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A boundary control problem for the steady self-propelled motion of a rigid body in a Navier–Stokes fluid

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Abstract

Consider a rigid body $\mathcal{S} \subset \mathbb{R}^3$ immersed in an infinitely extended Navier–Stokes fluid. We are interested in self-propelled motions of $\mathcal{S}$ in the steady state regime of the system rigid body-fluid, assuming that the mechanism used by the body to reach such a motion is modeled through a distribution of velocities v_* on $\partial\mathcal{S}$. If the velocity V of $\mathcal{S}$ is given, can we find v_* that generates V? We show that this can be solved as a control problem in which v_* is a six-dimensional control such that either $\operatorname{Supp} v_* \subset \Gamma$, an arbitrary nonempty open subset of $\partial\Omega$, or $v_* \cdot n|_{\partial\Omega} = 0$. We also show that one of the self-propelled conditions implies a better summability of the fluid velocity.

Keywords: 3-D Navier–Stokes equations; Exterior domain; Rotating body; Self-propelled motion; Boundary control; Asymptotic behavior

1. Introduction

Consider a rigid body $\mathcal{S} \subset \mathbb{R}^3$ immersed in a viscous incompressible fluid which fills the exterior domain $\Omega := \mathbb{R}^3 \setminus \mathcal{S}$. In this paper, we consider self-propelled motions of $\mathcal{S}$ in the steady state regime of the system rigid body-fluid, assuming, as in [13] and [14], that the mechanism used by the body to reach such a motion is modeled through a distribution of velocities v_* on $\partial\Omega = \partial\mathcal{S}$. In practice, such a velocity can be produced by propellers (submarines), deformations (fishes), cilia (micro-organisms), etc. In a reference frame attached to the rigid body, the

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system of equations modeling this mechanical system is

$$-\operatorname{div}\sigma(v,p) + (v-V)\cdot\nabla v + \omega\times v = 0 \quad \text{in } \Omega \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \tag{1.2}$$

$$v = V + v_* \quad \text{on } \partial\Omega \tag{1.3}$$

$$\lim_{|x|\to\infty} v = 0 \tag{1.4}$$

$$m\xi\times\omega + \int_{\partial\Omega}\left[-\sigma(v,p)n + (v_*\cdot n)(v_* + V + \omega\times x)\right]d\gamma = 0 \tag{1.5}$$

$$(I\omega)\times\omega + \int_{\partial\Omega} x\times\left[-\sigma(v,p)n + (v_*\cdot n)(v_* + V + \omega\times x)\right]d\gamma = 0 \tag{1.6}$$

where the quantities $v = v(x)$ and $p = p(x)$ represent, respectively, the velocity field and the pressure of the liquid and

$$V(x) = \xi + \omega\times x, \quad x\in\mathbb{R}^3$$

represents the velocity of the solid, as seen by an observer attached to $\mathcal{S}$. Moreover, we have denoted by $\sigma(v,p)$ the Cauchy stress tensor defined by

$$\sigma(v,p) := 2D(v) - p\mathbb{I}_3,$$

with $\mathbb{I}_3$ being the 3×3 identity matrix and $D(v)$ being the symmetric gradient defined by

$$D(v) := \frac{1}{2}\left((\nabla v) + (\nabla v)^\top\right),$$

so that $\operatorname{div}\sigma(v,p) = \Delta v - \nabla p$ since $\operatorname{div} v = 0$. In (1.5)–(1.6) the outward unit normal to $\partial\Omega$ is denoted by n. We also assume that the density of the rigid body is 1 and that its center of gravity is located at the origin:

$$m := \int_{\mathcal{S}} 1\,dx, \quad I := \int_{\mathcal{S}} (|x|^2\mathbb{I}_3 - x\otimes x)\,dx, \quad \int_{\mathcal{S}} x\,dx = 0.$$

The model (1.1)–(1.6) is inspired by Galdi [13–15], the equation (1.5) having been obtained as the net force exerted by the fluid on the solid (see the proof of Theorem 1.2) and the equation (1.6) being the corresponding balance of torques.

The problem we are interested in is the following one: assuming that ξ and ω are given, can we find v_*, v, p (in appropriate functional spaces) satisfying (1.1)–(1.6)? A positive answer to this question would show that the mechanism of locomotion expressed by the boundary velocity v_* allows the rigid body to move with the velocity $\xi + \omega\times x$. This can be seen as a control problem in which the velocity v_* on $\partial\Omega$ is the control of the problem. The corresponding problem for the steady, translational self-propelled motion of a symmetric body was solved by Galdi [13], but general case has remained open. When (1.1) is replaced by the classical Stokes system as the model with zero Reynolds number, this control problem was studied by Galdi [14, Sections 2 and 3]. In [24] San Martín, Takahashi and Tucsnak studied another quasi-steady control problem of finding a boundary control that achieves a final position with a prescribed velocity. We also mention some investigations of using boundary controls to minimize the drag about a three-dimensional translating body in an unsteady Navier–Stokes fluid, such as [12].

As pointed out by Galdi [13,14], we have various possibilities of finding a solution to our control problem. Among others, in this paper, we provide two sorts of solutions, both of which are physically relevant as well as interesting. One is a control v_* which vanishes outside a prescribed portion $\Gamma\subset\partial\Omega$ with arbitrary small $|\Gamma| > 0$, the other is a control v_* which is tangential to $\partial\Omega$. Both of them were found in [13,14,24] for problems of their own, where the Stokes approximation was adopted in the last two papers. In those literature a control v_* is chosen from a suitable finite dimensional (1-dimensional in [13], 6-dimensional in [14,24]) subspace, called the control space $\mathcal{C}$, of $L^2(\partial\Omega)$ which is constructed by use of the Stokes flow and depends only on geometric properties of the body $\mathcal{S}$ (thus the space $\mathcal{C}$ is independent of ξ,ω). We may expect such a space $\mathcal{C}$ for our problem as well, however, unlike the previous works it does not seem to be easy to find out a control v_* in terms only of the Stokes flow because of full generality of the problem. In order to get around the difficulty, in this paper, we construct a 6-dimensional subspace $\mathcal{C} = \mathcal{C}_{(\xi,\omega)}$ depending on a prescribed (ξ, ω) by using the adjoint system of the full linearized equation that involves $(\xi + \omega \times x) \cdot \nabla v - \omega \times v$, see (3.17) and (3.18) below, and then single out $v_* \in \mathcal{C}_{(\xi,\omega)}$ which provides a solution. The idea of finding v_* dependent of (ξ, ω) works well to solve the control problem under consideration and, to the best of our knowledge, our result first shows the existence of a control v_* in general nonlinear setting, where the rotation of the body is also taken into account, as long as $|\xi|$ and $|\omega|$ are sufficiently small. The main result of the paper can be stated as follows.

Theorem 1.1. *Assume $\partial\Omega$ is of class C^3. There exist constants $c_0, C > 0$ such that if $\xi, \omega \in \mathbb{R}^3$ satisfy*

$$|\xi| \leqslant c_0, \qquad |\omega| \leqslant c_0, \tag{1.7}$$

then problem (1.1)–(1.6) admits at least one solution (v, p, v_*) of class

$$\varpi v \in L^\infty(\Omega), \quad (\nabla v, p) \in W^{1,2}(\Omega), \quad v \in W^{2,2}_{loc}(\overline{\Omega}), \quad (v, p) \in C^\infty(\Omega), \quad v_* \in W^{3/2,2}(\partial\Omega) \tag{1.8}$$

with

$$\varpi(x) := \begin{cases} \left(1 + \left|x - \dfrac{\omega \times \xi}{|\omega|^2}\right|\right)\left(1 + 2\dfrac{|\omega \cdot \xi|}{|\omega|} s(x)\right), & \omega \neq 0, \\ (1 + |x|)\,(1 + 2(|\xi||x| + \xi \cdot x)), & \omega = 0, \end{cases} \tag{1.9}$$

where

$$s(x) := \left|x - \frac{\omega \times \xi}{|\omega|^2}\right| + \frac{\operatorname{sign}(\omega \cdot \xi)}{|\omega|}\, \omega \cdot x, \qquad \omega \neq 0,$$

and it satisfies

$$\sup_{x \in \Omega}[\varpi(x)|v(x)|] + \|(\nabla v, p)\|_{W^{1,2}(\Omega)} + \|v_*\|_{W^{3/2,2}(\partial\Omega)} \leqslant C\big(|(\xi, \omega)| + |(\xi, \omega)|^2\big). \tag{1.10}$$

Here, one can choose the boundary control v_* such that either $\operatorname{Supp} v_* \subset \Gamma$ *or $v_* \cdot n|_{\partial\Omega} = 0$, where Γ is an arbitrary small nonempty open subset of $\partial\Omega$ (with respect to the induced topology).*

The notion of solution to (1.1)–(1.4) is the same as introduced in [16], that is, it is basically of the so-called Leray class $\nabla v \in L^2(\Omega)$ with standard weak formulation, but the solution obtained in Theorem 1.1 actually becomes a strong (even smooth) one as described in (1.8), so that the boundary integrals in (1.5)–(1.6) make sense.

The weight function (1.9) looks complicated, however, if we replaced $|x - \frac{\omega \times \xi}{|\omega|^2}|$ just by $|x|$, then the constant C in (1.10) would depend on the ratio $\frac{|\xi|}{|\omega|}$ and the estimate (1.10) would become useless. Thus we should keep (1.9) as it is. Note that $s(x) \geqslant 0$ in (1.9), see Remark 2.2.

The last statement in Theorem 1.1 shows that there are two kinds of nontrivial controls (where the trivial control means $v_* = -V$, i.e. $(v, p) = (0, 0)$). As mentioned above, we could have some others for the same ξ, ω. What is more interesting would be to find an optimal control v_*, which minimizes (for instance but typically) the drag as in [12], that is, which attains

$$\inf \int_{\partial\Omega} \big(\sigma(v, p)n\big) \cdot v \, d\gamma,$$

where the infimum is taken over the set of all solutions of our control problem subject to a suitable side condition (which excludes the trivial solution). It should be emphasized that this admissible set is nonvoid at least for small (ξ, ω) by Theorem 1.1, which is certainly the first step toward our future research. In order to proceed to that, it is convenient to rewrite the drag functional as

$$\int_{\partial\Omega} \big(\sigma(v, p)n\big) \cdot v \, d\gamma = 2\int_{\Omega} |D(v)|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} (v_* \cdot n)|V + v_*|^2 \, d\gamma, \tag{1.11}$$

which is actually bounded from below provided that $\|v_*\|_{L^3(\partial\Omega)} \leqslant C$ as well as (1.7), however, we have to take care of the asymptotic behavior of (v, p) at infinity to justify (1.11) rigorously. By pointwise estimate (1.10) for ϖv we have the following summability property:

$$v \in L^{2+\varepsilon}(\Omega),\ \forall \varepsilon > 0, \qquad \text{if } (\omega = 0 \text{ and } \xi \neq 0) \text{ or } (\omega \cdot \xi \neq 0),$$
$$v \in L^{3+\varepsilon}(\Omega),\ \forall \varepsilon > 0, \qquad \text{if } \omega \neq 0 \text{ and } \omega \cdot \xi = 0,$$

which is indeed enough to justify the representation (1.11) of the drag (as in the proof of Lemma 4.2 below), but the self-propelled condition, especially (1.5), implies even faster decay of the solution and consequently improves its summability. In particular, it is worth while noting that the solution possesses finite kinetic energy: $v \in L^2(\Omega)$. This is not surprising because (1.5) tells us that the net force (momentum flux) exerted by the fluid to the rigid body vanishes, that is,

$$\int_{\partial\Omega} \big[\sigma(v,p) - v \otimes (v - V) - (\omega \times x) \otimes v\big] n \, d\gamma = 0, \tag{1.12}$$

where the left-hand side of (1.12) is consistent with the equation (1.1) of momentum which can be written in the divergence form: $\operatorname{div}[\sigma(v,p) - v \otimes (v - V) - (\omega \times x) \otimes v\big] = 0$. To be sure, let us observe that (1.5) is equivalent to (1.12); in fact, by (1.3) we have

$$\begin{aligned} N &:= \int_{\partial\Omega} [\sigma(v,p) - v \otimes (v - V) - (\omega \times x) \otimes v] n \, d\gamma \\ &= \int_{\partial\Omega} [\sigma(v,p)n - (v + \omega \times x)(v_* \cdot n)] \, d\gamma + \int_{\mathcal{S}} (\xi + \omega \times x) \cdot \nabla(\omega \times x) \, d\gamma \\ &= \int_{\partial\Omega} [\sigma(v,p)n - (v_* \cdot n)(v_* + V + \omega \times x)] \, d\gamma - m\xi \times \omega \end{aligned} \tag{1.13}$$

since $\int_{\mathcal{S}} x \, dx = 0$. We could claim that this observation provides another interpretation of (1.5).

To complete our study, we show that the solutions to (1.1)–(1.4) of class (1.8) (actually under less conditions) possess better summability provided $N = 0$ (even weaker condition $\omega \cdot N = 0$ is enough when $\omega \neq 0$). When the translation of the body is absent or orthogonal to the rotation, smallness of given solution is needed; indeed, for the solutions obtained in Theorem 1.1 this can be accomplished by (1.10) provided ξ and ω are taken still smaller. Our conclusion below is more or less known since the issue is closely related to asymptotic structure of the exterior Navier–Stokes flow near infinity. The classical case is that the body is purely translating ($\omega = 0, \xi \neq 0$), for which it is well known ([1,10,11], [16, Theorem X.8.1]) that the leading term of asymptotic expansion is given by the Oseen fundamental solution whose coefficient is N. In this case the result can be obtained even if assuming only $\nabla v \in L^2(\Omega)$ (the Leray class), however, the decay property of $v(x)$ like $|x|^{-1}$ is always assumed for any case in this paper. When the body is at rest ($\omega = 0, \xi = 0$), the leading term involves the nonlinear effect unlike the previous case and is given by a particular Landau solution (homogeneous Navier–Stokes flow of degree (-1)) U_N under smallness condition, where the set of all Landau solutions is parametrized as $\{U_b;\ b \in \mathbb{R}^3\}$, see Section 6 for details. This was proved by Korolev and Šverák [21]. Hence, for both cases ($\xi \neq 0, \xi = 0$) in which the rotation of the body is absent ($\omega = 0$), we know that N controls the rate of decay of the fluid velocity $v(x)$ at infinity. Compared with this, when the body is purely rotating ($\omega \neq 0, \xi = 0$), the leading term is given by U_b with $b = (\frac{\omega}{|\omega|} \cdot N)\frac{\omega}{|\omega|}$, that is, it is still a Landau solution, but this time the rate of decay of $v(x)$ can be controlled only by $\omega \cdot N$. This was proved first by Farwig and Hishida [8] in which the remainder has better summability, and then the result was refined by Farwig, Galdi and Kyed [6] in the sense that the remainder has better pointwise decay. Finally, when the body is translating as well as rotating ($\omega \neq 0, \xi \neq 0$), where ξ is parallel to ω, however, the general case can be reduced to this case or to the previous case as we will explain in the next section (see Galdi [16, Chapter VIII]), Kyed [23] proved that the leading term is given by the Oseen fundamental solution whose coefficient is $(\frac{\omega}{|\omega|} \cdot N)\frac{\omega}{|\omega|}$. Taking all the cases into account, we know almost everything, but there are things which are missing in the literature:

- when $\omega \neq 0$, all the papers [8,6,23] studied the case of no-slip boundary condition $v = V$ on $\partial\Omega$. In this paper, with the aid of the flux carrier, we discuss the asymptotic behavior without assuming any boundary condition on $\partial\Omega$. To do so, we need to develop analysis of the fundamental solution, see Lemma 6.1;
- when $\omega \neq 0$ and ξ is not parallel to ω, we need the reduction mentioned above by using the Mozzi–Chasles transform [16, Chapter VIII] to describe the conclusion completely.

The result is now summarized as follows. The statement is essentially independent of Theorem 1.1 and thus includes also the case $(\xi, \omega) = (0, 0)$ (due to [21]) although the solution obtained in Theorem 1.1 is the trivial one in that case.

Theorem 1.2. *Suppose* (v, p) *is a solution to (1.1)–(1.4) satisfying, in particular,* $\nabla v \in L^2(\Omega)$, $p \in L^2(\Omega)$ *and* $(1 + |x|)\, v \in L^\infty(\Omega)$.

1. *Case* $\omega = 0$, $\xi = 0$. *Let* $N = 0$, *that is,*
$$\int_{\partial\Omega} \left[\sigma(v, p)n - (v_* \cdot n)\, v_*\right] d\gamma = 0.$$
Then, for every $\varepsilon > 0$, *there is a constant* $\delta = \delta(\varepsilon) > 0$ *such that if* $\limsup_{|x|\to\infty} |x||v(x)| \leqslant \delta$, *then* $v \in L^{3/2+\varepsilon}(\Omega)$.
2. *Case* $\omega \neq 0$, $\omega \cdot \xi = 0$. *Let* $\omega \cdot N = 0$, *that is,*
$$\omega \cdot \int_{\partial\Omega} \left[\sigma(v, p)n - (v_* \cdot n)\, v_*\right] d\gamma = 0.$$
For every $\varepsilon > 0$, *there is a constant* $\delta = \delta(\varepsilon) > 0$ *such that if* $\limsup_{|x|\to\infty} |x||v(x)| \leqslant \delta$, *then* $v \in L^{3/2+\varepsilon}(\Omega)$.
3. *Case* $\omega = 0$, $\xi \neq 0$. *Let* $N = 0$, *that is,*
$$\int_{\partial\Omega} \left[\sigma(v, p)n - (v_* \cdot n)\,(v_* + \xi)\right] d\gamma = 0.$$
Then $v \in L^{3/2+\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$.
4. *Case* $\omega \cdot \xi \neq 0$. *Let* $\omega \cdot N = 0$, *that is,*
$$\omega \cdot \int_{\partial\Omega} \left[\sigma(v, p)n - (v_* \cdot n)\,(v_* + \xi)\right] d\gamma = 0.$$
Then $v \in L^{3/2+\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$.

In the last two cases, if we assume moreover that $\int_{\partial\Omega} v_* \cdot n\, d\gamma = 0$, *then* $v \in L^{4/3+\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$.

We note that N is understood as ${}_{W^{1/2,2}(\partial\Omega)}\langle 1, T(v, p)n\rangle_{W^{-1/2,2}(\partial\Omega)}$ and thus well-defined even under the condition of Theorem 1.2, where $T(v, p) := \sigma(v, p) - v \otimes (v - V) - (\omega \times x) \otimes v$, see (1.13); in fact, $T(v, p)n \in W^{-1/2,2}(\partial\Omega) := W^{1/2,2}(\partial\Omega)^*$ by the normal trace theorem since $T(v, p) \in L^2_{loc}(\overline{\Omega})$ and $\operatorname{div} T(v, p) = 0$. The last statement in Theorem 1.2 shows that for tangential boundary controls, that is, boundary velocities satisfying $v_* \cdot n = 0$, we have a better summability for the solutions of (1.1)–(1.6).

