Regularity in a one-phase free boundary problem for the fractional Laplacian

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Abstract

For a one-phase free boundary problem involving a fractional Laplacian, we prove that "flat free boundaries" are $C^{1,\alpha}$. We recover the regularity results of Caffarelli for viscosity solutions of the classical Bernoulli-type free boundary problem with the standard Laplacian.

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1. Introduction

The purpose of this paper is to answer a question left open in [7] on the regularity of free boundaries for the fractional Laplacian of order α – with $0 < \alpha < 1$, in the particular case $\alpha = 1/2$. Here is the setting: consider g a viscosity solution (this notion will be defined properly later) of the following free boundary problem in the ball $B_1 \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$,

$$\begin{cases} \Delta g = 0, & \text{in } B_1^+(g) := B_1 \setminus \big\{(x,0): g(x,0) = 0\big\}, \\ \dfrac{\partial g}{\partial U} = 1, & \text{on } F(g) := \partial_{\mathbb{R}^n}\big\{x \in \mathcal{B}_1: g(x,0) > 0\big\} \cap \mathcal{B}_1, \end{cases} \tag{1.1}$$

where

$$\frac{\partial g}{\partial U}(x_0) := \lim_{(t,z)\to(0,0)} \frac{g(x_0 + t\nu(x_0), z)}{U(t,z)}, \quad x_0 \in F(g) \tag{1.2}$$

and $\mathcal{B}_r \subset \mathbb{R}^n$ is the n-dimensional ball of radius r (centered at 0).

The function $U(t,z)$ is the harmonic extension of $\sqrt{t^+}$ to the upper half-plane $\mathbb{R}_+^2 = \{(t,z) \in \mathbb{R} \times \mathbb{R}, \ z > 0\}$, reflected evenly across $\{z = 0\}$. Precisely, after the polar change of coordinates

$$t = r\cos\theta, \qquad z = r\sin\theta, \quad r \geqslant 0, \ -\pi \leqslant \theta \leqslant \pi,$$

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U is given by

$$U(t, z) = r^{1/2} \cos \frac{\theta}{2}. \tag{1.3}$$

One can show that if a function $g \geqslant 0$ is harmonic in $B_1^+(g)$ and $F(g)$ is smooth around a point x_0 then $\frac{\partial g}{\partial U}(x_0)$ exists always and is finite. Here $\nu(x_0)$ denotes as usually the normal to $F(g)$ at x_0 pointing toward $\{x:\ g(x, 0) > 0\}$.

In this paper, we introduce the notion of viscosity solutions to (1.1) and prove the following result about the regularity of their free boundaries under appropriate flatness assumptions (for all the relevant definitions see Section 2).

Theorem 1.1. *There exists a universal constant $\bar{\epsilon} > 0$, such that if g is a viscosity solution to* (1.1) *satisfying*

$$\|g - U\|_{L^\infty(\overline{B}_1)} \leqslant \bar{\epsilon}, \tag{1.4}$$

and

$$\{x \in \mathcal{B}_1:\ x_n \leqslant -\bar{\epsilon}\} \subset \big\{x \in \mathcal{B}_1:\ g(x, 0) = 0\big\} \subset \{x \in \mathcal{B}_1:\ x_n \leqslant \bar{\epsilon}\}, \tag{1.5}$$

then $F(g)$ is $C^{1,\alpha}$ in $\mathcal{B}_{1/2}$.

Consequently $\frac{\partial g}{\partial U}$ exists and g is a classical solution to (1.1). Moreover, given a point x_0 on the free boundary $F(g)$ if one knows that a blow-up sequence of g around x_0 "converges" to the function U, then the flatness assumptions (1.4)–(1.5) are satisfied and hence the free boundary is $C^{1,\alpha}$ around that point.

Assumption (1.4) is a (slightly improved) nondegeneracy assumption which is usually true, and certainly satisfied in the framework of [7]. In any case it could be removed, but we keep it for simplicity.

The interest in our free boundary problem (1.1) arises from a natural generalization of the following classical Bernoulli-type one-phase free boundary problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = 1, & \text{on } \Omega \cap \partial\{u > 0\}, \end{cases} \tag{1.6}$$

with Ω a domain in $\mathbb{R}^n$. A pioneering investigation was that of Alt and Caffarelli [1] (variational context), and then Caffarelli [2–4] (viscosity solutions context). See also [8] for a complete survey.

A special class of viscosity solutions to (1.1) (with the constant 1 replaced by a precise constant A) is provided by minimizers to the energy functional

$$J(v, B_1) = \int\limits_{B_1} |\nabla v|^2 \, dx \, dz + \mathcal{L}_{\mathbb{R}^n}\big(\{v > 0\} \cap \mathbb{R}^n \cap B_1\big).$$

Such minimizers have been investigated by Caffarelli, Sire and the second author in [7], where general properties (optimal regularity, nondegeneracy, classification of global solutions), corresponding to those proved by Alt and Caffarelli in [1] for the Bernoulli-type problem (1.6), have been obtained.

As for the next issue, i.e. the regularity of the free boundary, here is what is proved in [7] in the setting of (1.1):

Let $u(x, y, z)$ be a solution of (1.1) *in $B_1 \subset \mathbb{R}^3$. Assume that the free boundary of u is a Lipschitz graph in $\mathcal{B}_1$. Then it is a C^1 graph in $\mathcal{B}_{1/2}$.*

The idea of this result is that (i) one can find two points on each side of 0 where the free boundary is flatter than what is dictated by the Lipschitz constant, (ii) this improvement could be propagated inside a small ball of controlled size. Thus the three-dimensionality of the problem (or, equivalently, the one-dimensionality of the free boundary) is heavily used. Moreover, this argument does not yield the extra Hölder regularity of the derivative – which we believe could itself yield C^∞ regularity of the free boundary. What we propose in this paper is to fill the gap between C^1 and $C^{1,\alpha}$, in arbitrary space dimension.

In view of the results in [7], one knows that the flatness assumptions (1.4), (1.5) in our main Theorem 1.1 are satisfied around each point of the reduced part of the free boundary of a minimizer (see Propositions 4.2 and Theorems 1.2, 1.3 in [7]). We thus obtain the following corollary to Theorem 1.1.

Corollary 1.2. *Let* v *be a local minimizer to*

$$J(v, B_1) = \int\limits_{B_1} |\nabla v|^2 \, dx \, dz + \mathcal{L}_{\mathbb{R}^n}\big(\{v > 0\} \cap \mathbb{R}^n \cap B_1\big).$$

Then the reduced part of the free boundary $F^*(v)$ *is* $C^{1,\alpha}$.

Let us now recall how the fractional Laplacian is involved in (1.1). Consider, for $\alpha \in (0, 1)$, the model (which generalizes (1.6))

$$\begin{cases} (-\Delta)^\alpha u = 0, & \text{in } \Omega \cap \{u > 0\}, \\ \lim\limits_{t \to 0^+} \dfrac{u(x_0 + t\nu(x_0))}{t^\alpha} = \text{const.}, & \text{on } \Omega \cap \partial\{u > 0\}, \end{cases} \tag{1.7}$$

with u defined on the whole $\mathbb{R}^n$ with prescribed values outside of Ω. Recall that, up to a normalization constant

$$(-\Delta)^\alpha u(x) = PV \int\limits_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} \, dy$$

where PV denotes the Cauchy principal value.

When studying local property of the free boundary in (1.7), the non-locality of the fractional Laplace makes the problem quite delicate. To avoid this "contrast" one can make use of an extension property proved by Caffarelli and Silvestre in [10] (see for example the work of Caffarelli, Savin and the second author [6], the paper [7], and the work of Caffarelli, Salsa and Silvestre [9] where this strategy has been employed). Precisely, let $u \in C^2(\mathbb{R}^n)$ and let v solve

$$\begin{cases} -\text{div}\big(z^\beta \nabla v\big) = 0, & \text{in } \mathbb{R}^{n+1}_+ = \big\{(x, z) \in \mathbb{R}^n \times \mathbb{R}, \ z > 0\big\}, \\ v(x, 0) = u(x), & \text{on } \mathbb{R}^n, \end{cases} \tag{1.8}$$

with $\beta = 1 - 2\alpha$. Then,

$$\big(-\Delta^\alpha\big)u(x) = -\lim_{z \to 0}\big(z^\beta v_z(x, z)\big). \tag{1.9}$$

After extending v evenly across the hyperplane $\{z = 0\}$, the first equation in (1.8) can be thought in the whole $\mathbb{R}^{n+1}$. In view of this formula, the focus shifts on the free boundary problem,

$$\begin{cases} -\text{div}\big(|z|^\beta \nabla v\big) = 0, & \text{in } B_1 \setminus \big\{(x, 0) : v(x, 0) = 0\big\}, \\ \lim\limits_{(t,z) \to (0,0)} \dfrac{v(x_0 + t\nu(x_0), z)}{U(t, z)} = \text{const.}, & \text{on } \partial_{\mathbb{R}^n}\big\{x : v(x, 0) > 0\big\} \cap \mathcal{B}_1, \end{cases} \tag{1.10}$$

where $U(t, z)$ solves (1.8) in $\mathbb{R}^2$ with $u(t) = (t^+)^\alpha$ and is extended evenly across $\{z = 0\}$.

For simplicity of exposition we have focused here on the case when $\alpha = 1/2$ (in which case the extension formula of [10] is a well-know fact). However our result can probably be extended to the general case $\alpha \in (0, 1)$.

Our definition of viscosity solution to (1.1) is similar to the one introduced by Caffarelli in [2,3] to deal with the problem (1.6). Indeed our result generalizes to this non-local setting the "flatness implies regularity" theory developed by Caffarelli in [3]. Let us also mention that Theorem 1.1 is probably optimal. Indeed, quite similarly to what happens for minimal surfaces, singular free boundaries for the Bernoulli-type problem (1.6) were discovered by Jerison and the first author [11].

Let us now describe our strategy to obtain Theorem 1.1. The main idea to prove Theorem 1.1 is to show that $F(g)$ enjoys an "improvement of flatness" property, that is if $F(g)$ oscillates ϵ away from a hyperplane in $\mathcal{B}_1$ (ϵ small), then in $\mathcal{B}_\rho$ (ρ universal) it oscillates $\epsilon\rho/2$ away from possibly a different hyperplane. To obtain this improvement of flatness, we use a compactness argument which goes as follows: assume one cannot do it however flat the free boundary is, then we blow it up in the x_n-direction, thus linearizing the problem into a limiting one, for which we prove that improvement of flatness holds – thus a contradiction. This scheme was used by Savin [13] to prove regularity of small solutions of fully nonlinear equations – including an elegant proof of the De Giorgi theorem for minimal surfaces. The key tool is a geometric Harnack inequality that localizes the free boundary well, and allows the passage to the limit under rescalings.

Such compactness arguments can also be found in Wang [14] for the regularity of the solutions of p-Laplace equations. More recently, the first author followed this strategy in [12] to provide a new proof of the Caffarelli "flat implies smooth" theory. The scheme is the same here, up to the fact that the construction of the sub-solution opening the way to the Harnack inequality is different from that of [12], and that the linear problem obtained eventually is non-standard (and interesting in its own).

The paper is organized as follows. In Section 2 we introduce the notion of viscosity solutions to (1.1) and we prove a basic comparison principle for such solutions. In Section 3 we explain how to interpret our solutions as perturbations of U after a "domain variation" in the e_n-direction and we present basic facts about such domain variations. Throughout the paper, this will be a convenient way of thinking about our viscosity solutions. In Section 4 we describe the linear problem associated to (1.1) and later in Section 8 we obtain a regularity result for its solutions. Section 5 contains some technical lemmas leading to the proof of Harnack inequality. In Section 6 we exhibit the proof of Harnack inequality using the barrier which we will construct later in Appendix A. Finally in Section 7 we provide the proof of the "improvement of flatness" property.

2. Definitions and basic lemmas

In this section we introduce notation and definitions which we will use throughout the paper and we prove a standard basic lemma (Comparison Principle).

A point $X \in \mathbb{R}^{n+1}$ will be denoted by $X = (x, z) \in \mathbb{R}^n \times \mathbb{R}$. We will also use the notation $x = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$. A ball in $\mathbb{R}^{n+1}$ with radius r and center X is denoted by $B_r(X)$ and for simplicity $B_r = B_r(0)$. Also we use $\mathcal{B}_r$ to denote the n-dimensional ball $B_r \cap \{z = 0\}$.

