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Existence of global solutions and global attractor for the third order Lugiato–Lefever equation on **T**

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Abstract

We show the existence of global solution and the global attractor in $L^2(\mathbf{T})$ for the third order Lugiato–Lefever equation on $\mathbf{T}$. Without damping and forcing terms, it has three conserved quantities, that is, the $L^2(\mathbf{T})$ norm, the momentum and the energy, but the leading term of the energy functional is not positive definite. So only the L^2 norm conservation is useful for the third order Lugiato–Lefever equation unlike the KdV and the cubic NLS equations. Therefore, it seems important and natural to construct the global attractor in $L^2(\mathbf{T})$. For the proof of the global attractor, we use the smoothing effect of cubic nonlinearity for the reduced equation.

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1. Introduction and main theorems

We consider the third order Lugiato–Lefever equation:

$$\partial_t u - \partial_x^3 u + i\alpha \partial_x^2 u + u + i|u|^2 u = f, \qquad t > 0, \quad x \in \mathbf{T}, \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{2}$$

where α is a real constant such that $2\alpha/3 \notin \mathbf{Z}$. In (1), all the parameters are normalized except for α. The case $\alpha > 0$ is called focusing and the case $\alpha < 0$ is called defocusing. In the physical context, the third order Lugiato–Lefever equation includes the detuning term $i\theta u$ ($\theta \in \mathbf{R}$) on the left hand side, but we omit the detuning term because it does not matter in this paper. From a mathematical point of view, it is more natural to regard equation (1) as the KdV type equation with second order dispersion. However, in most physical contexts, equation (1) is called the third order

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(or the generalized) Lugiato–Lefever equation and so in the present paper, we follow this convention. Recently the generalized Lugiato–Lefever equation has been attracting a great interest especially in the field of nonlinear optics. It appears as a mathematical model, for example, for Kerr frequency comb generation in a whispering gallery mode resonator [3], octave-spanning Kerr frequency comb in a micro-ring resonator [5], and cavity solitons in micro-ring resonator near zero group-dispersion [17]. An increasing attention among theoretical and experimental physicists in that field has been paid to the role of third order dispersion, i.e. the third order derivative in (1) (see [16,17] and [22]).

In this paper, we show the global well-posedness in $L^2(\mathbf{T})$ of the Cauchy problem (1) and (2) and investigate the nonlinear smoothing effect. This enables us to prove the existence of the global attractor in L^2 for flows generated by the third order Lugiato–Lefever equation (1). Without damping and forcing the solution u of (1) and (2) formally satisfies the following three conservations, that is, the mass, the momentum and the energy conservations for $t > 0$ (see [22, lines 7 to 10 on p. 2326]).

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \tag{3}$$

$$\mathrm{Im}\,(\partial_x u(t), u(t))) = \mathrm{Im}\,(\partial_x u_0, u_0), \tag{4}$$

$$\begin{aligned} &\mathrm{Im}\,(\partial_x^2 u(t), \partial_x u(t)) + \alpha\|\partial_x u(t)\|_{L^2}^2 - \frac{1}{2}\|u(t)\|_{L^4}^4 \\ &= \mathrm{Im}\,(\partial_x^2 u_0, \partial_x u_0) + \alpha\|\partial_x u_0\|_{L^2}^2 - \frac{1}{2}\|u_0\|_{L^4}^4, \end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ denotes the scalar product of $L^2(\mathbf{T})$. The energy functional defined as in (5) is neither positive definite nor negative definite, because it includes the L^2 scalar product of the second and the first derivatives of the solution. This suggests that the energy is not useful for controlling the global behavior of the solution. Therefore, we need to consider the global solution in L^2 and as a result, we need to construct the global attractor in L^2 instead of the H^1 global attractor. The construction of global attractor in L^2 causes a serious problem on the compactness of orbit.

In this paper, we prove the following two theorems concerning the well-posedness and the global attractor for the Cauchy problem (1) and (2).

Theorem 1.1. *(i) (Local existence) Assume that $u_0 \in L^2$ and $f \in C([0,\infty); L^2)$. Then, there exists a positive constant T such that the Cauchy problem (1) and (2) has a unique solution u on $[0,T]$ satisfying*

$$u \in C([0,T]; L^2)\bigcap L^4([0,T]\times\mathbf{T}).$$

(ii) (Global existence and a priori estimate) The solution given by part (i) can be extended to any positive times and satisfies the following identity

$$\|u(t)\|_{L^2}^2 = e^{-2t}\|u_0\|_{L^2}^2 + 2\int_0^t e^{-2(t-s)}\mathrm{Re}\,\big(u(s), f(s)\big)\,ds, \quad t > 0.$$

Theorem 1.2. *Assume that $2\alpha/3 \notin \mathbf{Z}$ and that f is a time-independent function in $L^2(\mathbf{T})$. The third order Lugiato–Lefever equation (1) has the global attractor in $L^2(\mathbf{T})$.*

Remark 1.3. (i) In Theorem 1.1, the external forcing term f is a function of variables t and x, while f is a time-independent function in Theorem 1.2. This is because equation (1) should be autonomous as we consider the global attractor in Theorem 1.2.

(ii) Theorem 1.1 holds for all $\alpha \in \mathbf{R}$, while our proof of Theorem 1.2 requires the assumption that $2\alpha/3 \notin \mathbf{Z}$. When $2\alpha/3 \in \mathbf{Z}$, it is open whether Proposition 3.5 in Section 3 holds or not. In that case, the nonresonance estimate (40) breaks down. The resonance structure for (1) is similar to that for the cubic nonlinear Schrödinger equation, but the combination of the third and the second order dispersions gives rise to a new resonance. This resonance seems to be a specific feature of the third order Lugiato–Lefever equation (1), because it never occurs for the modified KdV equation (see [2,21] and [24]) and for the cubic nonlinear Schrödinger equation (see [12] and [15]). It is an interesting problem what influence the resonance coming from the coupling of the third and the second order dispersions has on the regularity and the global behavior of the solution for (1).

To show the global well-posedness in $L^2(\mathbf{T})$, in Section 2, we prove the space-time integrability of solution for the linear inhomogeneous third order Schrödinger equation, which is called the Strichartz estimate. Theorem 1.1 is an immediate consequence of the Strichartz estimate. To show the existence of the global attractor, in Section 3, we prove a kind of the smoothing effect, which is not the same as that of the parabolic equation. Instead of the original equation (1), we consider the so-called reduced equations resulting from the removal of terms which give rise to the rapid oscillation of the solution (see (26) and (28) in Section 3). In [6] and [7], Erdoğan and Tzirakis use the smoothing effect of the Duhamel term to construct the global attractor for the KdV and the Zakharov equations (see also [13], [18] and [26]). However, the whole Duhamel term can not become more regular than the initial datum in the case of the third order Lugiato–Lefever equation (1), which is in sharp contrast to the KdV and the Zakharov equations (for more details, see Remark 3.9 in Section 3).

We now list notations which are used throughout this paper. For any $a \in \mathbf{C}$, we put $\langle a\rangle = 1 + |a|$. Let $U(t) = e^{t(\partial_x^3 - i\alpha\partial_x^2)}$. Let $\tilde{f}$ denote the Fourier transform of f in both the time and spatial variables. Let $\hat{f}$ denote the Fourier transform of f only in the spatial variable x or only in the time variable t. For $T > 0$, we put $\mathcal{T} = \min\{T, 1\}$. For b, $s \in \mathbf{R}$, we define the Fourier restriction norms $\|\cdot\|_{Y^{b,s}}$ and $\|\cdot\|_{\bar{Y}^{b,s}}$ as follows.

$$\|f\|_{Y^{b,s}} = \left\{ \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle k\rangle^{2s} \langle \tau + k^3 - \alpha k^2\rangle^{2b} |\tilde{f}(\tau,k)|^2 \, d\tau \right\}^{1/2},$$

$$\|f\|_{\bar{Y}^{b,s}} = \left\{ \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle k\rangle^{2s} \langle \tau + k^3 + \alpha k^2\rangle^{2b} |\tilde{f}(\tau,k)|^2 \, d\tau \right\}^{1/2}.$$

We also define the spaces $Y^{b,s}$ and $\bar{Y}^{b,s}$ by the completions of $C_0^\infty(\mathbf{R} \times \mathbf{T})$ in the norms $\|\cdot\|_{Y^{b,s}}$ and $\|\cdot\|_{\bar{Y}^{b,s}}$, respectively.

2. Strichartz' estimate for the linear third order Lugiato–Lefever equation

Theorem 1.1 follows from the standard contraction argument if we have proved the so-called Strichartz estimate for the linear third order Schrödinger equation (see, e.g., [2, Section 4 on pp. 135–142]). So, in this section, we only describe the proof of the Strichartz estimate and leave the proof of Theorem 1.1 to the reader.

