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Ann. I. H. Poincaré – AN 34 (2017) 1913–1923

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Analysis of the loss of boundary conditions for the diffusive Hamilton–Jacobi equation

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Received 3 October 2016; accepted 8 February 2017

Available online 21 March 2017

Abstract

We consider the diffusive Hamilton–Jacobi equation, with superquadratic Hamiltonian, homogeneous Dirichlet conditions and regular initial data. It is known from [4] (Barles–DaLio, 2004) that the problem admits a unique, continuous, global viscosity solution, which extends the classical solution in case gradient blowup occurs. We study the question of the possible loss of boundary conditions after gradient blowup, which seems to have remained an open problem till now.

Our results show that the issue strongly depends on the initial data and reveal a rather rich variety of phenomena. For any smooth bounded domain, we construct initial data such that the loss of boundary conditions occurs everywhere on the boundary, as well as initial data for which no loss of boundary conditions occurs in spite of gradient blowup. Actually, we show that the latter possibility is rather exceptional. More generally, we show that the set of the points where boundary conditions are lost, can be prescribed to be arbitrarily close to any given open subset of the boundary.

Keywords: Diffusive Hamilton–Jacobi equation; Viscosity solution; Gradient blow-up; Loss of boundary conditions

1. Introduction

We consider the initial-boundary value problem for the diffusive Hamilton–Jacobi equation:

$$\begin{cases} u_t - \Delta u = |\nabla u|^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Throughout this article, we assume that Ω is a $C^{2+\alpha}$ smooth bounded domain of $\mathbb{R}^n$, $p > 2$ and

$$u_0 \in X := \left\{ v \in C^1(\overline{\Omega});\ v \geq 0 \text{ and } v = 0 \text{ on } \partial\Omega \right\},$$

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http://dx.doi.org/10.1016/j.anihpc.2017.02.001

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endowed with the C^1 norm. This problem has been studied by many authors in the past decades (see e.g. [22, Chapter 40] and the references therein).

By standard theory [10], it is known that problem (1.1) admits a unique, maximal **classical C^1 solution** $u \geq 0$, such that $u \in C^{2,1}(\overline{\Omega} \times (0, T^*))$ and $u, \nabla u \in C(\overline{\Omega} \times [0, T^*))$. Here $T^* = T^*(u_0) \in (0, \infty]$ denotes its existence time and the differential equation and the boundary conditions are satisfied in the pointwise sense for $t \in (0, T^*)$. Moreover, the solution satisfies the maximum principle estimate

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad 0 < t < T^*,$$

and the classical C^1 solution can only cease to exist through **gradient blowup**:

$$T^* < \infty \quad \Longrightarrow \quad \lim_{t \to T^*} \|\nabla u(t)\|_\infty = \infty.$$

Actually, ∇u remains bounded away from the boundary and gradient blowup occurs only on $\partial\Omega$ (see [25]). Furthermore it is known (see, e.g., [1,2,23]) that $T^* < \infty$ whenever the initial data is suitably large. We also recall that this phenomenon does not occur when $1 \leq p \leq 2$.

On the other hand, it was proved in [4] that problem (1.1) admits a unique **global viscosity solution** $u \in C(\overline{\Omega} \times [0, \infty))$, where the boundary conditions have to be understood in the viscosity sense. Throughout this article, we shall denote this solution by u without risk of confusion, since the two solutions coincide on $[0, T^*)$. (The result in [4] is actually valid for any $u_0 \in C_0(\overline{\Omega})$, but this need not concern us here.) Moreover, u is actually smooth away from the boundary, namely

$$u \in C^{2,1}(\Omega \times (0, \infty))$$

and it solves the PDE in (1.1) in the classical sense in $\Omega \times (0, \infty)$ (see Section 3 for details). It was next proved in [20] that for $t > T_0 = T_0(\|u_0\|_\infty)$ sufficiently large, u is actually a classical solution again, namely $u \in C^{2,1}(\overline{\Omega} \times (T_0, \infty))$ with $u(\cdot, t) = 0$ on $\partial\Omega$.

When gradient blowup occurs, the question of possible loss of boundary conditions for $t \geq T^*$ (hence actually in $[T^*, T_0]$) has remained essentially open. Namely, it is unknown whether or not u satisfies the boundary conditions $u = 0$ on $\partial\Omega \times [T^*, T_0]$ in the classical sense. In what follows, we say that **loss of boundary conditions** occurs at a point $x_0 \in \partial\Omega$ if $u(x_0, t) > 0$ for some $t \geq T^*$.

