b$ is the width of the mouth of the symmetric nozzle.

Denote the symmetric axis by $I = \{(0, x_2) \mid x_2 \geq 0\}$ and $M^H = \{(x_1, H) \mid x_1 \geq b\}$.

Since, in this paper we will seek an impinging jet with non-positive vertical velocity in the whole flow field, the possible flow field is bounded by N, M^0, M^H and I, denoted as Ω.

The nozzle wall N and the ground M^0 are assumed to be solid, and thus

$$(u, v) \cdot \vec{n} = 0, \quad \text{on } N \cup I \cup M^0, \tag{1.4}$$

where $\vec{n}$ is the unit outer normal of the boundaries.

The free boundary Γ is a material surface, then velocity field still satisfies the perfect slip boundary condition (1.4) on Γ. Furthermore, for the dynamic condition on the free boundary, the classical assumption (neglecting the effects of gravity and surface tension) is that the pressure p is constant, say p_0 on the free boundary. Hence, due to the
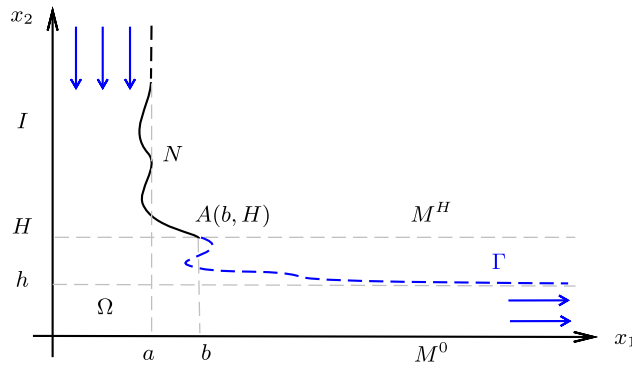
Fig. 2. Symmetric impinging jet flow.

Bernoulli's law for incompressible inviscid flows, the speed of the impinging jet remains a positive constant λ on the free boundary, namely,

$$\sqrt{u^2 + v^2} = \lambda \quad \text{on} \ \ \Gamma. \tag{1.5}$$

It follows from the continuity equation and the boundary condition (1.4) that the mass flux crossing any section S transversal to the x_2-direction with $x_2 \geq H$ remains a positive constant m_0, that is

$$\int_S (u, v) \cdot \vec{l} \, dS = m_0, \tag{1.6}$$

where $\vec{l}$ is the unit normal of S in the negative x_2-direction.

It should be noted that the two-dimensional steady incompressible Euler system is an elliptic–hyperbolic mixed-type system mathematically, which is the one of main differences to the incompressible irrotational flows. However, it's easy to see that there are two invariants along each streamlines for the steady inviscid flows, the vorticity and $\frac{u^2+v^2}{2} + p$. Hence, the strategy here is imposing the vorticity in the upstream replacing the irrotational condition, we can still formulate a single elliptic equation to the stream function in which the hyperbolic mode has been taken into account, as long as the streamlines are well-defined in flow field. On another side, the negativity of vertical velocity of the flows guarantees the well-definition of the streamlines. The similar idea has been applied to solve the ideal compressible subsonic flows in channel in [12–14,29]. Some ideas also inspired by A. Friedman [18] for incompressible cavity flows in rotational flows. Then, we can formulate the impinging jet flow problem as follows.

Impinging jet flow problem Given a semi-infinitely long nozzle N as above, the mass flux m_0 and vorticity $\omega_0 \leq 0$ of the incoming incompressible flows in the inlet, does there exist a unique impinging jet flow, such that a free streamline initiates smoothly from the endpoint of the nozzle and goes to infinity in x_1-direction, and the pressure remains p_0 on the free boundary?

Furthermore, we give the definition of the solution to the impinging jet flow problem in the following.

Definition 1.1. (A solution to the impinging jet flow problem) A quadruple (u, v, p, Γ) is called a solution to the impinging jet flow problem, provided that

(1) The smooth curve Γ is given by a function $x_1 = g(x_2) \in C^1((h, H])$ with

$$f(H) = g(H) \quad \text{(continuous fit condition)} \tag{1.7}$$

and

$$f'(H) = g'(H) \quad \text{(smooth fit condition)}, \tag{1.8}$$

h is the asymptotic height of impinging jet flow, which can be determined uniquely by

$$\lambda = \frac{m_0}{h} - \frac{1}{2}\omega_0 h. \tag{1.9}$$

(2) $(u, v, p) \in \left(C^{1,\alpha}(\Omega_0) \cap C^\alpha(\overline{\Omega}_0) \right)^3$ solves the steady incompressible Euler system (1.1), the boundary condition (1.4), and the mass flux condition (1.6), where $\Omega_0 = \{(x_1, x_2) \mid 0 \leq x_1 < f(x_2),$ for $x_2 \geq H; 0 \leq x_1 < g(x_2),$ for $0 < x_2 \leq H\}$.

(3) $v < 0$ in $\overline{\Omega}_0 \setminus M^0$.

(4) $\sqrt{u^2 + v^2} = \lambda$ and $p = p_0$ on Γ.

Remark 1.1. A point on M^H at which the free streamline of an inviscid flow initiates is called an initial point. Since the location of a free initial point is not known a priori, a condition is required for its determination. The classical condition for planar flows is the one due to M. Brillouin in [9] and H. Villat [26], requiring that the curvature of a free streamline at the free initial point be finite (so-called Villat–Brillouin condition). When the condition is satisfied, the streamline curvature is automatically equal to that of the endpoint of the nozzle. For this reason, free initial fitness is also called continuous and smooth fit conditions (1.7) and (1.8) here. In fact, in this paper, without imposing the initial point of the free streamline, the solution to the impinging jet problem is a family with the free initial point. Hence, we will show that the solution is actually unique with the continuous fit condition (1.7).

Our main results can be stated as follows.

Theorem 1.1. *Given a semi-infinitely long nozzle N, an incoming flow with mass flux $m_0 > 0$ and a constant vorticity $\omega_0 \leq 0$ in the upstream, with*

$$m_0 > \max\left(-\frac{\omega_0}{2} H^2, -\frac{\omega_0}{2} a^2 \right). \tag{1.10}$$

Then there exists a unique $\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ and a solution (u, v, p, Γ) to the impinging jet flow problem, which satisfies the conditions in the Definition 1.1.

Remark 1.2. The impinging jet flows established here possess a uniform direction, more precisely, the vertical velocity of the impinging jet is always negative except on the ground. Note that we do not impose the vertical velocity of the incoming flow in upstream, and it's easy to check that in fact the condition (1.10) ensures that the negativity of the vertical velocity of the impinging jet in the upstream and the positivity of the horizontal velocity in the downstream. On the other side, the condition (1.10) implies mathematically that for given m_0 and ω_0, λ is a decreasing function respect to the asymptotic height h, and h can be determined uniquely by λ, due to the relationship (1.9).

Remark 1.3. The important property that the negativity of the vertical velocity of the impinging jet guarantees the well-definedness of the streamline in rotational ideal flows. In another word, any streamline can not intersect each other, and any point in fluid field can be pulled back to the inlet of the nozzle along one streamline. Since the vorticity is an invariance along the each streamline in steady ideal flows, the vorticity of the impinging jet is well-defined in the whole fluid field for given the vorticity of the impinging flow in the inlet. Some similar ideas borrowed from the one in compressible subsonic flows in infinitely long nozzle in [12–15,28,29].

Remark 1.4. As we mentioned in Remark 1.2, λ is a strictly decreasing respect to the asymptotic height $h \in [0, H]$, under the assumption (1.10). Hence, the condition $\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ is natural and reasonable.

In fact, in Theorem 1.1, we can show that there exists a unique parameter $\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$, such that the continuous fit condition (1.7) holds. However, to obtain the uniqueness of the free boundary and the velocity field, we assume the nozzle wall N satisfies the following additional condition

$$x_1 = f(x_2) \text{ is a monotonic decreasing function.} \tag{1.11}$$

Theorem 1.2. *Suppose that the semi-infinitely long nozzle satisfies the additional condition (1.11), then the parameter λ and the solution (u, v, p, Γ) established in Theorem 1.1 is unique and the horizontal velocity $u > 0$ in $\overline{\Omega}_0 \setminus I$. Furthermore, the impinging jet flow satisfies the following asymptotic behavior in far fields,*

$$(u(x_1, x_2), v(x_1, x_2), p(x_1, x_2)) \to (0, v_0(x_1), p_1),$$

and

$$\nabla u \to 0, \quad \nabla v \to (\omega_0, 0), \quad \nabla p \to 0,$$

uniformly in any compact subset of $(0, a)$ as $x_2 \to +\infty$, where $v_0(x_1) = -\frac{m_0}{a} - \frac{1}{2}\omega_0 a + \omega_0 x_1$ and $p_1 = p_0 + \frac{\lambda^2}{2} - \frac{\left(\frac{m_0}{a} - \frac{1}{2}\omega_0 a\right)^2}{2}$.

Similarly,

$$(u(x_1, x_2), v(x_1, x_2), p(x_1, x_2)) \to (u_0(x_2), 0, p_0),$$

and

$$\nabla u \to (0, -\omega_0), \quad \nabla v \to 0, \quad \nabla p \to 0,$$

uniformly in any compact subset of $(0, h)$ as $x_1 \to +\infty$, where $u_0(x_2) = \lambda + \omega_0 h - \omega_0 x_2$, h is the asymptotic height of the impinging jet.

Remark 1.5. The asymptotic behaviors established in Theorem 1.2 gives that the pressure is constant in the inlet and the downstream, which seems to be reasonable due to the following reason. Assume that the flow satisfies the asymptotic behavior in the inlet as

$$(u(x_1, x_2), v(x_1, x_2), p(x_1, x_2)) \to (u_0(x_1), v_0(x_1), p_1(x_1)) \quad \text{as } x_2 \to +\infty,$$

and satisfies the far field conditions with high order compatibility conditions,

$$(\nabla u(x_1, x_2), \nabla v(x_1, x_2), \nabla p(x_1, x_2)) \to (\nabla u_0(x_1), \nabla v_0(x_1), \nabla p_1(x_1)) \quad \text{as } x_2 \to +\infty.$$

Due to the divergence-free condition, we have

$$u_0'(x_1) = 0.$$

Moreover, the conservation of momentum gives that

$$p_1'(x_1) = 0,$$

which implies that the pressure remains constant in the inlet. Similar result follows in the downstream. In fact, p_0 on the free streamline is the outside pressure, and $p_1 = p_0 + \frac{\lambda^2}{2} - \frac{1}{2}\left(\frac{m_0}{a} - \frac{1}{2}\omega_0 a\right)^2$ is the chamber pressure in the inlet of the channel.

The rest of the paper is organized as follows. In Section 2, we will introduce the mathematical setting of the impinging jet problem, such as stream function formulation to the incompressible inviscid rotational flows and the variational problem with a parameter λ. Furthermore, some preliminaries to the free boundary will be represented in Section 3. In Section 4, we will give the existence of the impinging jet flow problem via the variational method. Especially, we will show that there exists a unique parameter λ, such that the impinging jet satisfies the continuous fit condition. Finally, the uniqueness of the impinging jet flow problem is established in Section 5. In summary, the principal new ideas appear in this work center about the formulation of the hydrodynamical problem as a mathematical problem in the calculus of variations, the analysis of the free streamlines in rotational flow, such as the regularity and the continuous fit condition of free streamlines in rotational flows. Our results here are merely extension of the developments due to Garabedian, Lewy, Schiffer in [19] and Alt, Caffarelli, Friedman in [1,17], which were applied similarly in the existence proofs for jet and cavity flow given in [2–4,18]. The general background on jets, cavity flows and other problems with free streamlines are referred to the references [8,16,21,24,25,27].

2. Mathematical setting of the impinging jet problem

2.1. Stream function approach

In view of the continuity equation, there is a stream function ψ, such that

$$\frac{\partial \psi}{\partial x_1} = -v, \quad \frac{\partial \psi}{\partial x_2} = u. \tag{2.1}$$

It's easy to check that

$$(u, v) \cdot \nabla \omega = 0, \tag{2.2}$$

and

$$(u, v) \cdot \nabla \left(\frac{|\nabla \psi|^2}{2} + p \right) = 0 \tag{2.3}$$

where $\omega = \partial_{x_1} v - \partial_{x_2} u$ is the vorticity of the fluid in two dimensions, and it implies that the vorticity and $\frac{|\nabla \psi|^2}{2} + p$ remain a constant along each streamline. The formula (2.3) is so-called Bernoulli's law in steady incompressible flows. In this paper, we assume that the vorticity of the incoming flow is a given constant $\omega_0 \leq 0$, then the vorticity of the flow is also ω_0, as long as the streamlines are well-defined and can not intersect each other in the whole flow field. In fact, in this paper, we search an impinging jet flow with $v < 0$ in the fluid field and then the streamlines are well-defined in the whole flow field, and thus, the stream function satisfies

$$-\triangle \psi = \omega_0. \tag{2.4}$$

Without loss of generality, we impose the Dirichlet boundary value conditions as follows,

$$\psi = m_0 \text{ on } N \cup \Gamma, \text{ and } \psi = 0 \text{ on } I \cup M^0.$$

Thus, the free boundary of the impinging jet flow can be defined by

$$\Gamma = \Omega \cap \partial \{\psi < m_0\}. \tag{2.5}$$

On the free streamline Γ, the pressure is assumed to be a constant p_0, then it follows from Bernoulli's law that the speed remains a constant on Γ, namely,

$$|\nabla \psi| = \frac{\partial \psi}{\partial \nu} = \lambda, \tag{2.6}$$

where ν is outer unit normal of Γ.

Hence, we formulate the following boundary problem of the stream function as follows,

$$\begin{cases} -\Delta \psi = \omega_0, & \text{in } \Omega_0, \\ \frac{\partial \psi}{\partial \nu} = \lambda, & \text{on } \Gamma, \\ \psi = 0, & \text{on } I \cup M^0, \quad \psi = m_0, \quad \text{on } N \cup \Gamma, \end{cases} \tag{2.7}$$

where Ω_0 is bounded by N, I, M^0 and Γ.

Once the stream function is solved, the velocity field (u, v) can be obtained via (2.1) and the parameter λ can be solved by (2.6). Moreover, the asymptotic height h can be determined by (1.9) and the free boundary Γ can be obtained by the definition (2.5).

Finally, the pressure can be solved along the each streamlines as follows. For any point $(x_1, x_2) \in \Omega$, which can be pulled back along one streamline to the point $(\kappa(\psi), +\infty)$ in the inlet, we have

$$\psi = -\int_0^{\kappa(\psi)} v_0(s) ds,$$

where $v_0(x_1) = \lim_{x_2 \to +\infty} v(x_1, x_2)$. Then, $\kappa(\psi)$ satisfies the following initial value problem

$$\begin{cases} -\kappa'(\psi)v_0(\kappa(\psi)) = 1, \\ \kappa(0) = 0. \end{cases}$$

Hence, it follows from the Bernoulli's law that the pressure p can be solved as

$$p(x_1, x_2) = p_1 + \frac{v_0^2(\kappa(\psi))}{2} - \frac{u^2 + v^2}{2},$$

where the constant p_1 is the chamber pressure in the entrance of the nozzle.

