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On the inverse limit stability of endomorphisms

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Abstract

We present several results suggesting that the concept of C^1-inverse (limit structural) stability is free of singularity theory. An example of a robustly transitive, C^1-inverse stable endomorphism with a persistent critical set is given. We show that every C^1-inverse stable, axiom A endomorphism satisfies a certain strong transversality condition (T). We prove that every attractor–repeller endomorphism satisfying axiom A and condition (T) is C^1-inverse stable. The latter is applied to Hénon maps, rational functions and others. This leads us to conjecture that C^1-inverse stable endomorphisms are exactly those which satisfy axiom A and condition (T).

Résumé

Nous présentons différents résultats suggérant que le concept de C^1-stabilité (structurelle de la limite inverse) est indépendant de la théorie des singularités. Nous décrivons un exemple d'un endomorphisme robustement transitif et C^1-stable ayant un ensemble critique persistant. Nous montrons que tout endomorphisme axiome A et C^1-stable vérifie nécessairement une certaine condition de transversalité forte (T). Nous démontrons que tout endomorphisme attracteur–répulseur vérifiant la condition (T) est C^1-stable. Ce dernier résultat est appliqué, entre autres, aux applications de type Hénon et aux fractions rationnelles. Cela nous amène à conjecturer que les endomorphismes C^1-stables sont exactement ceux qui vérifient l'axiome A et la condition (T).

1. Introduction

There exist various concepts of stability for dynamical systems. When dealing with endomorphisms it makes sense to consider the inverse limit which is defined in the sequel. A C^1-*endomorphism* f is a C^1-map of a manifold M into itself, which is not necessarily bijective and which can have a nonempty *singular set* (formed by the points x s.t. the derivative $T_x f$ is not surjective). The *inverse limit set* of f is the space of the full orbits $(x_i)_i \in M^{\mathbb{Z}}$ of f. The dynamics induced by f on its inverse limit set is the shift. The endomorphism f is C^1-*inverse limit stable* if for every C^1 perturbation f' of f, the inverse limit set of f' is homeomorphic to the one of f via a homeomorphism which conjugates both induced dynamics and is C^0-close to the canonical inclusion.

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When the dynamics f is a diffeomorphism, the inverse limit set $\overleftarrow{M}_f$ is homeomorphic to the manifold M. The C^1-inverse limit stability of f is then equivalent to the C^1-*structural stability* of f: every C^1-perturbation of f is conjugated to f via a homeomorphism of M.

A great work was done by many authors to provide a satisfactory description of C^1-structurally stable diffeomorphisms, which starts with Anosov, Smale, Palis, de Melo, Robbin, and finishes with Robinson [14] and Mañé [9]. Such diffeomorphisms are those which satisfy axiom A and the strong transversality condition.

Almost the same description was accomplished for C^1-structurally stable flows by Robinson and Hayashi. The inverse limit set of a flow is a one dimensional foliation. The structural stability of a flow is also equivalent to the C^1-inverse stability. A flow ϕ is *structurally stable* if the foliation induced by ϕ is equivalent to the foliation induced by its perturbation, via a homeomorphism of M which is C^0-close to the identity.

The descriptions of the structurally stable maps for smoother topologies (C^r, C^∞, holomorphic ...) remain some of the hardest, fundamental, open questions in dynamics.

One of the difficulties occurring in the description of C^r-structurally stable smooth endomorphisms concerns the singularities. Indeed, a structurally stable map must display a stable singular set. But there is no satisfactory description of them in singularity theory.

This work suggests that the concept of inverse limit stability does not deal with singularity theory.

For other perspectives, let us mention that the concept of inverse limit stability is an area of great interest for semi-flows given by PDEs, although still at its infancy.

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2. Statement of the main results

Let f be a C^1-map of a compact manifold M into itself. The *inverse limit* of f is the set

$$\overleftarrow{M}_f := \{\underline{x} = (x_i)_{i\in\mathbb{Z}} \in M^{\mathbb{Z}}:\ f(x_i) = x_{i+1}\ \forall i \in \mathbb{Z}\}.$$

The set $\overleftarrow{M}_f$ is a compact metric space endowed with the metric:

$$d_1(\underline{x}, \underline{y}) := \sum_{n\in\mathbb{Z}} \frac{d(x_i, y_i)}{2^i},$$

with d the Riemannian metric on M. The map f induces the shift map $\overleftarrow{f}(\underline{x})_i = x_{i+1}$. We remark that $\overleftarrow{M}_f$ is equal to M and $\overleftarrow{f}$ is equal to f if f is bijective. The *global attractor* of f is defined as $M_f = \bigcap_{n\geqslant 0} f^n(M)$. For $j \in \mathbb{Z}$, let:

$$\pi_j : \underline{x} = (x_i)_i \in M^{\mathbb{Z}} \mapsto x_j \in M.$$

We note that $\pi_j \circ \overleftarrow{f} = f \circ \pi_j|\overleftarrow{M}_f$ and $\pi_j(\overleftarrow{M}_f) = M_f$.

Two endomorphisms f and f' are C^1-*inverse limit conjugated*, if there exists a homeomorphism h from $\overleftarrow{M}_f$ onto $\overleftarrow{M}_{f'}$, such that the following equality holds:

$$h \circ \overleftarrow{f} = \overleftarrow{f}' \circ h.$$

Definition 2.1. An endomorphism f is C^1-*inverse limit stable* or simply C^1-*inverse stable* if every C^1-perturbation f' of f is inverse limit conjugated to f.

Let K_f be a compact, f-invariant subset of M ($f(K_f) \subset K_f$). Then K_f is *hyperbolic* if there exist a section E^s of the Grassmannian of $TM|K_f$ and a Riemannian metric satisfying for every $x \in K_f$:

- $T_x f(E^s(x)) \subset E^s(f(x))$,
- the action $[T_x f]$ induced by f on the quotients $T_x M/E^s(x) \to T_{f(x)}M/E^s(f(x))$ is invertible,
- $\|T_x f|E^s(x)\| < 1$ and $\|[T_x f]^{-1}\| < 1$.

We notice that actually $E^s(\underline{x})$ depends only on $x_0 = \pi_0(\underline{x})$. It can be denoted by $E^s(x_0)$.

On the other hand, there exists a unique continuous family $(E^u(\underline{x}))_{\underline{x}}$ of subspaces $E^u(\underline{x}) \subset T_{x_0}M$, indexed by $\underline{x} \in \overleftarrow{K}_f := K_f^{\mathbb{Z}} \cap \overleftarrow{M}_f$, satisfying:

$$T_{x_0} f\big(E^u(\underline{x})\big) = E^u\big(\overleftarrow{f}(\underline{x})\big) \quad \text{and} \quad E^u(\underline{x}) \oplus E^s(x_0) = T_{x_0}M.$$

For $\epsilon > 0$, the ϵ-local stable set of $x \in K_f$ is:

$$W^s_\epsilon(x; f) = \big\{y \in M\colon \forall i \geqslant 0,\ d\big(f^i(x), f^i(y)\big) \leqslant \epsilon, \text{ and } d\big(f^i(x), f^i(y)\big) \to 0,\ i \to +\infty\big\}.$$

The ϵ-local unstable set of $\underline{x} \in \overleftarrow{K}_f$ is:

$$W^u_\epsilon(\underline{x}; \overleftarrow{f}) = \big\{\underline{y} \in \overleftarrow{M}_f\colon \forall i \leqslant 0,\ d(x_i, y_i) \leqslant \epsilon, \text{ and } d(x_i, y_i) \to 0,\ i \to -\infty\big\}.$$

We will write $W^u_\epsilon(\underline{x})$ and $W^s_\epsilon(x)$ for $W^u_\epsilon(\underline{x}; \overleftarrow{f})$ and $W^s_\epsilon(x; f)$ whenever the dynamics involved is clear. Let us justify why we have chosen $W^s_\epsilon(x)$ included in M whereas $W^u_\epsilon(\underline{x})$ is included in $\overleftarrow{M}_f$. One can prove that (for ϵ small enough) the local stable set is a submanifold whose tangent space at x equals $E^s(x_0)$; however the following is in general not a manifold (not even a lamination).

$$W^s_\epsilon(\underline{x}) = W^s_\epsilon(\underline{x}; \overleftarrow{f}) := \pi_0^{-1}\big(W^s_\epsilon(x; f)\big) \cap \overleftarrow{M}_f.$$

The ϵ-local unstable set is a manifold embedded into M by π_0; its tangent space at x_0 is equal to $E^u(\underline{x})$. In general the unstable manifold depends on the preorbit: the unstable sets of different orbits in $\pi_0^{-1}(x_0)$ are not necessarily equal.