The goal of this article is to give some answers to this question. A main conclusion is that loss of boundary conditions after gradient blowup **may or may not occur, depending on the initial data.** Furthermore, in case it occurs, the structure and size of the set of the points where boundary conditions are lost, strongly depends on the initial data. This is somewhat surprising and shows that the problem reveals a rather rich variety of phenomena.

Throughout this paper, we denote by φ_1 the first Dirichlet eigenfunction of $-\Delta$ in Ω, normalized by $\int_\Omega \varphi_1 \, dx = 1$.

2. Main results

For any $u_0 \in X$, we define the **loss of boundary conditions set** by

$$\mathcal{L}(u_0) = \big\{x_0 \in \partial\Omega,\ u(x_0, t) > 0 \text{ for some } t > 0\big\}.$$

Our first result shows that there exist initial data for which the loss of boundary conditions occurs **everywhere on $\partial\Omega$**, and moreover can be achieved at a common time.

Theorem 1. *Let $p > 2$. There exists $u_0 \in X$ such that $\mathcal{L}(u_0) = \partial\Omega$ and that, moreover,*

$$u(x, t_0) \geq c_0, \quad x \in \partial\Omega,$$

for some $t_0, c_0 > 0$. Furthermore, the same remains true for any $v_0 \in X$ with $v_0 \geq u_0$.

Our next result shows that, at the opposite, there are gradient blowup solutions for which **no loss of boundary conditions ever occurs.**

Theorem 2. *Let $p > 2$. There exists $u_0 \in X$ such that $T^*(u_0) < \infty$ and $\mathcal{L}(u_0) = \emptyset$, i.e.,*

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

At least in one space dimension, one can show that such u_0 are rather exceptional (see Remark 2.2 for more comments).

Theorem 3. *Let $p > 2$, $n = 1$ and let u_0 be as in Theorem 2. Let $v_0 \in X$, with $v_0 \not\equiv u_0$ and denote by v the corresponding global viscosity solution of (1.1).*

(i) If $v_0 \geq u_0$, then $\mathcal{L}(v_0) \neq \emptyset$.
(ii) If $v_0 \leq u_0$, then $T^*(v_0) = \infty$.

Our last two results are concerned with some "intermediate" ranges of u_0. We first consider initial data which are large in an integral sense (hence need not be the same as those in Theorem 1) and show that, as the size grows larger, the loss of boundary conditions occurs "near" every point of $\partial\Omega$.

Theorem 4. *Let $p > 2$. For any $\varepsilon > 0$, there exists a constant $M = M(\Omega, p, \varepsilon) > 0$ such that if $\int_\Omega u_0\varphi_1\,dx \geq M$, then for any $x_0 \in \partial\Omega$, we have $\mathcal{L}(u_0) \cap B_\varepsilon(x_0) \neq \emptyset$.*

Notice that, due to the continuity of u up to the boundary, $\mathcal{L}(u_0)$ is a (relatively) open subset of $\partial\Omega$. Our last result shows that one can find solutions for which the loss of boundary conditions occurs essentially only on a prescribed open subset of $\partial\Omega$, and at a common time.

Theorem 5. *Let $p > 2$. Let ω be any open set of $\mathbb{R}^n$ with $\omega \cap \partial\Omega \neq \emptyset$. Let $\varepsilon > 0$ and set $\omega_\varepsilon = \omega + B_\varepsilon(0)$. There exists $u_0 \in X$ such that*

$$\omega \cap \partial\Omega \subset \mathcal{L}(u_0) \subset \omega_\varepsilon \cap \partial\Omega$$

and such that, moreover,

$$u(x, t_0) \geq c_0, \quad x \in \omega \cap \partial\Omega, \tag{2.1}$$

for some $t_0, c_0 > 0$.

Remark 2.1. We note that some solutions with **single-point** gradient blowup on the boundary (at $T^*(u_0)$) may develop loss of boundary conditions on some **open subset** of the boundary after $T^*(u_0)$. Indeed, a closer inspection of the proof of Theorem 5 shows that one can construct u_0 which satisfy the conclusions of Theorem 5 and at the same time verify the assumptions of [15, Theorem 1.1], guaranteeing single-point gradient blowup on the boundary, for suitable domains of $\mathbb{R}^2$.

Remark 2.2. As shown by Theorem 3, the solutions constructed in Theorem 2 constitute (strong) **thresholds**, realizing the transition from global classical existence to loss of boundary conditions. This parallels the phenomenon of transition from global existence to (complete) blowup for the reaction–diffusion equation

$$u_t - \Delta u = u^p \tag{2.2}$$

(see [18,22], and the recent work [21] where the notion of *strong* threshold is studied). In this respect, gradient blowup without loss of boundary conditions plays the same role as "incomplete blowup" in the case of equation (2.2), which is the threshold behavior for supercritical p (i.e., $n \geq 3$ and $p > (n + 2)/(n - 2)$).