The plan of the paper is as follows. Section 2 contains the notation used throughout the paper and the results available in the literature for the generalized Oseen problem (which takes into account the rotation effect of $\mathcal{S}$) that are relevant for our work. In Section 3, we give the precise mathematical formulation of the control problem, introducing a set of adjoint problems which are used to define the control spaces. Then, in Section 4, we solve a linearized version of the control problem considering localized controls (in a portion of the boundary of $\mathcal{S}$) and tangential controls. The full non-linear control problem is solved in Section 5, by means of Banach fixed point Theorem, assuming that the data are suitably small. Finally, in Section 6, we show that the self-propelled condition (1.5), or equivalently (1.12), implies a better summability of the fluid velocity.

2. Notation and preliminary results on a generalized Oseen problem

In this paper, the usual notation is used for Lebesgue and Sobolev spaces on a domain $\mathcal{A}$, namely, $L^q(\mathcal{A})$ and $W^{m,q}(\mathcal{A})$, with norms $\|.\|_{q,\mathcal{A}}$ and $\|.\|_{m,q,\mathcal{A}}$, respectively. By $W^{m-\frac{1}{q},q}(\partial\mathcal{A})$ we indicate the trace space on the smooth boundary $\partial\mathcal{A}$ of $\mathcal{A}$, for functions from $W^{m,q}(\mathcal{A})$, equipped with the usual norm $\|.\|_{m-\frac{1}{q},q,\partial\mathcal{A}}$. The homogeneous Sobolev space of order (k,q) is defined by

$$D^{k,q}(\mathcal{A}) := \{u \in L^1_{loc}(\mathcal{A});\ D^\alpha u \in L^q(\mathcal{A}) \text{ for any multi-index } \alpha \text{ with } |\alpha| = k\}$$

with associated seminorm $|u|_{k,q,\mathcal{A}} = \sum_{|\alpha|=k} \|D^\alpha u\|_{q,\mathcal{A}}$, where $k \geqslant 1$ is an integer and $1 < q < \infty$. For a vector or second-order tensor field G and a positive function w defined on $\mathcal{A}$, we adopt the notation

$$\lceil G \rceil_{\alpha,w,\mathcal{A}} := \sup_{x\in\mathcal{A}} [w(x)^\alpha |G(x)|], \qquad \lceil G \rceil_\alpha := \lceil G \rceil_{\alpha,\,(1+|x|),\mathbb{R}^3} \tag{2.1}$$

for $\alpha \geqslant 0$. Throughout the paper we shall use the same font style to denote scalar, vector and tensor-valued functions and corresponding function spaces.

In what follows, $\mathcal{S}$ is a compact connected set, with nonempty interior, and we assume that $\Omega = \mathbb{R}^3 \setminus \mathcal{S}$ is a three-dimensional exterior domain. We will assume that the boundary $\partial\Omega$ of Ω is of class C^3. This is needed in Lemma 3.2 although Proposition 2.1 below holds provided $\partial\Omega$ is of class C^2.

Gathering several results in [16,18,19] by Galdi and Silvestre, and using suitable changes of variables, we can obtain existence, uniqueness and estimates for the general linear problem

$$\begin{aligned} &-\operatorname{div}\sigma(v,p) - (a + b\times x)\cdot\nabla v + b\times v = f \quad \text{in } \Omega \\ &\operatorname{div} v = 0 \quad \text{in } \Omega \\ &v = v_* \quad \text{on } \partial\Omega \\ &\lim_{|x|\to\infty} v = 0. \end{aligned} \tag{2.2}$$

We will need very specific estimates for the solution of problem (2.2), with a constant independent of a and b satisfying $|a|, |b| \leqslant B$ since this will be crucial for defining suitable smallness conditions when we apply the Banach fixed point theorem to solve the non-linear problem. When $b \neq 0$, the following theorem is essentially due to Galdi [16, Theorem VIII.6.1], in which both a and b are assumed to be parallel to $e_1 = (1,0,0)$. The general case is reduced to this particular case by the Mozzi–Chasles transform as explained in [16,18]. For later use it is convenient to summarize the result as follows.

Proposition 2.1. *We set*

$$w(x) := \begin{cases} \left(1 + \left|x - \dfrac{b\times a}{|b|^2}\right|\right)\left[1 + 2\left(\dfrac{|b\cdot a|}{|b|}\left|x - \dfrac{b\times a}{|b|^2}\right| + \dfrac{b\cdot a}{|b|^2}\, b\cdot x\right)\right], & b\neq 0, \\ (1+|x|)\,(1 + 2(|x||a| + a\cdot x))\,, & b = 0. \end{cases} \tag{2.3}$$

Assume that $f = \operatorname{div} F \in L^2(\Omega)$, *with*

$$\lceil F \rceil_{2,w,\Omega} := \sup_{x\in\Omega}\Big[w(x)^2 |F(x)|\Big] < \infty \tag{2.4}$$

and $v_* \in W^{3/2,2}(\partial\Omega)$. *Then, there exists a unique solution* (v,p) *to* (2.2) *with*

$$v \in D^{2,2}(\Omega)\cap D^{1,2}(\Omega)\cap L^2_{loc}(\overline{\Omega}), \qquad p \in W^{1,2}(\Omega), \tag{2.5}$$

$$\lceil v \rceil_{1,w,\Omega} := \sup_{x\in\Omega}[w(x)|v(x)|] < \infty \tag{2.6}$$

and

$$|v|_{2,2,\Omega} + |v|_{1,2,\Omega} + \lceil v \rceil_{1,w,\Omega} + \|p\|_{1,2,\Omega} \leqslant C(\|f\|_{2,\Omega} + \lceil F \rceil_{2,w,\Omega} + \|v_*\|_{3/2,2,\partial\Omega}), \tag{2.7}$$

where, for each $B > 0$, *one can choose a constant* $C = C(B) > 0$ *independent of* a *and* b *with* $|a|, |b| \in [0, B]$.

Proof. Let us consider the case $b \neq 0$. Let $M \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix that fulfills $M\frac{b}{|b|} = e_1$. By the transformation

$$\begin{aligned} &x' = Mx, \qquad \Omega' = M\Omega, \qquad v'(x') = Mv(M^\top x'), \qquad p'(x') = p(M^\top x'), \\ &v'_*(x') = Mv_*(M^\top x'), \qquad f'(x') = Mf(M^\top x'), \qquad F'(x') = (MFM^\top)(M^\top x'), \end{aligned} \tag{2.8}$$

so that $f' = \operatorname{div} F'$, we see that (2.2) can be written as

$$\begin{aligned} &-\operatorname{div}\sigma(v', p') - a'\cdot\nabla v' - |b|\{(e_1\times x')\cdot\nabla v' - e_1\times v'\} = f' \quad \text{in } \Omega' \\ &\operatorname{div} v' = 0 \quad \text{in } \Omega' \\ &v' = v'_* \quad \text{on } \partial\Omega' \\ &\lim_{|x'|\to\infty} v' = 0 \end{aligned} \tag{2.9}$$

where $a' = Ma$, and ∇ and div are differential operators with respect to x'. And then, following [16, Chapter VIII], we make further change of variables

$$\begin{aligned} &\widetilde{x} = x' - \frac{e_1\times a'}{|b|} = M\left(x - \frac{b\times a}{|b|^2}\right), \qquad \widetilde{\Omega} = \Omega' - \frac{e_1\times a'}{|b|} = M\left(\Omega - \frac{b\times a}{|b|^2}\right), \\ &\widetilde{v}(\widetilde{x}) = v'\left(\widetilde{x} + \frac{e_1\times a'}{|b|}\right) = Mv\left(M^\top\widetilde{x} + \frac{b\times a}{|b|^2}\right), \\ &\widetilde{p}(\widetilde{x}) = p'\left(\widetilde{x} + \frac{e_1\times a'}{|b|}\right) = p\left(M^\top\widetilde{x} + \frac{b\times a}{|b|^2}\right), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\widetilde{v}_*(\widetilde{x}) = v'_*\left(\widetilde{x} + \frac{e_1\times a'}{|b|}\right) = Mv_*\left(M^\top\widetilde{x} + \frac{b\times a}{|b|^2}\right), \\ &\widetilde{f}(\widetilde{x}) = f'\left(\widetilde{x} + \frac{e_1\times a'}{|b|}\right) = Mf\left(M^\top\widetilde{x} + \frac{b\times a}{|b|^2}\right), \qquad \widetilde{f} = \operatorname{div}\widetilde{F}, \\ &\widetilde{F}(\widetilde{x}) = F'\left(\widetilde{x} + \frac{e_1\times a'}{|b|}\right) = (MFM^\top)\left(M^\top\widetilde{x} + \frac{b\times a}{|b|^2}\right). \end{aligned} \tag{2.11}$$

Taking account of the relation $a' = (e_1\cdot a')e_1 + (e_1\times a')\times e_1$ in (2.9), we are led to

$$\begin{aligned} &-\operatorname{div}\sigma(\widetilde{v}, \widetilde{p}) - \mathcal{R}\,\partial_1\widetilde{v} - |b|\,\{(e_1\times\widetilde{x})\cdot\nabla\widetilde{v} - e_1\times\widetilde{v}\} = \widetilde{f} \quad \text{in } \widetilde{\Omega} \\ &\operatorname{div}\widetilde{v} = 0 \quad \text{in } \widetilde{\Omega} \\ &\widetilde{v} = \widetilde{v}_* \quad \text{on } \partial\widetilde{\Omega} \\ &\lim_{|\widetilde{x}|\to\infty} \widetilde{v} = 0 \end{aligned} \tag{2.12}$$

which is exactly (VIII.0.7) of [16], where

$$\mathcal{R} = e_1\cdot a' = \frac{b\cdot a}{|b|}.$$

Here, ∇ and div are differential operators with respect to $\widetilde{x}$ as well as $\partial_1 = \partial_{\widetilde{x}_1}$.

We can gather several results of Galdi [16]: from Theorem VIII.1.2 and Theorem VIII.2.1, there exists a unique weak solution, and from Theorem VIII.6.1 (which can be proved by the same cut-off technique after subtracting the flux carrier as in the proof of Theorem 1.2 in the present paper, see Section 6) we find an anisotropic pointwise decay estimate with wake property of this solution. Although $\mathcal{R} \geqslant 0$ is assumed in [16], it is obvious that these theorems still hold true for the other case $\mathcal{R} < 0$ as well. The only difference between those cases is the direction of the wake, which can be described by the following weight function:

$$\widetilde{w}(\widetilde{x}) := (1+|\widetilde{x}|)\big(1+2|\mathcal{R}|\widetilde{s}(\widetilde{x})\big), \qquad \widetilde{s}(\widetilde{x}) := \begin{cases} |\widetilde{x}| + \widetilde{x}_1, & \mathcal{R} > 0, \\ |\widetilde{x}| - \widetilde{x}_1, & \mathcal{R} < 0. \end{cases} \tag{2.13}$$

Let us define

$$w(x) := \widetilde{w}(\widetilde{x}) = \left(1 + \left|x - \frac{b\times a}{|b|^2}\right|\right)\left(1 + 2\frac{|b\cdot a|}{|b|}s(x)\right) \tag{2.14}$$

with

$$\begin{aligned} s(x) := \widetilde{s}(\widetilde{x}) &= \left|x - \frac{b\times a}{|b|^2}\right| + e_1\cdot\left(M\left(x - \frac{b\times a}{|b|^2}\right)\right) \\ &= \left|x - \frac{b\times a}{|b|^2}\right| + \frac{b}{|b|}\cdot\left(x - \frac{b\times a}{|b|^2}\right) \\ &= \left|x - \frac{b\times a}{|b|^2}\right| + \frac{b}{|b|}\cdot x \qquad \text{for } b\cdot a > 0, \end{aligned} \tag{2.15}$$

while

$$s(x) := \widetilde{s}(\widetilde{x}) = \left|x - \frac{b\times a}{|b|^2}\right| - \frac{b}{|b|}\cdot x \qquad \text{for } b\cdot a < 0. \tag{2.16}$$

We observe

$$\|\widetilde{f}\|_{2,\widetilde{\Omega}} = \|f\|_{2,\Omega}, \qquad \lceil \widetilde{F}\rceil_{2,\widetilde{w},\widetilde{\Omega}} := \sup_{\widetilde{x}\in\widetilde{\Omega}}\left[\widetilde{w}(\widetilde{x})^2|\widetilde{F}(\widetilde{x})|\right] = \lceil F\rceil_{2,w,\Omega} < \infty \tag{2.17}$$

by (2.4), as well as

$$\|\widetilde{v}_*\|_{3/2,2,\partial\widetilde{\Omega}} \leqslant C\|v_*\|_{3/2,2,\partial\Omega}. \tag{2.18}$$

In fact, the mapping $v_* \mapsto \widetilde{v}_*$ defined by (2.11) is isometric from $W^{k,2}(\partial\Omega)$ to $W^{k,2}(\partial\widetilde{\Omega})$ for $k = 1, 2$, which together with $W^{3/2,2}(\partial\mathcal{A}) = \left[W^{1,2}(\partial\mathcal{A}), W^{2,2}(\partial\mathcal{A})\right]_{1/2}$ for $\mathcal{A} = \Omega, \widetilde{\Omega}$ implies (2.18), where $[\cdot,\cdot]_{1/2}$ denotes the complex interpolation functor. By [16, Theorem VIII.6.1] problem (2.12) admits a unique solution

$$\begin{aligned} &\widetilde{v}\in D^{2,2}(\widetilde{\Omega})\cap D^{1,2}(\widetilde{\Omega})\cap L^2_{loc}(\overline{\widetilde{\Omega}}), \qquad \widetilde{p}\in W^{1,2}(\widetilde{\Omega}), \\ &\lceil \widetilde{v}\rceil_{1,\widetilde{w},\widetilde{\Omega}} := \sup_{\widetilde{x}\in\widetilde{\Omega}}[\widetilde{w}(\widetilde{x})|\widetilde{v}(\widetilde{x})|] < \infty \end{aligned}$$

which satisfies, by using (2.17) and (2.18),

$$\begin{aligned} |\widetilde{v}|_{2,2,\widetilde{\Omega}} + |\widetilde{v}|_{1,2,\widetilde{\Omega}} + \lceil \widetilde{v}\rceil_{1,\widetilde{w},\widetilde{\Omega}} + \|\widetilde{p}\|_{1,2,\widetilde{\Omega}} &\leqslant C\left(\|\widetilde{f}\|_{2,\widetilde{\Omega}} + \lceil\widetilde{F}\rceil_{2,\widetilde{w},\widetilde{\Omega}} + \|\widetilde{v}_*\|_{3/2,2,\partial\widetilde{\Omega}}\right) \\ &\leqslant C\left(\|f\|_{2,\Omega} + \lceil F\rceil_{2,w,\Omega} + \|v_*\|_{3/2,2,\partial\Omega}\right). \end{aligned}$$

Here, for each $B > 0$ the constant $C > 0$ can be taken independently of a and b provided $|a|, |b| \leqslant B$ (yielding $|\mathcal{R}|, |b| \leqslant B$). When we define (v, p) by (2.10), we observe

$$|v|_{k,2,\Omega} = |\widetilde{v}|_{k,2,\widetilde{\Omega}} \quad (k = 1,2), \qquad \lceil v\rceil_{1,w,\Omega} = \lceil\widetilde{v}\rceil_{1,\widetilde{w},\widetilde{\Omega}}, \qquad \|p\|_{1,2,\Omega} = \|\widetilde{p}\|_{1,2,\widetilde{\Omega}},$$

and conclude that (v, p) is the desired solution with estimate (2.7).

We will give a brief sketch for the other cases. When $b = 0$, $a \neq 0$, we take the orthogonal matrix $M \in \mathbb{R}^{3\times3}$ satisfying $M\frac{a}{|a|} = e_1$ and make the change of variables (2.8) to obtain

$$\begin{aligned} &-\operatorname{div}\sigma(v', p') - |a|\,\partial_1 v' = f' \quad \text{in } \Omega' \\ &\operatorname{div} v' = 0 \quad \text{in } \Omega' \\ &v' = v'_* \quad \text{on } \partial\Omega' \\ &\lim_{|x'|\to\infty} v' = 0. \end{aligned} \tag{2.19}$$

We first construct a unique solution of class (2.5). We then reduce (2.19) to the whole space problem by cut-off technique and use the pointwise decay of the Oseen fundamental solution, see (6.6) below. By using estimate of the Oseen potential representation given by Lemma VIII.3.5 of [16], which is traced back to Farwig [5], Kracmar, Novotny and Pokorny [22], we deduce the same result as in Theorem VIII.6.1 of [16], where the weight function is given by

$$w'(x') := (1+|x'|)(1+2|a|s'(x')), \qquad s'(x') := |x'| + x'_1.$$

In the original frame, they are transformed into

$$w(x) := w'(x') = (1+|x|)(1+2|a|s(x)), \qquad s(x) := s'(x') = |x| + e_1\cdot(Mx) = |x| + \frac{a}{|a|}\cdot x.$$

Finally, when $(a,b) = (0,0)$, the strategy of the proof of the corresponding result is the same as mentioned above, in which Lemma VIII.3.4 of [16] is employed for the Stokes potential representation. □

Remark 2.2. From Proposition 2.1 it follows that in the case $b\cdot a \neq 0$, as well as the classical case of $b = 0$ and $a \neq 0$, the "wake" behind the moving body can be described by (2.14), where (2.15)–(2.16) are unified as

$$s(x) = \left| x - \frac{b\times a}{|b|^2}\right| + \frac{\operatorname{sign}(b\cdot a)}{|b|}\, b\cdot x. \tag{2.20}$$

We note that $s(x) \geqslant 0$ in view of the second line of (2.15). In those cases above we deduce from the wake formula that for all $\varepsilon > 0$,

$$v \in L^{2+\varepsilon}(\Omega).$$

3. Formulation of the control problem. The adjoint systems

In this section, given the rigid body velocity $V = \xi + \omega\times x$, we propose two candidates of 6-dimensional subspace $\mathcal{C} = \mathcal{C}_{(\xi,\omega)}$ of $L^2(\partial\Omega)$, as mentioned in Section 1, from which we wish to take the boundary control v_*. For convenience, in what follows, we use the terminology "control space" for our space $\mathcal{C}$ as well.

