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Damping of particles interacting with a vibrating medium

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Abstract

We investigate the large time behavior of the solutions of a Vlasov–Fokker–Planck equation where particles are subjected to a confining external potential and a self-consistent potential intended to describe the interaction of the particles with their environment. The environment is seen as a medium vibrating in a direction transverse to particles' motion. We identify equilibrium states of the model and justify the asymptotic trend to equilibrium. The analysis relies on hypocoercivity techniques.

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1. Introduction

This work concerns the long-time behavior of the solution of the Vlasov equation

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (V + \Phi) \cdot \nabla_v F = \gamma \nabla_v \cdot (vF + \nabla_v F), \qquad t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d, \tag{1}$$

where Φ is self-consistently defined by the relations

$$\begin{cases} \Phi(t,x) = \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, & t \geq 0,\ x \in \mathbb{R}^d, \\ \big(\partial_{tt}^2 \Psi - c^2 \Delta_z \Psi\big)(t,x,z) = -\sigma_2(z) \displaystyle\int_{\mathbb{R}^d} \sigma_1(x-y)\rho(t,y)\,\mathrm{d}y, & t \geq 0,\ x \in \mathbb{R}^d,\ z \in \mathbb{R}^n, \\ \text{with } \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v. & \end{cases} \tag{2}$$

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The system is complemented with the initial data

$$F(0,x,v)=F_0(x,v),\qquad \Psi(0,x,z)=\Psi_0(x,z),\qquad \partial_t\Psi(0,x,z)=\Psi_1(x,z). \tag{3}$$

The parameters of the problem are set as follows

- $c>0$,
- $\sigma_1:\mathbb{R}^d\to[0,\infty)$ and $\sigma_2:\mathbb{R}^n\to[0,\infty)$ are radially symmetric C^∞ compactly supported functions,
- $V:\mathbb{R}^d\to\mathbb{R}$ is an external *confining* potential:

$$V\in C^0\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d),\qquad \lim_{|x|\to\infty}V(x)=\infty.$$

We will make the technical assumptions precise later on. A crucial role in the analysis will be played by the following entropy dissipation property

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(F\frac{v^2}{2}+F(V+\Phi)+F\ln(F)\Big)\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}(|\partial_t\Psi|^2+c^2|\nabla_z\Psi|^2)\,\mathrm{d}z\,\mathrm{d}x\Big\}\\ =-\gamma\int_{\mathbb{R}^d\times\mathbb{R}^d}|2\nabla_v\sqrt{F}+v\sqrt{F}|^2\,\mathrm{d}v\,\mathrm{d}x\le 0.\end{aligned} \tag{4}$$

The investigation of this problem is motivated by the work of S. De Bièvre and L. Bruneau [6] where a related model was introduced to describe the evolution of a single particle interacting with its environment. In [6] the particle is classically described by the pair position/velocity $(q(t),\dot q(t))$, and the dynamics is governed by

$$\begin{cases}\ddot q(t)=-\nabla V(q(t))-\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_1(q(t)-y)\,\sigma_2(z)\,\nabla_x\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z,\\ \partial^2_{tt}\Psi(t,x,z)-c^2\Delta_z\Psi(t,x,z)=-\sigma_2(z)\sigma_1(x-q(t)),\qquad x\in\mathbb{R}^d,\ z\in\mathbb{R}^n.\end{cases} \tag{5}$$

Such single particle description can be retrieved by setting $F(t,x,v)=\delta(x=q(t))\otimes\delta(v=\dot q(t))$ in (1), with $\gamma=0$. The dynamics can be thought of as if membranes continuously distributed transversely to the direction of the particle's motion — $z\in\mathbb{R}^n$ being transverse to $x\in\mathbb{R}^d$ — were activated by the passage of the particle, see Fig. 1 in [6]. The evolution of the system is, therefore, driven by energy exchanges between the particle and the membranes. We remark that the coupling between the particle and the membranes is embodied into the *product* $\sigma_1(x)\sigma_2(z)$, which appears symmetrically in the two equations of (5). This is crucial to establish Hamiltonian properties of (5) and its counterpart for the kinetic model, namely relation (4). The system is presented as a "dynamical Lorentz gas" and one is interested in asymptotic properties of the dynamics. This question has been further investigated in a series of papers by S. De Bièvre and his collaborators [1,9–11,22], that contains both theoretical results and convincing numerical experiments. On the one hand, the system has certain dissipative features: under certain circumstances (roughly speaking, $n=3$ and c large enough) the particle energy can be dissipated in the membranes, and the environment behaves like a friction force on the particle. In particular, when V is a confining potential with a (non-degenerate) minimum at q_0, then the particle stops at the location q_0 as time goes to ∞, see [6, Section 5, Theorem 4]. On the other hand, in [1,10] an approximated model is proposed, together with an interpretation of the dynamics in terms of random walk. This simplified framework permits to justify the approach to thermal equilibrium: the particle's momentum distribution is driven to a Maxwell–Boltzmann distribution.

We wish to revisit these questions in the framework of kinetic equations, where the description by the position/velocity pair is replaced by (1) when considering a distribution of particles in phase space $F(t,x,v)\ge 0$. More precisely, in the case $\gamma=0$, $\int_A F(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$ can be interpreted as the probability $\mathbb{P}\big((q(t),\dot q(t))\in A\big)$ when the initial state of the particle is distributed according to F_0. The analysis of existence and uniqueness of weak solutions for the non-linearly coupled problem (1)–(2), with $\gamma=0$, was established in [8], where it was also shown that a certain physical regime drives the solutions of (1)–(2) to solutions of the attractive Vlasov–Poisson system. It is likely that this approach can be combined to the analysis of the smoothing effect of the Fokker–Planck operator in [4,5] in order to investigate the well-posedness of the problem when $\gamma>0$. We will not elaborate more on this issue in this paper and, instead, focus on the long-time behavior of the solutions of (1)–(2). We treat the question by adding a dissipative structure through the Fokker–Planck term $\gamma \nabla_v \cdot (vF + \nabla_v F)$, with $\gamma > 0$, on the right hand side of (1). It corresponds to considering a large set of particles governed by (5) where, in addition, we add a friction term $-\gamma \dot{q}(t)$ and a Brownian motion term which can be attributed to the positive temperature of the medium [6,9]. We refer the reader to [29] for the analysis of the mean field regime that drives from the particles description to the Fokker–Planck equation, and to [17] for the adaptation to derive (1)–(2). Although this term drastically simplifies the objectives of [6], since the model with $\gamma = 0$ is supposed to contain by itself friction/dissipation mechanisms, the dissipative model already leads to non-trivial issues due to non-linear coupling of the interactions, and this first attempt on the PDE system (1)–(2) confirms the intuition that comes from the analysis of (5). Moreover, as a by-product, we are able to identify a family of stationary solutions of the system when $\gamma = 0$, which is less obvious than for the case of a single particle, and we establish that these solutions are linearly stable for the dissipationless model.

The paper is organized as follows. In Section 2, we will exhibit stationary solutions of (1)–(2). Having introduced the necessary notation, we give the statement of our main result, namely the convergence to equilibrium at exponential rate. As a preliminary step to understand the long-time behavior, it is convenient to discuss the so-called "diffusive scaling" for the problem (1)–(2). This is the object of Section 3. In Section 4, we investigate the large time behavior of the solutions. Our analysis relies on the assumption that the wave speed is sufficiently large. In this regime (1) appears, in some sense, as a perturbation of the linear Fokker–Planck equation with external potential V. In this context, the method recently presented in [13], and inspired from [19], based on hypocoercivity arguments becomes quite useful. We will follow such an approach where, roughly speaking, the goal is to define a suitable Lyapounov functional which combines the natural entropy of the problem and an additional inner product that allows us to control the hydrodynamic part of the solution. Furthermore, the solutions exhibited in Section 2 are also stationary solutions of the dissipationless model ($\gamma = 0$); we investigate their linearized stability in the Appendix.

2. Equilibrium states

We rewrite the Fokker–Planck operator, hereafter denoted by L, as follows

$$LF = \nabla_v \cdot (vF + \nabla_v F) = \nabla_v \cdot \Big(M \nabla_v \frac{F}{M} \Big), \qquad M(v) = (2\pi)^{-d/2} e^{-v^2/2}.$$

This form indicates the dissipative effect of this operator; in particular we have

$$\int_{\mathbb{R}^d} LF \frac{F}{M} \, \mathrm{d}v = -\int_{\mathbb{R}^d} M \Big| \nabla_v \Big(\frac{F}{M} \Big) \Big|^2 \mathrm{d}v,$$

which already shows that $\mathrm{Ker}(L) = \mathrm{Span}(M)$. We search for equilibrium solutions of (1)–(2), which means solutions independent of the time variable t, that make both the "transport part" and the "collisional part" of the equation vanish, namely we seek $\mathscr{M}_{\mathrm{eq}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, such that

$$\text{(a)} \quad L\mathscr{M}_{\mathrm{eq}} = 0, \qquad \text{(b)} \quad (v \cdot \nabla_x - \nabla_x (V + \Phi_{\mathrm{eq}}) \cdot \nabla_v) \mathscr{M}_{\mathrm{eq}} = 0.$$

Condition (b) is reached by any function depending on the total energy $v^2/2 + (V + \Phi_{\mathrm{eq}})(x)$, while, as said above, the kernel of L imposes a precise dependence with respect to the velocity variable. Combining (a) and (b), therefore, leads to

$$\mathscr{M}_{\mathrm{eq}}(x, v) = Z_{\mathrm{eq}} \exp\Big(-\frac{v^2}{2} - V(x) - \Phi_{\mathrm{eq}}(x) \Big).$$

In this formula, Z_{eq} is a normalizing factor. Indeed, (1) is mass preserving in the sense that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} F(t, x, v) \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(0, x, v) \, \mathrm{d}v \, \mathrm{d}x \stackrel{\mathrm{def}}{=} \mathfrak{m},$$

and therefore Z_{eq} is such that $\mathscr{M}_{\mathrm{eq}}$ has also mass $\mathfrak{m}$, which yields

$$Z_{\text{eq}} = \mathfrak{m} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-v^2/2 - V(x) - \Phi_{\text{eq}}(x)} \, \mathrm{d}v \, \mathrm{d}x \right)^{-1} = \frac{\mathfrak{m}}{(2\pi)^{d/2}} \left(\int_{\mathbb{R}^d} e^{-V(x) - \Phi_{\text{eq}}(x)} \, \mathrm{d}x \right)^{-1}.$$

However, we should take into account the non-linearity of the problem by revisiting the definition of the self-consistent potential in (2). Considering stationary solutions, (2) becomes

$$-c^2 \Delta_z \Psi_{\text{eq}}(x,z) = -\sigma_2(z)\, \sigma_1 * \rho_{\text{eq}}(x),$$
$$\rho_{\text{eq}}(x) = \int_{\mathbb{R}^d} \mathscr{M}_{\text{eq}}(x,v)\, \mathrm{d}v,$$
$$\Phi_{\text{eq}}(x) = \left(\sigma_1 * \int_{\mathbb{R}^n} \sigma_2(z) \Psi_{\text{eq}}(\cdot, z)\, \mathrm{d}z \right)(x).$$

For further purposes, it is convenient to keep in mind the following notation

$$\mathscr{M}_{\text{eq}}(x,v) = \rho_{\text{eq}}(x) M(v), \qquad \rho_{\text{eq}}(x) = (2\pi)^{d/2} Z_{\text{eq}} e^{-(\Phi_{\text{eq}} + V)(x)}.$$

For the stationary problem, the space variable x and the transverse variable z decouple. Let $\Upsilon : \mathbb{R}^n \to \mathbb{R}$ be the solution of

$$-\Delta_z \Upsilon = \sigma_2$$

(defined by the convolution of σ_2 by the fundamental solution of $(-\Delta)$ in $\mathbb{R}^n$). We obtain

$$\Psi_{\text{eq}}(x,z) = -\frac{1}{c^2} \Upsilon(z)\, \sigma_1 * \rho_{\text{eq}}(x).$$

It follows that the equilibrium potential satisfies

$$\Phi_{\text{eq}}(x) = -\frac{\Lambda}{c^2}\, \Sigma * \rho_{\text{eq}}(x)$$

where

$$\Lambda = \int_{\mathbb{R}^n} \sigma_2(z) \Upsilon(z)\, \mathrm{d}z, \qquad \Sigma = \sigma_1 * \sigma_1.$$

As far as $n \geq 3$, we justify the integration by parts that leads to

$$\Lambda = \int_{\mathbb{R}^n} |\nabla_z \Upsilon(z)|^2\, \mathrm{d}z \in (0, \infty).$$

Eventually, by combining the information, we are led to define the equilibrium potential as the solution of the nonlinear equation

$$\Phi_{\text{eq}}(x) = -\frac{(2\pi)^{d/2} \Lambda}{c^2} Z_{\text{eq}} \int_{\mathbb{R}^d} \Sigma(x-y) e^{-V(y) - \Phi_{\text{eq}}(y)}\, \mathrm{d}y. \tag{6}$$

This discussion motivates the introduction of the following mapping

$$\mathscr{T} : \Phi \mapsto -\Lambda Z[\Phi] \int_{\mathbb{R}^d} \Sigma(x-y) e^{-V(y) - \Phi(y)}\, \mathrm{d}y, \qquad Z[\Phi] = \frac{\mathfrak{m}}{(2\pi)^{d/2}} \left(\int_{\mathbb{R}^d} e^{-V(x) - \Phi(x)}\, \mathrm{d}x \right)^{-1},$$

and defining equilibrium states as fixed point of $\frac{1}{c^2}\mathscr{T}$. Before stating our first result, let us collect here the confining assumptions on the external potential:

$$e^{-V} \in L^1(\mathbb{R}^d). \tag{A1}$$

$$\liminf_{|x|\to\infty} \left(\left|\nabla_x V(x)\right|^2 - 2\Delta_x V(x)\right) > 0. \tag{A2}$$

$$\begin{aligned}&\text{There exists } c_1, c_2 > 0, \text{ and } 0 < c_3 < 1 \text{ such that}\\ &\Delta_x V \le c_1 + \frac{c_3}{2}\left|\nabla_x V\right|^2, \qquad \left|D_x^2 V\right| \le c_2\left(1 + \left|\nabla_x V\right|\right).\end{aligned} \tag{A3}$$

For the existence of equilibrium states, only (A1) will be useful; the other assumptions will be used for the analysis of the large time behavior.

