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Results of Ambrosetti–Prodi type for non-selfadjoint elliptic operators

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Abstract

The well-known Ambrosetti–Prodi theorem considers perturbations of the Dirichlet Laplacian by a nonlinear function whose derivative jumps over the principal eigenvalue of the operator. Various extensions of this landmark result were obtained for self-adjoint operators, in particular by Berger and Podolak, who gave a geometrical description of the solution set. In this text we show that similar theorems are valid for non-self-adjoint operators. In particular, we prove that the semilinear operator is a global fold. As a consequence, we obtain what appears to be the first exact multiplicity result for elliptic equations in non-divergence form. We employ techniques based on the maximum principle.

Keywords: Elliptic equations; Non-divergence form; Ambrosetti–Prodi; Multiplicity results; Global fold

1. Introduction

In this paper we study the solvability of the equation

$$-Lu = f(u) + g(x) \tag{1}$$

with a Dirichlet boundary condition in a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, where L is a uniformly elliptic operator in *non-divergence form* with bounded coefficients, f is a nonlinear function whose behavior at plus or minus infinity is different with respect to the first eigenvalue of L, and g is a given fixed function. Under such conditions the equation (1) is usually named of Ambrosetti–Prodi type, in honor of the celebrated work [3].

The large number of developments on Ambrosetti–Prodi type problems have gone, grosso modo, in two directions (more detailed statements and references will be given below): first, a precise count of solutions, a description of the solution set and of the action of the operator $-L - f(\cdot)$ on a natural function space are available if L is in divergence form, since then variational methods and theory of self-adjoint operators can be used; second, for more

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general operators L only fixed-point methods are available, and they lead to partial existence results in which just a lower bound on the number of solutions is given, as well as an incomplete description of the solution set.

In the present work we bridge this apparent gap, and show that for any operator L in non-divergence form with continuous second-order coefficients and for any nonlinearity f whose derivative has range containing $\lambda_1(L)$ and contained in a determined interval around $\lambda_1(L)$, we can precisely count the solutions and describe the action of $-L - f(\cdot)$ on $W^{2,p}(\Omega)$, for any $p \geq n$. Our approach, which is (necessarily) different from those in the previous works, uses techniques based on the maximum principle as well as elliptic regularity and results on the first eigenvalue of non-divergence form operators obtained by Berestycki, Nirenberg and Varadhan in [5].

To our knowledge, Theorem 1 below is the first result on exact multiplicity of solutions (i.e. exact number of solutions different from 0 or 1) for equations driven by an operator in non-divergence form.

Let us now give the detailed statement of our main result. We set

$$F(u) = -Lu - f(u), \qquad Lu := a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu = \mathrm{tr}(AD^2u) + b.\nabla u + cu,$$

where the coefficients $A(x), b(x), c(x)$ satisfy the following assumptions: for some constants $\Lambda \geq \lambda > 0$,

$$A \in C(\overline{\Omega}), \qquad \mathrm{spec}(A) \in [\lambda, \Lambda], \qquad |b|, |c| \leq \Lambda.$$

We denote the principal eigenvalue of $-L$ by $\lambda_1 = \lambda_1(L, \Omega) \in \mathbb{R}$ (necessarily simple, isolated) and a positive associated eigenfunction by ϕ_1 (see Section 2.1).

We consider Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the Ambrosetti–Prodi type hypothesis:

(AP)$_\mathbf{b}$ for some constants $a, b \in \mathbb{R}$, $a < \lambda_1 < b$, $a \leq \frac{f(x)-f(y)}{x-y} \leq b$ for $x \neq y$, and for some $M \geq 0$, we have $f(s) \geq \max\{bs - M, as - M\}$ for all $s \in \mathbb{R}$.

Throughout the paper we will assume $a = 0$ – without loss, since the problem does not change if we replace L by $L - a$, f by $f - a$, and b by $b - a$.

We also assume some convexity of f.

(C) The function f is convex on $\mathbb{R}$. Also, f is not in the form $f(s) = \lambda_1 s + \beta$, $\beta \in \mathbb{R}$, in a left or a right neighborhood of $s = 0$.

We set $X = \{u \in W^{2,p}(\Omega),\ u = 0 \text{ on } \partial\Omega\}$, $Y = L^p(\Omega)$ and consider the maps $L, F : X \to Y$. From now on, if $p > n$ when we say a constant depends on L we will mean it depends on n, p, λ, Λ, and a modulus of continuity of the coefficient matrix A. When $p = n$ we have less control on the constants, and they may depend on L in a more complicated way.

Theorem 1. *There exists $B = B(L, \Omega) > \lambda_1$ such that if f satisfies* (**C**) *and* (**AP**)$_\mathbf{b}$ *with $b < B$, then the operator $F(u) = -Lu - f(u)$ is a global topological fold from X to Y. More specifically, there exist (bi-Lipschitz) homeomorphisms $\Phi_1 : X \to X$, $\Phi_2 : Y \to Y$ and hyperplanes $W \subset X$, $Z \subset Y$, such that $X = W \oplus \mathbb{R}\phi_1$, $Y = Z \oplus \mathbb{R}\phi_1$, for which the restriction $(\Phi_2 \circ F \circ \Phi_1)|_W$ is a homeomorphism, and*

$$(\Phi_2 \circ F \circ \Phi_1)(w + t\phi_1) = -Lw - |t|\phi_1, \tag{2}$$

for any $t \in \mathbb{R}$, $w \in W$. For each $w_0 \in W$, $z_0 \in Z$, the map Φ_2 keeps the line $\{z_0 + t\phi_1, t \in \mathbb{R}\}$ invariant, while the map Φ_1 transforms $\{w_0 + t\phi_1, t \in \mathbb{R}\}$ into a curve which is asymptotically parallel to ϕ_1 for large values of $|t|$ (in the sense of (27), below).

In particular, the equation $F(u) = z_0 + t\phi_1$ has exactly 2, 1 or 0 solutions in X, according to whether t is respectively smaller than, equal to, or larger than a real number $\bar{t}(z_0)$.

The hypotheses on f are essentially optimal for this type of multiplicity, even in the simplest case $L = \Delta$, $f \in C^2(\mathbb{R})$. Indeed, it is well-known that if $\overline{\mathrm{Im}(f')}$ does not meet the spectrum of L then F is a homeomorphism, whereas when $\mathrm{Im}(f')$ contains more than one eigenvalue of L then there may be more than two solutions for some right-hand sides (see for instance [1], [26]). Furthermore, in the recent work [12] it is shown that, under $(\mathbf{AP})_\mathbf{b}$, even if f'' is negative just at one point then there are right-hand sides $z_0 + t\phi_1$ admitting at least four solutions.

We now discuss the main results on Ambrosetti–Prodi type problems obtained prior to Theorem 1. The original theorem assumes that f is a strictly convex C^2 function such that $f'(\mathbb{R}) = (a, b)$ contains the first but not the second eigenvalue of the Laplacian, see [3] and [23]. They prove that the critical set $\mathcal{C}$ of F with $L = \Delta$, defined from $\bar{X} = C^{2,\alpha}(\Omega) \cap C_0(\overline{\Omega})$ into $\bar{Y} = C^\alpha(\Omega)$, is a hypersurface homeomorphic to a hyperplane, which splits $\bar{X}$ into two disjoint components A, B, i.e. $\bar{X} = A \cup \mathcal{C} \cup B$. Ambrosetti and Prodi show that F is injective on $\mathcal{C}$ and $F(\mathcal{C})$ also generates a split of $\bar{Y}$, $\bar{Y} = S_0 \cup F(\mathcal{C}) \cup S_2$, into three connected components, in such a way that A and B are taken by F homeomorphically to S_2. Later Dancer [13], Berestycki [4], de Figueiredo and Solimini [14], [16], obtained extensions of that result for self-adjoint second order operators in divergence form, giving characterizations of the sets A and B in terms of the Morse index of their elements as critical points of the energy functional, or of the coercivity of the associated linearized operator.