For related results on the continuation of solutions after gradient blow-up, see [8,9,26]. We refer to [2,8,6,5,3,12, 24,25,11,15,19] for other aspects of gradient blowup phenomena, and to [13,14] for some physical background.

3. Preliminaries

We set $Q = \Omega \times (0, \infty)$ and denote the function distance to the boundary by

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

We set $\Omega_\eta = \{x \in \Omega;\ \delta(x) > \eta\}$ and recall that Ω_η is smooth for $\eta > 0$ small. Moreover, denoting by ν_η the outer normal unit vector and $d\sigma_\eta$ the surface measure on $\partial\Omega_\eta$, we have the property

$$\lim_{\eta\to 0} \int_{\partial\Omega_\eta} d\sigma_\eta = \int_{\partial\Omega} d\sigma \quad \text{and} \quad \lim_{\eta\to 0} \int_{\partial\Omega_\eta} V \cdot \nu_\eta \, d\sigma_\eta = \int_{\partial\Omega} V \cdot \nu \, d\sigma, \quad V \in (C(\overline{Q}))^n. \tag{3.1}$$

We shall also need the following uniform version of the Poincaré inequality (see, e.g., [17]). Let $k \in [1, \infty)$. For each $\varepsilon > 0$, there exists a constant $C = C(\Omega, \varepsilon, k) > 0$ such that

$$\int_\Omega |v|^k \le C \int_\Omega |\nabla v|^k \, dx, \qquad v \in \bigcup_{x_0 \in \partial\Omega} \{v \in W^{1,k}(\Omega);\ v_{|\partial\Omega \cap B_\varepsilon(x_0)} = 0\} \tag{3.2}$$

($v = 0$ being understood in the sense of traces if v is not continuous).

We now turn to properties of the unique global viscosity solution u of (1.1). We refer to [7,4] for more details about viscosity solutions theory. We first recall that the global viscosity solution can also be viewed as the limit of global classical solutions of regularized problems. Namely, for each integer $j \ge 1$, we set

$$F_j(\xi) = \min\bigl(|\xi|^p, j^{p-2}|\xi|^2\bigr), \quad \xi \in \mathbb{R}^n,$$

and, for $u_0 \in X$, consider the problem

$$\begin{cases} \partial_t u_j - \Delta u_j = F_j(\nabla u_j), & x \in \Omega,\ t > 0, \\ u_j(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u_j(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.3}$$

Since each F_j has at most quadratic growth, problem (3.3) admits a unique global classical solution $u_j \ge 0$. Moreover u_j is nondecreasing with respect to j by the comparison principle, and it is known (see [7] and [20]) that

$$\lim_{j\to\infty} u_j(x,t) = u(x,t), \qquad (x,t) \in Q.$$

As a consequence of this approximation procedure, one for instance easily recovers the maximum principle estimate

$$\|u(t) - v(t)\|_\infty \le \|u_0 - v_0\|_\infty, \quad t > 0 \tag{3.4}$$

for all $u_0, v_0 \in X$ (which yields in particular the continuous dependence in L^∞).

Next, as a consequence of uniform interior estimates for the approximating solutions u_j, one shows that

$$u \in C^{2,1}(Q) \tag{3.5}$$

and that u solves the PDE in (1.1) in the classical sense in Q. For that purpose, by standard parabolic regularity, it suffices to prove that ∇u_j is bounded on compact subsets of Q, independently of j. Such a bound can be proved by a Bernstein argument with cut-off (see e.g. [16] in the elliptic case and [25] in the parabolic case; more specifically, this follows from a simple modification of the proof of [25, Theorem 3.2]).

Moreover, we have the following time-derivative estimate.

Lemma 3.1. *Let $u_0 \in X$ and let u be the corresponding global viscosity solution of (1.1). Then, for all $t > 0$ we have $u_t(\cdot,t) \in L^\infty(\Omega)$. Moreover, for all $t_0 > 0$, there exists a constant $C(t_0) > 0$ such that*

$$\|u_t(t)\|_\infty \le C(t_0), \quad t \ge t_0. \tag{3.6}$$

Proof. We may assume without loss of generality that $t_0 \in (0, T^*(u_0))$. Let $t \geq t_0$ and $h > 0$. By estimate (3.4), we have

$$\|u(t+h) - u(t)\|_\infty \leq \|u(t_0+h) - u(t_0)\|_\infty.$$

Recall that $u \in C^{2,1}(\overline{\Omega} \times (0, T^*)) \cap C^{2,1}(\Omega \times (0, \infty))$. Dividing by h and letting $h \to 0$, we deduce that

$$\|u_t(t)\|_\infty \leq \|u_t(t_0)\|_\infty,$$

and the lemma is proved. □

On the other hand, we know that ∇u also satisfies the following Bernstein estimate: for each $\tau > 0$, there exists a constant $C(\tau) > 0$ such that

$$|\nabla u(x,t)| \leq C(\tau)\delta^{-1/(p-1)}(x), \quad x \in \Omega,\ t \geq \tau. \tag{3.7}$$