Theorem 2.1. *Let $n \ge 3$. Assume (A1). There exists $c_0 > 0$ such that for any $c > c_0$, the mapping $\frac{1}{c^2}\mathscr{T}$ admits a unique fixed point $\Phi \in C^0 \cap L^\infty(\mathbb{R}^d)$. If $0 < c \le c_0$, $\frac{1}{c^2}\mathscr{T}$ admits at least one fixed point.*

Proof. Let ρ be a non-negative function such that $\int_{\mathbb{R}^d} \rho \,\mathrm{d}x = \mathfrak{m}$. We also suppose that the product $\rho\, e^V$ belongs to L^∞. Then, $\hat{\Phi} : x \mapsto \hat{\Phi}(x) = -\frac{(2\pi)^{d/2}\Lambda}{c^2}\Sigma * \rho(x)$ is continuous and satisfies

$$0 \ge \hat{\Phi}(x) \ge -\frac{(2\pi)^{d/2}\Lambda}{c^2}\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m} \stackrel{\text{def}}{=} -\frac{\kappa}{c^2}.$$

It follows that, on the one hand

$$0 \le e^{-V} \le e^{-\hat{\Phi}-V} \le e^{\kappa/c^2}\, e^{-V}$$

and, on the other hand

$$\frac{\mathfrak{m}}{(2\pi)^{d/2}}\, e^{-\kappa/c^2}\left(\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x\right)^{-1} \le Z[\hat{\Phi}] \le \frac{\mathfrak{m}}{(2\pi)^{d/2}}\left(\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x\right)^{-1}. \tag{7}$$

By applying this reasoning to $\rho = (2\pi)^{d/2} Z[\Phi] e^{-V-\Phi}$, we conclude that $\frac{1}{c^2}\mathscr{T}$ leaves invariant the set

$$\mathscr{C} = \left\{\Phi \in C^0(\mathbb{R}^d),\ -\kappa/c^2 \le \Phi \le 0\right\}.$$

Furthermore, for Φ, $\Phi' \in \mathscr{C}$, we obtain (with obvious notation)

$$\left|\mathscr{T}(\Phi)(x) - \mathscr{T}(\Phi')(x)\right| = \left|-\Lambda\Sigma * (\rho - \rho')(x)\right| \le \Lambda\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\|\rho - \rho'\|_{L^1}$$

with

$$\begin{aligned}\left|\rho(x) - \rho'(x)\right| &= (2\pi)^{d/2} e^{-V(x)}\left|Z[\Phi]e^{-\Phi(x)} - Z[\Phi']e^{-\Phi'(x)}\right|\\ &\le (2\pi)^{d/2} e^{-V(x)}\left(e^{-\Phi(x)}\left|Z[\Phi] - Z[\Phi']\right| + Z[\Phi']\left|e^{-\Phi(x)} - e^{-\Phi'(x)}\right|\right).\end{aligned}$$

Since the elements of $\mathscr{C}$ are bounded, we find

$$\left|e^{-\Phi(x)} - e^{-\Phi'(x)}\right| \le e^{\kappa/c^2}\left|\Phi(x) - \Phi'(x)\right| \le e^{\kappa/c^2}\|\Phi - \Phi'\|_{L^\infty}.$$

Similarly, by using (7), we obtain

$$\left|Z[\Phi] - Z[\Phi']\right| \le e^{\kappa/c^2}\frac{\mathfrak{m}}{(2\pi)^{d/2}\|e^{-V}\|_{L^1}}\|\Phi - \Phi'\|_{L^\infty}.$$

Gathering these estimates we conclude that

$$\left|\mathscr{T}(\Phi)(x) - \mathscr{T}(\Phi')(x)\right| \le 2\Lambda\|\Sigma\|_\infty \mathfrak{m}\, e^{\kappa/c^2}\ \|\Phi - \Phi'\|_{L^\infty}.$$

Therefore $\frac{1}{c^2}\mathscr{T}$ is Lipschitz, with a constant that tends to 0 as $c \to \infty$; we conclude by a direct application of the Banach Fixed Point Theorem, provided c is large enough. Furthermore, we can also remark that, on the one hand, $\nabla_x\mathscr{T}(\Phi)$ is bounded uniformly for any $\Phi \in \mathscr{C}$, and, on the other hand, $\lim_{|x|\to\infty}\mathscr{T}(\Phi) = 0$, uniformly over $\Phi \in \mathscr{C}$, since these quantities can be dominated proportionally to $\int \Sigma(x-y)e^{-V(y)}\,\mathrm{d}y$. By virtue of the Ascoli Theorem, the mapping $\frac{1}{c^2}\mathscr{T}$ is therefore compact on $\mathscr{C}$. The Schauder Theorem proves that $\frac{1}{c^2}\mathscr{T}$ admits at least one fixed point in $\mathscr{C}$; however, uniqueness of the normalized equilibrium state is not guaranteed unless c is sufficiently large. □

Note that the regularity of Φ_{eq}, and that of $\rho_{\mathrm{eq}} = (2\pi)^{d/2} Z[\Phi_{\mathrm{eq}}]e^{-(\Phi_{\mathrm{eq}}+V)}$, is determined by the regularity of V and σ_1. Additionally observe that

$$0 \le \rho_{\mathrm{eq}} \le \frac{\mathfrak{m} e^{\kappa/c^2}}{\|e^{-V}\|_{L^1}} e^{-V}.$$

Remark 2.2. We can interpret the equilibrium states and the role of the smallness assumption on c in terms of the minimization of the following energy functional

$$\mathscr{E}[\rho] = \int_{\mathbb{R}^d} \Big(\rho \ln(\rho) - \frac{\Lambda}{2c^2}\rho\Sigma * \rho + V\rho\Big)\,\mathrm{d}x$$

over integrable non-negative functions with total mass $\mathfrak{m}$. Indeed, with the associated Euler–Lagrange equations we recover the definition of ρ_{eq} and Φ_{eq}. However, we observe that $\rho \mapsto \mathscr{E}[\rho]$ is strictly convex for c large enough, but due to the minus sign in front of the quadratic term, convexity might be lost for small c's.

Let us finish this section stating the main result of the paper which establishes the exponential trend to equilibrium, see Section 4.

Theorem 2.3. *Suppose $n = 3$, and let $\mathscr{E}_0, \mathfrak{m} > 0$ be fixed. We assume that the external potential satisfies* **(A1)**, **(A2)**, **(A3)**. *Then, there exists $c_1 \ge c_0 > 0$ and $\kappa > 0$ such that, for any $c \ge c_1$, and any datum in (3) that fulfills the conditions*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} F_0\,\mathrm{d}v\,\mathrm{d}x = \mathfrak{m}, \tag{A4}$$

$$\int_{\mathbb{R}^d\times\mathbb{R}^3} \big(|\nabla_z(\Psi_0 - \Psi_{\mathrm{eq}})|^2 + |\Psi_1|^2\big)\,\mathrm{d}z\,\mathrm{d}x \le \mathscr{E}_0, \tag{A5}$$

$$F_0 - \mathscr{M}_{\mathrm{eq}} \in L^2\Big(\mathbb{R}^d \times \mathbb{R}^d; \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\Big), \tag{A6}$$

$$\mathrm{supp}(\Psi_0 - \Psi_{\mathrm{eq}}, \Psi_1) \subset \mathbb{R}^d \times B(0, R_{\mathrm{I}}), \tag{A7}$$

we can find $M > 0$ such that the solution of (1)–(3) satisfies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|F(t,x,v) - \mathscr{M}_{\mathrm{eq}}(x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x \le Me^{-\kappa t}.$$

We point out the fact that all the constants c_1, κ, M can be explicitly computed by means of the data $\sigma_1, \sigma_2, F_0, \Psi_0, \Psi_1$. In particular, M is proportional to

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|F_0 - \mathscr{M}_{\mathrm{eq}}|^2}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^3} \big(|\nabla_z(\Psi_0 - \Psi_{\mathrm{eq}})|^2 + |\Psi_1|^2\big)\,\mathrm{d}z\,\mathrm{d}x,$$

which are the norms involved in **(A5)** and **(A6)**.

Remark 2.4. It is clear that the result can be extended to the full variety of collision operators considered in [13]; in particular it applies to, maybe less physical in this context, linear Boltzmann operators, see also [19]. Furthermore, as it will be clear within the proof, the regularity assumption on σ_1 and σ_2 can be relaxed and it is likely that the radial symmetry is not essential, but these assumptions stick to the framework introduced in [6].

3. Diffusion asymptotics

As is recalled in [13, Section 1.2], the intuition on the large time asymptotics can be motivated by investigating first a certain regime, where the PDEs system is rescaled by means of a relevant parameter $0 < \epsilon \ll 1$. Roughly speaking, we rescale the problem so that the Fokker–Planck term becomes stiff. It makes the relaxation effects strong. Since the flux of the equilibrium functions in $\mathrm{Ker}(L)$ vanishes, time and space scales should be appropriately rescaled in order to obtain a non-trivial problem in the limit $\epsilon \to 0$. It can formally be understood through the change of variables $t \to \epsilon^2 t$, $x \to \epsilon x$. Such regimes are usually referred to as "diffusive regimes" in kinetic theory since one is led to a diffusion equation for the limiting macroscopic density. We shall start by introducing more precisely the scaling of the equations, by means of the physical parameters of the model. We will also explain how to rescale the coupling term in the wave equation. Next, we can guess the asymptotic behavior by expanding formally the solution as a power series of the parameter ϵ. Finally, we state and prove the convergence of the solutions as ϵ tends to 0. Entropy dissipation is a crucial ingredient of the proof.

3.1. Scaling of the equations

We follow the dimension analysis proposed in [8]. Let T, L be time and space units, respectively. In dimensional form the right hand side of the kinetic equation should be written

$$\frac{1}{\tau}\nabla_v \cdot (vF + \mathscr{V}^2 \nabla_v F),$$

where the coefficient $\frac{1}{\tau}$ has the homogeneity of the inverse of time and $\mathscr{V}$ has the homogeneity of a velocity. We use $\mathscr{V}$ as a typical size for the velocity fluctuation (the thermal velocity in other contexts). The dimension of the particle distribution function is $\mathrm{L}^{-d}\mathscr{V}^{-d}$, so that $\int F \,\mathrm{d}v\,\mathrm{d}x$ is dimensionless. Considering the energy balance, $|\partial_t \Psi|^2 \,\mathrm{d}y\,\mathrm{d}x$ and $c^2|\nabla_y \Psi|^2 \,\mathrm{d}y\,\mathrm{d}x$ have the same dimension as $v^2 F \,\mathrm{d}v\,\mathrm{d}x$, that is the square of a velocity. With these remarks, we define dimensionless quantities, denoted by $\cdot'$, as follows

$$t = \mathrm{T}t', \qquad x = \mathrm{L}x', \qquad v = \mathscr{V}v',$$
$$\mathrm{L}^d \mathscr{V}^d F(t,x,v) = F'(t',x',v').$$

The external potential scales as $V(x) = \mathrm{U}V'(x')$, where U has the homogeneity of the square of the velocity. For the vibrating field, we set

$$y = \mathrm{L}y', \qquad \sqrt{\mathrm{L}^{d+n}}\,\Psi(t,x,z) = \mathrm{L}\,\Psi'(t',x',z').$$

Finally, the form functions are rescaled as follows

$$\sigma_1(x) = \varsigma_1 \sigma_1'(x'), \qquad \sigma_2(z) = \varsigma_2 \sigma_2'(z'),$$

with the suitable units for ς_1 and ς_2. In particular, we assume that the external potential and the self-consistent potential have the same order of magnitude, which leads to $\mathrm{U} = \varsigma_1\varsigma_2 L^{1+n/2+d/2}$ (note that the individual units of σ_1 and σ_2 do not really matter, the important quantity being their product). We can rewrite the PDE system in dimensionless form

$$\partial_t F + \frac{\mathscr{V}\mathrm{T}}{\mathrm{L}} v \cdot \nabla_x F - \frac{\mathrm{UT}}{\mathrm{L}\mathscr{V}} \nabla_x (V + \Phi) \cdot \nabla_v F = \frac{\mathrm{T}}{\tau} LF,$$
$$\partial_{tt}^2 \Psi - \left(\frac{c\mathrm{T}}{\mathrm{L}}\right)^2 \Delta_y \Psi(t,x,z) = -\frac{\mathrm{UT}^2}{\mathrm{L}^2} \sigma_2(z) \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_1(x-y) F(t,y,v)\,\mathrm{d}v\,\mathrm{d}z,$$

where we get rid of the $\cdot'$ for simplicity of notation. The self-consistent potential is given by

$$\Phi(t,x) = \int\limits_{\mathbb{R}^n \times \mathbb{R}^d} \sigma_2(y)\sigma_1(x-z)\Psi(t,z,y)\,\mathrm{d}z\,\mathrm{d}y.$$

We are interested in the regime where

$$\frac{\mathrm{T}}{\tau} = \frac{1}{\epsilon^2} = \frac{\mathrm{UT}^2}{\mathrm{L}^2}, \qquad \frac{\mathscr{V}\mathrm{T}}{\mathrm{L}} = \frac{1}{\epsilon} = \frac{c\mathrm{T}}{\mathrm{L}},$$

with $0 < \epsilon \ll 1$. Note that both the particles and the waves are "fast" compared to the velocity of observation, with speeds of the same order of magnitude. We arrive at

$$\begin{aligned} &\partial_t F_\epsilon + \frac{1}{\epsilon}(v \cdot \nabla_x - \nabla_x(V + \Phi_\epsilon) \cdot \nabla_v) F_\epsilon = \frac{1}{\epsilon^2} L F_\epsilon, \\ &\partial^2_{tt} \Psi_\epsilon - \frac{1}{\epsilon^2}\Delta_z \Psi_\epsilon(t,x,z) = -\frac{1}{\epsilon^2}\sigma_2(z) \int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma_1(x-y) F_\epsilon(t,y,v)\,\mathrm{d}v\,\mathrm{d}y, \end{aligned} \tag{8}$$

with $\Phi_\epsilon(t,x) = \left(\sigma_1 * \int_{\mathbb{R}^n} \sigma_2(z)\Psi_\epsilon(t,\cdot,z)\,\mathrm{d}z\right)(x)$. The entropy dissipation (4) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{v^2}{2} + V + \Phi_\epsilon + \ln(F_\epsilon)\right) F_\epsilon \,\mathrm{d}v\,\mathrm{d}x + \frac{\epsilon^2}{2} \int_{\mathbb{R}^n\times\mathbb{R}^d} |\partial_t \Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d} |\nabla_z \Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x \right) \\ &\quad = -\frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|2\nabla_v\sqrt{F_\epsilon} + v\sqrt{F_\epsilon}\right|^2 \mathrm{d}v\,\mathrm{d}x. \end{aligned}$$

3.2. *Formal asymptotic by Hilbert expansion*

In order to guess the asymptotic behavior of the system for small ϵ's, we expand the solution as follows

$$F_\epsilon = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \ldots$$

and we plug this expansion into (8). We identify terms arising with the same exponent of ϵ:

a) ϵ^{-2} terms: we get $LF^{(0)} = 0$ which yields $F^{(0)}(t,x,v) = \rho(t,x)M(v)$,
b) ϵ^{-1} terms: the relation $LF^{(1)} = (v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)F^{(0)} = vM(\nabla_x\rho + \rho\nabla_x(V+\Phi))$ yields $F^{(1)}(t,x,v) = -vM(v)(\nabla_x\rho(t,x) + \rho\nabla_x(V+\Phi)(t,x)) + \rho^{(1)}(t,x)M(v)$,
c) ϵ^0 terms: we obtain $LF^{(2)} = \partial_t F^{(0)} + (v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)F^{(1)}$. Integrating with respect to the velocity variable and taking into account the expression for $F^{(1)}$ obtained in Step b) lead to

$$\partial_t\rho - \nabla_x\cdot(\nabla_x\rho + \rho\nabla_x(V+\Phi)) = 0. \tag{9}$$

(Note that the term $\rho^{(1)}(t,x)M(v) \in \mathrm{Ker}(L)$ does not contribute to the equation.)