An endomorphism satisfies (*weak*) *axiom A* if the nonwandering set Ω_f of f is hyperbolic and equal to the closure of the set of periodic points.

In this work, we do not deal with *strong axiom A* endomorphisms which satisfy moreover that the action on each of the basic pieces of Ω_f is either expanding or injective. This stronger definition is relevant for structural stability [11,12], but it is conjectured below to be irrelevant for inverse stability. Note also that the present definition of axiom A endomorphisms is more general than the original one of [11] which assumes that the critical set does not intersect the nonwandering set.

We put $\overleftarrow{\Omega}_f := \Omega_f^{\mathbb{Z}} \cap \overleftarrow{M}_f$. In Section 4, we will see that $\overleftarrow{\Omega}_f$ is the nonwandering set of $\overleftarrow{f}$. Actually if the f-periodic points are dense in Ω_f then the $\overleftarrow{f}$-periodic points are dense in $\overleftarrow{\Omega}_f$. For the sets of the form $\pi_N^{-1}(B(x,\epsilon)) \cap \overleftarrow{\Omega}_f$, with $x \in \Omega_f$, $\epsilon > 0$ and $N \in \mathbb{Z}$, are elementary open sets of $\overleftarrow{\Omega}_f$ and contain periodic points.

Also if Ω_f is hyperbolic the restriction of $\overleftarrow{f}$ to $\overleftarrow{\Omega}_f$ is expansive. For the ϵ unstable manifold $W^u_\epsilon(\underline{x})$ intersects $W^s_\epsilon(\underline{x})$ at the unique point $\underline{x}$ since π_0 restricted to $W^u_\epsilon(\underline{x})$ is a homeomorphism and $\pi_0 W^u_\epsilon(\underline{x})$ intersects $W^s_\epsilon(\pi_0(\underline{x}))$ at the unique point $\pi_0(\underline{x})$, for every $\underline{x} \in \overleftarrow{\Omega}_f$.

Remark 2.2. Every axiom A endomorphism f has its inverse limit $\overleftarrow{f}$ which satisfies axiom A^* in the meaning of [4]; this implies ergodic properties such as the existence of the maximal entropy measure for $\overleftarrow{f}$ and so for f.

Definition 2.3. The dynamics f satisfies the *strong transversality condition* (T) if:

For all $n \geqslant 0$, $\underline{x} \in \overleftarrow{\Omega}_f$ and $y \in \Omega_f$, the map f^n restricted to $\pi_0 W^u_\epsilon(\underline{x})$ is transverse to $W^s_\epsilon(y)$. In other words, for every $z \in \pi_0 W^u_\epsilon(\underline{x}) \cap f^{-n}(W^s_\epsilon(y))$:

$$(T) \qquad T_z f^n\big(T_z \pi_0 W^u_\epsilon(\underline{x})\big) + T_{f^n(z)} W^s_\epsilon(y) = T_{f^n(z)} M.$$

Definition 2.4. *AS endomorphisms* are those which satisfy axiom A and the strong transversality condition (T).

A first result is:

Theorem 2.5. *Let M be a compact manifold and $f \in C^1(M, M)$. If f is C^1-inverse stable and satisfies axiom A, then the strong transversality condition holds for f.*

The second one concerns the converse:

Definition 2.6. An axiom A endomorphism f is *attractor–repeller* if Ω_f is included in the union of two compact hyperbolic sets R_f and A_f such that there exist:

(a) a neighborhood V_A of A_f *in* M satisfying $\bigcap_{n\geqslant 0} f^n(V_A) = A_f$,
(r) a neighborhood V_R of R_f *in* M_f satisfying $R_f = \bigcap_{n\geqslant 0} f^{-n}(V_R)$ and moreover $f^{-1}(R_f) = R_f$.

The set R_f is called a *repeller* and A_f an *attractor*.

One can show that if f is an endomorphism whose nonwandering set satisfies (a) and (r) with A_f and R_f hyperbolic, then f is axiom A and so attractor–repeller.

It is not true in general that $\Omega_f = A_f \sqcup R_f$ when f is attractor–repeller. Indeed, the endomorphism $f(x) = x^2 - 1$ defined in the real axis has its nonwandering set equal to the union of a (super)-attracting periodic point of period 2 and two repelling fixed points. Moreover, if R_f is defined as the union of the preimages of the fixed points, it is immediate that f is attractor–repeller.

Theorem 2.7. *Let M be a compact manifold and $f \in C^1(M, M)$. If f is an attractor–repeller endomorphism which satisfies the strong transversality condition, then f is C^1-inverse stable.*

It follows immediately from the theorem of Aoki, Moriyasu and Sumi in [1] that: if an endomorphism f is C^1-inverse stable and has no singularities in the nonwandering set, then f satisfies axiom A.

In Section 4, we will study the dynamics and the geometry of the inverse limit of every AS endomorphism. This enables the first author with A. Kocsard to generalize Theorem 2.7 to the general case of AS endomorphisms by adding analytical techniques to arguments of Section 5. It seems also possible to generalize Aoki–Moruriyasu–Sumi theorem to endomorphisms with singularity. This would complete the following conjecture (sketched in [13]):

Conjecture 2.8. *The C^1-inverse stable endomorphisms are exactly those which satisfy axiom A and the strong transversality condition (T).*

2.1. Application of Theorem 2.7

Example 2.9 *(Rational functions).* Let f be a rational function of the Riemann sphere. Let us suppose that all its critical points belong to basins of attracting periodic orbits, or equivalently that its Julia set is expanding. By Theorem 2.7, f is C^1-inverse stable. Note that C^1-perturbations of f may have very wild critical set. See [8] for a nice geometrical description of the inverse limit of f.

Example 2.10 *(One-dimensional dynamics and Hénon maps).* Kozlovski, Shen and van Strien showed that a (C^∞)-generic map f of the circle $\mathbb{S}^1$ is attractor–repeller [7], and so C^1-inverse limit stable, by Theorem 2.7.

Let $f'(\theta, y) = (f(\theta) + y, 0)$ be defined on the 2-torus $\mathbb{T}^2$ which enjoys of a canonical Abelian group structure. Aside finitely many attracting periodic points, the nonwandering set of f' consists of an expanding compact set of f times $\{0\}$. This product R is a hyperbolic set for f' and a repeller (for the restriction of f' to $M_{f'}$), as stated in Definition 2.6. It follows that f' satisfies the requirements of Theorem 2.7. This implies that if $g \in C^1(\mathbb{T}^2, \mathbb{R})$ is close to 0, then the inverse limits sets of f and of the map:

$$(\theta, y) \mapsto \big(f(\theta) + y, g(\theta, y)\big),$$

are conjugated.

For instance, take $f(x) = x^2 + c$ with $c \in (-2, 1/4)$ attractor–repeller on the one-point compactification of $\mathbb{R}$. The infinity is an attracting fixed point with basin bounded by the positive fixed point p of f and its preimage. Let ρ be a smooth function with compact support in $\mathbb{R}$ and equal to 1 on a neighborhood of $[-p, p]$.

For such a c and then b small enough, the global attractor of the Hénon map $(x, y) \mapsto (x^2 + c + y, bx)$ of $\mathbb{R}^2$, equal to the one of $(x, y) \mapsto (x^2 + c + y, \rho(x) \cdot b \cdot x)$ without the basin of $(\infty, 0)$, is conjugated to the inverse limit of $f|[-p, p]$.

The same example works with f a hyperbolic rational function of the sphere. This generalizes many results in this direction to the wide C^1-topology (see [5] which contains other references).

Example 2.11 *(C^1 robustly transitive, Anosov endomorphism with persistent critical set).* Przytycki showed that an Anosov endomorphism without singularities is inverse stable [12]. Latter Quandt generalized this for Anosov endomorphisms, possibly with singularities [13]. These results are consequences of Theorem 2.7.

The simplest known example of Anosov endomorphisms is action of linear maps on the quotient $\mathbb{R}^2/\mathbb{Z}^2$, for instance:

$$A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}, \quad n \in \{2, 3, \dots\}.$$

A constant map is a trivial example of an Anosov endomorphism. Let us construct an example of Anosov map whose singular set is persistently nonempty and whose nonwandering set is the whole manifold.

