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Ann. I. H. Poincaré – AN 34 (2017) 469–482

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

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A one-dimensional symmetry result for a class of nonlocal semilinear equations in the plane ☆

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Received 13 August 2015; received in revised form 21 January 2016; accepted 28 January 2016

Available online 20 February 2016

Abstract

We consider entire solutions to $\mathcal{L}u = f(u)$ in $\mathbb{R}^2$, where $\mathcal{L}$ is a nonlocal operator with translation invariant, even and compactly supported kernel K. Under different assumptions on the operator $\mathcal{L}$, we show that monotone solutions are necessarily one-dimensional. The proof is based on a Liouville type approach. A variational characterization of the stability notion is also given, extending our results in some cases to stable solutions.

MSC: 45A05; 47G10; 47B34; 35R11

Keywords: Integral operators; Convolution kernels; Nonlocal equations; Stable solutions; One-dimensional symmetry; De Giorgi Conjecture

1. Introduction

In this paper, we consider solutions of an integral equation driven by a nonlocal, linear operator of the form

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} \big(u(x) - u(y)\big)\, K(x - y)\, dy. \tag{1}$$

☆ The research leading to these results has received funding from the European Research Council Grant n. 321186 - ReaDi - "Reaction–Diffusion Equations, Propagation and Modelling" and n. 277749 - EPSILON - "Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities", the PRIN Grant n. 201274FYK7 "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations", the ANR "HAB" and "NONLOCAL" projects (ANR-14-CE25-0013), the Spanish Grant MTM2011-27739-C04-01 and the Catalan Grant 2009SGR345. Part of this work was carried out during a visit by F. Hamel to the Weierstraß Institute, whose hospitality is thankfully acknowledged.

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http://dx.doi.org/10.1016/j.anihpc.2016.01.001
0294-1449/

We suppose that K is a measurable and nonnegative kernel, such that $K(\zeta) = K(-\zeta)$ for a.e. $\zeta \in \mathbb{R}^n$. We consider both integrable and non-integrable kernels K.

We recall that in the past few years, there has been an intense activity in this type of operators, both for their mathematical interest and for their applications in concrete models. In particular, the fractional operators that we consider here can be seen as a compactly supported version of the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$ (and possibly arising from a more general kernel, which is not scale invariant and does not possess equivalent extended problems). Also, convolution operators are nowadays very popular, also in relation with biological models, see, among the others [26,27,30,32].

We consider here solutions u of the semilinear equation

$$\mathcal{L}u = f(u) \quad \text{in } \mathbb{R}^2. \tag{2}$$

Notice that, in the biological framework, the solution u of this equation is often thought as the density of a biological species and the nonlinearity f is a logistic map, which prescribes the birth and death rate of the population. In this setting, the nonlocal diffusion modeled by $\mathcal{L}$ is motivated by the long-range interactions between the individuals of the species.

The goal of this paper is to study the symmetry properties of solutions of (2) in the light of a famous conjecture of De Giorgi arising in elliptic partial differential equations, see [18]. The original problem consisted in the following question:

Conjecture 1.1. *Let u be a bounded solution of*

$$-\Delta u = u - u^3$$

in the whole of $\mathbb{R}^n$, with

$$\partial_{x_n} u(x) > 0 \text{ for any } x \in \mathbb{R}^n.$$

Then, u is necessarily one-dimensional, i.e. there exist $u_\star : \mathbb{R} \to \mathbb{R}$ and $\omega \in \mathbb{R}^n$ such that $u(x) = u_\star(\omega \cdot x)$, for any $x \in \mathbb{R}^n$, at least when $n \le 8$.

The literature has presented several variations of Conjecture 1.1: in particular, a weak form of it has been investigated when the additional assumption

$$\lim_{x_n \to \pm\infty} u(x_1, \dots, x_n) = \pm 1 \tag{3}$$

is added to the hypotheses. When the limit in (3) is uniform with respect to the variables $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, the version of Conjecture 1.1 obtained in this way is due to Gibbons and is related to problems in cosmology.

In spite of the intense activity of the problem, Conjecture 1.1 is still open in its generality. Up to now, Conjecture 1.1 is known to have a positive answer in dimension 2 and 3 (see [2,28] and also [1,5]) and a negative answer in dimension 9 and higher (see [20]). Also, the weak form of Conjecture 1.1 under the limit assumption in (3) was proved, up to the optimal dimension 8, in [35] (see also [25] for more general conditions at infinity), and the version of Conjecture 1.1 under a uniform limit assumption in (3) holds true in any dimension (see [3,6,23]). Since it is almost impossible to keep track in this short introduction of all the research developed on this important topic, we refer to [24] for further details and motivations.

The goal of this paper is to investigate whether results in the spirit of Conjecture 1.1 hold true when the Laplace operator is replaced by the nonlocal operator in (1). We remark that symmetry results in nonlocal settings have been obtained in [8–12,19,36], but all these works dealt with fractional operators with scaling properties at the origin and at infinity (and somehow with nice regularizing effects).

Also, some of the problems considered in the previous works rely on an extension property of the operator that brings the problem into a local (though higher dimensional and either singular or degenerate) problem (see however [7,15] where symmetry results for fractional problems have been obtained without extension techniques).