We now consider the following inhomogeneous linear Schrödinger equation with third order dispersion on one dimensional torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$.

$$\partial_t u - \partial_x^3 u + i\alpha\partial_x^2 u = f, \qquad t \in \mathbf{R}, \quad x \in \mathbf{T}, \tag{6}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{7}$$

where α is a real constant. We have the following L^4 space-time integrability estimate of solution for (6) and (7).

Theorem 2.1. *Let $T > 0$ and let $1/2 > b > 1/3$. Then, we have*

$$\|u\|_{L^4((-T,T)\times\mathbf{T})} \le CT^{1/2}\mathcal{T}^{-b}\big[\|u_0\|_{L^2(\mathbf{T})} + T^{1/2}\mathcal{T}^{-b}\|f\|_{L^{4/3}((-T,T)\times\mathbf{T})}\big], \tag{8}$$

where $\mathcal{T} = \min\{T, 1\}$ and C is a positive constant dependent only on b.

Remark 2.2. (i) Theorem 2.1 holds valid for all $\alpha \in \mathbf{R}$.

(ii) It is presumed that Theorem 2.1 may hold with the L^4 norm replaced by the L^p norm for some $p > 4$ on the left hand side of (8) for the same reason as it is conjectured for the Schrödinger equation of second order and the linear KdV equation (see [2]). The Strichartz estimate in the case of $\mathbf{T}$ is more complicated than that in the case of $\mathbf{R}$. For example, a sharp necessary condition for the Strichartz estimate in the $\mathbf{R}$ case follows directly from the scaling, but it is not the case with the Strichartz estimate on $\mathbf{T}$. The specific property of each equation only reflects on the lower bound of the index b as long as we consider the L^4 type Strichartz estimate (see, e.g., the proof of Proposition 2.4 in Section 2).

Theorem 2.1 is more or less known (for the Schrödinger equation and the linear KdV equation, see [2]), but there seems to be no literature which contains the statement and the proof of Theorem 2.1 explicitly. Moreover, the problem in the case of **T** has not been studied as well as in the case of **R**. So we present the proof of Theorem 2.1 in this paper.

In this section, we describe the proof Theorem 2.1. We begin with the following lemma about the estimate of the integral of the convolution type.

Lemma 2.3. *Let* $1/2 \geq b > 1/4$ *and* $0 < \varepsilon < 4b - 1$. *Then, for any* $a \in \mathbf{R}$, *we have*

$$\int_{-\infty}^{\infty} \frac{1}{\langle a - x\rangle^{2b} \langle x\rangle^{2b}}\, dx \leq \frac{C}{\langle a\rangle^{4b-1-\varepsilon}},$$

where C *is a positive constant independent of* a.

Proof. We denote the integral on the left hand side of the inequality by I. We split the integral into two parts as follows.

$$I = \int_{|x|\geq|a|/2} + \int_{|x|\leq|a|/2} =: I_1 + I_2.$$

When $|x| \geq |a|/2$, we have

$$\begin{aligned} I_1 &\leq \frac{C}{\langle 1 + |a|/2\rangle^{4b-1-\varepsilon}} \int_{-\infty}^{\infty} \frac{dt}{\langle x - a\rangle^{2b} \langle x\rangle^{-2b+1+\varepsilon}} \\ &\leq \frac{C}{\langle a\rangle^{4b-1-\varepsilon}}. \end{aligned}$$

Since $|x - a| \geq |a| - |x| \geq |a|/2$ for $|x| \leq |a|/2$, we have

$$\begin{aligned} I_2 &\leq \frac{C}{\langle 1 + |a|/2\rangle^{4b-1-\varepsilon}} \int_{-\infty}^{\infty} \frac{dt}{\langle x - a\rangle^{-2b+1+\varepsilon} \langle x\rangle^{2b}} \\ &\leq \frac{C}{\langle a\rangle^{4b-1-\varepsilon}}. \end{aligned}$$

Therefore, we obtain the desired inequality. □

We next prove the L^4 space-time estimate, which is a variant of the so-called Strichartz estimate for the linear KdV equation with second order dispersion.

Proposition 2.4. *Let* $b > 1/3$. *Then, we have*

$$\|f\|_{L^4(\mathbf{R}\times\mathbf{T})} \leq C\|f\|_{Y^{b,0}},$$

where C *is a positive constant dependent only on* b.

Proof. We follow the argument by Kenig, Ponce and Vega [14, the proof of Lemma 5.2] (see also [24, the proof of Lemma 2.1]). We divide the proof into the following two cases:

(Case 1) $2\alpha/3 \in \mathbf{Z}$,

(Case 2) $2\alpha/3 \notin \mathbf{Z}$.

From now on, we suppose Case 1, because Case 2 can be treated in the same way. We set $(f * g)(\tau) = \int_{\mathbf{R}} f(\tau - \tau_1)g(\tau_1)\, d\tau_1$. By the Parseval identity, we have

$$\begin{aligned}\|\widetilde{f\times f}\|^2_{L^2(\mathbf{R}\times\mathbf{T})} &\le C\sum_{k=-\infty}^{\infty}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\big|\tilde f(\cdot,k_1)*\tilde f(\cdot,k_2)\big|\Big)^2\,d\tau \\ &= C\sum_{k\neq 2\alpha/3}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\big|\tilde f(\cdot,k_1)*\tilde f(\cdot,k_2)\big|\Big)^2\,d\tau \\ &\quad + C\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=2\alpha/3}\big|\tilde f(\cdot,k_1)*\tilde f(\cdot,k_2)\big|\Big)^2\,d\tau =: I_1+I_2.\end{aligned} \tag{9}$$

We note that $k_1+k_2\neq 2\alpha/3$ in Case 2. So in Case 2, equation (15), which appears below, is always quadratic with respect to k_1. By the Schwarz inequality and the Minkowski inequality, we see that when $b>1/4$,

$$\begin{aligned} I_2 &\le C\int_{\mathbf{R}}\Big[\sum_{k_1=-\infty}^{\infty}\Big\{\langle\cdot+k_1^3-\alpha k_1^2\rangle^{-2b} \\ &\quad *\langle\cdot+(2\alpha/3-k_1)^3-\alpha(2\alpha/3-k_1)^2\rangle^{-2b}\Big\}^{1/2} \\ &\quad \times\Big\{\big(\langle\cdot+k_1^3-\alpha k_1^2\rangle^{2b}|\tilde f(\cdot,k_1)|^2\big) \\ &\quad *\big(\langle\cdot+(2\alpha/3-k_1)^3-\alpha(2\alpha/3-k_1)^2\rangle^{2b}|\tilde f(\cdot,-k_1+2\alpha/3)|^2\big)\Big\}^{1/2}\Big]^2\,d\tau \\ &\le C\Big[\sum_{k_1=-\infty}^{\infty}\Big\{\int_{\mathbf{R}^2}\langle\tau-\tau_1+k_1^3-\alpha k_1^2\rangle^{2b}|\tilde f(\tau-\tau_1,k_1)|^2 \\ &\quad \times\langle\tau_1+(2\alpha/3-k_1)^3-\alpha(2\alpha/3-k_1)^2\rangle^{2b}|\tilde f(\tau_1,2\alpha/3-k_1)|^2\,d\tau_1 d\tau\Big\}^{1/2}\Big]^2 \\ &\le C\|f\|^4_{Y^{b,0}}.\end{aligned}$$

Next we suppose that

$$\tilde g(\tau,k)=\tilde h(\tau,k)=0\quad(\tau\in\mathbf{R},\ k<0).$$

Then, for the estimate of I_1, it suffices to show that

$$\sum_{k\neq 2\alpha/3}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde g(\tau-\tau_1,k_1)||\tilde h(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau \le C\|g\|^2_{Y^{b,0}}\|h\|^2_{Y^{b,0}}, \tag{10}$$

$$\sum_{k\neq 2\alpha/3}\int_{\mathbf{R}}\Big(\sum_{k_1+k_2=k}\int_{\mathbf{R}}|\tilde g(\tau-\tau_1,k_1)||\tilde h(\tau_1,k_2)|\,d\tau_1\Big)^2\,d\tau \le C\|g\|^2_{\bar Y^{b,0}}\|h\|^2_{\bar Y^{b,0}} \tag{11}$$

for $b>5/16$. Indeed, if we write $f=f_1+f_2$ with $\tilde f_1(\tau,k)=\tilde f(\tau,k)$ $(k\ge 0)$ and $\tilde f_2(\tau,k)=\tilde f(\tau,k)$ $(k<0)$, the $L^2(\mathbf{T})$ norms of $(f_1)^2$, f_1f_2 and $(f_2)^2$ can be evaluated by virtue of the above estimate (10). This is because we have by the Parseval identity and the fact that $\tilde{\bar f}(\tau,k)=\overline{\tilde f}(-\tau,-k)$,

$$\|(f_2)^2\|_{L^2(\mathbf{R}\times\mathbf{T})}=\|(\bar f_2)^2\|_{L^2(\mathbf{R}\times\mathbf{T})}=\|(\tilde f_2)^-*(\tilde f_2)^-\|_{L^2(\mathbf{R}\times\mathbf{T})}, \tag{12}$$

$$\|f_1f_2\|_{L^2(\mathbf{R}\times\mathbf{T})}=\|f_1\bar f_2\|_{L^2(\mathbf{R}\times\mathbf{T})}\le\big\||\tilde f_1|*|(\tilde f_2)^-|\big\|_{L^2(\mathbf{R}\times\mathbf{T})}, \tag{13}$$

where $(\tilde f_2)^-(\tau,k)=\tilde f_2(-\tau,-k)$ and $\tilde f*\tilde g$ denotes the convolution in both τ and k of $\tilde f$ and $\tilde g$. Here, we note that if $f\in Y^{b,s}$, then $\mathcal{F}^{-1}(\tilde f)^-,\ \mathcal{F}^{-1}|(\tilde f)^-|\in\bar Y^{b,s}$, where $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f. Therefore, the right hand side of (12) can be estimated by (10) and the right hand side of (13) can be estimated by (10) and (11).