Begin with a linear map A of the plane as above. Close to the fixed point one can use linear coordinates to write the map as

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

where $0 < \mu < 1$ and $\lambda > 1$. Let ϵ be a positive constant and let Ψ be a nonnegative smooth function such that $\Psi(0) = 1$ and $\Psi(x) = 0$ for every $|x| > \epsilon$. Assume also that Ψ is an even function having a unique critical point in $(-\epsilon, \epsilon)$. Let φ be the C^1 function defined by: $\varphi(y) = 0$ for every $y \notin [0, \epsilon]$ and $\varphi'(y) = \sin(\frac{2\pi y}{\epsilon})$ for $y \in [0, \epsilon]$. Let f be the C^1-endomorphism of the torus equal to

$$f(x, y) = \big(\lambda x, \mu y - \Psi(x)\varphi(y)\big)$$

on the 2ϵ-neighborhood of 0 and to A off.

Let g be the real function $y \mapsto \mu y - \phi(y)$. There are regular points with different numbers of g-preimages. The same occurs for f. Consequently f has a singular set which is persistently nonempty.

We remark that f is Anosov. For the A-stable direction is still preserved and contracted; the action of Tf on the stable foliation normal bundle is still λ-expanding.

Moreover the stable leaves are irrational lines of the torus, uniformly contracted. Given a segment I contained in a stable leave and a number $A > 0$, there exists a positive integer k_0 such that any component of $f^{-k}(I)$ has length greater than A for every $k \geqslant k_0$. From this it comes that given nonempty open sets U and V, $f^{-k}(U)$ contains a sufficiently long segment, and thus it intersects V for every k large. In other words, f is mixing. It follows from Theorem 2.7 that every C^1 perturbation f' of f is inverse conjugated to f. As a map is transitive if and only if its inverse limit is transitive and projects on the whole manifold, we conclude that f' is transitive.

Example 2.12 *(Products).* Theorem 2.7 shows also the inverse stability of product of an Anosov endomorphism with an attractor–repeller endomorphism.

Example 2.13. (See Mañé and Pugh [10].) Mañé and Pugh gave an example of C^1-Ω-stable endomorphism for which the singular set persistently intersects an attracting basic piece. Their example is clearly not C^r-structurally stable but according to Theorem 2.7 it is C^1-inverse-limit stable.

3. Proof of Theorem 2.5

We begin this section with two well known facts of transversality theory.

Claim 3.1. *Let N_1 and N_2 be two embedded submanifolds of M.*

(i) *The set of maps $f \in C^1(M, M)$ such that $f^n|N_1$ is transverse to N_2 for every $n \geqslant 1$ is residual.*
(ii) *If f is a C^1 map and $f|N_1$ is not transverse to N_2, then there exists a C^1 perturbation f' of f such that $f'^{-1}(N_2) \cap N_1$ contains a submanifold whose codimension is less than the sum of the codimensions of N_1 and N_2.*

Let f be a C^1-inverse stable endomorphism satisfying axiom A. For each small perturbation f' of f, let $h(f')$ be the conjugacy between $\overleftarrow{M}_f$ and $\overleftarrow{M}_{f'}$.

We will assume by contradiction that the transversality condition fails to be true. This means that there exist $n \geqslant 0$, $\underline{x} \in \overleftarrow{\Omega}_f$, $y \in \Omega_f$ and $z \in f^{-n}(W^s_\epsilon(y; f)) \cap \pi_0 W^u_\epsilon(\underline{x}; \overleftarrow{f})$ such that Eq. (T) does not hold. Note first that $z \notin \Omega_f$, by hyperbolicity of the nonwandering set.

Moreover, by density of periodic orbits in $\overleftarrow{\Omega}_f$ after replacing f by a perturbation, we can assume that $\underline{x}$ and y are periodic points. To simplify the calculations, we can suppose that $\underline{x}$ and y are fixed points by considering an iterate of f.

The conjugacy $h(f')$ was asked to be close to the inclusion of $\overleftarrow{M}_f$ into $M^{\mathbb{Z}}$. By expansiveness of $\overleftarrow{\Omega}_f$, if a perturbation f' is equal to f at the nonwandering set Ω_f, then $h(f')$ is equal to the inclusion of $\overleftarrow{\Omega}_f$. We will produce perturbations f' and f'' of f that are equal to f on Ω_f.

The second item of Claim 3.1 can be used to produce a perturbation f' of f such that

$$f'^{-N}\left(W^s_\epsilon\left(y; f'\right)\right) \cap \pi_0 W^u_\epsilon\left(\underline{x}; \overleftarrow{f}'\right)$$

contains a submanifold of dimension $p > u + s - m$, where u is the dimension of $\pi_0 W^u_\epsilon(\underline{x})$, s is the dimension of $W^s_\epsilon(x)$ and m the dimension of the manifold M.

On the other hand, the first item of Claim 3.1 implies that for generic perturbations f'' of f, the restriction of $(f'')^k$ to $\pi_0 W^u_\epsilon(\underline{x}; \overleftarrow{f}'')$ is transverse to $W^s_\epsilon(y; f'')$ for every positive integer k.

If $\epsilon > 0$ is sufficiently small, the maps f' and f'' are injective restricted to the closures of $\pi_0 W^u_\epsilon(\underline{x}; f')$ and $\pi_0 W^u_\epsilon(\underline{x}; f'')$ respectively. This implies that the restrictions of π_0 to the closures of $W^u_\epsilon(\underline{x}; f')$ and $W^u_\epsilon(\underline{x}; f'')$ are homeomorphisms onto their images.

For $\underline{y} \in \pi_0^{-1}(\{y\})$, note that $\pi_0^{-1}(W^s(y; f'))$ and $\pi_0^{-1}(W^s(y; f''))$ are equal to the stable sets $W^s(\underline{y}; \overleftarrow{f}')$ and $W^s(\underline{y}; \overleftarrow{f}'')$ respectively.

Consequently $A' := W^s(\underline{y}; \overleftarrow{f}') \cap W^u_\epsilon(\underline{x}; \overleftarrow{f}')$ contains a manifold of dimension p whereas $A'' := W^u_\epsilon(\underline{x}; \overleftarrow{f}'') \cap W^s(y; \overleftarrow{f}'')$ is a (possibly disconnected) manifold of dimension $u + s - m < p$.

We assumed that f is inverse stable, so the map $\phi := h(f'')^{-1} \circ h(f')$ is a conjugacy between $\overleftarrow{f}'$ and $\overleftarrow{f}''$ which fixes $\underline{y}$ and $\underline{x}$. Thus ϕ must embed A' into $W^u(\underline{x}; \overleftarrow{f}'') \cap W^s(\underline{y}, \overleftarrow{f}'') = \bigcup_{n \geqslant 0} \overleftarrow{f}''^n(A'')$.

As $\overleftarrow{f}''$ is homeomorphism, a manifold of dimension p, contained in A', is embedded by ϕ into the manifold $\bigcup_{n \geqslant 0} \overleftarrow{f}''^n(A'')$ of dimension less than p. This is a contradiction.

4. General properties on axiom A endomorphisms

This section is devoted to the study of the geometry and the dynamics of the inverse limit of AS endomorphisms.

Let us first remark that $\overleftarrow{\Omega}_f := \Omega_f^{\mathbb{Z}} \cap \overleftarrow{M}_f$ is also the nonwandering set of $\overleftarrow{f}$. Indeed, an elementary open set V of $\overleftarrow{M}_f$ has the form $(\prod_{i<N} M \times U \times \prod_{i>N} M) \cap \overleftarrow{M}_f = \prod_{n \in \mathbb{Z}} f^{n-N}(U) \cap \overleftarrow{M}_f$, where U is an open set in M. Therefore $\overleftarrow{f}^n(V)$ intersects V for $n > 0$, iff $f^n(U)$ intersects U.