In those works the focus was the decomposition of domain and counterdomain of F in components on which the restriction of F acts injectively. On the other hand, with a view on the solvability of the equation for a given right-hand side, Berger and Podolak [7], and Berger and Church [6] used a global Lyapunov–Schmidt decomposition to give a geometric description of the map F: it is a topological fold from $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^2(\Omega)$. We note that the notion of a fold we use is a (global) Banach space version of the original concept introduced by Whitney [28], [22], in his study of generic maps from the plane to the plane.

In a nutshell, the works [7], [6] rely on the fundamental fact that the "vertical" lines $\{z + t\phi_1, t \in \mathbb{R}\} \subset Y$, $z \perp \phi_1$, when inverted by F, give rise to very special curves in the domain of F, the fibers. This follows from the global Lyapunov–Schmidt decomposition for F. Extensions of this approach (in [27], [11] and the references therein), still in the self-adjoint case, allow for less differentiability on the nonlinearity f, together with a larger choice of operators L. Thus, for example, folds are obtained for Schrödinger operators in bounded and unbounded domains, including the hydrogen atom and the quantum harmonic oscillator, the spectral (self-adjoint) fractional Laplacian.

When L is not self-adjoint, the convenient spectral estimates and integral representation of the equation are not available for constructing the Lyapunov–Schmidt decomposition. Prior to this work only topological methods, more precisely fixed-point theorems for Banach spaces, have been applied to nondivergence form equations (see [2], [15], [17], [18], [24], and the references in these papers; for a different approach to ODEs, see [22]). Topological methods cover a very large scope of problems, such as fully nonlinear equations or systems of equations, but have the important drawback that no exact count of solutions can be obtained and we are left with a rather poor description of the solvability of $F(u) = g$ for different right-hand sides. Specifically, these results always state that for every given right-hand side $z_0 + t\phi_1$ the problem has at least 2, at least 1 or 0 solutions according to whether t is respectively less, equal or larger than a real number $\bar{t}(z_0)$.

In Section 2 we will construct a global Lyapunov–Schmidt decomposition for F, for the first time in a non-divergence setting. The core of the construction is an elliptic estimate which can be interpreted as a bi-Lipschitz bound of F on "horizontal" subspaces of X, which is uniform in the "heights" of these subspaces (Proposition 4 and its consequence Theorem 6, below). This estimate allows us to construct fibers and to prove basic properties about their geometry and asymptotic behavior at infinity, implying also the properness of F.

The Lyapunov–Schmidt decomposition may be taken as a robust starting point for numerics, following ideas developed for the self-adjoint case in [10], [20], but we do not handle the issue in this paper. We could also allow less regular domains, for instance domains Ω that satisfy an exterior cone condition, but will also not consider these technicalities here.

The last Section 3 is dedicated to the proof of Theorem 1. We deviate from and simplify the earlier approaches, which identify critical points of F by computing derivatives of the so-called height function along fibers. Here we observe that our assumptions and properties of the principal eigenvalue of L and its positive eigenfunction suffice to prove that no point in the image of F has three preimages. From the existence of fibers and the properness of F, the fold structure is then deduced from the more general Proposition 10.

Finally, we remark that when L is self-adjoint, the optimal value for B in Theorem 1 is the second eigenvalue of L. On the other hand, a more general L might not even have a second real eigenvalue, and such a simple explicit lower bound for B is not available. However, if $p > n$ our proof does imply an explicit lower bound on B, depending on L and Ω, in terms of the constants which appear in the basic estimates of the elliptic theory.

2. The global Lyapunov–Schmidt decomposition

In this text, the letters C and c, indexed or not, denote positive constants which depend on the appropriate quantities and may change from line to line.

The convexity of the nonlinearity f plays no role in most of this section.

2.1. Preliminaries, basic results on principal eigenvalues

Let L be as in the introduction, and Ω be an arbitrary domain. We recall some basic facts about L from [5], related to maximum principles. The *principal eigenvalue* $\lambda_1(L,\Omega)$ is defined by

$$\lambda_1 = \lambda_1(L,\Omega) = \sup\left\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,n}_{\text{loc}}(\Omega) \text{ such that } \begin{array}{rl} (L+\lambda)\phi \le 0 & \text{in } \Omega \\ \phi > 0 & \text{in } \Omega \end{array}\right\}.$$

The associated eigenspace is spanned by the eigenvector $\phi_1 = \phi_1(L,\Omega) > 0$. It is also known that for the dual operator $L^* : Y^* = L^{\frac{n}{n-1}}(\Omega) \to X^*$, we have $\lambda_1(L^*,\Omega) = \lambda_1(L,\Omega)$ and $\phi_1^* = \phi_1(L^*,\Omega) > 0$ (see [8]).

Theorem 2 *(Theorem 2.3, [5]). Let ϕ be an eigenfunction of $-L$ with eigenvalue $\lambda \neq \lambda_1$. Then (i) $Re(\lambda) > \lambda_1$, (ii) if ϕ is real, then it changes sign in Ω.*

The following existence and uniqueness result holds.

Theorem 3 *(Theorem 1.2, [5], and Theorem 9.13 in [19]). If $\lambda_1 > 0$ then the map $L : X \to Y$ is a homeomorphism, and if $Lu = h$ then*

$$\|u\|_X \le \frac{C_{ABP}}{\lambda_1}\|h\|_Y$$

where C_{ABP} depends on n, λ, Λ, and Ω. If $h \le 0$, $h \not\equiv 0$ in Ω then $u > 0$ in Ω.

We will use the following characterization of λ_1.

Proposition 1 *(Corollary 1.1, [5]). If for some $A \in \mathbb{R}$ there exists a bounded $\phi \in W^{2,n}_{\text{loc}}(\Omega)$ such that $(L + A)\phi \ge 0$ in Ω, $\limsup_{x\to\partial\Omega} \phi(x) \le 0$, and ϕ is positive somewhere in Ω, then*

$$\lambda_1(L,\Omega) \le A.$$

The principal eigenvalue increases together with the zero-order coefficient of the operator, and decreases when the domain enlarges.

Proposition 2 *(Proposition 2.1, [5]). If $V(x) \ge 0$, $V \not\equiv 0$ is a bounded function in Ω, then $\lambda_1(L+V,\Omega) > \lambda_1(L,\Omega)$.*

Theorem 4 *(Theorem 2.4, [5]). Let $\Omega' \subset \Omega$ be an open subset and $\delta > 0$ be such that $|\Omega'| \le |\Omega| - \delta$. Then there exists $\eta = \eta(L,\Omega,\delta) > 0$ such that*

$$\lambda_1(L,\Omega') - \lambda_1(L,\Omega) \ge \eta.$$

In the sequel we will need the following fact.

Proposition 3. *There exists a constant $\widetilde{B} = \widetilde{B}(L,\Omega) > \lambda_1(L,\Omega)$ such that for any bounded function $V(x)$ with $V(x) \le \widetilde{B}$, the operator $\tilde{L}u = Lu + Vu$ has a nontrivial kernel if and only if 0 is the principal eigenvalue of $\tilde{L}$.*

Proof. We take $\widetilde{B} = \lambda_1(L,\Omega) + \eta$, where η is obtained from Theorem 4 with $\delta = |\Omega|/2$. Then for every open set $\Omega' \subset \Omega$ satisfying $|\Omega'| \le |\Omega|/2$, $\lambda_1(L,\Omega') > \widetilde{B}$.