We only show the estimate (10), since (11) can be proved in the same way as (10). We denote the left hand side of (10) by J and we have by the Schwarz inequality

$$J \le C \sum_{\substack{k\in\mathbf{Z}\\ k\ne 2\alpha/3}} \int_{\mathbf{R}} \Big(\sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0}} \int_{\mathbf{R}} \langle \tau-\tau_1+k_1^3-\alpha k_1^2\rangle^{-2b} \times \langle \tau_1+k_2^3-\alpha k_2^2\rangle^{-2b}\,d\tau_1\Big) \times \Big(\sum_{k_1+k_2=k} \int_{\mathbf{R}} \langle \tau-\tau_1+k_1^3-\alpha k_1^2\rangle^{2b} |\tilde g(\tau-\tau_1,k_1)|^2 \times \langle \tau_1+k_2^3-\alpha k_2^2\rangle^{2b}|\tilde h(\tau_1,k_2)|^2\,d\tau_1\Big)\,d\tau \le CM\|g\|_{Y^{b,0}}^2\|h\|_{Y^{b,0}}^2,$$

where

$$M = \sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{2\alpha/3\})} \Big[\sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0}} \int_{\mathbf{R}} \langle \tau-\tau_1+k_1^3-\alpha k_1^2\rangle^{-2b} \times \langle \tau_1+k_2^3-\alpha k_2^2\rangle^{-2b}\,d\tau_1\Big].$$

Consequently, for the proof of (10), it suffices to show that $M<\infty$. A simple computation and Lemma 2.3 yield

$$M \le C \sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{2\alpha/3\})} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0}} \Big\langle \tau+k_1^3-\alpha k_1^2 + (k-k_1)^3-\alpha(k-k_1)^2\Big\rangle^{-4b+1+\varepsilon} \tag{14}$$

for any ε with $0<\varepsilon<4(b-1/4)$

For each $(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{2\alpha/3\})$, we consider the following algebraic equation with respect to k_1, which corresponds to the formula inside the brackets on the right hand side of (14).

$$(2\alpha-3k)k_1^2 - k(2\alpha-3k)k_1 - k^3+\alpha k^2-\tau = 0. \tag{15}$$

We denote two roots of the algebraic equation (15) with respect to k_1 by β and γ, respectively. If the two roots are real, we write β and γ for those real roots. If the two roots are complex, that is, if $\beta=\bar\gamma$ and $\Im\beta\ne 0$, then we simply use the same notation β and γ for the real part of β and γ. In either case, there exist at most 8 k_1's such that

$$|k_1-\beta|<2 \text{ or } |k_1-\gamma|<2,$$

and we can choose $\eta>0$ so that for the other k_1's,

$$\big|k_1^2-kk_1-(2\alpha-3k)^{-1}(\tau+k^3-\alpha k^2)\big| \ge |(k_1-\beta)(k_1-\gamma)| \ge \eta\langle k_1-\beta\rangle\langle k_1-\gamma\rangle.$$

On the other hand, the condition $k_1\ge 0$ and $k-k_1\ge 0$ implies that $k\ge k_1\ge 0$. Since $|k-2\alpha/3|\ge 1$, we have

$$|k-2\alpha/3| \ge \frac12\Big[\min\Big\{\frac{3}{2|\alpha|},1\Big\}|k_1-2\alpha/3|+1\Big].$$

Indeed, we see that if $k\ge k_1\ge 2\alpha/3$, then $|k-2\alpha/3|\ge|k_1-2\alpha/3|$ and that if $k\ge 2\alpha/3\ge k_1\ge 0$, then $|k-2\alpha/3|\ge 3/(2\alpha)|k_1-2\alpha/3|$. Furthermore, we can choose $\varepsilon>0$ so small that $3(4b-1-\varepsilon)>1$. Therefore, the right hand side of (14) is bounded by the following:

$$C \sup_{(\tau,k)\in\mathbf{R}\times(\mathbf{Z}\setminus\{2\alpha/3\})} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 0\\ k\ne 0}} \frac{1}{\langle (k-2\alpha/3)(k_1-\beta)(k_1-\gamma)\rangle^{4b-1-\varepsilon}} \le C\sum_{k_1\in\mathbf{Z}} \frac{1}{\langle (|k_1|+1)(k_1-\beta)(k_1-\gamma)\rangle^{4b-1-\varepsilon}}$$

$$\le C\Big(8+\sum_{\substack{|k_1-\beta|\ge 2\\ |k_1-\gamma|\ge 2}}\frac{1}{\langle k_1\rangle^{4b-1-\varepsilon}\langle k_1-\beta\rangle^{4b-1-\varepsilon}\langle k_1-\gamma\rangle^{4b-1-\varepsilon}}\Big)$$
$$\le C\Big\{8+\Big(\sum_{k_1\in\mathbf{Z}}\frac{1}{\langle k_1\rangle^{3(4b-1-\varepsilon)}}\Big)^{1/3}\Big(\sum_{|k_1-\beta|\ge 2}\frac{1}{\langle k_1-\beta\rangle^{3(4b-1-\varepsilon)}}\Big)^{1/3}$$
$$\times\Big(\sum_{|k_1-\gamma|\ge 2}\frac{1}{\langle k_1-\gamma\rangle^{3(4b-1-\varepsilon)}}\Big)^{1/3}\Big\}<\infty,$$

since $3(4b-1-\varepsilon)>1$. This inequality shows that $M<\infty$ and so the proof is complete. □

Remark 2.5. (i) We use Lemma 2.3 to show (14) in the above proof of Proposition 2.4. Therefore, we need to assume that $b>1/4$, which corresponds to the Sobolev embedding in the time variable: $H^b(\mathbf{R})\subset L^4(\mathbf{R})$ $(b\ge 1/4)$.

(ii) The question of what happens as $\alpha\to 0$ seems to be interesting. For this passage to the limit, the above proof of Proposition 2.4 should be slightly modified as follows. We divide the summation into two cases $k\ne 0$ and $k=0$ on the right hand side of (9). By I_1 and I_2, we denote the terms corresponding to the summation over $k\ne 0$ and the summation over $k=0$, respectively. The estimate of I_2 is completely the same as above. For I_1, we note that if $k\ge k_1\ge 0$, then $|k-2\alpha/3|\ge \frac{1}{2}\big(|k_1-2\alpha/3|+1\big)$ for all α close to 0. The rest of the proof for I_1 proceeds without any change. The positive constant C on the right hand side of the inequality of Proposition 2.4 can be chosen independent of α as $\alpha\to 0$.

The following corollary is an immediate consequence of Proposition 2.4.

Corollary 2.6. *Let $T>0$ and let $1/2>b>1/3$. Then, we have*

$$\|U(\cdot)u_0\|_{L^4((-T,T)\times\mathbf{T})}\le CT^{1/2}\mathcal{T}^{-b}\|u_0\|_{L^2(\mathbf{T})},$$

where C is a positive constant dependent only on b.