A compact hyperbolic set $K \subset M$ has a *local product structure*, if there exists $\epsilon > 0$ small, such that for every pair of nearby points $\underline{x}, \underline{y} \in \overleftarrow{K}_f = K^{\mathbb{Z}} \cap \overleftarrow{M}_f$, the set $W^u_\epsilon(\underline{x})$ intersects $W^s_\epsilon(\underline{x})$ at a unique point $[\underline{x}, \underline{y}]$ which *belongs* to $\overleftarrow{K}_f$. Here ϵ is sufficiently small so that in particular $W^u_\epsilon(\underline{x})$ is embedded by π_0. The following is classical in the diffeomorphism case:

Lemma 4.1. *If K is a basic piece of an axiom A map f, then $\overleftarrow{K}_f$ has a local product structure.*

Proof. If $\underline{x}, \underline{y}$ are close enough then $\pi_0 W^u_\epsilon(\underline{x}; \overleftarrow{f})$ intersects $\pi_0 W^s_\epsilon(\underline{y}; \overleftarrow{f})$ at a unique point z. Thus for $\epsilon > 0$ small enough, it holds that for every pair of periodic points $\underline{x}', \underline{y}'$ close to $\underline{x}, \underline{y}$, the local unstable manifold $\pi_0 W^u_\epsilon(\underline{x}'; \overleftarrow{f})$ intersects $\pi_0 W^s_\epsilon(\underline{y}'; \overleftarrow{f})$ at a unique point z'. As $\pi_0 W^u_\epsilon(\underline{y}'; \overleftarrow{f})$ intersects $\pi_0 W^s_\epsilon(\underline{x}'; \overleftarrow{f})$, the point z' is nonwandering. Thus z is nonwandering and also its preimage $[\underline{x}, \underline{y}]$ by $\pi_0 | W^u_\epsilon(\underline{x}; \overleftarrow{f})$. □

The existence of a local product structure is useful to shadow:

Lemma 4.2. *A hyperbolic compact set K equipped with a local product structure satisfies the shadowing property*: *there exists $\eta > 0$ such that every η-pseudo-orbit $(x_i)_{i \in \mathbb{Z}}$ in an η-neighborhood of K is ϵ-shadowed by an orbit $(y_i)_{i \in \mathbb{Z}}$ in K.*

Proof. The proof of this lemma is treated as for diffeomorphisms [15, Prop. 8.20]. □

The following is going to be used many times.

Lemma 4.3. *If f satisfies axiom A, then $\overleftarrow{M}_f$ is equal to the union of the unstable manifolds of $\overleftarrow{\Omega}_f$'s points.*

Proof. Every point has its α-limit set in $\overleftarrow{\Omega}_f$, and so for $n \geqslant 0$ large, the points $\overleftarrow{f}^{-n}(\underline{x})$ are close to $\overleftarrow{\Omega}_f$. By shadowing we show that such $\overleftarrow{f}^{-n}(\underline{x})$ belong to local unstable manifolds of $\overleftarrow{\Omega}_f$'s points. □

As a consequence of the above lemma and condition (T), for every AS endomorphism f, every $x \in \Omega_f$, $n \geqslant 0$, and every $z \in f^{-n}(W^s_\epsilon(x)) \cap M_f$, it holds that:

$$T_z f^n(T_z M) + T_{f^n(z)}\big(W^s_\epsilon(x; f)\big) = T_{f^n(z)} M. \tag{4.1}$$

In other words, each iterate of f is transverse to each local stable manifold.

4.1. Spectral decomposition and filtration

Let f be a C^1-endomorphism of a compact manifold M, and $\overleftarrow{f}$ the associated homeomorphism of the inverse limit $\overleftarrow{M}_f$. Let $(\Lambda_i)_{i=1}^N$ be a family of disjoint, $\overleftarrow{f}$-invariant compact subsets of $\overleftarrow{M}_f$.

A *filtration adapted to* $(\Lambda_i)_i$ is a family of open sets $(U_i)_{i=0}^N$ which satisfies the following properties:

(i) $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_i \subset \cdots \subset U_N = \overleftarrow{M}_f$,
(ii) for every i, $cl(\overleftarrow{f}(U_i)) \subset U_i$,
(iii) for every $1 \leqslant i \leqslant N$, $\Lambda_i = \bigcap_{n\in\mathbb{Z}} \overleftarrow{f}^n(U_i \setminus U_{i-1})$.

We notice that if there exists a filtration adapted to $(\Lambda_i)_i$ then the limit set $\overleftarrow{L}_f$ of $\overleftarrow{f}$ is included in $\bigsqcup_i \Lambda_i$.

Given an $\overleftarrow{f}$-invariant, compact subset Λ, let us define:

$$W^s(\Lambda) := \big\{\underline{x} \in \overleftarrow{M}_f\colon d\big(\overleftarrow{f}^n(\underline{x}), \Lambda\big) \xrightarrow{+\infty} 0\big\}, \qquad W^u(\Lambda) := \big\{\underline{x} \in \overleftarrow{M}_f\colon d\big(\overleftarrow{f}^n(\underline{x}), \Lambda\big) \xrightarrow{-\infty} 0\big\}.$$

Given two $\overleftarrow{f}$-invariant, compact subsets Λ and Λ', we put $\Lambda \succ \Lambda'$ if $W^u(\Lambda) \setminus \Lambda$ intersects $W^s(\Lambda') \setminus \Lambda'$. We remark that if there exists a filtration adapted to $(\Lambda_i)_i$, then the following properties hold:

(a) the limit set $\overleftarrow{L}_f$ of $\overleftarrow{f}$ is included in $\bigsqcup_i \Lambda_i$,
(b) for all i, j, $\Lambda_i \succ \Lambda_j$ implies $i > j$.

Actually the two above properties are also sufficient conditions to have a filtration:

Proposition 4.4. *(See Theorem*[1] *2.3 of [15].) Let $(\Lambda_i)_{i=1}^N$ be a family of disjoint, $\overleftarrow{f}$-invariant compact subsets of $\overleftarrow{M}_f$. If properties* (a) *and* (b) *hold, then there exists a filtration adapted to $(\Lambda_i)_{i=1}^N$.*

As for attractor–repeller f, conditions (a) and (b) are satisfied for $\overleftarrow{R}_f \sqcup \overleftarrow{A}_f$, it holds:

Corollary 4.5. *There exists an open neighborhood U of $\overleftarrow{R}_f$ in $\overleftarrow{M}_f$ such that*:

$$\bigcap_{n\leqslant 0} \overleftarrow{f}^n(U) = \overleftarrow{R}_f, \qquad cl\big(\overleftarrow{f}^{-1}(U)\big) \subset U, \qquad \bigcup_{n\geqslant 0} \overleftarrow{f}^n(U) = \overleftarrow{M}_f \setminus A_f.$$

In general, for an axiom A endomorphism f, let $\bigsqcup_{i=1}^N \Omega_i$ be the splitting of Ω_f into maximal transitive sets. These sets are closed and f-stable: $f(\Omega_i) = \Omega_i$, for every i. Put $\overleftarrow{\Omega}_i := \Omega_i^{\mathbb{Z}} \cap \overleftarrow{M}_f$. We notice that $\overleftarrow{\Omega}_f = \bigsqcup_i \overleftarrow{\Omega}_i$. The families $(\Omega_i)_i$ and $(\overleftarrow{\Omega}_i)_i$ are called the *spectral decomposition* of respectively Ω_f and $\overleftarrow{\Omega}_f$.

[1] Actually the cited theorem asks $\overleftarrow{f}$ to be a homeomorphism of a *manifold*, but this fact is useful uniquely to choose $(U_i)_i$ among the submanifolds with boundary.

Proposition 4.6. *For every AS endomorphism f, we can index $(\overleftarrow{\Omega}_i)_i$ such that for all i, j, $\overleftarrow{\Omega}_i \succ \overleftarrow{\Omega}_j$ implies $i > j$.*

Proof. The proof is done as for diffeomorphisms: it follows from the fact that if $\overleftarrow{\Omega}_i \succ \overleftarrow{\Omega}_j$ then there exist periodic points $\underline{p} \in \overleftarrow{\Omega}_i$ and $\underline{q} \in \overleftarrow{\Omega}_j$ such that $W^u(\underline{p})$ intersects $W^s(\underline{q})$ and that for any $r \in \overleftarrow{\Omega}_j$, $W^u(\underline{q})$ intersects $W^s(\underline{r})$. □

Corollary 4.7. *There exists a filtration adapted to the spectral decomposition $(\overleftarrow{\Omega}_i)_i$ of every AS endomorphism.*

4.2. Stratifications of the inverse limit by laminations

We are going to study the geometry of the inverse limit $\overleftarrow{M}_f$ of AS endomorphisms. This set is in general not a manifold not even a lamination (see for instance Example 2.10). However we are going to stratify it by laminations. This will be suitable to construct the conjugacy in the attractor–repeller case. Let us recall some elements of the lamination theory applied to hyperbolic dynamical systems.

A *lamination* is a secondly countable metric space L locally modeled on open subsets U_i of products of $\mathbb{R}^n$ with locally compact metric spaces T_i (via homeomorphisms called *charts*) such that the changes of coordinates are of the form:

$$\phi_{ij} = \phi_j \circ \phi_i^{-1} : U_i \subset \mathbb{R}^n \times T_i \to U_j \subset \mathbb{R}^n \times T_j,$$
$$(x, t) \mapsto \big(g(x, t), \psi(x, t)\big),$$

where the partial derivative w.r.t. x of g exists and is a continuous function of both x and t, also ψ is locally constant w.r.t. x. A maximal atlas $\mathcal{L}$ of compatible charts is a *lamination structure* on L.