2.2. Variational approach

To solve the boundary value problem (2.7), we introduce the following variational problem with a parameter $\lambda > 0$. **The variational problem** (P_λ)**:** Define an admissible set as

$$K = \left\{ \psi \in H^1_{loc}(\Omega) \mid \psi = 0 \text{ on } I \cup M^0, \psi = m_0 \text{ on } N \cup M^H \right\},$$

and for any bounded domain $D \subset\subset \Omega$ define a functional

$$J_\lambda(\psi) = \int_D |\nabla\psi|^2 + \lambda^2 \chi_{\{\psi < m_0\} \cap E} - 2\omega_0(\psi - m_0) dx_1 dx_2, \tag{2.8}$$

where χ_A is the characteristic function of a set A and

$$E = \{(x_1, x_2) \mid x_1 > 0 \text{ and } 0 < x_2 < H\}. \tag{2.9}$$

Find a $\psi_\lambda \in K$ such that

$$J_\lambda(\psi_\lambda) = \min_{\psi \in K, \ \psi = \psi_\lambda \text{ on } \partial D} J_\lambda(\psi).$$

First, we will show that $\psi \leq m_0$ in the weak sense under the condition $\omega_0 \leq 0$.

Lemma 2.1. *For any minimizer ψ_λ to the variational problem (P_λ), $\psi_\lambda \leq m_0$ a.e. in Ω.*

Proof. Set $\psi_\lambda^\varepsilon(x_1, x_2) = \psi_\lambda(x_1, x_2) + \varepsilon \min(m_0 - \psi_\lambda(x_1, x_2), 0)$ for $\varepsilon \in (0, 1)$. It is clear that $\psi_\lambda^\varepsilon \in K$ and $\psi_\lambda^\varepsilon \leq \psi_\lambda$. Furthermore, one has

$$\psi_\lambda < m_0 \text{ if and only if } \psi_\lambda^\varepsilon < m_0.$$

Since ψ_λ is a minimizer to the variational problem (P_λ), we have

$$\begin{aligned} 0 &\geq J_\lambda(\psi_\lambda) - J_\lambda(\psi_\lambda^\varepsilon) \\ &= \int_D (|\nabla\psi_\lambda|^2 - |\nabla\psi_\lambda^\varepsilon|^2 - 2\omega_0(\psi_\lambda - \psi_\lambda^\varepsilon)) dx_1 dx_2 \\ &= (1 - (1-\varepsilon)^2) \int_D |\nabla \min(m_0 - \psi_\lambda, 0)|^2 dx_1 dx_2 + \int_D 2\omega_0 \varepsilon \min(m_0 - \psi_\lambda, 0) dx_1 dx_2. \end{aligned} \tag{2.10}$$

Thanks to the condition $\omega_0 \leq 0$ and (2.10), one has

$$(1 - (1-\varepsilon)^2) \int_D |\nabla \min(m_0 - \psi_\lambda, 0)|^2 dx_1 dx_2 \leq -\int_D 2\omega_0 \varepsilon \min(m_0 - \psi_\lambda, 0) dx_1 dx_2 \leq 0, \tag{2.11}$$

which implies that $\psi_\lambda \leq m_0$ a.e. in D, due to the arbitrariness of D, we complete the proof of Lemma 2.1. $\square$

Next, we will show that the minimizer ψ_λ satisfies the equation (2.4) in the flow field and $|\nabla\psi_\lambda| = \lambda$ on the free boundary Γ in the weak sense.

Proposition 2.2. *For any minimizer ψ_λ to the variational problem (P_λ), we have*

$$\Delta\psi_\lambda + \omega_0 \le 0 \ \text{ in } \ \Omega, \ \text{ and } \ \Delta\psi_\lambda + \omega_0 = 0 \ \text{ in } \ \Omega \cap \{\psi_\lambda < m_0\}.$$

Furthermore,

$$\lim_{\varepsilon\downarrow 0} \int\limits_{\partial\{\psi_\lambda < m_0-\varepsilon\}} \left(|\nabla\psi_\lambda|^2 - \lambda^2 - 2\omega_0(\psi_\lambda - m_0)\right)\eta\cdot\nu = 0, \tag{2.12}$$

for any 2-vector $\eta \in \left(C_0^1(E)\right)^2$, where ν is the normal vector to $\partial\{\psi_\lambda < m_0 - \varepsilon\}$.

Proof. First, we will show $\Delta\psi_\lambda + \omega_0 \le 0$ in Ω in weak sense. For any $D \subset\subset \Omega$ and nonnegative function $\xi \in C_0^1(D)$ and $\varepsilon > 0$ sufficiently small, we have

$$0 \le \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\left(J_\lambda(\psi_\lambda + \varepsilon\xi) - J_\lambda(\psi_\lambda)\right)$$
$$\le \int\limits_D (\nabla\psi_\lambda \cdot \nabla\xi - \omega_0\xi)\,dx_1 dx_2$$
$$= -\int\limits_D (\Delta\psi_\lambda + \omega_0)\,\xi\,dx_1 dx_2,$$

due to the integration by parts. Hence, the arbitrariness of D implies that $\Delta\psi_\lambda + \omega_0 \le 0$ in Ω in weak sense.

Next, we will show that if $\psi_\lambda \in K$ is a minimizer, then ψ_λ satisfies $\Delta\psi_\lambda + \omega_0 = 0$ in $\Omega \cap \{\psi_\lambda < m_0\}$ in weak sense. Indeed, taking any $\xi \in C_0^1(D \cap \{\psi_\lambda < m_0\})$, then $\psi_\lambda + \varepsilon\xi \in K$ for any sufficiently small $|\varepsilon|$, we obtain

$$\lim_{\varepsilon\to 0}\frac{J_\lambda(\psi_\lambda + \varepsilon\xi) - J_\lambda(\psi_\lambda)}{\varepsilon} = 0.$$

Then

$$0 = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{D\cap\{\psi_\lambda < m_0\}}\left(2\varepsilon\nabla\psi_\lambda\cdot\nabla\xi + \varepsilon^2\,|\nabla\xi|^2 - 2\varepsilon\omega_0\xi\right)dx_1 dx_2$$
$$= \int\limits_{D\cap\{\psi_\lambda < m_0\}}(2\nabla\psi_\lambda\cdot\nabla\xi - 2\omega_0\xi)\,dx_1 dx_2.$$

This implies that

$$\int\limits_{D\cap\{\psi_\lambda < m_0\}}(\Delta\psi_\lambda + \omega_0)\,\xi\,dx_1 dx_2 = 0,$$

due to integration by parts, which together with the arbitrariness of D gives that $\Delta\psi_\lambda + \omega_0 = 0$ in $\Omega \cap \{\psi_\lambda < m_0\}$ in weak sense.

Finally, we will prove the second part of this proposition. Let

$$\eta(x) = \eta(x_1, x_2) \in \left(C_0^1(E)\right)^2 \ \text{ and } \ \tau_\delta(x) = x + \delta\eta(x),$$

where δ is a real number and $|\delta| > 0$ is suitable small. Define $\psi_\lambda^\delta(\tau_\delta(x)) = \psi_\lambda(x)$ and it's easy to verify that $\psi_\lambda^\delta \in K$ and

$$D(\tau_\delta(x))^{-1} = (I + \delta\nabla\cdot\eta I - \delta D\eta)(\det D\tau_\delta)^{-1} \ \text{ and } \ \det D\tau_\delta = 1 + \delta\nabla\cdot\eta + o(\delta),$$

where I is the identity matrix.

Due to the fact that ψ_λ is a minimizer to the problem (P_λ), one gets

$$
\begin{aligned}
0 &\leq J_\lambda(\psi_\lambda^\delta) - J_\lambda(\psi_\lambda) \\
&= \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} \left(|\nabla \psi_\lambda (D\tau_\delta)^{-1})|^2 - 2\omega_0(\psi_\lambda^\delta - m_0) + \lambda^2 \right) \det D\tau_\delta dx_1 dx_2 \\
&\quad - \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} |\nabla \psi_\lambda|^2 - 2\omega_0(\psi_\lambda - m_0) + \lambda^2 dx_1 dx_2 \\
&= \delta \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} \left(|\nabla \psi_\lambda|^2 \nabla \cdot \eta - 2\nabla \psi_\lambda \cdot D\eta \cdot \nabla \psi_\lambda \right) dx_1 dx_2 \\
&\quad + \delta \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} \left(\lambda^2 - 2\omega_0(\psi_\lambda - m_0) \right) \nabla \cdot \eta dx_1 dx_2 + o(\delta).
\end{aligned}
\tag{2.13}
$$

In view of the arbitrariness of δ, the linear term of (2.13) in δ has to vanish, and then this gives that

$$
\begin{aligned}
0 &= \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} \left(|\nabla \psi_\lambda|^2 \nabla \cdot \eta - 2\nabla \psi_\lambda \cdot D\eta \cdot \nabla \psi_\lambda + (\lambda^2 - 2\omega_0(\psi_\lambda - m_0))\nabla \cdot \eta \right) dx_1 dx_2 \\
&= \int\limits_{\{\psi_\lambda < m_0\} \cap E \cap D} \nabla \cdot \left((|\nabla \psi_\lambda|^2 + \lambda^2 - 2\omega_0(\psi_\lambda - m_0))\eta - 2(\eta \cdot \nabla \psi_\lambda)\nabla \psi_\lambda \right) dx_1 dx_2 \\
&= \lim_{\varepsilon \downarrow 0} \int\limits_{\partial\{\psi_\lambda < m_0 - \varepsilon\} \cap E \cap D} \left((|\nabla \psi_\lambda|^2 + \lambda^2 - 2\omega_0(\psi_\lambda - m_0))\eta - 2(\eta \cdot \nabla \psi_\lambda)\nabla \psi_\lambda \right) \cdot v dS \\
&= \lim_{\varepsilon \downarrow 0} \int\limits_{\partial\{\psi_\lambda < m_0 - \varepsilon\} \cap E \cap D} \left((\lambda^2 - |\nabla \psi_\lambda|^2 - 2\omega_0(\psi_\lambda - m_0)) \right) \eta \cdot v dS,
\end{aligned}
\tag{2.14}
$$

where we have used the fact that $\nabla \psi_\lambda \parallel v$ on the free boundary $\{\psi_\lambda < m_0 - \varepsilon\} \cap E$. $\quad\square$

Finally, we introduce the regularity of the minimizer ψ_λ to the variational problem (P_λ).

Lemma 2.3. *The minimizer $\psi_\lambda \in C^{0,1}(D)$, that is, ψ_λ is Lipschitz continuous in any compact subsets of $D \subset\subset \Omega$. Furthermore, $\psi_\lambda \in C^{2,\alpha}$ in any compact subset of $\Omega \cap \{\psi_\lambda < m_0\}$.*

Proof. The Lipschitz continuity of the minimizer can be obtained by similar arguments in Lemma 2.4 in [18].

It follows from Proposition 2.2 that the minimizer ψ_λ satisfies the equation

$$-\Delta \psi_\lambda = \omega_0, \quad \text{in the weak sense in } \Omega \cap \{\psi_\lambda < m_0\}.$$

Thanks to the standard interior Schauder estimates to the linear elliptic equation in Chapter 8 in [20], one has

$$\psi_\lambda \in C^{2,\alpha} \quad \text{in any compact subset of } \Omega \cap \{\psi_\lambda < m_0\}. \quad\square$$

3. Preliminaries

In this section, we will introduce some important lemmas which are established by A. Friedman in [18], such as the nondegeneracy lemma and the nonoscillation lemma.

First, we introduce the nondegeneracy lemma, which plays an important role to investigate the properties of the free boundary, and the proof follows along the similar arguments in Lemma 2.5 in [18].

Lemma 3.1. *Let ψ_λ be a minimizer to the variational problem (P_λ). There exists a universal constant C^* such that, for any $X^0 \in \Omega$ with the disc $B_r(X^0) \subset \Omega$, satisfying*

$$\frac{1}{r} \fint_{\partial B_r(X^0)} (m_0 - \psi_\lambda) dS \geq \lambda C^*,$$

then $\psi_\lambda < m_0$ in $B_r(X^0)$, where r is enough small.

Lemma 3.2. *For any $\kappa \in (0,1)$, there exists a positive number $c^* = c^*(\kappa)$ such that for any minimizer ψ_λ, if $B_r(X^0) \subset \Omega$ and $r < -\frac{c^*\lambda}{\omega_0}$, $\omega_0 < 0$, then*

$$\frac{1}{r} \fint_{\partial B_{\kappa r}(X^0)} (m_0 - \psi_\lambda) dS \leq \lambda c^* \quad implies \quad \psi_\lambda = m_0 \quad in \ \ B_{\kappa r}(X^0).$$

Remark 3.1. For $\omega_0 = 0$, Lemma 3.2 can be obtained by using similar arguments in Lemma 2.4 in [4] for irrotational flows.

The following lemma implies that the free boundary $\Omega \cap \partial\{\psi_\lambda < m_0\}$ has Lebesgue measure zero.

Lemma 3.3. *Suppose $D \subset\subset \Omega$, and there exists a positive constant c and $0 < c < 1$, such that for any ball $B_r \subset D$ with center in the free boundary and r small enough, then*

$$c < \frac{\mathcal{L}^2 (B_r \cap \{\psi_\lambda < m_0\})}{\mathcal{L}^2 (B_r)} < 1 - c,$$

where $\mathcal{L}^2$ is the Lebesgue measure.

Next, the convergence of free boundary is stated, and we would like to refer the proof in §3.6 in [17]. Denote $\vartheta_n = m_0 - \psi_{\lambda_n}$, $\vartheta = m_0 - \psi_\lambda$.

Lemma 3.4. *Let U be an open bounded set in $\mathbb{R}^2$ and $\vartheta_n \in C^0(U)$ with*

$$\triangle \vartheta_n = \omega_0 \quad in \ \ U \cap \{\vartheta_n > 0\},$$

$$\frac{\partial \vartheta_n}{\partial \nu} = -\lambda \quad on \ \ U \cap \partial\{\vartheta_n > 0\},$$

$$U \cap \partial\{\vartheta_n > 0\} \ is \ C^{1,\alpha} \quad (0 < \alpha < 1),$$

where ν is the outward normal to the boundary $U \cap \partial\{\vartheta_n > 0\}$. If

$$\lambda_n \to \lambda \quad and \quad \vartheta_n \to \vartheta \quad uniformly \ in \ U, \ \ for \ some \ \ \vartheta \in H^1(U),$$

$$U \cap \{\vartheta_n > 0\} \to U \cap \{\vartheta > 0\} \quad in \ measure,$$

$$U \cap \partial\{\vartheta > 0\} \ is \ C^{1,\alpha} - smooth \quad (0 < \alpha < 1),$$

then ϑ satisfies

$$\begin{cases} \triangle \vartheta = \omega_0 \quad in \ \ U \cap \{\vartheta > 0\}, \\ \frac{\partial \vartheta}{\partial \nu} = -\lambda \quad on \ \ U \cap \partial\{\vartheta > 0\}, \end{cases}$$

in the weak sense.