Suppose $V(x) \le \widetilde{B}$ and $u \not\equiv 0$ with $Lu + V(x)u = 0$ in Ω, and $u = 0$ on $\partial\Omega$. We show that u does not change sign in Ω (and so by Theorem 2 it is a principal eigenfunction). Define

$$\Omega_u^+ := \{x \in \Omega : u(x) > 0\}, \quad \Omega_u^- := \{x \in \Omega : u(x) < 0\},$$

and assume by contradiction that $\Omega_u^+, \Omega_u^- \neq \Omega$. At least one of the sets Ω_u^+, Ω_u^- (say Ω_u^+) has measure smaller than or equal to $|\Omega|/2$. Use Proposition 1 with Ω replaced by Ω_u^+, and $\phi = u$ to obtain

$$Lu + \widetilde{B}u \ge Lu + V(x)u = 0, \qquad u > 0 \quad \text{in } \Omega_u^+, \qquad \text{and} \qquad \limsup_{x \to \partial\Omega_u^+} u(x) = 0$$

and hence $\lambda_1(L, \Omega_u^+) \le \widetilde{B}$, a contradiction. □

We quote a quantitative Hopf lemma [25], which extends results by Brezis–Cabre [9] for $L = \Delta$, and by Krylov [21], who obtained the interior estimate.

Theorem 5 *(Theorem 3.1, [25]). There exist $\varepsilon, c > 0$ depending on n, λ, Λ, p, and Ω such that, for each solution $u \in W^{2,p}(\Omega)$, $p > n$, of $Lu \le 0$, $u \ge 0$ in Ω,*

$$\inf_\Omega \frac{u}{d} \ge c \left(\int_\Omega (-Lu)^\varepsilon \right)^{1/\varepsilon},$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$.

2.2. The decomposition

We decompose X and Y in direct sums of *horizontal* and *slanted* subspaces,

$$X = W \oplus V, \quad Y = Z \oplus V,$$

where $Z = \mathrm{vect}(\phi_1^*)^\perp = (\mathbb{R}\phi_1^*)^\perp$, $W = Z \cap X$, $V = \mathrm{vect}(\phi_1) \subset X, Y$. Clearly $Z \cap V = \{0\}$, as $\phi_1, \phi_1^* > 0$. Contrary to the case when L is a self-adjoint operator, this decomposition is not necessarily orthogonal with respect to the inner product in $L^2(\Omega)$. For each $g \in Y$ we split

$$g = Pg + (I - P)g = z_g + h_g\phi_1 \in Z \oplus V$$

where P is the projection $P : Z \oplus V = Y \to Z \oplus V$, $z + v \mapsto z$. Throughout the text, the letters w and z will be reserved, respectively, for elements of the horizontal spaces W, Z. From the closed graph theorem, the norms on X and Y are equivalent to the direct sum norms, $\|g\|_Y \cong \|Pg\|_Z + \|(I - P)g\|_V$, and we change from one norm to the other without warning.

For each $g \in Y$, $g = z_g + h_g\phi_1$, we write the equation $F(u) = g$ as

$$F(w + t\phi_1) = z_g + h_g\phi_1 \qquad \text{for the unknowns} \quad w \in W, \ t \in \mathbb{R}.$$

For each fixed $t \in \mathbb{R}$ we set $F_t(w) = F(w + t\phi_1)$ and decompose the equation we want to solve as follows,

$$\begin{cases} PF_t(w) = z_g \\ (I - P)F_t(w) = h_g\phi_1 \end{cases} \quad \text{for the unknowns} \quad w \in W, \ t \in \mathbb{R}. \tag{3}$$

We will show in Proposition 5 below that the maps $PF_t : W \to Z$ are bi-Lipschitz homeomorphisms, uniformly in $t \in \mathbb{R}$ (bi-Lipschitz means the inverse is also Lipschitz). We may thus solve the first equation in (3); then from the second equation in (3) we can write h_g in terms of z_g and t.

Theorem 6. *There exists $B = B(L, \Omega) > \lambda_1$ such that if f satisfies* $(\mathbf{AP})_\mathbf{b}$ *with $b < B$, then the operator $F = -L - f(\cdot)$ admits a global Lyapunov–Schmidt decomposition: the map $\Psi : X \to Y$ defined by*

$$\Psi(w + t\phi_1) = PF_t(w) + t\phi_1,$$

is a bi-Lipschitz homeomorphism and, identifying $Z \oplus V$ and $Z \times \mathbb{R}$,

$$\tilde{F} = F \circ \Psi^{-1} : Z \times \mathbb{R} \to Z \times \mathbb{R} \qquad \text{is} \qquad \tilde{F}(z,t) = (z, \tilde{h}(z,t)),$$

where $\tilde{h}$ is the Lipschitz function given by

$$\tilde{h}(z,t) := \frac{\langle F(\Psi^{-1}(z + t\phi_1)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle}.$$

The following crucial coercivity bound for Ψ is proved in Section 2.4 below.

Proposition 4. *Let Ψ be as in the previous theorem. There exists a constant $B = B(L, \Omega) > \lambda_1$ such that if f satisfies* $(\mathbf{AP})_\mathbf{b}$ *with $b < B$, then for some $c = c(L, \Omega) > 0$ we have*

$$\|\Psi(u) - \Psi(\tilde{u})\|_Y \geq c\|u - \tilde{u}\|_X \quad \text{for all } u, \tilde{u} \in X.$$

We now turn to the proof of Theorem 6. The following proposition is the main step in this proof.

Proposition 5. *There exists $B = B(L, \Omega) > \lambda_1$ such that if f satisfies* $(\mathbf{AP})_\mathbf{b}$ *with $b < B$, then the maps $PF_t : W \to Z$ are bi-Lipschitz homeomorphisms, uniformly in t.*

Proof. For $w, \tilde{w} \in W$, $t \in \mathbb{R}$, Proposition 4 with $u = w + t\phi_1$, $\tilde{u} = \tilde{w} + t\phi_1$ gives

$$\|PF_t(w) - PF_t(\tilde{w})\|_Y \geq c\|w - \tilde{w}\|_X. \tag{4}$$

Hence PF_t is injective and its image is closed.

We will first prove Proposition 5 under the additional hypothesis that $f \in C^1$, which will let us use the implicit function theorem for F.

Since the functions in X are continuous in $\overline{\Omega}$, it is easy to see that $F : X \to Y$ is a C^1 function, with derivative at $u \in X$ given by

$$DF(u) : X \to Y \qquad \langle DF(u), v \rangle = -Lv - f'(u)v, \quad v \in X.$$

Similarly, $PF_t : W \to Z$ is C^1 and, for every $w \in W$,

$$D(PF_t)(w) : W \to Z, \quad \langle D(PF_t)(w), v \rangle = -Lv - P\big(f'(w + t\phi_1)v\big), \quad v \in W.$$

We now show that $D(PF_t)(w) = -L - Pf'(w + t\phi_1) : W \to Z$ is an isomorphism for every $w \in W$. From (4), it is injective. To prove surjectivity, it suffices to show that it is a Fredholm operator of index 0. Recall that we assume $a = 0$, by $(\mathbf{AP})_\mathbf{b}$ and the remark following it, and thus $\lambda_1 = \lambda_1(L, \Omega) > 0$, $L : X \to Y$ is an isomorphism (Theorem 3) and hence the restriction $L|_W : W \to Z$ is an isomorphism too. Also, the operator $v \in W \mapsto P\big(f'(w + t\phi_1)v\big) \in Z$ is compact, since $P : Y \to Z$ is continuous, f' is bounded, and W is compactly embedded in Z.

Thus $PF_t : W \to Z$ is a local diffeomorphism so, from the inverse function theorem, its image is open. Since the image is also closed, PF_t is surjective, hence, bijective. By (4) the inverse of PF_t is Lipschitz, uniformly in $t \in \mathbb{R}$. Finally, by $(\mathbf{AP})_\mathbf{b}$ and the definition of the projection P it is trivial to check that

$$\|PF_t(w) - PF_t(\tilde{w})\| \leq C\|w - \tilde{w}\|,$$

for some constant C which does not depend on t.