Let us write (1.1)–(1.6) in the following form

$$-\operatorname{div}\sigma(v,p) - V\cdot\nabla v + \omega\times v = f(v) \quad \text{in } \Omega \tag{3.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \tag{3.2}$$

$$v = V + v_* \quad \text{on } \partial\Omega \tag{3.3}$$

$$\lim_{|x|\to\infty} v = 0 \tag{3.4}$$

$$-\int_{\partial\Omega} [\sigma(v,p)n + (V\cdot n)v]d\gamma = \kappa(v_*) \tag{3.5}$$

$$-\int_{\partial\Omega} x\times[\sigma(v,p)n + (V\cdot n)v]\,d\gamma = \mu(v_*) \tag{3.6}$$

where

$$f(v) := -v\cdot\nabla v = \operatorname{div} F(v), \qquad F(v) := -v\otimes v, \tag{3.7}$$

$$\kappa(v_*) := -\int_{\partial\Omega}(v_*\cdot n)(v_* + V + \omega\times x)d\gamma - m\xi\times\omega - \int_{\partial\Omega}(V\cdot n)(V+v_*)d\gamma, \tag{3.8}$$

$$\mu(v_*) := -\int_{\partial\Omega} x\times(v_* + V + \omega\times x)(v_*\cdot n)d\gamma - (I\omega)\times\omega - \int_{\partial\Omega} x\times(V+v_*)(V\cdot n)d\gamma. \tag{3.9}$$

Indeed the formulation (3.5) and (3.6) might look artificial, but it depends on how to develop the linear theory in the next section (there are actually some other possible ways). In order to define the control space for our problem, we consider six auxiliary adjoint problems, associated with six elementary rigid body motion velocities. For each $i \in \{1,2,3\}$, let $(v^{(i)}, q^{(i)})$ be the solution of the generalized Oseen problem

$$
\begin{aligned}
&-\operatorname{div}\sigma(v^{(i)},q^{(i)})+(\xi+\omega\times x)\cdot\nabla v^{(i)}-\omega\times v^{(i)}=0 \quad \text{in } \Omega\\
&\operatorname{div} v^{(i)}=0 \quad \text{in } \Omega\\
&v^{(i)}=e_i \quad \text{on } \partial\Omega\\
&\lim_{|x|\to\infty} v^{(i)}=0
\end{aligned}
\tag{3.10}
$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We also consider the solutions $(V^{(i)}, Q^{(i)})$ of

$$
\begin{aligned}
&-\operatorname{div}\sigma(V^{(i)},Q^{(i)})+(\xi+\omega\times x)\cdot\nabla V^{(i)}-\omega\times V^{(i)}=0 \quad \text{in } \Omega\\
&\operatorname{div} V^{(i)}=0 \quad \text{in } \Omega\\
&V^{(i)}=e_i\times x \quad \text{on } \partial\Omega\\
&\lim_{|x|\to\infty} V^{(i)}=0
\end{aligned}
\tag{3.11}
$$

for $i\in\{1,2,3\}$. The above problems are well-posed, as a direct consequence of Proposition 2.1. One can obtain even smoothness of the solutions by the regularity theory for the classical Stokes system.

Lemma 3.1. *There exist unique smooth solutions $(v^{(i)}, q^{(i)})$ and $(V^{(i)}, Q^{(i)})$ of systems (3.10) and (3.11), respectively. Moreover, for each $B>0$ there exists a constant $C=C(B)>0$ independent of ξ and ω with $|\xi|, |\omega|\in[0,B]$ such that for $i\in\{1,2,3\}$*

$$|v^{(i)}|_{2,2,\Omega}+|v^{(i)}|_{1,2,\Omega}+\lceil v^{(i)}\rceil_{1,w,\Omega}+\|q^{(i)}\|_{1,2,\Omega}\leqslant C, \tag{3.12}$$

$$|V^{(i)}|_{2,2,\Omega}+|V^{(i)}|_{1,2,\Omega}+\lceil V^{(i)}\rceil_{1,w,\Omega}+\|Q^{(i)}\|_{1,2,\Omega}\leqslant C. \tag{3.13}$$

Here, $\lceil\cdot\rceil_{1,w,\Omega}$ is given by (2.6) and w is defined by (2.3) for $(a,b)=(-\xi,-\omega)$.

Assuming $\partial\Omega\in C^3$, we define

$$g^{(i)}:=\sigma(v^{(i)},q^{(i)})n \quad \text{on } \partial\Omega, \tag{3.14}$$

$$G^{(i)}:=\sigma(V^{(i)},Q^{(i)})n \quad \text{on } \partial\Omega. \tag{3.15}$$

This choice is inspired by Galdi [14, Section 2], in which the Stokes system was adopted instead of (3.10) and (3.11). Note that $g^{(i)}$ and $G^{(i)}$ depend on ξ,ω differently from [14], however, we have the following estimate.

Lemma 3.2. *Assume $\partial\Omega$ is of class C^3. For each $B>0$ there exists a constant $C=C(B)>0$ independent of ξ and ω with $|\xi|, |\omega|\in[0,B]$ such that $g^{(i)}, G^{(i)}\in W^{3/2,2}(\partial\Omega)$ with*

$$\|g^{(i)}\|_{3/2,2,\partial\Omega}+\|G^{(i)}\|_{3/2,2,\partial\Omega}\leqslant C \tag{3.16}$$

for any $i\in\{1,2,3\}$.

Proof. We can use Theorem IV.5.1 in [16, p. 276] to obtain

$$
\begin{aligned}
&\|v^{(i)}\|_{3,2,\Omega_R}+\|q^{(i)}\|_{2,2,\Omega_R}\\
&\leqslant c\left(\left\|(\xi+\omega\times x)\cdot\nabla v^{(i)}-\omega\times v^{(i)}\right\|_{1,2,\Omega_{R+1}}+\|e_i\|_{5/2,2,\partial\Omega_R}+\|v^{(i)}\|_{1,2,\Omega_{R+1}}+\|q^{(i)}\|_{2,\Omega_{R+1}}\right),
\end{aligned}
$$

where $\Omega_R:=\Omega\cap B_R$ and $B_R:=\{x\in\mathbb{R}^3;\ |x|<R\}$. Applying Proposition 2.1, in which L^∞ estimate is also involved in (2.7) through $\lceil v\rceil_{1,w,\Omega}$, we deduce from (2.7) that if $|\xi|,\ |\omega|\in[0,B]$, then there exists a constant $C=C(B)$ such that

$$\|v^{(i)}\|_{3,2,\Omega_R}+\|q^{(i)}\|_{2,2,\Omega_R}\leqslant C.$$

Since $\sigma(v^{(i)},q^{(i)})\in W^{2,2}(\Omega_R)$, the trace theorem yields (3.16). □

The control spaces we are going to consider for v_* are

$$\mathcal{C}_\chi := \operatorname{span}\{\chi g^{(i)}, \chi G^{(i)}\ ;\ i = 1, 2, 3\}, \tag{3.17}$$

$$\mathcal{C}_\tau := \operatorname{span}\{(g^{(i)} \times n) \times n, (G^{(i)} \times n) \times n\ ;\ i = 1, 2, 3\}, \tag{3.18}$$

where χ is a smooth function such that $\chi \geqslant 0$, with support in Γ, a nonempty open subset of $\partial\Omega$, and $\chi > 0$ on a nonempty open subset Γ_0 of Γ. We will see in Lemma 4.3 and in Lemma 4.7 that if $|\xi|$ and $|\omega|$ are small enough, then $\mathcal{C}_\chi$ and $\mathcal{C}_\tau$ are of dimension 6 because their Gram matrices are nonsingular. The control problem can be now formulated in the following way: Given $\xi, \omega \in \mathbb{R}^3$, find $v_* \in \mathcal{C}_\chi$ or $\mathcal{C}_\tau$ and (v, p) in appropriate Sobolev spaces, satisfying (3.1)–(3.6) and (3.7)–(3.9).

4. Linearized version of the problem

4.1. Localized boundary controls

In this section, we solve the following control problem for the linearized system: Given $f = \operatorname{div} F \in L^2(\Omega)$ satisfying (2.4) with w defined by (2.3) for $(a, b) = (\xi, \omega)$ and given $(\kappa, \mu) \in \mathbb{R}^6$, find $(\alpha, \beta) \in \mathbb{R}^6$ with $\alpha = (\alpha_j)$, $\beta = (\beta_j)$ and (u, p) such that

$$-\operatorname{div}\sigma(u, p) - (\xi + \omega \times x) \cdot \nabla u + \omega \times u = f \quad \text{in } \Omega \tag{4.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \tag{4.2}$$

$$u = V + \sum_{j=1}^{3} \left(\alpha_j \chi g^{(j)} + \beta_j \chi G^{(j)}\right) \quad \text{on } \partial\Omega \tag{4.3}$$

$$\lim_{|x|\to\infty} u = 0 \tag{4.4}$$

$$-\int_{\partial\Omega} [\sigma(u, p)n + (V \cdot n)u]\, d\gamma = \kappa \tag{4.5}$$

$$-\int_{\partial\Omega} x \times [\sigma(u, p)n + (V \cdot n)u]\, d\gamma = \mu \tag{4.6}$$

where $V = \xi + \omega \times x$. Recall that $g^{(j)}$ and $G^{(j)}$ are defined by (3.14)–(3.15). The linear control problem will be solved with the aid of the following auxiliary systems

$$\begin{cases} -\operatorname{div}\sigma(u^{(j)}, p^{(j)}) - (\xi + \omega \times x) \cdot \nabla u^{(j)} + \omega \times u^{(j)} = 0 \\ \operatorname{div} u^{(j)} = 0 \\ u^{(j)} = \chi g^{(j)} \quad \text{on } \partial\Omega \\ \lim\limits_{|x|\to\infty} u^{(j)} = 0, \end{cases} \tag{4.7}$$

$$\begin{cases} -\operatorname{div}\sigma(U^{(j)}, P^{(j)}) - (\xi + \omega \times x) \cdot \nabla U^{(j)} + \omega \times U^{(j)} = 0 \\ \operatorname{div} U^{(j)} = 0 \\ U^{(j)} = \chi G^{(j)} \quad \text{on } \partial\Omega \\ \lim\limits_{|x|\to\infty} U^{(j)} = 0, \end{cases} \tag{4.8}$$

and

$$\begin{cases} -\operatorname{div}\sigma(u_f, p_f) - (\xi + \omega \times x) \cdot \nabla u_f + \omega \times u_f = f \\ \operatorname{div} u_f = 0 \\ u_f = V \quad \text{on } \partial\Omega \\ \lim\limits_{|x|\to\infty} u_f = 0. \end{cases} \tag{4.9}$$

As a consequence of Proposition 2.1 and Lemma 3.2, the above problems are well-posed.

Lemma 4.1. *Define s and w by (2.3) for $(a,b)=(\xi,\omega)$. Let $B>0$.*

1. *For each $j\in\{1,2,3\}$, there exist unique solutions $(u^{(j)},p^{(j)})$ and $(U^{(j)},P^{(j)})$ of (4.7) and (4.8), respectively. They are of class (2.5)–(2.6) and there exists a constant $C=C(B)>0$ independent of ξ and ω with $|\xi|,|\omega|\in[0,B]$ such that*

$$\begin{aligned}&|u^{(j)}|_{2,2,\Omega}+|u^{(j)}|_{1,2,\Omega}+\lceil u^{(j)}\rceil_{1,w,\Omega}+\|p^{(j)}\|_{1,2,\Omega}\leqslant C,\\&|U^{(j)}|_{2,2,\Omega}+|U^{(j)}|_{1,2,\Omega}+\lceil U^{(j)}\rceil_{1,w,\Omega}+\|P^{(j)}\|_{1,2,\Omega}\leqslant C.\end{aligned}\tag{4.10}$$

2. *Suppose $f=\operatorname{div}F\in L^2(\Omega)$ with (2.4). Then there exists a unique solution (u_f,p_f) of (4.9). It is of class (2.5)–(2.6) and there exists a constant $C=C(B)>0$ independent of ξ and ω with $|\xi|,|\omega|\in[0,B]$ such that*

$$|u_f|_{2,2,\Omega}+|u_f|_{1,2,\Omega}+\lceil u_f\rceil_{1,w,\Omega}+\|p_f\|_{1,2,\Omega}\leqslant C\left(\|f\|_{2,\Omega}+\lceil F\rceil_{2,w,\Omega}+\|V\|_{3/2,2,\partial\Omega}\right).\tag{4.11}$$

Suppose $f=\operatorname{div}F$, $f'=\operatorname{div}F'\in L^2(\Omega)$ with F, F' satisfying (2.4). Let (u_f,p_f) and $(u_{f'},p_{f'})$ be respectively the corresponding solutions. Then

$$\begin{aligned}&|u_f-u_{f'}|_{2,2,\Omega}+|u_f-u_{f'}|_{1,2,\Omega}+\lceil u_f-u_{f'}\rceil_{1,w,\Omega}+\|p_f-p_{f'}\|_{1,2,\Omega}\\&\leqslant C\left(\|f-f'\|_{2,\Omega}+\lceil F-F'\rceil_{2,w,\Omega}\right),\end{aligned}\tag{4.12}$$

where C is the same constant as in (4.11).

Hereupon, we can seek a solution of (4.1)–(4.6) in the form

$$u:=u_f+\sum_{j=1}^{3}\left(\alpha_j u^{(j)}+\beta_j U^{(j)}\right),\qquad p:=p_f+\sum_{j=1}^{3}\left(\alpha_j p^{(j)}+\beta_j P^{(j)}\right).\tag{4.13}$$

It is clear that (u,p) satisfies (4.1)–(4.4). Therefore, it is a solution of (4.1)–(4.6) if and only if (4.5)–(4.6) holds true. Inserting (4.13) into these two equations yields

$$\begin{aligned}&-\sum_{j=1}^{3}\alpha_j\int_{\partial\Omega}[\sigma(u^{(j)},p^{(j)})n+(V\cdot n)u^{(j)}]\cdot e_i\,d\gamma-\sum_{j=1}^{3}\beta_j\int_{\partial\Omega}[\sigma(U^{(j)},P^{(j)})n+(V\cdot n)U^{(j)}]\cdot e_i\,d\gamma\\&=\kappa\cdot e_i+\int_{\partial\Omega}[\sigma(u_f,p_f)n+(V\cdot n)V]\cdot e_i\,d\gamma\quad(i=1,2,3),\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&-\sum_{j=1}^{3}\alpha_j\int_{\partial\Omega}[\sigma(u^{(j)},p^{(j)})n+(V\cdot n)u^{(j)}]\cdot(e_i\times x)\,d\gamma\\&-\sum_{j=1}^{3}\beta_j\int_{\partial\Omega}[\sigma(U^{(j)},P^{(j)})n+(V\cdot n)U^{(j)}]\cdot(e_i\times x)\,d\gamma\\&=\mu\cdot e_i+\int_{\partial\Omega}[\sigma(u_f,p_f)n+(V\cdot n)V]\cdot(e_i\times x)\,d\gamma\quad(i=1,2,3),\end{aligned}\tag{4.15}$$

where $u_f=V$ on $\partial\Omega$ is taken into account in the right-hand sides of (4.14)–(4.15).

Lemma 4.2. *For $i, j = 1, 2, 3$, we have*

$$\begin{aligned}
&\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V\cdot n)u^{(j)}]\cdot e_i \, d\gamma = \int_{\partial\Omega} \chi g^{(i)}\cdot g^{(j)}\, d\gamma,\\
&\int_{\partial\Omega} [\sigma(U^{(j)}, P^{(j)})n + (V\cdot n)U^{(j)}]\cdot e_i \, d\gamma = \int_{\partial\Omega} \chi g^{(i)}\cdot G^{(j)}\, d\gamma,\\
&\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V\cdot n)u^{(j)}]\cdot (e_i\times x) \, d\gamma = \int_{\partial\Omega} \chi G^{(i)}\cdot g^{(j)}\, d\gamma,\\
&\int_{\partial\Omega} [\sigma(U^{(j)}, P^{(j)})n + (V\cdot n)U^{(j)}]\cdot (e_i\times x) \, d\gamma = \int_{\partial\Omega} \chi G^{(i)}\cdot G^{(j)}\, d\gamma.
\end{aligned} \tag{4.16}$$

Proof. We only prove the first identity of (4.16) for the case $\omega \neq 0$. The other formulae are proved in a similar way. When $\omega = 0$, we have only to replace the cut-off function $\psi_R(x)$ given by (4.17) below just by $\psi(|x|/R)$. Consider a "cut-off" function $\psi_R \in C_0^\infty(\overline{\Omega})$ ($R \gg \mathrm{diam}(\mathcal{S}) + \frac{|\omega\times\xi|}{|\omega|^2}$) defined by

$$\psi_R(x) := \psi\left(\frac{|x - \frac{\omega\times\xi}{|\omega|^2}|}{R}\right), \tag{4.17}$$

with $\psi \in C^\infty([0,\infty))$, a non-increasing real function, such that $\psi(t) = 1$, $t \in [0,1]$ and $\psi(t) = 0$, $t \geqslant 2$. We have the standard properties

$$\begin{aligned}
&0 \leqslant \psi_R(x) \leqslant 1, \quad \text{for all } x\in\Omega,\\
&\lim_{R\to\infty} \psi_R(x) = 1 \quad \text{for all } x\in\Omega,\\
&\|\nabla\psi_R\|_\infty \leqslant \frac{C}{R},
\end{aligned} \tag{4.18}$$

where C is a positive constant independent of x and R. Moreover

$$\left(\omega\times\left(x - \frac{\omega\times\xi}{|\omega|^2}\right)\right)\cdot\nabla\psi_R(x) = 0.$$

Note that

$$\omega\times\left(x - \frac{\omega\times\xi}{|\omega|^2}\right) = \xi + \omega\times x - \frac{\xi\cdot\omega}{|\omega|}\frac{\omega}{|\omega|} = V(x) - \frac{\xi\cdot\omega}{|\omega|}\frac{\omega}{|\omega|}, \text{ for all } x\in\Omega,$$

and therefore

$$V\cdot\nabla\psi_R = \frac{\xi\cdot\omega}{|\omega|}\frac{\omega}{|\omega|}\cdot\nabla\psi_R.$$

Using the properties of ψ_R listed in (4.18), we get

$$V\cdot\nabla\psi_R = 0, \text{ if } \xi\cdot\omega = 0 \tag{4.19}$$

$$\|V\cdot\nabla\psi_R\|_\infty \leqslant \frac{C|\xi|}{R}, \text{ if } \xi\cdot\omega \neq 0. \tag{4.20}$$

Moreover, the support of $\nabla\psi_R$ is contained in $R \leqslant |x - \frac{\omega\times\xi}{|\omega|^2}| \leqslant 2R$. When $\xi \neq 0$, $\xi\cdot\omega = 0$ as well as $\omega \neq 0$, we do not have better summability of $v^{(i)}$ and $u^{(j)}$ such as $L^{2+\varepsilon}$, see Remark 2.2. If we used simply $\psi(|x|/R)$ in this case instead of (4.17), it would not be easy to treat the last term of (4.23) below. This is the reason why we adopt (4.17) which yields (4.19).