Let $v(X)$ be a continuous non-negative function in B_1. We associate to v the following sets:

$$B_1^+(v) := B_1 \setminus \{(x, 0): v(x, 0) = 0\} \subset \mathbb{R}^{n+1};$$
$$\mathcal{B}_1^+(v) := B_1^+(v) \cap \mathcal{B}_1 \subset \mathbb{R}^n;$$
$$F(v) := \partial_{\mathbb{R}^n} \mathcal{B}_1^+(v) \cap \mathcal{B}_1 \subset \mathbb{R}^n;$$
$$\mathcal{B}_1^0(v) := Int_{\mathbb{R}^n} \{x \in \mathbb{R}^n: v(x, 0) = 0\} \subset \mathbb{R}^n.$$

Often subsets of $\mathbb{R}^n$ are embedded in $\mathbb{R}^{n+1}$, as it will be clear from the context.

We may refer to $\mathcal{B}_1^0(v)$ as to the zero plate of v, while $F(v)$ is called the free boundary of v.

We consider the free boundary problem

$$\begin{cases} \Delta g = 0, & \text{in } B_1^+(g), \\ \dfrac{\partial g}{\partial U} = 1, & \text{on } F(g), \end{cases} \tag{2.1}$$

where

$$\frac{\partial g}{\partial U}(x_0) := \lim_{(t,z) \to (0,0)} \frac{g(x_0 + t\nu(x_0), z)}{U(t, z)}, \qquad X_0 = (x_0, 0) \in F(g).$$

Here $\nu(x_0)$ denotes the unit normal to $F(g)$ at x_0 pointing toward $\mathcal{B}_1^+(g)$ and U is the function defined in (1.3). Also, throughout the paper we call $U(X) := U(x_n, z)$.

We now introduce the notion of viscosity solutions to (2.1). First we need the following standard notion.

Definition 2.1. Given g, v continuous, we say that v touches g by below (resp. above) at $X_0 \in B_1$ if $g(X_0) = v(X_0)$, and

$$g(X) \geqslant v(X) \qquad \big(\text{resp.} g(X) \leqslant v(X)\big) \quad \text{in a neighborhood } O \text{ of } X_0.$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that v touches g strictly by below (resp. above).

Definition 2.2. We say that $v \in C^2(B_1)$ is a (strict) comparison subsolution to (2.1) if v is a non-negative function in B_1 which is even with respect to $\{z = 0\}$ and it satisfies

(i) $\Delta v \geqslant 0$ in $B_1^+(v)$;
(ii) $F(v)$ is C^2 and if $x_0 \in F(v)$ we have

$$v(x, z) = \alpha U\big((x - x_0) \cdot \nu(x_0), z\big) + o\big(\big|(x - x_0, z)\big|^{1/2}\big), \quad \text{as } (x, z) \to (x_0, 0),$$

with

$$\alpha \geqslant 1,$$

where $\nu(x_0)$ denotes the unit normal at x_0 to $F(v)$ pointing toward $\mathcal{B}_1^+(v)$;
(iii) Either v is not harmonic in $B_1^+(v)$ or $\alpha > 1$.

Similarly one can define a (strict) comparison supersolution.

Definition 2.3. We say that g is a viscosity solution to (2.1) if g is a continuous non-negative function in B_1 which is even with respect to $\{z = 0\}$ and it satisfies

(i) $\Delta g = 0$ in $B_1^+(g)$;
(ii) Any (strict) comparison subsolution (resp. supersolution) cannot touch g by below (resp. by above) at a point $X_0 = (x_0, 0) \in F(g)$.

Remark 2.4. By standard arguments, if g is a viscosity solution to (2.1) and $F(g)$ is C^1 then g is a classical solution of the free boundary problem (see for example Proposition 4.2 in [7]). Moreover, as remarked in the Introduction one can show that given any continuous function g which is harmonic in $B_1^+(g)$, then $\frac{\partial g}{\partial U}(x_0)$ exists at each point around which $F(g)$ is $C^{1,\alpha}$. These facts motivate our problem and the definition of viscosity solution.

Remark 2.5. We remark that if g is a viscosity solution to (2.1) in B_ρ, then

$$g_\rho(X) = \rho^{-1/2} g(\rho X), \quad X \in B_1$$

is a viscosity solution to (2.1) in B_1.

We finish this section by stating and proving a comparison principle for problem (2.1) which will be a key tool in the proof of Harnack inequality in Section 7.

Lemma 2.6 (*Comparison Principle*). *Let* $g, v_t \in C(\overline{B}_1)$ *be respectively a solution and a family of subsolutions to* (2.1), $t \in [0, 1]$. *Assume that*

(i) $v_0 \leqslant g$, *in* $\overline{B}_1$;
(ii) $v_t \leqslant g$ *on* ∂B_1 *for all* $t \in [0, 1]$;
(iii) $v_t < g$ *on* $\mathcal{F}(v_t)$ *which is the boundary in* ∂B_1 *of the set* $\partial \mathcal{B}_1^+(v_t) \cap \partial \mathcal{B}_1$, *for all* $t \in [0, 1]$;
(iv) $v_t(x)$ *is continuous in* $(x, t) \in \overline{B}_1 \times [0, 1]$ *and* $\overline{\mathcal{B}_1^+(v_t)}$ *is continuous in the Hausdorff metric.*

Then

$$v_t \leqslant g \quad \text{in } \overline{B}_1, \text{ for all } t \in [0, 1].$$

Proof. Let

$$A := \big\{t \in [0, 1]: v_t(x) \leqslant g(x) \text{ on } \overline{B}_1\big\}.$$

In view of (i) and (iv) A is closed and non-empty. Our claim will follow if we show that A is open. Let $t_0 \in A$, then $v_{t_0} \leqslant g$ on $\overline{B}_1$ and by the definition of viscosity solution

$$F(v_{t_0}) \cap F(g) = \emptyset.$$

Together with (iii) this implies that

$$\mathcal{B}_1^+(v_{t_0}) \subset \mathcal{B}_1^+(g), \qquad F(v_{t_0}) \cup \mathcal{F}(v_{t_0}) \subset \{x \in \overline{\mathcal{B}}_1 \colon g(x,0) > 0\}.$$

By (iv) this gives that for t close to t_0

$$\mathcal{B}_1^+(v_t) \subset \mathcal{B}_1^+(g), \qquad F(v_t) \cup \mathcal{F}(v_t) \subset \{x \in \overline{\mathcal{B}}_1 \colon g(x,0) > 0\}. \tag{2.2}$$

Call $D := B_1 \setminus (\mathcal{B}_1^0(v_t) \cup F(v_t))$. Combining (2.2) with assumption (ii) we get that

$$v_t \leqslant g \quad \text{on } \partial D,$$

and by the maximum principle the inequality holds also in D. Hence

$$v_t \leqslant g \quad \text{in } \overline{B}_1,$$

and $t \in A$ which shows that A is open. $\quad\square$

Corollary 2.7. *Let g be a solution to* (2.1) *and let v be a subsolution to* (2.1) *in B_2 which is strictly monotone increasing in the e_n-direction in $B_2^+(v)$. Call*

$$v_t(X) := v(X + te_n), \quad X \in B_1.$$

Assume that for $-1 \leqslant t_0 < t_1 \leqslant 1$

$$v_{t_0} \leqslant g, \quad \text{in } \overline{B}_1,$$

and

$$v_{t_1} \leqslant g \quad \text{on } \partial B_1, \qquad v_{t_1} < g \quad \text{on } \mathcal{F}(v_{t_1}).$$

Then

$$v_{t_1} \leqslant g \quad \text{in } \overline{B}_1.$$

3. The function $\tilde{g}$

Let g be a viscosity solution to (2.1). Throughout the paper, it will be convenient to interpret g as a perturbation of U via a domain variation in the e_n-direction. In this section we explain some basic facts about such domain variations.

Let $\epsilon > 0$ and let g be a continuous non-negative function in $\overline{B}_\rho$. Here and henceforth we denote by P and L the half-hyperplanes $P := \{X \in \mathbb{R}^{n+1} \colon x_n \leqslant 0, \ z = 0\}$ and $L := \{X \in \mathbb{R}^{n+1} \colon x_n = 0, \ z = 0\}$. To each $X \in \mathbb{R}^{n+1} \setminus P$ we associate $\tilde{g}_\epsilon(X) \subset \mathbb{R}$ via the formula

$$U(X) = g(X - \epsilon w e_n), \quad \forall w \in \tilde{g}_\epsilon(X). \tag{3.1}$$

We sometimes call $\tilde{g}_\epsilon$ the ϵ-domain variation associated to g. By abuse of notation, from now on we write $\tilde{g}_\epsilon(X)$ to denote any of the values in this set.

If g satisfies

$$U(X - \epsilon e_n) \leqslant g(X) \leqslant U(X + \epsilon e_n) \quad \text{in } B_\rho, \tag{3.2}$$

then

$$\tilde{g}_\epsilon(X) \in [-1, 1].$$

Indeed call

$$Y = X - \epsilon \tilde{g}_\epsilon(X) e_n, \quad X \in \mathbb{R}^{n+1} \setminus P.$$

Then according to (3.2),

$$U(Y - \epsilon e_n) \leqslant g(Y) = U\big(Y + \epsilon \tilde{g}_\epsilon(X) e_n\big) \leqslant U(Y + \epsilon e_n).$$

Since $U(Y + \epsilon \tilde{g}_\epsilon(X) e_n) = U(X) > 0$ our claim follows from the strict monotonicity of U in the e_n-direction (outside of P).

Moreover, under the assumption (3.2) for each $X \in B_{\rho-\epsilon} \setminus P$ there exists at least one value $\tilde{g}_\epsilon(X)$ such that

$$U(X) = g\big(X - \epsilon \tilde{g}_\epsilon(X)e_n\big). \tag{3.3}$$

Indeed, it follows from (3.2) that

$$g(X - \epsilon e_n) \leqslant U(X) \leqslant g(X + \epsilon e_n), \quad X \in B_{\rho-\epsilon}$$

and our claim follows by the continuity of $g(X - \delta\epsilon e_n)$, $\delta \in [-1, 1]$.

Thus if (3.2) holds, for all $\epsilon > 0$ we can associate to g a possibly multi-valued function $\tilde{g}_\epsilon$ defined at least on $B_{\rho-\epsilon} \setminus P$ and taking values in $[-1, 1]$ which satisfies (3.3). Moreover if g is strictly monotone in the e_n-direction in $B_\rho^+(g)$, then $\tilde{g}_\epsilon$ is single-valued.

The following elementary lemma will be used to obtain a useful comparison principle for the ϵ-domain variations of solutions to (2.1).

Lemma 3.1. *Let g, v be non-negative continuous functions in B_ρ. Assume that g satisfies the flatness condition* (3.2) *in B_ρ and that v is strictly increasing in the e_n-direction in $B_\rho^+(v)$. Then if*

$$v \leqslant g \quad in \ B_\rho,$$

and $\tilde{v}_\epsilon$ is defined on $B_{\rho-\epsilon} \setminus P$ we have that

$$\tilde{v}_\epsilon \leqslant \tilde{g}_\epsilon \quad on \ B_{\rho-\epsilon} \setminus P.$$

Vice versa, if $\tilde{v}_\epsilon$ is defined on $B_s \setminus P$ and

$$\tilde{v}_\epsilon \leqslant \tilde{g}_\epsilon \quad on \ B_s \setminus P,$$

then

$$v \leqslant g \quad on \ B_{s-\epsilon}.$$

Proof. The first implication is obvious. Indeed, assume by contradiction that $v \leqslant g$ in B_ρ and there exists $X \in B_{\rho-\epsilon} \setminus P$ such that

$$\tilde{v}_\epsilon(X) > \tilde{g}_\epsilon(X).$$

By the strict monotonicity of v in the e_n-direction in $B_\rho^+(v)$ we have that

$$0 < U(X) = g\big(X - \epsilon \tilde{g}_\epsilon(X)e_n\big) = v\big(X - \epsilon \tilde{v}_\epsilon(X)e_n\big) < v\big(X - \epsilon \tilde{g}_\epsilon(X)e_n\big).$$

Thus there exists $Y = X - \epsilon \tilde{g}_\epsilon(X)e_n \in B_\rho$ such that $g(Y) < v(Y)$, a contradiction. Vice versa, suppose that $\tilde{v}_\epsilon \leqslant \tilde{g}_\epsilon$ in $B_s \setminus P$. For a fixed $Y \in B_{s-\epsilon}$ we know by the flatness assumption (3.2) that

$$U(Y - \epsilon e_n) \leqslant g(Y) \leqslant U(Y + \epsilon e_n).$$

Thus, there exists $X \in B_s$ with $x_i = y_i$ for $i \neq n$ and $x_n \in [y_n - \epsilon e_n, y_n + \epsilon e_n]$ such that

$$g(Y) = U(X).$$

Suppose $g(Y) \neq 0$, then the identity above means that one of the possible values of $\tilde{g}_\epsilon(X) = \frac{y_n - x_n}{\epsilon}$. Again, using that v is increasing in the e_n-direction we get:

$$g(Y) = U(X) = v\big(X - \epsilon \tilde{v}_\epsilon(X)e_n\big) \geqslant v\big(X - \epsilon \tilde{g}_\epsilon(X)e_n\big) = v(Y), \quad Y \in B_s^+(g).$$

Thus the desired inequality holds in $B_s^+(g)$ and hence by continuity it holds in the full ball B_s. $\quad\square$

We now state and prove the desired comparison principle, which will follow immediately from the lemma above and Corollary 2.7.