For the general case of a Lipschitz function f satisfying $(\mathbf{AP})_\mathbf{b}$, we approximate f by smooth functions $f_k : \mathbb{R} \to \mathbb{R}$ which also satisfy $(\mathbf{AP})_\mathbf{b}$ and converge uniformly to f as $k \to \infty$. For instance, the bump function $\psi_\delta : \mathbb{R} \to \mathbb{R}$,

$$\psi_\delta(x) = \frac{1}{\delta}\psi(x/\delta),$$

with $\psi(x) = \chi_{[-1,1]}(x) \exp\big((|x|^2 - 1)^{-1}\big)$ yields the smooth functions

$$f_\delta(x) := \int_{\mathbb{R}} f(s)\psi_\delta(x-s)ds = \int_{\mathbb{R}} f(x-s)\psi_\delta(s)ds. \tag{5}$$

Since f is uniformly continuous, $f_\delta \to f$ uniformly as $\delta = 1/k \to 0$.

The associated maps $F_k : X \to Y$, $u \mapsto -Lu - f_k(u)$ are smooth and converge uniformly to F. Indeed, if $\epsilon > 0$ and N is such that for all $k \geq N$ and $s \in \mathbb{R}$ we have $|f_k(s) - f(s)| < \epsilon/|\Omega|^{\frac{1}{n}}$, then

$$\|F_k(u) - F(u)\|_Y = \|f_k(u) - f(u)\|_Y < (\epsilon/|\Omega|^{\frac{1}{n}})|\Omega|^{\frac{1}{n}} = \epsilon\ .$$

Thus the maps $PF_{k,t}(w) = PF_k(w + t\phi_1)$ are smooth diffeomorphisms which converge uniformly to an injective map PF_t with a closed image. Take $z \in Y$ and $w_k \in W$ for which $PF_{k,t}(w_k) = z$. Then we have

$$\|z - PF_t(w_k)\|_Y = \|PF_{k,t}(w_k) - PF_t(w_k)\|_Y \to 0 \quad \text{as } k \to \infty.$$

As the image of PF_t is closed, z is in the image of PF_t, i.e. PF_t is surjective. Uniform Lipschitz continuity for the inverses $(PF_t)^{-1} : Z \to W$ again follows from (4). □

We are ready to prove Theorem 6.

Proof. From the previous proposition, the maps Ψ and $\Phi = \Psi^{-1}$ are well defined. To see that Ψ is Lipschitz, take $u = w + t\phi_1, \tilde{u} = \tilde{w} + \tilde{t}\phi_1 \in W \oplus V$ and compute:

$$\begin{aligned}\|\Psi(u) - \Psi(\tilde{u})\|_Y &\leq C\big(\|PF_t(w) - PF_{\tilde{t}}(\tilde{w})\|_Y + |t-\tilde{t}|\big)\\ &\leq C\big(\|F(u) - F(\tilde{u})\|_Y + |t-\tilde{t}|\big) \leq C\big(\|u-\tilde{u}\|_Y + |t-\tilde{t}|\big)\ .\end{aligned}$$

To show that Φ is Lipschitz, for $t, \tilde{t} \in \mathbb{R}$ and $z, \tilde{z} \in Z$,

$$\begin{aligned}\|\Phi(z + t\phi_1) - \Phi(\tilde{z} + \tilde{t}\phi_1)\|_X &\leq \|(PF_t)^{-1}(z) - (PF_{\tilde{t}})^{-1}(\tilde{z}) + (t-\tilde{t})\phi_1\|_X\\ &\leq C\|z - \tilde{z} + (t-\tilde{t})\phi_1\|_Y \leq C(\|z-\tilde{z} + |t-\tilde{t}|)\ ,\end{aligned}$$

where for the second inequality we use Proposition 4 with

$$u = (PF_t)^{-1}(z) + t\phi_1, \quad \tilde{u} = (PF_{\tilde{t}})^{-1}(\tilde{z}) + \tilde{t}\phi_1\ .$$

From the definitions of F and Φ, $(F \circ \Phi)(z + t\phi_1) = z + \tilde{h}(z,t)\phi_1$, for some real number $\tilde{h}(z,t)$. Recall that $z \in Z$, so that z is orthogonal to ϕ_1^*. We must then have

$$\tilde{h}(z,t) = \frac{\langle (F \circ \Phi)(z + t\phi_1), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle}. \qquad \square$$

2.3. Fibers and heights, properness of F

In this section we assume f satisfies $\mathbf{(AP)_b}$ with $b < B$, where $B > \lambda_1$ is defined by Proposition 4.

Fix $z \in Z$. From the definition of Ψ and the results from the previous section, every horizontal affine subspace $W + t\phi_1$ is taken by F to a surface $F(W + t\phi_1)$ which projects homeomorphically onto Z. In particular, this surface meets each line $\{z + h\phi_1, h \in \mathbb{R}\} \subset Y$ at a single point $z + \tilde{h}(z,t)\phi_1$, which is the image of a point $w(z,t) + t\phi_1 \in W \oplus V$. Thus for each $z \in Z$ we can define the *fiber*

$$u_z(t) = u(z,t) := w(z,t) + t\phi_1 = \Phi(z + t\phi_1) = \Psi^{-1}(z + t\phi_1)$$

as the inverse of the slanted line $\{z + t\phi_1, t \in \mathbb{R}\} \subset Y$. In this way we also define the *height function* $\tilde{h} = \tilde{h}(z,t)$, by

$$F(u(z,t)) = -Lu(z,t) - f(u(z,t)) = z + \tilde{h}(z,t)\phi_1.$$

We rephrase some Lipschitz properties of F and Φ from the previous section.

Proposition 6. *For every $z \in Z$, the map $t \mapsto u(z,t) = \Phi(z + t\phi_1)$ is Lipschitz uniformly in z. The height $\tilde{h}(z,t)$ is Lipschitz in both z and t. The equation $F(u) = g = z_g + t_g\phi_1 \in Z \oplus V$ has as many solutions as the equation $\tilde{h}(z_g, t) = t_g$, for the unknown $t \in \mathbb{R}$.*

Proposition 7. *As $|t| \to \infty$, $\tilde{h}(z,t) \to -\infty$ uniformly in $z \in Z$.*

Proof. We expand the expression for $\tilde{h}(z,t)$ in Theorem 6, using $u(z,t) = w(z,t) + t\phi_1$, $w \in W$, $Lw \in Z$:

$$\begin{aligned}\tilde{h}(z,t) &= \frac{\langle F(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} \\ &= \frac{\langle -Lw(z,t) - tL\phi_1, \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} - \frac{\langle f(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} = \lambda_1 t - \frac{\langle f(u(z,t)), \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} \\ &\le \lambda_1 t + M \frac{\langle 1, \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} - b \frac{\langle w(z,t), \phi_1^* \rangle + t \langle \phi_1, \phi_1^* \rangle}{\langle \phi_1, \phi_1^* \rangle} \le (\lambda_1 - b)t + C, \end{aligned} \tag{6}$$

where we used $(\mathbf{AP})_\mathbf{b}$. Since $\lambda_1 < b$, for $t \to +\infty$ we have $\tilde{h}(z,t) \to -\infty$. The bound does not depend on $z \in Z$, implying uniform convergence. The case $t \to -\infty$ is similar: replace b by $a = 0$ in (6), again by $(\mathbf{AP})_\mathbf{b}$. □

Proposition 8. *The map $F : X \to Y$ is proper.*

Proof. From Theorem 6, it suffices to establish the properness of

$$\tilde{F} : Z \oplus V \to Z \oplus V \ , \quad (z,t) \mapsto (z, \tilde{h}(z,t)) \ .$$

Now, if $(z_k, \tilde{h}(z_k, t_k))$ is a convergent sequence then (z_k, t_k) is precompact, since by Proposition 7 the sequence $\{t_k\}$ is bounded. □

Next, we show that fibers at infinity are essentially parallel to ϕ_1, that is, $w(z,t)$ is $o(t)$ as $|t| \to \infty$. Here we use the convexity of f.