Proof. Let φ be a time cut-off function in $C_0^\infty(\mathbf{R})$ such that $\varphi(t)=1$ for $|t|\le 1$ and $\varphi(t)=0$ for $|t|\ge 2$. We put $\varphi_T(t)=\varphi(t/T)$ for $T>0$. We note that $\varphi_T(t)U(t)u_0\in Y^{b,0}$ for any $b\in\mathbf{R}$, since a simple computation yields

$$\widetilde{\varphi_T U(\cdot)u_0}=T\hat\varphi\big(T(\tau+k^3-\alpha k^2)\big)\hat u_0(k),$$

where $\hat{}$ denotes either the Fourier transform in the time variable or the Fourier coefficient in the spatial variable. Furthermore, for $b>0$,

$$\langle\tau\rangle^{2b}=\big(1+T^{-1}|T\tau|\big)^{2b}\le\mathcal{T}^{-2b}\langle T\tau\rangle^{2b}.$$

Therefore, for $1/2>b>0$, we have by the change of variables

$$\|\varphi_T U(\cdot)u_0\|^2_{Y^{b,0}}$$
$$=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\tau+k^3-\alpha k^2\rangle^{2b}|T\hat\varphi(T(\tau+k^3-\alpha k^2))\hat u_0(k)|^2\,d\tau$$
$$\le\Big(\sum_{k=-\infty}^{\infty}|\hat u_0(k)|^2\Big)\Big(\int_{-\infty}^{\infty}T\mathcal{T}^{-2b}\langle\tau\rangle^{2b}|\hat\varphi(\tau)|^2\,d\tau\Big)$$
$$\le CT\mathcal{T}^{-2b}\|u_0\|^2_{L^2(\mathbf{R})}.$$

Therefore, Proposition 2.4 implies Corollary 2.6. □

We are now in a position to show Theorem 2.1.

Proof of Theorem 2.1. Lemma 2.1 without external forcing f is reduced to Corollary 2.6. When $u_0 = 0$, it is sufficient to prove that

$$\left\| \int_0^t U(t-\tau) f(\tau)\, d\tau \right\|_{L^4((0,T)\times \mathbf{T})} \le C T \mathcal{T}^{-2b} \|f\|_{L^{4/3}((0,T)\times \mathbf{T})}, \tag{16}$$

where C is a positive constant dependent only on T. Because we can easily prove the estimate (16) on $(-T, 0)$ in the same way. From the Christ–Kiselev lemma (see [4] and [23, Lemma 3.1 on p. 2179]), it follows that the proof of (16) is reduced to that of the following inequality.

$$\left\| \int_0^T U(t-\tau) f(\tau)\, d\tau \right\|_{L^4((0,T)\times \mathbf{T})} \le C T \mathcal{T}^{-2b} \|f\|_{L^{4/3}((0,T)\times \mathbf{T})}, \tag{17}$$

where C is a positive constant dependent only on T. Then, Corollary 2.6 yields that

$$\begin{aligned} &\left\| \int_0^T U(t-\tau) f(\tau)\, d\tau \right\|_{L^4(0,T)\times \mathbf{T})} \\ &= \left\| U(t) \int_0^T U(-\tau) f(\tau)\, d\tau \right\|_{L^4(0,T)\times \mathbf{T})} \\ &\le C T^{1/2} \mathcal{T}^{-b} \left\| \int_0^T U(-\tau) f(\tau)\, d\tau \right\|_{L^2(\mathbf{T})}. \end{aligned} \tag{18}$$

Furthermore, we have by the Fubini theorem, Hölder's inequality and Corollary 2.6

$$\begin{aligned} \left| \left(\int_0^T U(-\tau) f(\tau)\, d\tau, v \right) \right| &= \left| \int_0^T \big(f(\tau), U(\tau) v \big)\, d\tau \right| \\ &\le \|f\|_{L^{3/4}((0,T)\times \mathbf{T})} \|U(\cdot) v\|_{L^4((0,T)\times \mathbf{T})} \\ &\le C T^{1/2} \mathcal{T}^{-b} \|f\|_{L^{4/3}((0,T)\times \mathbf{T})} \|v\|_{L^2(\mathbf{T})}, \quad v \in L^2(\mathbf{T}), \end{aligned} \tag{19}$$

where $(\cdot, \cdot)$ denotes the $L^2(\mathbf{T})$ scalar product and C is a positive constant dependent only on b. Accordingly, inequalities (18), (19) and the duality argument imply (17), which completes the proof of Theorem 2.1. □

Remark 2.7. When we use the Christ–Kiselev lemma to derive (16) from (17) in the above proof of Theorem 2.1, we can see explicitly how the right hand side of (16) depends on T and $\mathcal{T}$ (see, e.g., [23, Lemma 3.1 on p. 2179]).

3. Smoothing property and global attractor

In this section, we show the existence of the global attractor for the third order Lugiato–Lefever equation (1) (see Theorem 1.2). For that purpose, we investigate the smoothing effect of cubic nonlinearity (see Proposition 3.5 below). The general strategy to construct the global attractor consists of the following two steps. The first step is to show the absorbing set which absorbs all the trajectories starting from a bounded set in finite time. The second step is to show the compactness of the trajectory starting from the absorbing set. In nonlinear parabolic equations, one employs the smoothing effect to show the latter. But nonlinear dispersive equations such as the third order Lugiato–Lefever equation do not have exactly the same smoothing effect as nonlinear parabolic equations. In order to show the compactness of orbits, from the solution we separate its part smoothened by the nonresonant interaction of the cubic nonlinearity. Throughout this section, we assume that the external forcing term f is a time-independent function in $L^2(\mathbf{T})$.

We begin with the construction of the absorbing set. We multiply (1) by $\bar{u}$ and integrate the resulting equation in the spatial variable to have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{L^2}^2 &= -2\|u\|_{L^2}^2 + 2\mathrm{Re}\big(u(t), f\big) \qquad (20)\\ &\le -2\|u(t)\|_{L^2}^2 + 2\|u(t)\|_{L^2}\|f\|_{L^2} \le -2\|u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \|f\|_{L^2}^2\\ &\le -\|u(t)\|_{L^2}^2 + \|f\|_{L^2}^2, \quad t \ge 0.\end{aligned}$$

This formal calculation can be justified by the regularizing technique. Therefore, the integration in the time variable of the above inequality yields

$$\|u(t)\|_{L^2}^2 \le \big\{e^{-2t}\|u_0\|_{L^2}^2 + (1-e^{-2t})\|f\|_{L^2}^2\big\}, \quad t > 0. \qquad (21)$$

Inequality (21) implies the existence of the absorbing set in L^2 for (1). The closed ball in L^2 is a compact metrizable space equipped with the weak topology of L^2. Accordingly, if the semiflow generated by (1) were weakly continuous in L^2, it would follow from the well-known theorem on the existence of global attractor (see, e.g., [25, Theorem 1.1 on p. 23]) that there exists a unique global attractor $\mathcal{A}$ for the third order Lugiato–Lefever equation (1) in the weak topology of L^2 (see, e.g., Ghidaglia [10]). If it were true, we could show that all the trajectories of (1) would converge to the global attractor $\mathcal{A}$ in the strong topology of L^2, that is, $\mathcal{A}$ would be a global attractor in the strong topology of L^2. We could employ the argument due to Ball [1] and Ghidaglia [11]. However, it follows from the argument due to Molinet [19] and [20] that the weak continuity in L^2 of the solution map fails for the third order Lugiato–Lefever equation (1). In such a case, the use of the reduced equation is often helpful. Instead of (1), we consider the following equation from which the resonance terms breaking the weak continuity are removed.

$$\begin{aligned}&\partial_t v - \partial_x^3 v + i\alpha\partial_x^2 v + v + i\Big(|v|^2 - \frac{1}{\pi}\|v(t)\|_{L^2}^2\Big)v\\ &= e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds} f, \quad t > 0, \quad x \in \mathbf{T}.\end{aligned} \qquad (22)$$

Here, the solution u of (1) is linked to the solution v of (22) by the following transformation

$$u(t,x) = e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds} v(t,x), \qquad (23)$$

and the opposite is also true by the following transformation

$$v(t,x) = e^{\frac{i}{\pi}\int_0^t \|u(s)\|_{L^2}^2\,ds} u(t,x). \qquad (24)$$

We note that $|u(t,x)| = |v(t,x)|$ and so $\|u(t)\|_{L^2} = \|v(t)\|_{L^2}$. Let $S(t,s) : L^2 \to L^2$ $(t \ge s \ge 0)$ be the solution operator from initial data to solutions for (22). The operator $S(t,s)$ satisfies

$$S(t,s)S(s,r) = S(t,r) \quad (t \ge s \ge r \ge 0), \qquad S(t,t) = I \quad (t \ge 0), \qquad (25)$$

where I is the identity operator. When $f = 0$, the factor $\|u(t)\|_{L^2}$ does not appear in (22). In that case, the L^2 weak continuity of the flow map $S(t,s)$ holds, while it fails for the inhomogeneous case. Therefore, we can not expect the weak continuity in L^2 from the semiflow of the reduced equation (22).