This is proved in [25] for classical solutions, i.e., on $(0, T^*(u_0))$, but the proof remains valid for the global viscosity solution, using (3.5) along with estimate (3.6).

Finally, we give the following lemma, which will be useful for the proof of Theorem 3.

Lemma 3.2. *Let $u_0, v_0 \in X$ and $\lambda > 1$ be such that $v_0 \geq \lambda u_0$ and denote by u, v the corresponding global viscosity solutions of (1.1). Then*

$$v \geq \lambda u \quad \text{in } \overline{\Omega} \times (0, \infty).$$

Proof. Let $j \geq 1$ and let u_j, v_j be the solutions of the approximating problems (3.3). Setting $\underline{u}_j = \lambda u_j$, we see that

$$\begin{aligned}
\partial_t \underline{u}_j - \Delta \underline{u}_j - F_j(\nabla \underline{u}_j) &= \lambda\Big[\min\big(|\nabla u_j|^p, j^{p-2}|\nabla u_j|^2\big) - \min\big(\lambda^{p-1}|\nabla u_j|^p, j^{p-2}\lambda|\nabla u_j|^2\big)\Big] \\
&\leq \lambda\Big[\min\big(|\nabla u_j|^p, j^{p-2}|\nabla u_j|^2\big) - \min\big(|\nabla u_j|^p, j^{p-2}|\nabla u_j|^2\big)\Big] \\
&= 0 = \partial_t v_j - \Delta v_j - F_j(\nabla v_j)
\end{aligned}$$

in Q. We deduce from the comparison principle that $\underline{u}_j \leq v_j$ in Q and the result follows by passing to the limit $j \to \infty$. □

4. Proof of Theorem 4

We first prove Theorem 4, since the result is (independently) used in the proof of Theorem 1. We adapt eigenfunction arguments used in [1,23] to prove gradient blowup for weak or classical solutions. It turns out that these arguments can be modified to establish the loss of boundary conditions for global viscosity solutions, making use of the Bernstein estimate (3.7).

Recall that we denote by φ_1 the first Dirichlet eigenfunction of $-\Delta$ in Ω, normalized by $\int_\Omega \varphi_1\, dx = 1$ and let $\lambda_1 > 0$ be the corresponding eigenvalue. Let $0 < \tau < t < \infty$ and let $\eta > 0$ small. Since we only have $u \in C^{2,1}(Q) \cap C(\overline{Q})$, we cannot directly integrate in Ω. Instead, we multiply the PDE in (1.1) by φ_1 and integrate by parts on Ω_η. This yields

$$\begin{aligned}
\Big[\int_{\Omega_\eta} u\varphi_1\, dx\Big]_\tau^t &= \int_\tau^t \int_{\Omega_\eta} \varphi_1 \Delta u\, dx ds + \int_\tau^t \int_{\Omega_\eta} |\nabla u|^p \varphi_1\, dx ds \\
&= \int_\tau^t \int_{\Omega_\eta} u \Delta\varphi_1\, dx ds + \int_\tau^t \int_{\partial\Omega_\eta} (\varphi_1 \nabla u - u\nabla\varphi_1)\cdot \nu_\eta\, d\sigma_\eta ds + \int_\tau^t \int_{\Omega_\eta} |\nabla u|^p \varphi_1\, dx ds.
\end{aligned}$$

Recall that

$$c_1\delta(x) \leq \varphi_1(x) \leq c_2\delta(x), \qquad x \in \Omega, \tag{4.1}$$

and

$$\int_\Omega \delta^{-\beta}(x)\,dx < \infty, \quad \text{for all } \beta \in (0,1) \tag{4.2}$$

(see e.g. [23]). Using (3.7), (4.1) and (3.1), we obtain

$$\begin{aligned}\Big|\int_\tau^t \int_{\partial\Omega_\eta} \varphi_1 \nabla u \cdot \nu_\eta \, d\sigma_\eta ds\Big| &\le C(\tau)t \int_{\partial\Omega_\eta} \delta^{(p-2)/(p-1)}(x)\,d\sigma_\eta \\ &\le C(\tau)t\eta^{(p-2)/(p-1)} \int_{\partial\Omega_\eta} d\sigma_\eta \\ &\le C(\tau)t\eta^{(p-2)/(p-1)} \to 0, \quad \text{as } \eta \to 0.\end{aligned}$$