Let us multiply the first equation of (4.7) by $\psi_R v^{(i)}$, where $v^{(i)}$ is the solution of (3.10), to obtain

$$-\int_\Omega \psi_R v^{(i)} \cdot \operatorname{div}\sigma(u^{(j)}, p^{(j)})\,dx - \int_\Omega \psi_R v^{(i)} \cdot \left[V\cdot\nabla u^{(j)} - \omega\times u^{(j)}\right]dx = 0, \tag{4.21}$$

and let us multiply the first equation of (3.10) by $\psi_R u^{(j)}$, where $u^{(j)}$ is the solution of (4.7):

$$-\int_\Omega \psi_R u^{(j)} \cdot \operatorname{div}\sigma(v^{(i)}, q^{(i)})\,dx + \int_\Omega \psi_R u^{(j)} \cdot \left[V\cdot\nabla v^{(i)} - \omega\times v^{(i)}\right]dx = 0. \tag{4.22}$$

Integrating (4.21) and (4.22) by parts and combining them, we obtain

$$\begin{aligned}
&-\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V\cdot n)u^{(j)}]\cdot(\psi_R v^{(i)})\,d\gamma + \int_{\partial\Omega} \big(\sigma(v^{(i)}, q^{(i)})n\big)\cdot(\psi_R u^{(j)})\,d\gamma \\
&= \int_\Omega \Big[-\big(\nabla\psi_R\cdot\nabla u^{(j)}\big)\cdot v^{(i)} + \big(\nabla\psi_R\cdot\nabla v^{(i)}\big)\cdot u^{(j)} - \big(v^{(i)}\cdot\nabla u^{(j)}\big)\cdot\nabla\psi_R + \big(u^{(j)}\cdot\nabla v^{(i)}\big)\cdot\nabla\psi_R\Big]dx \\
&\quad + \int_\Omega \big(p^{(j)} v^{(i)}\cdot\nabla\psi_R - q^{(i)} u^{(j)}\cdot\nabla\psi_R\big)\,dx - \int_\Omega V\cdot\nabla\psi_R\big(v^{(i)}\cdot u^{(j)}\big)\,dx.
\end{aligned} \tag{4.23}$$

We recall that from Lemma 3.1 and Lemma 4.1,

$$|v^{(i)}(x)| + |u^{(j)}(x)| \leqslant \frac{C}{w(x)} \leqslant \frac{C}{1+|x|}, \qquad \forall x\in\Omega,$$

where w is defined by (2.3) with $(a,b) = (\xi,\omega)$. If $\omega\cdot\xi \neq 0$, we have even better summability

$$v^{(i)}, u^{(j)} \in L^{2+\varepsilon}(\Omega), \quad \forall\varepsilon > 0, \tag{4.24}$$

see Remark 2.2. Using the above properties of the functions $u^{(j)}$ and $v^{(i)}$ and (4.18) for $\nabla\psi_R$, we get

$$\begin{aligned}
\left|\int_\Omega \big(\nabla\psi_R\cdot\nabla u^{(j)}\big)\cdot v^{(i)}\,dx\right| &\leqslant \frac{C}{R}\|\nabla u^{(j)}\|_{2,\Omega}\left(\int_{R<|x-\frac{\omega\times\xi}{|\omega|^2}|<2R} \frac{1}{(1+|x|)^2}dx\right)^{\frac12} \\
&\leqslant C\|\nabla u^{(j)}\|_{2,\Omega} R^{-1/2}.
\end{aligned}$$

In a similar way

$$\begin{aligned}
&\left|\int_\Omega q^{(i)}u^{(j)}\cdot\nabla\psi_R\,dx\right| + \left|\int_\Omega p^{(j)}v^{(i)}\cdot\nabla\psi_R\,dx\right| + \left|\int_\Omega \big(\nabla\psi_R\cdot\nabla v^{(i)}\big)\cdot u^{(j)}\,dx\right| \\
&\quad + \left|\int_\Omega \big(v^{(i)}\cdot\nabla u^{(j)}\big)\cdot\nabla\psi_R\,dx\right| + \left|\int_\Omega \big(u^{(j)}\cdot\nabla v^{(i)}\big)\cdot\nabla\psi_R\,dx\right| \leqslant CR^{-1/2}.
\end{aligned}$$

Finally, if $\omega\cdot\xi = 0$ then, from (4.19) it follows

$$\int_\Omega V\cdot\nabla\psi_R\left(v^{(i)}\cdot u^{(j)}\right)dx = 0,$$

else estimates (4.20) and (4.24) imply

$$\left|\int_\Omega V\cdot\nabla\psi_R\left(v^{(i)}\cdot u^{(j)}\right)\,dx\right| \leqslant \frac{C|\xi|}{R}\int_{R<|x-\frac{\omega\times\xi}{|\omega|^2}|<2R}\left|v^{(i)}\cdot u^{(j)}\right|\,dx$$
$$\leqslant \frac{C|\xi|}{R}\left\|v^{(i)}\right\|_{2+\varepsilon,\Omega}\left\|u^{(j)}\right\|_{2+\varepsilon,\Omega}R^{\frac{3\varepsilon}{2+\varepsilon}}.$$

Letting $R\to\infty$ in (4.23) yields the first identity of (4.16). □

The identities established in Lemma 4.2 allow us to write (4.14), (4.15) in the form

$$A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\delta\\ \eta\end{bmatrix}, \tag{4.25}$$

where

$$\delta_i := -\kappa\cdot e_i-\int_{\partial\Omega}[\sigma(u_f,p_f)n+(V\cdot n)V]\cdot e_i\,d\gamma \quad (i=1,2,3), \tag{4.26}$$

$$\eta_i := -\mu\cdot e_i-\int_{\partial\Omega}[\sigma(u_f,p_f)n+(V\cdot n)V]\cdot (e_i\times x)\,d\gamma \quad (i=1,2,3), \tag{4.27}$$

and $A=(A_{i,j})\in\mathbb{R}^{6\times 6}$ is defined by

$$\begin{aligned}
A_{i,j}&=\int_{\partial\Omega}\chi g^{(i)}\cdot g^{(j)}\,d\gamma \qquad (i,j\leqslant 3),\\
A_{i,j}&=\int_{\partial\Omega}\chi g^{(i)}\cdot G^{(j-3)}\,d\gamma \qquad (i\leqslant 3,\ j\geqslant 4),\\
A_{i,j}&=\int_{\partial\Omega}\chi G^{(i-3)}\cdot g^{(j)}\,d\gamma \qquad (i\geqslant 4,\ j\leqslant 3),\\
A_{i,j}&=\int_{\partial\Omega}\chi G^{(i-3)}\cdot G^{(j-3)}\,d\gamma \qquad (i,j\geqslant 4).
\end{aligned} \tag{4.28}$$

Lemma 4.3. *The matrix A defined by (4.28) is symmetric nonnegative. Furthermore, there exist positive constants c_1, K such that if*

$$|\xi|\leqslant c_1, \qquad |\omega|\leqslant c_1, \tag{4.29}$$

then A is invertible with

$$\|A^{-1}\|\leqslant K, \tag{4.30}$$

where $\|\cdot\|=\|\cdot\|_{\mathcal{L}(\mathbb{R}^6)}$, and K is independent of ξ,ω with $|\xi|$, $|\omega|\leqslant c_1$.

Remark 4.4. The matrix A is the Gram matrix of the family $\left\{\sqrt{\chi}g^{(i)},\ \sqrt{\chi}G^{(i)}\ ;\ i=1,2,3\right\}$. Thus, since A is invertible for (ξ,ω) with (4.29), it yields that the family is linearly independent. This fact implies that the family $\mathcal{C}_\chi$ defined by (3.17) is linearly independent.

Proof of Lemma 4.3. Let $(\alpha,\beta)\in\mathbb{R}^6$ with $\alpha=(\alpha_i)$, $\beta=(\beta_i)$ and set

$$(v,q):=\sum_{i=1}^{3}\alpha_i(v^{(i)},q^{(i)})+\sum_{i=1}^{3}\beta_i(V^{(i)},Q^{(i)}).$$

From (3.10)–(3.11), (3.14)–(3.15) and (4.28) it follows that this pair is a smooth solution to

$$\begin{aligned} &-\operatorname{div}\sigma(v,q) + (\xi + \omega\times x)\cdot\nabla v - \omega\times v = 0 \\ &\operatorname{div} v = 0 \\ &v = \alpha + \beta\times x \quad \text{on } \partial\Omega \\ &\lim_{|x|\to\infty} v = 0 \end{aligned} \tag{4.31}$$

and that

$$A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\cdot\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \int_{\partial\Omega} \chi\,|\sigma(v,q)n|^2\,d\gamma \geqslant 0, \qquad \forall(\alpha,\beta)\in\mathbb{R}^6. \tag{4.32}$$

Let us now show that A is invertible in the case $(\xi,\omega)=(0,0)$. If

$$A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\cdot\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = 0$$

then, from (4.32) it follows that v satisfies

$$\begin{aligned} &-\operatorname{div}\sigma(v,q) = 0 \\ &\operatorname{div} v = 0 \\ &v = \alpha + \beta\times x \quad \text{on } \partial\Omega \\ &\lim_{|x|\to\infty} v = 0 \\ &\sigma(v,q)n = 0 \quad \text{on } \Gamma_0\subset\partial\Omega, \end{aligned}$$

where χ is assumed to be positive on a nonempty open subset Γ_0 of $\partial\Omega$. The following procedure is classical, see San Martín, Takahashi and Tucsnak [24, Lemma 4.1]. We consider $\widetilde{v}(x) := v(x) - (\alpha + \beta\times x)$, which is a weak solution of

$$\begin{aligned} &-\operatorname{div}\sigma(\widetilde{v},q) = 0 \quad \text{in } \Omega \\ &\operatorname{div}\widetilde{v} = 0 \quad \text{in } \Omega \\ &\widetilde{v} = 0 \quad \text{on } \partial\Omega \\ &\sigma(\widetilde{v},q)n = 0 \quad \text{on } \Gamma_0\subset\partial\Omega. \end{aligned} \tag{4.33}$$

Indeed $\widetilde{v}$ is growing for $|x|\to\infty$, but the argument below works well no matter how $\widetilde{v}$ behaves at infinity. We can extend Ω by adding a small open subset $\mathcal{E}\subset\mathcal{S}$ (with respect to the induced topology) such that $\mathcal{E}\cap\partial\mathcal{S}\ (\neq\emptyset)$ is strictly included into Γ_0 and that in the domain $\widetilde{\Omega} := \Omega\cup\mathcal{E}$ the function $\widetilde{v}$ is a weak solution of

$$\begin{aligned} &-\operatorname{div}\sigma(\widetilde{v},q) = 0 \quad \text{in } \widetilde{\Omega} \\ &\operatorname{div}\widetilde{v} = 0 \quad \text{in } \widetilde{\Omega} \\ &\widetilde{v} = 0 \quad \text{in } \mathcal{E} \end{aligned} \tag{4.34}$$

where $(\widetilde{v},q)$ is understood as extension to $\widetilde{\Omega}$ by setting zero outside Ω. Using the unique continuation property for the Stokes system due to [4], we deduce that $\widetilde{v}=0$ in Ω and thus $v(x)=\alpha+\beta\times x$ for $x\in\Omega$. Since $\lim_{|x|\to\infty} v(x) = 0$ we conclude that $\alpha=\beta=0$. This implies that A is definite positive for $(\xi,\omega)=(0,0)$.

To prove that A is invertible and that (4.30) holds true, we show that the mapping

$$(\xi,\omega)\mapsto A = A_{(\xi,\omega)} \tag{4.35}$$

is continuous at $(0,0)$. Consider a sequence

$$\lim_{k\to\infty}(\xi_k,\omega_k) = (0,0) \tag{4.36}$$

and denote by $(v_k^{(i)}, q_k^{(i)})$ and $(V_k^{(i)}, Q_k^{(i)})$ the solutions of systems (3.10) and (3.11), respectively, associated with (ξ_k, ω_k). From Lemma 3.1, we have that

$$|v_k^{(i)}|_{2,2,\Omega} + |v_k^{(i)}|_{1,2,\Omega} + \|q_k^{(i)}\|_{1,2,\Omega} \leqslant C,$$
$$|V_k^{(i)}|_{2,2,\Omega} + |V_k^{(i)}|_{1,2,\Omega} + \|Q_k^{(i)}\|_{1,2,\Omega} \leqslant C.$$

Since $v_k^{(i)}$ and $V_k^{(i)}$ tend to zero as $|x| \to \infty$, from a classical embedding inequality we also deduce (see, for instance, [16, Theorem II.6.1])

$$\|v_k^{(i)}\|_{6,\Omega} \leqslant C, \quad \|V_k^{(i)}\|_{6,\Omega} \leqslant C.$$

As a consequence, we find a pair $(v^{(i)}, q^{(i)})$ such that, up to a subsequence,

$$\begin{aligned} & v_k^{(i)} \to v^{(i)} \quad \text{in } L^2_{loc}(\overline{\Omega}), \\ & \left(\nabla v_k^{(i)}, q_k^{(i)}\right) \rightharpoonup \left(\nabla v^{(i)}, q^{(i)}\right) \quad \text{weakly in } W^{1,2}(\Omega), \\ & v_k^{(i)} \rightharpoonup v^{(i)} \quad \text{weakly in } L^6(\Omega). \end{aligned} \tag{4.37}$$

Using (4.37) together with (4.36), we deduce that $(v^{(i)}, q^{(i)})$ is a weak solution (see for instance [16, Definition V.1.1] but eventually a smooth solution) of

$$\begin{aligned} & -\operatorname{div}\sigma(v^{(i)}, q^{(i)}) = 0 \quad \text{in } \Omega \\ & \operatorname{div} v^{(i)} = 0 \quad \text{in } \Omega \\ & v^{(i)} = e_i \quad \text{on } \partial\Omega \\ & \lim_{|x|\to\infty} v^{(i)} = 0, \end{aligned} \tag{4.38}$$

where the boundary condition on $\partial\Omega$ follows from the trace estimate

$$\|v_k^{(i)} - v^{(i)}\|_{2,\partial\Omega} \leqslant C\|v_k^{(i)} - v^{(i)}\|_{2,\Omega_R}^{1/2}\|v_k^{(i)} - v^{(i)}\|_{1,2,\Omega_R}^{1/2} \to 0 \quad (k \to \infty), \qquad \Omega_R = \Omega \cap B_R,$$

while the boundary condition at infinity is satisfied in L^6-sense and even pointwise, see [16, Theorem V.3.1]. By uniqueness of solutions to (4.38) [16, Theorem V.3.4], (4.37) holds for the whole sequence $(v_k^{(i)}, q_k^{(i)})$ (not only for a subsequence). Using classical compactness results, we also deduce from (4.37) that $(\nabla v_k^{(i)}, q_k^{(i)}) \to (\nabla v^{(i)}, q^{(i)})$ strongly in $L^2_{loc}(\overline{\Omega})$. This together with the trace estimate implies that

$$\begin{aligned} & \|\sigma(v_k^{(i)}, q_k^{(i)})n - \sigma(v^{(i)}, q^{(i)})n\|_{2,\partial\Omega} \\ & \leqslant C\|\nabla v_k^{(i)} - \nabla v^{(i)}\|_{2,\Omega_R}^{1/2}\|\nabla v_k^{(i)} - \nabla v^{(i)}\|_{1,2,\Omega_R}^{1/2} \\ & \quad + C\|q_k^{(i)} - q^{(i)}\|_{2,\Omega_R}^{1/2}\|q_k^{(i)} - q^{(i)}\|_{1,2,\Omega_R}^{1/2} \to 0 \quad (k \to \infty). \end{aligned} \tag{4.39}$$

By the same reasoning, we also have

$$\|\sigma(V_k^{(i)}, Q_k^{(i)})n - \sigma(V^{(i)}, Q^{(i)})n\|_{2,\partial\Omega} \to 0 \quad (k \to \infty), \tag{4.40}$$

where $\{V^{(i)}, Q^{(i)}\}$ denotes the solution to (3.11) with $(\xi, \omega) = (0, 0)$. In view of (3.14)–(3.15) and (4.28), the convergence (4.39) and (4.40) yield the continuity at $(0, 0)$ of the mapping defined by (4.35). Now it follows from the argument by use of the Neumann series that if

$$\|A_{(\xi,\omega)} - A_{(0,0)}\| \leqslant \frac{1}{2\|A_{(0,0)}^{-1}\|}$$

then $A_{(\xi,\omega)}$ is invertible and $\|A_{(\xi,\omega)}^{-1}\| \leqslant 2\|A_{(0,0)}^{-1}\|$. Hence, the condition (4.29) with sufficiently small c_1 implies (4.30). □

We are now in a position to give a result on solvability of the control problem for the linearized system in which the control v_* is taken from $\mathcal{C}_\chi$ defined by (3.17), that is,

$$v_* = \sum_{j=1}^{3} \left(\alpha_j \chi g^{(j)} + \beta_j \chi G^{(j)} \right).$$

Proposition 4.5. *Suppose $(\xi, \omega) \in \mathbb{R}^6$ satisfies (4.29). Let w be the function defined by (2.3) for $(a, b) = (\xi, \omega)$.*

1. *Given $f = \operatorname{div} F \in L^2(\Omega)$ with (2.4) and given $(\kappa, \mu) \in \mathbb{R}^6$, problem (4.1)–(4.6) admits a unique solution (α, β, u, p) of class*
$$(\alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3, \qquad u \in D^{2,2}(\Omega) \cap D^{1,2}(\Omega) \cap L^2_{loc}(\overline{\Omega}), \qquad wu \in L^\infty(\Omega), \qquad p \in W^{1,2}(\Omega) \tag{4.41}$$
subject to
$$\begin{aligned} &|(\alpha, \beta)| + |u|_{2,2,\Omega} + |u|_{1,2,\Omega} + \lceil u \rceil_{1,w,\Omega} + \|p\|_{1,2,\Omega} \\ &\leqslant C\big(|(\kappa, \mu)| + \|f\|_{2,\Omega} + \lceil F \rceil_{2,w,\Omega} + |(\xi, \omega)| + |(\xi, \omega)|^2\big), \end{aligned} \tag{4.42}$$
where the constant $C > 0$ is independent of ξ, ω, κ, μ and f.
2. *Let $f = \operatorname{div} F$, $f' = \operatorname{div} F' \in L^2(\Omega)$ with F, F' satisfying (2.4), and let $(\kappa, \mu), (\kappa', \mu') \in \mathbb{R}^6$. Then the solutions (α, β, u, p) and $(\alpha', \beta', u', p')$ obtained above fulfill*
$$\begin{aligned} &|(\alpha - \alpha', \beta - \beta')| + |u - u'|_{2,2,\Omega} + |u - u'|_{1,2,\Omega} + \lceil u - u' \rceil_{1,w,\Omega} + \|p - p'\|_{1,2,\Omega} \\ &\leqslant C\big(|(\kappa - \kappa', \mu - \mu')| + \|f - f'\|_{2,\Omega} + \lceil F - F' \rceil_{2,w,\Omega}\big), \end{aligned} \tag{4.43}$$
where C is the same constant as in (4.42).