In this sense, as far as we know, this paper is the first one to take into account kernels that are compactly supported, for which the above regularization techniques do not always hold and for which equivalent local problems are not available. Moreover, the strategy used in our proof is different from the ones already exploited in the nonlocal setting, since it relies directly on a technique introduced by [5] and refined in [2], which reduced the symmetry property of the level sets of a solution to a Liouville type property for an associated equation (of course, differently from the classical case, we will have to deal with equations, and in fact inequalities, of integral type, in which the appropriate simplifications are more involved).

In this paper, we prove the following one-dimensional result in dimension 2. The case of dimension 3, following the approach of Ambrosio and Cabré in the local case for instance would require deeper analysis of optimal energy estimates. Here, and throughout the paper, B_r denotes the open Euclidean ball with radius $r > 0$ and centered at the origin, $B_r(x) = x + B_r$, and χ_E denotes the characteristic function of a set E.

Theorem 1.2. *Let $n = 2$ and let $\mathcal{L}$ be an operator of the form (1), with K satisfying either*

$$m_0 \chi_{B_{r_0}}(\zeta) \le K(\zeta) \le M_0 \chi_{B_{R_0}}(\zeta) \tag{4}$$

or

$$m_0 \chi_{B_{r_0}}(\zeta) \le |\zeta|^{2+2s} K(\zeta) \le M_0 \chi_{B_{R_0}}(\zeta), \tag{5}$$

for any $\zeta \in \mathbb{R}^2$, for some fixed $M_0 \ge m_0 > 0$, $R_0 \ge r_0 > 0$, and $0 < s < 1$ in (5). Let u be a solution of (2), with $u \in C^1(\mathbb{R}^2)$ and $f \in C^{1,\alpha}(\mathbb{R})$. Assume that

$$\partial_{x_2} u(x) > 0 \text{ for any } x \in \mathbb{R}^2. \tag{6}$$

Then, u is necessarily one-dimensional.

The assumptions in (4) and (5) correspond, respectively, to the case of an integrable kernel of convolution type and to the case of a cutoff fractional kernel. For the existence and further properties of one-dimensional solutions of (2) under quite general conditions, see Theorem 3.1(b) in [4], and [14,16]. As far as assumption (5) is concerned, there is no direct reference on the existence of one-dimensional solutions. However, an adaptation of the techniques in [33] could lead to such a result.

We recall that if condition (5) (or, more generally, (H1) below) is assumed, one needs to interpret (1) in the principal value sense, i.e., as customary,

$$\begin{aligned}\mathcal{L}u(x) &:= \text{P.V.} \int_{\mathbb{R}^n} \big(u(x) - u(y)\big) K(x-y)\, dy \\ &:= \lim_{r \to 0} \int_{\mathbb{R}^n \setminus B_r(x)} \big(u(x) - u(y)\big) K(x-y)\, dy.\end{aligned}$$

As a matter of fact, our proof of Theorem 1.2 does not use any special structure of the kernel K, but only relies on the following facts: the kernel K has compact support, and the operator $\mathcal{L}$ satisfies a Harnack inequality. More precisely, we need:

(H1) The operator $\mathcal{L}$ is of the form (1), with the kernel K satisfying $K \ge 0$, $K(\zeta) = K(-\zeta)$ and $K(\zeta) \ge m_0 \chi_{B_{r_0}}(\zeta)$ in $\mathbb{R}^2$ for some $m_0 > 0$ and $r_0 > 0$. Moreover, K has *compact support* in B_{R_0} for some $R_0 > 0$, that is,

$$K \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R_0},$$

and

$$\int_{B_{R_0}} |\zeta|^2 K(\zeta) d\zeta < \infty.$$

(H2) The operator $\mathcal{L}$ satisfies the following *Harnack inequality*: if φ is continuous and positive in $\mathbb{R}^2$ and is a weak solution to $\mathcal{L}\varphi + c(x)\varphi = 0$ in B_R, with $c(x) \in L^\infty(B_1)$ and $\|c\|_{L^\infty(B_R)} \le b$, then

$$\sup_{B_{R/2}} \varphi \le C \inf_{B_{R/2}} \varphi$$

for some constant C depending on $\mathcal{L}$ and b, but independent of φ.

Under these assumptions, we have the following.

Theorem 1.3. *Let $n = 2$, let $\mathcal{L}$ be an operator of the form (1), with K and $\mathcal{L}$ satisfying* (H1) *and* (H2), *and let u be a solution of (2), with $u \in C^1(\mathbb{R}^2)$ and $f \in C^1(\mathbb{R})$. Assume that*

$$\partial_{x_2} u(x) > 0 \text{ for any } x \in \mathbb{R}^2.$$

If K is not integrable, assume in addition that $u \in C^3(\mathbb{R}^2)$. Then, u is necessarily one-dimensional.

When (4) holds, then (H2) follows from the results of Coville (more precisely, Corollary 1.7 in [17]). Similarly, when (5) is in force, then (H2) follows from a suitable generalization of the results in [21] (see Remark 1.5 below). Thus, thanks to the results in [17,21], Theorem 1.2 follows from Theorem 1.3 — the only difference being the regularity assumed on the solution u.

Notice that when the kernel K is non-integrable at the origin, then one expects the operator $\mathcal{L}$ to be regularizing, and thus bounded solutions u to (2) to be at least C^1 (recall that f is $C^1(\mathbb{R})$). Moreover, when f is smooth, then u is expected to be smooth. However, in case that K is integrable at the origin as in (4), then it is not clear if all bounded solutions are in $C^1(\mathbb{R}^2)$, and this is why we need to take this assumption in Theorem 1.2.