A *plaque* is a component of $\phi_i^{-1}(\mathbb{R}^n \times \{t\})$ for a chart ϕ_i and $t \in T_i$. The *leaf* of $x \in L$ is the union of all the plaques which contain x. A leaf has a structure of manifold of dimension n. The *tangent space* $T\mathcal{L}$ of $\mathcal{L}$ is the vector bundle over L whose fiber $T_x\mathcal{L}$ at $x \in L$ is the tangent space at x of its leaf.

Proposition 4.8. *Let K be a hyperbolic compact, invariant set of an endomorphism f which has a local product structure. Let $\overleftarrow{K} := K^{\mathbb{Z}} \cap \overleftarrow{M}_f$. Then the local unstable manifolds $(W^u_\epsilon(\underline{x}))_{\underline{x} \in \overleftarrow{K}}$ form the plaques of a lamination on $W^u_\epsilon(\overleftarrow{K}) := \bigcup_{\underline{x} \in \overleftarrow{K}} W^u_\epsilon(\underline{x})$. Moreover the local stable manifolds $(W^s_\epsilon(x))_{x \in K}$ form the plaques of a lamination on $W^s_\epsilon(K) := \bigcup_{x \in K} W^s_\epsilon(x)$.*

Proof. The proofs of both statements are similar and so, we shall only show the one regarding $W^u_\epsilon(\overleftarrow{K})$. Let us express some charts of neighborhoods of any $\underline{x} \in \overleftarrow{K}$ that span the laminar structure on $W^u_\epsilon(\overleftarrow{K})$. By the local product structure, for every $\underline{y} \in \overleftarrow{K}$ close to $\underline{x}$, the intersection of $W^u_\epsilon(\underline{y})$ with $W^s_\epsilon(\underline{x})$ is a point $t = [\underline{y}, \underline{x}]$ in $\overleftarrow{K}$. Also we can find a family of homeomorphisms $(\phi_t)_t$ which depend continuously on t and send $W^u_\epsilon(t)$ onto $\mathbb{R}^d$. We notice that the map:

$$\underline{y} \mapsto \big(\phi_t(\underline{y}), t\big) \in \mathbb{R}^d \times W^s_\epsilon(\underline{x})$$

is a homeomorphism which is a chart of lamination. □

Corollary 4.9. *Let Ω_i be a basic piece of an AS endomorphism f. Then the unstable manifolds of points in $\overleftarrow{\Omega}_i$ form the leaves of a lamination on $W^u(\overleftarrow{\Omega}_i)$.*

Proof. We recall that there are open sets U_i and U_{i-1} such that:

$$\overleftarrow{\Omega}_i = \bigcap_n \overleftarrow{f}^n(U_i \setminus U_{i-1}), \qquad cl\big(\overleftarrow{f}(U_i)\big) \subset U_i, \qquad cl\big(\overleftarrow{f}(U_{i-1})\big) \subset U_{i-1}.$$

First let us notice that $W^u(\overleftarrow{\Omega}_i) \subset U_i$. By $\overleftarrow{f}$-invariance of $W^u(\overleftarrow{\Omega}_i)$, it is included in $\bigcap_{n\in\mathbb{Z}} \overleftarrow{f}^n(U_i)$.

Let $x \in W^u(\overleftarrow{\Omega}_i)$, there exists $N \geqslant 0$ such that $\overleftarrow{f}^{-N}(x)$ is included in the interior of U^c_{i-1}. It is also the case for a neighborhood U of x in $W^u(\overleftarrow{\Omega}_i)$:

$$f^{-N}(U) \subset U^c_{i-1} \cap W^u(\overleftarrow{\Omega}_i) \subset U^c_{i-1} \cap \bigcap_{n\in\mathbb{Z}} \overleftarrow{f}^n(U_i).$$

As the decreasing sequence of compact sets $(\overleftarrow{f}^{-p}(U_{i-1}^c) \cap \bigcap_{n\in\mathbb{Z}} \overleftarrow{f}^n(U_i))_{p\geqslant 0}$ converges to $\overleftarrow{\Omega}_i$ in the Hausdorff topology, for p large, $\overleftarrow{f}^{-p}(U)$ lies in a small neighborhood of $\overleftarrow{\Omega}_i$. By shadowing, $\overleftarrow{f}^{-p}(U)$ is included in $W^u_\epsilon(\overleftarrow{\Omega}_i)$. As $\overleftarrow{f}^p$ is a homeomorphism, we can push forward the lamination structure on U. □

The above proof can be also applied for attractor–repeller endomorphisms. Indeed, it uses only the existence of a filtration and of the local product structure (in order to shadow).

Let us recall that by Lemma 4.3, if f is attractor–repeller or AS, then the unstable manifolds of the nonwandering set form a partition of the inverse limit $\overleftarrow{M}_f$. Thus from the results above, the inverse limit $\overleftarrow{M}_f$ is stratified by a finite number of laminations, the leaves of which are unstable manifolds.

As a consequence of the strong transversality, it follows also a similar partition by stable manifolds of a neighborhood of M_f. However, as this construction holds on the manifold M and not on the inverse limit, let us first recall some general definitions and facts about transversality.

We recall that a continuous map g from a lamination $\mathcal{L}$ to a manifold M is *of class* C^1 if its restriction to every plaque of $\mathcal{L}$ is a C^1 map of manifolds and the induced map $Tg : T\mathcal{L} \to TM$ is continuous: the restriction $T_x g : T_x\mathcal{L} \to T_{g(x)}M$ depends continuously on x even transversally.

For instance the restriction of π_0 to any of the above laminations by unstable manifold is of class C^1.

Let $\mathcal{L}'$ be a lamination embedded into M. The lamination $\mathcal{L}$ is *transverse to* $\mathcal{L}'$ *via* g if for every $x \in \mathcal{L}$ such that $g(x)$ belongs to $\mathcal{L}'$, the following equality holds:

$$Tg(T_x\mathcal{L}) + T_{g(x)}\mathcal{L}' = T_{g(x)}M.$$

Claim 4.10. *There exists a lamination $\mathcal{L} \pitchfork_g \mathcal{L}'$ on $\mathcal{L} \cap g^{-1}(\mathcal{L}')$ the plaques of which are intersections of $\mathcal{L}$-plaques with g-preimages of $\mathcal{L}'$-plaques.*

Proof. Let us construct a chart of $\mathcal{L} \pitchfork_g \mathcal{L}'$ for distinguished open sets which cover $\mathcal{L} \cap g^{-1}(\mathcal{L}')$. Let $x \in \mathcal{L} \cap g^{-1}(\mathcal{L}')$, let $\phi : U \to \mathbb{R}^d \times T$ be an $\mathcal{L}$-chart of a neighborhood U of x and let $\phi' : U \to \mathbb{R}^{d'} \times T'$ be an $\mathcal{L}'$-chart of a neighborhood U' of $g(x)$.

For each $t \in T$, let $T'(t)$ be the set of $t' \in T'$ s.t. $g \circ \phi^{-1}(\mathbb{R}^d \times \{t\})$ intersects $\phi'^{-1}(\mathbb{R}^{d'} \times \{t'\})$. Let $P_{t,t'}$ be the g-pull back of this intersection. We notice that $P_{t,t'}$ depends continuously on $(t,t') \in \bigsqcup_{t\in T} T'(t)$ as a C^1-manifold of M. By restricting U, $P_{t,t'}$ is diffeomorphic to $\mathbb{R}^{d+d'-n}$, via a map $\phi_{t,t'} : P_{t,t'} \to \mathbb{R}^{d+d'-n}$ which depends continuously on t, t'. This provides a chart:

$$U \cap g^{-1}(U') \to \mathbb{R}^{d+d'-n} \times \bigsqcup_{t\in T} T'(t),$$
$$x \in P_{t,t'} \to (\phi_{t,t'}(x), (t,t')). \qquad \square$$

The above claim is useful for the following proposition:

Proposition 4.11. *For every AS endomorphism f, for every basic piece Ω_i, there exists a neighborhood V_i of $W^s(\Omega_i) \cap M_f$ which supports a lamination by stable manifolds.*

If f is furthermore attractor–repeller, then a neighborhood of $W^s(A_f) \cap M_f$ supports a lamination by stable manifolds.

Proof. We are going to prove only the second statement on attractor–repeller endomorphisms f since the proof of the first statement is a simple combination of it with the argument using filtration of Corollary 4.9.