Finally, we introduce the following nonoscillation lemma and the uniformly bounded gradient lemma, which is a crucial part to study the property of the free boundary in next section.
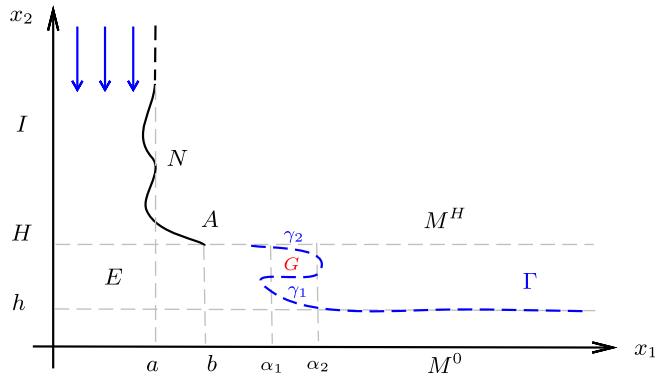
Fig. 3. The domain G.

Lemma 3.5. *Let G be a domain in $E \cap \{\psi_\lambda < m_0\}$ bounded by two disjointed arcs γ_1, γ_2 of free boundary, $\{x_1 = \alpha_1\}$ and $\{x_1 = \alpha_2\}$ (see Fig. 3). Suppose the γ_i lies in $\{\alpha_1 < x_1 < \alpha_2\}$ with endpoints (α_1, β_i) and (α_2, ζ_i) for $i = 1, 2$, and $\mathrm{dist}(A, G) \geq c > 0$, then*

$$\frac{(\alpha_2 - \alpha_1)^2}{1 + (\alpha_2 - \alpha_1)^2} \leq C \max \left\{ |\beta_1 - \beta_2|^2, |\zeta_1 - \zeta_2|^2 \right\},$$

where C depends only on λ, m_0, ω_0 and c.

Proof. Set $\widetilde{h} = \max\{|\beta_1 - \beta_2|, |\zeta_1 - \zeta_2|\}$. It follows from the definition of G that $-\triangle\psi_\lambda = \omega_0$ in G. Green's formula implies that

$$\begin{aligned}
\int_G (x_1 - \alpha_1)\omega_0 dx_1 dx_2 &= \int_G (x_1 - \alpha_1)\triangle(m_0 - \psi_\lambda) dx_1 dx_2 \\
&= -\int_{\partial G} (x_1 - \alpha_1)\frac{\partial \psi_\lambda}{\partial \nu} dS - \int_{\partial G} (m_0 - \psi_\lambda)\frac{\partial x_1}{\partial \nu} dS.
\end{aligned} \tag{3.1}$$

Since γ_1 and γ_2 are two arcs of the free boundary, we obtain

$$\begin{aligned}
\int_{\gamma_1 \cup \gamma_2} \lambda(x_1 - \alpha_1) dS &= -(\alpha_2 - \alpha_1) \int_{\partial G \cap \{x_1 = \alpha_2\}} \frac{\partial \psi_\lambda}{\partial \nu} dS - \int_G \omega_0(x_1 - \alpha_1) dS \\
&\quad - \int_{\partial G \cap (\{x_1 = \alpha_1\} \cup \{x_1 = \alpha_2\})} (m_0 - \psi_\lambda)\frac{\partial x_1}{\partial \nu} dS \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.2}$$

Thanks to the Lipschitz continuity of ψ_λ, we have

$$I_1 = -(\alpha_2 - \alpha_1) \int_{\partial G \cap \{x_1 = \alpha_2\}} \frac{\partial \psi_\lambda}{\partial \nu} dS \leq C(\alpha_2 - \alpha_1)\widetilde{h}. \tag{3.3}$$

For I_2 and I_3, one gets

$$I_2 = -\int_G \omega_0(x_1 - \alpha_1) dx_1 dx_2 \leq -\omega_0(\alpha_2 - \alpha_1) \int_G dx_1 dx_2 \leq -\omega_0(\alpha_2 - \alpha_1)^2 \widetilde{h}, \tag{3.4}$$
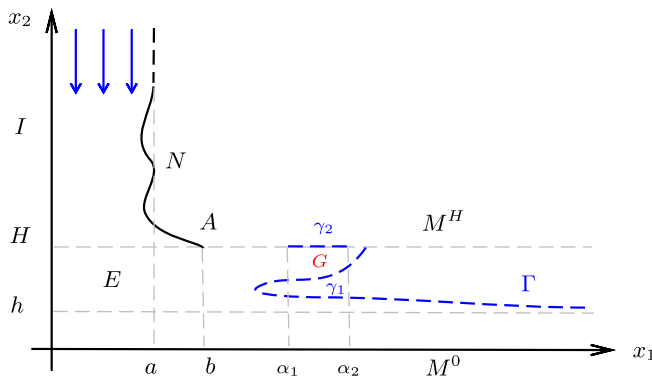
Fig. 4. The domain G in special case.

and

$$I_3 = - \int_{\partial G \cap \{x_1 = \alpha_1, x_1 = \alpha_2\}} (m_0 - \psi_\lambda) \frac{\partial x_1}{\partial \nu} dS \le C\widetilde{h}^2, \tag{3.5}$$

where we have used the fact

$$m_0 - \psi_\lambda \le C\widetilde{h}, \quad C \text{ is the Lipschitz constant.}$$

On another hand,

$$\int_{\gamma_1 \cup \gamma_2} \lambda(x_1 - \alpha_1) dS \ge \lambda(\alpha_2 - \alpha_1)^2, \tag{3.6}$$

and then, we have

$$\begin{aligned}
\lambda(\alpha_2 - \alpha_1)^2 &\le C(\alpha_2 - \alpha_1)\widetilde{h} - \omega_0(\alpha_2 - \alpha_1)^2\widetilde{h} + C\widetilde{h}^2 \\
&\le \frac{\lambda}{2}(\alpha_2 - \alpha_1)^2 + C\widetilde{h}^2 + C(\alpha_2 - \alpha_1)^2\widetilde{h}^2.
\end{aligned} \tag{3.7}$$

Hence

$$\frac{(\alpha_2 - \alpha_1)^2}{1 + (\alpha_2 - \alpha_1)^2} \le C \max \left\{ |\beta_1 - \beta_2|^2, |\zeta_1 - \zeta_2|^2 \right\}, \tag{3.8}$$

where C depends only on λ, ω_0, m_0 and c.   □

Remark 3.2. The nonoscillation lemma remains true if one of the arcs γ_2 is a line segment on M^H (see Fig. 4), provided that

$$\frac{\partial \psi_\lambda}{\partial \nu} \ge \lambda \text{ on } \gamma_2. \tag{3.9}$$

In fact, with the aid the condition (3.9), (3.2) can be written as follows

$$\int_{\gamma_1 \cup \gamma_2} \lambda(x_1 - \alpha_1) dS \le I_1 + I_2 + I_3. \tag{3.10}$$

It is easy to verify that the Lemma 3.5 is true under the condition $\frac{\partial \psi_\lambda}{\partial \nu} \ge \lambda$ on γ_2. Actually, we can check the fact (3.9) in Lemma 4.7.

Lemma 3.6. *Let $X^0 = (x_1^0, x_2^0)$ be a free boundary point in Ω and let G be a compact subset of Ω and contain X^0. Then*

$$|\nabla \psi_\lambda| \le C \quad in \ \ G \cap \{\psi_\lambda < m_0\},$$

where C depends only on λ, ω_0, c_0 and G, but it is independent of m_0.

Proof. Set $d_0 = dist(G, \partial\Omega)$. For any $X \in G \cap \{\psi_\lambda < m_0\}$, there are some points $X^1, \cdots, X^n = X$ in G (n depends on d_0) such that

$$X^k \in B_{r_0}(X^{k-1}) \quad \text{for } k = 1, ..., n, \quad r_0 = \frac{d_0}{4}.$$

We choose $n_0 \in \{1, 2, ..., n\}$ be the largest number such that $B_{2r_0}(X^{n_0})$ contains a free boundary point $\tilde{X}$. The existence of n_0 follows from the fact that $B_{2r_0}(X^1)$ contains X^0. Then $m_0 - \psi_\lambda - \frac{\omega_0}{2}(x_1 - x_1^0)^2$ is harmonic in $B_{2r_0}(X^{k-1})$ for $k \ge n_0 + 2$, and Harnack's inequality gives that

$$m_0 - \psi_\lambda(X^k) - \frac{\omega_0}{2}(x_1^k - x_1^0)^2 \le C\left(m_0 - \psi_\lambda(X^{k-1}) - \frac{\omega_0}{2}(x_1^{k-1} - x_1^0)^2\right).$$

Since $\triangle(m_0 - \psi_\lambda) - \omega_0 \ge 0$, we also have

$$
\begin{aligned}
& m_0 - \psi_\lambda(X^{n_0+1}) \\
& \le - \int_{B_{4r_0}(\tilde{X})} \omega_0 G_{X^{n_0+1}} dx_1 dx_2 - \int_{\partial B_{4r_0}(\tilde{X})} \frac{\partial G_{X^{n_0+1}}}{\partial \nu}(m_0 - \psi_\lambda) dS \\
& \le - C\omega_0 r_0^2 + C \fint_{\partial B_{4r_0}(\tilde{X})} (m_0 - \psi_\lambda) dS \\
& \le C\omega_0 r_0^2 + C r_0 \\
& \le C,
\end{aligned}
\tag{3.11}
$$

where we have used Lemma 3.1 and the constant C depends only on λ, ω_0, c_0 and G. Therefore, $m_0 - \psi_\lambda(X) \le C$ for any $X \in G$. With the aid of the Lipschitz continuity of $\psi_\lambda(X)$, we have

$$|\nabla \psi_\lambda(X)| \le C \left(1 + \sup_{X \in G}(m_0 - \psi_\lambda(X))\right) \le C,$$

where the constant C depends only on λ, ω_0, c_0 and G, but independent of m_0. $\quad\square$

4. Existence of the impinging jet flows

In this section, we establish the existence of the impinging jet flows via the variational approach.

Firstly, for any large $L > \max\{H, b\}$, denote the segments as follows

$$N_L = N \cap \{x_2 < L\}, \ I_L = \{(0, x_2) \mid 0 < x_2 < L\}, \ M_L^H = \{(x_1, H) \mid b < x_1 < L\},$$

$$M_L^0 = \{(x_1, 0) \mid 0 < x_1 < L\}, \quad T_L = \{(x_1, L) \mid 0 < x_1 < f(L)\},$$

and

$$E_L^h = \Omega_L \cap \{0 < x_2 < h\}, \quad \sigma_L = \{(L, x_2) \mid 0 < x_2 < H\}.$$

It's easy to see that functional $J_\lambda(\psi)$ is unbounded for any $\psi \in K$, then we truncate Ω by Ω_L and the corresponding domain E_L as follows (see Fig. 5)

Ω_L is bounded by $N_L, I_L, M_L^H, M_L^0, T_L$ and σ_L, $E_L = \{(x_1, x_2) \in \Omega_L \mid 0 < x_2 < H\}$.

Set

$$\phi_h(x_2) = \begin{cases} m_0 & \text{for } h < x_2 < H, \\ \phi(x_2) & \text{for } 0 \le x_2 \le h, \end{cases} \tag{4.1}$$
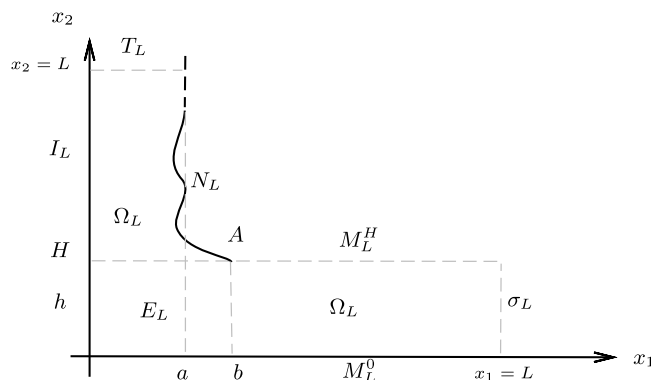
Fig. 5. Truncated domain.

where

$$\phi(x_2) = -\frac{1}{2}\omega_0 x_2^2 + (\lambda + \omega_0 h)x_2, \tag{4.2}$$

$$\Psi_L(x_1) = \min\left\{m_0, -\frac{1}{2}\omega_0 x_1^2 + v_0^L x_1\right\}, \tag{4.3}$$

and $v_0^L = \max\left\{0, \frac{m_0}{f(L)} + \frac{1}{2}\omega_0 f(L)\right\}$.

Define a truncated variational problem with the parameter λ as follows.

Truncated variational problem $(P_{\lambda,L})$. Find a $\psi_{\lambda,L} \in K_L$ such that

$$J_{\lambda,L}(\psi_{\lambda,L}) = \min_{\psi \in K_L} J_{\lambda,L}(\psi),$$

where

$$J_{\lambda,L}(\psi) = \int_{\Omega_L} |\nabla\psi|^2 + \lambda^2 \chi_{\{\psi<m_0\}\cap E_L} - 2\omega_0(\psi - m_0)dx_1 dx_2, \tag{4.4}$$

with the admissible set

$$K_L = \{\psi \in H^1(\Omega_L) \mid \psi = \psi_0 \text{ on } \partial\Omega_L\},$$

where

$$\psi_0 = \begin{cases} m_0 & \text{on } N_L \cup M_L^H, \\ 0 & \text{on } I_L \cup M_L^0, \\ \Psi_L(x_1) & \text{on } T_L, \\ \phi_h(x_2) & \text{on } \sigma_L. \end{cases} \tag{4.5}$$

Next, we will establish the existence and uniqueness of the minimizer $\psi_{\lambda,L}$ to the truncated variational problem $(P_{\lambda,L})$ for any $\lambda > 0$ and any $L > \max\{H, b\}$.

Theorem 4.1. *There exists a solution $\psi_{\lambda,L}(x_1, x_2)$ to the truncated variational problem $(P_{\lambda,L})$ and*

$$\psi(x_1, x_2) \leq \phi_h(x_2) \text{ in } \Omega_L. \tag{4.6}$$

Furthermore,

$$\psi(x_1, x_2) < \phi_h(x_1, x_2) < m_0 \text{ in } E_L^h.$$

Proof. First, the existence of the minimizer follows from the standard variational method. Denote $\psi(x_1, x_2) = \psi_{\lambda,L}(x_1, x_2)$ for simplicity and let ψ_k be a minimizing sequence, it follows from the definition of the minimizer that

$\{\psi_k\}$ is bounded in $H^1(\Omega_L)$.