Lemma 3.2. *Let g, v be respectively a solution and a subsolution to (2.1) in B_2, with v strictly increasing in the e_n-direction in $B_2^+(v)$. Assume that g satisfies the flatness assumption (3.2) in B_2 for $\epsilon > 0$ small and that $\tilde{v}_\epsilon$ is defined in $B_{2-\epsilon} \setminus P$ and satisfies*

$$|\tilde{v}_\epsilon| \leqslant C.$$

If,

$$\tilde{v}_\epsilon + c \leqslant \tilde{g}_\epsilon \quad in \ (B_{3/2} \setminus \overline{B}_{1/2}) \setminus P, \tag{3.4}$$

then

$$\tilde{v}_\epsilon + c \leqslant \tilde{g}_\epsilon \quad in \ B_{3/2} \setminus P. \tag{3.5}$$

Proof. We wish to apply Corollary 2.7 to the functions g and

$$v_{\epsilon t} = v(X + \epsilon t e_n).$$

We need to verify that for some $t_0 < t_1 = c$

$$v_{\epsilon t_0} \leqslant g \quad in \ \overline{B}_1, \tag{3.6}$$

and for all $\delta > 0$ and small

$$v_{\epsilon t_1} \leqslant g \quad on \ \partial B_1, \qquad v_{\epsilon t_1} < g \quad on \ \mathcal{F}(v_{\epsilon(t_1 - \delta)}). \tag{3.7}$$

Then our corollary implies

$$v_{\epsilon(t_1 - \delta)} \leqslant g \quad in \ \overline{B}_1.$$

By letting δ go to 0, we obtain that

$$v_{\epsilon t_1} \leqslant g \quad in \ \overline{B}_1,$$

which in view of Lemma 3.1 gives

$$\widetilde{(v_{\epsilon t_1})}_\epsilon \leqslant \tilde{g}_\epsilon \quad in \ B_{1-\epsilon} \setminus P,$$

assuming that the ϵ-domain variation on the left-hand side exists on $B_{1-\epsilon} \setminus P$. On the other hand, it is easy to verify that on such set

$$\widetilde{(v_{\epsilon t})}_\epsilon(X) = \tilde{v}_\epsilon(X) + t, \tag{3.8}$$

and hence we have

$$\tilde{v}_\epsilon + c = \tilde{v}_\epsilon + t_1 \leqslant \tilde{g}_\epsilon \quad in \ B_{1-\epsilon} \setminus P,$$

which gives the desired conclusion. We are left with the proof of (3.6)–(3.7).

In view of Lemma 3.1, in order to obtain (3.6) it suffices to show that

$$\widetilde{(v_{\epsilon t_0})}_\epsilon \leqslant \tilde{g}_\epsilon, \quad in \ B_{1+\epsilon} \setminus P,$$

which by (3.8) becomes

$$\tilde{v}_\epsilon + t_0 \leqslant \tilde{g}_\epsilon, \quad in \ B_{1+\epsilon} \setminus P.$$

This last inequality holds trivially since $\tilde{g}_\epsilon$ and $\tilde{v}_\epsilon$ are bounded.

For (3.7), notice that the first inequality follows easily from our assumption (3.4) together with (3.8) and Lemma 3.1. More precisely we have that

$$v_{\epsilon t_1} \leqslant g \quad in \ B_{\frac{3}{2}-\epsilon} \setminus B_{\frac{1}{2}+\epsilon}.$$

In particular, from the strict monotonicity of v in the e_n-direction in $B_2^+(v)$ we have that

$$v_{\epsilon t_1} > 0 \quad on \ \mathcal{F}(v_{\epsilon(t_1 - \delta)}),$$

which combined with the previous inequality gives that

$$g > 0 \quad \text{on } \mathcal{F}(v_{\epsilon(t_1 - \delta)}),$$

that is the second condition in (3.7). □

Finally, given $\epsilon > 0$ small and a Lipschitz function $\tilde{\varphi}$ defined on $B_\rho(\overline{X})$, with values in $[-1, 1]$, then there exists a unique function φ_ϵ defined at least on $B_{\rho-\epsilon}(\overline{X})$ such that

$$U(X) = \varphi_\epsilon\big(X - \epsilon\tilde{\varphi}(X)e_n\big), \quad X \in B_\rho(\overline{X}). \tag{3.9}$$

Moreover such function φ_ϵ is increasing in the e_n-direction.

With a similar argument as in Lemma 3.1 we can conclude that if g satisfies the flatness assumption (3.2) in B_1 and $\tilde{\varphi}$ is as above then (say $\rho, \epsilon < 1/4$, $\overline{X} \in B_{1/2}$,)

$$\tilde{\varphi} \leqslant \tilde{g}_\epsilon \quad \text{in } B_\rho(\overline{X}) \setminus P \quad \Rightarrow \quad \varphi_\epsilon \leqslant g \quad \text{in } B_{\rho-\epsilon}(\overline{X}). \tag{3.10}$$

We will use this fact in the proof of our improvement of flatness theorem.

4. The linearized problem

We introduce here the linearized problem associated to (2.1). Here and later U_n denotes the x_n-derivative of the function U defined in (1.3). Recall also that we denote by P and L the half-hyperplanes $P := \{X \in \mathbb{R}^{n+1}: x_n \leqslant 0, z = 0\}$ and $L := \{X \in \mathbb{R}^{n+1}: x_n = 0, z = 0\}$.

Given $w \in C(B_1)$ and $X_0 = (x_0', 0, 0) \in B_1 \cap L$, we call

$$|\nabla_r w|(X_0) := \lim_{(x_n, z) \to (0,0)} \frac{w(x_0', x_n, z) - w(x_0', 0, 0)}{r}, \quad r^2 = x_n^2 + z^2.$$

Once the change of unknowns (3.1) has been done, the linearized problem associated to (2.1) is

$$\begin{cases} \Delta(U_n w) = 0, & \text{in } B_1 \setminus P, \\ |\nabla_r w| = 0, & \text{on } B_1 \cap L. \end{cases} \tag{4.1}$$

As we will show later in Section 8, if $w \in C(B_1)$ satisfies

$$\Delta(U_n w) = 0 \quad \text{in } B_1 \setminus P,$$

w is even with respect to $\{z = 0\}$, and w is smooth in the x'-direction, then given $X_0 = (x_0', 0, 0) \in B_1 \cap L$,

$$w(X) = w(X_0) + a \cdot \big(x' - x_0'\big) + br + O\big(|x' - x_0'|^2 + r^{3/2}\big), \tag{4.2}$$

with $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$ depending on X_0.

This motivates our notion of viscosity solution for this problem which we define below.

Definition 4.1. We say that w is a solution to (4.1) if $w \in C(B_1)$, w is even with respect to $\{z = 0\}$ and it satisfies

(i) $\Delta(U_n w) = 0$ in $B_1 \setminus P$;
(ii) Let ϕ be continuous around $X_0 = (x_0', 0, 0) \in B_1 \cap L$ and satisfy

$$\phi(X) = \phi(X_0) + a(X_0) \cdot \big(x' - x_0'\big) + b(X_0)r + O\big(|x' - x_0'|^2 + r^{3/2}\big),$$

with

$$b(X_0) \neq 0.$$

If $b(X_0) > 0$ then ϕ cannot touch w by below at X_0, and if $b(X_0) < 0$ then ϕ cannot touch w by above at X_0.

In Section 8, we will show the following main regularity result about viscosity solutions to (4.1).

Theorem 4.2 *(Improvement of flatness). There exists a universal constant C such that if w is a viscosity solution to (8.1) in B_1 with*

$$-1 \leqslant w(X) \leqslant 1 \quad \text{in } B_1,$$

then

$$a_0 \cdot x' - C|X|^{3/2} \leqslant w(X) - w(0) \leqslant a_0 \cdot x' + C|X|^{3/2},$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

We conclude this short section with a remark which we will use in the proof of the theorem above.

Lemma 4.3. *Let $w_1, w_2 \in C(B_1)$ satisfy*

$$\Delta(U_n w_i) = 0, \quad \text{in } B_1 \setminus P, \ i = 1, 2.$$

Then w_1 and w_2 cannot touch (either by above or below) on $P \setminus L$, unless they coincide.

Proof. Assume by contradiction that

$$w_1(X_0) = w_2(X_0), \quad X_0 \in P \setminus L,$$

and

$$w_1 \geqslant w_2, \quad \text{in } B_\rho(X_0).$$

Then $U_n(w_1 - w_2)$ is a non-negative harmonic function in $B_1 \setminus P$ which vanishes continuously on $P \setminus L$. Hence unless $w_1 = w_2$, by the boundary Harnack inequality (in the appropriate domain),

$$U_n(w_1 - w_2) \geqslant \delta U_n \quad \text{in } B_{\rho/2}(X_0) \cap \{z > 0\},$$

for some small positive constant δ. Thus

$$w_1 - w_2 \geqslant \delta \quad \text{in } B_{\rho/2}(X_0) \cap \{z > 0\},$$

and by continuity

$$(w_1 - w_2)(X_0) > 0,$$

a contradiction. $\square$

5. Properties of U

The first two lemmas in this section describe properties of U which will be used in the proof of Harnack inequality, and in particular when constructing the barriers which are used in that proof.

The third lemma, which is incorporated here since its proof uses similar arguments to the proof of the first two lemmas, allows us to replace the assumptions in our main Theorem 1.1 with a more standard "flatness" assumption of the form

$$U(X - \epsilon e_n) \leqslant g(X) \leqslant U(X + \epsilon e_n), \quad \text{in } B_1.$$

Lemma 5.1. *Let $g \in C(B_2)$, $g \geqslant 0$ be a harmonic function in $B_2^+(g)$ and let $\overline{X} = \frac{3}{2} e_n$. Assume that*

$$g \geqslant U \quad \text{in } B_2, \qquad g(\overline{X}) - U(\overline{X}) \geqslant \delta_0$$

for some $\delta_0 > 0$, then

$$g \geqslant (1 + c\delta_0)U \quad \text{in } B_1 \tag{5.1}$$

for a small universal constant c. In particular, for any $0 < \epsilon < 2$

$$U(X + \epsilon e_n) \geqslant (1 + c\epsilon)U(X) \quad \text{in } B_1, \tag{5.2}$$

with c small universal.

Proof. Call g^* the harmonic function in

$$D = B_{3/2} \setminus \{x \in \mathcal{B}_{3/2} : x_n \leqslant 0\},$$

such that

$$g^* = g \quad \text{on } \partial B_{3/2}, \qquad g^* = 0 \quad \text{on } \{x \in \mathcal{B}_{3/2} : x_n \leqslant 0\}.$$

Then by the maximum principle

$$g \geqslant g^* \quad \text{on } \overline{B}_{3/2},$$

and it suffices to show that (5.1) holds with g^* on the left-hand side.

Since $g \geqslant U$ in B_2 we have

$$g^* - U = g - U \geqslant 0 \quad \text{on } \partial B_{3/2}, \qquad g^* - U = 0 \quad \text{on } \{x \in \mathcal{B}_{3/2} : x_n \leqslant 0\},$$

and hence $g^* - U \geqslant 0$ in D where it is also harmonic. Moreover, from the assumption $g(\overline{X}) - U(\overline{X}) \geqslant \delta_0$ we get by Harnack inequality that

$$g^* - U = g - U \geqslant c_0 \delta_0 \quad \text{on } \partial B_{3/2} \cap B_{1/4}(\overline{X}).$$

Thus

$$g^*(\widetilde{X}) - U(\widetilde{X}) \geqslant c_1 \delta_0, \quad \text{at some } \widetilde{X} \in B_1 \cap D.$$

Thus, by the boundary Harnack inequality we get that for $c > 0$ universal,

$$g^* - U \geqslant c_2 \frac{g^*(\widetilde{X}) - U(\widetilde{X})}{U(\widetilde{X})} U \geqslant c \delta_0 U \quad \text{in } B_1,$$

as desired.