We recall that by Proposition 4.8, the set $W^s_\epsilon(A_f)$ supports a structure of lamination by local stable manifolds. On the other hand, the manifold M is a lamination formed by a single leaf. We wish to use Claim 4.10 with $g = f^n$, $\mathcal{L} := W^s_\epsilon(A_f)$ and $\mathcal{L}' = M$.

However f^n is not necessarily transverse to $W^s_\epsilon(A_f)$ of M_f, and so $W^s(A_f) = \bigcup_{n\geqslant 0} f^{-n}(W^s_\epsilon(A_f))$ is in general not endowed with a structure of lamination.

Nevertheless, by Eq. (4.1), transversality occurs at a neighborhood U_n of $M_f \cap f^{-n}(W^s_\epsilon(A_f))$. This implies the existence of a structure of lamination $\mathcal{L}^s$ on $U \cap W^s(A_f)$, with $U := \bigcup_{n\geqslant 0} U_n$. □

Thus $\overleftarrow{M}_f$ is stratified by laminations by unstable manifolds and a superset of M_f is stratified by laminations by stable manifolds. We can even show a frontier condition on these partitions [2].

By transversality condition (T), these laminations are pairwise transverse via π_0. Indeed every $\underline{x}$ in one of these unstable laminations is equal to an iterate $\overleftarrow{f}^n(\underline{y})$ with $\underline{y} \in W^u_\epsilon(\overleftarrow{\Omega})$ and $f^n \circ \pi_0$ is equal to $\pi_0 \circ \overleftarrow{f}^n$. As the unstable and stable laminations are left invariant by $\overleftarrow{f}$ and f respectively, it follows that their transverse intersection is left invariant by $\overleftarrow{f}$. This leads to the following:

Proposition 4.12. *For every AS endomorphism f, every basic pieces Ω_i and Ω_j, the transverse intersection $W^u(\overleftarrow{\Omega}_i) \pitchfork_{\pi_0} W^s(\Omega_j)$ defines a lamination on a subset of $\overleftarrow{M}_f$ left invariant by $\overleftarrow{f}$. The supports of the laminations $(W^u(\overleftarrow{\Omega}_i) \pitchfork_{\pi_0} W^s(\Omega_j))_{i,j}$ form a partition of $\overleftarrow{M}_f$.*

Of course the same holds for attractor–repeller endomorphisms which satisfy condition (T), let us introduce a few notations.

The space $\overleftarrow{M}_f$ is stratified by the three following laminations:

- the 0-dimensional lamination $W^u(\overleftarrow{A}_f) \pitchfork_{\pi_0} W^s(A_f)$ (leaves are points) on $\overleftarrow{A}_f$,
- the 0-dimensional lamination $W^u(\overleftarrow{R}_f) \pitchfork_{\pi_0} W^s(R_f)$ on $\overleftarrow{R}_f$,
- the lamination $\mathcal{L}_f := W^u(\overleftarrow{R}_f) \pitchfork_{\pi_0} W^s(\overleftarrow{A}_f)$ on $\overleftarrow{M}_f \setminus (\overleftarrow{A}_f \sqcup \overleftarrow{R}_f)$.

These laminations are going to be useful to prove structural stability theorems. In [14], structural stability is shown by using Robin's metric on $\overleftarrow{M}_f$:

$$d_\infty(\underline{x}, \underline{y}) = \sup_{i\in\mathbb{Z}} d(x_i, y_i).$$

This metric is actually equivalent to a Riemannian metric on the leaves on the laminations associated to every AS endomorphism f:

Proposition 4.13. *For $\epsilon > 0$ small enough, for every i, j, for every leaf L of $W^u(\overleftarrow{\Omega}_i) \pitchfork_{\pi_0} W^s(\Omega_j)$, the following hold*:

(i) *for every $x \in L$, $y \notin L$, $d_\infty(x, y) \geqslant \epsilon$,*
(ii) *the metric d_∞ restricted to L is equivalent to a Riemannian metric on L.*

Proof. The first statement is an immediate consequence of the expensiveness of the basic pieces.

Let us prove the second statement. As $\overleftarrow{f}$ is an isometry for d_∞, it is sufficient to prove this for $\underline{x} \in W^u_\epsilon(\overleftarrow{\Omega}_i)$. We recall that for ϵ small enough, the restriction $\pi_0|W^u_\epsilon(\underline{x})$ is a C^1-embedding. Then for $\underline{y}$ d_∞-close to $\underline{x}$, x_0 is d-close to y_0. By hyperbolicity of Ω_i, the sequence $(d(x_{-k}, y_{-k}))_{k\geqslant 0}$ decreases exponentially fast to 0. Therefore:

$$d_\infty(\underline{x}, \underline{y}) = \sup_{k\geqslant 0} d(x_k, y_k).$$

For N large enough, the sequence $(d(x_{k+N}, y_{k+N}))_{k\geqslant 0}$ decreases exponentially fast to 0. Therefore:

$$d_\infty(\underline{x}, \underline{y}) = \sup_{0\leqslant k\leqslant N} d(x_k, y_k) = \max_{0\leqslant k\leqslant N} d\big(f^k(x_0), f^k(y_0)\big). \qquad \square$$

5. Proof of structural stability Theorem 2.7

We construct the conjugacy for perturbations of an attractor–repeller endomorphism f which satisfies the strong transversality condition (T).

By Proposition 1 of [13], the hyperbolic continuity theorem holds for the inverse limit of hyperbolic sets. In particular

Corollary 5.1. *For f' C^1-close to f there exists an embedding h of $\overleftarrow{A}_f \sqcup \overleftarrow{R}_f$ onto $\overleftarrow{A}_{f'} \sqcup \overleftarrow{R}_{f'} \subset \overleftarrow{M}_{f'}$, and such that $\overleftarrow{f}' \circ h = h \circ \overleftarrow{f}|\overleftarrow{R}_f \sqcup \overleftarrow{A}_f$. Also h is C^0-close to the canonical inclusion of $\overleftarrow{A}_f \sqcup \overleftarrow{R}_f$ into $M^{\mathbb{Z}}$.*

We are going to extend the conjugacy h to $\overleftarrow{M}_f$. First we need the following:

Proposition 5.2. *For f' C^1-close to f, its inverse limit satisfies $\overleftarrow{M}_{f'} = W^u(\overleftarrow{R}_{f'}) \cup \overleftarrow{A}_{f'}$.*

Proof. For the proof of this proposition, it is convenient to extend canonically the dynamics of $\overleftarrow{f}$ and $\overleftarrow{f}'$ to $M^{\mathbb{Z}}$. Let V_1 and V_2 be small open neighborhoods in $M^{\mathbb{Z}}$ of $\overleftarrow{R}_f$ and $\overleftarrow{A}_f$ respectively. We recall that $\overleftarrow{R}_f$ has a product structure, and so $\overleftarrow{R}_{f'}$ does. By shadowing, V_1 and V_2 satisfy:

$$\bigcap_{n\geqslant 0} \overleftarrow{f}'^n(V_1) \subset W^u_\epsilon(\overleftarrow{R}_{f'}) \quad \text{and} \quad \bigcap_{n\geqslant 0} \overleftarrow{f}'^n(V_2) = W^u_\epsilon(\overleftarrow{A}_{f'}) = \overleftarrow{A}_{f'}.$$

As the ω-limit set is included in $R_f \sqcup A_f$, by compactness there exists N large such that $\overleftarrow{f}^N(M^{\mathbb{Z}}) \subset V_1 \cup \overleftarrow{f}^{-N}(V_2)$. Consequently $\overleftarrow{M}_{f'} \subset \overleftarrow{f}'^N(M^{\mathbb{Z}}) \subset V_1 \cup \overleftarrow{f}'^{-N}(V_2)$, for f' close enough to f. Using the $\overleftarrow{f}'$ invariance of $\overleftarrow{M}_{f'}$, the latter is included in $W^u(\overleftarrow{R}_{f'}) \cup \overleftarrow{A}_{f'}$. □

Now it is possible to follow a similar construction as [6].

By Corollary 4.5, there exists an arbitrarily small open neighborhood U of $\overleftarrow{R}_f$ in $\overleftarrow{M}_f$ such that:

$$cl\big(\overleftarrow{f}^{-1}(U)\big) \subset U \quad \text{and} \quad \bigcap_{n\geqslant 0} \overleftarrow{f}^{-n}(U) = \overleftarrow{R}_f.$$

By shadowing, a sufficiently small U is included in $W^u_\epsilon(\overleftarrow{R}_f)$, where $\epsilon > 0$ is supposed sufficiently small so that $\pi_0 | W^u_{2\epsilon}(\underline{x})$ is a C^1-embedding, for every $\underline{x} \in \overleftarrow{R}_f$. Let $D_f := U \setminus \overleftarrow{f}^{-1}(U)$.