Therefore, there exist two functions $\psi \in K_L$ and $\gamma \in L^\infty(\Omega_L)$ such that for a subsequence,

$\psi_k \rightarrow \psi$ weakly in $H^1(\Omega_L)$,

$\psi_k \rightarrow \psi$ a.e. in Ω_L,

and

$\chi_{\{\psi_k < m_0\} \cap E_L} \rightarrow \gamma$ weakly star in $L^\infty(\Omega_L)$ and $0 \leq \gamma \leq 1$.

Moreover, there is a function $f \in L^1(\Omega_L)$ such that for a subsequence,

$$\left(|\nabla \psi_k|^2 - 2\omega_0(\psi_k - m_0) + \lambda^2 \chi_{\{\psi_k < m_0\} \cap E_L}\right) \rightarrow f \text{ weakly in } L^1(\Omega_L),$$

and

$$\int_{\Omega_L} f dx_1 dx_2 \leq \liminf_{k \rightarrow \infty} J_{\lambda,L}(\psi_k).$$

In view of the definitions of f and γ, we have

$\gamma = 1$ a.e. in $E_L \cap \{\psi < m_0\}$ and $f = |\nabla \psi|^2 - 2\omega_0(\psi - m_0) + \lambda^2 \gamma$.

It follows from the weakly lower semicontinuity of $J_{\lambda,L}(\psi_k)$ with respect to ψ_k that

$$J_{\lambda,L}(\psi) \leq \int_{\Omega_L} |\nabla \psi|^2 - 2\omega_0(\psi - m_0) + \lambda^2 \gamma \, dx_1 dx_2 = \int_{\Omega_L} f dx_1 dx_2 \leq \liminf_{k \rightarrow \infty} J_{\lambda,L}(\psi_k),$$

which implies that ψ is a minimizer to the truncated variational problem $(P_{\lambda,L})$.

By virtue of the definition of $\phi_h(x_2)$ in (4.1), to obtain (4.6), it suffices to prove that

$\psi(x_1, x_2) \leq \phi_h(x_2)$ in E_L^h.

In fact, consider an auxiliary function

$$\phi_h^\tau(x_2) = -\frac{1}{2}\omega_0 x_2^2 + (\lambda + \omega_0 h)x_2 + \tau,$$

with a parameter $\tau \geq 0$. Taking τ be suitable large such that $\psi \leq \phi_h^\tau$ in E_L^h. Decrease τ and denote the smallest nonnegative value of τ by τ_0, such that $\psi \leq \phi_h^{\tau_0}$ holds throughout E_L^h. We claim that $\tau_0 = 0$.

Indeed, suppose $\tau_0 > 0$. In view of the definition of $\phi_h^{\tau_0}$, it is easy to check that $\psi(x_1, x_2) < \phi_h^{\tau_0}(x_2)$ on ∂E_L^h. Therefore, the equality $\psi = \phi_h^{\tau_0}$ has to hold at some point $X^0 = (x_1^0, x_2^0) \in E_L^h$. Since

$-\triangle \psi = \omega_0$ and $-\triangle \phi_h^{\tau_0} = \omega_0$ in $E_L^h \cap \{0 < \psi < m_0\}$,

the maximum principle implies that X^0 must be a free boundary point. It follows from Corollary 3.13 in [18] that the free boundary Γ is analysis in the fluid field, thanks to the Hopf lemma, we have

$$\frac{\partial(\psi - \phi_h^{\tau_0})}{\partial \nu} > 0 \text{ at } X^0,$$

where ν is outer normal vector of the free boundary. Therefore

$$\lambda = \frac{\partial \psi}{\partial \nu} < \frac{\partial \phi_h^{\tau_0}}{\partial \nu} = \lambda + \omega_0 h - \omega_0 x_2^0 \text{ at } X^0,$$

which leads to a contradiction, since $0 < x_2^0 < h$. Thus $\tau_0 = 0$ and we complete the proof of the second part of Theorem 4.1.

Next, we claim that $\psi < \phi_h < m_0$ in E_L^h. Indeed, it follows from the definition of $\phi_h(x_2)$ that

$$\begin{cases} -\triangle\psi = \omega_0 \quad \text{and} \quad -\triangle\phi_h = \omega_0 \quad \text{in} \quad E_L^h, \\ \psi \leq \phi_h \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial E_L^h. \end{cases} \tag{4.7}$$

The strong minimum principle implies that

$$\psi(x_1, x_2) < \phi_h(x_2) < m_0 \quad \text{in} \quad E_L^h.$$

Hence, we complete the proof of Theorem 4.1. □

Theorem 4.1 gives that $\psi_{\lambda,L}(x_1, x_2) \leq \phi_h(x_2)$ in Ω_L, which plays a crucial role to obtain the monotonicity of $\psi_{\lambda,L}(x_1, x_2)$ respect to x_1.

Theorem 4.2. *The truncated variational problem $(P_{\lambda,L})$ has a unique minimizer $\psi_{\lambda,L}$ for given $\lambda > 0$. Furthermore $\psi_{\lambda,L}(x_1, x_2) \geq \psi_{\lambda,L}(\tilde{x}_1, x_2)$ in Ω_L for any $x_1 > \tilde{x}_1$.*

Proof. Denote $\psi(x_1, x_2) = \psi_{\lambda,L}(x_1, x_2)$ for simplicity. Suppose that ψ_1 and ψ_2 be two minimizers to the truncated variational problem $(P_{\lambda,L})$ and set $\psi_1^\varepsilon(x_1, x_2) = \psi_1(x_1 - \varepsilon, x_2)$ for small $\varepsilon > 0$.

It is clear that

ψ_1^ε is also a minimizer to the functional $J_{\lambda,L}^\varepsilon$ in the admissible set K_L^ε,

where $\Omega_L^\varepsilon = \{(x_1, x_2) \mid (x_1 - \varepsilon, x_2) \in \Omega_L\}$ and

$$K_L^\varepsilon = \left\{ \psi^\varepsilon \in H^1(\Omega_L^\varepsilon) \mid \psi^\varepsilon = \psi_0^\varepsilon = \psi_0(x_1 - \varepsilon, x_2) \text{ on } \partial\Omega_L^\varepsilon \right\}.$$

Extend the functions $\psi_1^\varepsilon(x_1, x_2)$ and $\psi_2(x_1, x_2)$ as

$$\psi_1^\varepsilon(x_1, x_2) = 0 \text{ for } \Omega_L \cap \{x_1 < \varepsilon\}, \tag{4.8}$$

and

$$\psi_2(x_1, x_2) = \begin{cases} m_0 & \text{for } \{(x_1, x_2) \mid H < x_2 < L, f(x_2) < x_1 < f(x_2) + \varepsilon\}, \\ \phi_h(x_2) & \text{for } \{(x_1, x_2) \mid 0 < x_2 < H, L < x_1 < L + \varepsilon\}. \end{cases} \tag{4.9}$$

It is easy to check that

$$\psi_1^\varepsilon \vee \psi_2 = \max\{\psi_1^\varepsilon(x_1, x_2), \psi_2(x_1, x_2)\} \in K_L,$$

and

$$\psi_1^\varepsilon \wedge \psi_2 = \min\{\psi_1^\varepsilon(x_1, x_2), \psi_2(x_1, x_2)\} \in K_L^\varepsilon.$$

We claim that

$$J_{\lambda,L}^\varepsilon(\psi_1^\varepsilon) + J_{\lambda,L}(\psi_2) = J_{\lambda,L}(\psi_1^\varepsilon \vee \psi_2) + J_{\lambda,L}^\varepsilon(\psi_1^\varepsilon \wedge \psi_2). \tag{4.10}$$

In fact, it suffices to verify that

$$\int_{\Omega_L^\varepsilon} \left(|\nabla\psi_1^\varepsilon|^2 - |\nabla(\psi_1^\varepsilon \wedge \psi_2)|^2 \right) dx_1 dx_2 = \int_{\Omega_L} \left(|\nabla(\psi_1^\varepsilon \vee \psi_2)|^2 - |\nabla\psi_2|^2 \right) dx_1 dx_2, \tag{4.11}$$

$$\int_{\Omega_L^\varepsilon \cap E_L^\varepsilon} \left(\chi_{\{\psi_1^\varepsilon < m_0\}} - \chi_{\{(\psi_1^\varepsilon \wedge \psi_2) < m_0\}} \right) dx_1 dx_2 = \int_{\Omega_L \cap E_L} \left(\chi_{\{(\psi_1^\varepsilon \vee \psi_2) < m_0\}} - \chi_{\{\psi_2 < m_0\}} \right) dx_1 dx_2, \tag{4.12}$$

and

$$\int_{\Omega_L^\varepsilon} 2\omega_0 \left(\psi_1^\varepsilon - \psi_1^\varepsilon \wedge \psi_2 \right) dx_1 dx_2 = \int_{\Omega_L} 2\omega_0 \left(\psi_1^\varepsilon \vee \psi_2 - \psi_2 \right) dx_1 dx_2. \tag{4.13}$$

To verify the equation (4.11), we have

$$
\int_{\Omega_L^\varepsilon} \left(|\nabla \psi_1^\varepsilon|^2 - |\nabla(\psi_1^\varepsilon \wedge \psi_2)|^2 \right) dx_1 dx_2
$$

$$
= \int_{\Omega_L} |\nabla \psi_1|^2 dx_1 dx_2 - \int_{\Omega_L^\varepsilon \cap \{\psi_1^\varepsilon > \psi_2\}} |\nabla \psi_2|^2 dx_1 dx_2 - \int_{\Omega_L^\varepsilon \cap \{\psi_1^\varepsilon \leq \psi_2\}} |\nabla \psi_1^\varepsilon|^2 dx_1 dx_2
$$

$$
= \int_{\Omega_L} |\nabla \psi_1|^2 dx_1 dx_2 - \int_{\Omega_L \cap \{\psi_1^\varepsilon > \psi_2\}} |\nabla \psi_2|^2 dx_1 dx_2 - \int_{\Omega_L \cap \{\psi_1 \leq \psi_2\}} |\nabla \psi_1|^2 dx_1 dx_2 \tag{4.14}
$$

$$
= \int_{\Omega_L \cap \{\psi_1 > \psi_2\}} |\nabla \psi_1|^2 dx_1 dx_2 - \int_{\Omega_L \cap \{\psi_1^\varepsilon > \psi_2\}} |\nabla \psi_2|^2 dx_1 dx_2,
$$

and

$$
\int_{\Omega_L} \left(|\nabla \psi_1^\varepsilon \vee \psi_2|^2 - |\nabla \psi_2|^2 \right) dx_1 dx_2
$$

$$
= \int_{\Omega_L \cap \{\psi_1^\varepsilon > \psi_2\}} |\nabla \psi_1^\varepsilon|^2 dx_1 dx_2 + \int_{\Omega_L \cap \{\psi_1^\varepsilon \leq \psi_2\}} |\nabla \psi_2|^2 dx_1 dx_2 - \int_{\Omega_L} |\nabla \psi_2|^2 dx_1 dx_2 \tag{4.15}
$$

$$
= \int_{\Omega_L \cap \{\psi_1 > \psi_2\}} |\nabla \psi_1|^2 dx_1 dx_2 - \int_{\Omega_L \cap \{\psi_1^\varepsilon > \psi_2\}} |\nabla \psi_2|^2 dx_1 dx_2.
$$

Consequently, the equality (4.11) follows immediately.

Similarly, we can check the equalities (4.12) and (4.13) indeed hold.

Since ψ_1^ε and ψ_2 are the minimizers to the variational problems $(P_{\lambda,L}^\varepsilon)$ and $(P_{\lambda,L})$, respectively, the fact (4.10) gives

$$
J_{\lambda,L}^\varepsilon(\psi_1^\varepsilon) = J_{\lambda,L}^\varepsilon(\psi_1^\varepsilon \wedge \psi_2) \quad \text{and} \quad J_{\lambda,L}(\psi_2) = J_{\lambda,L}(\psi_1^\varepsilon \vee \psi_2).
$$

In view of the definition of $\psi_1^\varepsilon(x_1, x_2)$ and Theorem 4.1, we conclude that $\psi_1^\varepsilon(x_1, x_2) < \psi_2(x_1, x_2)$ near σ_L. Furthermore, we claim that

$$
\psi_1^\varepsilon < \psi_2 \text{ in the connected component } \Omega_L^0 \text{ of } \Omega_L \cap \{\psi_2 < m_0\} \text{ near } \sigma_L.
$$

Indeed, if the assertion is not true, then there exists a disc $\overline{B} \subset \Omega_L \cap \{\psi_2 < m_0\}$ and one has

$$
\begin{cases} \psi_1^\varepsilon < \psi_2 & \text{in} \quad B, \\ \psi_1^\varepsilon = \psi_2 & \text{at a point} \quad X^0 \in \partial B, \end{cases} \tag{4.16}
$$

where ν is the outer normal to ∂B at X^0.

The strong maximum principle implies that

$$
\frac{\partial}{\partial \nu}(\psi_1^\varepsilon - \psi_2) > 0 \text{ at } X^0 \in \partial B.
$$

Hence, there exists a smooth curve γ_{X^0} passing through X^0 such that

$$
\psi_1^\varepsilon < \psi_2 \text{ on the side } B \text{ of } \gamma_{X^0},
$$

and

$$
\psi_1^\varepsilon > \psi_2 \text{ on the other side } C \text{ of } \gamma_{X^0}.
$$

We have

$$\frac{\partial}{\partial \nu}(\psi_1^\varepsilon \vee \psi_2 - \psi_2)(X) \to \frac{\partial}{\partial \nu}(\psi_2 - \psi_2)(X^0) = 0, \quad X \in B \text{ and } X \to X^0,$$

and

$$\frac{\partial}{\partial \nu}(\psi_1^\varepsilon \vee \psi_2 - \psi_2)(X) \to \frac{\partial}{\partial \nu}(\psi_1^\varepsilon - \psi_2)(X^0) > 0, \quad X \in C \text{ and } X \to X^0,$$

where ν is the outer normal to ∂B at X^0.

This implies that $\psi_1^\varepsilon \vee \psi_2$ is not C^1 in a neighborhood of X^0. However, it follows from the fact $J_{\lambda,L}(\psi_2) = J_{\lambda,L}(\psi_1^\varepsilon \vee \psi_2)$ that $\psi_1^\varepsilon \vee \psi_2$ is a minimizer to the truncated variational problem $(P_{\lambda,L})$, $\psi_1^\varepsilon \vee \psi_2$ should be $C^{2,\alpha}$ in a neighborhood of $X^0 \in \Omega_L \cap \{\psi_2 < m_0\}$. This leads a contradiction.