Remark 1.4. Notice that one can produce a C^1 solution by the following argument: rewrite equation (2) into the following form:

$$\int_{\mathbb{R}^n} u(y)K(x-y)dy = u(x) - f(u(x)).$$

Hence if K is C^1, then the left hand-side of the equation is also C^1. Therefore, assuming that the map $r \to r - f(r)$ is invertible with a C^1 inverse, leads to a C^1 solution u.

Remark 1.5. Thanks to the results of [21], the Harnack inequality holds for fractional truncated kernels as in (5) — see (2.2)–(2.3) in [21]. Moreover, a straightforward adaptation of their proof allows to take into account the (bounded) zero order term $c(x)$, and thus condition (H2) is satisfied for kernels K satisfying (5).

Harnack inequalities for general nonlocal operators $\mathcal{L}$ have been widely studied and are known for different classes of kernels K; see for instance a rather general form of the Harnack inequality in [21]. Notice that in our case, we need a Harnack inequality with a zero order term in the equation. It has been proved when the integral operator is the pure fractional Laplacian in [13] and refined in [37]. It is by now well known that the Harnack inequality may fail depending on the kernel K under consideration, and a characterization of the classes of kernels for which it holds is out of the scope of this paper. Notice indeed that condition (4) is stronger than (H1), but under the general assumption (H1) then the Harnack inequality in (H2) is not known, and thus needs to be assumed in Theorem 1.3.

The rest of the paper is devoted to the proof of Theorems 1.2 and 1.3. In particular, Section 2 will present the proof these results, making use of suitable algebraic identities and a Liouville type result in a nonlocal setting. Then, in Section 3 we will consider the extension of Theorem 1.3 to stable (instead of monotone) solutions, giving also a variational characterization of stability.[1]

2. Proof of Theorems 1.2 and 1.3

The proofs of Theorems 1.2 and 1.3 are exactly the same. We will prove them at the same time. The first step towards the proof of these results is a suitable algebraic computation, that we express in this result:

[1] This paper is the outcome of two parallel and independent projects developed at the same time for these two classes of operators, see [29,34]. Since the motivation and the techniques used are similar, we thought that it was simpler to merge the two projects into a single, and comprehensive, paper.

Lemma 2.1. *Let u be as in Theorem 1.2 or 1.3. Let $u_i := \partial_{x_i} u$, for $i \in \{1,2\}$, and*

$$v(x) := \frac{u_1(x)}{u_2(x)}. \tag{7}$$

Also, let $\tau \in C_0^\infty(\mathbb{R}^2)$. Then

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \\ &\qquad = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\big(\tau^2(x)-\tau^2(y)\big)\, v(y)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy.\end{aligned} \tag{8}$$

Proof. First, notice that in case (5), since $f \in C^{1,\alpha}$ then $u \in C^{1+2s+\alpha}(\mathbb{R}^2)$. This means that in all cases — either (4) or (5) or (H1) —, the derivatives u_i are regular enough so that $\mathcal{L}u_i$ is well defined pointwise, and hence all the following integrals converge.

We observe that, for any g and h regular enough,

$$\begin{aligned}2\int_{\mathbb{R}^2} \mathcal{L}h(x)\, g(x)\, dx &= 2\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} \big(h(x)-h(y)\big) K(x-y)\, dy\right] g(x)\, dx \\ &= \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} \big(h(x)-h(y)\big) K(x-y)\, dy\right] g(x)\, dx \\ &\quad + \int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2} \big(h(y)-h(x)\big) K(x-y)\, dx\right] g(y)\, dy \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(h(x)-h(y)\big)\big(g(x)-g(y)\big) K(x-y)\, dx\, dy.\end{aligned} \tag{9}$$

By (2), we have that

$$\begin{aligned}f'\big(u(x)\big)\, u_i(x) &= \partial_{x_i}\big(f\big(u(x)\big)\big) \\ &= \partial_{x_i}\big(\mathcal{L}u(x)\big) = \partial_{x_i}\left(\int_{\mathbb{R}^2} \big(u(x)-u(x-\zeta)\big) K(\zeta)\, d\zeta\right) \\ &= \int_{\mathbb{R}^2} \big(u_i(x)-u_i(x-\zeta)\big) K(\zeta)\, d\zeta \\ &= \mathcal{L}u_i(x).\end{aligned} \tag{10}$$

Accordingly,

$$\begin{aligned}&f'(u)\, u_1 u_2 = \big(\mathcal{L}u_1\big)\, u_2 \\ \text{and}\quad &f'(u)\, u_1 u_2 = \big(\mathcal{L}u_2\big)\, u_1.\end{aligned}$$

By subtracting these two identities and using (7), we obtain

$$0 = \big(\mathcal{L}u_1\big)\, u_2 - \big(\mathcal{L}u_2\big)\, u_1 = \big(\mathcal{L}(vu_2)\big)\, u_2 - \big(\mathcal{L}u_2\big)\,(vu_2).$$