Lemma 1. *For every $z \in Z$,*

$$\lim_{|t|\to\infty} \| \frac{w(z,t)}{t} \|_X = \lim_{|t|\to\infty} \left\| \frac{u(z,t)}{t} - \phi_1 \right\|_X = 0.$$

Proof. Fix $z \in Z$. By Proposition 4, for some $C > 0$,

$$\begin{aligned} o(t) + \left\| \frac{PF_t(0)}{t} \right\|_Y &\ge \left\| \frac{z}{t} - \frac{PF_t(0)}{t} \right\|_Y \\ &= \frac{1}{|t|} \| PF_t(w(z,t)) - PF_t(0) \|_Y \ge C \left\| \frac{w(z,t)}{t} \right\|_X, \end{aligned}$$

so it suffices to prove that

$$\frac{1}{|t|} \| PF_t(0) \|_Y = \frac{1}{|t|} \| Pf(t\phi_1(x)) \|_Y \to 0 \qquad \text{as } t \to \pm\infty.$$

Say $t \to +\infty$. Since f is convex, $(f(t) - f(0))/t$ is nondecreasing and bounded (by $(\mathbf{AP})_\mathbf{b}$), hence convergent to some number $\tilde{b} \le b$. In the limit, the expression

$$\frac{f(t\phi_1(x))}{t} = \frac{f(t\phi_1(x))}{t\phi_1(x)} \phi_1(x)$$

converges pointwise to $\tilde{b}\phi_1(x)$, whose projection is the origin. The result follows by dominated convergence. □

2.4. Proof of Proposition 4

The proposition is proved if we find numbers $\rho, c_0 \in (0,1]$, depending only on L and Ω, such that if f satisfies $(\mathbf{AP})_\mathbf{b}$ with $b = \lambda_1 + \rho$, then for every $u, \tilde{u} \in X$,

$$\| \Psi(u) - \Psi(\tilde{u}) \|_Y \ge c_0 \, \| u - \tilde{u} \|_X . \tag{7}$$

We use the product norms $\|u\| = \|w\| + |t| \|\phi_1\|$, if $u = w + t\phi_1$, $w \in \text{vect}(\phi_1^*)^\perp$ and normalize ϕ_1 so that $\|\phi_1\|_X = 1$.

Fix $u = w + t\phi_1, \tilde{u} = \tilde{w} + \tilde{t}\phi_1 \in X, u \neq \tilde{u}$. By the definition of Ψ,

$$\Psi(u) - \Psi(\tilde{u}) = L(w - \tilde{w}) + P(f(u) - f(\tilde{u})) + (t - \tilde{t})\phi_1.$$

Set

$$v := \frac{w - \tilde{w}}{\|w - \tilde{w}\|_X} \in W, \qquad \tau := \frac{|t - \tilde{t}|}{\|u - \tilde{u}\|_X} \in [0, 1],$$

and

$$\begin{aligned}\psi &:= (1-\tau)Lv + \frac{P(f(u) - f(\tilde{u}))}{\|u - \tilde{u}\|_X} \\ &= \frac{L(w - \tilde{w})}{\|u - \tilde{u}\|_X} + \frac{P(f(u) - f(\tilde{u}))}{\|u - \tilde{u}\|_X}\end{aligned} \tag{8}$$

With this notation, statement (7), equivalent to Proposition 4, becomes

$$\|\psi\|_{L^p(\Omega)} + \tau \geq c_0(L, \Omega). \tag{9}$$

From now on we assume that $\tau \leq 1/2$ (else (9) holds with $c_0 = 1/2$). Set

$$q(x) := \begin{cases} \dfrac{f(u(x)) - f(\tilde{u}(x))}{u(x) - \tilde{u}(x)} & \text{if} \quad u(x) \neq \tilde{u}(x) \\ 0 & \text{if} \quad u(x) = \tilde{u}(x). \end{cases}$$

From Lemma 1.1 in [5], $\lambda_1 \leq C(L, \Omega)$. Thus $(\mathbf{AP})_\mathbf{b}$ implies $q \in L^\infty(\Omega)$, with

$$0 \leq q \leq \lambda_1 + \rho \leq C_1(L, \Omega).$$

Observe that

$$\frac{f(u) - f(\tilde{u})}{\|u - \tilde{u}\|_X} = q(x)\left((1-\tau)v + \tau\phi_1\right), \tag{10}$$

and hence, for $C_2 = C_2(L, \Omega) = \|P\|\, C_1$,

$$\left\| \frac{P(f(u) - f(\tilde{u}))}{\|u - \tilde{u}\|_X} \right\|_Y \leq C_2\left(\|v\|_Y + \tau\right). \tag{11}$$

We now apply the classical $W^{2,p}$-estimate (see for instance Theorem 9.13 in [19]) to (8), seen as an elliptic equation satisfied by v. Thus, if $C_3 = C_3(L, \Omega)$ is the constant from that estimate, by using (11) we get

$$\begin{aligned}1 = \|v\|_X &\leq \frac{C_3}{1-\tau}\left(\|\psi\|_Y + C_2(\|v\|_Y + \tau) + \|v\|_{L^\infty(\Omega)}\right) \\ &\leq C_4(\|\psi\|_Y + \tau + \|v\|_{L^\infty(\Omega)})\end{aligned} \tag{12}$$

for some $C_4 = C_4(L, \Omega)$, where we also used $\|v\|_Y = \|v\|_{L^p(\Omega)} \leq |\Omega|^{1/p}\|v\|_{L^\infty(\Omega)}$.

From now on we suppose $\|\psi\|_Y + \tau \leq 1/(2C_4)$ (else (9) holds, by setting $c_0 = 1/(2C_4)$). Then by (12)

$$\|v\|_{L^\infty(\Omega)} \geq \frac{1}{2C_4} =: c_1. \tag{13}$$

On the other hand we also have, by the embedding $X \hookrightarrow C^{0,\alpha}(\Omega)$ for some fixed $\alpha < 1$, that, for some $C_5 = C_5(\Omega)$,

$$\|v\|_{C^\alpha(\Omega)} \leq C_5\|v\|_X = C_5. \tag{14}$$

If $p > n$ we have more, since then $X \hookrightarrow C^{1,\alpha}(\Omega)$ for $\alpha \in (0, 1 - n/p)$, and

$$\|v\|_{C^{1,\alpha}(\Omega)} \leq C_5'\|v\|_X = C_5'. \tag{15}$$

This estimate and $v = 0$ on $\partial\Omega$ imply that v/d is Hölder continuous:

$$\left\|\frac{v}{d}\right\|_{C^\alpha(\Omega)} \leq C_5'', \quad \text{for} \quad d(x) = \text{dist}(x, \partial\Omega)\,. \tag{16}$$

Alternatively, the last estimate can be deduced in a standard fashion from the general Harnack inequality for v/d, proved in [25]. Clearly, (16) holds if v is replaced by its positive or negative parts v^+ and v^-, which are compositions of v and a Lipschitz function with Lipschitz constant equal to one.

We now establish a useful property of v.

Lemma 2. *There exist constants $\epsilon, \nu > 0$ depending only on L and Ω, and subdomains $\omega_1, \omega_2 \subset \Omega$ with measures $|\omega_1|, |\omega_2| \geq \nu$, such that*

$$v \geq \epsilon \quad \text{in } \omega_1, \qquad \text{and} \qquad v \leq -\epsilon \quad \text{in } \omega_2.$$

Proof. We first record the following

Fact. If $g \in C^\alpha(\Omega)$ is such that $\|g\|_{C^\alpha(\Omega)} \leq A$ and $g(x_0) \geq a > 0$ (resp. $g(x_0) \leq -a < 0$) for some $x_0 \in \overline{\Omega}$, then

$$g \geq \frac{a}{2} \left(\text{resp. } g \leq -\frac{a}{2}\right) \quad \text{in } B_\nu(x_0) \cap \Omega, \qquad \text{where } \nu = \left(\frac{a}{2A}\right)^{1/\alpha}.$$

This is immediate from the definition of the Hölder seminorm

$$g(x_0) - g(x) \leq \|v\|_{C^\alpha} |x - x_0|^\alpha, \qquad \text{i.e.} \qquad g(x) \geq g(x_0) - \|g\|_{C^\alpha} |x - x_0|^\alpha.$$

We prove Lemma 2. By (13), there exists $x_1 \in \Omega$ such that either $v(x_1) \geq c_1$ or $v(x_2) \leq -c_1$. Say the first happens. Then the fact above and (14) imply $v \geq \epsilon_1 = c_1/2$ in $\omega_1 = B_{\bar{\nu}_1}(x_1) \cap \Omega = B_{\bar{\nu}_1}(x_1)$, where $\bar{\nu}_1 = (c_1/(2C_5))^{1/\alpha}$. It is clear that

$$|\omega_1| = |B_{\bar{\nu}_1}(x_1)| \geq \nu_1 > 0,$$

for some ν_1 which depends only on $\bar{\nu}_1$ and n, i.e. on L and Ω.