We notice that $\bigcup_{n\in\mathbb{Z}} \overleftarrow{f}^n(D_f) = W^u(\overleftarrow{R}_f) \setminus \overleftarrow{R}_f$. A domain with this last property is called a *fundamental domain for* $W^u(\overleftarrow{R}_f)$.

Let $\partial^- D_f := \overleftarrow{f}^{-1}(cl(D_f) \setminus D_f)$. In the last section we are going to prove the following:

Lemma 5.3. *For f' sufficiently C^1-close to f, there exists a homeomorphism $h_\#$ from a small open neighborhood V of $cl(D_f)$ into $\mathcal{L}_{f'}$ such that*:

(i) *the map $\pi_0 \circ h_\#$ is C^1-close to π_0,*
(ii) *for all $\underline{x} \in \overleftarrow{A}_f$, $\underline{y} \in \overleftarrow{R}_f$, $\underline{z} \in W^s(\underline{x}; \overleftarrow{f}) \cap W^u(\underline{y}; \overleftarrow{f}) \cap D_f$, the point $h_\#(\underline{z})$ belongs to $W^s(h(\underline{x}); \overleftarrow{f}') \cap W^u(h(\underline{y}); \overleftarrow{f}')$,*
(iii) *for every $\underline{z} \in \partial^- D_f$, we have $h_\# \circ \overleftarrow{f}(\underline{z}) = \overleftarrow{f}' \circ h_\#(\underline{z})$.*

We define h on $\mathcal{L}_f$ via the following expression:

$$h : \underline{x} \in \mathcal{L}_f \mapsto \overleftarrow{f}'^n \circ h_\# \circ \overleftarrow{f}^{-n}(\underline{x}), \quad \text{if } \underline{x} \in \overleftarrow{f}^n(D_f),\ n \in \mathbb{Z}.$$

We notice that for every $\underline{x} \in \mathcal{L}_f$, we have:

$$\overleftarrow{f}' \circ h(\underline{x}) = h \circ \overleftarrow{f}(\underline{x}).$$

This expression complements the above definition of h on $\overleftarrow{R}_f$ and $\overleftarrow{A}_f$ as hyperbolic continuation.

It is easy to see that the restriction of h to $\mathcal{L}_f$ is continuous. Moreover for every $\underline{x} \in \overleftarrow{R}_f$, $\underline{y} \in \overleftarrow{A}_f$, the map h sends $W^s(\underline{x}; \overleftarrow{f}) \cap W^u(\underline{y}; \overleftarrow{f})$ into $W^s(h(\underline{x}); \overleftarrow{f}') \cap W^u(h(\underline{y}); \overleftarrow{f}')$. As moreover $\pi_0 \circ h$ is C^1-close to π_0, the map h is injective.

To prove that h sends $\overleftarrow{M}_f$ onto $\overleftarrow{M}_{f'}$, we need to prove first the global continuity of h. In order to do so, it remains only to show that the definition of h on $\mathcal{L}_f$ and the definition of h on $\overleftarrow{R}_f \sqcup \overleftarrow{A}_f$ fit together continuously.

Proof of the continuity at the repeller. Let $(\underline{x}^n)_{n\geqslant 0}$ be a sequence of points in $\mathcal{L}_f$ converging to $\underline{x} \in \overleftarrow{R}_f$. We want to show that $(h(\underline{x}^n))_{n\geqslant 0}$ converges to $h(\underline{x})$.

The union of $W^u_\epsilon(\underline{x}'; \overleftarrow{f})$ among $\underline{x}' \in W^s_\epsilon(\underline{x}; \overleftarrow{f})$ is a distinguish neighborhood of $\underline{x}$ in the lamination $W^u(\overleftarrow{R}_f)$.

Thus, for n large, there exists $\underline{x}'^n \in W^s_\epsilon(\underline{x}; \overleftarrow{f})$ such that $\underline{x}^n$ belongs to $W^u_\epsilon(\underline{x}'^n; \overleftarrow{f})$. Actually for n large, the point $\underline{x}^n$ is much closer to $\underline{x}'^n$ than ϵ. Also $(x'^n)_n$ converges to $\underline{x}$.

As each $\underline{x}'^n$ is in $\overleftarrow{R}_f$, for n sufficiently large, the point $h(\underline{x}^n)$ belongs to $W^u_\epsilon(h(\underline{x}'^n); \overleftarrow{f}')$. By continuity of $h|\overleftarrow{R}_f$ and of the holonomy of $W^u_\epsilon(\overleftarrow{R}_{f'})$, any limit point $\underline{z}$ of $(h(\underline{x}^n))_{n\geqslant 0}$ belongs to $W^u_\epsilon(h(\underline{x}); \overleftarrow{f}')$.

We can do the same proof for the sequence $(\overleftarrow{f}^k(\underline{x}^n))_{n\geqslant 0}$, from which we get that any limit point of $(h\circ\overleftarrow{f}^k(\underline{x}^n))_{n\geqslant 0}$ belongs to $W^u_\epsilon(h(\overleftarrow{f}^k(\underline{x}')); \overleftarrow{f}')$. By using the equality $h\circ\overleftarrow{f}^k(\underline{x}^n) = \overleftarrow{f}'^k\circ h(\underline{x}^n)$ and the continuity of $\overleftarrow{f}'$, we note that the iterate $\overleftarrow{f}'^k(\underline{z})$ is a limit point of $(h\circ\overleftarrow{f}^k(\underline{x}^n))_n$. Thus $\overleftarrow{f}'^k(\underline{z})$ belongs to $W^u_\epsilon(h\circ\overleftarrow{f}^k(\underline{x}); \overleftarrow{f}') = W^u_\epsilon(\overleftarrow{f}'^k\circ h(\underline{x}); \overleftarrow{f}')$ for every $k\geqslant 0$. By expansion along the unstable manifolds, the point $\underline{z}$ must be $h(\underline{x})$. □

Proof of the continuity at the attractor. Let $(\underline{x}^n)_{n\geqslant 0}$ be a sequence of $\mathcal{L}$ approaching to $\underline{x}\in\overleftarrow{A}_f$. We are going to show that $(h(\underline{x}^n))_{n\geqslant 0}$ approaches $h(\underline{x})$, by the same way as above, but this time we work on M.

Indeed, by taking a distinguished neighborhood of the lamination $W^s(A_f)$, we have that any limit point $\underline{z}$ of $(h(\underline{x}^n))_{n\geqslant 0}$ satisfies that $\pi_0(\underline{z})$ belongs to $W^s_\epsilon(\pi_0\circ h(\underline{x}); f')$. The same holds for $\pi_0\circ\overleftarrow{f}^{-k}(\underline{z}) = \pi_{-k}(\underline{z})$: it belongs to $W^s_\epsilon(\pi_{-k}\circ h(\underline{x}); f')$, for every $k\geqslant 0$. By contraction of the stable manifold, this means that $\pi_{-k}(\underline{z})$ is equal to $\pi_{-k}\circ h(\underline{x})$ for every $k\geqslant 0$. In other words, $\underline{z}$ is equal to $h(\underline{x})$. □

Surjectivity of *h*. The proof is not obvious since $W^u(\overleftarrow{R}_{f'})$ is not always connected and lands in the space $\overleftarrow{M}_f$ which is not necessarily a manifold and not even a lamination, but as we saw stratified by laminations. The surjectivity of h is a consequence of the following proposition:

Proposition 5.4. *Let f be an AS C^1-endomorphism. Let f' be a C^1-perturbation of f. If there exists an injective map*:

$$h : \overleftarrow{M}_f \to \overleftarrow{M}_{f'},$$

which is C^0-close to the canonical inclusion $\overleftarrow{M}_f \hookrightarrow M^{\mathbb{Z}}$ and satisfies $\overleftarrow{f}'\circ h = h\circ\overleftarrow{f}$, then h is surjective.

Proof. For every basic piece Ω_i of f, let us show that the image of h contains the unstable set of the corresponding basic piece $\overleftarrow{\Omega}'_i$ of f' given by hyperbolic continuity (given by [13]). As the $\overleftarrow{M}_{f'} = \bigsqcup_i W^u(\overleftarrow{\Omega}'_i)$ by Proposition 5.2, the map h turns out to be surjective.