Now, we multiply the previous equality by $2\tau^2 v$ and we integrate over $\mathbb{R}^2$. Recalling (9) together with vu_2, we conclude that

$$
\begin{aligned}
0 &= 2\int_{\mathbb{R}^2} \mathcal{L}(vu_2)(x)\,\tau^2(x)v(x)u_2(x)\,dx - 2\int_{\mathbb{R}^2} \mathcal{L}u_2(x)\,\tau^2(x)v^2(x)u_2(x)\,dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)u_2(x)-v(y)u_2(y)\big)\big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,K(x-y)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\big(\tau^2(x)v^2(x)u_2(x)-\tau^2(y)v^2(y)u_2(y)\big)K(x-y)\,dx\,dy \\
&=: I_1 - I_2.
\end{aligned}
$$

By writing

$$
v(x)u_2(x)-v(y)u_2(y) = \big(u_2(x)-u_2(y)\big)\,v(x) + \big(v(x)-v(y)\big)\,u_2(y),
$$

we see that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,v(x)\,K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,u_2(y)\,K(x-y)\,dx\,dy.
\end{aligned} \tag{11}
$$

In the same way, if we write

$$
\begin{aligned}
\tau^2(x)v^2(x)u_2(x)-\tau^2(y)v^2(y)u_2(y) &= \big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,v(x) \\
&\quad + \big(v(x)-v(y)\big)\,\tau^2(y)v(y)u_2(y),
\end{aligned}
$$

we get that

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,v(x)\,K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\big(v(x)-v(y)\big)\,\tau^2(y)v(y)u_2(y)\,K(x-y)\,dx\,dy.
\end{aligned} \tag{12}
$$

By (11) and (12), after a simplification we obtain that

$$
\begin{aligned}
I_1 - I_2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\big(\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y)\big)\,u_2(y)\,K(x-y)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(u_2(x)-u_2(y)\big)\big(v(x)-v(y)\big)\,\tau^2(y)v(y)u_2(y)\,K(x-y)\,dx\,dy.
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
\tau^2(x)v(x)u_2(x)-\tau^2(y)v(y)u_2(y) &= \big(v(x)-v(y)\big)\,\tau^2(x)\,u_2(x) + \\
&\quad + \big(\tau^2(x)-\tau^2(y)\big)\,u_2(x)\,v(y) + \big(u_2(x)-u_2(y)\big)\,\tau^2(y)\,v(y),
\end{aligned}
$$

and so

$$
\begin{aligned}
I_1 - I_2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)^2\,\tau^2(x)\,u_2(x)\,u_2(y)\,K(x-y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x)-v(y)\big)\big(\tau^2(x)-\tau^2(y)\big)\,v(y)\,u_2(x)\,u_2(y)\,K(x-y)\,dx\,dy.
\end{aligned}
$$

This proves (8). □

Now we use a Liouville type approach to prove that solutions v of the integral equation in (8) are necessarily constant (and this is the only step in which the assumption that the ambient space is $\mathbb{R}^2$ plays a crucial role):

Lemma 2.2. *Let u be as in Theorem 1.2 or 1.3, and let $v = u_1/u_2$. Then v is constant.*

Proof. First, by the previous Lemma v satisfies (8) for all $\tau \in C_c^\infty(\mathbb{R}^2)$.

Let $R > 1$, to be taken arbitrarily large in the sequel. Let $\tau := \tau_R \in C_0^\infty(B_{2R})$, such that $0 \le \tau \le 1$ in $\mathbb{R}^2$, $\tau = 1$ in B_R and

$$|\nabla\tau| \le C R^{-1}, \tag{13}$$

for some $C > 0$ independent of $R > 1$. Throughout the proof, C will denote a positive constant which may change from a line to another, but which is independent of $R > 1$. Using (8), and recalling (4), (6) and the support properties of τ, we observe that

$$\begin{aligned} 0 \le J_1 &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \\ &\le \iint_{\mathcal{R}_R} \big|v(x)-v(y)\big| \, \big|\tau(x)-\tau(y)\big| \, \big|\tau(x)+\tau(y)\big| \, |v(y)|\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \\ &=: J_2, \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\mathcal{R}_R := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } |x-y| \le R_0\} \cap \mathcal{S}_R \qquad \text{and} \\ &\mathcal{S}_R := \Big((B_{2R} \times B_{2R}) \setminus (B_R \times B_R)\Big) \cup \Big(B_{2R} \times (\mathbb{R}^2 \setminus B_{2R})\Big) \\ &\qquad \cup \Big((\mathbb{R}^2 \setminus B_{2R}) \times B_{2R}\Big). \end{aligned}$$

Moreover, making use of the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} J_2^2 \le &\iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \big(\tau(x) + \tau(y)\big)^2\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \\ &\cdot \iint_{\mathcal{R}_R} \big(\tau(x) - \tau(y)\big)^2\, v^2(y)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy. \end{aligned} \tag{15}$$

Now we notice that

$$u_2(x) \le C\, u_2(y) \tag{16}$$

for any $(x,y) \in \mathcal{R}_R$, for a suitable $C > 0$, possibly depending on R_0 but independent of $R > 1$ and $(x,y) \in \mathcal{R}_R$. This is a consequence of (10) with $f'(u) \in L^\infty(\mathbb{R}^2)$ and of assumption (H2) applied recursively to some shifts of the continuous and positive function u_2.