The self consistent potential is determined by considering the leading terms in the wave equation. We arrive at the relation

$$\Phi(t,x) = -\Lambda \int_{\mathbb{R}^d} \Sigma(x-y)\rho(t,y)\,\mathrm{d}y. \tag{10}$$

Therefore, as $\epsilon \to 0$, we expect that $F_\epsilon(t,x,v)$ converges to $\rho(t,x)M(v)$, with ρ solution of the system (9)–(10).

Theorem 3.1. *We assume $n \geq 3$. We slightly strengthen the confining assumption* (**A1**)*, by assuming $e^{-\nu V/2} \in L^1(\mathbb{R}^d)$ for some $\nu \in (0,1/2)$. Let us denote*

$$\begin{aligned} \mathscr{K}_0 = \sup_{\epsilon>0} \Bigg\{ &\int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon(0,x,v)\left(1 + |\ln(F_\epsilon)(0,x,v)| + V(x) + \frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x \\ &+ \frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_z\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x \Bigg\} \end{aligned} \tag{11}$$

which is assumed to be finite. Then, up to a subsequence, the solutions $F_\epsilon(t,x,v)$ of (8) converge as $\epsilon \to 0$ to $\rho(t,x)M(v)$, with ρ solution of (9)–(10), complemented with the initial data $\rho(0,x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} F_\epsilon(0,x,v)\,\mathrm{d}v$ (in the sense of the weak convergence in $L^1(\mathbb{R}^d)$). The convergence holds strongly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, while $\rho_\epsilon = \int_{\mathbb{R}^d} F_\epsilon\,\mathrm{d}v$ converges to ρ strongly in $L^1((0,T)\times\mathbb{R}^d)$ and in $C([0,T];L^1(\mathbb{R}^d)$-weak).

Remark 3.2. Since it can be shown that the problem (9)–(10), admits a unique solution ρ for a given initial data $\rho_0 \in L^1(\mathbb{R}^d)$, the entire sequence F_ϵ converges to ρM if, in addition to (11), we have $\int_{\mathbb{R}^d} F_\epsilon(0,x,v)\,\mathrm{d}v \rightharpoonup \rho_0$ weakly in $L^1(\mathbb{R}^d)$.

Stationary solutions of (9) satisfy

$$\nabla_x \rho_{\mathrm{eq}} + \rho_{\mathrm{eq}}\nabla_x(V+\Phi_{\mathrm{eq}}) = 0, \qquad \Phi_{\mathrm{eq}} = -\Lambda\Sigma * \rho_{\mathrm{eq}}.$$

Of course, due to the scaling the parameter c has disappeared but this problem has exactly the same form as the one discussed in Section 2. As a matter of fact Theorem 2.1 can be rephrased by saying that there exists a unique stationary solution satisfying $\int \rho_{\mathrm{eq}}\,\mathrm{d}x = \mathfrak{m}$ provided $\Lambda\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}$ is small enough. It can be interpreted as a condition on the coefficients σ_1,σ_2 for instance. It is therefore a natural question to wonder whether the solutions of (9) with a given mass converge to the corresponding stationary state.

Corollary 3.3. *Let $n \geq 3$. We suppose that V is uniformly convex: there exists $\alpha > 0$ such that for any $x \in \mathbb{R}^d$, and any $\xi \in \mathbb{R}^d$, we have $\sum_{i,j=1}^d \partial^2_{x_i x_j} V(x)\xi_j\xi_i \geq \alpha|\xi|^2$. We can find $\lambda_0, \kappa > 0$ such that if $\Lambda\|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)}\mathfrak{m} < \lambda_0$, any solution ρ of (9) with initial data $\rho_0 \geq 0$ such that*

$$\int_{\mathbb{R}^d} \rho_0\,\mathrm{d}x = \mathfrak{m}, \qquad \int_{\mathbb{R}^d} \frac{|\rho_0-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x < \infty$$

satisfies

$$\int_{\mathbb{R}^d} \frac{|\rho(t,x)-\rho_{\mathrm{eq}}(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x \leq e^{-\kappa t}\int_{\mathbb{R}^d} \frac{|\rho_0-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x.$$

The convexity assumption on V clearly implies (**A1**). Furthermore, in the case where the diffusion coefficient in (9) is a mere constant this condition implies the Sobolev inequality

$$C\int_{\mathbb{R}^d} \left|g(x) - \int_{\mathbb{R}^d} g(y)\frac{e^{-V(y)}}{\bar\mu}\,\mathrm{d}y\right|^2 \frac{e^{-V(x)}}{\bar\mu}\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla_x g|^2\,\frac{e^{-V}}{\bar\mu}\,\mathrm{d}x \tag{12}$$

with $\bar\mu = \int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x$. This is indeed the simplest case where the diffusion operator

$$\nabla_x\cdot(\nabla_x\rho + \rho\nabla_x V) = \nabla_x\cdot\Big(e^{-V}\nabla_x\Big(\frac{\rho}{e^{-V}}\Big)\Big)$$

satisfies the so-called Bakry–Emery condition. Consequently, the solutions to the linear equation $\partial_t\rho = \nabla_x\cdot(\nabla_x\rho + \rho\nabla_x V)$ can be shown to converge exponentially fast to the equilibrium $\mathfrak{m}\,e^{-V}/\bar\mu$. We refer the reader to [2] for an overview of these techniques. The remarkable fact is that the smallness condition on the parameters of the non-linear problem ensures that the latter inherits the dissipative structure of the linear equation.

3.3. Proof of Theorem 3.1

In what follows, the initial data $(x,v)\mapsto F_\epsilon(0,x,v)$ is denoted as $F_{\epsilon,0}$. We start by establishing uniform estimates by first recalling mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon\,\mathrm{d}v\,\mathrm{d}x = 0.$$

Additionally, we have identified a entropy-energy functional which is dissipated by the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\limits_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon\Big(\ln(F_\epsilon)+\frac{v^2}{2}+(V+\Phi_\epsilon)\Big)\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n}\big(\epsilon^2|\partial_t\Psi_\epsilon|^2+|\nabla_z\Psi_\epsilon|^2\big)\,\mathrm{d}z\,\mathrm{d}x\right\}$$
$$=-\frac{1}{\epsilon^2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\big|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\big|^2\,\mathrm{d}v\,\mathrm{d}x.$$

The contributions of the terms containing $F_\epsilon\ln(F_\epsilon)$ and $F_\epsilon\Phi_\epsilon$ are not signed, we fix this now. We observe that the self-consistent potential energy can be dominated, using Sobolev's inequality [23, Th. 8.3], as follows

$$\Big|\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Phi_\epsilon F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\Big|\le\|F_\epsilon(t,\cdot)\|_{L^1}\|\Phi_\epsilon(t,\cdot)\|_{L^\infty}$$
$$\le\|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\int\limits_{\mathbb{R}^d}\Big|\int\limits_{\mathbb{R}^n}\sigma_2(z)\Psi_\epsilon(t,x,z)\,\mathrm{d}z\Big|^2\,\mathrm{d}x\Big)^{1/2}$$
$$\le\|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\Big(\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^n}|\Psi_\epsilon(t,x,z)|^{2n/(n-2)}\,\mathrm{d}z\Big)^{(n-2)/n}\,\mathrm{d}x\Big)^{1/2}$$
$$\le\|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\nabla_z\Psi_\epsilon(t,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}$$
$$\le\|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)}+\frac{1}{4}\|\nabla_z\Psi_\epsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}.$$

Also, for a given nonnegative map $(x,v)\mapsto\omega(x,v)$ the particles entropy may be estimated as

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon|\ln(F_\epsilon)|\,\mathrm{d}v\,\mathrm{d}x=\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x-2\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\big(\mathbf{1}_{\{e^{-\omega}\le F_\epsilon\le 1\}}+\mathbf{1}_{\{F_\epsilon<e^{-\omega}\}}\big)\,\mathrm{d}v\,\mathrm{d}x$$
$$\le\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x+2\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\,\omega\,\mathrm{d}v\,\mathrm{d}x+\frac{4}{e}\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\omega/2}\,\mathrm{d}v\,\mathrm{d}x.$$

In particular, for $\omega(x,v)=\nu(V(x)+v^2/2)$, with $0<\nu<1/2$, it follows that

$$\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon|\ln(F_\epsilon)|\,\mathrm{d}v\,\mathrm{d}x\le\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x+\frac{4}{e}\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x$$
$$+2\nu\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Big(V(x)+\frac{v^2}{2}\Big)F_\epsilon\,\mathrm{d}v\,\mathrm{d}x.$$

Gathering the previous estimates we arrive at

$$0\le\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon|\ln(F_\epsilon)|\,\mathrm{d}v\,\mathrm{d}x+(1-2\nu)\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Big(V(x)+\frac{v^2}{2}\Big)F_\epsilon\,\mathrm{d}v\,\mathrm{d}x$$
$$+\frac{\epsilon^2}{2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x+\frac{1}{4}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x+\frac{1}{\epsilon^2}\int_0^t\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\big|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\big|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$
$$\le\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x+\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Big(V(x)+\frac{v^2}{2}+\Phi_\epsilon\Big)F_\epsilon\,\mathrm{d}v\,\mathrm{d}x$$

$$+\frac{\epsilon^2}{2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x+\frac{1}{2}\int\limits_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x+\frac{1}{\epsilon^2}\int\limits_0^t\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\left|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\right|^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$

$$+\frac{4}{e}\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x+\|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)}$$

$$\le\mathscr{K}_0+\frac{4}{e}\int\limits_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x+\|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)}.$$

These manipulations prove the following statement.

Proposition 3.4. *Let the assumptions of Theorem 3.1 be fulfilled. Then, the following assertions hold uniformly with respect to* $\epsilon>0$

i) $F_\epsilon\big(|\ln(F_\epsilon)|+V(x)+v^2\big)$ *is bounded in* $L^\infty([0,\infty];L^1(\mathbb{R}^d\times\mathbb{R}^d))$,
ii) $\epsilon\,\partial_t\Psi_\epsilon$ *and* $\nabla_z\Psi_\epsilon$ *are bounded in* $L^\infty([0,\infty];L^2(\mathbb{R}^d\times\mathbb{R}^n))$,
iii) Φ_ϵ *and* $\nabla_x\Phi_\epsilon$ *are bounded in* $L^\infty((0,\infty)\times\mathbb{R}^d)$,
iv) $D_\epsilon\overset{\text{def}}{=}\frac{1}{\epsilon}(2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon})$ *is bounded in* $L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$.

Let $0<T<\infty$. By virtue of the Dunford–Pettis theorem, see [16, Section 7.3.2], it follows that

$$F_\epsilon\rightharpoonup F\text{ weakly in }L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d).$$

Furthermore, the control of the particles kinetic energy allows us to additionally justify that

$$\rho_\epsilon=\int\limits_{\mathbb{R}^d}F_\epsilon\,\mathrm{d}v\rightharpoonup\rho=\int\limits_{\mathbb{R}^d}F\,\mathrm{d}v\text{ weakly in }L^1((0,T)\times\mathbb{R}^d).$$

Let us integrate the kinetic equation in (8) with respect to the velocity variable. We get, on the one hand

$$\partial_t\rho_\epsilon+\nabla_x\cdot J_\epsilon=0, \tag{13}$$

and, on the other hand, after multiplying the same equation by v

$$\epsilon^2\partial_t J_\epsilon+\nabla_x\cdot\mathbb{P}_\epsilon+\rho_\epsilon\nabla_x(V+\Phi_\epsilon)=-J_\epsilon. \tag{14}$$

In these equations we have denoted the momentum and the kinetic pressure as

$$J_\epsilon=\frac{1}{\epsilon}\int\limits_{\mathbb{R}^d}vF_\epsilon\,\mathrm{d}v,\qquad\mathbb{P}_\epsilon=\int\limits_{\mathbb{R}^d}v\otimes v\,F_\epsilon\,\mathrm{d}v.$$

Lemma 3.5. *The sequence* $\big(J_\epsilon\big)_{\epsilon>0}$ *is bounded in* $L^2(0,T;L^1(\mathbb{R}^d))$. *Furthermore, one can write* $\mathbb{P}_\epsilon=\rho_\epsilon\mathbb{I}+\mathbb{R}_\epsilon$, *where* $\mathbb{R}_\epsilon\to0$ *as* $\epsilon\to0$ *strongly in* $L^2(0,T;L^1(\mathbb{R}^d))$.

Proof. Note that the momentum can be written as

$$J_\epsilon=\int\limits_{\mathbb{R}^d}D_\epsilon\sqrt{F_\epsilon}\,\mathrm{d}v.$$

Thus, it can be estimated by a direct application of the Cauchy–Schwarz inequality and Proposition 3.4. Similarly, for the kinetic pressure we write

$$\mathbb{P}_\epsilon = \int_{\mathbb{R}^d} v\sqrt{F_\epsilon} \otimes (v\sqrt{F_\epsilon} + 2\nabla_v\sqrt{F_\epsilon})\,\mathrm{d}v - \int_{\mathbb{R}^d} v \otimes \nabla_v F_\epsilon\,\mathrm{d}v$$
$$= \underbrace{\epsilon \int_{\mathbb{R}^d} v\sqrt{F_\epsilon} \otimes D_\epsilon\,\mathrm{d}v}_{\stackrel{\text{def}}{=}\mathbb{R}_\epsilon} + \mathbb{I}\int_{\mathbb{R}^d} F_\epsilon\,\mathrm{d}v$$

where we have used integration by parts in the last integral. The Cauchy–Schwarz inequality allows us to estimate

$$\int_0^T \left(\int_{\mathbb{R}^d} |\mathbb{R}_\epsilon|\,\mathrm{d}x\right)^2 \mathrm{d}t \le \epsilon \left(\sup_t \int_{\mathbb{R}^d\times\mathbb{R}^d} v^2 F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right)\left(\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right),$$

from which one concludes using the bounds of Proposition 3.4. □

Owing to Lemma 3.5, we can assume that J_ϵ admits a limit J, say in $\mathscr{M}^1((0,T)\times\mathbb{R}^d)$. We also note that $\rho_\epsilon\nabla_x\Phi_\epsilon$ is bounded in $L^1((0,T)\times\mathbb{R}^d)$ by virtue of Proposition 3.4. Thus, letting ϵ decrease towards 0 in (13) and (14) yields

$$\partial_t\rho + \nabla_x\cdot J = 0,$$
$$-(J + \nabla_x\rho + \rho\nabla_x V) = \lim_{\epsilon\to 0}\rho_\epsilon\nabla_x\Phi_\epsilon.$$

Thus, it only remains the task of identifying the limit of the nonlinear term in the last equation. By using the estimates in Proposition 3.4 and Sobolev's embedding theorem, we can also assume that Ψ_ϵ admits a weak limit, say in $L^2((0,T)\times\mathbb{R}^d; L^{2n/(n-2)}(\mathbb{R}^n))$. In the limit $\epsilon\to 0$ the wave equation in (8) becomes

$$-\Delta_z\Psi(t,x,z) = -\sigma_2(z)\int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y.$$

Therefore, it follows that $\Psi(t,x,z) = -\Upsilon(z)\ \int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y$, and as a consequence, the self-consistent potential converges to

$$\Phi(t,x) = -\Lambda\int_{\mathbb{R}^d}\Sigma(x-y)\rho(t,y)\,\mathrm{d}y,$$

say, weakly-$\star$ in $L^\infty((0,T)\times\mathbb{R}^d)$. A similar conclusion applies to $\nabla_x\Phi(t,x)$. Furthermore, owing to the regularity of σ_1 ($\sigma_1\in W^{2,\infty}(\mathbb{R}^d)$), we have the following property

$$\sup_{\epsilon>0}|\nabla_x\Phi_\epsilon(t,x+h) - \nabla_x\Phi_\epsilon(t,x)| \xrightarrow[|h|\to 0]{} 0.$$

Using equation (13) and Lemma 3.5 it follows that $\partial_t\rho_\epsilon$ is bounded in $L^1([0,T];W^{-1,1}(\mathbb{R}^d))$. Combining these properties it is possible to apply directly the compactness statement given in [24, Lemma 5.1, p. 12] which ensures that $\rho_\epsilon\nabla_x\Phi_\epsilon \rightharpoonup \rho\nabla_x\Phi$ weakly in $L^1((0,T)\times\mathbb{R}^d)$. Thus, we conclude that

$$-(J + \nabla_x\rho + \rho\nabla_x V) = \rho\nabla_x\Phi, \qquad \Phi = -\Lambda\int_{\mathbb{R}^d}\Sigma(x-y)\rho(t,y)\,\mathrm{d}y.$$

Note also that the bound on $\partial_t\rho_\epsilon$ also implies that ρ_ϵ is compact in $C([0,T];L^1(\mathbb{R}^d)\text{-weak})$, that is to say the family $\{\int_{\mathbb{R}^d}\rho_\epsilon(t,x)\chi(x)\,\mathrm{d}x,\ \epsilon>0\}$ is relatively compact in $C([0,T])$ for any fixed $\chi\in L^\infty(\mathbb{R}^d)$. In particular, the initial data also makes sense for the limiting equation. With these arguments, we have justified the convergence of ρ_ϵ to ρ, solution of (9)–(10).