In particular, if $g(X) = U(X + \epsilon e_n)$ the assumptions of the lemma are satisfied. Indeed U is monotone increasing in the e_n-direction thus $U(X + \epsilon e_n) \geqslant U(X)$ in B_2. Moreover,

$$U(\overline{X} + \epsilon e_n) - U(\overline{X}) = U_t(\overline{X} + \lambda e_n)\epsilon \geqslant c'\epsilon, \quad \lambda \in (0, \epsilon),$$

with c' universal. $\quad \square$

Lemma 5.2. *For any $\epsilon > 0$ small, given $2\epsilon < \bar{\delta} < 1$, there exists a constant $C > 0$ depending on $\bar{\delta}$ such that*

$$U(t + \epsilon, z) \leqslant (1 + C\epsilon)U(t, z) \quad \text{in } \overline{B}_1 \setminus B_{\bar{\delta}} \subset \mathbb{R}^2.$$

Proof. In this lemma B_ρ denotes a ball of radius ρ in $\mathbb{R}^2$.

Since U is monotone increasing in the t-direction, $U(t + \epsilon, z) - U(t, z)$ is non-negative and harmonic in the set $D_{\bar{\delta}} := (B_2 \setminus \{t \in (-2, 2) : t \leqslant 0\}) \setminus \overline{B}_{\bar{\delta}/2}$. Moreover,

$$U(3/2 + \epsilon, 0) - U(3/2, 0) = U_t(t, 0)\epsilon \leqslant C_0\epsilon, \quad t \in (3/2, 3/2 + \epsilon),$$

with C_0 universal. By the boundary Harnack inequality in $D_{\bar{\delta}}$,

$$U(t + \epsilon, z) - U(t, z) \leqslant C_1 \frac{U(3/2 + \epsilon, 0) - U(3/2, 0)}{U(3/2, 0)} U(t, z) \leqslant C\epsilon U(t, z) \quad \text{in } \overline{B}_1 \setminus B_{\bar{\delta}},$$

as desired. $\quad \square$

Lemma 5.3. *Let $g \in C(\overline{B}_2)$, $g \geqslant 0$ be a harmonic function in $B_2^+(g)$ satisfying*

$$\|g - U\|_{L^\infty(\overline{B}_2)} \leqslant \delta, \tag{5.3}$$

and

$$\{x \in \mathcal{B}_2 : x_n \leqslant -\delta\} \subset \{x \in \mathcal{B}_2 : g(x, 0) = 0\} \subset \{x \in \mathcal{B}_2 : x_n \leqslant \delta\},$$

with $\delta > 0$ small universal. Then

$$U(X - \epsilon e_n) \leqslant g(X) \leqslant U(X + \epsilon e_n) \quad \text{in } B_1, \tag{5.4}$$

for some $\epsilon = K\delta$, K universal.

Proof. Let $\bar{g}$ be the harmonic function in

$$B_2^\delta := B_2 \setminus \{x \in \mathcal{B}_2 : x_n \leqslant -\delta\},$$

such that

$$\bar{g} = g \quad \text{on } \partial B_2, \qquad \bar{g} = 0 \quad \text{on } \{x \in \mathcal{B}_2 : x_n \leqslant -\delta\}.$$

Since g is subharmonic in B_2^δ, the maximum principle gives us

$$g \leqslant \bar{g} \quad \text{on } \overline{B}_2.$$

We need to show that for $K > 0$ universal,

$$\bar{g} \leqslant U(X + K\delta e_n) \quad \text{in } B_1. \tag{5.5}$$

(The lower bound follows from a similar argument.) Since U is monotone increasing in the e_n-direction and it satisfies (5.3) we get that

$$U(X + \delta e_n) \geqslant U(X) \geqslant g(X) - \delta \quad \text{on } \overline{B}_2,$$

and hence

$$U(X + \delta e_n) \geqslant \bar{g}(X) - \delta \quad \text{on } \partial B_2.$$

By the maximum principle in the domain B_2^δ we get that this inequality holds in B_2 and hence

$$\bar{g}(X) - U(X + \delta e_n) \leqslant \delta \quad \text{in } B_2. \tag{5.6}$$

Let g^* be the harmonic function in $B_{3/2} \setminus \{x \in \mathcal{B}_{3/2} : x_n \leqslant -\delta\}$ such that

$$g^* = \delta \quad \text{on } \partial B_{3/2}, \qquad g^* = 0 \quad \text{on } \{x \in \mathcal{B}_{3/2} : x_n \leqslant -\delta\}.$$

Clearly

$$0 \leqslant g^* \leqslant \delta.$$

Then by the boundary Harnack inequality, say for $\overline{X} = e_n$

$$g^*(X) \leqslant \overline{C} \frac{g^*(\overline{X})}{U(\overline{X} + \delta e_n)} U(X + \delta e_n) \leqslant C\delta U(X + \delta e_n) \quad \text{in } B_1, \tag{5.7}$$

with $C > 0$ universal. Moreover, in view of (5.6) again by the maximum principle we have

$$\bar{g}(X) - U(X + \delta e_n) \leqslant g^*(X) \quad \text{in } B_{3/2}.$$

This inequality together with (5.7) gives that

$$\bar{g}(X) \leqslant (1 + C\delta)U(X + \delta e_n) \quad \text{in } B_1.$$

By (5.2) (applied to a translate of U) we have that for $K > 1$

$$(1 + C\delta)U(X + \delta e_n) \leqslant \frac{1 + C\delta}{1 + cK\delta} U(X + K\delta e_n) \leqslant U(X + K\delta e_n) \quad \text{in } B_1,$$

as long as K is large enough. Combining these two last inequalities we obtain the desired claim (5.5). $\quad\square$

6. Harnack inequality

In this section we state and prove a Harnack type inequality for solutions to our free boundary problem (2.1).

Theorem 6.1 *(Harnack inequality). There exists $\bar{\epsilon} > 0$ such that if g solves* (2.1) *and it satisfies*

$$U(X + \epsilon a_0 e_n) \leqslant g(X) \leqslant U(X + \epsilon b_0 e_n) \quad \text{in } B_\rho(X^*), \tag{6.1}$$

with

$$\epsilon(b_0 - a_0) \leqslant \bar{\epsilon}\rho,$$

then

$$U(X + \epsilon a_1 e_n) \leqslant g(X) \leqslant U(X + \epsilon b_1 e_n) \quad \text{in } B_{\eta\rho}(X^*), \tag{6.2}$$

with

$$a_0 \leqslant a_1 \leqslant b_1 \leqslant b_0, \qquad (b_1 - a_1) \leqslant (1 - \eta)(b_0 - a_0),$$

for a small universal constant η.

From this statement we get the desired corollary to be used in the proof of our main result. Precisely, if g satisfies (6.1) with say $\rho = 1/2$, then we can apply Harnack inequality repeatedly and obtain

$$U(X + \epsilon a_m e_n) \leqslant g(X) \leqslant U(X + \epsilon b_m e_n) \quad \text{in } B_{\frac{1}{2}\eta^m}(X^*),$$

with

$$b_m - a_m \leqslant (b_0 - a_0)(1 - \eta)^m, \tag{6.3}$$

for all m's such that

$$2\epsilon(1 - \eta)^m \eta^{-m}(b_0 - a_0) \leqslant \bar{\epsilon}. \tag{6.4}$$

This implies that for all such m's, the function $\tilde{g}_\epsilon$ defined in Section 2.3 satisfies

$$a_m \leqslant \tilde{g}_\epsilon(X) \leqslant b_m, \quad \text{in } B_{\frac{1}{2}\eta^m - \epsilon}(X^*) \setminus P, \tag{6.5}$$

with a_m, b_m as in (6.3). Let A_ϵ be the following set

$$A_\epsilon := \big\{ (X, \tilde{g}_\epsilon(X)) \colon X \in B_{1-\epsilon} \setminus P \big\} \subset \mathbb{R}^{n+1} \times [a_0, b_0]. \tag{6.6}$$

Since $\tilde{g}_\epsilon$ may be multi-valued, we mean that given X all pairs $(X, \tilde{g}_\epsilon(X))$ belong to A_ϵ for all possible values of $\tilde{g}_\epsilon(X)$. In view of (6.5) we then get

$$A_\epsilon \cap \big(B_{\frac{1}{2}\eta^m - \epsilon}(X^*) \times [a_0, b_0] \big) \subset B_{\frac{1}{2}\eta^m - \epsilon}(X^*) \times [a_m, b_m], \tag{6.7}$$

with a_m, b_m as in (6.3) for all m's such that (6.4) holds.

Thus we get the following corollary.

Corollary 6.2. *If*

$$U(X - \epsilon e_n) \leqslant g(X) \leqslant U(X + \epsilon e_n) \quad \text{in } B_1,$$

with $\epsilon \leqslant \bar{\epsilon}/2$, given $m_0 > 0$ such that

$$2\epsilon(1 - \eta)^{m_0} \eta^{-m_0} \leqslant \bar{\epsilon},$$

then the set $A_\epsilon \cap (B_{1/2} \times [-1, 1])$ is above the graph of a function $y = a_\epsilon(X)$ and it is below the graph of a function $y = b_\epsilon(X)$ with

$$b_\epsilon - a_\epsilon \leqslant 2(1 - \eta)^{m_0 - 1},$$

and a_ϵ, b_ϵ having a modulus of continuity bounded by the Hölder function αt^β for α, β depending only on η.

The proof of Harnack inequality will easily follow from the lemma below.

Lemma 6.3. *There exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon \leqslant \bar{\epsilon}$ if g is a solution to (2.1) in B_1 such that*

$$g(X) \geqslant U(X) \quad \text{in } B_{1/2}, \tag{6.8}$$

and at $\overline{X} \in B_{1/8}(\frac{1}{4}e_n)$

$$g(\overline{X}) \geqslant U(\overline{X} + \epsilon e_n), \tag{6.9}$$

then

$$g(X) \geqslant U(X + \tau \epsilon e_n) \quad \text{in } B_\delta, \tag{6.10}$$

for universal constants τ, δ. Similarly, if

$$g(X) \leqslant U(X) \quad \text{in } B_{1/2},$$

and

$$g(\overline{X}) \leqslant U(\overline{X} - \epsilon e_n),$$

then

$$g(X) \leqslant U(X - \tau \epsilon e_n) \quad \text{in } B_\delta.$$

The main tool in the proof of Lemma 6.3 will be the following family of radial subsolutions. Let $R > 0$ and denote

$$V_R(t, z) = U(t, z)\left((n-1)\frac{t}{R} + 1\right).$$

Then set

$$v_R(X) = V_R\left(R - \sqrt{|x'|^2 + (x_n - R)^2}, z\right), \tag{6.11}$$

that is we obtain the $(n+1)$-dimensional function v_R by rotating the 2-dimensional function V_R around $(0, R, z)$.

Proposition 6.4. *If R is large enough, the function $v_R(X)$ is a comparison subsolution to (2.1) in B_2 which is strictly monotone increasing in the e_n-direction in $B_2^+(v_R)$. Moreover, there exists a function $\tilde{v}_R$ such that*

$$U(X) = v_R\left(X - \tilde{v}_R(X)e_n\right) \quad \text{in } B_1 \setminus P,$$

and

$$\left|\tilde{v}_R(X) - \gamma_R(X)\right| \leqslant \frac{C}{R^2}|X|^2, \quad \gamma_R(X) = -\frac{|x'|^2}{2R} + 2(n-1)\frac{x_n r}{R},$$

with $r = \sqrt{x_n^2 + z^2}$ and C universal.

The proof of Proposition 6.4 follows from long and tedious computations and we postpone it till Appendix A. Using the estimate for $\tilde{v}_R$ in Proposition 6.4 and Lemma 3.1, we also obtain the following corollary which will be crucial for the proof of Lemma 6.3. Its proof is again presented in Appendix A.

Corollary 6.5. *There exist δ, c_0, C_0, C_1 universal constants, such that*

$$v_R\left(X + \frac{c_0}{R}e_n\right) \leqslant \left(1 + \frac{C_0}{R}\right)U(X), \quad \text{in } \overline{B}_1 \setminus B_{1/4}, \tag{6.12}$$

with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n)) \cap \overline{B}_1 \setminus B_{1/4}$,

$$v_R\left(X + \frac{c_0}{R}e_n\right) \geqslant U\left(X + \frac{c_0}{2R}e_n\right), \quad \text{in } B_\delta, \tag{6.13}$$

$$v_R\left(X - \frac{C_1}{R}e_n\right) \leqslant U(X), \quad \text{in } \overline{B}_1. \tag{6.14}$$

We are now ready to present the proof of Lemma 6.3.