From (13), (16) and the assumption $v\, u_2 \in L^\infty(\mathbb{R}^2)$, we obtain that, for any $(x,y) \in \mathcal{R}_R$,

$$\big(\tau(x) - \tau(y)\big)^2 v^2(y)\, u_2(x)\, u_2(y) \le C R^{-2} |x-y|^2\, v^2(y)\, u_2^2(y) \le C R^{-2} |x-y|^2,$$

for some $C > 0$ independent of $R > 1$ (the constant C in the last term may be larger than the one in the second term). Hence, by (4), (H1) and the symmetry in the (x,y) variables,

$$\begin{aligned} &\iint_{\mathcal{R}_R} \big(\tau(x) - \tau(y)\big)^2 v^2(y)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \\ &\qquad \le C\, R^{-2} \iint_{\mathcal{R}_R} |x-y|^2\, K(x-y)\, dx\, dy \end{aligned}$$

$$\leq 2C R^{-2} \int_{B_{2R}} \left[\int_{B_{R_0}} |z|^2 K(z)\, dz \right] dx \leq C,$$

for some $C > 0$. Therefore, recalling (15),

$$J_2^2 \leq C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \big(\tau(x) + \tau(y)\big)^2 u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy. \tag{17}$$

Hence, since

$$\big(\tau(x) + \tau(y)\big)^2 = \tau^2(x) + \tau^2(y) + 2\tau(x)\,\tau(y) \leq 2\tau^2(x) + 2\tau^2(y),$$

we can use the symmetric role played by x and y in (17) and obtain that

$$J_2^2 \leq C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy,$$

up to renaming $C > 0$. So, we insert this information into (14) and we conclude that

$$\begin{aligned} &\left[\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \right]^2 = J_1^2 \\ &\quad \leq J_2^2 \leq C \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy, \end{aligned} \tag{18}$$

for some $C > 0$.

Since $\mathcal{R}_R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ and u_2 and K are nonnegative, we can simplify the estimate in (18) by writing

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \big(v(x) - v(y)\big)^2 \tau^2(x)\, u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \leq C.$$

In particular, since $\tau = 1$ in B_R,

$$\iint_{B_R \times B_R} \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy \leq C.$$

Since C is independent of R, we can send $R \to +\infty$ in this estimate and obtain that the map

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (x, y) \mapsto \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, K(x-y)$$

belongs to $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

Using this and the fact that $\mathcal{R}_R$ approaches the empty set as $R \to +\infty$, we conclude from Lebesgue's dominated convergence theorem that

$$\lim_{R \to +\infty} \iint_{\mathcal{R}_R} \big(v(x) - v(y)\big)^2 u_2(x)\, u_2(y)\, K(x-y)\, dx\, dy = 0.$$

Therefore, going back to (18) and recalling the properties of $\tau = \tau_R$,

$$\left[\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x)-v(y)\big)^2 u_2(x)\,u_2(y)\,K(x-y)\,dx\,dy \right]^2$$

$$= \lim_{R\to+\infty} \left[\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \big(v(x)-v(y)\big)^2 \tau^2(x)\,u_2(x)\,u_2(y)\,K(x-y)\,dx\,dy \right]^2$$

$$\le \lim_{R\to+\infty} C \iint\limits_{\mathcal{R}_R} \big(v(x)-v(y)\big)^2 \tau^2(x)\; u_2(x)\,u_2(y)\,K(x-y)\,dx\,dy.$$

$$= 0.$$

This and (6) imply that $\big(v(x)-v(y)\big)^2 K(x-y) = 0$ for a.e. $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$. Hence, recalling assumption (H1), we have that $v(x)=v(y)$ for any $x\in\mathbb{R}^2$ and any $y\in B_{r_0}(x)$. As a consequence, the set $\{y\in\mathbb{R}^2 \text{ s.t. } v(y)=v(0)\}$ is open and closed in $\mathbb{R}^2$, and so, by connectedness, we obtain that v is constant. □

By combining Lemmata 2.1 and 2.2, we can finish the proof of Theorems 1.2 and 1.3:

Completion of the proof of Theorems 1.2 and 1.3. Using first Lemma 2.1 and then Lemma 2.2, we obtain that v is constant, where v is as in (7). Let us say that $v(x)=a$ for some $a\in\mathbb{R}$. So we define $\omega := \frac{(a,1)}{\sqrt{a^2+1}}$ and we observe that

$$\nabla u(x) = u_2(x)\,(v(x),1) = u_2(x)\,\sqrt{a^2+1}\,\omega.$$

Thus, if $\omega\cdot y=0$ then

$$u(x+y)-u(x) = \int_0^1 \nabla u(x+ty)\cdot y\,dt = \int_0^1 u_2(x+ty)\,\sqrt{a^2+1}\,\omega\cdot y\,dt = 0.$$

Therefore, if we set $u_\star(t) := u(t\omega)$ for any $t\in\mathbb{R}$, and we write any $x\in\mathbb{R}^2$ as

$$x = (\omega\cdot x)\,\omega + y_x$$

with $\omega\cdot y_x = 0$, we conclude that

$$u(x) = u\left((\omega\cdot x)\,\omega + y_x\right) = u\left((\omega\cdot x)\,\omega\right) = u_\star\left(\omega\cdot x\right).$$

This completes the proof of Theorem 1.3. □

It is an interesting open problem to investigate if symmetry results in the spirit of Theorems 1.2 and 1.3 hold true in higher dimension.