Also we note that, for all $t > 0$, we have

$$\int_\Omega |\nabla u(t)|^p \varphi_1\,dx \le C(t)\int_\Omega \delta^{-p/(p-1)}(x)\delta(x)\,dx = C(t)\int_\Omega \delta^{-1/(p-1)}(x)\,dx < \infty, \tag{4.3}$$

owing to (3.7), (4.1) and (4.2). Using (3.1) and the fact that $u \in C(\overline{Q})$, we may pass to the limit $\eta \to 0$ to get

$$\Big[\int_\Omega u\varphi_1\,dx\Big]_\tau^t = -\lambda_1 \int_\tau^t \int_\Omega u\varphi_1\,dxds - \int_\tau^t \int_{\partial\Omega} u\partial_\nu \varphi_1\,d\sigma ds + \int_\tau^t \int_\Omega |\nabla u|^p \varphi_1\,dxds.$$

Using $u \ge 0$ and $\partial_\nu \varphi_1 \le 0$ on $\partial\Omega$, and then passing to the limit $\tau \to 0$, we get, for all $t > 0$,

$$\int_\Omega u(t)\varphi_1\,dx \ge \int_\Omega u_0\varphi_1\,dx + \int_0^t \int_\Omega |\nabla u|^p \varphi_1\,dxds - \lambda_1 \int_0^t \int_\Omega u\varphi_1\,dxds, \tag{4.4}$$

hence in particular the finiteness of the integral of the gradient term in (4.4). Let $k \in [1, p/2)$. By Hölder's inequality, we have

$$\int_\Omega |\nabla u|^k\,dx = \int_\Omega |\nabla u|^k \varphi_1^{k/p} \varphi_1^{-k/p}\,dx \le \Big(\int_\Omega |\nabla u|^p \varphi_1\,dx\Big)^{k/p} \Big(\int_\Omega \varphi_1^{-k/(p-k)}\,dx\Big)^{(p-k)/p},$$

hence

$$\Big(\int_\Omega |\nabla u|^k\,dx\Big)^{p/k} \le C(k)\int_\Omega |\nabla u|^p \varphi_1\,dx, \tag{4.5}$$

owing to (4.1) and (4.2). In particular, in view of (4.3) and of $u \in C(\overline{Q})$, we have

$$u(t) \in W^{1,k}(\Omega), \quad \text{for all } t > 0. \tag{4.6}$$

Now assume that there exist $\varepsilon > 0$ and $x_0 \in \partial\Omega$ such that $u = 0$ on $(\partial\Omega \cap B_\varepsilon(x_0)) \times (0,\infty)$. Fixing any $k \in (1, p/2)$, and taking (4.6) into account, we may apply the Poincaré inequality (3.2). This along with Hölder's inequality and (4.5) yields

$$\Big(\int_\Omega u\varphi_1\,dx\Big)^p \le \Big(\int_\Omega u^k\,dx\Big)^{p/k} \le C(\varepsilon)\Big(\int_\Omega |\nabla u|^k\,dx\Big)^{p/k} \le C(\varepsilon)\int_\Omega |\nabla u|^p \varphi_1\,dx.$$

Going back to (4.4), it follows that, for all $t > 0$,

$$\int_\Omega u(t)\varphi_1\,dx \ge \int_\Omega u_0\varphi_1\,dx + c_0 \int_0^t \Big[\Big(\int_\Omega u\varphi_1\,dx\Big)^p - c_1^p\Big]ds, \tag{4.7}$$

for some constants $c_0, c_1 > 0$ depending on ε. Assume that

$$\int_\Omega u_0\varphi_1\,dx \ge 2c_1.$$

It then easily follows from (4.7) that $\int_\Omega u(t)\varphi_1\,dx \ge \int_\Omega u_0\varphi_1\,dx \ge 2c_1$ for all $t > 0$. Consequently

$$\int_\Omega u(t)\varphi_1\,dx \ge \int_\Omega u_0\varphi_1\,dx + \frac{c_0}{2}\int_0^t\Big(\int_\Omega u\varphi_1\,dx\Big)^p ds =: H(t), \quad t > 0,$$

hence $H'(t) \ge (c_0/2)H^p$, which implies the finite time blowup of $\int_\Omega u(t)\varphi_1\,dx$. But this is a contradiction with the fact that $u \in C(\overline{Q})$ (or with the estimate $\int_\Omega u(t)\varphi_1\,dx \le \|u(t)\|_\infty \le \|u_0\|_\infty$). $\square$