Proof. From Lemma 4.3 and (4.25)–(4.27) we deduce that, under the smallness condition (4.29), there exists a unique (α, β) such that (4.1)–(4.6) holds with (u, p) of the form (4.13). By the trace estimate we have

$$|(\alpha, \beta)| \leqslant K|(\delta, \eta)| \leqslant C \left(|(\kappa, \mu)| + \|(\nabla u_f, p_f)\|_{1,2,\Omega} + \|V\|^2_{2,\partial\Omega} \right),$$

which together with (4.11) implies that

$$|(\alpha, \beta)| \leqslant C \left(|(\kappa, \mu)| + \|f\|_{2,\Omega} + \lceil F \rceil_{2,w,\Omega} + |(\xi, \omega)| + |(\xi, \omega)|^2 \right), \tag{4.44}$$

where we have used $\|V\|_{3/2,2,\partial\Omega} = \|\xi + \omega \times x\|_{3/2,2,\partial\Omega} \leqslant C|(\xi, \omega)|$. In view of (4.13) we collect (4.10), (4.11) and (4.44) to obtain (4.42). Concerning the difference between two solutions, we use (4.12) in place of (4.11) to get (4.43). □

4.2. Tangential boundary controls

In this case, our aim is to use a control v_* that is tangential on $\partial\Omega$. More precisely, the control space is (3.18), where $g^{(i)}$ and $G^{(i)}$ are defined by (3.14) and (3.15). We replace (4.3) by

$$u = V + \sum_{j=1}^{3} \left\{ \alpha_j (g^{(j)} \times n) \times n + \beta_j (G^{(j)} \times n) \times n \right\} \quad \text{on } \partial\Omega, \tag{4.45}$$

and, accordingly, $(u^{(j)}, p^{(j)})$ and $(U^{(j)}, P^{(j)})$ are respectively solutions (as in the first assertion of Lemma 4.1) to

$$\begin{cases} -\operatorname{div} \sigma(u^{(j)}, p^{(j)}) - (\xi + \omega \times x) \cdot \nabla u^{(j)} + \omega \times u^{(j)} = 0 \\ \operatorname{div} u^{(j)} = 0 \\ u^{(j)} = (g^{(j)} \times n) \times n \quad \text{on } \partial\Omega \\ \lim_{|x| \to \infty} u^{(j)} = 0, \end{cases} \tag{4.46}$$

$$\begin{cases} -\operatorname{div}\sigma(U^{(j)}, P^{(j)}) - (\xi + \omega \times x) \cdot \nabla U^{(j)} + \omega \times U^{(j)} = 0 \\ \operatorname{div} U^{(j)} = 0 \\ U^{(j)} = (G^{(j)} \times n) \times n \quad \text{on } \partial\Omega \\ \lim_{|x|\to\infty} U^{(j)} = 0, \end{cases} \tag{4.47}$$

in place of (4.7)–(4.8).

We look for a solution of the form (4.13) and we arrive as in the previous subsection at system (4.14)–(4.15). Lemma 4.2 is then transformed into

Lemma 4.6. *For $i, j = 1, 2, 3$, we have*

$$\begin{aligned} &\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V \cdot n)u^{(j)}] \cdot e_i \, d\gamma = -\int_{\partial\Omega} \Big[(g^{(i)} \times n) \times n\Big] \cdot \Big[(g^{(j)} \times n) \times n\Big] \, d\gamma, \\ &\int_{\partial\Omega} [\sigma(U^{(j)}, P^{(j)})n + (V \cdot n)U^{(j)}] \cdot e_i \, d\gamma = -\int_{\partial\Omega} \Big[(g^{(i)} \times n) \times n\Big] \cdot \Big[(G^{(j)} \times n) \times n\Big] \, d\gamma, \\ &\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V \cdot n)u^{(j)}] \cdot (e_i \times x) \, d\gamma = -\int_{\partial\Omega} \Big[(G^{(i)} \times n) \times n\Big] \cdot \Big[(g^{(j)} \times n) \times n\Big] \, d\gamma, \\ &\int_{\partial\Omega} [\sigma(U^{(j)}, P^{(j)})n + (V \cdot n)U^{(j)}] \cdot (e_i \times x) \, d\gamma = -\int_{\partial\Omega} \Big[(G^{(i)} \times n) \times n\Big] \cdot \Big[(G^{(j)} \times n) \times n\Big] \, d\gamma. \end{aligned} \tag{4.48}$$

Proof. The proof of (4.23) is exactly the same as in the proof of Lemma 4.2. This time, letting $R \to \infty$ in (4.23) yields

$$\int_{\partial\Omega} [\sigma(u^{(j)}, p^{(j)})n + (V \cdot n)u^{(j)}] \cdot e_i \, d\gamma = \int_{\partial\Omega} g^{(i)} \cdot [(g^{(j)} \times n) \times n] \, d\gamma.$$

Using the relation

$$g^{(i)} = (n \cdot g^{(i)})n + (n \times g^{(i)}) \times n,$$

we find the first identity of (4.48). The other formulae can be verified similarly. □

We are thus led to the linear system (4.25) with the matrix $A = (A_{i,j}) \in \mathbb{R}^{6\times 6}$ defined by

$$\begin{aligned} A_{i,j} &= -\int_{\partial\Omega} \Big[(g^{(i)} \times n) \times n\Big] \cdot \Big[(g^{(j)} \times n) \times n\Big] \, d\gamma \qquad (i, j \leqslant 3), \\ A_{i,j} &= -\int_{\partial\Omega} \Big[(g^{(i)} \times n) \times n\Big] \cdot \Big[(G^{(j-3)} \times n) \times n\Big] \, d\gamma \qquad (i \leqslant 3, j \geqslant 4), \\ A_{i,j} &= -\int_{\partial\Omega} \Big[(G^{(i-3)} \times n) \times n\Big] \cdot \Big[(g^{(j)} \times n) \times n\Big] \, d\gamma \qquad (i \geqslant 4, j \leqslant 3), \\ A_{i,j} &= -\int_{\partial\Omega} \Big[(G^{(i-3)} \times n) \times n\Big] \cdot \Big[(G^{(j-3)} \times n) \times n\Big] \, d\gamma \qquad (i, j \geqslant 4). \end{aligned} \tag{4.49}$$

Lemma 4.7. *The matrix A defined by (4.49) is symmetric nonpositive. Furthermore, there exist positive constants c_1', K' such that if (4.29) holds true with c_1 replaced by c_1', then A is invertible with*

$$\|A^{-1}\| \leqslant K', \tag{4.50}$$

where $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\mathbb{R}^6)}$, and K' is independent of ξ, ω with $|\xi|, |\omega| \leqslant c_1'$.

Proof. The invertibility of $A_{(0,0)}$ is proved by Galdi [14, Lemma 2.1]. In view of (4.49) with (3.14)–(3.15), the convergence (4.39) and (4.40) tell us that the mapping (4.35) is continuous at $(0,0)$. As in the proof of Lemma 4.3, we obtain the assertion. □

Now it is obvious that Proposition 4.5 holds true for the control problem (4.1), (4.2), (4.45), (4.4)–(4.6) as well. Since the statement is exactly the same, we do not repeat it.

5. Solution of the nonlinear control problem

In this section we combine the formulation of the control problem given in Section 3 with Proposition 4.5 to prove Theorem 1.1. Let us set

$$\mathcal{X} := \Big\{(v,\alpha,\beta) \in D^{1,2}(\Omega)\times\mathbb{R}^3\times\mathbb{R}^3\ ;\ \lceil v\rceil_{1,w,\Omega} < \infty\Big\}$$

endowed with the norm

$$\|(v,\alpha,\beta)\|_{\mathcal{X}} := |v|_{1,2,\Omega} + \lceil v\rceil_{1,w,\Omega} + |(\alpha,\beta)|.$$

Note that the space $\mathcal{X}$ depends on ξ and ω through the weight function $w(x)$ defined by (2.3) for $(a,b)=(\xi,\omega)$. It is a Banach space with the norm defined above.

Proof of Theorem 1.1. In order to solve (3.1)–(3.6) with (3.7)–(3.9), we intend to find a fixed point of the map $\mathcal{Z}$ defined below. Given $(v,\alpha,\beta)\in\mathcal{X}$, we consider $f(v)=\operatorname{div} F(v)$, $\kappa(v_*)$ and $\mu(v_*)$, given respectively by (3.7)–(3.9), with

$$v_* = \sum_{j=1}^{3}\Big(\alpha_j\chi g^{(j)} + \beta_j\chi G^{(j)}\Big) \in \mathcal{C}_\chi, \tag{5.1}$$

or

$$v_* = \sum_{j=1}^{3}\Big\{\alpha_j(g^{(j)}\times n)\times n + \beta_j(G^{(j)}\times n)\times n\Big\} \in \mathcal{C}_\tau. \tag{5.2}$$

Since

$$\|f(v)\|_{2,\Omega} + \lceil F(v)\rceil_{2,w,\Omega} \leqslant \|v\|_{\infty,\Omega}\|\nabla v\|_{2,\Omega} + \|wv\|_{\infty,\Omega}^2 \leqslant \big(|v|_{1,2,\Omega} + \lceil v\rceil_{1,w,\Omega}\big)\lceil v\rceil_{1,w,\Omega}, \tag{5.3}$$

and since (3.16) yields

$$\Big|\big(\kappa(v_*),\mu(v_*)\big)\Big| \leqslant C|(\alpha,\beta)|^2 + C|(\xi,\omega)|^2, \tag{5.4}$$

where $C=C(B)>0$ is independent of ξ,ω with $|\xi|,|\omega|\in[0,B]$, one can apply Proposition 4.5 to $f=f(v)$, $\kappa=\kappa(v_*)$ and $\mu=\mu(v_*)$ under the condition $|\xi|,|\omega|\in[0,c_1]$, see (4.29). By $\mathcal{Z}(v,\alpha,\beta)$ we denote the solution obtained by Proposition 4.5. Combining (4.42) with (5.3)–(5.4), we find

$$\|\mathcal{Z}(v,\alpha,\beta)\|_{\mathcal{X}} \leqslant C_2\big(|(\xi,\omega)| + |(\xi,\omega)|^2\big) + C_3\|(v,\alpha,\beta)\|_{\mathcal{X}}^2 \tag{5.5}$$

with some constants $C_2, C_3>0$. Suppose

$$|(\xi,\omega)| + |(\xi,\omega)|^2 \leqslant \frac{1}{4C_2C_3} \tag{5.6}$$

and set

$$L := 2C_2\big(|(\xi,\omega)| + |(\xi,\omega)|^2\big). \tag{5.7}$$

Then it easily follows from (5.5) that $\|(v,\alpha,\beta)\|_{\mathcal{X}}\leqslant L$ implies $\|\mathcal{Z}(f,\alpha,\beta)\|_{\mathcal{X}}\leqslant L$.

We next show that the map $\mathcal{Z}$ is contractive from this closed ball

$$\mathcal{X}_L := \{(v,\alpha,\beta)\in\mathcal{X};\ \|(v,\alpha,\beta)\|_{\mathcal{X}}\leqslant L\}$$

into $\mathcal{X}_L$. Let $(v, \alpha, \beta), (v', \alpha', \beta') \in \mathcal{X}_L$. Then we have

$$\begin{aligned}&\|f(v) - f(v')\|_{2,\Omega} + \lceil F(v) - F(v')\rceil_{2,w,\Omega}\\ &\leqslant \big(|v|_{1,2,\Omega} + \lceil v\rceil_{1,w,\Omega} + \lceil v'\rceil_{1,w,\Omega}\big)\lceil v - v'\rceil_{1,w,\Omega} + \lceil v'\rceil_{1,w,\Omega}|v - v'|_{1,2,\Omega}\end{aligned} \tag{5.8}$$

$$\big|\big(\kappa(v_*) - \kappa(v_*'), \mu(v_*) - \mu(v_*')\big)\big| \leqslant C\big(|(\alpha,\beta)| + |(\alpha',\beta')| + |(\xi,\omega)|\big)|(\alpha-\alpha', \beta-\beta')| \tag{5.9}$$

where v_* and v_*' denote the control functions given by (5.1) or (5.2) with (α, β) and (α', β'), respectively. We then combine (4.43) with (5.8)–(5.9) to deduce

$$\|\mathcal{Z}(v,\alpha,\beta) - \mathcal{Z}(v',\alpha',\beta')\|_{\mathcal{X}} \leqslant C_4 L\|(v,\alpha,\beta) - (v',\alpha',\beta')\|_{\mathcal{X}}.$$

Let us take $c_0 > 0$ so small that (1.7) implies not only (4.29), (5.6) but also

$$C_4 L = 2C_2C_4\big(|(\xi,\omega)| + |(\xi,\omega)|^2\big) < 1,$$

see (5.7). Then the map $\mathcal{Z}$ admits a unique fixed point $(v, \alpha, \beta) \in \mathcal{X}_L$, which together with (5.1) or (5.2) provides the desired solution. By Proposition 4.5 we know that $(v, \alpha, \beta) = \mathcal{Z}(v, \alpha, \beta)$ and the associated pressure p belong to the additional class (4.41). We gather $\|(v, \alpha, \beta)\|_{\mathcal{X}} \leqslant L$ with (5.7), Lemma 3.2, (4.42), (5.3) and (5.4) to obtain (1.10). Finally, the interior regularity theory for the classical Stokes system [16, Theorem IV.4.1] and the bootstrap argument lead us to $(v, p) \in C^\infty(\Omega)$. We have thus completed the proof. □

6. On the asymptotic behavior and summability of solutions

In this section we will discuss the asymptotic behavior of solutions at infinity for (1.1)–(1.2), independently of Theorem 1.1, without assuming any boundary condition on $\partial\Omega$ when they enjoy $N = 0$ or $\omega \cdot N = 0$, see (1.13). And then, as an application, it is at once shown that the self-propelled condition (1.5) implies faster decay of solutions constructed in Theorem 1.1. Our starting point is that a solution to (1.1)–(1.2) (only these two equations) with

$$(\nabla v, p) \in L^2(\Omega), \qquad (1 + |x|)\, v \in L^\infty(\Omega) \tag{6.1}$$

is given. Then, as in the end of the proof of Theorem 1.1 given by the previous section, the regularity theory for the Stokes system yields

$$(v, p) \in C^\infty(\Omega). \tag{6.2}$$

As explained in Section 1, Theorem 1.2 for the case $\omega = 0$ is completely covered by previous literature and so, in what follows, we will concentrate ourselves on the other case $\omega \neq 0$.

As is standard, the proof of Theorem 1.2 is done by cut-off procedure after subtracting the flux carrier, see (6.24) below. In order to recover the solenoidal condition, we need a correction term, whose support can be compact because the total flux through $\partial\Omega$ vanishes by this subtraction. We then analyze the whole space problem and the point is that the information about the net force N goes to the external force of the equation of momentum, see (6.32) with (6.34) below. We follow in principle the argument developed by [8,6,23] (in which no-slip boundary condition is imposed) and two cases $\omega \cdot \xi \neq 0$ and $\omega \cdot \xi = 0$ are discussed independently as we will soon describe.

Unlike the case of no-slip boundary condition, the flux carrier mentioned above brings the external force with noncompact support in the whole space problem. For the case $\omega \cdot \xi \neq 0$, in spite of this change, we will make it clear how the argument of Kyed [23] still works. The reason why his argument does not work for the other case $\omega \cdot \xi = 0$ is that the following claim is no longer true if we replace the Oseen fundamental solution $\mathcal{E}_{Os}(x)$ by the Stokes fundamental solution $\mathcal{E}_{St}(x)$, see (6.5) and (6.6):

$$\mathcal{E}_{Os} * \operatorname{div}(\widetilde{u} \otimes \widetilde{u}) \in L^r, \qquad \forall r \in (4/3, \infty) \tag{6.3}$$

under the condition (6.1), where $\widetilde{u}$ is a suitable modification of v by (6.27), (6.31) and (6.35) below. The only thing we can obtain for the case $\omega \cdot \xi = 0$ is that $\mathcal{E}_{St} * \operatorname{div}(\widetilde{u} \otimes \widetilde{u}) \in L^r$ for all $r \in (3, \infty)$ under the condition (6.1); indeed, we have no gain compared with (6.1). This suggests that the leading term does not come from the linear part when $\omega \cdot \xi = 0$, while the leading term is given by $(\frac{\omega}{|\omega|} \cdot N)\mathcal{E}_{Os}(x)\frac{\omega}{|\omega|}$ when $\omega \cdot \xi \neq 0$.

We turn to the case $\omega \cdot \xi = 0$. Then, as in Farwig and Hishida [8], it is possible to show that the leading term is given by a member of the Landau solutions. It is known (Korolev and Šverák [21, Section 3]) that the class of those solutions can be parametrized as $\{U_b;\ b \in \mathbb{R}^3\}$ by vectorial parameter b, which denotes the axis of symmetry of U_b, and coincides with the family of all self-similar solutions (that is, homogeneous solutions of degree (-1)) to the Navier–Stokes system in $\mathbb{R}^3 \setminus \{0\}$. The member U_b together with the associated pressure P_b (which is homogeneous of degree (-2)) satisfies

$$-\Delta U_b + \nabla P_b + U_b \cdot \nabla U_b = b\delta_0, \qquad \operatorname{div} U_b = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

where δ_0 denotes the Dirac measure supported in the origin. Since $U_b(x) \to 0$ pointwise in $\mathbb{R}^3 \setminus \{0\}$ as $|b| \to 0$, one may regard $U_0 = 0$. The Landau solution with parameter $b = (\frac{\omega}{|\omega|} \cdot N)\frac{\omega}{|\omega|}$ provides the leading term of the flow under consideration as in the case of no-slip boundary condition [8], however, we do not derive the asymptotic expansion here. Instead, under the condition $\omega \cdot N = 0$ we directly deduce $v \in L^{3/2+\varepsilon}$ as long as $\limsup_{|x|\to\infty} |x||v(x)|$ is small enough. We can do that by making full use of the Lorentz space as in [8], but in this paper we adopt another framework with use of less function spaces.

We begin with introducing several fundamental solutions, which play an important role. First of all, we recall the fundamental solution

$$\mathcal{E}_{La}(x) = \frac{1}{4\pi|x|} \tag{6.4}$$

of the Laplace operator $-\Delta$. The fundamental solution of the classical Stokes system is given by

$$\mathcal{E}_{St}(x) = \frac{1}{8\pi}\left(\frac{\mathbb{I}_3}{|x|} + \frac{x \otimes x}{|x|^3}\right), \qquad \mathcal{P}(x) = -\nabla \mathcal{E}_{La}(x) = \frac{x}{4\pi|x|^3}. \tag{6.5}$$

At the stage of the whole space problem, see (6.32) below, we will reduce our consideration for general case (ξ, ω) with $\omega \neq 0$ to the case where both ω and ξ are parallel to $e_1 = (1, 0, 0)$ as performed in Section 2. It thus suffices to provide the representation formulae of the fundamental solutions of the Oseen and rotating Stokes systems below for this particular case. Let $\mathcal{R} \in \mathbb{R} \setminus \{0\}$. The velocity part of fundamental solution of the Oseen system

$$-\Delta u + \nabla p - \mathcal{R}\,\partial_1 u = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3$$

is given by

$$\mathcal{E}_{Os}(x) = \left(\Delta\,\mathbb{I}_3 - \nabla^2\right)\Psi(x), \qquad \Psi(x) = \frac{1}{4\pi|\mathcal{R}|}\int_0^{\frac{|\mathcal{R}|}{2}s(x)} \frac{1 - e^{-\tau}}{\tau}\,d\tau, \tag{6.6}$$

together with the same $\mathcal{P}$ as in (6.5) for the pressure part, where

$$s(x) = \begin{cases} |x| + x_1, & \mathcal{R} > 0, \\ |x| - x_1, & \mathcal{R} < 0. \end{cases}$$

Let us also introduce

$$\mathcal{E}_{RSt}(x, y) = \int_0^\infty O_\omega(t)^\top K(O_\omega(t)x - y, t)\,dt, \tag{6.7}$$

where $K(x, t)$ is the fundamental solution of unsteady Stokes system given by

$$K(x, t) = G(x, t)\,\mathbb{I}_3 + \int_t^\infty \nabla^2 G(x, s)\,ds, \qquad G(x, t) = (4\pi t)^{-3/2} e^{-|x|^2/4t},$$

and

$$O_\omega(t) = O(|\omega| t), \qquad O(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}. \tag{6.8}$$

Note that the fundamental solution (6.5) of Stokes system is represented as

$$\mathcal{E}_{St}(x)=\int_0^\infty K(x,t)\,dt,$$

which can be compared with (6.7). We know (see [7,9]) that $(\mathcal{E}_{RSt}(x,y),\mathcal{P}(x-y))$ with $\mathcal{P}$ given by (6.5) is a fundamental solution of the rotating Stokes system

$$-\Delta u+\nabla p-|\omega|\{(e_1\times x)\cdot\nabla u-e_1\times u\}=f,\qquad \operatorname{div}u=0\qquad \text{in } \mathbb{R}^3. \tag{6.9}$$

Analysis of the fundamental solution (6.7) was developed by Farwig and Hishida [7], but we need a bit more. When the support of f is assumed to be compact in (6.9), the result of [7] can be directly applied, however, that is not the case here. But we do not intend to optimize the condition on f. For later use, the following result is enough. Although pointwise estimate (6.15) was already proved by Farwig, Galdi and Kyed [6, (2.12)], our proof is somewhat different from theirs.