We take the Fourier transform of (22) to have the following equation.

$$\begin{aligned}&\partial_t \hat{v}(t,k) + \big(i(k^3 - \alpha k^2) + 1\big)\hat{v}(t,k) \qquad (26)\\ &+ i \sum_{\substack{k_1+k_2+k_3=k\\ (k_1+k_2)(k_2+k_3)\neq 0}} \hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3) - i|\hat{v}(t,k)|^2 v(t,k)\\ &= \hat{f}(k)e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds}, \qquad t > 0.\end{aligned}$$

We further define $w(t,x)$ as follows.

$$\hat{w}(t,k) = e^{-i\int_0^t |\hat{v}(s,k)|^2\,ds}\hat{v}(t,k). \qquad (27)$$

Then, $\hat{w}(t,k)$ satisfies

$$\begin{aligned} &\partial_t \hat{w}(t,k) + \big(i(k^3-\alpha k^2)+1\big)\hat{w}(t,k) \\ &+ i e^{i\int_0^t |\hat{v}(s,k)|^2\,ds} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)\neq 0}} \hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3) \\ &= \hat{f}(k) e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds - i\int_0^t |\hat{v}(s,k)|^2\,ds}, \qquad t>0. \end{aligned} \tag{28}$$

Here, we note that the nonlinear term on the left hand side of (28) is expressed in terms of the solution v and not of w, because it is easier to evaluate the nonlinear interaction of v than that of w. We now estimate the solution w of (28) through the estimate of the solution v to (22). The analogous argument is used in [21] and [24] for the modified KdV equation. We note that $|\hat{w}(t,k)| = |\hat{v}(t,k)|$ and so $\|w(t)\|_{L^2} = \|v(t)\|_{L^2}$. Therefore, by (28), we obtain

$$\begin{aligned} \hat{w}(t,k) &= e^{-(i(k^3-\alpha k^2)+1)t}\hat{u}_0 \\ &- i\int_0^t e^{-(i(k^3-\alpha k^2)+1)(t-s)} e^{i\int_0^s |\hat{v}(\tau,k)|^2\,d\tau} \\ &\times \Big[\sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_2+k_3)\neq 0}} \hat{v}(s,k_1)\hat{\bar{v}}(s,k_2)\hat{v}(s,k_3)\Big]\,ds \\ &+ \hat{f}(k)\int_0^t e^{-(i(k^3-\alpha k^2)+1)(t-s) - \frac{i}{\pi}\int_0^s \|v(\tau)\|_{L^2}^2\,d\tau - i\int_0^s |\hat{v}(\tau,k)|^2\,d\tau}\,ds, \qquad t>0. \end{aligned} \tag{29}$$

On the right hand side of (29), the first term converges to 0 as $t \to \infty$, and so all what we have to do is to investigate the second term and the third term, from which we can expect the smoothing property.

We first look into the third term. By $\hat{J}(t,k)$, we denote the third term on the right hand side of (29). We now show that the orbit of $J(t)$ is precompact in L^2. The integration by parts yields

$$\begin{aligned} \hat{J} &= \hat{f}(k)\Big[\big(i(k^3-\alpha k^2)+1\big)^{-1} e^{-(i(k^3-\alpha k^2)+1)(t-s) - \frac{i}{\pi}\int_0^s \|v(\tau)\|_{L^2}^2\,d\tau - i\int_0^s |\hat{v}(\tau,k)|^2\,d\tau}\Big]_{s=0}^{s=t} \\ &\quad - \big(i(k^3-\alpha k^2)+1\big)^{-1}\hat{f}(k)\int_0^t \Big(-\frac{i}{\pi}\|v(s)\|_{L^2}^2 - i|\hat{v}(s,k)|^2\Big) \\ &\quad \times e^{-(i(k^3-\alpha k^2)+1)(t-s) - \frac{i}{\pi}\int_0^s \|v(\tau)\|_{L^2}^2\,d\tau - i\int_0^s |\hat{v}(\tau,k)|^2\,d\tau}\,ds \\ &= \hat{f}(k)\big(i(k^3-\alpha k^2)+1\big)^{-1}\Big[e^{-\frac{i}{\pi}\int_0^t \|v(\tau)\|_{L^2}^2\,d\tau - i\int_0^t |\hat{v}(\tau,k)|^2\,d\tau} - e^{-(i(k^3-\alpha k^2)+1)t} \\ &\quad - \int_0^t \Big(-\frac{i}{\pi}\|v(s)\|_{L^2}^2 - i|\hat{v}(s,k)|^2\Big) e^{-(i(k^3-\alpha k^2)+1)(t-s) - \frac{i}{\pi}\int_0^s \|v(\tau)\|_{L^2}^2\,d\tau - i\int_0^s |\hat{v}(\tau,k)|^2\,d\tau}\,ds\Big]. \end{aligned}$$

The following lemma will be useful when one proves a set of orbits of rapidly oscillating solutions is precompact.

Lemma 3.1. *Let A and B be two positive constants. We define two sets X_1 in $C([0,\infty); L^2)$ and X_2 in $C([0,\infty); \ell^2)$ as follows.*

$$X_1 = \Big\{g(t) \in C([0,\infty); L^2) \;\Big|\; \sup_{t\ge 0}\|g(t)\|_{L^2} \le A\Big\},$$

$$\begin{aligned} X_2 = \Big\{&h(t) = (\cdots, h_{-1}(t), h_0(t), h_{+1}(t), \cdots) \;\Big| \\ &h_k(t) \in C([0,\infty); \mathbf{C}), \quad \sup_{k\in\mathbf{Z}, t\ge 0} |h_k(t)| \le B\Big\}. \end{aligned}$$

For any $\eta > 0$, $D \subset X_1$ and $E \subset X_2$, let $F(t)$ be an L^2-valued function on $[0,\infty) \times D \times E$ such that

$$F(t,g,h)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\langle k\rangle^{-\eta}\hat{g}(t,k)h_k(t)e^{ikx},\quad t\geq 0,\ g\in D,\ h\in E.$$

We put $G=\{F(t,g,h)|\ t\geq 0,\ g\in D,\ h\in E\}$. Then, the set G is precompact in L^2.

Proof. We put

$$\tilde{F}(t,g,h)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\hat{g}(t,k)h_k(t)e^{ikx},\quad t\geq 0,\ g\in D,\ h\in E.$$

Let $\tilde{G}=\{\tilde{F}(t,g,h)|\ t\geq 0,\ g\in D,\ h\in E\}$. By the Parseval identity, we note that

$$\begin{aligned}\sup_{\substack{t\geq 0,\\ g\in D,\\ h\in E}}\|\tilde{F}(t,g,h)\|_{L^2}^2&\leq\sup_{\substack{t\geq 0,\\ g\in D,\\ h\in E}}\Big[\sum_{k=-\infty}^{\infty}|\hat{g}(t,k)|^2|h_k(t)|^2\Big]\\ &\leq\big[\sup_{g\in D}(\sup_{t\geq 0}\|g(t)\|_{L^2}^2)\big]\big[\sup_{h\in E}(\sup_{\substack{t\geq 0,\\ k\in\mathbf{Z}}}|h_k(t)|^2)\big]\leq A^2B^2.\end{aligned}$$

Hence, the set $\tilde{G}$ is a bounded set in L^2. Now we define the linear operator $T_\eta: L^2\to L^2$ as follows.

$$T_\eta:\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbf{Z}}\hat{v}(k)e^{ikx}\mapsto\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbf{Z}}\langle k\rangle^{-\eta}\hat{v}(k)e^{ikx},\qquad v\in L^2(\mathbf{T}).$$

Obviously, the operator T_η is a compact operator in L^2. Therefore, the image $T_\eta(\tilde{G})$ of the set $\tilde{G}$ by T_η is precompact in L^2. Since $G=T_\eta(\tilde{G})$, this completes the proof of Lemma 3.1. □

Remark 3.2. In fact, the above proof of Lemma 3.1 does not require the continuity in L^2 of $g(t)$. The notion of the compactness for orbits is different from that of the continuity for solutions.

Lemma 3.1 ensures that the orbit of J, that is, $\{J(t)|\ t\geq 0\}$ is precompact in L^2.