In fact, by using techniques elaborated in [14,25] it is possible to improve the nature of the convergence and to show that ρ_ϵ converges strongly to ρ in $L^1((0,T)\times\mathbb{R}^d)$ and F_ϵ converges strongly to ρM in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$.

The reasoning combines renormalization and average lemma techniques. One of the main difficulties relies on the fact that a suitable version of the average lemma is not available for a sequence of particle distribution functions weakly compact in L^1 solving a kinetic equation with velocity derivatives on the right hand side. Let us sketch the arguments. We start by setting $\beta_\delta(s) = \frac{s}{1+\delta s}$. On the one hand, by virtue of the equi-integrability of $\left(F_\epsilon\right)_{\epsilon>0}$ we have

$$\lim_{\delta\to 0}\sup_{\epsilon>0}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} |\beta_\delta(F_\epsilon) - F_\epsilon|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{15}$$

Indeed, observe that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} |\beta_\delta(F_\epsilon) - F_\epsilon|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\delta F_\epsilon^2}{1+\delta F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{F_\epsilon\le\mu} \frac{\delta F_\epsilon^2}{1+\delta F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int_{F_\epsilon\ge\mu} F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{\delta\mu}{1+\delta\mu}\sup_{\epsilon>0}\int F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \sup_{\epsilon>0}\int_{F_\epsilon\ge\mu} F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The last term can be made arbitrarily small choosing μ sufficiently large, and then, we let δ decrease towards zero. On the other hand, we shall use the renormalized equation

$$(\epsilon\partial_t + v\cdot\nabla_x)\beta_\delta(F_\epsilon) = h_{\delta,\epsilon} + \nabla_v\cdot g_{\delta,\epsilon}$$

where

$$\begin{aligned} g_{\delta,\epsilon} &= \nabla_x(V+\Phi_\epsilon)\beta_\delta(F_\epsilon) + \frac{1}{\epsilon}\beta_\delta'(F_\epsilon)M\nabla_v\left(\frac{F_\epsilon}{M}\right) \\ &= \nabla_x(V+\Phi_\epsilon)\beta_\delta(F_\epsilon) + \beta_\delta'(F_\epsilon)\sqrt{F_\epsilon}\,D_\epsilon\,, \\ h_{\delta,\epsilon} &= -\frac{1}{\epsilon}\beta_\delta''(F_\epsilon)M\nabla_v\left(\frac{F_\epsilon}{M}\right)\cdot\nabla_v F_\epsilon \\ &= -2F_\epsilon\beta_\delta''(F_\epsilon)\;D_\epsilon\cdot\nabla_v\sqrt{F_\epsilon}\,. \end{aligned}$$

In these equations we have used the fact

$$\frac{1}{\epsilon}M\nabla_v\left(\frac{F_\epsilon}{M}\right) = \frac{1}{\epsilon}(\nabla_v F_\epsilon + vF_\epsilon) = \sqrt{F_\epsilon}\,D_\epsilon.$$

For any $\delta > 0$ fixed, the sequence $\left(\beta_\delta(F_\epsilon)\right)_{\epsilon>0}$ is bounded in $L^1\cap L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Since $s\mapsto\beta_\delta'(s)$ is bounded, the sequence $\left(g_{\delta,\epsilon}\right)_{\epsilon>0}$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Moreover, using integration by parts one notices that

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v\sqrt{F_\epsilon}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \frac{\epsilon^2}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d}|D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} v^2F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} v\sqrt{F_\epsilon}\cdot\nabla_v\sqrt{F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{\epsilon^2}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d}|D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \frac{d}{2}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore $\nabla_v\sqrt{F_\epsilon}$ is bounded in $L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, and, since $s\mapsto s\beta_\delta''(s)$ is bounded (with a bound depending on δ), the sequence $\left(h_{\delta,\epsilon}\right)_{\epsilon>0}$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. The average lemma then leads to the following compactness property

$$\sup_{\epsilon>0} \int_{(0,T)\times\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \beta_\delta(F_\epsilon)(t,x+h,v)\zeta(v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \beta_\delta(F_\epsilon)(t,x,v)\zeta(v)\,\mathrm{d}v \right| \chi(t,x)\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[|h|\to 0]{} 0,$$

which holds for any $0 < T < \infty$, $\zeta \in C_c^\infty(\mathbb{R}^d)$ and nonnegative $\chi \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$. We refer the reader to [12, Th. 3, Th. 6], [27, Th. 2] and [25, Appendix B]. Since the transport operator has an ϵ in front of the time derivative, the average lemma provides a gain with respect to the space variable only. Using the fact that $(V(x)+v^2)F_\epsilon$ is bounded in L^1, with $V(x)\to\infty$ as $|x|\to\infty$, we can extend previous compactness property on the whole space and for any bounded test function, non-necessarily compactly supported. Hence, by using (15), it holds that

$$\sup_{\epsilon>0} \int_{(0,T)\times\mathbb{R}^d} \left|\rho_\epsilon(t,x+h) - \rho_\epsilon(t,x)\right| \mathrm{d}x\,\mathrm{d}t \xrightarrow[|h|\to 0]{} 0.$$

We conclude to the strong convergence of sequence $(\rho_\epsilon)_{\epsilon>0}$ by combining this information and the fact that $(\partial_t\rho_\epsilon)_{\epsilon>0}$ is bounded in $L^1\big([0,T]; W^{-1,1}(\mathbb{R}^d)\big)$, see Appendix B. The convergence of $(F_\epsilon)_{\epsilon>0}$ towards ρM is proved by using the estimate on D_ϵ, the logarithmic Sobolev inequality and the Cziszar–Kullback inequality. Indeed, it is now clear that $\rho_\epsilon M$ tends to ρM strongly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Therefore it remains to show that

$$\lim_{\epsilon\to 0} \int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |F_\epsilon(t,x,v) - \rho_\epsilon(t,x)M(v)|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The Csiszar–Kullback–Pinsker inequality [7,21], implies that

$$\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |F_\epsilon - \rho_\epsilon M|\,\mathrm{d}v\,\mathrm{d}x \right)^2 \le \frac{1}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon \ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right) \mathrm{d}v\,\mathrm{d}x.$$

By using the logarithmic Sobolev inequality, see e.g. [23, Theorem 8.14], the integrand of the right hand side is itself dominated as follows

$$\begin{aligned} 0 \le \int_{\mathbb{R}^d} \left\{ \frac{F_\epsilon}{\rho_\epsilon M} \ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right) - \frac{F_\epsilon}{\rho_\epsilon M} + 1 \right\} \rho_\epsilon\, M\,\mathrm{d}v &= \int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon \ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right) \mathrm{d}v \\ &\le 2 \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v \sqrt{\frac{F_\epsilon}{M}}\right|^2 M\,\mathrm{d}v \\ &\le \frac{\epsilon^2}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} |D_\epsilon|^2\,\mathrm{d}v. \end{aligned}$$

Integrating with respect to space and time variable we conclude that

$$\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |F_\epsilon - \rho_\epsilon M|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le \frac{\epsilon\sqrt{T}}{2} \|D_\epsilon\|_{L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)}.$$

Since $\|D_\epsilon\|_{L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)}$ is bounded uniformly with respect to ϵ, it ends the proof. □

3.4. Proof of Corollary 3.3

We start by rewriting (9) as follows

$$\partial_t\rho - \nabla_x\cdot\left(\rho_{\mathrm{eq}}\nabla_x\left(\frac{\rho}{\rho_{\mathrm{eq}}}\right)\right) - \nabla_x\cdot\big(\rho\nabla_x(\Phi-\Phi_{\mathrm{eq}})\big) = 0.$$

Furthermore, we observe that

$$\int_{\mathbb{R}^d} \frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{\rho^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x - 2\int_{\mathbb{R}^d} \rho\,\mathrm{d}x + \int_{\mathbb{R}^d} \rho_{\mathrm{eq}}\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{\rho^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x - \mathfrak{m}.$$

Therefore, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{\rho^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2 \rho_{\mathrm{eq}}\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho\nabla_x(\Phi-\Phi_{\mathrm{eq}})\cdot\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\,\mathrm{d}x.\end{aligned}$$

The last integral can be cast as

$$\begin{aligned}&\Lambda\int_{\mathbb{R}^d} \rho\nabla_x\Sigma*(\rho-\rho_{\mathrm{eq}})\cdot\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\,\mathrm{d}x \\ &= \Lambda\int_{\mathbb{R}^d} \rho_{\mathrm{eq}}\nabla_x\Sigma*(\rho-\rho_{\mathrm{eq}})\cdot\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\,\mathrm{d}x + \Lambda\int_{\mathbb{R}^d} (\rho-\rho_{\mathrm{eq}})\nabla_x\Sigma*(\rho-\rho_{\mathrm{eq}})\cdot\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\,\mathrm{d}x,\end{aligned}$$

where we denote by I and J the two terms of this splitting, respectively. We have, on the one hand,

$$\begin{aligned}|\mathrm{I}| &\le \Lambda\|\nabla_x\Sigma*(\rho-\rho_{\mathrm{eq}})\|_{L^\infty(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2} \\ &\le \Lambda\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\|\rho-\rho_{\mathrm{eq}}\|_{L^1(\mathbb{R}^d)}\sqrt{\mathfrak{m}}\left(\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2} \\ &\le \Lambda\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}\left(\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2},\end{aligned}$$

and, on the other hand,

$$|\mathrm{J}| \le 2\Lambda\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}\left(\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2},$$

since $\|\nabla_x\Sigma*(\rho-\rho_{\mathrm{eq}})\|_{L^\infty(\mathbb{R}^d)}\| \le \|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\|(\|\rho\|_{L^1(\mathbb{R}^d)} + \|\rho_{\mathrm{eq}}\|_{L^1(\mathbb{R}^d)})$. Thus, we arrive at the inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x + \int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x \\ &\quad\le 3\Lambda\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}\left(\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

The final step relies on the following statement.

Lemma 3.6. *There exists a constant $\Omega > 0$ such that*

$$\Omega\int_{\mathbb{R}^d}\left|g(x) - \int_{\mathbb{R}^d} g(y)\frac{\rho_{\mathrm{eq}}(y)}{\mathfrak{m}}\,\mathrm{d}y\right|^2\frac{\rho_{\mathrm{eq}}(x)}{\mathfrak{m}}\,\mathrm{d}x \le \int_{\mathbb{R}^d}|\nabla_x g(x)|^2\,\frac{\rho_{\mathrm{eq}}(x)}{\mathfrak{m}}\,\mathrm{d}x$$

holds.

Indeed, owing to Lemma 3.6, we get

$$\Omega \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x \le \int_{\mathbb{R}^d} \left|\nabla_x \left(\frac{\rho}{\rho_{\mathrm{eq}}}\right)\right|^2 \rho_{\mathrm{eq}} \,\mathrm{d}x.$$

Let us denote $\mathscr{A} = 6\Lambda \|\nabla_x \Sigma\|_{L^\infty(\mathbb{R}^d)} \mathfrak{m}$. By using Cauchy–Schwarz and Young inequalities, for any $0 < \nu < 2$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x &\le (-2+\nu) \int_{\mathbb{R}^d} \left|\nabla_x \left(\frac{\rho}{\rho_{\mathrm{eq}}}\right)\right|^2 \rho_{\mathrm{eq}} \,\mathrm{d}x + \frac{\mathscr{A}^2}{4\nu} \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x \\
&\le \left((-2+\nu)\Omega + \frac{\mathscr{A}^2}{4\nu}\right) \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x.
\end{aligned}$$

Optimizing with respect to ν yields $\nu = \frac{\mathscr{A}}{2\sqrt{\Omega}}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x \le \left(-2 + \frac{\mathscr{A}\sqrt{\Omega}}{2}\right) \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}} \,\mathrm{d}x.$$

Therefore, we conclude ρ converges to ρ_{eq} with exponential rate $\kappa = 2\Omega - \frac{\mathscr{A}\sqrt{\Omega}}{2}$ provided the smallness condition $\mathscr{A} \le 4\sqrt{\Omega}$ is fulfilled. It finishes the proof of Corollary 3.3, up to the justification of Lemma 3.6. □

Proof of Lemma 3.6. Lemma 3.6 is an extension of (12) for the state ρ_{eq}, which is seen as a perturbation of $\frac{\mathfrak{m}}{\bar{\mu}} e^{-V}$. Since $\Lambda\Sigma * \rho_{\mathrm{eq}} \ge 0$, we can write

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_{\mathrm{eq}} \,\mathrm{d}x &= \frac{1}{Z_{\mathrm{eq}}} \int_{\mathbb{R}^d} |\nabla_x g|^2 e^{-V + \Lambda\Sigma * \rho_{\mathrm{eq}}} \,\mathrm{d}x \\
&\ge \frac{1}{Z_{\mathrm{eq}}} \int_{\mathbb{R}^d} |\nabla_x g|^2 e^{-V} \,\mathrm{d}x \\
&\ge \frac{C}{Z_{\mathrm{eq}}} \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \frac{e^{-V(y)}}{\bar{\mu}} \,\mathrm{d}y \right|^2 e^{-V(x)} \,\mathrm{d}x,
\end{aligned}$$

by using (12). Next, $\Lambda\Sigma * \rho_{\mathrm{eq}} \le \Lambda \|\Sigma * \rho_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}^d)} \le \Lambda \|\Sigma\|_{L^\infty(\mathbb{R}^d)} \mathfrak{m}$ implies

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_{\mathrm{eq}} \,\mathrm{d}x &\ge \frac{C}{Z_{\mathrm{eq}} e^{\Lambda \|\Sigma * \rho_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}^d)}}} \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \frac{e^{-V(y)}}{\bar{\mu}} \,\mathrm{d}y \right|^2 e^{-V(x) + \Lambda\Sigma * \rho_{\mathrm{eq}}(x)} \,\mathrm{d}x \\
&\ge \Omega \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \frac{e^{-V(y)}}{\bar{\mu}} \,\mathrm{d}y \right|^2 \rho_{\mathrm{eq}}(x) \,\mathrm{d}x
\end{aligned}$$

with $\Omega = C e^{-\Lambda \mathfrak{m} \|\Sigma\|_{L^\infty(\mathbb{R}^d)}}$. However, $\frac{\rho_{\mathrm{eq}}(x)}{\mathfrak{m}} \,\mathrm{d}x$ is a probability measure and we can check that, for any probability measure $\mathrm{d}\mu$, the function $X \mapsto \int_{\mathbb{R}^d} |g(x) - X|^2 \,\mathrm{d}\mu(x)$ reaches its minimum for $X = \int_{\mathbb{R}^d} g(x) \,\mathrm{d}\mu(x)$. We conclude that

$$\int_{\mathbb{R}^d} |\nabla_x g|^2 \rho_{\mathrm{eq}} \,\mathrm{d}x \ge \Omega \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \rho_{\mathrm{eq}}(y) \,\mathrm{d}y \right|^2 \rho_{\mathrm{eq}}(x) \,\mathrm{d}x$$

holds.