Lemma 6.1. *Suppose $\omega=|\omega|e_1\neq 0$.*

1. *Let*

$$f\in L^r(\mathbb{R}^3)\cap L^s(\mathbb{R}^3)$$

for some (r,s) satisfying $1<r<3/2<s\leqslant 2$. Then the potential representations

$$u(x)=\int_{\mathbb{R}^3}\mathcal{E}_{RSt}(x,y)f(y)\,dy,\qquad p(x)=\int_{\mathbb{R}^3}\mathcal{P}(x-y)f(y)\,dy \tag{6.10}$$

are well-defined as

$$u\in L^\infty(\mathbb{R}^3),\qquad p\in L^{r_*}(\mathbb{R}^3)\cap L^{s_*}(\mathbb{R}^3) \tag{6.11}$$

and the pair (u,p) is a solution to (6.9) in the sense of distributions, where $1/r_*=1/r-1/3$ and $1/s_*=1/s-1/3$.

2. *Suppose*

$$\begin{cases} f=f_0+\operatorname{div}F,\\ f_0\in L^\infty(\mathbb{R}^3) \text{ with compact support},\qquad (1+|x|)^\alpha F\in L^\infty(\mathbb{R}^3),\\ \operatorname{div}F\in L^1(\mathbb{R}^3)\cap L^s(\mathbb{R}^3)\end{cases} \tag{6.12}$$

for some $\alpha\in(2,3]$ and $s\in(3/2,2]$. Then the solution (6.10) enjoys the asymptotic representation

$$u(x)=\left(e_1\cdot\int_{\mathbb{R}^3}f(y)\,dy\right)\mathcal{E}_{St}(x)e_1+\begin{cases}O(|x|^{-2}\log|x|), & \alpha=3,\\ O(|x|^{-\alpha+1}), & 2<\alpha<3,\end{cases}\qquad \text{as } |x|\to\infty, \tag{6.13}$$

where $\mathcal{E}_{St}(x)$ is the Stokes fundamental solution (6.5).

3. *Assume (6.12) for some $\alpha\in(2,3)$ and $s\in(3/2,2]$. If in particular $f_0=0$, then the solution (6.10) is of class*

$$(1+|x|)^{\alpha-1}u\in L^\infty(\mathbb{R}^3),\qquad u\in D^{1,2}(\mathbb{R}^3)\subset L^6(\mathbb{R}^3),\qquad p\in L^2(\mathbb{R}^3) \tag{6.14}$$

and there is a constant $C=C(\alpha)>0$ such that

$$\lceil u\rceil_{\alpha-1}+|u|_{1,2,\mathbb{R}^3}+\|u\|_{6,\mathbb{R}^3}+\|p\|_{2,\mathbb{R}^3}\leqslant C\lceil F\rceil_\alpha, \tag{6.15}$$

where the abbreviation $\lceil F\rceil_\alpha:=\lceil F\rceil_{\alpha,(1+|x|),\mathbb{R}^3}$ is used for simplicity of notation, see (2.1). Furthermore, it is a unique solution to (6.9) within the class $(u,p)\in L^6(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$.

Proof. We verify the first assertion when $f \in L^r(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$ for some r, s specified above. By the Hardy–Littlewood–Sobolev inequality, it is obvious that $p \in L^{r_*}(\mathbb{R}^3) \cap L^{s_*}(\mathbb{R}^3)$ with

$$\|p\|_{r_*,\mathbb{R}^3} \leqslant C\|f\|_{r,\mathbb{R}^3}, \qquad \|p\|_{s_*,\mathbb{R}^3} \leqslant C\|f\|_{s,\mathbb{R}^3}. \tag{6.16}$$

By Tf we denote the right-hand side of the first formula of (6.10), which is written as

$$\big(\mathcal{F}(Tf)\big)(\zeta) = \int_0^\infty O_\omega(t)^\top e^{-|\zeta|^2 t}\left(\mathbb{I}_3 - \frac{(O_\omega(t)\zeta)\otimes(O_\omega(t)\zeta)}{|\zeta|^2}\right)(\mathcal{F}f)(O_\omega(t)\zeta)\,dt$$

in the Fourier side, where $\mathcal{F}$ stands for the Fourier transform. Let us consider

$$\int_{\mathbb{R}^3} |\big(\mathcal{F}(Tf)\big)(\zeta)|\,d\zeta \leqslant C\left(\int_{|\zeta|\leqslant 1} + \int_{|\zeta|>1}\right)\frac{|(\mathcal{F}f)(\zeta)|}{|\zeta|^2}\,d\zeta =: I_1 + I_2.$$

Since $1 < r < 3/2 < s \leqslant 2$, we have

$$I_1 \leqslant C\|\mathcal{F}f\|_{r',\mathbb{R}^3_\zeta}\left(\int_{|\zeta|\leqslant 1}\frac{d\zeta}{|\zeta|^{2r}}\right)^{1/r} \leqslant C\|f\|_{r,\mathbb{R}^3},$$

$$I_2 \leqslant C\|\mathcal{F}f\|_{s',\mathbb{R}^3_\zeta}\left(\int_{|\zeta|>1}\frac{d\zeta}{|\zeta|^{2s}}\right)^{1/s} \leqslant C\|f\|_{s,\mathbb{R}^3},$$

where $1/r' + 1/r = 1$ and $1/s' + 1/s = 1$. We thus obtain $\mathcal{F}(Tf) \in L^1(\mathbb{R}^3_\zeta)$ and, therefore, $Tf \in L^\infty(\mathbb{R}^3)$ subject to

$$\|Tf\|_{\infty,\mathbb{R}^3} \leqslant C\big(\|f\|_{r,\mathbb{R}^3} + \|f\|_{s,\mathbb{R}^3}\big).$$

Given $f \in L^r(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$, we take $f_j \in C_0^\infty(\mathbb{R}^3)$ such that $f_j \to f$ in $L^r(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$; then, $Tf_j \to Tf$ in $L^\infty(\mathbb{R}^3)$ and $\mathcal{P} * f_j \to \mathcal{P} * f$ in $L^{r_*}(\mathbb{R}^3) \cap L^{s_*}(\mathbb{R}^3)$ as $j \to \infty$. Since $(Tf_j, \mathcal{P} * f_j)$ is a solution to (6.9), so is $(Tf, \mathcal{P} * f)$.

Let us prove the second assertion. Set

$$\mathcal{H}(x) := \frac{1}{8\pi|x|^3}\begin{bmatrix} |x|^2 + x_1^2 & 0 & 0 \\ x_2x_1 & 0 & 0 \\ x_3x_1 & 0 & 0 \end{bmatrix}.$$

Then it follows from [7, Section 4] that

$$|\mathcal{E}_{RSt}(x,y) - \mathcal{H}(x)| \leqslant \frac{C|y|}{|x|^2} + \frac{C}{|\omega||x|^3} \qquad \text{for } |x| > 2|y|, \tag{6.17}$$

which together with the assumption on f_0 immediately implies that

$$\int_{\mathbb{R}^3} \mathcal{E}_{RSt}(x,y) f_0(y)\,dy = \left(e_1 \cdot \int_{\mathbb{R}^3} f_0(y)\,dy\right)\mathcal{E}_{St}(x)e_1 + O(|x|^{-2}) \tag{6.18}$$

as $|x| \to \infty$ (see also [8, Lemma 3.7, Lemma 3.8]). Let $\psi \in C^\infty([0,\infty))$ be the same cut-off function as in the beginning of the proof of Lemma 4.2 and set $\psi_R(x) = \psi(|x|/R)$. Then we have

$$\int_{\mathbb{R}^3} (\psi_R \operatorname{div} F + F \cdot \nabla\psi_R)(y)\,dy = \int_{\mathbb{R}^3} \operatorname{div}(\psi_R F)(y)\,dy = 0.$$

Since (6.12) yields

$$\left| \int_{\mathbb{R}^3} (F \cdot \nabla \psi_R)(y)\, dy \right| \leqslant \frac{C}{R^{\alpha-2}} \to 0 \quad (R \to \infty)$$

and since $\operatorname{div} F \in L^1(\mathbb{R}^3)$, we find $\int_{\mathbb{R}^3} \operatorname{div} F\, dy = 0$, so that

$$\int_{\mathbb{R}^3} f(y)\, dy = \int_{\mathbb{R}^3} f_0(y)\, dy. \tag{6.19}$$

It thus suffices to show that

$$u_1(x) := \int_{\mathbb{R}^3} \mathcal{E}_{RSt}(x,y)\, (\operatorname{div} F)(y)\, dy = \begin{cases} O(|x|^{-2} \log |x|), & \alpha = 3, \\ O(|x|^{-\alpha+1}), & 2 < \alpha < 3, \end{cases} \tag{6.20}$$

as $|x| \to \infty$. Similarly to [7, (2.11)], we have

$$|\mathcal{E}_{RSt}(x,y)| \leqslant \frac{C}{|x|}, \qquad |\nabla_y \mathcal{E}_{RSt}(x,y)| \leqslant \frac{C}{|x|^2} \qquad \text{for } |x| > 2|y|, \tag{6.21}$$

$$|\mathcal{E}_{RSt}(x,y)| \leqslant \frac{C}{|y|}, \qquad |\nabla_y \mathcal{E}_{RSt}(x,y)| \leqslant \frac{C}{|y|^2} \qquad \text{for } |y| > 2|x|. \tag{6.22}$$

By the Fubini theorem together with a simple transformation we find

$$\begin{aligned} &\int_{|y|\leqslant 2|x|} |\nabla_y \mathcal{E}_{RSt}(x,y)|\, dy \\ &\leqslant \int_{|y|\leqslant 2|x|} \int_0^\infty |(\nabla K)(O_\omega(t)x - y, t)|\, dt\, dy \\ &\leqslant C \int_{|y|\leqslant 2|x|} \int_0^\infty \left(t^{-2} e^{-|O_\omega(t)x-y|^2/8t} + \int_t^\infty s^{-3} e^{-|O_\omega(t)x-y|^2/8s}\, ds \right) dt\, dy \\ &= C \int_{|y|\leqslant 2|x|} \int_0^\infty \left(t^{-2} e^{-|x-y|^2/8t} + \int_t^\infty s^{-3} e^{-|x-y|^2/8s}\, ds \right) dt\, dy \\ &= C \int_{|y|\leqslant 2|x|} \frac{dy}{|x-y|^2} \leqslant C|x|. \end{aligned} \tag{6.23}$$

By (6.22) together with (6.12) for $\alpha > 2$ one can justify the following integration by parts and then split the integral into three parts

$$\begin{aligned} u_1(x) &= -\int_{\mathbb{R}^3} \nabla_y \mathcal{E}_{RSt}(x,y) : F(y)\, dy \\ &= \int_{|y|<|x|/2} + \int_{|x|/2\leqslant|y|\leqslant 2|x|} + \int_{|y|>2|x|} =: u_{11}(x) + u_{12}(x) + u_{13}(x). \end{aligned}$$

Combining (6.21)–(6.23) with (6.12), we get

$$|u_{11}(x)| \leqslant \frac{C\lceil F\rceil_\alpha}{|x|^2}\int\limits_{|y|<|x|/2}\frac{dy}{(1+|y|)^\alpha} \leqslant \begin{cases} \dfrac{C\lceil F\rceil_\alpha}{|x|^2}\log\Big(1+\dfrac{|x|}{2}\Big), & \alpha=3,\\ \dfrac{C\lceil F\rceil_\alpha}{(3-\alpha)|x|^2}\Big(1+\dfrac{|x|}{2}\Big)^{3-\alpha}, & 2<\alpha<3,\end{cases}$$

$$|u_{12}(x)| \leqslant \frac{C\lceil F\rceil_\alpha}{(1+\frac{|x|}{2})^\alpha}\int\limits_{|y|\leqslant 2|x|}|\nabla_y\mathcal{E}_{RSt}(x,y)|\,dy \leqslant C\lceil F\rceil_\alpha|x|\Big(1+\frac{|x|}{2}\Big)^{-\alpha},$$

$$|u_{13}(x)| \leqslant C\lceil F\rceil_\alpha\int\limits_{|y|>2|x|}|y|^{-2}(1+|y|)^{-\alpha}\,dy = \frac{C\lceil F\rceil_\alpha}{\alpha-1}(1+2|x|)^{-\alpha+1}.$$

Summing up, we obtain (6.20), which together with (6.18)–(6.19) concludes (6.13).

Finally, we show the third assertion. Among three estimates of $u_1(x)$ above, the only problem is the boundedness of $u_{11}(x)$, however, one has only to estimate this term in the other way

$$\frac{\lceil F\rceil_\alpha}{|x|^2}\int_0^{|x|/2}\frac{\rho^2}{(1+\rho)^\alpha}\,d\rho \leqslant \frac{\lceil F\rceil_\alpha}{4}\int_0^{|x|/2}\frac{d\rho}{(1+\rho)^\alpha} \leqslant \frac{C\lceil F\rceil_\alpha}{\alpha-1}$$

near $x=0$. We thus obtain

$$\lceil u\rceil_{\alpha-1} \leqslant C\lceil F\rceil_\alpha.$$

Since $F\in L^2(\mathbb{R}^3)$, one can employ [20, Theorem 2.1], [16, Theorem VIII.1.2] to find that (6.9) admits a solution (u',p') of class

$$u'\in D^{1,2}(\mathbb{R}^3)\subset L^6(\mathbb{R}^3), \qquad p'\in L^2(\mathbb{R}^3)$$

with

$$|u'|_{1,2,\mathbb{R}^3}+\|u'\|_{6,\mathbb{R}^3}+\|p'\|_{2,\mathbb{R}^3} \leqslant C\|F\|_{2,\mathbb{R}^3} \leqslant C\lceil F\rceil_\alpha.$$

We here note that $C_0^\infty(\mathbb{R}^3)$ is dense in $D^{1,2}(\mathbb{R}^3)$ and hence the embedding relation $\|g\|_{6,\mathbb{R}^3}\leqslant C\|\nabla g\|_{2,\mathbb{R}^3}=C|g|_{1,2,\mathbb{R}^3}$ holds for all $g\in D^{1,2}(\mathbb{R}^3)$. Let us identify (u',p') with (u,p) given by (6.10). Set $(v,q):=(u-u',p-p')\in\mathcal{S}'(\mathbb{R}^3)$, which fulfills

$$-\Delta v+\nabla q-|\omega|\{(e_1\times x)\cdot\nabla v-e_1\times v\}=0, \qquad \operatorname{div} v=0 \qquad \text{in } \mathbb{R}^3,$$

where $\mathcal{S}'(\mathbb{R}^3)$ denotes the class of tempered distributions. Since

$$\operatorname{div}[(e_1\times x)\cdot\nabla v-e_1\times v]=(e_1\times x)\cdot\nabla\operatorname{div} v=0$$

so that $\Delta q=0$ and since $q\in L^{s_*}(\mathbb{R}^3)+L^2(\mathbb{R}^3)$, we get $q=0$, which leads to

$$-\Delta v-|\omega|\{(e_1\times x)\cdot\nabla v-e_1\times v\}=0 \qquad \text{in } \mathbb{R}^3.$$

As shown in [9, p. 311], [20, Lemma 4.2], the Fourier transform $\mathcal{F}v$ is supported in the origin. Hence v is a polynomial vector field, which concludes $v=0$ since $u'\in L^6(\mathbb{R}^3)$ as well as $(1+|x|)^{\alpha-1}u\in L^\infty(\mathbb{R}^3)$. The final statement on uniqueness is obvious by the same reasoning as above. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. As we have mentioned, we only consider here the cases 2 and 4. We thus assume throughout that $\omega\neq 0$ and $\omega\cdot N=0$. Given (v,p), a solution to (1.1)–(1.2) with (6.1) (and, as a consequence, (6.2) as well), we set

$$\Phi=\int\limits_{\partial\Omega}v\cdot n\,d\gamma.$$

We fix $x_0 \in \operatorname{int}\mathcal{S}$ and use (6.4) to introduce the flux carrier $\mathcal{W} \in L^{3/2+\varepsilon}(\Omega)$ by

$$\mathcal{W}(x) = \Phi\nabla\mathcal{E}_{La}(x-x_0) = \frac{-\Phi(x-x_0)}{4\pi|x-x_0|^3}, \tag{6.24}$$

which satisfies

$$\begin{cases} \operatorname{div}\mathcal{W} = 0, \qquad \Delta\mathcal{W} = 0, \qquad \xi\cdot\nabla\mathcal{W} = \nabla(\xi\cdot\mathcal{W}), \\ \big(\omega\times(x-x_0)\big)\cdot\nabla\mathcal{W} = \omega\times\mathcal{W}, \qquad \mathcal{W}\cdot\nabla\mathcal{W} = \nabla(\tfrac{1}{2}|\mathcal{W}|^2) \end{cases} \tag{6.25}$$

in $\mathbb{R}^3\setminus\{x_0\}$ as well as

$$\int_{\partial\Omega} \mathcal{W}\cdot n\, d\gamma = \Phi. \tag{6.26}$$

Note that we do not always claim $0 \in \operatorname{int}\mathcal{S}$ without loss because the axis of rotation runs through the origin and because the equation (1.1) changes by translation. We set

$$\hat{v} = v - \mathcal{W}, \qquad \hat{p} = p - (\xi + \omega\times x_0)\cdot\mathcal{W} - \frac{1}{2}|\mathcal{W}|^2. \tag{6.27}$$