Recall that $v \in Z$, which means that $\langle v, \phi_1^* \rangle = 0$ where $\phi_1^* > 0$ is the principal eigenfunction of the dual operator. In other words, $\int_\Omega v^+ \phi_1^* = \int_\Omega v^- \phi_1^*$. We assume ϕ_1^* is normalized so that $\int_\Omega \phi_1^* = 1$, and estimate

$$\begin{aligned}\sup_\Omega v^- \geq \int_\Omega v^- \phi_1^* &= \int_\Omega v^+ \phi_1^* \\ &\geq \frac{c_1}{2} \int_{\omega_1} \phi_1^* \geq \frac{c_1}{2} \inf_{x \in \Omega} \int_{B_{\bar{\nu}_1}(x) \cap \Omega} \phi_1^* =: c_2.\end{aligned}$$

Note that the positive constant c_2 depends only on c_1, ν_1, Ω, and ϕ_1^*, and therefore only on the operator L and the domain Ω. This proves Lemma 2.

However, it would certainly be nice to know that the constants in our estimates depend only on bounds on the coefficients of L – see the remark on the constants which precedes Theorem 1. The last estimate does not immediately give us such control, because of the rather obscure behavior of ϕ_1^*.

By working a bit more we will now show that a more explicit lower bound for $\sup_\Omega v^-$ can be obtained if $p > n$, in terms of the constants in the basic elliptic estimates (the ABP inequality, the various forms of the Harnack inequality and the regularity estimates). This eventually yields the statements on the dependence of the constants, given before Theorem 1.

We introduce the auxiliary function $\zeta \in X$, the solution of

$$\begin{cases} L\zeta = -\chi(x) & \text{in} \quad \Omega \\ \quad \zeta = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{17}$$

where $\chi(x) = \chi_{\omega_1}(x)$ denotes the indicator function of the set $\omega_1 = B_{\nu_1}(x_1)$. This equation has a solution since $\lambda_1 = \lambda_1(L, \Omega) > 0$ so $L : X \to Y$ is an isomorphism (Theorem 3), by $(\mathbf{AP})_\mathbf{b}$ and the remark following it.

By applying Theorem 5 to (17) we get, for some $\bar{c} = \bar{c}(L, \Omega)$,

$$\zeta \geq \bar{c} |\omega_1|^{1/\varepsilon} d \geq \bar{c} \nu_1^{1/\varepsilon} d \qquad \text{in } \Omega.$$

Normalize now ϕ_1^* so that $\langle \phi_1^*, d\rangle = \int_\Omega \phi_1^* d = 1$. We have the chain of estimates

$$\begin{aligned}\sup_\Omega \frac{v^-}{d} &\geq \int_\Omega \frac{v^-}{d}\phi_1^* d = \int_\Omega v^-\phi_1^* = \int_\Omega v^+\phi_1^* \\ &\geq \frac{c_1}{2}\int_{\omega_1}\phi_1^* = \frac{c_1}{2}\langle \phi_1^*, \chi\rangle = \frac{c_1}{2}\langle \phi_1^*, -L\zeta\rangle = \frac{c_1}{2}\langle -L^*\phi_1^*, \zeta\rangle \\ &= \lambda_1 \frac{c_1}{2}\int_\Omega \phi_1^*\zeta \geq \lambda_1\frac{c_1}{2}\bar{c}\nu_1^{1/\varepsilon}\int_\Omega \phi_1^* d = \lambda_1 \frac{c_1}{2}\bar{c}\nu_1^{1/\varepsilon} := c_2.\end{aligned}$$

Thus $\sup_\Omega \frac{v^-}{d} > c_2 = c_2(L,\Omega) > 0$. We now apply the fact above to $\frac{v^-}{d}$, by using (16), to find a point $x_2 \in \overline{\Omega}$ and some $\bar{\nu}_2 > 0$ such that $\frac{v^-}{d} > c_2/2$ in $\hat{\omega}_2 = B_{\bar{\nu}_2}(x_2)\cap\Omega$. Set

$$\omega_2 = \hat{\omega}_2 \cap \{x\in\Omega \,:\, d(x) \geq \bar{\nu}_2/2\}.$$

The measure of ω_2 is controlled below by a constant $\nu_2 > 0$ which depends only on Ω and $\bar{\nu}_2$, and we have

$$v^- \geq \frac{c_2}{2}d \geq \frac{c_2\bar{\nu}_2}{4} \quad \text{in } \omega_2.$$

Lemma 2 is proved. □

We continue with the proof of Proposition 4. Define

$$\rho := \min\left\{1, \frac{\eta_1}{2}, \frac{\eta_2}{2}\right\},$$

where $\eta_i = \eta_i(L,\Omega) > 0$ is determined by Theorem 4, applied with $\Omega' = \Omega\setminus\bar{\omega}_i$ (ω_i are given by Lemma 2).

By the definition of P there exists $s\in\mathbb{R}$ such that (recall also (10))

$$\begin{aligned}P\frac{f(u)-f(\tilde{u})}{\|u-\tilde{u}\|_X} &= \frac{f(u)-f(\tilde{u})}{\|u-\tilde{u}\|_X} + s\phi_1 \\ &= q(x)\left((1-\tau)v + \tau\phi_1\right) + s\phi_1.\end{aligned}$$

Then (8) can be written as

$$Lv + q(x)v = \frac{1}{1-\tau}\psi - \frac{\tau}{1-\tau}q(x)\phi_1 - s\phi_1. \tag{18}$$

Assume first that $s \leq 0$.

Since $q \leq \lambda_1 + \rho \leq C_1(L,\Omega)$, $\|\phi_1\|_{L^\infty(\Omega)} \leq C_5\|\phi_1\|_X = C_5$, we get from (18)

$$Lv + (\lambda_1+\rho)v \geq -2|\psi| - 2C_5(\lambda_1+\rho)\tau \geq -C_6(|\psi|+\tau) \quad \text{in } \Omega. \tag{19}$$

Set

$$\widetilde{L} = L + \lambda_1 + \rho, \qquad \xi = C_6(|\psi|+\tau).$$

To summarize, we have

$$\begin{cases} \widetilde{L}v \geq -\xi & \text{in } \Omega \\ v \geq \epsilon & \text{in } \omega_1 \\ v \leq -\epsilon & \text{in } \omega_2 \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

In addition,

$$\begin{aligned}\lambda_1(\widetilde{L}, \Omega\setminus\bar{\omega}_i) &= \lambda_1(L,\Omega\setminus\bar{\omega}_i) - (\lambda_1+\rho) \\ &\geq \lambda_1 + \eta_i - (\lambda_1+\rho) \geq \frac{\eta_i}{2},\end{aligned}$$

by the choice of ρ and Theorem 4. Hence by Theorem 3 we can solve the problem

$$\begin{cases} \widetilde{L}\zeta = \xi & \text{in} \quad \Omega \setminus \bar{\omega}_2 \\ \zeta = 0 & \text{on} \quad \partial(\Omega \setminus \bar{\omega}_2), \end{cases} \tag{21}$$

and obtain

$$\|\zeta\|_{L^\infty(\Omega\setminus\bar{\omega}_2)} \le \frac{C_{ABP}}{\eta_2/2}\|\xi\|_{L^n(\Omega)} \le \frac{C_{ABP}}{\eta_2/2}|\Omega|^{1/n-1/p}\|\xi\|_{L^p(\Omega)} =: C_7\|\xi\|_{L^p(\Omega)}.$$

Assume by contradiction that

$$\|\xi\|_{L^p(\Omega)} < \frac{\epsilon}{C_7}. \tag{22}$$

Then the function $v = v + \zeta$ satisfies

$$\widetilde{L}v \ge 0 \quad \text{in } \Omega \setminus \bar{\omega}_2, \qquad v \le 0 \quad \text{on } \partial(\Omega \setminus \bar{\omega}_2), \qquad v > 0 \quad \text{in } \omega_1 \subset \Omega \setminus \bar{\omega}_2.$$

Thus Proposition 1 implies $\lambda_1(\widetilde{L}, \Omega \setminus \bar{\omega}_2) \le 0$, contradicting $\lambda_1(\widetilde{L}, \Omega \setminus \bar{\omega}_2) \ge \eta_2/2$.