3. Stable solutions and extension of the main results

We discuss here the extension of Theorems 1.2 and 1.3 to the more general context of bounded *stable* solutions u of (2) in the whole space $\mathbb{R}^n$ with $n\ge 2$. In the case of second order equations, there are two equivalent definitions of stability: a variational one and a non-variational one. In case of nonlocal operators (1), these two different definitions read as follows.

(S1) The following inequality holds

$$\frac{1}{2}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \big(\xi(x)-\xi(x+y)\big)^2 K(y)\,dy\,dx \ge \int\limits_{\mathbb{R}^n} f'(u)\xi^2$$

for every $\xi\in C_c^\infty(\mathbb{R}^n)$. That is, the second variation of the energy functional associated to (2) is nonnegative under perturbations with compact support in $\mathbb{R}^n$.

(S2) There exists a positive continuous solution $\varphi > 0$ to the linearized equation

$$\mathcal{L}\varphi = f'(u)\varphi \quad \text{in } \mathbb{R}^n. \tag{19}$$

For completeness, we observe that a more general version of Theorems 1.2 and 1.3 holds true, namely if we replace assumption (6) with the following non-variational stability condition (S2).

Theorem 3.1. *Let $n = 2$ and $\mathcal{L}$ be an operator of the form (1), with K satisfying either (4), or (5), or* (H1)–(H2). *Let u be a solution of (2), with $u \in C^1(\mathbb{R}^2)$ and $f \in C^{1,\alpha}(\mathbb{R})$, and with $u \in C^3(\mathbb{R}^2)$ in case* (H1)–(H2). *Assume that u is stable, in the sense of* (S2). *Then, u is necessarily one-dimensional.*

Notice that, in this setting, Theorems 1.2 and 1.3 are a particular case of Theorem 3.1, choosing $\varphi := u_2 = \partial_{x_2} u$ and recalling (10).

The proof of Theorem 3.1 is exactly the one of Theorem 1.3, with only a technical difference: instead of (7), one has to define, for $i \in \{1, 2\}$,

$$v(x) := \frac{u_i(x)}{\varphi(x)}.$$

Then the proof of Theorem 1.3 goes through (replacing u_2 with φ when necessary) and implies that v is constant, i.e. $u_i = a_i\varphi$, for some $a_i \in \mathbb{R}$. This gives that $\nabla u(x) = \varphi(x)\,(a_1, a_2)$, which in turn implies the one-dimensional symmetry of u.

Given the result in Theorem 3.1, we discuss next the equivalence between the two definitions of stability (S1) and (S2). We will always assume that the kernel K satisfies assumption (H1).

Proposition 3.2. *Let $n \geq 1$ and $\mathcal{L}$ be any operator of the form (1). Let u be a bounded solution of (2) in the whole of $\mathbb{R}^n$ with $f \in C^1(\mathbb{R})$. Assume that the kernel K satisfies assumption* (H1). *Then,* (S2) $\Longrightarrow$ (S1).

Proof. Let $\xi \in C_0^\infty(\mathbb{R}^n)$. Using ξ^2/φ as a test function in the equation $\mathcal{L}\varphi = f'(u)\varphi$, we find

$$\int_{\mathbb{R}^n} f'(u)\xi^2 = \int_{\mathbb{R}^n} \frac{\xi^2}{\varphi}\,\mathcal{L}\varphi.$$

Next, we use (9) (which holds in $\mathbb{R}^n$ as in $\mathbb{R}^2$) to see that at least at the formal level for any function v and w such that $\mathcal{L}w$ is well defined and v belongs to $L^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} v\,\mathcal{L}w = \frac{B(v, w)}{2},$$

where

$$B(v, w) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(v(x) - v(y)\big)\big(w(x) - w(y)\big)K(x - y)\,dx\,dy.$$

We find (recall that φ is such that $\mathcal{L}\varphi$ exists and ξ is compactly supported)

$$2\int_{\mathbb{R}^n} f'(u)\xi^2 = B\left(\varphi, \xi^2/\varphi\right).$$

Now, it is immediate to check that

$$\frac{\xi^2(x)}{\varphi(x)} - \frac{\xi^2(y)}{\varphi(y)} = \left(\xi^2(x) - \xi^2(y)\right)\frac{\varphi(x) + \varphi(y)}{2\varphi(x)\varphi(y)} - (\varphi(x) - \varphi(y))\,\frac{\xi^2(x) + \xi^2(y)}{2\varphi(x)\varphi(y)},$$

and this yields

$$2\int_{\mathbb{R}^n} f'(u)\xi^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\varphi(x)-\varphi(y)\big)\Big(\xi^2(x)-\xi^2(y)\Big)\frac{\varphi(x)+\varphi(y)}{2\varphi(x)\varphi(y)}K(x-y)dx\,dy$$
$$-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\varphi(x)-\varphi(y)\big)^2\frac{\xi^2(x)+\xi^2(y)}{2\varphi(x)\varphi(y)}K(x-y)dx\,dy.$$

Let us now show that

$$\Theta(x,y) := \big(\varphi(x)-\varphi(y)\big)\Big(\xi^2(x)-\xi^2(y)\Big)\frac{\varphi(x)+\varphi(y)}{2\varphi(x)\varphi(y)} - \big(\varphi(x)-\varphi(y)\big)^2\frac{\xi^2(x)+\xi^2(y)}{2\varphi(x)\varphi(y)} \le \big(\xi(x)-\xi(y)\big)^2. \tag{20}$$

Once this is proved, then we will have

$$2\int_{\mathbb{R}^n} f'(u)\xi^2 \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\xi(x)-\xi(y)\big)^2K(x-y)dx\,dy,$$

and thus the result will be proved.