Since the part $\partial \Omega_L \cap \{\psi_2 < m_0\}$ of $\partial \Omega_L$ is a connected arc, it follows from the maximum principle that the minimizer ψ can not attain the maximum m_0 in $\Omega_L \cap \{\psi_2 < m_0\}$, this gives that $\Omega_L \cap \{\psi_2 < m_0\}$ must touch $\partial \Omega_L \cap \{\psi_2 < m_0\}$. We conclude that Ω_L^0 coincides with $\Omega_L \cap \{\psi_2 < m_0\}$. Consequently,

$$\psi_1^\varepsilon \leq \psi_2 \quad \text{in} \quad \Omega_L.$$

Furthermore, we have

$$\psi_1 \leq \psi_2 \qquad \text{in } \Omega_L, \text{ as } \varepsilon \to 0.$$

Similarly, we can show

$$\psi_1 \geq \psi_2 \quad \text{in} \quad \Omega_L.$$

Hence, we complete the proof of uniqueness.

Specially, taking $\psi = \psi_1 = \psi_2$ in previous arguments, we have

$$\psi(x_1 - \varepsilon, x_2) = \psi^\varepsilon(x_1, x_2) \leq \psi(x_1, x_2) \quad \text{in} \quad \Omega_L,$$

which implies that $\psi_{\lambda,L}(x_1, x_2) \geq \psi_{\lambda,L}(\tilde{x}_1, x_2)$ in Ω_L for $x_1 > \tilde{x}_1$. $\quad\square$

Since $\psi_{\lambda,L}(x_1, x_2)$ is a monotonic increasing function with respect to x_1, there exists a map $x_1 = g_{\lambda,L}(x_2)$ in $0 < x_2 \leq H$, such that

$$\{0 < \psi_{\lambda,L} < m_0\} \cap \{0 < x_2 < H\} = \{(x_1, x_2) \mid 0 < x_2 < H, 0 < x_1 < g_{\lambda,L}(x_2)\}.$$

Lemma 4.3. $g_{\lambda,L}(x_2)$ has at most one limit point as $x_2 \uparrow \tilde{x}_2$ or $x_2 \downarrow \tilde{x}_2$, where $\tilde{x}_2 \in (h, H]$.

Proof. First, consider the case $\tilde{x}_2 < H$. Suppose that there are two limit points as $x_2 \uparrow \tilde{x}_2$ and denoted by $\tilde{x}_1^1$ and $\tilde{x}_1^2$ with $\tilde{x}_1^2 < \tilde{x}_1^1$.

Due to the definition of the free boundary and the monotonicity $\psi_{\lambda,L}(x_1, x_2)$ with respect to x_1, we can find two sequences $\{x_2^n\}_{n=1}^\infty$ and $\{\bar{x}_2^n\}_{n=1}^\infty$ such that $x_2^n \uparrow \tilde{x}_2$, $\bar{x}_2^n \uparrow \tilde{x}_2$ and

$$\psi_{\lambda,L}(x_1, x_2^n) = m_0 \quad \text{and} \quad \psi_{\lambda,L}(x_1, \bar{x}_2^n) < m_0 \tag{4.17}$$

for $\left| x_1 - \frac{\tilde{x}_1^1 + \tilde{x}_1^2}{2} \right| < \frac{\tilde{x}_1^1 - \tilde{x}_1^2}{4}$, $x_2^n < \bar{x}_2^n < x_2^{n+1}$. Lemma 2.3 implies that $\psi_{\lambda,L}(x_1, x_2)$ is Lipschitz continuous in neighborhood of the segment joining $\left(\frac{\tilde{x}_1^1 + 3\tilde{x}_1^2}{4}, \tilde{x}_2 \right)$ to $\left(\frac{3\tilde{x}_1^1 + \tilde{x}_1^2}{4}, \tilde{x}_2 \right)$.

Denote by $E_n \subset E_L \cap \{\psi_{\lambda,L} < m_0\}$ bounded by the arcs

$$x_1 = \frac{3\tilde{x}_1^1 + \tilde{x}_1^2}{4}, \quad x_1 = \frac{\tilde{x}_1^1 + 3\tilde{x}_1^2}{4}, \quad \bar{x}_2 = \bar{x}_2^n(x_1) \text{ and } x_2 = x_2^n(x_1),$$

where $(x_1, x_2^n(x_1))$ and $(x_1, \bar{x}_2^n(x_1))$ are free boundary points and $x_2^n(x_1) < \bar{x}_2^n(x_1)$ with

$$h_n = \sup_{x_1}\{\bar{x}_2^n(x_1) - x_2^n(x_1)\} \to 0 \text{ as } n \to \infty.$$

The existence of domains E_n follows from (4.17).

Thanks to the nonoscillation Lemma 3.5, we have

$$\frac{(\tilde{x}_1^1 - \tilde{x}_1^2)^2}{1 + (\tilde{x}_1^1 - \tilde{x}_1^2)^2} \le C h_n^2,$$

which contradicts to the assumption $\tilde{x}_1^2 < \tilde{x}_1^1$ for sufficiently large n.

So we obtain the fact

$$g_{\lambda,L}(\tilde{x}_2 - 0) = \lim_{x_2^n \to \tilde{x}_2^-} g_{\lambda,L}(x_2).$$

Next, by similar arguments, we have

$$g_{\lambda,L}(\tilde{x}_2 + 0) = \lim_{x_2^n \to \tilde{x}_2^+} g_{\lambda,L}(x_2),$$

for any $\tilde{x}_2 \in (0, H)$ and

$$g_{\lambda,L}(H - 0) = \lim_{x_2^n \to H^-} g_{\lambda,L}(x_2). \quad \square$$

Lemma 4.4. $g_{\lambda,L}(x_2)$ *is a continuous function in* $(h, H]$ *with values in* $(0, L]$.

Proof. Lemma 4.3 implies that

$$\lim_{x_2 \to \tilde{x}_2^+} g_{\lambda,L}(x_2) \text{ and } \lim_{x_2 \to \tilde{x}_2^-} g_{\lambda,L}(x_2) \text{ exist, for any } \tilde{x}_2 \in (0, H).$$

Denote

$$g_{\lambda,L}(\tilde{x}_2 + 0) = \lim_{x_2 \to \tilde{x}_2^+} g_{\lambda,L}(x_2) \text{ and } g_{\lambda,L}(\tilde{x}_2 - 0) = \lim_{x_2 \to \tilde{x}_2^-} g_{\lambda,L}(x_2).$$

It suffices to prove that

$$g_{\lambda,L}(\tilde{x}_2 + 0) = g_{\lambda,L}(\tilde{x}_2 - 0) = g_{\lambda,L}(\tilde{x}_2) \text{ for any } \tilde{x}_2 \in (0, H).$$

Suppose on the contrary that there exists a point $\tilde{x}_2 \in (0, H)$ such that $g_{\lambda,L}(\tilde{x}_2 - 0) \ne g_{\lambda,L}(\tilde{x}_2)$, and without loss of generality we assume $g_{\lambda,L}(\tilde{x}_2 - 0) < g_{\lambda,L}(\tilde{x}_2)$. Then, there exist two positive constants $\varepsilon > 0$ and $\delta > 0$ such that there exists a strip as

$$F_{\varepsilon,\delta} = \{\tilde{x}_2 < x_2 < \tilde{x}_2 + \delta, g_{\lambda,L}(\tilde{x}_2 - 0) + \varepsilon < x_1 < g_{\lambda,L}(\tilde{x}_2) - \varepsilon\},$$

and

$$\begin{cases} -\triangle \psi_{\lambda,L} = \omega_0 & \text{in } F_{\varepsilon,\delta}, \\ \psi_{\lambda,L} = m_0, \ \frac{\partial \psi_{\lambda,L}}{\partial x_2} = -\lambda & \text{on } \{(x_1, \tilde{x}_2) \mid g_{\lambda,L}(\tilde{x}_2 - 0) + \varepsilon < x_1 < g_{\lambda,L}(\tilde{x}_2) - \varepsilon\}. \end{cases} \tag{4.18}$$

There is a unique solution to the problem (4.18), as

$$\psi_{\lambda,L}(x_1, x_2) = -\frac{1}{2}\omega_0(x_2^2 - \tilde{x}_2^2) - (\lambda - \omega_0\tilde{x}_2)(x_2 - \tilde{x}_2) + m_0 \text{ in } F_{\varepsilon,\delta}. \tag{4.19}$$

Thus, we claim that it follows from the Cauchy–Kowalewski theorem and unique continuation that

$$\psi_{\lambda,L}(x_2) = -\frac{1}{2}\omega_0(x_2^2 - \tilde{x}_2^2) - (\lambda - \omega_0\tilde{x}_2)(x_2 - \tilde{x}_2) + m_0 \text{ in } E_{\tilde{x}_2}, \tag{4.20}$$

where $E_{\tilde{x}_2} = \{0 < x_1 < L, \tilde{x}_2 < x_2 < \tilde{x}_2 + \delta\}$.

In fact, denote

$$w(x_2) = -\frac{1}{2}\omega_0(x_2^2 - \tilde{x}_2^2) - (\lambda - \omega_0\tilde{x}_2)(x_2 - \tilde{x}_2) + m_0 \ \text{ in } \ E_{\tilde{x}_2},$$

and suppose not, we consider two cases as follows.

Case 1. There exists a point $X^0 = (x_1^0, x_2^0) \in E_{\tilde{x}_2}$ with $x_1^0 \le g_{\lambda, L}(\tilde{x}_2 - 0) + \varepsilon$, such that

$$\psi_{\lambda, L}(X^0) < w(x_2^0).$$

Denote the domain G bounded by $x_2 = \tilde{x}_2$, $x_2 = \tilde{x}_2 + \delta$, $x_1 = \frac{g_{\lambda, L}(\tilde{x}_2 - 0) + g_{\lambda, L}(\tilde{x}_2)}{2}$ and $x_1 = \min\{x_1^0, g_{\lambda, L}(x_2)\}$. In view of the monotonicity of $\psi_{\lambda, L}$ with respect to x_1, we have

$$\begin{cases} \Delta\psi_{\lambda, L} = \Delta w = -\omega_0 & \text{in } G, \\ \psi_{\lambda, L} \le w & \text{on } \partial G. \end{cases}$$

Using the strong maximum principle gives that

$$\psi_{\lambda, L} < w \ \text{ in } G,$$

which is a contradiction to

$$\psi_{\lambda, L} = w \ \text{ in } \ F_{\varepsilon, \delta} \cap \left\{ x_1 \le \frac{g_{\lambda, L}(\tilde{x}_2 - 0) + g_{\lambda, L}(\tilde{x}_2)}{2} \right\}.$$

Case 2. There exists a point $X^0 = (x_1^0, x_2^0) \in E_{\tilde{x}_2}$ with $x_1^0 \ge g_{\lambda, L}(\tilde{x}_2) - \varepsilon$, such that

$$\psi_{\lambda, L}(X^0) > w(x_2^0).$$

Denote the domain G bounded by $x_2 = \tilde{x}_2$, $x_2 = \tilde{x}_2 + \delta$, $x_1 = \frac{g_{\lambda, L}(\tilde{x}_2 - 0) + g_{\lambda, L}(\tilde{x}_2)}{2}$ and $x_1 = \max\{x_1^0, g_{\lambda, L}(x_2)\}$. Similarly, we have

$$\begin{cases} \Delta\psi_{\lambda, L} = \Delta w = -\omega_0 & \text{in } G, \\ \psi_{\lambda, L} \ge w & \text{on } \partial G, \end{cases}$$

and

$$\psi_{\lambda, L} > w \ \text{ in } G,$$

which leads to a contradiction with

$$\psi_{\lambda, L} = w \ \text{ in } \ F_{\varepsilon, \delta} \cap \left\{ x_1 \ge \frac{g_{\lambda, L}(\tilde{x}_2 - 0) + g_{\lambda, L}(\tilde{x}_2)}{2} \right\}.$$

Therefore, we complete the proof of the claim (4.20). However, this contradicts with the fact $\psi_{\lambda, L}(0, \tilde{x}_2) = 0$. Hence, $g_{\lambda, L}(x_2)$ is a continuous function in $(h, H]$. $\quad\square$

Set

$$\lambda(h_0) = \frac{m_0}{h_0} - \frac{\omega_0}{2}h_0 \ \text{ for } \ h_0 \in (0, H],$$

then the condition (1.10) ensures that $\lambda'(h_0) = -\frac{m_0}{h_0^2} - \frac{\omega_0}{2} < 0$ for $h_0 \in (0, H)$. This implies that $\lambda \ge \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ and h_0 can be determined uniquely by λ for given m_0 and ω_0. Next, we discuss the location of initial point of the free boundary $g_{\lambda, L}(H)$ when the parameter $\lambda - \frac{m_0}{H} + \frac{1}{2}\omega_0 H$ is sufficiently small or large.

Lemma 4.5. $g_{\lambda, L}(H) \to L$, as $\lambda \downarrow \frac{m_0}{H} - \frac{1}{2}\omega_0 H$.

Proof. Suppose not, then there exists a subsequence $\lambda_k > \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ such that

$$\lim_{\lambda_k \to \frac{m_0}{H} - \frac{1}{2}\omega_0 H} g_{\lambda_k, L}(H) = \tilde{a} < L.$$

In view of the condition $\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ and $\lambda'(h) < 0$ for $h \in (0, H)$, one has

$$\phi(x_2) = -\frac{1}{2}\omega_0 x_2^2 + (\lambda + \omega_0 H)x_2 < -\frac{1}{2}\omega_0 H^2 + \left(\frac{m_0}{H} + \frac{1}{2}\omega_0 H\right)H = m_0, \quad \text{in } E_L. \tag{4.21}$$

Theorem 4.1 implies that

$$\psi_{\lambda, L}(x_1, x_2) \leq \min\{m_0, \phi(x_2)\} < m_0 \quad \text{for } x_2 \in (0, H). \tag{4.22}$$

Thanks to Lemma 3.4, we have

$$\psi_{\lambda, L} = m_0 \quad \text{and} \quad \frac{\partial \psi_{\lambda, L}}{\partial x_2} = \lambda \quad \text{on } \{(x_1, H), \tilde{a} < x_1 < L\}.$$

It follows from the Cauchy–Kowalewski theorem and unique continuation that

$$\psi_{\lambda, L}(x_2) = -\frac{1}{2}\omega_0(x_2^2 - H^2) + (\lambda + \omega_0 H)(x_2 - H) + m_0, \tag{4.23}$$

which is a contradiction to $\psi_{\lambda, L}(0, H) = 0$. $\square$

Lemma 4.6. $g_{\lambda, L}(H) < b$, for sufficiently large $\lambda > \frac{m_0}{H} - \frac{1}{2}\omega_0 H$.

Proof. Suppose on the contrary, $g_{\lambda, L}(H) \geq b$.

Case 1. There exist the free boundary points in Ω_L.