Let $\hat{\epsilon} > \epsilon > 0$ and suppose that h is sufficiently close to the canonical inclusion, such that by shadowing and expansiveness, for every $\underline{x}\in\overleftarrow{\Omega}_i$, the map h sends the local unstable manifold $W^u_\epsilon(\underline{x})$ to a subset of $W^u_{\hat{\epsilon}}(h(\underline{x}))$ which contains $h(\underline{x})$. As h is a homeomorphism onto its image, its restriction to this manifold is a homeomorphism onto its image which is a manifold of the same dimension. Thus $h(W^u_\epsilon(\underline{x}))$ is an open neighborhood of $h(\underline{x})$ in $W^u_{\hat{\epsilon}}(h(\underline{x}))$. By compactness of $\overleftarrow{\Omega}_i$, there exists $\eta > 0$ such that for every $\underline{x}\in\overleftarrow{\Omega}_i$, the open set $h(W^u_\epsilon(\underline{x}))$ contains $W^u_\eta(h(\underline{x}))$. This implies that the image of h is a superset of $W^u_\eta(\overleftarrow{\Omega}'_i)$ which contains the fundamental domain $W^u_\eta(\overleftarrow{\Omega}'_i)\setminus\overleftarrow{f}'^{-1}(W^u_\eta(\overleftarrow{\Omega}'_i))$ for $W^u(\overleftarrow{\Omega}'_i)$. By conjugacy the image of h contains $W^u(\overleftarrow{\Omega}'_i)$. □

Proof of Lemma 5.3. Let V be a precompact, open neighborhood of D_f in $W^u_\epsilon(\overleftarrow{R}_f)\setminus\overleftarrow{R}_f\subset\mathcal{L}_f$.

Lemma 5.5. *There exists $I : V\supset D_f\to\mathcal{L}_{f'}$ a homeomorphism onto its image such that*:

- *For every $\underline{z}\in V$, the point $I(\underline{z})$ belongs to $W^s(h(\underline{x}); \overleftarrow{f}')\cap W^u(h(\underline{y}); \overleftarrow{f}')$ if $\underline{z}$ belongs to $W^s(\underline{x}; \overleftarrow{f})\cap W^u(\underline{y}; \overleftarrow{f})$, with $\underline{x}\in\overleftarrow{R}_f$ and $\underline{y}\in\overleftarrow{A}_f$,*
- *the map $i_0 := \pi_0\circ I$ is C^1-close to $\pi_0|V$ when f' is close to f.*

Proof. Let $N\geqslant 0$ be such that $\overleftarrow{f}^N(V)$ has its closure in $W^s_\epsilon(\overleftarrow{A}_f)$. Let us first notice that the images by π_0 of $W^u_\epsilon(h(\underline{x}); \overleftarrow{f}')$ and $W^s_\epsilon(h(\underline{y}); \overleftarrow{f}')$ depend continuously on f', $\underline{x}$ and $\underline{y}$ for the C^1-topologies.

For $\underline{z}\in V$, let $L_{\underline{z}}$ be the set of pairs $(\underline{x},\underline{y})\in\overleftarrow{R}_f\times\overleftarrow{A}_f$ such that $\underline{z}$ belongs to $W^u_\epsilon(\underline{x}; \overleftarrow{f})\cap\overleftarrow{f}^{-N}(W^s_\epsilon(\underline{x}; \overleftarrow{f}))$. Put:

$$\mathcal{L}_z := \bigcup_{(\underline{x},\underline{y})\in L_{\underline{z}}} W^u_\epsilon(\underline{x}; \overleftarrow{f})\cap\overleftarrow{f}^{-N}\big(W^s_\epsilon(\underline{y}; \overleftarrow{f})\big) \quad\text{and}\quad \mathcal{L}'_z := \bigcup_{(\underline{x},\underline{y})\in L_{\underline{z}}} W^u_\epsilon\big(h(\underline{x}); \overleftarrow{f}'\big)\cap\overleftarrow{f}'^{-N}\big(W^s_\epsilon\big(h(\underline{y}); \overleftarrow{f}'\big)\big).$$

We remark that $\mathcal{L}'_z$ and $\mathcal{L}_z$ are manifolds, and $\mathcal{L}_z$ contains the $\mathcal{L}_f|V$-leaf of $\underline{z}$. Indeed π_0 embeds $\mathcal{L}_z$ since it embeds $W^u_{2\epsilon}(\underline{x})$, for every $\underline{x}\in\overleftarrow{R}_f$.

To prove the lemma, we endow the lamination $\mathcal{L}_f|V$ immersed by π_0 with a tubular neighborhood, that is a family of C^1-disks $(D_{\underline{z}'})_{\underline{z}'\in\mathcal{L}_f}$ embedded into M, such that for $z \in V$:

- $D_{\underline{z}}$ is transverse to $\pi_0\mathcal{L}_{\underline{z}}$ and satisfies $D_{\underline{z}} \pitchfork_{\pi_0} \mathcal{L}_{\underline{z}} = \{\underline{z}\}$,
- the disks of each small $\mathcal{L}_f$-plaque form the leaves of a C^1-foliation of an open subset of M,
- these foliations depend C^1-continuously transversally on $\mathcal{L}_f|V$.

By [3, Prop. 1.5], any C^1-immersed lamination has a tubular neighborhood.

For f' sufficiently close to f, the submanifold $\pi_0\mathcal{L}'_{\underline{z}}$ intersects $D_{\underline{z}}$ at a unique point $i_0(\underline{z})$, for every $\underline{z} \in V$. By transversality, the map $i_0 : V \to M$ is of class C^1.

We put $I(\underline{z}) := (\pi_0|W^u_\epsilon(h(\underline{x}); \overleftarrow{f}'))^{-1}(i_0(\underline{z}))$, with $\underline{z} \in W^u_\epsilon(\underline{x}; \overleftarrow{f})$. Such a map satisfies the required properties. □

Let W be a small neighborhood of $\partial^- D_f$ such that the closures of W and $\overleftarrow{f}(W)$ are disjoint and included in V.

Let us modify I to a map $h_\#$ which satisfies moreover that for every $\underline{z} \in W$:

$$h_\# \circ \overleftarrow{f}(\underline{z}) = \overleftarrow{f}' \circ h_\#(\underline{z}).$$

We define $h_\#$ on $\overleftarrow{f}(W)$ as equal to I and on W as equal to $h_1 := \overleftarrow{f}'^{-1} \circ I \circ \overleftarrow{f}$.

Between, $h_\#$ will be such that it respects the lamination $\mathcal{L}_{f'}$ and remains C^1-close to I.

To this end, let us define a map $h_2 : V \to M$ equal to i_0 on $\overleftarrow{f}(W)$ and to $\pi_0 \circ h_1$ on W.

Take a C^1-function ρ equal to 1 on W with support in a small neighborhood $\hat{W}$ of W (disjoint from $\overleftarrow{f}(\hat{W})$) in V. Let exp be the exponential map associated to a Riemannian metric of M.

Put:

$$h_2 : \underline{z} \in V \mapsto \begin{cases} \exp_{i_0(\underline{z})}[\rho(\underline{z}) \cdot \exp^{-1}_{i_0(\underline{z})}(\pi_0 \circ h_1(\underline{z}))] & \text{if } \underline{z} \in \hat{W}, \\ i_0(\underline{z}) & \text{otherwise.} \end{cases}$$

The map h_2 is of class C^1 as composition of C^1-maps. Moreover it is C^1-close to π_0 since i_0 and $\pi_0 \circ h_1$ are C^1-close to π_0. In particular, for f' close to f, h_2 is an immersion of the lamination $\mathcal{L}_f|V$. We notice that h_2 sends $\mathcal{L}_f$ plaques included in $W \cup \overleftarrow{f}(W)$ into the π_0-image of $\mathcal{L}_{f'}$-plaques.

In order to construct the map $h_\# : D_f \to \mathcal{L}_{f'}$ from h_2, we take a tubular neighborhood $(D_{\underline{z}})_{\underline{z}\in V}$ of $\mathcal{L}_f$ (see the definition in the proof of the above lemma).

For $\underline{z} \in V$, the point $h_2(\underline{z})$ is close to $\pi_0(\underline{z})$ and so belongs to a unique disk $D_{\underline{z}'}$ with $\underline{z}' \in \mathcal{L}_{\underline{z}}$. Also $\pi_0\mathcal{L}'_{\underline{z}}$ intersects $D_{\underline{z}'}$ at a unique point. Let $h_\#(\underline{z})$ be the preimage of this point by $\pi_0|\mathcal{L}'_{\underline{z}}$.

We note that $h_\#$ sends each $\mathcal{L}_f$-plaque included in V into an $\mathcal{L}_{f'}$-plaque. By smoothness of the holonomy between two transverse sections of a C^1-foliation, the map $\pi_0 \circ h_\#$ is of class C^1. This concludes the proof of Lemma 5.3. □

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