Proof of Lemma 6.3. We prove the first statement. In view of (6.9)

$$g(\overline{X}) - U(\overline{X}) \geqslant U(\overline{X} + \epsilon e_n) - U(\overline{X}) = \partial_t U(\overline{X} + \lambda e_n)\epsilon \geqslant c\epsilon, \quad \lambda \in (0, \epsilon).$$

As in Lemma 5.1 we then get

$$g(X) \geqslant (1 + c'\epsilon)U(X) \quad \text{in } \overline{B}_{1/4}. \tag{6.15}$$

Now let

$$R = \frac{C_0}{c'\epsilon},$$

where from now on the C_i, c_i are the constants in Corollary 6.5. Then, for ϵ small enough v_R is a subsolution to (2.1) in B_2 which is monotone increasing in the e_n-direction and it also satisfies (6.12)–(6.14). We now wish to apply the comparison principle as stated in Corollary 2.7. Let

$$v_R^t(X) = v_R(X + te_n), \quad X \in B_1,$$

then according to (6.14),

$$v_R^{t_0} \leqslant U \leqslant g \quad \text{in } \overline{B}_{1/4}, \text{ with } t_0 = -C_1/R.$$

Moreover, from (6.12) and (6.15) we get that for our choice of R,

$$v_R^{t_1} \leqslant (1 + c'\epsilon)U \leqslant g \quad \text{on } \partial B_{1/4}, \text{ with } t_1 = c_0/R,$$

with strict inequality on $F(v_R^{t_1}) \cap \partial B_{1/4}$. In particular

$$g > 0 \quad \text{on } \mathcal{F}(v_R^{t_1}) \text{ in } \partial B_{1/4}.$$

Thus we can apply Corollary 2.7 in the ball $B_{1/4}$ to obtain

$$v_R^{t_1} \leqslant g \quad \text{in } B_{1/4}.$$

From (6.13) we have that

$$U\left(X + \frac{c_1}{R}e_n\right) \leqslant v_R^{t_1}(X) \leqslant g(X) \quad \text{on } B_\delta$$

which is the desired claim (6.10) with $\tau = \frac{c_1 c'}{C_0}$. $\quad \square$

We now present the proof of the Harnack inequality.

Proof of Theorem 6.1. Without loss of generality, we can assume $a_0 = -1, b_0 = 1$. Also, in view of Remark 2.5 we can take $\rho = 1$ (thus $2\epsilon \leqslant \bar{\epsilon}$).

We distinguish several cases. In what follows $\bar{\epsilon}$ and δ denote the universal constants in Lemma 6.3.

Case 1. If

$$d(X^*, \{x_n \geqslant \epsilon, z = 0\}) > \delta/16,$$

then $U(X - \epsilon e_n) > 0$ in $B_{\delta/16}(X^*) \subset B_1(X^*)$. Thus the functions $U(X - \epsilon e_n), U(X + \epsilon e_n)$ and $g(X)$ are positive and harmonic in $B_{\delta/16}(X^*)$. Assume that (the other case is treated similarly)

$$g(X^*) \geqslant U(X^*).$$

Then,

$$g(X^*) \geqslant U(X^*) = U(X^* - \epsilon e_n) + U_n(X^* - \lambda\epsilon e_n)\epsilon, \quad \lambda \in (0, 1).$$

Since U_n is positive and harmonic in $B_{\delta/16}(X^*)$ and for $\bar{\epsilon} < \delta/16$

$$X^* - \lambda\epsilon e_n \in B_{\delta/32}(X^*),$$

we can apply Harnack inequality to conclude that

$$g(X^*) \geqslant U(X^* - \epsilon e_n) + cU_n(X^*)\epsilon,$$

for c small universal.

Then again by Harnack inequality in $B_{\delta/16}(X^*)$ for $g(X) - U(X - \epsilon e_n) \geqslant 0$ we get that for c' universal

$$g(X) \geqslant U(X - \epsilon e_n) + c' U_n(X^*)\epsilon, \quad \text{in } B_{\delta/32}(X^*).$$

By a similar argument, for $\bar\epsilon < \delta/32$

$$U(X - (1 - \eta)\epsilon e_n) - U(X - \epsilon e_n) \leqslant C U_n(X^*)\eta\epsilon, \quad \text{in } B_{\delta/64}(X^*),$$

with C universal.

Thus, combining these two last inequalities we obtain that for $\eta = \min\{c'/C, \delta/64\}$

$$g(X) \geqslant U(X - (1 - \eta)\epsilon e_n), \quad \text{in } B_\eta(X^*),$$

as desired.

Case 2. If

$$d(X^*, \{x_n = -\epsilon, \; z = 0\}) \leqslant \delta/2,$$

we wish to apply Lemma 6.3. Then (for $\bar\epsilon < \delta/4$)

$$g(X) \geqslant U(X - \epsilon e_n) \quad \text{in } B_{1/2}(\epsilon e_n) \subset B_1(X^*).$$

Let $\overline{X} = \frac{1}{4}e_n$ and assume that (the other case follows similarly)

$$g(\overline{X}) \geqslant U(\overline{X}).$$

Since (for $\bar\epsilon$ small)

$$\overline{X} \in B_{1/8}\left(\left(\frac{1}{4} + \epsilon\right)e_n\right),$$

we can apply Lemma 6.3 and conclude that

$$g(X) \geqslant U(X - (1 - \eta)\epsilon e_n), \quad \text{in } B_\delta(\epsilon e_n).$$

Thus the desired improvement holds by choosing $\eta < \delta/4$. Indeed for such η and $\bar\epsilon < \delta/4$ we have that $d(X^*, \{x_n = \epsilon, \; z = 0\}) \leqslant 3\delta/4$ and hence

$$B_\eta(X^*) \subset B_\delta(\epsilon e_n).$$

Case 3. If

$$d(X^*, \{x_n = -\epsilon, \; z = 0\}) > \delta/2 \quad \text{and} \quad d(X^*, \{x_n \geqslant \epsilon, \; z = 0\}) \leqslant \delta/16,$$

then the functions $U(X - \epsilon e_n), U(X + \epsilon e_n)$ and $g(X)$ are positive and harmonic in the half-ball $B_{\delta/4}(\widetilde{X}) \cap \{z > 0\}$ for some $\widetilde{X} \in \{x_n \leqslant -\epsilon, \; z = 0\}$ and they all vanish continuously on $B_{\delta/4}(\widetilde{X}) \cap \{z = 0\}$. Thus we can repeat a similar argument as in Case 1, by using the boundary Harnack inequality. Precisely, let $\overline{X} = \widetilde{X} + \frac{\delta}{6}e_{n+1}$ and assume that (the other case is treated similarly)

$$g(\overline{X}) \geqslant U(\overline{X}).$$

Then,

$$g(\overline{X}) \geqslant U(\overline{X}) = U(\overline{X} - \epsilon e_n) + U_n(\overline{X} - \lambda\epsilon e_n)\epsilon, \quad \lambda \in (0, 1).$$

By Harnack inequality for U_n in the ball $B_{2\epsilon}(\overline{X}) \subset B_{\delta/4}(\widetilde{X}) \cap \{z > 0\}$ (with $\bar\epsilon < \delta/12$) we conclude that

$$g(\overline{X}) \geqslant U(\overline{X} - \epsilon e_n) + c U_n(\overline{X})\epsilon, \tag{6.16}$$

for c small universal.

Then by boundary Harnack inequality in $B_{\delta/4}(\widetilde{X}) \cap \{z > 0\}$, for the functions $g(X) - U(X - \epsilon e_n)$ and $U_n(X)$ we get that for c' universal

$$g(X) \geqslant U(X - \epsilon e_n) + c' U_n(X)\epsilon, \quad \text{in } B_{\delta/8}(\widetilde{X}) \cap \{z \geqslant 0\}. \tag{6.17}$$

Thus to obtain the desired claim it is enough to choose η small such that for $X \in B_{\delta/8}(\widetilde{X}) \cap \{z \geqslant 0\}$

$$U(X - \epsilon e_n) + c' U_n(X)\epsilon \geqslant U\big(X - (1 - \eta)\epsilon e_n\big).$$

By a similar argument as above

$$U\big(\overline{X} - (1 - \eta)\epsilon e_n\big) - U(\overline{X} - \epsilon e_n) \leqslant C U_n(\overline{X})\eta\epsilon,$$

and hence by boundary Harnack inequality,

$$U\big(X - (1 - \eta)\epsilon e_n\big) - U(X - \epsilon e_n) \leqslant C' U_n(X)\eta\epsilon, \quad \text{in } B_{\delta/8}(\widetilde{X}) \cap \{z \geqslant 0\}.$$

Combining this inequality with (6.17) we obtain that for $\eta = c'/C'$

$$g(X) \geqslant U\big(X - (1 - \eta)\epsilon e_n\big), \quad \text{in } B_{\delta/8}(\widetilde{X}) \cap \{z \geqslant 0\}.$$

Since all the functions involved are even with respect to $\{z = 0\}$ and for $\eta < \delta/16$

$$B_\eta(X^*) \subset B_{\delta/8}(\widetilde{X}),$$

our proof is complete.　$\square$

7. Improvement of flatness

In this section we state the improvement of flatness property for solutions to (2.1) and we provide its proof. Our main Theorem 1.1 follows from Theorem 7.1 and Lemma 5.3.

Theorem 7.1 (*Improvement of flatness*). *There exist $\bar{\epsilon} > 0$ and $\rho > 0$ universal constants such that for all $0 < \epsilon \leqslant \bar{\epsilon}$ if g solves (2.1) with $0 \in F(g)$ and it satisfies*

$$U(X - \epsilon e_n) \leqslant g(X) \leqslant U(X + \epsilon e_n) \quad \text{in } B_1, \tag{7.1}$$

then

$$U\left(x \cdot \nu - \frac{\epsilon}{2}\rho, z\right) \leqslant g(X) \leqslant U\left(x \cdot \nu + \frac{\epsilon}{2}\rho, z\right) \quad \text{in } B_\rho, \tag{7.2}$$

for some direction $\nu \in \mathbb{R}^n$, $|\nu| = 1$.

The proof of Theorem 7.1 will easily follow from the next four lemmas.

Lemma 7.2. *Let g be a solution to (2.1) with $0 \in F(g)$ and satisfying (7.1). Assume that the corresponding $\tilde{g}_\epsilon$ satisfies*

$$a_0 \cdot x' - \frac{1}{4}\rho \leqslant \tilde{g}_\epsilon(X) \leqslant a_0 \cdot x' + \frac{1}{4}\rho \quad \text{in } B_{2\rho} \setminus P, \tag{7.3}$$

for some $a_0 \in \mathbb{R}^{n-1}$. Then if $\epsilon \leqslant \bar{\epsilon}(a_0, \rho)$, g satisfies (7.2) in B_ρ.

Proof. We prove that the lower bound holds (the upper bound can be proved similarly).

Let,

$$\nu = \big(\nu', \nu_n\big) := \frac{(0, 1) + \epsilon(a_0, 0)}{\sqrt{1 + \epsilon^2 a_0^2}},$$

and call

$$u(X) = U\left(x \cdot \nu - \frac{\epsilon}{2}\rho, z\right).$$

Notice that since $\nu_n > 0$, u is strictly monotone increasing in the e_n-direction say in $B_{2\rho}^+(u)$. Also, we can easily compute $\tilde{u}_\epsilon$ by its definition. Indeed, the identity

$$u\big(X - \epsilon \tilde{u}_\epsilon(X) e_n\big) = U(X), \quad X \in \mathbb{R}^{n+1} \setminus P$$

reads as

$$U\left(x' \cdot v' + x_n v_n - \epsilon \tilde{u}_\epsilon(X) v_n - \frac{\epsilon}{2}\rho, z\right) = U(x_n, z),$$

and hence

$$\tilde{u}_\epsilon(X) = \frac{x' \cdot v' + (v_n - 1)x_n}{\epsilon v_n} - \frac{\rho}{2v_n}. \tag{7.4}$$

Thus, according to Lemma 3.1 it suffices to show that

$$\tilde{u}_\epsilon \leqslant \tilde{g}_\epsilon \quad \text{in } B_{\rho+\epsilon} \setminus P,$$

and hence in view of (7.3) we must show that

$$\tilde{u}_\epsilon(X) \leqslant a_0 \cdot x' - \frac{1}{4}\rho \quad \text{in } B_{\rho+\epsilon} \setminus P.$$

From the choice of v we see that

$$\frac{v'}{\epsilon v_n} = a_0,$$

and

$$\frac{|v_n - 1|}{\epsilon v_n} = \frac{1 - v_n}{\epsilon v_n} \leqslant \epsilon a_0^2.$$

Thus, in view of the formula (7.4) the desired inequality reduces to

$$x' \cdot a_0 + 2\rho\epsilon a_0^2 - \frac{\rho}{2} \leqslant x' \cdot a_0 - \frac{\rho}{4},$$

which is trivially satisfied for ϵ small enough (depending on a_0, ρ).   □

The next lemma follows immediately from Corollary 6.2 to Harnack inequality.