Since the free boundary Γ connects the point $(g_{\lambda, L}(H), H)$ to the point (L, h), there exists a free boundary point $X^0 = (x_1^0, x_2^0)$ and $r > 0$ independent of λ such that

$$\text{either } B_r(X^0) \subset \Omega_L \cap \{x_2 > h/2\} \text{ or } B_r(X^0) \subset \{x_1 > a\} \cap \{x_2 > h/2\}. \tag{4.24}$$

It follows from nondegeneracy Lemma 3.1 near the free boundary point for (4.24) that

$$\frac{m_0}{r} \geq \frac{1}{r} \fint_{\partial B_r(X^0)} (m_0 - \psi_{\lambda, L}) dS \geq \lambda C^*,$$

we obtain a contradiction for sufficiently large λ.

Case 2. There is no free boundary point in Ω_L.

Let $X^0 = (x_1^0, H)$, $L > x_1^0 > b$. Similarly, we can derive a contradiction by using the nondegeneracy lemma about the boundary point X^0 when λ is sufficiently large. $\square$

Lemma 4.7. If $g_{\lambda, L}(H) > b$ then $\frac{\partial \psi_{\lambda, L}}{\partial \nu} \geq \lambda$ on the segment $\Lambda = \{(x_1, H) \mid b < x_1 < g_{\lambda, L}(H)\}$.

Proof. Along the similar arguments in Theorem 2.2, choose $\eta(x) \in \left(C^1(E_L)\right)^2$ such that

$$\eta = 0 \text{ on } \partial E_L \setminus \Lambda \text{ and } E_L \cap \partial\{\psi_{\lambda, L} < m_0\}, \quad \eta \cdot \nu \leq 0 \text{ on } \Lambda,$$

and set $\tau_\delta = x + \delta\eta(x)$, $\psi_{\lambda, L}^\delta(\tau_\delta(x)) = \psi_{\lambda, L}(x)$ for some small parameter $\delta > 0$.

Since $\psi_{\lambda, L}$ is the minimizer to the truncated variational problem $(P_{\lambda, L})$, we have

$$0 \leq J(\psi_{\lambda, L}^\delta) - J(\psi_{\lambda, L})$$

$$= \delta \int_{\{\psi_{\lambda, L} < m_0\} \cap E_L} \left(|\nabla \psi_{\lambda, L}|^2 \nabla \cdot \eta - 2\nabla \psi_{\lambda, L} \cdot D\eta \cdot \nabla \psi_{\lambda, L}\right) dx_1 dx_2$$

$$+ \delta \int_{\{\psi_{\lambda, L} < m_0\} \cap E_L} (\lambda^2 - 2\omega_0(\psi_{\lambda, L} - m_0))\nabla \cdot \eta \, dx_1 dx_2 + o(\delta). \tag{4.25}$$

Using the similar arguments in (2.14) and the equality (4.25) give that

$$
\begin{aligned}
0 \le \int_{\partial(\{\psi_{\lambda,L} < m_0\} \cap E_L)} & \left(\lambda^2 - |\nabla \psi_{\lambda,L}|^2 \right) \eta \cdot \nu \, dS \\
& = \int_\Lambda \left(\lambda^2 - |\nabla \psi_{\lambda,L}|^2 \right) \eta \cdot \nu \, dS,
\end{aligned}
\tag{4.26}
$$

which implies that $\frac{\partial \psi}{\partial \nu} \ge \lambda$ on Λ, due to $\eta \cdot \nu \le 0$ on Λ.   □

To check the continuous fit condition to the free boundary, namely, there exists a λ, such that $g_{\lambda,L}(H) = b$, we will show the continuous dependence of $\psi_{\lambda,L}$ with respect to λ. First, we give the following theorem on the convergence of the solution and the free boundary.

Theorem 4.8.

$$
\psi_{\lambda_n,L} \to \psi_{\lambda,L} \ \text{ weakly in } \ H^1(\Omega_L) \ \text{and a.e. in} \ \Omega_L,
$$

and

$$
g_{\lambda_n,L}(x_2) \to g_{\lambda,L}(x_2) \ \text{ for any } \ x_2 \in (h, H],
$$

as $\lambda_n \to \lambda$.

Proof. Since $0 \le \psi_{\lambda,L} \le m_0$, there exists a subsequence of $\psi_{\lambda_n,L}$ such that

$$
\psi_{\lambda_n,L} \to \omega \ \text{ weakly in } \ H^1(\Omega_L) \ \text{and a.e. in} \ \Omega_L.
$$

We claim that ω is in fact a minimizer to the truncated variational problem $(P_{\lambda,L})$. Denote $\psi^n(x_1, x_2) = \psi_{\lambda_n,L}(x_1, x_2)$.

Step 1. $\partial\{\psi^n < m_0\} \cap \Omega_L \to \partial\{\omega < m_0\} \cap \Omega_L$ in the Hausdorff distance.

First, we recall the definition of Hausdorff distance $d(A, D)$ between two sets A and D as

$$
d(A, D) = \inf \left\{ \varepsilon > 0 \mid D \subset \bigcup_{X \in A} B_\varepsilon(X) \ \text{ and } \ A \subset \bigcup_{X \in D} B_\varepsilon(X) \right\}.
$$

For any $X \in \Omega_L$ and the ball $B_r(X)$ with small $r > 0$, if $V_r = B_r(X) \cap \partial\{\omega < m_0\} = \varnothing$ and $\omega = m_0$ in $B_r(X)$, then the minimizer ψ^n to the truncated variational problem $(P_{\lambda_n,L})$ satisfies that $m_0 - \psi^n$ is small in $B_r(X)$ for sufficiently large n. It follows from Lemma 3.4 for ψ^n that $\psi^n = m_0$ in $B_{\frac{r}{2}}(X)$.

If $\omega < m_0$ in $B_r(X)$, it is clear that $\psi^n < m_0$ in $B_{\frac{r}{2}}(X)$. Then the both cases imply that $B_{\frac{r}{2}}(X) \cap \partial\{\psi^n < m_0\} = \varnothing$ for sufficiently large n.

On the other hand, if $V_r^n = B_r(X) \cap \partial\{\psi^n < m_0\} = \varnothing$ for sufficiently large n, we can obtain that $B_{\frac{r}{2}}(X) \cap \partial\{\omega < m_0\} = \varnothing$ by similar arguments. Hence, we have the convergence of the free boundary in the Hausdorff distance.

Step 2. $\chi_{\{\psi^n < m_0\} \cap E_L} \to \chi_{\{\omega < m_0\} \cap E_L}$ in $L^1(\Omega_L)$.

In view of Lemma 3.1 and Lemma 3.2, taking the limit $n \to \infty$, we can deduce that Lemma 3.1 and Lemma 3.2 still hold for ω. Hence, we can obtain that the result in Lemma 3.3 is still valid to ω by using Lemma 3.1 and Lemma 3.2, which implies that

$$
\partial\{\omega < m_0\} \text{ has Lebesgue measure zero.}
\tag{4.27}
$$

Let U_r be an r-neighborhood of $\partial\{\omega < m_0\}$ and denote $\mathcal{M}_{r,L} = |\Omega_L \cap U_r|$. It follows from Lemma 3.3 that

$$
\mathcal{M}_{r,L} \downarrow 0 \quad \text{as } r \downarrow 0.
\tag{4.28}
$$

Hence, for sufficiently large n, we have

$$
\int_{\Omega_L} \left| \chi_{\{\psi^n < m_0\} \cap E_L} - \chi_{\{\omega < m_0\} \cap E_L} \right| dx_1 dx_2 \le \int_{\Omega_L \cap U_r} dx_1 dx_2 = \mathcal{M}_{r,L}.
\tag{4.29}
$$

This together with (4.28) gives that $\chi_{\{\psi^n < m_0\} \cap E_L} \to \chi_{\{\omega < m_0\} \cap E_L}$ in $L^1(\Omega_L)$.

Step 3. $\nabla \psi^n \to \nabla \omega$ a.e. in Ω_L.

Let E be any compact subset of $\Omega_L \cap (\{\omega < m_0\} \cup \text{int}\{\omega = m_0\})$, since $-\Delta \psi^n = \omega_0$ in E, and $\nabla \psi^n \to \nabla \omega$ uniformly in any compact subset E of $\Omega_L \cap (\{\omega < m_0\} \cup \text{int}\{\omega = m_0\})$, the result in (4.27) implies that $\nabla \psi^n \to \nabla \omega$ a.e. in Ω_L,

Step 4. ω is a minimizer to the truncated variational problem $(P_{\lambda, L})$.

It suffices to verify that

$$J_{\lambda, L}(\omega) \le J_{\lambda, L}(v), \tag{4.30}$$

for any function $v - \omega \in H_0^1(\Omega_L)$.

Obviously, $v \in K_L$ and set

$$v^n = v + (1 - \eta)(\psi^n - \omega),$$

where $\eta \in C_0^1(\Omega_L)$ and $0 \le \eta \le 1$. It's easy to see that $v^n \in K_L$.

Since ψ^n is a minimizer to the truncated variational problem $(P_{\lambda_n, L})$, one has

$$
\begin{aligned}
&\int_{\Omega_L} |\nabla \psi^n|^2 + \lambda_n^2 \chi_{\{\psi^n < m_0\} \cap E_L} - 2\omega_0(\psi^n - m_0) dx_1 dx_2 \\
&\le \int_{\Omega_L} |\nabla v^n|^2 + \lambda_n^2 \chi_{\{v^n < m_0\} \cap E_L} - 2\omega_0(v^n - m_0) dx_1 dx_2.
\end{aligned}
\tag{4.31}
$$

Taking $n \to \infty$ and using the convergence results in Step 2 and Step 3 give that

$$
\begin{aligned}
&\int_{\Omega_L} |\nabla \omega|^2 + \lambda^2 \chi_{\{\omega < m_0\} \cap E_L} - 2\omega_0(\omega - m_0) dx_1 dx_2 \\
&\le \int_{\Omega_L \cap \{\eta = 1\}} |\nabla v|^2 + \lambda^2 \chi_{\{v < m_0\} \cap E_L} - 2\omega_0(v - m_0) dx_1 dx_2 \\
&\quad + \int_{\Omega_L \cap \{0 \le \eta < 1\}} |\nabla v|^2 + \lambda^2 \chi_{\{v < m_0\} \cap E_L} - 2\omega_0(v - m_0) dx_1 dx_2.
\end{aligned}
\tag{4.32}
$$

Set a sequence $\{\Omega_L^j\}_{j=1}^\infty$ such that

$$\Omega_L^j \to \Omega_L, \quad \text{and} \ \Omega_L^j \subset \Omega_L^{j+1} \ \text{for any} \ j \ge 1,$$

and a cut-off function $\eta_j \in C_0^1(\Omega_L^j)$ such that $\eta_j = 1$ in Ω_L^j, and $0 \le \eta_j \le 1$. Replacing η by η_j in (4.32), we have

$$
\begin{aligned}
&\int_{\Omega_L} |\nabla \omega|^2 + \lambda^2 \chi_{\{\omega < m_0\} \cap E_L} - 2\omega_0(\omega - m_0) dx_1 dx_2 \\
&\le \int_{\Omega_L} |\nabla v|^2 + \lambda^2 \chi_{\{v < m_0\} \cap E_L} - 2\omega_0(v - m_0) dx_1 dx_2.
\end{aligned}
\tag{4.33}
$$

Due to the arbitrariness of $v \in K_L$, this implies that ω is a minimizer to the truncated variational problem $(P_{\lambda, L})$. Hence, thanks to the uniqueness of the minimizer to the truncated variational problem $(P_{\lambda, L})$ in Theorem 4.1, we have $\psi_{\lambda, L} = \omega$, then the first convergence result in Theorem 4.8 is obtained.

Next, we will show the second part of this theorem.

Consider $x_2^0 \in (h, H)$ firstly, there exists a subsequence still labeled by $g_{\lambda_n, L}(x_2^0)$ such that

$$g_{\lambda_n, L}(x_2^0) \to g_{\lambda, L}(x_2^0) \quad \text{as} \ n \to \infty,$$

where the point $X_n^0 = (g_{\lambda_n, L}(x_2^0), x_2^0)$ is a free boundary point of the minimizer $\psi_{\lambda_n, L}$.

Then, it suffices to show that the limit point $X^0 = (g_{\lambda,L}(x_2^0), x_2^0)$ is a free boundary point of $\psi_{\lambda,L}$.
With the aid of Lemma 3.1 and Lemma 3.2, we have

$$c \leq \frac{1}{r} \fint_{\partial B_r(X_n^0)} (m_0 - \psi_{\lambda_n,L}) dS \leq C \quad \text{for some} \ \ C > c > 0,$$

where r is sufficiently small such that $0 < r < r_0$, r_0 is independent of n.

Taking $n \to \infty$, we obtain

$$c \leq \frac{1}{r} \fint_{\partial B_r(X^0)} (m_0 - \psi_{\lambda,L}) dS \leq C,$$

which implies that $X^0 = (g_{\lambda,L}(x_2^0), x_2^0)$ is a free boundary point of $\psi_{\lambda,L}$. Hence,

$$g_{\lambda_n,L}(x_2^0) \to g_{\lambda,L}(x_2^0) \quad \text{for any} \ \ x_2^0 \in (h, H).$$

Finally, we will show that $g_{\lambda_n,L}(H) \to g_{\lambda,L}(H)$ as $n \to +\infty$.

Suppose not, there exists a subsequence still labeled by λ_n such that $g_{\lambda_n,L}(H) \to g_{\lambda,L}(H) + \beta$, $\beta \neq 0$.

Case 1. $\beta < 0$. Due to the convergence of the free boundary in Lemma 3.4, we have

$$\frac{\partial \psi_{\lambda,L}(x_1, H)}{\partial x_2} = \lambda \quad \text{if} \ \ g_{\lambda,L}(H) + \beta < x_1 < g_{\lambda,L}(H).$$

It follows from the uniqueness of the Cauchy problem that

$$\psi_{\lambda,L} = -\frac{1}{2}\omega_0(x_2^2 - H^2) + (\lambda + \omega_0 H)(x_2 - H) + m_0 \quad \text{in} \ \ V,$$

where V is a Ω_L-neighborhood of the streamline $\{(x_1, H) \mid g_{\lambda,L}(H) + \beta < x_1 < g_{\lambda,L}(H)\}$. In view of the unique continuation, we obtain $\psi_{\lambda,L} = m_0$ on the segment $\{(x_1, H) \mid 0 < x_1 < L\}$, which is impossible.

Case 2. $\beta > 0$ and $g_{\lambda,L}(H) < b$.

It follows from the Theorem 2.2 that

$$\frac{\partial \psi_{\lambda,L}(x_1, H+0)}{\partial x_2} = \lambda \quad \text{if} \ \ g_{\lambda,L}(H) < x_1 < b,$$

which also leads to a contradiction.

Case 3. $\beta > 0$ and $g_{\lambda,L}(H) > b$.