5. Proof of Theorem 1

We shall modify an argument from [15] based on a radial auxiliary problem and a scaling argument. Consider the auxiliary problem

$$\begin{cases} V_t - \Delta V = |\nabla V|^p, & x \in B_1(0),\ t > 0, \\ V(x,t) = 0, & x \in \partial B_1(0),\ t > 0, \\ V(x,0) = V_0(x), & x \in B_1(0), \end{cases} \tag{5.1}$$

where $V_0 \in C^1(\overline{B}_1(0))$, with V_0 radially symmetric and supported in $B_{1/2}(0)$. As a consequence of Theorem 4, proved in the previous section, we may choose V_0 such that loss of boundary conditions occurs for V. Since V is radially symmetric, it follows that there exist $t_0, c_0 > 0$ such that

$$V(x,t_0) = c_0 \quad \text{for all } x \in \partial B_1(0).$$

Next, since $\partial\Omega$ is smooth, one can find $\rho > 0$ such that, for all $x_0 \in \partial\Omega$, there exists $x_1 = x_1(x_0)$ such that

$$B_\rho(x_1) \subset \Omega \quad \text{and} \quad \partial\Omega \cap \partial B_\rho(x_1) = \{x_0\}. \tag{5.2}$$

We now use the scale invariance of the equation and set

$$w(x_0; x,t) = \rho^\beta V\big(\rho^{-1}(x-x_1), \rho^{-2}t\big), \quad (x,t) \in \overline{B}_\rho(x_1) \times [0,\infty),$$

with $\beta = (p-2)/(p-1)$. A straightforward computation shows that $w(x_0;\cdot,\cdot)$ is the solution of (5.1) with $B_1(0)$ replaced by $B_\rho(x_1)$ and V_0 replaced by $\rho^\beta V_0(\rho^{-1}(x-x_1))$. Now choose $u_0 \in X$ such that

$$u_0 \ge \rho^\beta\|V_0\|_\infty \quad \text{in } \Omega'_\rho := \{x \in \Omega;\ \rho/2 \le \delta(x) \le 3\rho/2\}.$$

For any $x_0 \in \partial\Omega$, the function $w(x_0;\cdot,0)$ is supported in $B_{\rho/2}(x_1) \subset \Omega'_\rho$, owing to (5.2), hence $u_0 \ge w(x_0;\cdot,0)$ in $\overline{B}_\rho(x_1)$. By the comparison principle, it follows that $u \ge w(x_0;\cdot,\cdot)$ in $\overline{B}_\rho(x_1) \times [0,\infty)$, hence in particular,

$$u(x_0,\rho^2 t_0) \ge w(x_0;x_0,\rho^2 t_0) = \rho^\beta V\big(\rho^{-1}(x_0-x_1), t_0\big) = c_0\rho^\beta > 0.$$

The conclusion for u_0 follows. The assertion for $v_0 \ge u_0$ is then an immediate consequence of the comparison principle. $\square$

6. Proof of Theorem 2

Fix $\phi \in X$, $\phi \not\equiv 0$ and, for $\lambda > 0$, consider u_λ the solution of (1.1) with initial data $u_0 = \lambda\phi$. By, e.g., [23] we know that $T^*(\lambda\phi) = \infty$ for λ small and $T^*(\lambda\phi) < \infty$ for λ large. We may thus define

$$\lambda^* = \inf\{\lambda > 0;\ T^*(\lambda\phi) < \infty\} \in (0,\infty).$$

We shall prove that u_{λ^*} has the desired properties.

First, since $u_\lambda = 0$ on $\partial\Omega \times (0, \infty)$ for all $\lambda \in (0, \lambda^*)$, it follows from the L^∞ continuous dependence estimate (3.4) that

$$u_{\lambda^*} = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Next, by [23], the trivial solution is asymptotically stable in X. Namely, there exists $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ such that, for any $v_0 \in X$,

$$\|v_0\|_X \le \varepsilon_0 \quad \Longrightarrow \quad T^*(v_0) = \infty \quad \text{and} \quad \lim_{t\to\infty} \|v(t)\|_X = 0.$$

On the other hand, by [20], we know that there exists $t_0 > 0$ such that

$$u_{\lambda^*}(t) \in X \quad \text{for all } t \ge t_0 \quad \text{and} \quad \lim_{t\to\infty} \|u_{\lambda^*}(t)\|_X = 0.$$

Consequently, there exists $t_1 > t_0$ such that $\|u_{\lambda^*}(t_1)\|_X < \varepsilon_0(\Omega, p)$.