Lemma 7.3. *Let $\epsilon_k \to 0$ and let g_k be a sequence of solutions to (2.1) with $0 \in F(g_k)$ satisfying*

$$U(X - \epsilon_k e_n) \leqslant g_k(X) \leqslant U(X + \epsilon_k e_n) \quad \text{in } B_1. \tag{7.5}$$

Denote by $\tilde{g}_k$ the ϵ_k-domain variation of g_k. Then the sequence of sets

$$A_k := \left\{\left(X, \tilde{g}_k(X)\right): X \in B_{1-\epsilon_k} \setminus P\right\}$$

has a subsequence that converges uniformly (in Hausdorff distance) in $B_{1/2} \setminus P$ to the graph

$$A_\infty := \left\{\left(X, \tilde{g}_\infty(X)\right): X \in B_{1/2} \setminus P\right\},$$

where $\tilde{g}_\infty$ is a Holder continuous function.

From here on $\tilde{g}_\infty$ will denote the function from Lemma 7.3.

Lemma 7.4. *The function $\tilde{g}_\infty$ satisfies the linearized problem (4.1) in $B_{1/2}$.*

Proof. We start by showing that $U_n \tilde{g}_\infty$ is harmonic in $B_{1/2} \setminus P$.

Let $\tilde{\varphi}$ be a C^2 function which touches $\tilde{g}_\infty$ strictly by below at $X_0 \in B_{1/2} \setminus P$. We need to show that

$$\Delta(U_n \tilde{\varphi})(X_0) \leqslant 0. \tag{7.6}$$

Since by the previous lemma, the sequence A_k converges uniformly to A_∞ in $B_{1/2} \setminus P$ we conclude that there exist a sequence of constants $c_k \to 0$ and a sequence of points $X_k \in B_{1/2} \setminus P$, $X_k \to X_0$ such that $\tilde{\varphi}_k := \tilde{\varphi} + c_k$ touches $\tilde{g}_k$ by below at X_k for all k large enough.

Define the function φ_k by the following identity

$$\varphi_k\big(X - \epsilon_k \tilde{\varphi}_k(X)e_n\big) = U(X). \tag{7.7}$$

Then according to (3.10) φ_k touches g_k by below at $Y_k = X_k - \epsilon_k \tilde{\varphi}_k(X_k)e_n \in B_1^+(g_k)$, for k large enough. Thus, since g_k satisfies (2.1) in B_1 it follows that

$$\Delta \varphi_k(Y_k) \leqslant 0. \tag{7.8}$$

Let us compute $\Delta \varphi_k(Y_k)$. Since $\tilde{\varphi}$ is smooth, for any Y in a neighborhood of Y_k we can find a unique $X = X(Y)$ such that

$$Y = X - \epsilon_k \tilde{\varphi}_k(X)e_n. \tag{7.9}$$

Thus (7.7) reads

$$\varphi_k(Y) = U\big(X(Y)\big),$$

with $Y_i = X_i$ if $i \neq n$ and

$$\frac{\partial X_j}{\partial Y_i} = \delta_{ij}, \quad \text{when } j \neq n.$$

Using these identities we can compute that

$$\Delta \varphi_k(Y) = U_n(X)\Delta X_n(Y) + \sum_{j \neq n}\left(U_{jj}(X) + 2U_{jn}(X)\frac{\partial X_n}{\partial Y_j}\right) + U_{nn}(X)|\nabla X_n|^2(Y). \tag{7.10}$$

From (7.9) we have that

$$D_X Y = I - \epsilon_k D_X(\tilde{\varphi}_k e_n).$$

Thus, since $\tilde{\varphi}_k = \tilde{\varphi} + c_k$

$$D_Y X = I + \epsilon_k D_X(\tilde{\varphi} e_n) + O\big(\epsilon_k^2\big),$$

with a constant depending only on the C^2-norm of $\tilde{\varphi}$.

It follows that

$$\frac{\partial X_n}{\partial Y_j} = \delta_{jn} + \epsilon_k \partial_j \tilde{\varphi}(X) + O\big(\epsilon_k^2\big). \tag{7.11}$$

Hence

$$|\nabla X_n|^2(Y) = 1 + 2\epsilon_k \partial_n \tilde{\varphi}(X) + O\big(\epsilon_k^2\big), \tag{7.12}$$

and also,

$$\frac{\partial^2 X_n}{\partial Y_j^2} = \epsilon_k \sum_i \partial_{ji}\tilde{\varphi}\frac{\partial X_i}{\partial Y_j} + O\big(\epsilon_k^2\big) = \epsilon_k \sum_{i \neq n}\partial_{ji}\tilde{\varphi}\delta_{ij} + \epsilon_k \partial_{jn}\tilde{\varphi}\frac{\partial X_n}{\partial Y_j} + O\big(\epsilon_k^2\big),$$

from which we obtain that

$$\Delta X_n = \epsilon_k \Delta \tilde{\varphi} + O\big(\epsilon_k^2\big). \tag{7.13}$$

Combining (7.10) with (7.12) and (7.13) we get that

$$\Delta \varphi_k(Y) = \Delta U(X) + \epsilon_k U_n \Delta \tilde{\varphi} + 2\epsilon_k \nabla \tilde{\varphi} \cdot \nabla U_n + O\big(\epsilon_k^2\big)\big(U_n(X) + U_{nn}(X)\big).$$

Using (7.8) together with the fact that U is harmonic at X_k we conclude that

$$0 \geqslant \Delta(U_n \tilde{\varphi})(X_k) + O\big(\epsilon_k^2\big)\big(U_n(X_k) + U_{nn}(X_k)\big).$$

The desired inequality (7.6) follows by letting $k \to +\infty$.

Next we need to show that

$$|\nabla_r \tilde{g}_\infty|(X_0) = 0, \quad X_0 = \big(x_0', 0, 0\big) \in B_{1/2} \cap L,$$

in the viscosity sense of Definition 4.1.

Assume by contradiction that there exists a function ϕ which touches $\tilde{g}_\infty$ by below at $X_0 = (x_0', 0, 0) \in B_{1/2} \cap L$ and such that

$$\phi(X) = \phi(X_0) + a(X_0) \cdot (x' - x_0') + b(X_0)r + O(|x' - x_0'|^2 + r^{3/2}),$$

with

$$b(X_0) > 0.$$

Then we can find constants $\alpha, \delta, \bar{r}$ and a point $Y' = (y_0', 0, 0) \in B_2$ depending on ϕ such that the polynomial

$$q(X) = \phi(X_0) - \frac{\alpha}{2}|x' - y_0'|^2 + 2\alpha(n-1)x_n r$$

touches ϕ by below at X_0 in a tubular neighborhood $N_{\bar{r}} = \{|x' - x_0'| \leqslant \bar{r}, r \leqslant \bar{r}\}$ of X_0, with

$$\phi - q \geqslant \delta > 0, \quad \text{on } N_{\bar{r}} \setminus N_{\bar{r}/2}.$$

This implies that

$$\tilde{g}_\infty - q \geqslant \delta > 0, \quad \text{on } N_{\bar{r}} \setminus N_{\bar{r}/2}, \tag{7.14}$$

and

$$\tilde{g}_\infty(X_0) - q(X_0) = 0. \tag{7.15}$$

In particular,

$$\left|\tilde{g}_\infty(X_k) - q(X_k)\right| \to 0, \quad X_k \in N_{\bar{r}} \setminus P, X_k \to X_0. \tag{7.16}$$

Now, let us choose $R_k = 1/(\alpha\epsilon_k)$ and let us define

$$w_k(X) = v_{R_k}(X - Y' + \epsilon_k \phi(X_0)e_n), \quad Y' = (y_0', 0, 0),$$

with v_R the function defined in Proposition 6.4. Then the ϵ_k-domain variation of w_k, which we call $\tilde{w}_k$, can be easily computed from the definition

$$w_k(X - \epsilon_k \tilde{w}_k(X)e_n) = U(X).$$

Indeed, since U is constant in the x'-direction, this identity is equivalent to

$$v_{R_k}(X - Y' + \epsilon_k \phi(X_0)e_n - \epsilon_k \tilde{w}_k(X)e_n) = U(X - Y'),$$

which in view of Proposition 6.4 gives us

$$\tilde{v}_{R_k}(X - Y') = \epsilon_k(\tilde{w}_k(X) - \phi(X_0)).$$

From the choice of R_k, the formula for q and (A.2), we then conclude that

$$\tilde{w}_k(X) = q(X) + \alpha^2 \epsilon_k O(|X - Y'|^2),$$

and hence

$$|\tilde{w}_k - q| \leqslant C\epsilon_k \quad \text{in } N_{\bar{r}} \setminus P. \tag{7.17}$$

Thus, from the uniform convergence of A_k to A_∞ and (7.14)–(7.17) we get that for all k large enough

$$\tilde{g}_k - \tilde{w}_k \geqslant \frac{\delta}{2} \quad \text{in } (N_{\bar{r}} \setminus N_{\bar{r}/2}) \setminus P. \tag{7.18}$$

Similarly, from the uniform convergence of A_k to A_∞ and (7.17)–(7.16) we get that for k large

$$\tilde{g}_k(X_k) - \tilde{w}_k(X_k) \leqslant \frac{\delta}{4}, \quad \text{for some sequence } X_k \in N_{\bar{r}} \setminus P, X_k \to X_0. \tag{7.19}$$

On the other hand, it follows from Lemma 3.2 and (7.18) that

$$\tilde{g}_k - \tilde{w}_k \geqslant \frac{\delta}{2} \quad \text{in } N_{\bar{r}} \setminus P,$$

which contradicts (7.19). $\square$

The lemmas above allow us to reduce the regularity question for our free boundary problem (2.1) to the regularity of the linear problem (4.1). We will analyze such question in the next section and we will consequently obtain the following lemma, which we use here to conclude the proof of our improvement of flatness theorem.

Lemma 7.5. *There exists a universal constant $\rho > 0$ such that $\tilde{g}_\infty$ satisfies*

$$a_0 \cdot x' - \frac{1}{8}\rho \leqslant \tilde{g}_\infty(X) \leqslant a_0 \cdot x' + \frac{1}{8}\rho \quad in\ B_{2\rho}, \tag{7.20}$$

for a vector $a_0 \in \mathbb{R}^{n-1}$.

We are now ready to prove our main theorem, by combining all the lemmas above.

Proof of Theorem 7.1. Let ρ be the universal constant from Lemma 7.5 and assume by contradiction that we can find a sequence $\epsilon_k \to 0$ and a sequence g_k of solutions to (2.1) in B_1 such that g_k satisfies (7.1), i.e.

$$U(X - \epsilon_k e_n) \leqslant g_k(X) \leqslant U(X + \epsilon_k e_n) \quad in\ B_1, \tag{7.21}$$

but it does not satisfy the conclusion of the theorem.

Denote by $\tilde{g}_k$ the ϵ_k-domain variation of g_k. Then by Lemma 7.3 the sequence of sets

$$A_k := \left\{ \left(X, \tilde{g}_k(X) \right)\colon X \in B_{1-\epsilon_k} \setminus P \right\}$$

converges uniformly (up to extracting a subsequence) in $B_{1/2} \setminus P$ to the graph

$$A_\infty := \left\{ \left(X, \tilde{g}_\infty(X) \right)\colon X \in B_{1/2} \setminus P \right\},$$

where $\tilde{g}_\infty$ is a Holder continuous function in $B_{1/2}$. By Lemma 7.4, the function $\tilde{g}_\infty$ solves the linearized problem (4.1) and hence by Lemma 7.5 $\tilde{g}_\infty$ satisfies

$$a_0 \cdot x' - \frac{1}{8}\rho \leqslant \tilde{g}_\infty(X) \leqslant a_0 \cdot x' + \frac{1}{8}\rho \quad in\ B_{2\rho}, \tag{7.22}$$

with $a_0 \in \mathbb{R}^{n-1}$.