We point out the fact that Ω depends on Λ, Σ and $\mathfrak{m}$, and the condition $\mathscr{A} \le 4\sqrt{\Omega}$ met above still can be interpreted as a smallness condition for the product $\Lambda \|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)} \mathfrak{m}$, since $X \mapsto X e^X$ tends to 0 as $X \to 0$. □

4. Asymptotic trend to equilibrium

We restrict the discussion to the case where, given the total mass $\mathfrak{m}$, the equilibrium $\mathscr{M}_{\rm eq}$ is uniquely defined. We rewrite the problem by considering fluctuation

$$F = \mathscr{M}_{\rm eq} + f, \qquad \Phi = \Phi_{\rm eq} + \phi, \qquad \Psi = \Psi_{\rm eq} + \psi$$

where we remind the reader that $\rho_{\rm eq}(x) = (2\pi)^{d/2} Z[\Phi_{\rm eq}] e^{-V(x)-\Phi_{\rm eq}(x)}$, and

$$\Psi_{\rm eq}(x,z) = -\frac{1}{c^2}\Upsilon(z)\sigma_1 * \rho_{\rm eq}(x), \quad \text{with } -c^2\Delta_z\Upsilon = \sigma_2.$$

We define the operator

$$T_{\rm eq} = v\cdot\nabla_x - \nabla_x(\Phi_{\rm eq}+V)\cdot\nabla_v.$$

We obtain the coupled system for the fluctuations

$$\begin{aligned}
&(\partial_t + T_{\rm eq} - L) f = \nabla_x\phi\cdot\nabla_v\mathscr{M}_{\rm eq} + \nabla_x\phi\cdot\nabla_v f,\\
&(\partial^2_{tt} - c^2\Delta_z)\psi(t,x,z) = -\sigma_2(z)\int_{\mathbb{R}^d}\sigma_1(x-y)\varrho(t,y)\,\mathrm{d}y,\\
&\phi(t,x) = \int_{\mathbb{R}^n\times\mathbb{R}^d}\sigma_1(x-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y
\end{aligned}\tag{16}$$

where $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. The problem is complemented with initial conditions

$$f\Big|_{t=0} = f_0 = F_0 - \mathscr{M}_{\rm eq}, \qquad (\psi,\partial_t\psi)\Big|_{t=0} = (\psi_0,\psi_1).$$

Note that from the definition we have

$$\|\varrho(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|f(t,\cdot)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \le 2\,\mathfrak{m}, \qquad \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

A crucial role is played by the entropy dissipation, which casts as follows

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f^2}{\mathscr{M}_{\rm eq}}\,\mathrm{d}v\,\mathrm{d}x &= -\gamma\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathscr{M}_{\rm eq}\Big|\nabla_v\frac{f}{\mathscr{M}_{\rm eq}}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\rm eq}}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Let us now introduce the following useful notation and observations:

- For $f\in L^1(\mathbb{R}^d)$, let $\langle f\rangle \stackrel{\rm def}{=} \int_{\mathbb{R}^d} f\,\mathrm{d}v$,
- and $Pf(v) \stackrel{\rm def}{=} \langle f\rangle\, M(v)$ stands for the projection onto $\mathrm{Ker}(L)$.
- Entropy dissipation makes $L^2\big(\mathbb{R}^d\times\mathbb{R}^d; \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\rm eq}(x,v)}\big)$ a suitable functional space, and we denote by $(\cdot|\cdot)$ its inner product.
- Since we work with fluctuation we consider the closed subspace

$$H = \Big\{f\in L^2\Big(\mathbb{R}^d\times\mathbb{R}^d; \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\rm eq}(x,v)}\Big), \int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}v\,\mathrm{d}x = (f|\mathscr{M}_{\rm eq}) = 0\Big\}$$

  endowed with the norm $\|f\|_H = \sqrt{(f|f)}$.

- We also remark that

$$T_{\rm eq}^\star = -T_{\rm eq}, \qquad P^\star = P, \qquad PT_{\rm eq}P = 0. \tag{17}$$

The last equality in (17) comes from the fact that $\langle vM\rangle = 0$ after noticing that

$$T_{\rm eq}Pf(x,v) = vM(v)\cdot\big(\nabla_x\langle f\rangle(x) + \langle f\rangle(x)\nabla_x(\Phi_{\rm eq}+V)(x)\big).$$

The idea of the proof of Theorem 2.3 consists in constructing a new functional $\mathscr{H}$ such that $\mathscr{H} \simeq \|f\|_H^2$ and identifying some number $\theta > 0$ with $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} \le -\theta\mathscr{H}$. Such an inequality can be obtained in the linear framework, see [13,19]; in our case, however, the inequality will contain remainder terms that can be controlled by assuming c sufficiently large. The new functional is constructed by involving a certain operator A that combines appropriately the projection P and the transport operator $T_{\rm eq}$.

Lemma 4.1 *([13]). Define the operator* $A \overset{\rm def}{=} (1 + (T_{\rm eq}P)^\star(T_{\rm eq}P))^{-1}(T_{\rm eq}P)^\star$. *Then, we have*

- $\mathrm{Ran}(A) \subset \mathrm{Ran}(P) \subset \mathrm{Ker}(L)$, *so that* $LA = 0$ *and* $PA = A$.
- $\|Af\|_H \le \frac{1}{2}\|(1-P)f\|_H$ *and* $\|T_{\rm eq}Af\|_H \le \|(1-P)f\|_H$.

Proof. For the sake of completeness we collect the arguments from [13]. Owing to (17), we can rewrite

$$A = -(I - PT_{\rm eq}^2P)^{-1}PT_{\rm eq}.$$

Let us denote $Af = g$, thus,

$$g - PT_{\rm eq}^2Pg = -PT_{\rm eq}f \tag{18}$$

which can be cast as $g = P(T_{\rm eq}^2Pg - T_{\rm eq}f)$. It proves $g \in \mathrm{Ran}(P)$, and thus $LA = 0$. Furthermore, by using (17), we get

$$\begin{aligned}\|g\|_H^2 + \|T_{\rm eq}Pg\|_H^2 = (g - PT_{\rm eq}^2Pg|g) &= -(PT_{\rm eq}f|g)\\ &= (f|T_{\rm eq}Pg) = ((1-P)f|T_{\rm eq}Pg) + (Pf|T_{\rm eq}Pg)\\ &= ((1-P)f|T_{\rm eq}Pg) + 0 \le \|(1-P)f\|_H\|T_{\rm eq}Pg\|_H\\ &\le \frac{1}{2\alpha^2}\|(1-P)f\|_H^2 + \frac{\alpha^2}{2}\|T_{\rm eq}Pg\|_H^2.\end{aligned}$$

It yields (by successively taking $\alpha = \sqrt{2}$ and $\alpha = 1$)

$$\begin{aligned}&\|Af\|_H = \|g\|_H \le \frac{1}{2}\|(1-P)f\|_H,\\ &\|T_{\rm eq}Af\|_H = \|T_{\rm eq}PAf\|_H = \|T_{\rm eq}Pg\|_H \le \|(1-P)f\|_H. \qquad \square\end{aligned}$$

In contrast to [13,19], here we are dealing with a nonlinear problem. In order to handle the nonlinear terms involving fluctuations, additional estimates on the adjoint operator $A^\star$ will be needed. In what follows, we will frequently use the following simple fact: let $x \mapsto U(x)$ be a field depending only on the space variable, then

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |vM(v)\cdot U(x)|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\rm eq}(x,v)} &= \int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v\right)}_{=\mathbb{I}} U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\rm eq}(x)}\\ &= \int_{\mathbb{R}^d}\frac{|U(x)|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x.\end{aligned}$$

Lemma 4.2. *The following estimates hold for the adjoint operator*

$$\|\nabla_v A^\star f\|_H \le \sqrt{\frac{d+1}{2}}\, \|f\|_H\,, \text{ and } \|vA^\star f\|_H \le \sqrt{\frac{d+2}{2}}\, \|f\|_H.$$

Proof. We have $A^\star = T_{\rm eq} P(I - PT_{\rm eq}^2 P)^{-1}$. Let $g = (I - PT_{\rm eq}^2 P)^{-1} f$, so that $A^\star f = T_{\rm eq} P g$. We already know that

$$\frac{1}{2}\|g\|_H^2 + \|T_{\rm eq} P g\|_H^2 \le \frac{1}{2}\|f\|_H^2$$

holds since taking the inner product of $(I - PT_{\rm eq}^2 P)g = f$ with g yields $\|g\|_H^2 + \|T_{\rm eq} P g\|_H^2 = (f|g) \le \frac{1}{2}(\|f\|_H^2 + \|g\|_H^2)$. Next, we compute

$$\begin{aligned} &T_{\rm eq} P g(x,v) = T_{\rm eq}\big(M(v)\langle g\rangle(x)\big) = vM(v)\cdot U(x),\\ &\text{with } U(x) = \nabla_x\langle g\rangle(x) + \nabla_x(\Phi_{\rm eq} + V)\langle g\rangle(x). \end{aligned} \tag{19}$$

On the one hand, we observe that

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|\langle g\rangle(x)|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x &\le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{|g(x,v)|^2}{M(v)}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} M(v)\,\mathrm{d}v\right)\frac{\mathrm{d}x}{\rho_{\rm eq}(x)}\\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|g(x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,\mathrm{d}v\,\mathrm{d}x = \|g\|_H^2 \le \|f\|_H^2. \end{aligned}$$

On the other hand, since $\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v = \mathbb{I}$, it follows that

$$\begin{aligned} \|A^\star f\|_H^2 = \|T_{\rm eq} P g\|_H^2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d} M^2(v)|v\cdot U(x)|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\rm eq}(x,v)}\\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v\right)U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\rm eq}(x)}\\ &= \int_{\mathbb{R}^d}\frac{|U(x)|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x \le \frac{1}{2}\|f\|_H^2. \end{aligned}$$

Now, we turn to the velocity derivative

$$\begin{aligned} \nabla_v A^\star f(x,v) &= \nabla_v\big(T_{\rm eq} P g\big)(x,v) = T_{\rm eq}\big(\nabla_v P g\big)(x,v) + \nabla_x P g(x,v)\\ &= T_{\rm eq}\big(-vM(v)\langle g\rangle(x)\big) + \nabla_x\big(M(v)\langle g\rangle(x)\big)\\ &= \big(\mathbb{I} - v\otimes v\big)M(v)\big(\nabla_x\langle g\rangle(x) + \nabla_x(\Phi_{\rm eq} + V)\langle g\rangle(x)\big)\\ &= \big(\mathbb{I} - v\otimes v\big)M(v)U(x). \end{aligned}$$

We are thus led to evaluate

$$\begin{aligned} &\int_{\mathbb{R}^d\times\mathbb{R}^d} (\mathbb{I} - v\otimes v)M(v)U(x)\cdot(\mathbb{I} - v\otimes v)M(v)U(x)\frac{\mathrm{d}v\,\mathrm{d}x}{(2\pi)^{d/2}Z_{\rm eq}\mathscr{M}_{\rm eq}(x,v)}\\ &\quad = \int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d}(\mathbb{I} - v\otimes v)^2 M(v)\,\mathrm{d}v\right)}_{=(d+1)\mathbb{I}} U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\rm eq}(x)} \end{aligned}$$

$$= (d+1) \int_{\mathbb{R}^d} |U(x)|^2 \frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}.$$

We conclude that

$$\|\nabla_v A^\star f\|_H^2 = (d+1)\|T_{\mathrm{eq}} P g\|_H^2 \le \frac{d+1}{2}\|f\|_H^2.$$

Finally, we also note that

$$\begin{aligned}
\|v A^\star f\|_H^2 = \|v\, T_{\mathrm{eq}} P g\|_H^2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \big|v M(v)\big(v\cdot U(x)\big)\big|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)} \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \big|v\otimes v\, M(v) U(x)\big|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)} \\
&= \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} (v\otimes v)^2 M(v)\,\mathrm{d}v\right)}_{=(d+2)\mathbb{I}} U(x)\cdot U(x) \\
&= (d+2)\int_{\mathbb{R}^d} \frac{|U(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x = (d+2)\|T_{\mathrm{eq}} P g\|_H^2 \le \frac{d+2}{2}\|f\|_H^2. \qquad \square
\end{aligned}$$

Let $0<\eta<1$ a suitable parameter to be determined in the sequel and define

$$\mathscr{H} \overset{\mathrm{def}}{=} \frac{1}{2}\|f\|_H^2 + \eta(Af|f).$$

By using Lemma 4.1, we note that

$$\frac{1}{2}(1-\eta)\|f\|_H^2 \le \mathscr{H} = \frac{1}{2}\|f\|_H^2 + \eta(Af|f) \le \frac{1}{2}(1+\eta)\|f\|_H^2. \tag{20}$$

Furthermore, recall that $(Af|Lf) = (LAf|f) = 0$ and $(Af|T_{\mathrm{eq}}f) = (T_{\mathrm{eq}}Af|f)$ since $T_{\mathrm{eq}}\mathscr{M}_{\mathrm{eq}} = 0$. Thus, using the equation for the fluctuation (16) and the fact that time derivative commutes with the operator A we can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Af|f) = &-(AT_{\mathrm{eq}}f|f) + (ALf|f) + (T_{\mathrm{eq}}Af|f) \\
&+ \big(A[\nabla_x\phi\cdot\nabla_v(\mathscr{M}_{\mathrm{eq}}+f)]|f\big) + \big(Af|\nabla_x\phi\cdot\nabla_v(\mathscr{M}_{\mathrm{eq}}+f)\big).
\end{aligned}$$

Thus, we are led to estimate the four terms on the right hand side of the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \tag{21}$$

with

$$\begin{aligned}
\mathrm{I} &= (Lf|f) - \eta(AT_{\mathrm{eq}}Pf|f) = (Lf|f) - \eta(AT_{\mathrm{eq}}Pf|Pf), \\
\mathrm{II} &= -\eta(AT_{\mathrm{eq}}(1-P)f|f) + \eta(ALf|f) + \eta(T_{\mathrm{eq}}Af|f), \\
\mathrm{III} &= +\eta\big(A[\nabla_x\phi\cdot\nabla_v f]|f\big) + \eta\big(Af|\nabla_x\phi\cdot\nabla_v f\big), \\
\mathrm{IV} &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \eta\big(A[\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}]|f\big) + \eta\big(Af|\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}\big).
\end{aligned}$$

On the right hand side of (21), the terms I and II already appear in the linear analysis of [13]. They are handled using the same arguments as those given in this reference.