Now assume for contradiction that $T^*(\lambda^*\phi) = \infty$. Then by continuous dependence in X of classical solutions, there exists $\eta > 0$ such that

$$\text{if } |\lambda - \lambda^*| < \eta, \text{ then } T^*(\lambda\phi) > t_1 \text{ and } \|u_\lambda(t_1)\|_X < \varepsilon_0.$$

By the above asymptotic stability property, it follows in particular that $T^*(\lambda\phi) = \infty$ for all $\lambda \in (\lambda^*, \lambda^* + \eta)$. But this contradicts the definition of λ^*. The proof is complete. $\square$

7. Proof of Theorem 3

Assume without loss of generality that $\Omega = (-1, 1)$. Set

$$\beta = (p-2)/(p-1), \qquad c_p = (p-2)^{-1}(p-1)^{\frac{p-2}{p-1}}.$$

For any $w_0 \in X$, denoting by w the corresponding solution of (1.1), we know from [6] and [22, Theorem 40.14] that, if $T^*(w_0) < \infty$, then

$$\lim_{x\to x_0} \frac{w(x, T^*(w_0))}{\delta^\beta(x)} = c_p, \quad \text{for some } x_0 \in \{-1, 1\}. \tag{7.1}$$

Next, for any $t > 0$ and any $x_0 \in \{-1, 1\}$, we claim that

$$w(x_0, t) = 0 \quad \Longrightarrow \quad \limsup_{x\to x_0} \frac{w(x, t)}{\delta^\beta(x)} \le c_p. \tag{7.2}$$

Consider the case $x_0 = -1$ (the other case being similar). For a fixed $t > 0$, we let

$$y(x) = (w_x(x, t) - C_1(x + 1))_+,$$

where $C_1 = C(t)$ is given by (3.6). The function y satisfies

$$y' + y^p = (w_{xx} - C_1)\chi_{\{w_x > C_1(x+1)\}} + (w_x - C_1(x + 1))_+^p, \quad \text{for a.e. } x \in (-1, 0].$$

For a.e. $x \in (-1, 0]$ such that $w_x(x, t) > C_1(x + 1)$, we thus have

$$(y' + y^p)(x) \le (w_{xx} - C_1 + |w_x|^p)(x, t) \le 0$$

by (1.1) and (3.6). Therefore, we have $y' + y^p \le 0$ a.e. on $(-1, 0]$. By integration, it follows that $y(x) \le ((p-1)(x+1))^{-\frac{1}{p-1}}$, hence $w_x(x, t) \le ((p-1)(x+1))^{-\frac{1}{p-1}} + C_1$ on $(-1, 0]$. Assuming $w(-1, t) = 0$, a further integration then yields

$$w(x, t) \le c_p(x + 1)^\beta + C_1(x + 1), \qquad x \in (-1, 0],$$

and claim (7.2) is proved.

Let now $u_0 \in X$ be such that $T^*(u_0) < \infty$ and

$$u = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{7.3}$$

We first prove assertion (i) and consider $v_0 \in X$ such that $v_0 \geq u_0$ and $v_0 \not\equiv u_0$. Pick t_0 such that $0 < t_0 < T^*(v_0) \leq T^*(u_0)$. It follows easily from the Hopf Lemma that $v(\cdot, t_0) \geq \lambda u(\cdot, t_0)$ in Ω for some $\lambda > 1$. By Lemma 3.2, we deduce that $v \geq \lambda u$ in $\overline{\Omega} \times [t_0, \infty)$. Next applying (7.1) with $w = u$, it follows that there exists $x_0 \in \{-1, 1\}$ such that

$$\limsup_{x \to x_0} \frac{v(x, T^*(u_0))}{\delta^\beta(x)} \geq \lambda c_p > c_p.$$

As a consequence of (7.2) applied with $w = v$, we deduce that $v(x_0, T^*(u_0)) > 0$.

To prove assertion (ii), we consider $v_0 \in X$ such that $v_0 \leq u_0$ and $v_0 \not\equiv u_0$. Picking t_0 such that $0 < t_0 < T^*(u_0) \leq T^*(v_0)$ and arguing similarly as before, we deduce that $v \leq \lambda u$ in $\overline{\Omega} \times [t_0, \infty)$ for some $\lambda < 1$. By (7.3) and (7.2) applied with $w = u$, for any $t \geq t_0$ and any $x_0 \in \{-1, 1\}$, it follows that

$$\limsup_{x \to x_0} \frac{v(x, t)}{\delta^\beta(x)} \leq \lambda c_p < c_p.$$

As a consequence of (7.1) applied with $w = v$, we deduce that $T^*(v_0) = \infty$. The result is proved. □