We then see from (6.1)–(6.2) and (6.25)–(6.26) that the pair $(\hat{v}, \hat{p})$ obeys

$$\begin{aligned} &-\operatorname{div}\sigma(\hat{v},\hat{p}) + (\hat{v}-\xi-\omega\times x)\cdot\nabla\hat{v} + \omega\times\hat{v} = -v\cdot\nabla\mathcal{W} - \mathcal{W}\cdot\nabla v \qquad \text{in } \Omega, \\ &\operatorname{div}\hat{v} = 0 \qquad \text{in } \Omega, \\ &\int_{\partial\Omega} \hat{v}\cdot n\, d\gamma = 0, \end{aligned} \tag{6.28}$$

and satisfies

$$(\nabla\hat{v}, \hat{p}) \in L^2(\Omega), \qquad (1+|x|)\,\hat{v} \in L^\infty(\Omega), \qquad (\hat{v}, \hat{p}) \in C^\infty(\Omega). \tag{6.29}$$

We fix $R_0 > 0$ such that $\mathcal{S} \subset B_{R_0}$. Let $R \in [R_0, \infty)$ be a parameter to be determined later (one may take $R = R_0$ when $\omega\cdot\xi \neq 0$, while when $\omega\cdot\xi = 0$ we have to be more precise in the choice of R, see (6.57) below). We take $\phi_R \in C^\infty(\mathbb{R}^3; [0,1])$ such that

$$1-\phi_R \in C_0^\infty(B_{3R}), \qquad \phi_R(x) = 0 \quad (x \in \overline{B_{2R}}), \qquad \|\nabla\phi_R\|_{\infty,\mathbb{R}^3} \leqslant \frac{C}{R}. \tag{6.30}$$

We set

$$u := \phi_R\hat{v} - \mathcal{B}[\hat{v}\cdot\nabla\phi_R], \qquad q := \phi_R\hat{p}, \tag{6.31}$$

where $\mathcal{B}$ denotes the Bogovskii operator [2,3,16] in the domain

$$B_{R,3R} := \{x \in \mathbb{R}^3;\ R < |x| < 3R\}.$$

Since $\int_{B_{R,3R}} \hat{v}\cdot\nabla\phi_R\, dx = 0$ follows from $\int_{\partial\Omega} \hat{v}\cdot n\, d\gamma = 0$, we have $\operatorname{div}\mathcal{B}[\hat{v}\cdot\nabla\phi_R] = \hat{v}\cdot\nabla\phi_R$. We thus obtain

$$\begin{aligned} &-\operatorname{div}\sigma(u,q) + (u-\xi-\omega\times x)\cdot\nabla u + \omega\times u = g \qquad \text{in } \mathbb{R}^3, \\ &\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3, \\ &\lim_{|x|\to\infty} u = 0 \end{aligned} \tag{6.32}$$

where (6.1)–(6.2) and (6.24) imply that

$$\begin{cases} g = g_0 - \operatorname{div} G, \qquad g_0 \in C_0^\infty(B_{R,3R}), \\ G := \phi_R\,(\mathcal{W}\otimes v + v\otimes\mathcal{W}) \in C^\infty(\mathbb{R}^3), \qquad (1+|x|)^3\, G \in L^\infty(\mathbb{R}^3), \\ \operatorname{div} G \in L^r(\mathbb{R}^3) \quad \text{for all } r \in [1,2]. \end{cases} \tag{6.33}$$

We do not need any exact form of g_0. A key observation is

$$\int_{\mathbb{R}^3} g(y)\,dy = N + \Phi(\omega \times x_0), \qquad \omega \cdot \int_{\mathbb{R}^3} g(y)\,dy = \omega \cdot N = 0, \tag{6.34}$$

see (1.13), which follows only from the structure of the equation (6.32). Indeed, for $\rho > 3R + |x_0|$ we set $B_\rho(x_0) := \{y \in \mathbb{R}^3; |y - x_0| < \rho\} \supset B_{3R}$ and then we see from (6.32) that

$$\begin{aligned}
&\int_{B_\rho(x_0)} g(y)\,dy \\
&= -\int_{B_\rho(x_0)} \mathrm{div}[\sigma(u,q) - u \otimes (u - \xi - \omega \times y) - (\omega \times y) \otimes u]\,dy \\
&= -\int_{|y-x_0|=\rho} \big[\sigma(\hat{v},\hat{p}) - \hat{v} \otimes (\hat{v} - \xi - \omega \times y) - (\omega \times y) \otimes \hat{v}\big] \frac{y - x_0}{\rho}\,d\gamma_y \\
&= -\int_{|y-x_0|=\rho} \big[\sigma(v,p) - v \otimes (v - \xi - \omega \times y) - (\omega \times y) \otimes v\big] \frac{y - x_0}{\rho}\,d\gamma_y \\
&\quad + \int_{|y-x_0|=\rho} \Big[\sigma\Big(\mathcal{W},\ (\xi + \omega \times x_0) \cdot \mathcal{W} + \frac{1}{2}|\mathcal{W}|^2\Big) \\
&\quad + \mathcal{W} \otimes \mathcal{W} + \mathcal{W} \otimes (\xi + \omega \times y) - (\omega \times y) \otimes \mathcal{W}\Big] \frac{y - x_0}{\rho}\,d\gamma_y \\
&\quad - \int_{|y-x_0|=\rho} \big[v \otimes \mathcal{W} + \mathcal{W} \otimes v\big] \frac{y - x_0}{\rho}\,d\gamma_y.
\end{aligned}$$

By (1.1) we have

$$\begin{aligned}
&-\int_{|y-x_0|=\rho} \big[\sigma(v,p) - v \otimes (v - \xi - \omega \times y) - (\omega \times y) \otimes v\big] \frac{y - x_0}{\rho}\,d\gamma_y \\
&= \int_{\partial\Omega} \big[\sigma(v,p) - v \otimes (v - \xi - \omega \times y) - (\omega \times y) \otimes v\big] n\,d\gamma_y = N.
\end{aligned}$$

We thus obtain

$$\int_{B_\rho(x_0)} g(y)\,dy = N + J_1 + J_2 + J_3 + J_4 + J_5 + J_6,$$

where (6.25) leads us to

$$\begin{aligned}
J_1 &:= 2\int_{|y-x_0|=\rho} D(\mathcal{W}) \frac{y - x_0}{\rho}\,d\gamma_y \\
&= 2\int_{|y-x_0|=\rho} (\nabla \mathcal{W}) \frac{y - x_0}{\rho}\,d\gamma_y = \frac{\Phi}{\pi\rho^4} \int_{|y-x_0|=\rho} (y - x_0)\,d\gamma_y = 0, \\
J_2 &:= \int_{|y-x_0|=\rho} [-\xi \cdot \mathcal{W}\mathbb{I}_3 + \mathcal{W} \otimes \xi] \frac{y - x_0}{\rho}\,d\gamma_y \\
&= \frac{1}{\rho} \int_{|y-x_0|=\rho} \big[-\big(\xi \cdot \mathcal{W}\big)(y - x_0) + \mathcal{W}\big(\xi \cdot (y - x_0)\big)\big]\,d\gamma_y = 0,
\end{aligned}$$

$$
\begin{aligned}
J_3 &:= \int_{|y-x_0|=\rho} \left[-(\omega\times x_0)\cdot \mathcal{W}\mathbb{I}_3 + \mathcal{W}\otimes(\omega\times y)\right]\frac{y-x_0}{\rho}\,d\gamma_y \\
&= \frac{\Phi}{4\pi\rho^4}\int_{|y-x_0|=\rho}\left[\big((\omega\times x_0)\cdot y\big)(y-x_0) + \big((\omega\times y)\cdot x_0\big)(y-x_0)\right]d\gamma_y = 0, \\
J_4 &:= -\int_{|y-x_0|=\rho}(\omega\times y)\otimes \mathcal{W}\,\frac{y-x_0}{\rho}\,d\gamma_y \\
&= \frac{\Phi}{4\pi\rho^2}\int_{|y-x_0|=\rho}\left\{\omega\times(y-x_0)+\omega\times x_0\right\}d\gamma_y = \Phi(\omega\times x_0), \\
J_5 &:= \int_{|y-x_0|=\rho}\left[-\frac{|\mathcal{W}|^2}{2}\mathbb{I}_3 + \mathcal{W}\otimes\mathcal{W}\right]\frac{y-x_0}{\rho}\,d\gamma_y \\
&= \frac{\Phi^2}{32\pi^2\rho^5}\int_{|y-x_0|=\rho}(y-x_0)\,d\gamma_y = 0, \\
J_6 &:= -\int_{|y-x_0|=\rho}\left[v\otimes\mathcal{W} + \mathcal{W}\otimes v\right]\frac{y-x_0}{\rho}\,d\gamma_y.
\end{aligned}
$$

Since $v(x) = O(|x|^{-1})$ and $\mathcal{W}(x) = O(|x|^{-2})$, we find that the integral J_6 goes to zero as $\rho\to\infty$. This yields (6.34).

It is convenient to reduce the whole space problem (6.32) to an equivalent one in which both rotation and translation are parallel to $e_1 = (1,0,0)$. Let $M\in\mathbb{R}^{3\times 3}$ be an orthogonal matrix that fulfills $M\frac{\omega}{|\omega|} = e_1$. As in the proof of Proposition 2.1, by the transformation

$$
x' = Mx, \qquad u'(x') = Mu(M^\top x'), \qquad q'(x') = q(M^\top x'), \qquad g'(x') = Mg(M^\top x'),
$$

we are led to

$$
\begin{aligned}
&-\operatorname{div}\sigma(u',q') + u'\cdot\nabla u' - \xi'\cdot\nabla u' - |\omega|\{(e_1\times x')\cdot\nabla u' - e_1\times u'\} = g' \quad \text{in } \mathbb{R}^3_{x'} \\
&\operatorname{div} u' = 0 \quad \text{in } \mathbb{R}^3_{x'} \\
&\lim_{|x'|\to\infty} u' = 0
\end{aligned}
$$

where $\xi' = M\xi$, and ∇ and div are differential operators with respect to x'. And then, by the translation

$$
\begin{aligned}
\widetilde{x} &= x' - \frac{e_1\times\xi'}{|\omega|} = M\Big(x - \frac{\omega\times\xi}{|\omega|^2}\Big), \\
\widetilde{u}(\widetilde{x}) &= u'\left(\widetilde{x} + \frac{e_1\times\xi'}{|\omega|}\right) = Mu\left(M^\top\widetilde{x} + \frac{\omega\times\xi}{|\omega|^2}\right), \\
\widetilde{q}(\widetilde{x}) &= q'\left(\widetilde{x} + \frac{e_1\times\xi'}{|\omega|}\right) = q\left(M^\top\widetilde{x} + \frac{\omega\times\xi}{|\omega|^2}\right), \\
\widetilde{g}(\widetilde{x}) &= g'\left(\widetilde{x} + \frac{e_1\times\xi'}{|\omega|}\right) = Mg\left(M^\top\widetilde{x} + \frac{\omega\times\xi}{|\omega|^2}\right),
\end{aligned}
\tag{6.35}
$$

we obtain

$$
\begin{aligned}
&-\operatorname{div}\sigma(\widetilde{u},\widetilde{q}) + \widetilde{u}\cdot\nabla\widetilde{u} - \mathcal{R}\,\partial_1\widetilde{u} - |\omega|\{(e_1\times\widetilde{x})\cdot\nabla\widetilde{u} - e_1\times\widetilde{u}\} = \widetilde{g} \quad \text{in } \mathbb{R}^3_{\widetilde{x}} \\
&\operatorname{div}\widetilde{u} = 0 \quad \text{in } \mathbb{R}^3_{\widetilde{x}} \\
&\lim_{|\widetilde{x}|\to\infty}\widetilde{u} = 0
\end{aligned}
\tag{6.36}
$$

where

$$\mathcal{R} = e_1 \cdot \xi' = \frac{\omega \cdot \xi}{|\omega|}$$

and ∇ and div are differential operators with respect to $\widetilde{x}$. Then it follows from (6.34) that

$$e_1 \cdot \int_{\mathbb{R}^3} \widetilde{g}(\widetilde{x})\, d\widetilde{x} = \left(M \frac{\omega}{|\omega|} \right) \cdot \left(M \int_{\mathbb{R}^3} g(x)\, dx \right) = \frac{\omega}{|\omega|} \cdot N = 0. \tag{6.37}$$

Look at the properties described in (6.33) for $g = g_0 - \operatorname{div} G$. We take the same transformation $\widetilde{g}_0$ of g_0 as above and

$$\widetilde{G}(\widetilde{x}) = \left(M G M^\top \right) \left(M^\top \widetilde{x} + \frac{\omega \times \xi}{|\omega|^2} \right),$$

so that

$$\begin{cases} \widetilde{g} = \widetilde{g}_0 - \operatorname{div} \widetilde{G}, & \widetilde{g}_0 \in C_0^\infty(\mathbb{R}^3), \\ \widetilde{G} \in C^\infty(\mathbb{R}^3), & (1 + |\widetilde{x}|)^3\, \widetilde{G} \in L^\infty(\mathbb{R}^3), \qquad \operatorname{div} \widetilde{G} \in L^r(\mathbb{R}^3) \quad \text{for all } r \in [1, 2]. \end{cases} \tag{6.38}$$

By (6.29), (6.31) and (6.35) together with properties of the Bogovskii operator, we have

$$(\nabla \widetilde{u}, \widetilde{q}) \in L^2(\mathbb{R}^3), \qquad (\widetilde{u}, \widetilde{q}) \in C^\infty(\mathbb{R}^3). \tag{6.39}$$

Furthermore, when we take

$$R \geqslant \max \left\{ 2R_0,\ 1 + \frac{|\xi|}{|\omega|} \right\} \tag{6.40}$$

in (6.30)–(6.31), we find

$$\lceil \widetilde{u} \rceil_1 \leqslant C_0 \left(\sup_{x \in \mathbb{R}^3 \setminus B_R} |x||v(x)| + \frac{|\Phi|}{R} \right) \tag{6.41}$$

with some constant $C_0 > 0$ independent of R satisfying (6.40), where the abbreviation (2.1) is used. In fact, we employ the Gagliardo–Nirenberg inequality with fixed $r \in (3, \infty)$, the Poincaré inequality and L^r-estimate of the Bogovskii operator (where the estimate is dilation invariant, see Borchers and Sohr [3, Theorem 2.10]) to obtain

$$\begin{aligned} \| |x| \mathcal{B}[\hat{v} \cdot \nabla \phi_R] \|_{\infty, B_{R,3R}} &\leqslant C R \| \mathcal{B}[\hat{v} \cdot \nabla \phi_R] \|_{r,\mathbb{R}^3}^{1-3/r} \| \nabla \mathcal{B}[\hat{v} \cdot \nabla \phi_R] \|_{r,\mathbb{R}^3}^{3/r} \\ &\leqslant C R^{2-3/r} \| \nabla \mathcal{B}[\hat{v} \cdot \nabla \phi_R] \|_{r, B_{R,3R}} \\ &\leqslant C R^{2-3/r} \| \hat{v} \cdot \nabla \phi_R \|_{r, B_{R,3R}} \\ &\leqslant C R^{1-3/r} \| \hat{v} \|_{L^r(B_{R,3R})} \\ &\leqslant C \sup_{x \in \mathbb{R}^3 \setminus B_R} |x||\hat{v}(x)| \end{aligned}$$

which implies

$$\sup_{x \in \mathbb{R}^3 \setminus B_R} |x||u(x)| \leqslant C \sup_{x \in \mathbb{R}^3 \setminus B_R} |x||v(x) - \mathcal{W}(x)|. \tag{6.42}$$

Since $x_0 \in \operatorname{int} \mathcal{S}$, so that $|x_0| < R_0$, the flux carrier (6.24) can be estimated as

$$|\mathcal{W}(x)| = \frac{|\Phi|}{4\pi |x - x_0|^2} < \frac{|\Phi|}{\pi |x|^2} \qquad \text{for } |x| \geqslant R \geqslant 2R_0, \tag{6.43}$$

see (6.40). We use (6.40) again to observe

$$\sup_{\widetilde{x} \in \mathbb{R}^3} (1 + |\widetilde{x}|) |\widetilde{u}(\widetilde{x})| = \sup_{x \in \mathbb{R}^3} \left(1 + \left| x - \frac{\omega \times \xi}{|\omega|^2} \right| \right) |u(x)| \leqslant 2 \sup_{x \in \mathbb{R}^3 \setminus B_R} |x||u(x)|,$$

which combined with (6.42)–(6.43) concludes (6.41).

Let us divide our study into two cases: $\omega \cdot \xi \neq 0$ and $\omega \cdot \xi = 0$. We note that (6.41) is needed only for the latter.

Case $\omega \cdot \xi \neq 0$. The argument of Kyed [23] still works well in this case although the support of g is not compact. We will briefly describe the change which is not obvious. First of all, by Galdi and Kyed [17, Theorem 4.4] the Leray class (6.1) implies that $v \in L^{2+\varepsilon}(\Omega)$ for every $\varepsilon > 0$ and, therefore, $\widetilde{u} \in L^{2+\varepsilon}(\mathbb{R}^3)$ by using the L^q-estimate of the Bogovskii operator $\mathcal{B}$, see [2,3,16].

Following [23], we consider

$$w(\widetilde{x},t) = O_\omega(t)\widetilde{u}(O_\omega(t)^\top \widetilde{x}), \qquad r(\widetilde{x},t) = \widetilde{q}(O_\omega(t)^\top \widetilde{x}), \qquad h(\widetilde{x},t) = h_0(\widetilde{x},t) - \operatorname{div} H(\widetilde{x},t),$$
$$h_0(\widetilde{x},t) = O_\omega(t)\widetilde{g}_0(O_\omega(t)^\top \widetilde{x}), \qquad H(\widetilde{x},t) = \left(O_\omega(t)\widetilde{G}\,O_\omega(t)^\top\right)(O_\omega(t)^\top \widetilde{x}),$$

with use of the same rotation matrix $O_\omega(t)$ given by (6.8). Then (w,r) is a time-periodic flow with period $\frac{2\pi}{|\omega|}$ to the system

$$\partial_t w - \Delta w + \nabla r - \mathcal{R}\,\partial_1 w = h - w \cdot \nabla w \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}/\left(\tfrac{2\pi}{|\omega|}\mathbb{Z}\right)$$
$$\operatorname{div} w = 0 \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}/\left(\tfrac{2\pi}{|\omega|}\mathbb{Z}\right)$$
$$\lim_{|\widetilde{x}|\to\infty} w = 0.$$

The point of observation due to [23] is that the leading term of $\widetilde{u}$ at infinity comes from the average

$$\overline{w}(\widetilde{x}) = \frac{|\omega|}{2\pi}\int_0^{2\pi/|\omega|} w(\widetilde{x},t)\,dt, \tag{6.44}$$

because $\widetilde{u} - \overline{w} \in L^q(\mathbb{R}^3)$ for all $q \in (1,2]$, which follows from [23, Lemma 2.2] and $\widetilde{g} - \widetilde{u} \cdot \nabla \widetilde{u} \in L^q(\mathbb{R}^3)$ for such q, see (6.38). Here, $\overline{w}$ together with the associated pressure

$$\overline{r}(\widetilde{x}) = \frac{|\omega|}{2\pi}\int_0^{2\pi/|\omega|} r(\widetilde{x},t)\,dt$$

can be regarded as the solution to the Oseen system

$$-\Delta \overline{w} + \nabla \overline{r} - \mathcal{R}\,\partial_1 \overline{w} = \overline{h} - \operatorname{div} \overline{K} \quad \text{in } \mathbb{R}^3$$
$$\operatorname{div} \overline{w} = 0 \quad \text{in } \mathbb{R}^3$$
$$\lim_{|\widetilde{x}|\to\infty} \overline{w} = 0$$

with

$$\overline{h} = \overline{h_0} - \operatorname{div} \overline{H},$$

where $\overline{h_0}$, $\overline{H}$ and $\overline{K}$ are defined respectively by the average of $h_0(\cdot,t)$, $H(\cdot,t)$ and $(w \otimes w)(\cdot,t)$ over the period as in (6.44). The case of absence of $\overline{H}$ was discussed by [23]. The only change here is that $\operatorname{div} \overline{H}$ is treated as follows.