It follows from Lemma 4.7 that we have

$$\frac{\partial \psi_{\lambda_n,L}(x_1, H-0)}{\partial x_2} \geq \lambda_n \quad \text{on} \ \ \left\{x_2 = H, b < x_1 < g_{\lambda,L}(H) + \frac{7\beta}{8}\right\}.$$

Denote G_n as a connected domain, bounded by $x_1 = g_{\lambda,L}(H) + \frac{\beta}{4}$, $x_1 = g_{\lambda,L}(H) + \frac{3\beta}{4}$, $x_2 = H$ and $x_1 = g_{\lambda_n,L}(x_2)$.
Set $x_2^n = \sup\left\{x_2 \mid g_{\lambda_n,L}(x_2) = g_{\lambda,L}(H) + \frac{\beta}{4}\right\}$, then one has

$$H - x_2^n \to 0 \quad \text{as} \ \ n \to +\infty.$$

It follows from the nonoscillation Lemma 3.5 and Remark 3.2 for $\psi_{\lambda_n,L}$ that

$$\frac{\beta^2}{4 + \beta^2} \leq C(H - x_2^n)^2,$$

which leads a contradiction for sufficiently large n. $\square$

Lemma 4.5 and Lemma 4.6 yield directly the continuous fit condition, namely, there exists a $\lambda_L \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$ such that

$$g_{\lambda_L,L}(H) = b.$$

Denote $h_L = \sup\{\bar{x}_2 \mid \psi_{\lambda_L,L} < m_0 \text{ in } E_L \cap \{x_2 < \bar{x}_2\}\}$ and the corresponding solution $\psi_{\lambda_L,L}$ satisfies

$$\psi_{\lambda_L,L}(x_1, x_2) < m_0 \text{ in } E_L, \text{ if and only if } 0 < x_1 < g_{\lambda_L,L}(x_2). \tag{4.34}$$

4.1. Proof of Theorem 1.1

In view of (4.34), there exist a subsequence $L_k \to \infty$, a constant λ and a funciton ψ_λ, such that

$$\lambda_{L_k} \to \lambda, \quad \psi_{\lambda_{L_k}, L_k} \to \psi_\lambda, \quad h_{L_k} \to \bar{h} \quad \text{uniformly in any compact subsets of } \bar{\Omega},$$

and

$$v_0^{L_k} \to v_0 \quad \text{uniformly,}$$

where $v_0 = \frac{m_0}{a} + \frac{1}{2}\omega_0 a$.

Moreover, $\psi_\lambda(x_1, x_2)$ is a minimizer to the variational problem (P_λ) and $\psi_\lambda(x_1, x_2)$ is a monotonic increasing with respect to x_1. Thus, there exists a function $x_1 = g_\lambda(x_2)$ such that

$$\psi_\lambda(x_1, x_2) < m_0 \quad \text{in } E \text{ if and only if } x_1 \in (0, g_\lambda(x_2)), \tag{4.35}$$

and

$$\psi_\lambda(x_1, x_2) < m_0 \quad \text{for } x_2 \in (0, \bar{h}). \tag{4.36}$$

Along the similar arguments in the proof of Theorem 4.8, one has

$$g_\lambda(x_2) = \lim_{L_k \to \infty} g_{\lambda_{L_k}, L_k}(x_2) \quad \text{for } x_2 \in (\bar{h}, H] \text{ and } g_\lambda(H) = \lim_{L_k \to \infty} g_{\lambda_{L_k}, L_k}(H).$$

Lemma 4.9. *The function $g_\lambda(x_2)$ is finite valued for $\bar{h} < x_2 \leq H$ and $\bar{h}$ satisfies (1.9), that is to say $\bar{h} = h$ is the asymptotic height.*

Proof. Suppose that $\bigcup_k (\beta_k, \alpha_k)$ is the union of maximal intervals such that $g_\lambda(x_2)$ is finite valued in the union set and $\beta_k \geq \alpha_{k+1}$ for any k. We claim that the number of the intervals is finite.

Indeed, if not, there exists a subsequence $(\beta_{k_i}, \alpha_{k_i})$ such that

$$\beta_{k_i} \geq \alpha_{k_{i+1}}, \quad \alpha_{k_i} - \beta_{k_i} \to 0 \quad \text{as } i \to \infty.$$

It follows from the definition of the interval $(\beta_{k_i}, \alpha_{k_i})$ that

$$g_\lambda(\alpha_{k_i} - 0) = \lim_{x_2 \to \alpha_{k_i}^-} g_\lambda(x_2) = +\infty \quad \text{and} \quad g_\lambda(\beta_{k_i} + 0) = \lim_{x_2 \to \beta_{k_i}^+} g_\lambda(x_2) = +\infty. \tag{4.37}$$

Set

$$G_i \subset \left\{ (x_1, x_2) \mid x_1 < g_\lambda(x_2), x_2 \in \left(\frac{\alpha_{k_i} + \beta_{k_i}}{2}, \frac{\alpha_{k_{i+1}} + \beta_{k_{i+1}}}{2} \right) \right\},$$

and G_i satisfies

$$\forall \, X = (x_1, x_2) \in G_i, \quad \exists \, \bar{x}_2 \text{ such that } x_1 = g_\lambda(\bar{x}_2) \text{ and } \bar{X} = (x_1, \bar{x}_2) \in \Gamma.$$

However, we can derive a contradiction in G_i by using the nonoscillation Lemma 3.5 for sufficiently large i. Then, we conclude that the number of the intervals is finite.

Let

$$\alpha_1 = H, \quad \beta_k > \alpha_{k+1}, \quad k = 1, 2, ..., m.$$

It follows from the Corollary 3.13 in [18] that the free boundary satisfies the flatness condition for $x_2 \in (\bar{h}, \bar{h} + \varepsilon)$ for sufficiently small ε. Then, there exists a sufficiently large $x_1^0 > 0$ such that the free boundary can be written in the form

$$x_2 = k(x_1) \quad \text{in } \Omega \cap \{x_1 > x_1^0\}, \tag{4.38}$$

where the function $k(x_1)$ is a monotonic function.

In view of the definition of $\bar{h}$ and the flatness property of the free boundary, one has

$$k(x_1) \to \bar{h} \ \text{ and } \ k'(x_1) \to 0 \ \text{ as } \ x_1 \to +\infty.$$

Thanks to the standard elliptic estimates, we have

$$D^j[\psi_\lambda(x_1, x_2) - \phi_h(x_1, x_2)] \to 0 \ \text{ in } \ \Omega \cap \{0 < x_2 < k(x_1)\} \ \text{ as } \ x_1 \to +\infty, \ j = 0, 1, 2,$$

which implies that $\bar{h}$ satisfies (1.9), that is $\bar{h} = h$.

Next, we will show the number of the interval (β_k, α_k) is one and $\beta_1 = h$. Suppose not, the number of the interval (β_j, α_j) is more than 1, namely, $m > 1$.

Thus, there exists a domain $\Omega_1 \subset \Omega$,

$$\Omega_1 = \{k(x_1) < x_2 < \tilde{k}(x_1), x_1 \geq x_1^0\} \ \text{ for large } x_1^0 > 0,$$

where $x_2 = k(x_1)$ and $x_2 = \tilde{k}(x_1)$ are the free boundary arcs, $x_2 = k(x_1)$ is finite valued in (β_i, α_i) and $x_2 = \tilde{k}(x_1)$ is finite valued in (β_j, α_j) $(i \neq j)$.

It follows from the blow-up argument that

$$\psi_\lambda(x_1, x_2) \to \varphi(x_2) \ \text{ as } \ x_1 \to +\infty, (x_1, x_2) \in \Omega_1,$$

and $\varphi(x_2)$ satisfies that

$$\begin{cases} -\triangle\varphi = \omega_0 \ \text{ if } \alpha < x_2 < \beta, \\ \varphi(\alpha) = \varphi(\beta) = m_0, \\ -\varphi'(\alpha) = \varphi'(\beta) = \lambda, \end{cases} \tag{4.39}$$

where $\beta = \lim_{x_1 \to \infty} k(x_1)$ and $\alpha = \lim_{x_1 \to \infty} \tilde{k}(x_1)$. It is easy to show that the problem (4.39) has a unique solution as

$$\varphi(x_2) = -\frac{1}{2}\omega_0 x_2^2 + (\lambda + \omega_0\beta)x_2 + m_0 - \frac{1}{2}\omega_0\alpha\beta,$$

where $\lambda = -\frac{\omega_0}{2}(\beta - \alpha)$.

The assumption $m > 1$ implies that $\beta - \alpha < H - h$. Furthermore, we have

$$\frac{m_0}{h} - \frac{\omega_0}{2}h = \lambda = -\frac{1}{2}\omega_0(\beta - \alpha) < -\frac{1}{2}\omega_0(H - h), \tag{4.40}$$

that is

$$m_0 < \omega_0 h^2 - \frac{1}{2}\omega_0 H h = \omega_0\left(h - \frac{H}{4}\right)^2 - \frac{1}{16}\omega_0 H^2 \leq -\frac{1}{16}\omega_0 H^2. \tag{4.41}$$

which contradicts with the condition $m_0 > -\frac{1}{2}\omega_0 H^2$ in Theorem 1.1. $\square$

Next, we will show the uniqueness of the parameter λ.

Proposition 4.10. *There exists a unique $\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H$, such that the free streamline Γ satisfies the continuous fit and smooth fit condition.*

Proof. Suppose that $(\psi_\lambda, \Gamma, \lambda)$ and $(\tilde{\psi}_{\tilde{\lambda}}, \tilde{\Gamma}, \tilde{\lambda})$ be two solutions to the impinging jet flow problem with $\Gamma : x_1 = g(x_2)$, $\tilde{\Gamma} : x_1 = \tilde{g}(x_2)$ and $g(H) = \tilde{g}(H) = b$. We claim that $\lambda = \tilde{\lambda}$.

Indeed, without loss of generality suppose that $\lambda > \tilde{\lambda}$.

In view of the relation (1.9) and the monotonicity between λ and h, we have

$$h < \tilde{h}, \tag{4.42}$$

where h and $\tilde{h}$ are asymptotic heights of the two impinging jets, respectively.

Set $\tilde{\psi}_{\tilde{\lambda}}^{\varepsilon}(x_1, x_2) = \tilde{\psi}_{\tilde{\lambda}}(x_1 - \varepsilon, x_2)$ and choose $\varepsilon \geq 0$ to be the smallest number denoted by ε_0, such that $\psi_\lambda(x_1, x_2) \geq \tilde{\psi}_{\tilde{\lambda}}^{\varepsilon_0}(x_1, x_2)$ in Ω and $\psi_\lambda(X^0) = \tilde{\psi}_{\tilde{\lambda}}^{\varepsilon_0}(X^0)$ for some $X_0 \in \bar{\Omega}$. Next, consider the following two cases for ε_0.

Case 1. If $\varepsilon_0 > 0$ then the maximum principle gives that $X^0 \notin \Omega \cap \{\psi_\lambda < m_0\}$ and $|X_0| \leq C$, then $X^0 \in \Gamma \cap \tilde{\Gamma}^{\varepsilon_0}$. Due to the choice of ε_0, we have

$$\begin{cases} -\Delta\psi_\lambda = -\Delta\tilde{\psi}_{\tilde{\lambda}}^{\varepsilon_0} = \omega_0 & \text{in } \Omega \cap \{\psi_\lambda < m_0\}, \\ \psi_\lambda = \tilde{\psi}_{\tilde{\lambda}}^{\varepsilon_0} = m_0 & \text{at } X^0 \in \Gamma \cap \tilde{\Gamma}^{\varepsilon_0}, \end{cases} \tag{4.43}$$

the maximum principle implies that

$$\frac{\partial\psi_\lambda}{\partial\nu} < \frac{\partial\tilde{\psi}_{\tilde{\lambda}}^{\varepsilon_0}}{\partial\nu} \quad \text{at } X^0, \quad \nu \text{ is outer normal vector,}$$

which implies that $\lambda < \tilde{\lambda}$, we derive a contradiction.

Case 2. If $\varepsilon_0 = 0$, we can choose $X^0 = A$. Then we have $\psi_\lambda(x_1, x_2) < \tilde{\psi}_{\tilde{\lambda}}(x_1, x_2)$ in $\Omega \cap \{\psi_\lambda < m_0\}$, it follows from the results in Corollary 11.5 in [17] that

$$\lambda = \frac{\partial\psi_\lambda}{\partial\nu} \leq \frac{\partial\tilde{\psi}_{\tilde{\lambda}}}{\partial\nu} = \tilde{\lambda} \quad \text{at } A,$$

which leads a contradiction to our assumption $\lambda > \tilde{\lambda}$.

Moreover, the free boundaries are not only continuous, but also smooth at the endpoints of the nozzle. The proof is similar to the problem in [4,18], we omit it here.

Hence, we complete the proof of Proposition 4.10. $\square$

Finally, we will prove the property (3) in Definition 1.1, namely, the vertical velocity of impinging jet flow established here is indeed negative in $\overline{\Omega_0} \setminus M^0$. Consider $v = -\frac{\partial\psi_\lambda}{\partial x_1}$ in Ω_0 and satisfies the equation

$$\Delta v = 0 \quad \text{in } \Omega_0.$$

It suffices to prove $v(x_1, x_2) < 0$ in any compact subset of Ω_0.

For any compact subset $D \subset\subset \Omega_0$ with smooth boundary, we have

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ v = -\frac{\partial\psi_\lambda}{\partial x_1} \leq 0 & \text{on } \partial D, \end{cases} \tag{4.44}$$

the maximum principle gives that $v(x_1, x_2) < 0$ in D.

Note that along $N \cup \Gamma$, $\psi = m_0$, it follows from the boundary condition (1.4) that

$$\partial_{x_1}\psi_\lambda(F(x_2), x_2)F'(x_2) + \partial_{x_2}\psi_\lambda(F(x_2), x_2) = 0,$$

where

$$F(x_2) = \begin{cases} f(x_2), & (F(x_2), x_2) \in N, \\ g_\lambda(x_2), & (F(x_2), x_2) \in \Gamma, \end{cases}$$

is a C^1 function.

Therefore, the outer normal derivative satisfies

$$\frac{\partial\psi_\lambda}{\partial\nu}(F(x_2), x_2) = \partial_{x_1}\psi_\lambda(F(x_2), x_2)\sqrt{1 + F'(x_2)^2}.$$

On the other hand, ψ_λ attains its minimum on $N \cup \Gamma$, it follows from the Hopf Lemma that

$$v = -\frac{\partial\psi_\lambda(x_1, x_2)}{\partial x_1} < 0, \quad \text{on } \partial\Omega_0 \setminus M^0.$$

So we obtain that the vertical velocity $v < 0$ in $\overline{\Omega_0} \setminus M^0$.

Now, we can conclude that there exists a (u, v, p, Γ) satisfies

$$\lambda \geq \frac{m_0}{H} - \frac{1}{2}\omega_0 H, \ \Gamma : x_1 = g_\lambda(x_2), \ u = \partial_{x_2}\psi_\lambda, \ v = -\partial_{x_1}\psi_\lambda,$$

and p is determined by the Bernoulli's law, which is a solution to the impinging jet problem. Hence, we complete the proof of Theorem 1.1.