8. Proof of Theorem 5

First, following [15, Lemma 2.3], we fix a smooth function $h \geq 0$ in $\mathbb{R}^n$ such that

$$h(x) = \begin{cases} 1, & x \in \omega \\ 0, & x \in \mathbb{R}^n \setminus \omega_\varepsilon \end{cases} \tag{8.1}$$

and we consider the elliptic problem

$$\begin{cases} -\Delta \psi = 1, & x \in \Omega, \\ \psi = h, & x \in \partial\Omega. \end{cases} \tag{8.2}$$

We have

$$-\Delta(c_1 \psi) = c_1 \geq |\nabla(c_1 \psi)|^p, \quad x \in \Omega, \tag{8.3}$$

with $c_1 := \|\nabla \psi\|_{L^\infty(\Omega)}^{-p/(p-1)} > 0$.

Next, by the continuity of ψ in $\overline{\Omega}$, we may find $\rho \in (0, \varepsilon/3)$ such that

$$\psi \geq 1/2 \quad \text{in } \{x \in \overline{\Omega};\ \mathrm{dist}(x, \omega \cap \partial\Omega) \leq 2\rho\}. \tag{8.4}$$

Taking ρ smaller and owing to the regularity of $\partial\Omega$, we may also assume that for all $x_0 \in \partial\Omega$, there exists a point $x_1 = x_1(x_0)$ such that

$$B_\rho(x_1) \subset \Omega \quad \text{and} \quad \partial\Omega \cap \partial B_\rho(x_1) = \{x_0\}. \tag{8.5}$$

Now let V_0 be as in the proof of Theorem 1. Taking ρ even smaller, we may also assume that

$$\rho^\beta \|V_0\|_\infty < \frac{c_1}{2}.$$

We may thus choose $u_0 \in X$ such that

$$u_0 = 0 \quad \text{in } \{x \in \overline{\Omega};\ \mathrm{dist}(x, \omega \cap \partial\Omega) \geq 2\rho\},$$
$$u_0 \leq \frac{c_1}{2} \quad \text{in } \{x \in \overline{\Omega};\ \mathrm{dist}(x, \omega \cap \partial\Omega) < 2\rho\}$$

and

$$u_0 \geq \rho^\beta \|V_0\|_\infty \quad \text{in } \Omega''_\rho := \{x \in \Omega;\ \delta(x) \geq \rho/2 \text{ and } \mathrm{dist}(x, \omega \cap \partial\Omega) \leq 3\rho/2\}. \tag{8.6}$$

In particular, in view of (8.4), we have

$$u_0 \leq c_1 \psi \quad \text{in } \overline{\Omega}.$$

By (8.3) and the comparison principle, we deduce that

$$u \le c_1\psi \quad \text{in } \overline{\Omega} \times (0,\infty),$$

hence in particular $\mathcal{L}(u_0) \subset \omega_\varepsilon \cap \partial\Omega$, due to (8.1), (8.2).

On the other hand, for each $x_0 \in \omega \cap \partial\Omega$, we can prove the loss of boundary conditions at x_0 by arguing similarly as in the proof of Theorem 1. Namely, recalling (8.5), we set

$$w(x_0; x, t) = \rho^\beta V\big(\rho^{-1}(x - x_1), \rho^{-2}t\big), \qquad (x,t) \in \overline{B}_\rho(x_1) \times [0,\infty),$$

where V is as in the proof of Theorem 1 and $\beta = (p-2)/(p-1)$. The function $w(x_0; \cdot, 0)$ is supported in $B_{\rho/2}(x_1) \subset \Omega''_\rho$, owing to (8.5), hence $u_0 \ge w(x_0; \cdot, 0)$ in $\overline{B}_\rho(x_1)$ by (8.6). By the comparison principle, it follows that $u \ge w(x_0; \cdot, \cdot)$ in $\overline{B}_\rho(x_1) \times [0,\infty)$, hence in particular,

$$u(x_0, \rho^2 t_0) \ge w(x_0; x_0, \rho^2 t_0) = \rho^\beta V\big(\rho^{-1}(x_0 - x_1), t_0\big) = c_0\rho^\beta > 0.$$

Therefore, $\omega \cap \partial\Omega \subset \mathcal{L}(u_0)$ and (2.1) holds. The theorem is proved. □

Conflict of interest statement

No conflict of interest.

Acknowledgements

Most of this work was done during a visit of Ph. Souplet at the Dipartimento di Matematica of Università di Roma Tor Vergata in April 2016. He wishes to thank this institution for the kind hospitality.

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