Hence (22) fails, which is what we wanted to prove, since

$$C_6(\|\psi\|_{L^p(\Omega)} + |\Omega|^{1/p}\tau) \ge \|\xi\|_{L^p(\Omega)} \ge \frac{\epsilon}{C_7}$$

implies (9) by taking

$$c_0 := \min\left\{\frac{1}{2}, \frac{1}{2C_4}, \frac{\epsilon}{C_6C_7\max\{1, |\Omega|^{1/p}\}}\right\}.$$

If $s \ge 0$, instead of (19) we have

$$L(-v) + (\lambda_1 + \rho)(-v) \ge -C_6(|\psi| + \tau)$$

so we can repeat the same argument, interchanging ω_1 and ω_2.

Proposition 4 is proved. □

3. Proof of Theorem 1

Take $B = B(L, \Omega)$ in the hypothesis of Theorem 1 to be the minimum of the constants $\widetilde{B}$ and B, defined in Proposition 3 and Proposition 4, respectively.

Proposition 9. *Under the hypotheses of Theorem 1, no point of Y has three preimages under F.*

Proof. Such preimages would have to lie in the same fiber, that is, for some $z \in Y$ there exist $t_1 < t_2 < t_3$ and $u_i = u_z(t_i) = w_i + t_i\phi_1 \in W_X \oplus V$ with $F(u_i) = z + t\phi_1$ for a common height t. Then

$$-L(u_2 - u_1) - (f(u_2) - f(u_1)) = 0\,, \quad -L(u_3 - u_2) - (f(u_3) - f(u_2)) = 0.$$

We consider the potentials

$$V_{i,j}(x) := \begin{cases} \dfrac{f(u_i(x)) - f(u_j(x))}{u_i(x) - u_j(x)} & \text{if} \quad u_j(x) \ne u_i(x) \\ 0 & \text{if} \quad u_i(x) = u_j(x). \end{cases}$$

Clearly $a = 0 \le V_{i,j} \le b < B$, and

$$\big(-L - V_{2,1}\big)(u_2 - u_1) = 0\,, \quad \big(-L - V_{3,2}\big)(u_3 - u_2) = 0 \qquad \text{in } \Omega. \tag{23}$$

By Proposition 3, $u_3 - u_2$ and $u_2 - u_1$ are principal eigenfunctions and do not change sign throughout Ω. They are positive: indeed, as $\langle w_i - w_j, \phi_1^*\rangle = 0$,

$$\langle u_i - u_j, \phi_1^*\rangle = \lambda_1\langle (t_i - t_j)\phi_1, \phi_1^*\rangle = \lambda_1(t_i - t_j)\langle \phi_1, \phi_1^*\rangle > 0.$$

Hence $u_3 > u_2 > u_1$ in Ω and the potentials $V_{2,1}$, $V_{3,2}$ are continuous.

The convexity of f implies that for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$$\alpha_1 < \alpha_2 < \alpha_3 \Longrightarrow \frac{f(\alpha_2) - f(\alpha_1)}{\alpha_2 - \alpha_1} \leq \frac{f(\alpha_3) - f(\alpha_2)}{\alpha_3 - \alpha_2} .$$

If equality happens, the function f is affine in $[\alpha_1, \alpha_3]$.

Set $\alpha_i = u_i(x)$ to obtain $V_{2,1}(x) \leq V_{3,2}(x)$ for $x \in \Omega$. From Proposition 2 and the fact that 0 is the principal eigenvalue of both $-L - V_{2,1}$ and $-L - V_{3,2}$, we must have $V_{2,1} \equiv V_{3,2}$ in Ω. Thus, by the continuity of u_i and f,

$$f(t) = \alpha t + \beta , \quad \text{for } t \in I = [\inf u_1, \sup u_3] ,$$

so that $V_{2,1} = V_{3,2} = \alpha$, and $\alpha = \lambda_1$ by (23) and $u_2 - u_1 > 0$. Also $u_i(x) = 0$ for $x \in \partial\Omega$, so that $0 \in I$. This is a contradiction with (**C**). □

The second hypothesis in (**C**) is indeed necessary. If for instance $f(s) = \lambda_1 s + \beta$ in some interval $(0, M)$ then for $t \in (0, M/\max \phi_1)$,

$$F(t\phi_1) = -Lt\phi_1 - f(t\phi_1) = t\lambda_1\phi_1 - t\lambda_1\phi_1 - \beta = -\beta,$$

that is, the equation $F(u) = -\beta$ has a full segment of solutions.

Proposition 10. *For a Banach space E, consider the continuous proper map*

$$G : E \times \mathbb{R} \to E \times \mathbb{R} , \quad (e, t) \mapsto (e, g(e, t)) .$$

Suppose that no point in $E \times \mathbb{R}$ has three preimages under G. If some point has two preimages, G is a global fold, that is, there are homeomorphisms

$$\sigma_1, \sigma_2 : E \times \mathbb{R} \to E \times \mathbb{R} , \quad \sigma_1(e, t) = (e, g_1(e, t)) , \quad \sigma_2(e, t) = (e, g_2(e, t)),$$

such that $(\sigma_2 \circ G \circ \sigma_1)(e, t) = (e, -|t|)$. Otherwise G is a homeomorphism.

Proof. The argument breaks in simple steps.

Step 1: Height functions $g(e, .)$ may have only four distinct topological types.

By properness, on each vertical line $l_e = \{(e, t), t \in \mathbb{R}\}$, $e \in E$,

$$\lim_{t\to\infty} g(e, t) = \pm\infty , \quad \lim_{t\to-\infty} g(e, t) = \pm\infty$$

where the signs of both limits are not necessarily the same: there are two possibilities in which they are the same and two in which they are different.

By hypothesis, there are no three points in a vertical line l_e in the domain taken to the same point by G. A continuous real function which has infinite limits at plus and minus infinity and which sends no three points to the same image has exactly one of the following four properties (to which we will refer as *types*): (i) it is strictly increasing on $\mathbb{R}$; (ii) it is strictly decreasing on $\mathbb{R}$; (iii) it is strictly increasing to the left of some real number and strictly decreasing to the right of that number; (iv) it is strictly decreasing to the left of some real number and strictly increasing to the right of that number. Observe that the type of a function is determined by the signs of its limits at plus and minus infinity.

Step 2: All heights of G are of the same type.

By a connectivity argument, it suffices to prove that, for a fixed e_0, there is a neighborhood N of e_0 for which all height functions $g(e, .)$ for $e \in N$ have the same limit. For example, suppose by contradiction that $e_k \to e_0$ are such that

$$(+) \quad \lim_{t\to\infty} g(e_k, t) = \infty , \qquad (-) \quad \lim_{t\to\infty} g(e_0, t) = -\infty .$$

By properness, the inverse of the compact set $K = \{(e_k, 0)_k\} \cup \{(e_0, 0)\}$ is a compact set, and therefore lies in $\cup_k (e_k \times [-M, M])$ for some $M \in \mathbb{R}$. Then by the property (+), we must have $g(e_k, M + 1) \geq 0$ for each k, and thus $g(e_0, M + 1) \geq 0$, contradicting (−).