5. Uniqueness of the impinging jet flow

In this section, we will show the uniqueness of the impinging jet flow, namely, the solution (u, v, p, Γ) to the impinging jet flow problem is unique under the additional condition (1.11) on the nozzle wall N. Moreover, the asymptotic behavior in upstream and downstream are obtained in this section.

First, we will establish the monotonicity of ψ_λ with respect to x_2 under the condition (1.11). Recall the minimizer $\psi_{\lambda,L}$ to the truncated variational problem again and extend the function $\Psi_L(x_1)$ as follows

$$\tilde{\Psi}_L(x_1) = \begin{cases} \Psi_L(x_1) & \text{if } 0 \leq x_1 \leq f(L), \\ m_0 & \text{if } f(L) \leq x_1 \leq L, \end{cases} \tag{5.1}$$

where $\Psi_L(x_1) = \min\left\{m_0, -\frac{1}{2}\omega_0 x_1^2 + v_0^L x_1\right\}$ and $v_0^L = \max\left\{0, \frac{m_0}{f(L)} + \frac{1}{2}\omega_0 f(L)\right\}$.

We claim that

$$0 \leq \psi_{\lambda,L}(x_1, x_2) \leq \tilde{\Psi}_L(x_1) \ \text{in} \ \Omega_L. \tag{5.2}$$

The monotonicity (1.11) of the nozzle wall N and the definition of $\tilde{\Psi}_L(x_1)$ implies that it suffices to prove $\psi_{\lambda,L}(x_1, x_2) \leq \tilde{\Psi}(x_1)$ in G_L, where $G_L = \Omega_L \cap \{0 \leq x_1 \leq f(L)\}$.

On the other hand, taking a smallest $\tau_0 \geq 0$, such that

$$\psi_{\lambda,L}(x_1, x_2) \leq \tilde{\Psi}_L(x_1) + \tau_0 \ \text{in} \ G_L.$$

Since $\psi_{\lambda,L}(x_1, x_2) \leq \tilde{\Psi}_L(x_1)$ on ∂G_L, similar to the proof of Theorem 4.1, we can obtain $\tau_0 = 0$. Therefore, (5.2) holds.

Next, we give the monotonicity of $\psi_{\lambda,L}$ with respect to x_2.

Proposition 5.1. *Suppose the condition* (1.11) *holds, then the solution* $\psi_{\lambda,L}(x_1, x_2)$ *to the truncated variational problem* $(P_{\lambda,L})$ *is monotonic with respect to* x_2*, that is to say,*

$$\psi_{\lambda,L}(x_1, x_2) \geq \psi_{\lambda,L}(x_1, \tilde{x}_2) \ \text{in} \ \Omega_L, \ \text{for any} \ x_2 > \tilde{x}_2. \tag{5.3}$$

Proof. The proof is similar to the one of Lemma 4.2 under the condition (5.2), and we omit it here. □

Thus the solution to the truncated problem $(P_{\lambda,L})$ satisfies that $\psi_{\lambda,L}(x_1, x_2)$ is monotonic increasing with respect to x_1 and x_2. Similar to the previous Lemma 4.3 and Lemma 4.4, we can show that the free boundary of the truncated problem $(P_{\lambda,L})$ can be described by a monotonic function $x_2 = k_{\lambda,L}(x_1)$. Similar to Section 4, there exists a sequence $\psi_{\lambda,L_k} \to \psi_\lambda$ in any compact subset of $\overline{\Omega}$, if $L_k \to +\infty$, and the free boundary Γ is given by a function $x_2 = k_\lambda(x_1) \in C^1(h, H]$.

Proposition 5.2. *Suppose that the semi-infinitely long nozzle satisfies the additional condition* (1.11)*, then the solution* ψ_λ *established in Section 4 is unique.*

Proof. Assume that ψ_λ and $\tilde{\psi}_{\tilde{\lambda}}$ are two solutions to the impinging jet flow. In view of the monotonicity of free boundary in the previous arguments, then we may assume that the free boundaries of ψ_λ and $\tilde{\psi}_{\tilde{\lambda}}$ can be denoted as follows

$$\Gamma : x_2 = k_\lambda(x_1) \ \text{and} \ \tilde{\Gamma} : x_2 = \tilde{k}_\lambda(x_1).$$

In Section 4, we obtain the uniqueness of λ, namely, $\lambda = \tilde{\lambda}$, which gives that

$$\lim_{x_1 \to \infty} k_\lambda(x_1) = \lim_{x_1 \to \infty} \tilde{k}_\lambda(x_1) = h.$$

Suppose $\psi_\lambda(x_1, x_2) \neq \tilde{\psi}_\lambda(x_1, x_2)$ in Ω. Without loss of generality, we assume that there exists some $\tilde{x}_1 \in (b, +\infty)$ such that

$$k_\lambda(\tilde{x}_1) > \tilde{k}_\lambda(\tilde{x}_1). \tag{5.4}$$

Set

$$\psi_\lambda^\varepsilon(x_1, x_2) = \psi_\lambda(x_1, x_2 + \varepsilon) \ \text{ for } \ \varepsilon \geq 0,$$

and denote ε_0 as to be the smallest number such that $\psi_\lambda^{\varepsilon_0}(x_1, x_2) \geq \tilde{\psi}_\lambda(x_1, x_2)$ in Ω and there exists a point $X^0 \in \bar{\Omega}$ with $\psi_\lambda^{\varepsilon_0}(X^0) = \tilde{\psi}_\lambda(X^0)$. The assumption (5.4) implies that $\varepsilon_0 > 0$, then the maximum principle gives that $X^0 \notin \Omega \cap \{\psi_\lambda^{\varepsilon_0} < m_0\}$ and $|X_0| \leq C$, we conclude that $X^0 \in \Gamma^{\varepsilon_0} \cap \tilde{\Gamma}$. In view of the choice of ε_0, we have

$$\begin{cases} -\Delta \psi_\lambda^{\varepsilon_0} = -\Delta \tilde{\psi}_\lambda = \omega_0 & \text{in } \Omega \cap \{\psi_\lambda^{\varepsilon_0} < m_0\}, \\ \psi_\lambda^{\varepsilon_0} = \tilde{\psi}_\lambda = m_0 & \text{at } X^0 \in \Gamma^{\varepsilon_0} \cap \tilde{\Gamma}, \end{cases} \tag{5.5}$$

and the maximum principle implies that

$$\lambda = \frac{\partial \psi_\lambda^{\varepsilon_0}}{\partial \nu} < \frac{\partial \tilde{\psi}_\lambda}{\partial \nu} = \lambda \ \text{ at } \ X^0, \ \ \nu \text{ is outer normal vector.}$$

That is a contradiction. Therefore, we obtain the uniqueness of ψ_λ. Finally, the uniqueness of the free boundary Γ is done, due to the definition (2.5). $\quad\square$

Proposition 5.3. *The impinging jet flow satisfies the asymptotic behavior at the far fields, namely*

$$(u, v, p) \to \begin{cases} (0, v_0(x_1), p_1), & \text{uniformly in any compact subset of } (0, a), \text{ as } x_2 \to +\infty, \\ (u_0(x_2), 0, p_0), & \text{uniformly in any compact subset of } (0, h), \text{ as } x_1 \to +\infty, \end{cases} \tag{5.6}$$

where $p_1 = p_0 + \frac{\lambda^2}{2} - \frac{\left(\frac{m_0}{a} - \frac{1}{2}\omega_0 a\right)^2}{2}$, $v_0(x_1) = -\frac{m_0}{a} - \frac{1}{2}\omega_0 a + \omega_0 x_1$ *and* $u_0(x_2) = \lambda + \omega_0 h - \omega_0 x_2$.
Furthermore,

$$\nabla u \to 0, \ \ \nabla v \to (\omega_0, 0), \ \ \nabla p \to 0, \ \text{uniformly in any compact subset of } (0, a), \tag{5.7}$$

as $x_2 \to +\infty$, *and*

$$\nabla u \to (0, -\omega_0), \ \ \nabla v \to 0, \ \ \nabla p \to 0, \ \text{uniformly in any compact subset of } (0, h), \tag{5.8}$$

as $x_1 \to +\infty$.

Proof. Define the function $\psi_\lambda^n(x_1, x_2) = \psi_\lambda(x_1, x_2 + n)$ for $x_2 > -\frac{n}{2}$ and n is sufficiently large. In view of the property of the nozzle N in (1.2) and (1.3), it follows from the elliptic estimates that we have

$$\|\psi_\lambda^n\|_{C^{2,\alpha}(G)} \leq C(G) \ \text{ for sufficiently large } n, \tag{5.9}$$

where G is any compact subset of $S = (0, a) \times (-\infty, +\infty)$.

It follows from Arzela–Ascoli Lemma that there exists a subsequence still labeled by ψ_λ^n, such that

$$\psi_\lambda^n \to \psi_0 \ \text{ uniformly in } \ C^{2,\alpha}(G), \tag{5.10}$$

for any $G \Subset S$. Furthermore, ψ_0 satisfies the equation

$$\begin{cases} -\Delta \psi_0 = \omega_0 \ \text{ in } \ S, \\ \psi_0(0, x_2) = 0, \ \ \psi_0(a, x_2) = m_0, \\ 0 \leq \psi_0(x_1, x_2) \leq -\frac{\omega_0}{2}x_1^2 + v_0 x_1 \ \text{ in } \ S. \end{cases} \tag{5.11}$$

We can solve the problem (5.11) that

$$\psi_0 = \Psi(x_1) = -\frac{\omega_0}{2}x_1^2 + v_0 x_1, \quad \text{in } S. \tag{5.12}$$

Hence, (5.10) and (5.12) give that

$$\nabla \psi_\lambda = (-v(x_1, x_2), u(x_1, x_2)) \to (-\omega_0 x_1 + v_0, 0), \tag{5.13}$$

uniformly for any $x_1 \in (0, a)$ as $x_2 \to +\infty$.

Furthermore, we have

$$\nabla u \to 0, \quad \nabla v \to (\omega_0, 0), \quad \nabla p \to 0, \quad \text{uniformly in any compact subset of } (0, a),$$

as $x_2 \to +\infty$.

Next, we consider the asymptotic behavior in the downstream. It follows from Corollary 3.13 in [18] that the free boundary $x_2 = k_\lambda(x_1)$ satisfies

$$x_2 = k_\lambda(x_1) \in C^{1,\alpha} \text{ for } x_2 \in (h, h + \varepsilon) \text{ and } (x_1, x_2) \in \Omega \cap \{x_1 > x_1^0\},$$

for small $\varepsilon > 0$ and large $x_1^0 > 0$. Moreover, the flatness condition and Corollary 3.13 in [18] imply that

$$k_\lambda'(x_1) \to 0 \text{ as } x_1 \to +\infty, \text{ and } |k_\lambda^{(j)}(x_1)| \le C \text{ for } j = 2, 3.$$

Similarly, we define the function $\psi_\lambda^n(x_1, x_2) = \psi_\lambda(x_1 + n, x_2)$ for $x_1 > -\frac{n}{2}$ and sufficiently large n. Using the standard elliptic estimate, we have

$$\|\psi_\lambda^n\|_{C^{2,\alpha}(G)} \le C(G) \text{ for sufficiently large } n, \tag{5.14}$$

where G is any compact subset of $S_1 = (-\infty, +\infty) \times (0, h)$ with h satisfies (1.9).

It follows from Arzela–Ascoli Lemma that there exists a subsequence ψ_λ^n such that

$$\psi_\lambda^n \to \bar{\psi} \text{ in } C^{2,\alpha}(G), \tag{5.15}$$

for any compact $G \Subset S_1$. Furthermore, $\bar{\psi}$ solves uniquely the following problem (5.16).

$$\begin{cases} -\Delta \bar{\psi} = \omega_0 \text{ in } S_1, \\ \bar{\psi}(x_1, h) = m_0, \bar{\psi}(x_1, 0) = 0, \\ \bar{\psi}(x_1, x_2) \le \phi_h(x_2) \text{ in } S_1. \end{cases} \tag{5.16}$$

Obviously,

$$\bar{\psi} = \phi(x_2) = -\frac{1}{2}\omega_0 x_2^2 + (\lambda + \omega_0 h)x_2, \quad \text{in } S_1. \tag{5.17}$$

Hence, (5.15) and (5.17) give that

$$\nabla \psi_\lambda = (-v(x_1, x_2), u(x_1, x_2)) \to (0, \lambda - \omega_0 h + \omega_0 x_2), \tag{5.18}$$

uniformly for any $x_2 \in (0, h)$ as $x_1 \to +\infty$.

Furthermore, we have

$$\nabla u \to (0, -\omega_0), \quad \nabla v \to 0, \quad \nabla p \to 0, \quad \text{uniformly in any compact subset of } (0, h),$$

as $x_1 \to +\infty$. $\quad\square$

Proposition 5.4. *Suppose that the condition* (1.11) *holds, we have*

$$u = \frac{\partial \psi_\lambda}{\partial x_2} > 0 \quad \text{in } \overline{\Omega_0} \setminus I. \tag{5.19}$$

Proof. We consider $u = \frac{\partial \psi_\lambda}{\partial x_2}$ in Ω_0 and satisfies the equation

$$\Delta u = 0, \quad \text{in } \Omega_0.$$

It suffices to prove $u(x_1, x_2) > 0$ in any compact subset of Ω_0.

For any compact subset $D \Subset \Omega_0$ with smooth boundary, we have

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = \frac{\partial \psi_\lambda}{\partial x_2} \geq 0 & \text{on } \partial D. \end{cases} \tag{5.20}$$

The maximum principle gives that $u(x_1, x_2) > 0$ in D.

Note that along $N \cup \Gamma$, $\psi_\lambda = m_0$, using the boundary condition (1.4), that is to see that

$$\partial_{x_1} \psi_\lambda(, x_1, H(x_1)) + \partial_{x_2} \psi_\lambda(x_1, H(x_1)) H'(x_1) = 0,$$

where $H(x_1) \in C^1$ and

$$H(x_1) = \begin{cases} f^{-1}(x_1), & (f(x_2), x_2) \in N, \\ k_\lambda(x_1), & (x_1, k_\lambda(x_1)) \in \Gamma. \end{cases}$$

Therefore, the outer normal derivative satisfies

$$\frac{\partial \psi_\lambda}{\partial \nu}(x_1, H(x_1)) = \partial_{x_2} \psi_\lambda(x_1, H(x_1)) \sqrt{1 + H'(x_1)^2}.$$

On the one hand, ψ_λ attains its minimum of at $N \cup \Gamma$, it follows from the Hopf Lemma that

$$u = \frac{\partial \psi_\lambda(x_1, x_2)}{\partial x_2} > 0, \quad \text{on } \partial \Omega_0 \setminus I.$$

So we obtain that the horizontal velocity $u > 0$ in $\overline{\Omega_0} \setminus I$. $\quad \square$

Hence, we complete the proof of Theorem 1.2.

Conflict of interest statement

The authors declare that they have no conflict of interest.

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