We next show that the second term on the right hand side of (29) has the smoothing property. We first show the time local estimate of the solution v to (26), which, together with the damping effect, yields the time global estimate of the solution w to (29). For that purpose, we consider the following integral equation associated with (26) whose initial datum is prescribed at $t=r>0$.

$$\begin{aligned}\hat{v}(t,k)&=e^{-i(k^3-\alpha k^2)t}\hat{v}_r+\int_r^t e^{-i(k^3-\alpha k^2)(t-s)}\hat{v}(s,k)\,ds\\ &-i\int_r^t e^{-i(k^3-\alpha k^2)(t-s)}\Big[\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}}\hat{v}(s,k_1)\hat{\bar{v}}(s,k_2)\hat{v}(s,k_3)\\ &\qquad-|\hat{v}(s,k)|^2\hat{v}(s,k)\Big]\,ds\\ &+\hat{f}(k)\int_r^t e^{-i(k^3-\alpha k^2)(t-s)-\frac{i}{\pi}\int_r^s\|v(\tau)\|_{L^2}^2\,d\tau}\,ds,\quad t>r>0,\end{aligned}\tag{30}$$

where

$$\hat{v}_r(k)=\hat{u}(r,k)e^{\frac{i}{\pi}\int_0^r\|u(s)\|_{L^2}^2\,ds}.$$

Here, we note that the damping term is treated as an inhomogeneous term in (30). By (21), we easily see that

$$\|v_r(k)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + \|f\|_{L^2}^2, \quad r \ge 0. \tag{31}$$

We have the following identity, which represents the resonance in the cubic nonlinear interaction.

$$\begin{aligned}
&(\tau + k^3 - \alpha k^2) - (\tau_1 + k_1^3 - \alpha k_1^2)\\
&- (\tau_2 + k_2^3 + \alpha k_2^2) - (\tau_3 + k_3^3 - \alpha k_3^2)\\
&= (k_1 + k_2 + k_3)^3 - \alpha(k_1 + k_2 + k_3)^2\\
&- (k_1^3 + k_2^3 + k_3^3) + \alpha(k_1^2 - k_2^2 + k_3^2)\\
&= 3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1 - 2\alpha/3),
\end{aligned} \tag{32}$$

where

$$\tau = \tau_1 + \tau_2 + \tau_3, \quad k = k_1 + k_2 + k_3.$$

This identity implies that no resonance occurs in the third term on the right hand side of (30), which leads to the smoothing property of the second term on the right hand side of (29). We define function φ on $\mathbf{R}$ as follows.

$$\varphi(t) = \begin{cases} 1, & 0 < t < 1,\\ 0, & \text{otherwise.} \end{cases}$$

For $r, T > 0$ and $g \in C([0,\infty); L^2)$, we put

$$\begin{aligned}
&\varphi_{r,T}(t) = \varphi((t-r)/T),\\
&\psi_{r,T,g}(t, D_x)u = \frac{e^t}{\sqrt{2\pi}}\varphi_{r,T}(t)\sum_{k=-\infty}^{\infty} e^{i\int_0^t |\hat{g}(s,k)|^2\,ds}\hat{u}(t,k)e^{ikx}.
\end{aligned}$$

We easily see that for $\tau \in \mathbf{R}$,

$$|\hat{\varphi}_{r,T}(\tau)| \le 2T\langle T\tau\rangle^{-1}, \tag{33}$$

$$|\tilde{\psi}_{r,T,g}(\tau,k)| \le 2(1 + \sup_{t\ge 0}\|g(t)\|_{L^2})e^{r+T}\langle\tau\rangle^{-1}, \tag{34}$$

since simple computations yield, for $r, T > 0$ and $g \in C([0,\infty); L^2)$,

$$\begin{aligned}
\hat{\varphi}_{r,T}(\tau) &= \frac{e^{-ir\tau}\left(1 - e^{-iT\tau}\right)}{i\tau},\\
\tilde{\psi}_{r,T,g}(\tau,k) &= \frac{e^r\left(e^T e^{-i(r+T)\tau + i\int_0^{r+T}|\hat{g}(s,k)|^2\,ds} - e^{-ir\tau + i\int_0^r |\hat{g}(s,k)|^2\,ds}\right)}{1 - i\tau}\\
&+ \frac{i}{1 - i\tau}\int_r^{r+T}\left(e^{t - it\tau + i\int_0^t |\hat{g}(s,k)|^2\,ds}\right)|\hat{g}(t,k)|^2 dt.
\end{aligned}$$

The following lemma follows directly from (33) and (34).

Lemma 3.3. *Let $s \in \mathbf{R}$, $1 \ge T > 0$, $r, b, \eta > 0$ and let $g \in C([0,\infty); L^2)$. Assume that $b + \eta < 1/2$ and $b + 2\eta \ge 1/2$. Then, we have*

$$\|\varphi_{r,T}u\|_{Y^{s,b}} \le CT^{1/2-(b+\eta)}\|u\|_{Y^{s,b+\eta}}, \tag{35}$$

$$\|\psi_{r,T,g}u\|_{Y^{s,b}} \le Ce^{r+T}(1 + \sup_{t\ge 0}\|g(t)\|_{L^2})\|u\|_{Y^{s,b+\eta}}. \tag{36}$$

Proof. Lemma 3.3 essentially follows from the Sobolev embedding with respect to the time variable t. We only show (35) because (36) can be treated in the same way. We may assume $s = 0$ without loss of generality. Let $U(t) = e^{t(\partial_x^3 - i\alpha\partial_x^2)}$.

$$\|\varphi_{r,T}u\|_{Y^{0,b}} \le C\|\langle D_t\rangle^b U(-t)\varphi_{r,T}u\|^2_{L^2(\mathbf{R}^2)}$$
$$= C\int_{\mathbf{T}}\int_{\mathbf{R}} \big|\langle D_t\rangle^b U(-t)\varphi_{r,T}u\big|^2 dt dx = C\int_{\mathbf{T}} I(x)\, dx, \tag{37}$$

where $I = \int_{\mathbf{R}}\big|\langle D_t\rangle^b U(-t)\varphi_{r,T}u\big|^2 dt$. We put $g = U(-t)\varphi_{r,T}u$ and we regard g as a function of the time variable t by fixing the spatial variable x for the moment. Let $\langle D_t\rangle = \mathcal{F}^{-1}\langle\tau\rangle\mathcal{F}$. So, by the Plancherel theorem and (33), we have

$$I = C\int_{\mathbf{R}}\Big|\langle\tau\rangle^b \int_{\mathbf{R}} T\langle T(\tau-\tau')\rangle^{-1}\hat{g}(\tau')\, d\tau'\Big|^2 d\tau$$
$$\le C\Big[\int_{\mathbf{R}}\Big|\int_{\mathbf{R}}\big\{\big(\langle\tau-\tau'\rangle^b T\langle T(\tau-\tau')\rangle^{-1}\big)\hat{g}(\tau')$$
$$+ T\langle T(\tau-\tau')\rangle^{-1}\big(\langle\tau'\rangle^b\hat{g}(\tau')\big)\big\}\, d\tau'\Big|^2 d\tau\Big]$$

We put $\hat{G}(\tau) = T\langle T\tau\rangle^{-1}$ and $\hat{H}(\tau) = |\hat{g}(\tau)|$. Then, we have

$$G \in H^s(\mathbf{R}), \quad s < 1/2,$$
$$\|\langle D_t\rangle^{b+\eta}G(\cdot/T)\|_{L^2} = T^{1/2-(b+\eta)}\|\langle D_t\rangle^{b+\eta}G(\cdot)\|_{L^2}.$$

Therefore, we obtain

$$\sqrt{I} \le C\Big\{\big\|\big(\langle D_t\rangle^b G(\cdot/T)\big)H\big\|_{L^2(\mathbf{R})}$$
$$+ \big\|G(\cdot/T)\big(\langle D_t\rangle^b H\big)\big\|_{L^2(\mathbf{R})}\Big\}$$
$$\le C\Big\{\|\langle D_t\rangle^b G(\cdot/T)\|_{L^{2/(1-2\eta)}}\|H\|_{L^{1/\eta}}$$
$$+ \|G(\cdot/T)\|_{L^{1/\eta}}\|\langle D_t\rangle^b H\|_{L^{2/(1-2\eta)}}\Big\}$$
$$\le CT^{1/2-(b+\eta)}\|\langle D_t\rangle^{b+\eta}H\|_{L^2}.$$

At the last inequality, we have used the Sobolev embeddings $H^\eta \subset L^{2/(1-2\eta)}$ and $H^{b+\eta} \subset L^{1/\eta}$ for the one dimensional case, since $b+\eta < 1/2$ and $b + 2\eta \ge 1/2$. Thus, the application of the above inequality to the right hand side of (37) leads to the estimate (35). □

We next state the following fundamental lemma concerning the trilinear estimate in terms of the Fourier restriction norms (see, e.g., [2] and [14]).

Lemma 3.4. *Let $r > 0$, $1 \ge T > 0$, $b > 1/3$ and $\eta > 0$. Assume that $b + \eta < 1/2$ and $b + 2\eta \ge 1/2$. Then, there exist two positive constants α and C such that*

$$\big\|\varphi_{r,T}(u\bar{v}w)\big\|_{Y^{0,-b}} \le CT^\alpha\|u\|_{Y^{0,b+\eta}}\|v\|_{Y^{0,b+\eta}}\|w\|_{Y^{0,b+\eta}},$$

where α and C depend only on b.

We apply Lemma 3.4 to (30) to show that there exist $M, T > 0$ such that

$$\|\varphi_{r,T}v\|_{Y^{0,b+\eta}} \le M, \quad r > 0, \tag{38}$$

where M and T are two positive constants depending only on $\|u_0\|_{L^2}$, $\|f\|_{L^2}$, b and η. We are in a position to prove the following proposition concerning the smoothing property of the second term on the right hand side of (29).