From the uniform convergence of A_k to A_∞, we get that for all k large enough

$$a_0 \cdot x' - \frac{1}{4}\rho \leqslant \tilde{g}_k(X) \leqslant a_0 \cdot x' + \frac{1}{4}\rho \quad in\ B_{2\rho} \setminus P, \tag{7.23}$$

and hence from Lemma 7.2, the g_k satisfy the conclusion of our theorem (for k large). We have thus reached a contradiction. $\square$

8. The regularity of the linearized problem

The purpose of this section is to prove an improvement of flatness result for viscosity solutions to the linearized problem associated to (2.1), that is

$$\begin{cases} \Delta(U_n w) = 0, & in\ B_1 \setminus P, \\ |\nabla_r w| = 0, & on\ B_1 \cap L, \end{cases} \tag{8.1}$$

where we recall that for $X_0 = (x_0', 0, 0) \in B_1 \cap L$, we set

$$|\nabla_r w|(X_0) := \lim_{(x_n, z) \to (0,0)} \frac{w(x_0', x_n, z) - w(x_0', 0, 0)}{r}, \quad r^2 = x_n^2 + z^2.$$

We remark that if we restrict this linear problem to the class of functions $w(X) = \tilde{w}(x', r)$ that depend only on (x', r) then the problem reduces to the classical Neumann problem

$$\begin{cases} \Delta \tilde{w} = 0, & in\ B_1^+, \\ \tilde{w}_r = 0, & on\ \{r = 0\}. \end{cases}$$

The following is our main theorem.

Theorem 8.1. *Given a boundary data $\bar{h} \in C(\partial B_1)$, $|\bar{h}| \leqslant 1$, which is even with respect to $\{z = 0\}$, there exists a unique classical solution h to (8.1) such that $h \in C(\overline{B}_1)$, $h = \bar{h}$ on ∂B_1, h is even with respect to $\{z = 0\}$ and it satisfies*

$$\left| h(X) - h(X_0) - a' \cdot \left(x' - x_0' \right) \right| \leqslant C \left(\left| x' - x_0' \right|^2 + r^{3/2} \right), \quad X_0 \in B_{1/2} \cap L, \tag{8.2}$$

for a universal constant C and a vector $a' \in \mathbb{R}^{n-1}$ depending on X_0.

As a corollary of the theorem above we obtain the desired regularity result, as stated also in Section 3.

Theorem 8.2 *(Improvement of flatness). There exists a universal constant C such that if w is a viscosity solution to (8.1) in B_1 with*

$$-1 \leqslant w(X) \leqslant 1 \quad \text{in } B_1,$$

then

$$a_0 \cdot x' - C|X|^{3/2} \leqslant w(X) - w(0) \leqslant a_0 \cdot x' + C|X|^{3/2}, \tag{8.3}$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

Proof. Let h be the unique solution to (8.1) in $B_{1/2}$ with boundary data w. We will prove that $w = h$ in $B_{1/2}$ and hence it satisfies the desired estimate in view of (8.2). Denote by

$$\bar{h}_\epsilon := h - \epsilon + \epsilon^2 r.$$

Then, for ϵ small

$$\bar{h}_\epsilon < w \quad \text{on } \partial B_{1/2}.$$

We wish to prove that

$$\bar{h}_\epsilon \leqslant w \quad \text{in } B_{1/2}. \tag{8.4}$$

Now, notice that $\bar{h}_\epsilon$ (and all its translations) is a classical strict subsolution to (8.1) that is

$$\begin{cases} \Delta(U_n \bar{h}_\epsilon) = 0, & \text{in } B_{1/2} \setminus P, \\ |\nabla_r \bar{h}_\epsilon| > 0, & \text{on } B_{1/2} \cap L. \end{cases} \tag{8.5}$$

Since w is bounded, for t large enough $\bar{h}_\epsilon - t$ lies strictly below w. We let $t \to 0$ and show that the first contact point cannot occur for $t \geqslant 0$. Indeed since $\bar{h}_\epsilon - t$ is a strict subsolution which is strictly below w on $\partial B_{1/2}$ then no touching can occur either in $B_{1/2} \setminus P$ or on $B_{1/2} \cap L$. We only need to check that no touching occurs on $P \setminus L$. This follows from Lemma 4.3.

Thus (8.4) holds. Passing to the limit as $\epsilon \to 0$ we get that

$$h \leqslant w \quad \text{in } B_{1/2}.$$

Similarly we also obtain that

$$h \geqslant w \quad \text{in } B_{1/2},$$

and the desired equality holds. $\quad \square$

The existence of the classical solution of Theorem 8.1 will be achieved via a variational approach in the appropriate weighted Sobolev space.

We say that $h \in H^1(U_n^2 dX, B_1)$ is a minimizer to the energy functional

$$J(h) := \int\limits_{B_1} U_n^2 |\nabla h|^2 \, dX,$$

if

$$J(h) \leqslant J(h + \phi), \quad \forall \phi \in C_0^\infty(B_1).$$

Since J is strictly convex this is equivalent to

$$\lim_{\epsilon \to 0} \frac{J(h) - J(h + \epsilon\phi)}{\epsilon} = 0, \quad \forall \phi \in C_0^\infty(B_1),$$

which is satisfied if and only if

$$\int_{B_1} U_n^2 \nabla h \cdot \nabla \phi \, dX = 0, \quad \forall \phi \in C_0^\infty(B_1).$$

As remarked above, if we restrict to the space of functions h which are axisymmetric with respect to L then the energy above reduces to the Dirichlet energy.

We start with a few standard facts about minimizers of J. First, h solves the equation

$$\operatorname{div}(U_n^2 \nabla h) = 0 \quad \text{in } B_1,$$

which is uniformly elliptic in any compact subset of $B_1 \setminus P$ where U_n is bounded. In particular $h \in C^\infty(B_1 \setminus P)$, and we easily obtain the following lemma.

Lemma 8.3. *Let h be a minimizer to J in B_1, then*

$$\Delta(U_n h) = 0 \quad \text{in } B_1 \setminus P.$$

Proof. Since h is smooth in $B_1 \setminus P$, from

$$\operatorname{div}(U_n^2 \nabla h) = 0 \quad \text{in } B_1,$$

we obtain that

$$U_n^2 \Delta h + 2 \sum_{i=1}^{n+1} U_n U_{ni} h_i = 0 \quad \text{in } B_1 \setminus P.$$

Since $U_n > 0$ and $\Delta U = 0$ in $B_1 \setminus P$ the identity above is equivalent to

$$\Delta(U_n h) = U_n \Delta h + 2 \nabla U_n \cdot \nabla h = 0 \quad \text{in } B_1 \setminus P,$$

as desired. $\quad\square$

The next lemma contains a characterization of minimizer, which will be useful later in this section.

Lemma 8.4. *Let $h \in C(B_1)$ be a solution to*

$$\Delta(U_n h) = 0 \quad \text{in } B_1 \setminus P, \tag{8.6}$$

and assume that

$$\lim_{r \to 0} h_r(x', x_n, z) = b(x'),$$

with $b(x')$ a continuous function. Then h is a minimizer to J in B_1 if and only if $b \equiv 0$.

Proof. By integration by parts and the computation in Lemma 8.3 the identity

$$\int_{B_1} U_n^2 \nabla h \cdot \nabla \phi \, dX = 0, \quad \forall \phi \in C_0^\infty(B_1),$$

is equivalent to the following two conditions

$$\Delta(U_n h) = 0 \quad \text{in } B_1 \setminus P, \tag{8.7}$$

and

$$\lim_{\delta \to 0} \int_{\partial C_\delta \cap B_1} U_n^2 \phi \nabla h \cdot \nu \, d\sigma = 0, \tag{8.8}$$

where C_δ is the cylinder $\{r \leqslant \delta\}$ and ν the inward unit normal to C_δ.

Here we use that

$$\lim_{\epsilon \to 0} \int_{\{|z|=\epsilon\} \cap (B_1 \setminus C_\delta)} U_n^2 \phi h_\nu \, d\sigma = 0.$$

Indeed, in the set $\{|z| = \epsilon\} \cap (B_1 \setminus C_\delta)$ we have

$$U_n \leqslant C\epsilon,$$

and

$$\left| \nabla(U_n h) \right|, |\nabla U_n| \leqslant C,$$

from which it follows that

$$U_n |\nabla h| \leqslant C.$$

In conclusion we need to show that (8.8) is equivalent to $b(x') = 0$.
Indeed,

$$\int_{\partial C_\delta \cap B_1} U_n^2 \phi \nabla h \cdot \nu \, d\sigma = \frac{1}{\delta} \int_{\partial C_\delta \cap B_1} \cos^2\left(\frac{\theta}{2}\right) h_r \phi \, d\sigma$$

$$= \int_{\partial C_1 \cap B_1} \cos^2\left(\frac{\theta}{2}\right) (h_r \phi)\left(X', \delta \cos\theta, \delta \sin\theta\right) dx' \, d\theta,$$

hence

$$\lim_{\delta \to 0} \int_{\partial C_\delta \cap B_1} U_n^2 \phi \nabla h \cdot \nu \, d\sigma = \pi \int_L b(x') \phi(x', 0, 0) \, dx'$$

and our claim clearly follows. $\quad\square$

The next lemma follows by standard arguments, hence we omit its proof.

Lemma 8.5 (*Comparison Principle*). *Let h_1, h_2 be minimizers to J in B_1. If*

$$h_1 \geqslant h_2 \quad \text{a.e in } B_1 \setminus B_\rho,$$

then

$$h_1 \geqslant h_2 \quad \text{a.e. in } B_1.$$

Finally one of the main ingredients in the proof of Theorem 8.1 is the following Harnack inequality.

Lemma 8.6 (*Harnack inequality*). *Let h be a minimizer to J in B_1 which is even with respect to $\{z = 0\}$. Then $h \in C^\alpha(B_{1/2})$ and*

$$[h]_{C^\alpha(B_{1/2})} \leqslant C,$$

with C universal.

The proof of this lemma follows the same lines as the proof of Harnack inequality (Theorem 6.1) for the free boundary problem (2.1). We briefly sketch it in what follows.

Sketch of the proof of Lemma 8.6. The key step consists in proving the following claim, which plays the same role as Lemma 6.3 in the proof of Theorem 6.1. The remaining ingredients are the standard Harnack inequality and boundary Harnack inequality for harmonic functions.

Claim. *There exist universal constants δ, c such that if $h \geqslant 0$ a.e. in B_1 and*

$$h\left(\frac{1}{4}e_n\right) \geqslant 1,$$

then

$$h \geqslant c \quad \text{a.e. in } B_\delta.$$

As in the proof of Lemma 6.3, since minimizers satisfy the comparison principle Lemma 8.5, the claim will follow if we provide the right family of comparison minimizers. This family plays the same role as the v_R's in Lemma 6.3 and it is obtained by translations and multiplication by constants of the following function

$$v(X) := -\frac{|x'|^2}{n-1} + 2x_n r.$$

We need to show that v is a minimizer to J in B_1. To do so we prove that v satisfies Lemma 8.4.

To prove that

$$\Delta(U_n v) = 0 \quad \text{in } B_1 \setminus P,$$

we use that $2rU_n = U$ and that U, U_n are harmonic outside of P and do not depend on x'. Thus

$$\Delta(U_n v) = -\Delta\left(\frac{|x'|^2}{n-1}U_n\right) + \Delta(x_n U) = -2U_n + 2U_n = 0.$$

Finally the fact that

$$\lim_{r \to 0} v_r(x', x_n, z) = 0,$$

follows immediately from the definition of v. $\quad\square$

Since our linear problem is invariant under translations in the x'-direction, we see that discrete differences of the form

$$h(X + \tau) - h(X),$$

with τ in the x'-direction are also minimizers. Now by standard arguments (see [5]) we obtain the following corollary.

Corollary 8.7. *Let h be a minimizer to J in B_1 which is even with respect to $\{z = 0\}$. Then $D_{x'}^\beta h \in C^\alpha(B_{1/2})$ and*

$$\left[D_{x'}^\beta h\right]_{C^\alpha(B_{1/2})} \leqslant C,$$

with C depending on β.

We are now ready to prove our main theorem.

Proof of Theorem 8.1. We divide our proof in several steps.

Step 1. In this step, we show the existence of a classical solution to our problem, which achieves the boundary data continuously.

Assume without loss of generality that $\bar{h} \in C^\infty(\partial B_1)$. The general case when $\bar{h} \in C(\partial B_1)$ follows by approximation and the comparison principle.

We minimize $J(\cdot)$ among all functions h with boundary data $\bar{h}$, which are even with respect to $\{z = 0\}$. From Lemma 8.3 we have that

$$\Delta(U_n h) = 0 \quad \text{in } B_1 \setminus P.$$

In Steps 2–3 we will show that h satisfies the estimate (8.2) and in particular

$$h(x_0', x_n, x) - h(x_0', 0, 0) = O(r^{3/2}), \quad X_0 = (x_0', 0, 0) \in B_1 \cap L$$

which gives

$$|\nabla_r h| = 0 \quad \text{on } B_1 \cap L,$$

where we recall that

$$|\nabla_r h|(X_0) = \lim_{(x_n, z) \to (0,0)} \frac{h(x_0', x_n, z) - h(x_0', 0, 0)}{r}, \quad r^2 = x_n^2 + z^2.$$

Notice that since $|\bar{h}| \leqslant 1$, also $|h| \leqslant 1$ and $h, D_{x'}h \in C^{0,\alpha}(B_1)$ in view of Lemma 8.6 and its corollary.