As mentioned above, $\overline{H}$ is given by

$$\overline{H}(\widetilde{x}) = \frac{|\omega|}{2\pi}\int_0^{2\pi/|\omega|} \left(O_\omega(t)\widetilde{G}\,O_\omega(t)^\top\right)(O_\omega(t)^\top \widetilde{x})\,dt,$$

whose properties follow from those of $\widetilde{G}$, see (6.38). Let $\mathcal{E}_{Os}$ be the Oseen fundamental solution (6.6). Since $\mathcal{E}_{Os} \in L^q(\mathbb{R}^3)$ for $q \in (2,3)$, see [16, Chapter VII], the Hausdorff–Young inequality implies that the convolution $U := \mathcal{E}_{Os} * (\operatorname{div} \overline{H})$ is well-defined in $L^r(\mathbb{R}^3)$ for $r \in (2,\infty]$. By $\overline{H}(\widetilde{x}) = O(|\widetilde{x}|^{-3})$ and by $\overline{H} \in L^s(\mathbb{R}^3)$ for all $s \in (1,\infty]$, one can justify integration by parts to obtain $U = (\nabla \mathcal{E}_{Os}) * \overline{H}$, which belongs to $L^r(\mathbb{R}^3)$ for even better $r \in (4/3,\infty]$ on account of $\nabla \mathcal{E}_{Os} \in L^q(\mathbb{R}^3)$ for $q \in (4/3,3/2)$. In this way, $\operatorname{div} \overline{H}$ brings better summability and that is also the case for $\operatorname{div} \overline{K}$, see (6.3) in which $\widetilde{u} \otimes \widetilde{u}$ is replaced by $\overline{K}$ (they have the same summability). As a consequence, the leading term of $\widetilde{u}$ arises from $\mathcal{E}_{Os} * \overline{h_0}$. Since $\int_{\mathbb{R}^3} \widetilde{g}\, dy = \int_{\mathbb{R}^3} \widetilde{g}_0\, dy$ by the same reasoning as in (6.19), we can conclude the asymptotic expansion

$$\widetilde{u}(\widetilde{x}) = \left(e_1 \cdot \int_{\mathbb{R}^3} \widetilde{g}(y)\, dy \right) \mathcal{E}_{Os}(\widetilde{x}) e_1 + \mathcal{U}(\widetilde{x}), \tag{6.45}$$

where the remainder possesses better summability $\mathcal{U} \in L^q(\mathbb{R}^3 \setminus B_L)$ for all $q \in (4/3, 2]$ and $L > 0$; hence, by virtue of (6.37), $\widetilde{u}$ enjoys such summability and thus the relation (6.35) leads us to

$$\int_{\left| x - \frac{\omega \times \xi}{|\omega|^2} \right| > L} |u(x)|^q\, dx < \infty.$$

Since u is smooth, we have $u \in L^{4/3+\varepsilon}(\mathbb{R}^3)$. This together with (6.31) yields $\hat{v} = v - \mathcal{W} \in L^{4/3+\varepsilon}(\Omega)$ and thereby $v \in L^{3/2+\varepsilon}(\Omega)$ unless $\Phi = 0$, while $v \in L^{4/3+\varepsilon}(\Omega)$ if in particular $\Phi = 0$.

Case $\omega \cdot \xi = 0$. For $\alpha > 0$ we set

$$\mathcal{M}_\alpha(\mathbb{R}^3) := \left\{ f \in D^{1,2}(\mathbb{R}^3)\ ;\ \lceil f \rceil_\alpha < \infty \right\} \tag{6.46}$$

which is a Banach space endowed with the norm

$$\|f\|_{\mathcal{M}_\alpha} := |f|_{1,2,\mathbb{R}^3} + \lceil f \rceil_\alpha,$$

where the abbreviation (2.1) is used. Let us consider the auxiliary linear system

$$\begin{aligned} &-\operatorname{div} \sigma(U, Q) - |\omega| \{(e_1 \times \widetilde{x}) \cdot \nabla U - e_1 \times U\} = \widetilde{g} - U \cdot \nabla \widetilde{u} \quad \text{in } \mathbb{R}^3_{\widetilde{x}} \\ &\operatorname{div} U = 0 \quad \text{in } \mathbb{R}^3_{\widetilde{x}}. \end{aligned} \tag{6.47}$$

We know from (6.36) with $\mathcal{R} = 0$ that $(\widetilde{u}, \widetilde{q})$ itself is a solution to (6.47) of class (6.39) together with $\lceil \widetilde{u} \rceil_1 < \infty$, to be more precise, (6.41). We fix $\varepsilon > 0$ arbitrarily small. For (6.47) our task is to show

(i) uniqueness in the space $\mathcal{M}_1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$;
(ii) existence in the space $\mathcal{M}_\gamma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with $\dfrac{3}{\frac{3}{2} + \varepsilon} < \gamma < 2$

provided $\lceil \widetilde{u} \rceil_1$ is sufficiently small (the smallness condition in (ii) will depend on $\varepsilon > 0$). Once we have these results, the only solution $\widetilde{u}$ must belong to $\mathcal{M}_\gamma(\mathbb{R}^3)$ and, therefore,

$$\sup_{x \in \mathbb{R}^3} \left(1 + \left| x - \frac{\omega \times \xi}{|\omega|^2} \right| \right)^\gamma |u(x)| < \infty.$$

We thus obtain $u \in L^{3/2+\varepsilon}(\mathbb{R}^3)$, which concludes $v = \hat{v} + \mathcal{W} \in L^{3/2+\varepsilon}(\Omega)$.

Let us start with the proof of uniqueness (i). Suppose

$$(U^{(1)}, Q^{(1)}),\ (U^{(2)}, Q^{(2)}) \in \mathcal{M}_1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

are two solutions of (6.47) and set $U := U^{(1)} - U^{(2)}$, $Q := Q^{(1)} - Q^{(2)}$. Then

$$\begin{aligned} &-\operatorname{div} \sigma(U, Q) - |\omega| \{(e_1 \times \widetilde{x}) \cdot \nabla U - e_1 \times U\} = -U \cdot \nabla \widetilde{u} \qquad \text{in } \mathbb{R}^3_{\widetilde{x}} \\ &\operatorname{div} U = 0 \qquad \text{in } \mathbb{R}^3_{\widetilde{x}}. \end{aligned} \tag{6.48}$$

As in the proof of Lemma 4.2, we consider the truncation function $\psi_R(x) = \psi(|x|/R)$, where $\psi \in C^\infty([0, \infty))$ is the same function as in the beginning of the proof of Lemma 4.2, multiply (6.48) by $\psi_R U$, use the properties

$$\nabla U,\ Q \in L^2(\mathbb{R}^3), \qquad \lceil U \rceil_1 < \infty, \qquad \lceil \widetilde{u} \rceil_1 < \infty,$$

and let $R \to \infty$ to deduce that

$$\|\nabla U\|_{2,\mathbb{R}^3}^2 = \int_{\mathbb{R}^3} \widetilde{u} \cdot (U \cdot \nabla U)\, d\widetilde{x}.$$

By the Hardy inequality we get

$$\|\nabla U\|_{2,\mathbb{R}^3}^2 \leqslant 2 \sup_{\widetilde{x}\in\mathbb{R}^3} \big[|\widetilde{x}||\widetilde{u}(\widetilde{x})|\big] \|\nabla U\|_{2,\mathbb{R}^3}^2.$$

We thus conclude that $U^{(1)} = U^{(2)}$, $Q^{(1)} = Q^{(2)}$ under the condition

$$\lceil \widetilde{u} \rceil_1 < \frac{1}{2}. \tag{6.49}$$

We next consider the existence result (ii). Let us consider the solution (u_0, q_0) of

$$\begin{aligned} &-\operatorname{div}\sigma(u_0, q_0) - |\omega|\{(e_1 \times \widetilde{x}) \cdot \nabla u_0 - e_1 \times u_0\} = \widetilde{g} \quad \text{in } \mathbb{R}^3_{\widetilde{x}} \\ &\operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^3_{\widetilde{x}}. \end{aligned} \tag{6.50}$$

By (6.38) we can apply the second assertion of Lemma 6.1 and take account of (6.37) to find that the solution u_0 given by (6.10) (with $f = \widetilde{g}$) enjoys

$$u_0(\widetilde{x}) = O(|\widetilde{x}|^{-2} \log |\widetilde{x}|) \qquad \text{as } |\widetilde{x}| \to \infty. \tag{6.51}$$

Since $\widetilde{g} \in C^\infty(\mathbb{R}^3)$, the regularity theory for the Stokes system implies that $u_0 \in C^\infty(\mathbb{R}^3) \subset L^\infty_{loc}(\mathbb{R}^3)$, which combined with (6.51) yields

$$(1 + |\widetilde{x}|)^\gamma u_0 \in L^\infty(\mathbb{R}^3), \tag{6.52}$$

while the associated pressure is of class

$$q_0 \in L^s(\mathbb{R}^3), \qquad \forall s \in (3/2, 6], \tag{6.53}$$

which follows from (6.16) with r close to 1 and $s = 2$. Since $C_0^\infty(\mathbb{R}^3)$ is dense in $D^{1,2}(\mathbb{R}^3)$, we have the embedding relation $D^{1,2}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ to regard $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3);\ \nabla u \in L^2(\mathbb{R}^3)\}$, which is a Banach space with the norm $\|\nabla(\cdot)\|_{2,\mathbb{R}^3} = |\cdot|_{1,2,\mathbb{R}^3}$. Having this in mind, we denote by $D^{-1,2}(\mathbb{R}^3)$ the dual space of $D^{1,2}(\mathbb{R}^3)$. In view of (6.38) again, since $\widetilde{g}_0 \in L^{6/5}(\mathbb{R}^3) \subset D^{-1,2}(\mathbb{R}^3)$ and since $\widetilde{G} \in L^2(\mathbb{R}^3)$, we have $\widetilde{g} \in D^{-1,2}(\mathbb{R}^3)$. Therefore, by [20, Theorem 2.1], [16, Theorem VIII.1.2] problem (6.50) admits a solution $u_0' \in D^{1,2}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, $q_0' \in L^2(\mathbb{R}^3)$. By the same argument as in the end of the proof of Lemma 6.1, we see that $(u_0', q_0') = (u_0, q_0)$ and that it is the only solution to (6.50) within the class $L^6(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. We thus obtain

$$u_0 \in \mathcal{M}_\gamma(\mathbb{R}^3), \qquad q_0 \in L^2(\mathbb{R}^3). \tag{6.54}$$

By $T : f \mapsto u$ we denote the solution operator for (6.9) defined by the third assertion of Lemma 6.1 ($u = Tf$ has the representation (6.10)). Given $U \in \mathcal{M}_\gamma(\mathbb{R}^3)$, we deduce from (6.39) and (6.41) that $f = -U \cdot \nabla\widetilde{u} = -\operatorname{div}(\widetilde{u} \otimes U)$ satisfies (6.12) with $\alpha = \gamma + 1$, $s = 2$ and $f_0 = 0$:

$$\lceil \widetilde{u} \otimes U \rceil_{\gamma+1} \leqslant \lceil \widetilde{u} \rceil_1 \lceil U \rceil_\gamma, \qquad U \cdot \nabla\widetilde{u} \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \tag{6.55}$$

This together with (6.54) shows that the mapping $\mathcal{Z} : \mathcal{M}_\gamma(\mathbb{R}^3) \to \mathcal{M}_\gamma(\mathbb{R}^3)$ is well-defined by

$$\mathcal{Z}U := u_0 - T\left(\operatorname{div}(\widetilde{u} \otimes U)\right)$$

and that the solution of (6.47) can be understood as the fixed point of $\mathcal{Z}$. Let $U^{(1)}, U^{(2)} \in \mathcal{M}_\gamma(\mathbb{R}^3)$. By (6.15) it is easily seen that

$$\begin{aligned} \|\mathcal{Z}U^{(1)} - \mathcal{Z}U^{(2)}\|_{\mathcal{M}_\gamma} = \left\|T\left(\operatorname{div}\left(\widetilde{u} \otimes \left(U^{(1)} - U^{(2)}\right)\right)\right)\right\|_{\mathcal{M}_\gamma} &\leqslant C_* \lceil \widetilde{u} \otimes \left(U^{(1)} - U^{(2)}\right) \rceil_{\gamma+1} \\ &\leqslant C_* \lceil \widetilde{u} \rceil_1 \|U^{(1)} - U^{(2)}\|_{\mathcal{M}_\gamma} \end{aligned}$$

with some constant $C_* = C_*(\gamma) > 0$. This implies that $\mathcal{Z}$ is a contraction mapping and thus provides a solution $U \in \mathcal{M}_\gamma(\mathbb{R}^3)$, together with the pressure $Q = q_0 - \mathcal{P} * \big(\mathrm{div}(\widetilde{u} \otimes U)\big) \in L^2(\mathbb{R}^3)$, to (6.47) provided

$$\lceil \widetilde{u} \rceil_1 < \frac{1}{C_*}. \tag{6.56}$$

Set

$$\delta = \delta(\varepsilon) = \frac{\eta}{4C_0}, \qquad \eta = \eta(\varepsilon) = \min\left\{\frac{1}{2}, \frac{1}{C_*}\right\}$$

where C_0 and $C_* = C_*(\gamma)$ are the constants in (6.41) and (6.56), respectively, and γ is taken such that $\frac{3}{3/2+\varepsilon} < \gamma < 2$. Suppose that

$$\limsup_{|x|\to\infty} |x||v(x)| \leqslant \delta.$$

Then there is a constant

$$R_1 = R_1(\varepsilon) \geqslant \max\left\{2R_0, 1 + \frac{|\xi|}{|\omega|}\right\},$$

see (6.40), such that

$$|x||v(x)| < 2\delta = \frac{\eta}{2C_0} \qquad \text{for } |x| \geqslant R_1.$$

By virtue of (6.41) we take

$$R = R(\varepsilon) := \max\left\{R_1, \frac{2C_0|\Phi|}{\eta}\right\} \tag{6.57}$$

to accomplish both (6.49) and (6.56), which completes the proof. □

Conflict of interest statement

There is no conflict of interest.

References

[1] K.I. Babenko, On stationary solutions of the problem of flow past a body of a viscous incompressible fluid, Math. USSR Sb. 20 (1973) 1–25.
[2] M.E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Sov. Math. Dokl. 20 (1979) 1094–1098.
[3] W. Borchers, H. Sohr, On the equations rot $v = g$ and div $u = f$ with zero boundary conditions, Hokkaido Math. J. 19 (1990) 67–87.
[4] C. Fabre, G. Lebeau, Prolongement unique des solutions de l'équation de Stokes, Commun. Partial Differ. Equ. 21 (3–4) (1996) 573–596.
[5] R. Farwig, The stationary exterior 3D-problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces, Math. Z. 211 (1992) 409–447.
[6] R. Farwig, G.P. Galdi, M. Kyed, Asymptotic structure of a Leray solution to the Navier–Stokes flow around a rotating body, Pac. J. Math. 253 (2011) 367–382.
[7] R. Farwig, T. Hishida, Asymptotic profile of steady Stokes flow around a rotating obstacle, Manuscr. Math. 136 (2011) 315–338.
[8] R. Farwig, T. Hishida, Leading term at infinity of steady Navier–Stokes flow around a rotating obstacle, Math. Nachr. 284 (2011) 2065–2077.
[9] R. Farwig, T. Hishida, D. Müller, L^q-theory of a singular "winding" integral operator arising from fluid dynamics, Pac. J. Math. 215 (2004) 297–312.
[10] R. Finn, Estimates at infinity for stationary solutions of the Navier–Stokes equations, Bull. Math. Soc. Sci. Math. Phys. R.P. Roumanie 3 (1959) 387–418.
[11] R. Finn, On the exterior stationary problem for the Navier–Stokes equations, and associated perturbation problems, Arch. Ration. Mech. Anal. 19 (1965) 363–406.
[12] A.V. Fursikov, M.D. Gunzburger, L.S. Hou, Optimal boundary control for the evolutionary Navier–Stokes system: the three-dimensional case, SIAM J. Control Optim. 43 (6) (2005) 2191–2232.
[13] G.P. Galdi, On the steady, translational self-propelled motion of a symmetric body in a Navier–Stokes fluid, in: Quaderni di Matematica della II Universita di Napoli, vol. 1, 1997, pp. 97–169.
[14] G.P. Galdi, On the steady self-propelled motion of a body in a viscous incompressible fluid, Arch. Ration. Mech. Anal. 148 (1999) 53–88.

[15] G.P. Galdi, On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications, in: Handbook of Mathematical Fluid Dynamics, vol. 1, 2002, pp. 655–679.
[16] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Steady-State Problem, second ed., Springer, 2011.
[17] G.P. Galdi, M. Kyed, Steady-state Navier–Stokes flows past a rotating obstacle: Leray solutions are physically reasonable, Arch. Ration. Mech. Anal. 200 (2011) 21–58.
[18] G.P. Galdi, A.L. Silvestre, The steady motion of a Navier–Stokes liquid around a rigid body, Arch. Ration. Mech. Anal. 184 (2007) 371–400.
[19] G.P. Galdi, A.L. Silvestre, Further results on steady-state flow of a Navier–Stokes liquid around a rigid body. Existence of the wake, RIMS Kôkyûroku Bessatsu B1 (2007) 127–143.
[20] T. Hishida, L^q estimates of weak solutions to the stationary Stokes equations around a rotating body, J. Math. Soc. Jpn. 58 (2006) 743–767.
[21] A. Korolev, V. Šverák, On the large-distance asymptotics of steady state solutions of the Navier–Stokes equations in 3D exterior domains, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 28 (2011) 303–313.
[22] S. Kracmar, A. Novotny, M. Pokorny, Estimates of Oseen kernels in weighted L^p spaces, J. Math. Soc. Jpn. 53 (2001) 59–111.
[23] M. Kyed, On the asymptotic structure of a Navier–Stokes flow past a rotating body, J. Math. Soc. Jpn. 66 (2014) 1–16.
[24] J. San Martín, T. Takahashi, M. Tucsnak, A control theoretic approach to the swimming of microscopic organisms, Q. Appl. Math. 65 (2007) 405–424.