To establish (20), it is convenient to write Θ as

$$\Theta(x,y) = 2\big(\varphi(x)-\varphi(y)\big)\big(\xi(x)-\xi(y)\big)\frac{\xi(x)+\xi(y)}{\varphi(x)+\varphi(y)}\cdot\frac{\big(\varphi(x)+\varphi(y)\big)^2}{4\varphi(x)\varphi(y)}$$
$$-\big(\varphi(x)-\varphi(y)\big)^2\cdot\left(\frac{\xi(x)+\xi(y)}{\varphi(x)+\varphi(y)}\right)^2\frac{2\xi^2(x)+2\xi^2(y)}{\big(\xi(x)+\xi(y)\big)^2}\cdot\frac{\big(\varphi(x)+\varphi(y)\big)^2}{4\varphi(x)\varphi(y)}.$$

Now, using the inequality

$$2\big(\varphi(x)-\varphi(y)\big)\big(\xi(x)-\xi(y)\big)\frac{\xi(x)+\xi(y)}{\varphi(x)+\varphi(y)} \le \big(\xi(x)-\xi(y)\big)^2 + \big(\varphi(x)-\varphi(y)\big)^2\cdot\left(\frac{\xi(x)+\xi(y)}{\varphi(x)+\varphi(y)}\right)^2,$$

we find

$$\Theta(x,y) \le \big(\xi(x)-\xi(y)\big)^2\frac{\big(\varphi(x)+\varphi(y)\big)^2}{4\varphi(x)\varphi(y)} +$$
$$+\big(\varphi(x)-\varphi(y)\big)^2\cdot\left(\frac{\xi(x)+\xi(y)}{\varphi(x)+\varphi(y)}\right)^2\cdot\frac{\big(\varphi(x)+\varphi(y)\big)^2}{4\varphi(x)\varphi(y)}\cdot\left\{1-\frac{2\xi^2(x)+2\xi^2(y)}{\big(\xi(x)+\xi(y)\big)^2}\right\}.$$

But since

$$1-\frac{2\xi^2(x)+2\xi^2(y)}{\big(\xi(x)+\xi(y)\big)^2} = -\frac{\big(\xi(x)-\xi(y)\big)^2}{\big(\xi(x)+\xi(y)\big)^2},$$

we obtain

$$\Theta(x,y) \le \big(\xi(x)-\xi(y)\big)^2\frac{\big(\varphi(x)+\varphi(y)\big)^2}{4\varphi(x)\varphi(y)} - \big(\varphi(x)-\varphi(y)\big)^2\cdot\frac{\big(\xi(x)-\xi(y)\big)^2}{4\varphi(x)\varphi(y)}$$
$$= \frac{\big(\xi(x)-\xi(y)\big)^2}{4\varphi(x)\varphi(y)}\left\{\big(\varphi(x)+\varphi(y)\big)^2-\big(\varphi(x)-\varphi(y)\big)^2\right\}$$
$$= \big(\xi(x)-\xi(y)\big)^2.$$

Hence (20) is proved, and the result follows. □

Notice that the previous proposition holds for *any* operator of the form (1), with no additional assumptions on K. However, we do not know if the two stability conditions (S1) and (S2) are equivalent for all operators $\mathcal{L}$. Indeed, in order to show the other implication (S1) $\Longrightarrow$ (S2), we need some additional assumptions. Namely, we need:

if $w \in L^\infty(\mathbb{R}^n)$ is any weak solution to $\mathcal{L}w = g$ in B_1, with $g \in L^\infty(B_1)$, then

$$\|w\|_{C^\alpha(B_{1/2})} \le C\big(\|g\|_{L^\infty(B_1)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big)$$

for some constants $\alpha \in (0, 1]$ and $C > 0$ independent of w and g. (21)

and

the space $H_K(\mathbb{R}^n)$, defined as the closure of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|w\|^2_{H_K(\mathbb{R}^n)} := \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(w(x) - w(y)\big)^2 K(x-y)\,dx\,dy$$

is compactly embedded in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. (22)

Remark 3.3. These two assumptions (21)–(22) are satisfied for all kernels satisfying (5). Indeed, the C^α estimate (21) can be found in [13, Section 14], while the compact embedding (22) follows easily in two steps: fix $p \in \mathbb{R}^n$ and use (5) to have compactness in $L^2(B_{r_0/2}(p))$; then use a standard covering argument to have the compact embedding in B_R (for any $R > 0$). See, for instance, [31] and [22, Theorem 7.1] for further details on the compact embeddings.

Using (21)–(22), we have the following.