The cornerstone of the proof consists in observing that $(Lf|f) - \eta(AT_{\rm eq}Pf|Pf)$ is the dissipative contribution, owing to Poincaré's inequalities. Indeed, on the one hand, there exists $\Xi > 0$ such that

$$\Xi \int_{\mathbb{R}^d} \frac{|f(v) - \langle f\rangle M(v)|^2}{M(v)}\,\mathrm{d}v \le \int_{\mathbb{R}^d} \left|\nabla_v\Big(\frac{f(v)}{M(v)}\Big)\right|^2 M(v)\,\mathrm{d}v,$$

see e.g. [2, Cor. 2.18]. We deduce that

$$\Xi\|f - Pf\|_H^2 \le -(Lf|f). \tag{22}$$

On the other hand, by assumption on the external potential V, there exists $\Xi' > 0$ such that the following Poincaré's inequality holds

$$\Xi' \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|Pf(x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,\mathrm{d}v\,\mathrm{d}x \le \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|T_{\rm eq}Pf(x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,\mathrm{d}v\,\mathrm{d}x. \tag{23}$$

To see this, note that the left hand side in inequality (23) recasts as

$$\Xi' \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{M^2(v)|\langle f\rangle(x)|^2}{M(v)\rho_{\rm eq}(x)}\,\mathrm{d}v\,\mathrm{d}x = \Xi' \int_{\mathbb{R}^d} \frac{|\langle f\rangle(x)|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x.$$

For the right hand side in (23), we observe that

$$\begin{aligned}\int_{\mathbb{R}^d} \left|T_{\rm eq}Pf(x,v)\right|^2 \frac{\mathrm{d}v}{M(v)} &= \int_{\mathbb{R}^d} \left|(vM(v)\cdot\big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x\Phi_{\rm eq}(x)\big)\right|^2 \frac{\mathrm{d}v}{M(v)}\\ &= \int_{\mathbb{R}^d} v\otimes v\, M(v)\,\mathrm{d}v\,\big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\rm eq}+V)(x)\big)\cdot\big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\rm eq}+V)(x)\big)\\ &= \left|\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\rm eq}+V)(x)\right|^2.\end{aligned}$$

Then, the Poincaré inequality (23) writes as

$$\Xi' \int_{\mathbb{R}^d} \frac{|\langle f\rangle(x)|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x \le \int_{\mathbb{R}^d} \frac{\left|\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\rm eq}+V)(x)\right|^2}{\rho_{\rm eq}(x)}\,\mathrm{d}x.$$

We set $u(x) = \langle f\rangle(x)e^{(\Phi_{\rm eq}+V)(x)/2}$, and (23) reduces to the more standard expression

$$\Xi' \int_{\mathbb{R}^d} |u|^2\,\mathrm{d}x \le \int_{\mathbb{R}^d} \Big(|\nabla_x u|^2 + |u|^2\Big(\frac{1}{4}|\nabla_x(\Phi_{\rm eq}+V)|^2 - \frac{1}{2}\Delta_x(\Phi_{\rm eq}+V)\Big)\Big)\,\mathrm{d}x$$

where we recognize a spectral property of the Schrödinger operator associated to the potential $\frac{1}{4}|\nabla_x(\Phi_{\rm eq}+V)|^2 - \frac{1}{2}\Delta_x(\Phi_{\rm eq}+V)$. The Poincaré inequality (23) is therefore a consequence of (**A2**), see [26]. The next step appeals to the following elementary statement.

Lemma 4.3. *Let S be a self-adjoint operator on a Hilbert space H, with dense domain $D(S)$. Assume there exists $\lambda > 0$ such that $(Sx|x) \ge \lambda\|\xi\|^2$ holds for any $\xi \in D(S)$. Then, $1+S$ is invertible and $((1+S)^{-1}S\xi|\xi) \ge \frac{\lambda}{1+\lambda}\|\xi\|^2$.*

Proof. Clearly we have $((1+S)\xi|\xi) \ge (1+\lambda)\|\xi\|^2$. In particular $\|(1+S)\xi\| \ge (1+\lambda)\|\xi\|$ holds for any $\xi \in D(S)$. The inequality obviously implies that $(1+S)$ is injective. Next, if $(1+S)x_n \to y \in H$, then $\big(x_n\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus it converges to some $x \in H$. For any $\xi \in D(S^\star)$, we get $((1+S)x_n|\xi) = (x_n|(1+S^\star)\xi) \to (y|\xi) = (x|(1+S^\star)\xi) = ((1+S)x|\xi)$. Since $D(1+S^\star)$ is dense (it contains $D(1+S)$ provided S is symmetric), this extends to any $\xi \in H$. Hence $y = (1+S)x$ and $\mathrm{Ran}(1+S)$ is closed. Finally, let $\xi \in \overline{\mathrm{Ran}(1+S)}^\perp = \mathrm{Ran}(1+S)^\perp$: for any $x \in D(S)$, we have $((1+S)x|\xi) = 0 = (x|(1+S^\star)\xi)$. Since the domain $D(S)$ is dense, we actually have $(x|(1+S^\star)\xi) = 0$

for any $x \in H$. Now, we use the fact that S is self-adjoint: $S = S^\star$ (which means that their domains coincide $D(S^\star) = D(S)$ and $(1 + S^\star)\xi = (1 + S)\xi$). Hence, we can use this relation with $x = \xi$, together with the coercivity estimate; it proves that $\xi = 0$. Thus $(1 + S)$ is invertible. Using the coercivity estimate inequality with $\xi = (1 + S)^{-1}\zeta$ proves that its inverse satisfies $\|(1 + S)^{-1}\| \le \frac{1}{1+\lambda}$. Now, for any $\xi \in D(S)$, we compute

$$((1 + S)^{-1}S\xi|\xi) = ((1 + S)^{-1}(1 + S - 1)\xi|\xi) = \|\xi\|^2 - ((1 + S)^{-1}\xi|\xi).$$

The Cauchy–Schwarz inequality leads to

$$((1 + S)^{-1}S\xi|\xi) \ge \|\xi\|^2 - \|\xi\|\|(1 + S)^{-1}\xi\| \ge \Big(1 - \frac{1}{1+\lambda}\Big)\|\xi\|^2 = \frac{\lambda}{1+\lambda}\,\|\xi\|^2. \qquad \square$$

We refer the reader to [18] for complete details on the case considered here. To be specific, we shall take $S = (T_{\text{eq}}P)^\star T_{\text{eq}}P$ in the Hilbert space $H = \text{Ran}(P)$. By using (17) and (19), we get

$$Sg(x,v) = -\nabla_x \cdot \Big(\rho_{\text{eq}}\nabla_x \frac{\langle g\rangle}{\rho_{\text{eq}}}\Big)(x)\; M(v).$$

For g_1 and g_2 in H, we have $(g_1|g_2) = (\langle g_1\rangle M|\langle g_2\rangle M)$ and we check that

$$(Sg_1|g_2) = -\int_{\mathbb{R}^d} \nabla_x \cdot \Big(\rho_{\text{eq}}\nabla_x \frac{\langle g_1\rangle}{\rho_{\text{eq}}}\Big)\frac{\langle g_2\rangle}{\rho_{\text{eq}}}\,\mathrm{d}x. \tag{24}$$

The Fokker–Planck-like operator $g \mapsto \nabla_x \cdot \big(\rho_{\text{eq}}\nabla_x \frac{g}{\rho_{\text{eq}}}\big)$ is clearly self-adjoint for the inner product $L^2(\mathbb{R}^d, \mathrm{d}x/\rho_{\text{eq}})$, so is S by virtue of (24).

Consequently, using Lemma 4.3 it follows with (23) that

$$(AT_{\text{eq}}Pf|Pf) \ge \frac{\Xi'}{1+\Xi'}\|Pf\|_H^2. \tag{25}$$

We keep in mind previous observations for they will be used to estimate the term I. Proceeding as in [13], we obtain

$$\text{II} \le \sqrt{C}\eta\|Pf\|_H\|(1-P)f\|_H \le \frac{\Xi}{4}\|(1-P)f\|_H^2 + \frac{\eta^2}{\Xi}C\|Pf\|_H^2, \tag{26}$$

for a certain constant $C > 0$. Indeed, Cauchy–Schwarz inequality and Lemma 4.1 already prove that $\big|(T_{\text{eq}}Af|f)\big| \le \|(1-P)f\|_H\|f\|_H$. Next, we remark that

$$PT_{\text{eq}}f(x,v) = M(v)\langle T_{\text{eq}}f\rangle(x) = M(v)\nabla_x \cdot \langle vf\rangle(x).$$

Therefore, using integration by parts we obtain

$$PT_{\text{eq}}Lf(x,v) = M(v)\nabla_x \cdot \langle vLf\rangle(x) = -M(v)\nabla_x \cdot \langle vf\rangle(x) = -PT_{\text{eq}}f(x,v).$$

Thus, it follows that $AL = -A$. Using again Lemma 4.1 we deduce that

$$\big|(ALf|f)\big| = \big|(Af|f)\big| \le \frac{1}{2}\|(1-P)f\|_H\|f\|_H.$$

It remains to justify that there exists $C_1 > 0$ such that

$$\big|(AT_{\text{eq}}(1-P)f|f)\big| \le C_1\|(1-P)f\|_H\|f\|_H,$$

which is the most delicate part of the proof of (26). The boundedness of $AT_{\text{eq}}(1-P)$ can be rephrased in terms of regularity analysis for the solution u of the elliptic problem

$$\rho_{\text{eq}}u - \nabla_x \cdot (\rho_{\text{eq}}\nabla_x u) = \rho, \qquad \rho_{\text{eq}} = \rho_{\text{eq}}(x).$$

We need to justify the regularization

$$\int_{\mathbb{R}^d} \big|\rho_{\text{eq}}\,D_x^2 u\big|^2\,\mathrm{d}x \le C\int_{\mathbb{R}^d} |\rho|^2\,\mathrm{d}x.$$

That (**A3**) allows us to establish this inequality is the object of [13, Section 2, see in particular Prop. 5 and the comments with assumption (H4.1)]. The constant C in (26) can be estimated as $C = (3/2 + C_1)^2$.

We are left with the task of estimating the coupling terms III and IV. The following observation is crucial for the analysis; in particular, it will allow us to make the contribution of the nonlinear terms small. Owing to the linearity of the wave equation, we can write $\psi = \psi_{\mathrm{I}} + \psi_{\mathrm{S}}$ with ψ_{I} the solution of the free wave equation with (ψ_0, ψ_1) as initial data, namely

$$\begin{aligned} &(\partial^2_{tt} - c^2\Delta_z)\psi_{\mathrm{I}} = 0, \\ &\psi_{\mathrm{I}}\Big|_{t=0} = \psi_0, \qquad \partial_t\psi_{\mathrm{I}}\Big|_{t=0} = \psi_1, \end{aligned} \tag{27}$$

and ψ_{S} the solution of the wave equation with 0 as initial data and $-\sigma_2(z)\sigma_1 * \varrho(t,x)$ as source

$$\begin{aligned} &(\partial^2_{tt} - c^2\Delta_z)\psi_{\mathrm{S}} = -\sigma_2(z)\sigma_1 * \varrho(t,x), \\ &\psi_{\mathrm{S}}\Big|_{t=0} = 0, \qquad \partial_t\psi_{\mathrm{I}}\Big|_{t=0} = 0. \end{aligned} \tag{28}$$

Accordingly, the self-consistent potential ϕ splits into two parts

$$\phi = \phi_{\mathrm{I}} + \phi_{\mathrm{S}} \tag{29}$$

with

$$\phi_{\mathrm{I}}(t,x) = \int_{\mathbb{R}^d} \sigma_1(x-y)\left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_{\mathrm{I}}(t,y,z)\,\mathrm{d}z\right)\mathrm{d}y \tag{30}$$

and

$$\phi_{\mathrm{S}}(t,x) = \int_{\mathbb{R}^d} \sigma_1(x-y)\left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_{\mathrm{S}}(t,y,z)\,\mathrm{d}z\right)\mathrm{d}y. \tag{31}$$

The latter can be rewritten as

$$\begin{aligned} &\phi_{\mathrm{S}}(t,x) = -\int_0^t \int_{\mathbb{R}^d} p(t-s)\Sigma(x-y)\varrho(s,y)\,\mathrm{d}y\,\mathrm{d}s, \\ &p(s) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\sin(sc|\xi|)}{c|\xi|}\,|\widehat{\sigma}_2(\xi)|^2\,\mathrm{d}\xi \end{aligned} \tag{32}$$

where $\widehat{\cdot}$ stands for the Fourier transform (see e.g. [28, Chap. I, formula (1.14)]). We can start with the following rough estimate, which appeals to assumptions (**A5**).