Proposition 3.5. *Let $1/2 > b > 1/3$ and $\eta > 0$. Let v be the solution of (26). Assume that $b + 2\eta < 1/2$ and $b + 3\eta \ge 1/2$. We put*

$$\hat{N}(t,k) = \int_0^t e^{-(i(k^3-\alpha k^2)+1)(t-s)} e^{i\int_0^s |\hat{v}(s',k)|^2\,ds'} \times \Big[\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} \hat{v}(s,k_1)\hat{\bar{v}}(s,k_2)\hat{v}(s,k_3)\Big]\,ds.$$

Then, there exists a positive constant C such that

$$\sup_{t\geq 0}\big\|(1-\partial_x^2)^{\eta/2}N(t)\big\|_{L^2} \leq CM^3,$$

where $C = C(\|u_0\|_{L^2}, \|f\|_{L^2}, b, \eta)$ and M is defined as in (38).

Remark 3.6. We note that there exist constants b and η satisfying all the assumptions in Proposition 3.5. For example, we can choose $b = 1/2 - 3\varepsilon$ and $\eta = \varepsilon$ for any ε with $1/18 > \varepsilon > 0$.

Proof of Proposition 3.5. We prove Proposition 3.5 in a slightly more general form. Let $u, v, w \in Y^{0,b+2\eta}$ and let $g \in C([0,\infty); L^2)$ with $\sup_{t\geq 0}\|g(t)\|_{L^2} < \infty$. By $(\cdot,\cdot)$, we denote the scalar product of $L^2(\mathbf{T})$. We put

$$V(u,v,w) = \sum_{k=-\infty}^{\infty}\Big\{\sum_{k_2\in\mathbf{Z}}\big(\hat{u}(-k_2)\hat{\bar{v}}(k_2)\hat{w}(k) + \hat{u}(k)\hat{\bar{v}}(k_2)\hat{w}(-k_2)\big) - \hat{u}(k)\hat{\bar{v}}(-k)\hat{w}(k)\Big\}e^{ikx}.$$

Let $t_0 > 0$ be arbitrarily fixed and let $t_0 \geq r + T > r > 0$. We first note that for any $h \in L^2(\mathbf{T})$,

$$\Big(\int_r^{r+T} \sum_{k=-\infty}^{\infty} e^{-(i(k^3-\alpha k^2)+1)(t_0-s)+ikx} e^{i\int_0^s |\hat{g}(s',k)|^2\,ds'} \times \sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} \hat{u}(s,k_1)\hat{\bar{v}}(s,k_2)\hat{w}(s,k_3)\,ds,\ h\Big)$$
$$= \int_{\mathbf{R}} \big(\varphi_{r,T}(u\bar{v}w - V(u,v,w)),\ \psi^*_{r,T,g} e^{s(\partial_x^3 - i\alpha\partial_x^2)}\tilde{h}\big)\,ds,$$

where $\tilde{h} = e^{-t_0(\partial_x^3 - i\alpha\partial_x^2)}h$ and $\psi^*_{r,T,g}$ is the adjoint operator of $\psi_{r,T,g}(t, D_x)$. Since $\tilde{h} \in L^2(\mathbf{T})$, it suffices to show that for $r > 0$ and $1 \geq T > 0$,

$$\Big|\int_{\mathbf{R}} \big(\varphi_{r,T}(u\bar{v}w - V(u,v,w)),\ \psi^*_{r,T,g}z\big)\,dt\Big| \leq CT^{3(1/2-(b+2\eta))}e^{r+T}(1 + \sup_{t\geq 0}\|g(t)\|_{L^2}) \times \|u\|_{Y^{0,b+2\eta}}\|v\|_{Y^{0,b+2\eta}}\|w\|_{Y^{0,b+2\eta}}\|z\|_{Y^{-\eta,b+2\eta}}. \tag{39}$$

Indeed, if we have proved (39), then we have by duality

$$\Big(\sum_{k=-\infty}^{\infty}\langle k\rangle^{2\eta}\Big|\int_r^{r+T} e^{-(i(k^3-\alpha k^2)+1)(t-s)} e^{i\int_0^s |\hat{v}(s',k)|^2\,ds'} \times \sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} \hat{w}(s,k_1)\hat{\bar{w}}(s,k_2)\hat{w}(s,k_3)\,ds\Big|^2\Big)^{1/2}$$
$$\leq CT^{3(1/2-(b+2\eta))}e^{-(t-(r+T))}\|\varphi_{r,T}w\|^3_{Y^{0,b+2\eta}}, \qquad t \geq r+T > r > 0.$$

At the last inequality, we have used (39) with u, v, w and z replaced by $\varphi_{r,T}u$, $\varphi_{r,T}v$, $\varphi_{r,T}w$ and $\varphi_{r,T}z$, respectively. For $a \in \mathbf{R}$, we denote by $[a]$ the greatest integer that does not exceed a. Let T be given by (38) and $n = [t/T]$. For a, $b \in \mathbf{R}$, we put $a \wedge b = \min\{a, b\}$. Accordingly, we obtain by (38)

$$\|(1-\partial_x^2)^{\eta/2}N(t)\|_{L^2} \le C\sum_{j=1}^{n+1}\Big(\sum_{k=-\infty}^{\infty}\langle k\rangle^{2\eta}\Big|\int\limits_{(j-1)T}^{t\wedge(jT)} e^{-(i(k^3-\alpha k^2)+1)(t-s)}$$

$$\times e^{i\int_0^s |\hat{v}(s',k)|^2\,ds'} \sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} \hat{v}(s,k_1)\hat{\bar{v}}(s,k_2)\hat{v}(s,k_3)\,ds\Big|^2\Big)^{1/2}$$

$$\le C\Big(e^{-(t-nT)} + \sum_{j=1}^{n} e^{-(n-j)T}\Big)T^{3(1/2-(b+2\eta))}\|\varphi_{r,T}v\|^3_{Y^{0,b+2\eta}} \le CM^3, \quad t \ge 0,$$

which shows Proposition 3.5.

It remains only to show (39). To simplify the notation, we write u, v, w and z for $\varphi_{r,T}u$, $\varphi_{r,T}v$, $\varphi_{r,T}w$ and $\psi_{r,T,g}z$, respectively. Then, by the Plancherel theorem, we have

$$\Big|\int\limits_{\mathbf{R}}(u\bar{v}w - V(u,v,w), z)\,dt\Big|$$

$$\le C\int\limits_{\tau_1+\tau_2+\tau_3=\tau}\sum_{k=-\infty}^{\infty}\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} |\tilde{u}(\tau_1,k_1)||\tilde{\bar{v}}(\tau_2,k_2)||\tilde{w}(\tau_3,k_3)|$$

$$\times|\tilde{z}(\tau,k)|\,d\tau_1 d\tau_2 d\tau =: A.$$

We put

$$D = \max\big\{|\tau + k^3 - \alpha k^2|, |\tau_1 + k_1^3 - \alpha k_1^2|, |\tau_2 + k_2^3 - \alpha k_2^2|, |\tau_3 + k_3^3 - \alpha k_3^2|\big\}.$$

By the identity (32), we first have

$$\exists c > 0;\ \ D \ge c|k|. \tag{40}$$

We divide the integral region into the following four parts.

(Region I) $D = |\tau + k^3 - \alpha k^2|$,
(Region II) $D = |\tau_1 + k_1^3 - \alpha k_1^2|$,
(Region III) $D = |\tau_2 + k_2^3 - \alpha k_2^2|$,
(Region IV) $D = |\tau_3 + k_3^3 - \alpha k_3^2|$.

We only consider Region I, since the other regions can be treated in the same way. Let A_I denote the contribution of Region I to the integral A. By the Plancherel theorem, we have

$$A_I \le C\int\limits_{\tau_1+\tau_2+\tau_3=\tau}\sum_{k=-\infty}^{\infty}\sum_{\substack{k_1+k_2+k_3=k\\(k_1+k_2)(k_2+k_3)\neq 0}} |\tilde{u}(\tau,k_1)|$$

$$\times|\tilde{\bar{v}}(\tau,k_2)||\tilde{w}(\tau,k_3)|$$

$$\times(\langle\tau + k^3 - \alpha k^2\rangle^{\eta}|\hat{z}(\tau,k)|)\langle k\rangle^{-\eta/2}\,d\tau_1 d\tau_2 d\tau$$

$$\le C\Big|\int\limits_{\mathbf{R}}(f_1 f_2 f_3, f_4)dt\Big|, \tag{41}$$

where $\tilde{f}_1(\tau,k) = |\tilde{u}(\tau,k)|$, $\tilde{f}_2(\tau,k) = |\tilde{\bar{v}}(\tau,k)|$, $\tilde{f}_3(\tau,k) = |\tilde{w}(\tau,k)|$ and $\tilde{f}_4(\tau,k) = \langle k\rangle^{-\eta}\langle\tau + k^3 - \alpha k^2\rangle^{\eta}|\tilde{z}(\tau,k)|$. Therefore, the application of Proposition 2.4 to (41) yields

$$A_I \le C\|u\|_{Y^{0,b}}\|v\|_{Y^{0,b}}\|w\|_{Y^{0,b}}\|z\|_{Y^{-\eta/2,b+\eta}}.$$