Proposition 3.4. *Let $n \ge 1$ and $\mathcal{L}$ be any operator of the form (1) with kernel K satisfying (5). Let u be any bounded solution of (2) in the whole of $\mathbb{R}^n$, with $f \in C^{1,\alpha}(\mathbb{R})$. Then,* (S1) $\Longrightarrow$ (S2)

Proof. Let $R > 0$ and consider the quadratic form

$$\mathcal{Q}_R(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\xi(x) - \xi(y)\big)^2 K(x-y)\,dx\,dy - \int_{B_R} f'(u)\xi^2\,dx,$$

for $\xi \in C_0^\infty(\mathbb{R}^n)$. Let $H_K(\mathbb{R}^n)$ be as in (22) and λ_R be the infimum of $\mathcal{Q}_R$ among the class $\mathcal{S}_R$ defined by

$$\mathcal{S}_R := \left\{ \xi \in H_K(\mathbb{R}^n) \text{ s.t. } \xi = 0 \text{ in } \mathbb{R}^n \setminus B_R \text{ and } \int_{B_R} \xi^2 = 1 \right\}.$$

Since the functional $\mathcal{Q}_R$ is bounded from below in $\mathcal{S}_R$ (recall that $f'(u)$ is bounded) and thanks to the compactness assumption in (22), we see that its infimum λ_R is attained for a function $\phi_R \in \mathcal{S}_R$. Moreover, by assumption (S1), we have

$$\lambda_R \ge 0. \tag{23}$$

Also, we can assume that $\phi_R \ge 0$, since if ϕ is minimizer then $|\phi|$ is also a minimizer. Thus, the function $\phi_R \ge 0$ is a solution, not identically zero, of the problem

$$\begin{cases} \mathcal{L}\phi_R = f'(u)\phi_R + \lambda_R\phi_R & \text{in } B_R, \\ \phi_R = 0 & \text{in } \mathbb{R}^n \setminus B_R. \end{cases}$$

It follows from the strong maximum principle for integro-differential operators (remember that K satisfies (5)) that ϕ_R is continuous in $\mathbb{R}^n$ and $\phi_R > 0$ in B_R. On the other hand, for any $0 < R < R'$ we have

$$\int_{B_{R'}} \phi_R\,\mathcal{L}\phi_{R'} = \int_{B_{R'}} \phi_{R'}\,\mathcal{L}\phi_R < \int_{B_R} \phi_{R'}\,\mathcal{L}\phi_R.$$

The equality above is a consequence of (9) (in $\mathbb{R}^n$), while the inequality follows from the fact that $\phi_R = 0$ in $B_{R'} \setminus B_R$, and thus $\mathcal{L}\phi_R < 0$ in that annulus. Hence, using the equations for ϕ_R and $\phi_{R'}$ we deduce that

$$\lambda_{R'} \int_{B_R} \phi_R \phi_{R'} < \lambda_R \int_{B_R} \phi_R \phi_{R'}.$$

Therefore, $\lambda_{R'} < \lambda_R$ for any $R' > R > 0$. From this and (23), it follows that $\lambda_R > 0$ for all $R > 0$.

Now consider the problem

$$\begin{cases} \mathcal{L}\varphi_R = f'(u)\varphi_R & \text{in } B_R, \\ \varphi_R = c_R & \text{in } \mathbb{R}^n \setminus B_R, \end{cases} \tag{24}$$

for any fixed $c_R > 0$. The solution to this problem can be found by writing $\psi_R = \varphi_R - c_R$, which solves

$$\begin{cases} \mathcal{L}\psi_R = f'(u)\psi_R + c_R f'(u) & \text{in } B_R, \\ \psi_R = 0 & \text{in } \mathbb{R}^n \setminus B_R. \end{cases}$$

It is immediate to check that the energy functional associated to this last problem is bounded from below and coercive, thanks to the inequality $\lambda_R > 0$. Therefore, ψ_R and φ_R exist.

Next we claim that $\varphi_R > 0$ in B_R. To show this, we use φ_R^- as a test function for the equation for φ_R. We find

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(\varphi_R(x) - \varphi_R(y)\big)\big(\varphi_R^-(x) - \varphi_R^-(y)\big) K(x-y)\, dx\, dy \\ &= \int_{B_R} f'(u)\varphi_R \varphi_R^- \\ &= -\int_{B_R} f'(u)|\varphi_R^-|^2. \end{aligned}$$

Now, since

$$\big(\varphi_R(x) - \varphi_R(y)\big)\big(\varphi_R^-(x) - \varphi_R^-(y)\big) \le -\big(\varphi_R^-(x) - \varphi_R^-(y)\big)^2,$$

this yields

$$\mathcal{Q}_R(\varphi_R^-) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(\varphi_R^-(x) - \varphi_R^-(y)\big)^2 K(x-y)\, dx\, dy - \int_{B_R} f'(u)|\varphi_R^-|^2\, dx \le 0.$$

Since $\lambda_R > 0$, this means that φ_R^- vanishes identically, and thus $\varphi_R \ge 0$. Since K satisfies (5), φ_R is then continuous and positive in $\mathbb{R}^n$. The above arguments also imply that the solution φ_R of (24) is unique, whence $(1/c_R)\varphi_R$ is actually independent of $R > 0$. Therefore, one can choose the constant $c_R > 0$ so that $\varphi_R(0) = 1$. Then, by the Hölder regularity in (21) and the Harnack inequality in (H2), we have that, for a sequence $(R_k)_{k \in \mathbb{N}} \to +\infty$, the functions φ_{R_k} converge to a continuous function $\varphi > 0$ in $\mathbb{R}^n$ and satisfying (19). □

Conflict of interest statement

There is no conflict of interest.

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