Lemma 4.4. *The potential can be estimated by using the following properties:*

i) *Let ϕ_{I} be defined by (30) with ψ_0 and ψ_1 of finite energy:*

$$\int_{\mathbb{R}^d\times\mathbb{R}^n} \left(|\psi_1(x,z)|^2 + c^2|\nabla_z\psi_0(x,z)|^2\right)\mathrm{d}z\,\mathrm{d}x = \mathscr{E}_{\mathrm{I}} < \infty.$$

Then, we have

$$\begin{aligned} &|\phi_{\mathrm{I}}(t,x)| \le \frac{1}{c}\,\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\mathrm{I}}}, \\ &|\nabla_x\phi_{\mathrm{I}}(t,x)| \le \frac{1}{c}\,\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\mathrm{I}}}. \end{aligned}$$

ii) *Let* $\varrho \in L^\infty(0,\infty; L^1(\mathbb{R}^d))$ *and let* ϕ_S *be defined by* (32)*. Then, we have*

$$|\phi_S(t,x)| \le \frac{\Pi_0}{c^2}\|\Sigma\|_{L^\infty}\|\varrho\|_{L^\infty(0,t;L^1(\mathbb{R}^d))}, \quad |\nabla_x\phi_S(t,x)| \le \frac{\Pi_0}{c^2}\|\nabla_x\Sigma\|_{L^\infty}\|\varrho\|_{L^\infty(0,t;L^1(\mathbb{R}^d))},$$

where

$$\Pi_0 = \frac{1}{(2\pi)^n}\int_0^\infty \left| \int_{\mathbb{R}^n} \frac{\sin(t|\xi|)}{|\xi|}\, |\widehat{\sigma}_2(\xi)|^2\,\mathrm{d}\xi \right| \mathrm{d}t \in (0,\infty).$$

Proof. The solution of (27) satisfies the energy conservation

$$\int_{\mathbb{R}^d\times\mathbb{R}^n} \left(|\partial_t\psi_I(t,x,z)|^2 + c^2|\nabla_z\psi_I(t,x,z)|^2\right)\mathrm{d}z\,\mathrm{d}x = \mathscr{E}_I.$$

Next, we use Hölder inequality together with the Sobolev inequality to estimate

$$\begin{aligned}
|\phi_I(t,x)| &\le \left(\int_{\mathbb{R}^n} |\sigma_2(z)|^{2n/(n+2)}\right)^{(n+2)/2n} \int_{\mathbb{R}^d} \sigma_1(x-y)\left(\int_{\mathbb{R}^n} |\psi_I(t,y,z)|^{2n/(n-2)}\right)^{(n-2)/2n} \mathrm{d}y \\
&\le \left(\int_{\mathbb{R}^n} |\sigma_2(z)|^{2n/(n+2)}\right)^{(n+2)/2n} \int_{\mathbb{R}^d} \sigma_1(x-y)\left(\int_{\mathbb{R}^n} |\nabla_z\psi_I(t,y,z)|^{2}\right)^{1/2} \mathrm{d}y \\
&\le \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_z\psi_I(t,y,z)|^2\,\mathrm{d}z\,\mathrm{d}y\right)^{1/2} \\
&\le \frac{1}{c}\,\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\sqrt{\mathscr{E}_I}.
\end{aligned}$$

We proceed similarly to estimate $\nabla_x\phi_I$.

The estimate on ϕ_S is immediate once it is known that $t \mapsto p(t) \in L^1((0,\infty))$, with norm proportional to $1/c^2$. The claim is the object of [8, Lemma 4.4]. For the sake of completeness we sketch the proof in Appendix C. □

Let us proceed to control the nonlinear terms in the entropy estimate starting with term III.

- Owing to Lemma 4.1, we have $Af(x,v) = PAf(x,v) = \langle Af\rangle(x)M(v)$. Hence the product $\left(Af|\nabla_x\phi\cdot\nabla_v f\right)$ vanishes since it can be cast as

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\langle Af\rangle(x)M(v)\nabla_x\phi(x)\cdot\nabla_v f(x,v)}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \langle Af\rangle(x)\nabla_x\phi(x)\cdot\left(\int_{\mathbb{R}^d} \nabla_v f(x,v)\,\mathrm{d}v\right)\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)} = 0.$$

- Next, duality implies that

$$\left(A[\nabla_x\phi\cdot\nabla_v f]|f\right) = -\left(f\nabla_x\phi|\nabla_v A^\star f\right) - \left(f\nabla_x\phi|vA^\star f\right).$$

  Thus, invoking Lemma 4.2, it is concluded that

$$|\left(f\nabla_x\phi|\nabla_v A^\star f\right)| \le \left(\sqrt{\frac{d+1}{2}} + \sqrt{\frac{d+2}{2}}\right)\|f\|_H^2\|\nabla_x\phi\|_{L^\infty}.$$

Coming back to the estimates in Lemma 4.4, we conclude

$$\mathrm{III} \le \eta\Big(\frac{\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\mathrm{I}}}}{c} + \frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty}}{c^2}\Big)\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\ \|f\|_H^2.$$

We continue now with the control of the term IV. Observe that $PA = A$ and $P(v\mathscr{M}_{\mathrm{eq}}) = 0$ imply $\big(Af|\nabla_x\phi\cdot v\mathscr{M}_{\mathrm{eq}}\big) = \big(PAf|\nabla_x\phi\cdot v\mathscr{M}_{\mathrm{eq}}\big) = 0$. Thus, the last term in IV vanishes

$$\big(Af|\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}\big) = -\big(Af|\nabla_x\phi\cdot v\mathscr{M}_{\mathrm{eq}}\big) = 0.$$

Additionally, we can use Lemma 4.2 to obtain

$$\begin{aligned}
\big|(A[\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}]|f)\big| &= \big|(\mathscr{M}_{\mathrm{eq}}\nabla_x\phi|\nabla_v A^\star f + vA^\star f)\big| \\
&\le \|\mathscr{M}_{\mathrm{eq}}\nabla_x\phi\|_H\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H \\
&\le \|\mathscr{M}_{\mathrm{eq}}\nabla_x\phi\|_H\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H \\
&\le \Bigg(\int_{\mathbb{R}^d}|\nabla_x\phi(t,x)|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Bigg)^{1/2}\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H.
\end{aligned} \tag{33}$$

Let us postpone for a moment the estimation of this last quantity and instead consider the integrals

$$-\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x. \tag{34}$$

Note that they are associated to the energy exchanges, since their sum (34) can be shown to be equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f\phi\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^n}(|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2)\,\mathrm{d}z\,\mathrm{d}x.$$

The Cauchy–Schwarz inequality permits us to evaluate

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x\Big| &= \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi\cdot v\sqrt{\mathscr{M}_{\mathrm{eq}}}\,\frac{f}{\sqrt{\mathscr{M}_{\mathrm{eq}}}}\,\mathrm{d}v\,\mathrm{d}x\Big| \\
&\le \Bigg(\int_{\mathbb{R}^d}\big|\nabla_x\phi(t,x)\big|^2\Bigg(\int_{\mathbb{R}^d}v^2M(v)\,\mathrm{d}v\Bigg)\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Bigg)^{1/2}\|f\|_H \\
&\le \Bigg(\int_{\mathbb{R}^d}\big|\nabla_x\phi(t,x)\big|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Bigg)^{1/2}\|f\|_H.
\end{aligned} \tag{35}$$

The second contribution in (34) can be estimated using the entropy dissipation. Indeed, note that

$$(Lf|f) = -\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\nabla_v\Big(\frac{f}{\mathscr{M}_{\mathrm{eq}}}\Big)\Big|^2\mathscr{M}_{\mathrm{eq}}\,\mathrm{d}v\,\mathrm{d}x = -\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\nabla_v f + vf\Big|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}}.$$

Therefore, using the Cauchy–Schwarz inequality and Lemma 4.4 we are led to

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi\frac{f}{\sqrt{\mathscr{M}_{\mathrm{eq}}}}\cdot\frac{\nabla_v f + vf}{\sqrt{\mathscr{M}_{\mathrm{eq}}}}\,\mathrm{d}v\,\mathrm{d}x\Big|$$
$$\le \|\nabla_x\phi\|_{L^\infty}\|f\|_H\sqrt{-(Lf|f)}$$

$$\leq \Big(\frac{\|\nabla_x \sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \sqrt{\mathscr{E}_{\mathrm{I}}}}{c} + \frac{2\mathfrak{m}\Pi_0 \|\nabla_x \Sigma\|_{L^\infty}}{c^2} \Big) \|f\|_H \sqrt{-(Lf|f)}$$
$$\leq -\frac{1}{2}(Lf|f) + \frac{1}{2}\Big(\frac{\|\nabla_x \sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \sqrt{\mathscr{E}_{\mathrm{I}}}}{c} + \frac{2\mathfrak{m}\Pi_0 \|\nabla_x \Sigma\|_{L^\infty}}{c^2} \Big)^2 \|f\|_H^2.$$

In (33) and (35), we need to estimate $\int \mathscr{M}_{\mathrm{eq}} |\nabla_x \phi|^2 \,\mathrm{d}v \,\mathrm{d}x$. Lemma 4.4 tells us that this quantity is uniformly bounded, but we need a more refined estimate that takes into account the finite speed of wave propagation. To this end, from now on, we restrict to the specific case $n = 3$.

Lemma 4.5. *We assume $n = 3$ and $\mathrm{supp}(\sigma_2) \subset B(0, R_2)$.*

i) *We suppose that (A7) is fulfilled. Let ϕ_{I} be defined by (32). Then, there exists $\Gamma, S_0 > 0$, that depends on the assumptions on (ψ_0, ψ_1) (but that does not depend on $c \geq c_0$) such that*

$$|\nabla_x \phi_{\mathrm{I}}(t,x)| \leq \Gamma\, \mathbf{1}_{\{c\,t \leq S_0\}}(t).$$

ii) *Let $\varrho \in L^\infty(0,\infty; L^1(\mathbb{R}^d))$ and let ϕ_{S} be defined by (32). For $0 \leq t \leq T < \infty$, we set*

$$\tau(t) = \max\{0, t - 2R_2/c\}.$$

Then, we have

$$|\nabla_x \phi_{\mathrm{S}}(t,x)| \leq \frac{\Lambda}{c^2} \|\nabla_x \Sigma\|_{L^\infty} \|\varrho\|_{L^\infty(\tau(t),t;L^1(\mathbb{R}^d))}.$$

Proof. We use Kirchhoff's formula, see e.g. [15, Eq. (22), Chapter 2.4, p. 73], for the solution of (27)

$$\psi_{\mathrm{I}}(t,x,z) = \frac{1}{4\pi c^2 t^2} \int\limits_{|z-z'|=ct} \Big(t\,\psi_1(x,z') + \psi_0(x,z') + \nabla_z \psi_0(x,z') \cdot (z'-z) \Big) \,\mathrm{d}S(z')$$

with $\mathrm{d}S$ the Lebesgue measure on the sphere. We use the support assumption (A7) as follows. Observe that $\psi_{\mathrm{I}}(t,x,z)$, as a function of $z \in \mathbb{R}^3$, is supported in the annulus

$$\big\{ z \in \mathbb{R}^3,\ ct - R_{\mathrm{I}} \leq |z| \leq ct + R_{\mathrm{I}} \big\}.$$

Accordingly, the product $\psi_{\mathrm{I}}(t,x,z)\sigma_2(z)$ vanishes for

$$ct \geq R_{\mathrm{I}} + R_2 \overset{\mathrm{def}}{=} S_0 \in (0,\infty).$$

Then, by using the Cauchy–Schwarz inequality, we get

$$\big|\nabla_x \phi_{\mathrm{I}}(t,x)\big| \leq \|\nabla_x \sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^2(\mathbb{R}^3)} \Bigg(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^3} \big|\psi_{\mathrm{I}}(t,y,z)\big|^2 \,\mathrm{d}z \,\mathrm{d}y \Bigg)^{1/2} \mathbf{1}_{\{ct \leq S_0\}}.$$

Next, for estimating the L^2-norm of ψ_{I}, we use the Cauchy–Schwarz inequality again

$$|\psi_{\mathrm{I}}(t,x,z)|^2 \leq \Big(\frac{1}{4\pi c^2 t^2} \Big)^2 \int\limits_{|z'|=ct} \mathrm{d}S(z') \int\limits_{|z'|=ct} \Big(t\,\psi_1(x,z-z') + \psi_0(x,z-z') + \nabla_z \psi_0(x,z-z') \cdot z' \Big)^2 \,\mathrm{d}S(z')$$
$$\leq \frac{3(1+(1+c)t)^2}{4\pi^2 c^2 t^2} \int\limits_{|z'|=ct} \Big(|\psi_1|^2 + |\psi_0|^2 + |\nabla_z \psi_0|^2 \Big)(x, z-z') \,\mathrm{d}S(z').$$

We integrate over x, z and we obtain

$$\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^3} |\psi_{\mathrm{I}}(t,x,z)|^2 \,\mathrm{d}z \,\mathrm{d}x \leq 3(1+(1+c)t)^2 \int\limits_{|z'|=ct} \Bigg(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^3} \Big(|\psi_1|^2 + |\psi_0|^2 + |\nabla_z \psi_0|^2 \Big)(x, z-z') \,\mathrm{d}z \,\mathrm{d}x \Bigg) \frac{\mathrm{d}S(z')}{4\pi c^2 t^2}$$
$$\leq 3(1+(1+c)t)^2 \Big(\|\psi_1\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^3)} + \|\psi_0\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^3)} + \|\nabla_z \psi_0\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^3)} \Big).$$

Furthermore, since $z \mapsto \psi_0(x,z)$ is compactly supported in the ball $B(0,R_{\rm I})$, we can apply the Poincaré estimate $\|\psi_0(x,\cdot)\|^2_{L^2(\mathbb{R}^3)} \le C(R_{\rm I})\|\nabla_z\psi_0(x,\cdot)\|^2_{L^2(\mathbb{R}^3)}$ and finally we conclude that

$$\begin{aligned}\big|\nabla_x\phi_{\rm I}(t,x)\big| &\le \sqrt{3(1+C(R_{\rm I}))\mathscr{E}_0}\,(1+(1+c)t)\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^3)}\,\mathbf{1}_{\{ct\le S_0\}}\\ &\le \underbrace{\sqrt{3(1+C(R_{\rm I}))\mathscr{E}_0}\,\Big(1+S_0\Big(\frac{1}{c_0}+1\Big)\Big)\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^3)}}_{\stackrel{\rm def}{=}\Gamma}\,\mathbf{1}_{\{ct\le S_0\}}\end{aligned}$$

holds since $c \ge c_0$.