We apply Lemma 3.3 to the right hand side of the above inequality with u, v, w and z replaced by $\varphi_{r,T}u$, $\varphi_{r,T}v$, $\varphi_{r,T}w$ and $\psi_{r,T,g}z$, respectively to have

$$A_I \le CT^{3(1/2-(b+2\eta))}e^{r+T}(1+\sup_{t\ge 0}\|g(t)\|_{L^2}) \\ \times \|u\|_{Y^{0,b+2\eta}}\|v\|_{Y^{0,b+2\eta}}\|w\|_{Y^{0,b+2\eta}}\|z\|_{Y^{-\eta/2,b+2\eta}}$$

for $r>0$ and $1\ge T>0$. At the last inequality, we have used Lemma 3.3 with η replaced by 2η for $\varphi_{r,T}u$, $\varphi_{r,T}v$, $\varphi_{r,T}w$ and with b replaced by $b+\eta$ for $\psi_{r,T,g}z$. This shows (39) for Region I. □

Remark 3.7. (i) By inspecting the above proof of Proposition 3.5, we can show that the constant $C=C(\cdot,\cdot,b,\eta)$ appearing on the right hand side of the inequality is a locally bounded function of the first and the second variables for each fixed b and η.

(ii) The above proof of (39) is slightly redundant, because we have applied (35) to $\varphi_{r,T}u$, $\varphi_{r,T}v$ and $\varphi_{r,T}w$. In fact, unless we derive the factor $T^{3(1/2-(b+\eta))}$ in (39), we do not have to use (35). But we need to apply (36) to $\psi_{r,T,g}z$ to have the factor e^{r+T}.

Finally, by combining Lemma 3.1, Proposition 3.5 and (21), we can see that for any $R>0$, the following set is precompact in L^2.

$$\Big\{\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\hat{N}(t,k)\exp\Big(-i\int_0^t\|w(s)\|_{L^2}^2\,ds+i\int_0^t|\hat{w}(s,k)|^2\,ds+ikx\Big)\ \Big| \\ w \text{ is a solution of (28) with } w(0)=u_0 \text{ and } \|u_0\|_{L^2}\le R,\quad t\ge 0\Big\}.$$

Moreover, for any $R>0$, the following set is also precompact in L^2, as is already proved.

$$\Big\{\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\hat{J}(t,k)\exp\Big(-i\int_0^t\|w(s)\|_{L^2}^2\,ds+i\int_0^t|\hat{w}(s,k)|^2\,ds+ikx\Big)\ \Big| \\ w \text{ is a solution of (28) with } w(0)=u_0 \text{ and } \|u_0\|_{L^2}\le R,\quad t\ge 0\Big\}.$$

The two sets mentioned above correspond to the sets of orbits transformed by (23) and (27) from the nonlinear Duhamel term and the inhomogeneous term of (29), respectively. The rest of the solution u, that is, the homogeneous term $e^{t(\partial_x^3-i\alpha\partial_x^2-1)}u_0$ multiplied by the gauge factors converges to 0 as $t\to\infty$. Therefore, by Theorem 1.1 on p. 23 in [25] and Remark 3.7 (i), we can conclude the existence of the global attractor in L^2 for the third order Lugiato–Lefever equation (1).

The above proof of Proposition 3.5 is partly applicable to the original equation (1). Indeed, by the proof of Proposition 3.5 and Remark 3.6, we obtain the following corollary.

Corollary 3.8. *Assume that $1/18>\eta>0$ and $u_0\in L^2(\mathbf{T})$. Let u be the solution of (1) and (2) given by part (i) of Theorem 1.1. Then, we have*

$$i\int_0^t U(t-t')\big[\big(|u|^2u\big)(t')-R(t')\big]\,dt' \\ \in C([0,\infty);H^\eta(\mathbf{T})),$$

where

$$R(t,x)=\frac{1}{\pi}\|u(t)\|_{L^2}^2u(t,x)+\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbf{Z}}\big(|\hat{u}(t,k)|^2\hat{u}(t,k)\big)e^{ikx}.$$

Remark 3.9. Corollary 3.8 implies that the Duhamel term from which the resonant part is removed has a smoothing effect. This kind of smoothing property is known for some other nonlinear dispersive equations (see, e.g., [6–8,12] and [24]). Especially, in [6–8], Erdoğan and Tzirakis study the global attractor by using the nonlinear smoothing effect for the KdV and the Zakharov equations. But the smoothing effect of the third order Lugiato–Lefever equation (1) is different from that of the KdV and the Zakharov equations. To be more specific, the last term $|\hat{v}(t,k)|^2\hat{v}(t,k)$ on the right hand side of (26) is the worst term as far as the regularity of the solution with initial data in $L^2(\mathbf{T})$ is concerned. This doubly resonant term does not appear in such quadratic nonlinear dispersive equations as the KdV and the Zakharov equations. On the other hand, if one considers the solution with initial datum in $H^s(\mathbf{T})$, $s > 0$, then the term $|\hat{v}(t,k)|^2\hat{v}(t,k)$ also has a smoothing property (see, e.g., [9, Theorem 4.1 on p. 112] for the cubic nonlinear Schrödinger equation). So one does not have the difficulty with the term $|\hat{v}(t,k)|^2\hat{v}(t,k)$ as long as one considers the regular solution of the third order Lugiato–Lefever equation (1). However, in that case, no conservation law corresponding to such regularity is available, as was already pointed out.

By inspecting the above proof of Theorem 1.2, we can have a quantitative result on the regularity of the global attractor. To be more precise, for any $r > 0$, we have by (29), Proposition 3.5, Remark 3.6 and the proof of Theorem 1.2

$$\sup_{\substack{t\geq 0\\ \|u_0\|_{L^2}\leq r}} \|w(t) - e^{t(\partial_x^3 - i\alpha\partial_x^2 - 1)}u_0\|_{H^\eta} < \infty \quad (0 < \eta < 1/18).$$

The transformations (23) and (27) convert this regularity property of w to that of u as follows.

$$\sup_{\substack{t\geq 0\\ \|u_0\|_{L^2}\leq r}} \|u(t) - V(t)u_0\|_{H^\eta} < \infty \quad (0 < \eta < 1/18), \tag{42}$$

where

$$V(t)u_0 = \sum_{k\in\mathbf{Z}} e^{-t(i(k^3-\alpha k^2)+1) - i\int_0^t \|u(s)\|_{L^2}^2\,ds + i\int_0^t |\hat{u}(s,k)|^2\,ds + ikx}\hat{u}_0(k).$$

Since we have

$$\sup_{\|u_0\|_{L^2}\leq r} \|V(t)u_0\|_{L^2} \to 0 \quad (t\to\infty)$$

for any $r > 0$, the following corollary concerning the regularity of the global attractor is an immediate consequence of (42).

Corollary 3.10. *The global attractor given by Theorem 1.2 is a compact set in $H^\eta(\mathbf{T})$ for $1/18 > \eta > 0$.*

Concluding remark After this paper was submitted, we became aware of the paper by Molinet [20] concerning the global attractor in L^2 of the cubic nonlinear Schrödinger equation with damping and forcing terms. The new ingredient of his proof is the application of the argument by Ball [1] to the weak limit equation keeping the same structure as the original equation. It would be possible to apply the proof by Molinet [20] to our problem for $2\alpha/3 \notin \mathbf{Z}$. But the smoothing property we have proved in this paper is stronger than that in the paper [20]. Furthermore, it seems difficult to apply Molinet's proof to the case $2\alpha/3 \in \mathbf{Z}\backslash\{0\}$ for the same reason as in our proof. Indeed, when $2\alpha/3 \in \mathbf{Z}\backslash\{0\}$, equation (26) is written as follows.

$$\begin{aligned}
&\partial_t\hat{v}(t,k) + \big(i(k^3 - \alpha k^2) + 1\big)\hat{v}(t,k) \\
&+ i\sum_{\substack{k_1+k_2+k_3=k\\ (k_1+k_2)(k_2+k_3)\neq 0\\ k_3+k_1\neq 2\alpha/3}} \hat{v}(t,k_1)\hat{\bar{v}}(t,k_2)\hat{v}(t,k_3) - i|\hat{v}(t,k)|^2 v(t,k) \\
&+ i\sum_{k_1\in\mathbf{Z}} \hat{v}(t,k_1)\hat{v}(t,2\alpha/3 - k_1)\hat{\bar{v}}(t,k-2\alpha/3) \\
&= \hat{f}(k)e^{-\frac{i}{\pi}\int_0^t \|v(s)\|_{L^2}^2\,ds}, \qquad t > 0.
\end{aligned} \tag{43}$$

The last term on the left hand side of (43) prevents us from applying not only our proof but also the proof by Molinet [20].

Conflict of interest statement

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