If one $g(e,.)$ is of the first two types, G is a homeomorphism. For the rest of the proof, we suppose that $g(e,.)$ is of the third type: in particular each function $g(e,.)$ is strictly unimodal, that is, $g(e,t)$ is strictly increasing for $t < T$, and strictly decreasing for $t > T$, for some $T = T(e) \in \mathbb{R}$.

Step 3: Maxima of height functions, as well as points where they are attained, vary continuously across vertical lines.

Let $T(e)$ be the value of $t \in \mathbb{R}$ at which $g(e,t)$ attains its maximum. The map $e \in E \mapsto T(e)$ is well defined by the unimodality.

We show the continuity of T at an arbitrary $e_0 \in E$. Set $T_0 = T(e_0)$ and take $\epsilon > 0$. For L, R satisfying

$$L < T_0 < R\,, \quad T_0 - L < \epsilon/2\,, \quad R - T_0 < \epsilon/2\,,$$

take $d > 0$ so that

$$g(e_0, T_0) - g(e_0, L) > d\,, \quad g(e_0, T_0) - g(e_0, R) > d\,.$$

By the continuity of G, there is $\delta > 0$ for which, if $|e - e_0| < \delta$, then

$$|g(e,L) - g(e_0,L)|\,, \quad |g(e,T_0) - g(e_0,T_0)|\,, \quad |g(e,R) - g(e_0,R)| < d/3\,.$$

If $|e - e_0| < \delta$, $g(e,T_0)$ is larger than $g(e,L)$ and $g(e,R)$: for example, to estimate $g(e,T_0) - g(e,L)$, write

$$g(e,T_0) - g(e_0,T_0) + g(e_0,T_0) - g(e_0,L) + g(e_0,L) - g(e,L) > -d/3 + d - d/3 = d/3\,.$$

Thus the point $T(e)$ where $g(e,t)$ attains its maximum is still between L and R, by the unimodality of $g(e,.)$. Since $R - L < \epsilon$, we also have $|T(e) - T(e_0)| < \epsilon$. The continuity of the maximal value $z \in E \mapsto g(e,T(e))$ is now immediate.

Step 4: The global normal form.

The homeomorphisms

$$\tau_1, \tau_2 : E \times \mathbb{R} \to E \times \mathbb{R}\,, \quad \tau_1(e,t) = (e, t + T(e))\,, \quad \tau_2(e,s) = (e, s - g(e,T(e)))$$

yield the map $\tilde{G} = \tau_2 \circ G \circ \tau_1$, whose critical set $\tilde{C}$ together with its image $\tilde{G}(\tilde{C})$ coincide with the horizontal plane $E \times \{0\}$. In addition, $\tilde{G}|_{\tilde{C}}$ is the identity. Moreover, the restrictions of $\tilde{G}$ on the half-spaces

$$\tilde{G}_- : E \times (-\infty, 0] \to E \times (-\infty, 0] \quad \text{and} \quad \tilde{G}_+ : E \times [0, \infty) \to E \times (-\infty, 0]$$

are also homeomorphisms.

Set $\nu(z,t) = (z,-t)$. The juxtaposition of the maps $\tilde{G}_-$ and $\nu \circ \tilde{G}_+$ along $E \times \{0\}$ is a homeomorphism $j : E \times \mathbb{R} \to E \times \mathbb{R}$, and it is easy to see that $\tilde{G} \circ j^{-1} : E \times \mathbb{R} \to E \times \mathbb{R}$ takes (e,t) to $(e, -|t|)$.

The proposition is proved, setting $\sigma_1 = \tau_1 \circ j^{-1}$, and $\sigma_2 = \tau_2$. □

We finally complete the proof of Theorem 1.

Proof of Theorem 1. Let $\tilde{F} = F \circ \Psi^{-1} : Z \times \mathbb{R} \to Z \times \mathbb{R}$ be the map defined in Theorem 6. From Proposition 9 no point has three preimages under F, and hence under $\tilde{F}$.

From the previous proposition, $\tilde{F}$ is either a homeomorphism or a global fold. It is not a homeomorphism, since from Proposition 7 on both extremes of each fiber there are points which have the same image under F.

Let σ_1, σ_2 be the maps given by Proposition 10, applied to $\tilde{F}$. Define the map $\tilde{\psi} : X = W \oplus \mathbb{R}\phi_1 \to Y = Z \oplus \mathbb{R}\phi_1$ by $\tilde{\psi}(w + t\phi_1) = -Lw + t\phi_1$.

Finally, we set

$$\Phi_1 = \Psi^{-1} \circ \sigma_1 \circ \tilde{\psi} : X \to X, \qquad \Phi_2 = \sigma_2 : Y \to Y.$$

With this definition and Proposition 10, we easily check that (2) holds.

Obviously Φ_2 leaves vertical lines invariant, by the definition of σ_2. To show the asymptotic property of Φ_1, observe that by the definition of this map for each fixed $w \in W$ the point $\Phi_1(w + t\phi_1)$ is on the fiber generated by

$z = -Lw$, and

$$\Phi_1(w + t\phi_1) = \Psi^{-1}(-Lw, \hat{t}), \tag{24}$$

where $\hat{t} = \hat{t}(w, t)$ is the number for which there exists a point $\hat{w} \in W$ such that

$$F(\hat{w} + (\hat{t} + c_1)\phi_1) = \begin{cases} -Lw + (t - c_2)\phi_1 & \text{if} \quad t \le c_2, \quad \text{and} \quad \hat{t} < 0 \\ -Lw - (t - c_2)\phi_1 & \text{if} \quad t \ge c_2, \quad \text{and} \quad \hat{t} > 0. \end{cases} \tag{25}$$

Here c_1, c_2 are real constants whose values are irrelevant to our computation below (they depend only on w, and are related to the maximum of the height function on the fiber generated by $z = -Lw$). By (25) and the properness of F it is clear that $\lim_{t\to\pm\infty} \hat{t} = \pm\infty$.

We are going to show that

$$\lim_{t\to-\infty} \frac{t}{\hat{t}} = \lambda_1 - \tilde{a}, \qquad \lim_{t\to\infty} \frac{t}{\hat{t}} = \tilde{b} - \lambda_1, \tag{26}$$

where $\tilde{a} := \lim_{s\to-\infty} \frac{f(s)}{s} < \lambda_1$, $\tilde{b} := \lim_{s\to\infty} \frac{f(s)}{s} > \lambda_1$ (see the proof of Lemma 1), from which we infer the asymptotics

$$\lim_{t\to-\infty} \frac{\Phi_1(w + t\phi_1)}{t} = \frac{1}{\lambda_1 - \tilde{a}}\phi_1, \qquad \lim_{t\to\infty} \frac{\Phi_1(w + t\phi_1)}{t} = \frac{1}{\tilde{b} - \lambda_1}\phi_1 \tag{27}$$

in X, thanks to (24) and Lemma 1.

Exactly like in the proof of Lemma 1 we can show that

$$\lim_{|\hat{t}|\to\infty} \|\frac{\hat{w}}{\hat{t}}\|_X \le C \lim_{|\hat{t}|\to\infty} \frac{1}{|\hat{t}|} \|PF_{\hat{t}+c_1}(\hat{w}) - PF_{\hat{t}+c_1}(0)\| = 0.$$

Writing (25) in the form

$$-L(\hat{w} + (\hat{t} + c_1)\phi_1) - V(x)(\hat{w} + (\hat{t} + c_1)\phi_1) = -Lw \pm (t - c_2)\phi_1$$

where $V(x) = f(\hat{w} + (\hat{t} + c_1)\phi_1)/(\hat{w} + (\hat{t} + c_1)\phi_1)$ converges to $\tilde{b}$ as $t \to \infty$ (resp. to $\tilde{a}$ as $t \to -\infty$), multiplying by ϕ_1^* and integrating, dividing by $\hat{t}$ and letting $\hat{t} \to \pm\infty$, we arrive at (26). □

Conflict of interest statement

There is no conflict of interest.

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