Similarly, the solution of (28) is given by (see [15, Eq. (44), Chapter 2.4, p. 82])

$$\psi_{\rm S}(t,x,z) = \frac{1}{4\pi c^2}\int\limits_{|z-z'|\le ct}\frac{\sigma_2(z')\,\sigma_1 * \varrho(t-|z-z'|/c,x)}{|z-z'|}\,{\rm d}z'.$$

The product $\sigma_2(z)\sigma_2(z')$ does not vanish as long as $\max\{|z|,|z'|\}\le R_2$, which implies $|z-z'|\le 2R_2$. As a consequence, when the product $\sigma_2(z)\psi_{\rm S}(t,x,z)$ does not vanish only the values of the density $\varrho(s,\cdot)$ for $\tau(t)\le s\le t$ are relevant. More precisely, coming back to (31), we get

$$\begin{aligned}\big|\nabla_x\phi_{\rm S}(t,x)\big| &= \Big|\frac{1}{c^2}\int\limits_{|z-z'|\le ct}\frac{\sigma_2(z)\,\sigma_2(z')}{4\pi|z-z'|}\nabla_x\Sigma * \varrho(t-|z-z'|/c,x)\,{\rm d}z'\,{\rm d}z\Big|\\ &\le \frac{\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}}{c^2}\sup_{\tau(t)\le s\le t}\|\varrho(s,\cdot)\|_{L^1(\mathbb{R}^d)}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\sigma_2(z)\sigma_2(z')}{4\pi|z-z'|}\,{\rm d}z'\,{\rm d}z\\ &\le \frac{\Lambda}{c^2}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\sup_{\tau(t)\le s\le t}\|\varrho(s,\cdot)\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

where we used for the last integral that $(-\Delta_z)\Upsilon = \sigma_2$, so that

$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\sigma_2(z)\sigma_2(z')}{4\pi|z-z'|}\,{\rm d}z'\,{\rm d}z = \int\limits_{\mathbb{R}^3}\sigma_2\Upsilon\,{\rm d}z = \int\limits_{\mathbb{R}^3}(-\Delta_z)\Upsilon\Upsilon\,{\rm d}z = \int\limits_{\mathbb{R}^3}|\nabla_z\Upsilon|^2\,{\rm d}z = \Lambda. \qquad \square$$

Finally, with Lemma 4.5, we are able to estimate (33) and (35). Indeed, we shall use the obvious inequality

$$\int\limits_{\mathbb{R}^d}|\varrho|\,{\rm d}x \le \int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f|}{\sqrt{\mathscr{M}_{\rm eq}}}\sqrt{\mathscr{M}_{\rm eq}}\,{\rm d}v\,{\rm d}x \le \sqrt{\mathfrak{m}}\|f\|_H.$$

We arrive at (mind the condition $c > c_0$)

$$|{\rm IV}| \le -\frac{1}{2}(Lf|f) + \frac{Q}{c^2}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar\Gamma\|f\|_H\mathbf{1}_{\{ct\le S_0\}}$$

where we have set

$$\begin{aligned}Q &\stackrel{\rm def}{=} \Big(1+\eta\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\Big)\Lambda\mathfrak{m}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\\ &\quad + \frac{1}{2}\Big(\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{6/5}(\mathbb{R}^3)}\sqrt{\mathscr{E}_{\rm I}} + \frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}}{c_0}\Big)^2,\end{aligned}$$

and

$$\bar\Gamma = \Gamma\Big(1+\eta\sqrt{\mathfrak{m}}\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\Big).$$

Gathering the information all together it is concluded that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} \le \frac{1}{2}(Lf|f) - \eta\,(AT_{\mathrm{eq}}Pf|Pf) + \frac{\Xi}{4}\|(1-P)f\|_H^2 + \\ + \frac{\eta^2}{\Xi}C\|Pf\|^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H \mathbf{1}_{\{ct\le S_0\}}$$

holds with

$$\bar{Q} = \frac{1}{c_0}Q + \eta\Big(\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{6/5}(\mathbb{R}^3)}\sqrt{\mathscr{E}_{\mathrm{I}}} + \frac{2\mathfrak{m}\Pi_0}{c_0}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\Big)\Big(\sqrt{\frac{d+1}{2}} + \sqrt{\frac{d+2}{2}}\Big).$$

Poincaré inequalities, see (22) and (25), allow us to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} \le -\frac{\Xi}{2}\|(1-P)f\|_H^2 - \eta\frac{\Xi'}{1+\Xi'}\Big(1-\eta\frac{C(1+\Xi')}{\Xi\Xi'}\Big)\|Pf\|_H^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H\mathbf{1}_{\{ct\le S_0\}}.$$

Choosing η sufficiently small ($0<\eta<\min\big\{1, \frac{\Xi\Xi'}{C(1+\Xi')}\big\}$), we can use (20) to define $\theta=\theta(\eta)>0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t) &\le -2\theta\|f(t,\cdot)\|_H^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H\mathbf{1}_{\{ct\le S_0\}} \\ &\le -2\theta\|f(t,\cdot)\|_H^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \theta\|f\|_H^2 + \frac{\bar{\Gamma}^2}{4\theta}\mathbf{1}_{\{ct\le S_0\}} \\ &\le -\frac{2\theta}{1-\eta}\mathscr{H}(t) + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \frac{\bar{\Gamma}^2}{4\theta}\mathbf{1}_{\{ct\le S_0\}}.\end{aligned}$$

This last inequality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\bar{\theta}t}\mathscr{H}(t)\Big) \le \frac{\bar{Q}}{c}e^{\bar{\theta}t}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \frac{\bar{\Gamma}^2}{4\theta}e^{\bar{\theta}t}\mathbf{1}_{\{ct\le S_0\}},$$

where we have set $\bar{\theta} = \frac{2\theta}{1-\eta}$. We integrate over $0\le t\le\tau$ and we make use of (20) again to obtain

$$\frac{1-\eta}{2}\,e^{\bar{\theta}\tau}\|f(\tau,\cdot)\|_H^2 \le e^{\bar{\theta}\tau}\mathscr{H}(\tau) \le \mathscr{H}(0) + \frac{\bar{\Gamma}^2}{4\theta\bar{\theta}}(e^{\bar{\theta}S_0/c}-1) + \frac{\bar{Q}}{c}\int_0^\tau e^{\bar{\theta}s}\sup_{\tau(s)\le\sigma\le s}\|f(\sigma,\cdot)\|_H^2\,\mathrm{d}s.$$

Setting

$$M(t) \stackrel{\text{def}}{=} \sup_{0\le s\le t} e^{\bar{\theta}s}\|f(s,\cdot)\|_H^2,$$

we are led to

$$\frac{1-\eta}{2}\,M(t) \le \mathscr{H}(0) + \frac{\bar{\Gamma}^2}{4\theta\bar{\theta}}(e^{\bar{\theta}S_0/c}-1) + \frac{\bar{Q}}{c}\int_0^t e^{2\bar{\theta}R_2/c}M(s)\,\mathrm{d}s.$$

Grönwall lemma readily implies that the estimate

$$\|f(t,\cdot)\|_H^2 \le \frac{2}{1-\eta}\Big(\mathscr{H}(0) + \frac{\bar{\Gamma}^2}{4\theta\bar{\theta}}(e^{\bar{\theta}S_0/c}-1)\Big)\exp\Big(-\Big(\bar{\theta} - \frac{2\bar{Q}e^{\bar{\theta}R_2/c}}{(1-\eta)c}\Big)t\Big)$$

holds. This completes the proof of Theorem 2.3. □

Remark 4.6. The main argument in Lemma 4.5 relies on the evaluation of the support of the solution of the wave equation by means of Huygens' principle. The analysis can be extended to odd space dimensions $n\ge 3$, at the price of more intricate formulae for ψ_{I} and ψ_{S}, see [15, Eq. (31), Chapter 2.4, p. 77]. Details are left to the reader. Arguments that make the case $n=3$ particularly relevant on physical grounds are presented in [6].

Conflict of interest statement

No conflict of interest.

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Appendix A. Linearized stability for the dissipationless model

By construction $\mathscr{M}_{\text{eq}}(x, v)$ is still a solution of the Vlasov-Wave equation (1)–(2) in the case where $\gamma = 0$. Let us consider the *linearized problem*

$$\begin{aligned}
&(\partial_t + T_{\text{eq}}) f = \nabla_x \phi \cdot \nabla_v \mathscr{M}_{\text{eq}} = -v \mathscr{M}_{\text{eq}} \cdot \nabla_x \phi, \\
&\phi(t, x) = \sigma_1 * \left(\int_{\mathbb{R}^n} \sigma_2(z) \psi(t, \cdot, z)\, \mathrm{d}z \right)(x), \\
&(\partial_{tt}^2 - c^2 \Delta_z) \psi(t, x, z) = -\sigma_2(z) \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_1(x - y) f(t, y, v)\, \mathrm{d}v\, \mathrm{d}y.
\end{aligned} \tag{A.1}$$

The linear stability can be established by adapting the reasoning in [3] for the gravitational Vlasov–Poisson system.

Theorem A.1. *We suppose $n \geq 3$. There exists $c_1 \geq c_0 > 0$ such that the following assertion holds true for any $c > c_1$: for any $\epsilon > 0$, there exists $\eta > 0$ such that if the initial data for (A.1) satisfies*

$$\|f(0, \cdot)\|_H + \|\partial_t \psi(0, \cdot)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^d)} + c\|\nabla_z \psi(0, \cdot)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^d)} \leq \eta,$$

then, for the solution of (A.1) we have $\|f(t, \cdot)\|_H \leq \epsilon$.

Proof. We check that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(t, x, v)|^2}{\mathscr{M}_{\text{eq}}(x, v)}\, \mathrm{d}v\, \mathrm{d}x + \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(t, x)\ f(t, x, v)\, \mathrm{d}v\, \mathrm{d}x \right.$$
$$\left. + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} \left(|\partial_t \psi|^2 + c^2 |\nabla_z \psi|^2 \right)(t, x, z)\, \mathrm{d}z\, \mathrm{d}x \right\} = 0.$$

By using the Sobolev embedding, see [23, Th. 8.3] we can estimate the contribution of the potential energy as follows

$$\begin{aligned}
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi f\, \mathrm{d}v\, \mathrm{d}x \right| &\leq \|f(t, \cdot)\|_{L^1} \|\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \|f(t, \cdot)\|_H \|\sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^n} |\psi(t, x, z)|^{2n/(n-2)}\, \mathrm{d}z \right)^{(n-2)/n} \mathrm{d}x \right)^{1/2} \\
&\leq \|f(t, \cdot)\|_H \|\sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \|\nabla_z \psi(t, \cdot)\|_{L^2(\mathbb{R}^d \times \mathbb{R}^n)} \\
&\leq \frac{1}{4} \|f(t, \cdot)\|_H^2 + \left(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c} \right)^2 c^2 \|\nabla_z \psi(t, \cdot)\|_{L^2(\mathbb{R}^d \times \mathbb{R}^n)}^2.
\end{aligned}$$

Coming back to the energy conservation, we are led to the inequalities

$$\frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(t,x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,{\rm d}v\,{\rm d}x+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\psi(t,x,z)|^2\,{\rm d}z\,{\rm d}x$$
$$+\Big(\frac{1}{2}-\Big(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c}\Big)^2\Big)c^2\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi(t,x,z)|^2\,{\rm d}z\,{\rm d}x$$
$$\le\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(t,x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,{\rm d}v\,{\rm d}x+\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x)\ f(t,x,v)\,{\rm d}v\,{\rm d}x$$
$$+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\big(|\partial_t\psi|^2+c^2|\nabla_z\psi|^2\big)(t,x,z)\,{\rm d}z\,{\rm d}x$$
$$\le\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(0,x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,{\rm d}v\,{\rm d}x+\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(0,x)\ f(0,x,v)\,{\rm d}v\,{\rm d}x$$
$$+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\big(|\partial_t\psi|^2+c^2|\nabla_z\psi|^2\big)(0,x,z)\,{\rm d}z\,{\rm d}x$$
$$\le\frac{3}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(0,x,v)|^2}{\mathscr{M}_{\rm eq}(x,v)}\,{\rm d}v\,{\rm d}x+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\psi(0,x,v)|^2\,{\rm d}z\,{\rm d}x$$
$$+\Big(\frac{1}{2}+\Big(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c}\Big)^2\Big)c^2\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi|^2(0,x,z)\,{\rm d}z\,{\rm d}x.$$

This estimate allows us to conclude by choosing $c_1=\sqrt{2}\|\sigma_1\||_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}$. □

Appendix B. A compactness lemma

In Section 3.3, we made use of the following claim.

Lemma B.1. *Let $\big(u_n\big)_{n\in\mathbb{N}}$ be a sequence defined on $(0,T)\times\mathbb{R}^N$ such that:*

i) We can find a non-decreasing function $\omega:(0,\infty)\to(0,\infty)$ such that

$$\sup_n\int_0^T\int_{\mathbb{R}^N}|u_n(t,x+h)-u_n(t,x)|\,{\rm d}x\,{\rm d}t\le\omega(|h|)\xrightarrow[|h|\to 0]{}0,$$

ii) $\partial_t u_n=\sum_{|\alpha|\le k}\partial_x^\alpha g_n^{(\alpha)}$, *with* $\sup_{n,\alpha}\|g_n^{(\alpha)}\|_{L^1((0,T)\times\mathbb{R}^N)}=M<\infty.$

Then, $\big(u_n\big)_{n\in\mathbb{N}}$ is relatively compact in $L^1_{\rm loc}((0,T)\times\mathbb{R}^N)$.

Proof. Let $\big(\zeta^\delta\big)_{\delta>0}$ be a sequence of mollifiers:

$$0\le\zeta^\delta(x)\le 1,\qquad\int\zeta^\delta(x)\,{\rm d}x=1,\qquad{\rm supp}(\zeta^\delta)\subset B(0,\delta).$$

We set $u_n^\delta(t,x)=\int\zeta^\delta(x-y)u_n(t,y)\,{\rm d}y=\int\zeta^\delta(y)u_n(t,x-y)\,{\rm d}y$. Owing to i), we get

$$\int_0^T\int_{\mathbb{R}^N}|u_n^\delta(t,x)-u_n(t,x)|\,{\rm d}x\,{\rm d}t\le\int\zeta^\delta(y)\left(\int_0^T\int_{\mathbb{R}^N}|u_n(t,x-y)-u_n(t,x)|\,{\rm d}x\,{\rm d}t\right){\rm d}y\le\omega(\delta).$$

In other words u_n^δ converges in $L^1((0,T)\times\mathbb{R}^N)$ as $\delta\to 0$, uniformly with respect to n. We are going to conclude by showing the compactness in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$ of the family $\{u_n^\delta,\ n\in\mathbb{N}\}$, for $\delta>0$ fixed. It is clear that

$$\sup_n \int_0^T\int_{\mathbb{R}^N} |u_n^\delta(t,x+h)-u_n^\delta(t,x)|\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[|h|\to 0]{} 0$$

holds. Next, we observe that (possibly extending the functions by 0 out of $(0,T)$)

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^N} |u_n^\delta(t+\tau,x)-u_n^\delta(t,x)|\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^N}\left|\int \zeta^\delta(x-y)\left(\int_t^{t+\tau}\partial_t u_n(s,y)\,\mathrm{d}s\right)\mathrm{d}y\right|\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^N}\left|\sum_{|\alpha|\le k}\int\int_t^{t+\tau}(\partial^\alpha\zeta^\delta)(x-y)\; g_n^{(\alpha)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\mathrm{d}x\,\mathrm{d}t\\
&\le k\|\zeta^\delta\|_{W^{k,\infty}}\int_0^T\int_t^{t+\tau}|g_n^{(\alpha)}(s,y)|\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t \le C\tau.
\end{aligned}$$

The conclusion follows by virtue of the Kolmogorov–Riesz–Fréchet criterion [16, Th. 7.56]. □

Appendix C. Proof of Lemma 4.4

Let us set

$$q(t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\sin(t|\xi|)}{|\xi|}|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi.$$

The Lebesgue theorem tells us that $t\mapsto q(t)$ is continuous on $[0,\infty)$. Since σ_2 is radially symmetric, we have

$$\begin{aligned}
q(t) &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty \sin(tr)r^{n-2}|\widehat{\sigma_2}(re_1)|^2\,\mathrm{d}r\\
&= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty \frac{\cos(tr)}{t}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\right]\mathrm{d}r\\
&= -\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty \frac{\sin(tr)}{t^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\right]\mathrm{d}r.
\end{aligned}$$

Therefore, q is integrable as a consequence of the following estimate

$$|q(t)|\le\frac{K}{t^2}\quad\text{with}\quad K=\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty\left|\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\right]\right|\mathrm{d}r<\infty.$$

Note added to the proof

Since the completion of this work, we learnt that a similar analysis has been performed for the Vlasov–Poisson–Fokker–Planck system by F. Hérau and L. Thomann. The result of [20] has the same flavor, namely the existence-uniqueness of a normalized equilibrium state, obtained as a solution of a nonlinear integro-differential equation (Poisson–Emden equation), and the asymptotic trend to equilibrium, with an exponential rate. The approach is also perturbative, in the sense that the results hold provided the coupling parameter in the Poisson equation is small enough.

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