We now show that h achieves the boundary data $\bar{h}$ continuously. Indeed this follows by classical elliptic theory if we restrict to $\partial B_1 \setminus P$.

If $X_0 \in \partial B_1 \cap (P \setminus L)$ then in a small neighborhood of X_0 intersected with $B_1 \cap \{z > 0\}$ the function $U_n h$ is harmonic continuous up to the boundary and vanishes continuously on $\{z = 0\}$ (since h is bounded). The continuity of h at X_0 then follows from standard boundary regularity for the harmonic function $U_n h$.

Finally, on the set $\partial B_1 \cap L$ as in the case of Laplace equation, it suffices to construct at each point X_0 a local barrier (minimizer) for h which is zero at X_0 and strictly negative in a neighborhood of X_0. Such a barrier is given by (see Lemma 8.4)

$$(x' - x_0') \cdot x_0'.$$

Step 2. In this step we wish to prove that

$$|h(x', x_n, x) - h(x', 0, 0) - b(x')r| \leqslant Cr^{3/2}, \quad (x', 0, 0) \in B_{1/2} \cap L, \tag{8.9}$$

$$|h_r(x', x_n, z) - b(x')| \leqslant Cr^{1/2}, \quad (x', 0, 0) \in B_{1/2} \cap L, \tag{8.10}$$

with C universal and $b(x')$ a Lipschitz function.

Indeed, h solves

$$\Delta(U_n h) = 0 \quad \text{in } B_1 \setminus P.$$

Since U_n is independent on x' we can rewrite this equation as

$$\Delta_{x_n, z}(U_n h) = -U_n \Delta_{x'} h, \tag{8.11}$$

and according to Lemma 8.6 we have that

$$\Delta_{x'} h \in C^\alpha(B_{1/2}),$$

with universal bound. Thus, for each fixed x', we need to investigate the 2-dimensional problem

$$\Delta(U_t h) = U_t f, \quad \text{in } B_{1/2} \setminus \{t \leqslant 0, \ z = 0\}$$

with

$$f \in C^\alpha(B_{1/2}),$$

and h bounded. Without loss of generality, for a fixed x' we may assume $h(x', 0, 0) = 0$.

Let $H(t, z)$ be the solution to the problem

$$\Delta H = U_t f, \quad \text{in } B_{1/2} \setminus \{t \leqslant 0, \ z = 0\},$$

such that

$$H = U_t h \quad \text{on } \partial B_{1/2}, \qquad H = 0 \quad \text{on } B_{1/2} \cap \{t \leqslant 0, \ z = 0\}.$$

We wish to prove that

$$U_t h = H. \tag{8.12}$$

First notice that

$$\Delta(H - U_t h) = 0 \quad \text{in } B_{1/2} \setminus \{t \leqslant 0, \ z = 0\},$$

and

$$H = 0 \quad \text{on } \partial B_{1/2} \cup \left(B_{1/2} \cap \{t < 0, \ z = 0\} \right).$$

We claim that

$$\lim_{(t,z) \to (0,0)} \frac{H - U_t h}{U_t} = \lim_{(t,z) \to (0,0)} \frac{H}{U_t} = 0. \tag{8.13}$$

If the claim holds, then given any $\epsilon > 0$

$$-\epsilon U_t \leqslant H - U_t h \leqslant \epsilon U_t, \quad \text{in } B_\delta$$

with $\delta = \delta(\epsilon)$. Then by the maximum principle the inequality above holds in the whole $B_{1/2}$ and by letting $\epsilon \to 0$ we obtain (8.12).

To prove the claim (8.13) we show that H satisfies the following

$$\left| H(t, z) - aU(t, z) \right| \leqslant C_0 r^{1/2} U(t, z), \quad r^2 = t^2 + z^2, \ a \in \mathbb{R}, \tag{8.14}$$

with C_0 universal.

To do so, we consider the holomorphic transformation

$$\Phi : (s, y) \to (t, z) = \left(\frac{1}{2}(s^2 - y^2), sy \right)$$

which maps $B_1 \cap \{s > 0\}$ into $B_{1/2} \setminus \{t \leqslant 0, \ z = 0\}$ and call

$$\widetilde{H}(s, y) = H(t, z), \qquad \tilde{f}(s, y) = f(t, z).$$

Then, easy computations show that

$$\Delta \widetilde{H} = s \tilde{f} \quad \text{in } B_1 \cap \{s > 0\}, \qquad \widetilde{H} = 0 \quad \text{on } B_1 \cap \{s = 0\}.$$

Since the right-hand side is C^α we conclude that $\widetilde{H} \in C^{2,\alpha}$. In particular $\widetilde{H}_s$ satisfies

$$\left| \widetilde{H}_s(s, y) - a \right| \leqslant C_0 \left| (s, y) \right|, \quad a = \widetilde{H}_s(0, 0)$$

with C_0 universal. Integrating this inequality between 0 and s and using that $\widetilde{H} = 0$ on $B_1 \cap \{s = 0\}$ we get that

$$\left| \widetilde{H}(s, y) - as \right| \leqslant C_0 s \left| (s, y) \right|.$$

In terms of H, this equation gives us

$$|H - aU| \leqslant C_0 r^{1/2} U \tag{8.15}$$

as desired.

Thus (8.12) and (8.14) hold and by combining them and using that $U/U_t = 2r$ we get that

$$|h - 2ar| \leqslant 2C_0 r^{3/2},$$

which is the desired estimate (8.9) i.e. (recall that above we assumed $h(x', 0, 0) = 0$)

$$\left| h(x', x_n, x) - h(x', 0, 0) - b(x')r \right| \leqslant 2C_0 r^{3/2},$$

with

$$b(x') = 2\widetilde{H}_s(x', 0, 0).$$

We remark that $b(x')$ is Lipschitz. Indeed, notice that the derivatives h_i, $i = 1, \ldots, n - 1$, still satisfy the same Eq. (8.11) as h, where the C^α-norm of the right-hand side has a universal bound. Thus, we can argue as above to conclude that

$$\left| \partial_i \widetilde{H}_s(x', 0, 0) \right| \leqslant C,$$

which together with the formula for $b(x')$ shows that $b(x')$ is a Lipschitz function.

Finally, to obtain the second of our estimates (8.10) we proceed similarly as above.

Since $U_t h = H$ one can compute easily that

$$U_t h_r = H_r + \frac{1}{2}\frac{H}{r}. \tag{8.16}$$

Moreover, after our holomorphic transformation

$$2r H_r(t, z) = s\widetilde{H}_s(s, y) + y\widetilde{H}_y(s, y). \tag{8.17}$$

As observed above,

$$\left|\widetilde{H}_s - a(x')\right| \leqslant C|(s, y)|,$$

and similarly since $\widetilde{H} = 0$ on $B_1 \cap \{s = 0\}$

$$|\widetilde{H}_y| \leqslant Cs.$$

These two inequalities combined with (8.17) give us

$$\left|2r H_r - a(x')U\right| \leqslant Cr^{1/2}U. \tag{8.18}$$

Combining (8.16) with (8.15)–(8.18) we obtain (8.10) as desired.

Step 3. In this step we show that h satisfies (8.2).

In view of Lemma 8.4 and estimate (8.10) we obtain that $b(x') = 0$. Since $b(x') = 0$ then (8.9) reads

$$\left|h(x', x_n, z) - h(x', 0, 0)\right| \leqslant Cr^{3/2}.$$

Since h is C^∞ in the x'-direction, we have that for a given $X_0 \in B_{1/4} \cap L$

$$\left|h(x', 0, 0) - h(x'_0, 0, 0) - a' \cdot (x' - x'_0)\right| \leqslant C|x' - x'_0|^2,$$

which combined with the previous inequality gives us the desired bound (8.2). $\quad\square$

Acknowledgement

Appendix A

The purpose of this section is to provide the proof of Proposition 6.4 and Corollary 6.5 which have been used in the proof of Harnack inequality in Section 6. For the reader's convenience we recall their statements.

Let $R > 0$ and denote

$$V_R(t, z) = U(t, z)\left((n-1)\frac{t}{R} + 1\right).$$

Then set

$$v_R(X) = V_R\left(R - \sqrt{|x'|^2 + (x_n - R)^2}, z\right).$$

Proposition 6.4. *If R is large enough, the function $v_R(X)$ is a comparison subsolution to (2.1) in B_2 which is strictly monotone increasing in the e_n-direction in $B_2^+(v_R)$. Moreover, there exists a function $\tilde{v}_R$ such that*

$$U(X) = v_R\left(X - \tilde{v}_R(X)e_n\right), \quad \text{in } B_1 \setminus P \tag{A.1}$$

and

$$\left|\tilde{v}_R(X) - \gamma_R(X)\right| \leqslant \frac{C}{R^2}|X|^2, \qquad \gamma_R(X) = -\frac{|x'|^2}{2R} + 2(n-1)\frac{x_n r}{R}, \tag{A.2}$$

with $r = \sqrt{x_n^2 + z^2}$ and C universal.

Proof. We divide the proof of this proposition in two steps.

Step 1. In this step we show that v_R is a comparison subsolution in B_2 which is monotone in the e_n-direction.

First we need to show that v_R is subharmonic (but not harmonic) in $B_2^+(v_R)$. From the formula for v_R we see immediately that $(R > 2)$

$$B_2^+(v_R) = B_2 \setminus \left(\mathcal{B}_2 \setminus \overline{\mathcal{B}}_R(Re_n) \right).$$

One can easily compute that on such set,

$$\Delta v_R(X) = \left((\partial_{tt} + \partial_{zz}) V_R \right)(R - \rho, z) - \frac{n-1}{\rho} \partial_t V_R(R - \rho, z),$$

where for simplicity we call

$$\rho := \sqrt{|x'|^2 + (x_n - R)^2}.$$

Also for (t, z) outside the set $\{(t, 0): t \leqslant 0\}$

$$\Delta_{t,z} V_R(t, z) = (\partial_{tt} + \partial_{zz}) V_R(t, z) = \frac{2(n-1)}{R} \partial_t U(t, z) + \left(1 + (n-1)\frac{t}{R} \right) \Delta_{t,z} U(t, z)$$

$$= \frac{2(n-1)}{R} \partial_t U(t, z),$$

and

$$\partial_t V_R(t, z) = \left(1 + (n-1)\frac{t}{R} \right) \partial_t U(t, z) + \frac{n-1}{R} U(t, z). \tag{A.3}$$

Thus to show that Δv_R is subharmonic in $B_2^+(v_R)$ we need to prove that in such set

$$\frac{2(n-1)}{R} \partial_t U - \frac{n-1}{\rho} \left[\left(1 + (n-1)\frac{R-\rho}{R} \right) \partial_t U + \frac{n-1}{R} U \right] \geqslant 0,$$

where U and $\partial_t U$ are evaluated at $(R - \rho, z)$.

Set $t = R - \rho$, then straightforward computations reduce the inequality above to

$$(n-1)\left[2(R-t) - R - (n-1)^2 t \right] \partial_t U(t, z) - (n-1)^2 U(t, z) \geqslant 0.$$

Using that $\partial_t U(t, z) = U(t, z)/(2r)$ with $r^2 = t^2 + z^2$, this inequality becomes

$$R \geqslant 2t + (n-1)^2 t + 2(n-1)r.$$

This last inequality is easily satisfied for R large enough, since $t, r \leqslant 3$.

Now we prove that v_R satisfies the free boundary condition in Definition 2.2. First observe that

$$F(v_R) = \partial \mathcal{B}_R(Re_n, 0) \cap B_2,$$

and hence it is smooth. By the radial symmetry it is enough to show that the free boundary condition is satisfied at $0 \in F(v_R)$ that is

$$v_R(x, z) = \alpha U(x_n, z) + o\left(|(x, z)|^{1/2} \right), \quad \text{as } (x, z) \to (0, 0), \tag{A.4}$$

with $\alpha \geqslant 1$.

First notice since U is Holder continuous with exponent $1/2$, it follows from the formula for V_R that

$$\left| V_R(t, z) - V_R(t_0, z) \right| \leqslant C|t - t_0|^{1/2} \quad \text{for } |t - t_0| \leqslant 1.$$

Thus for $(x, z) \in B_s$, s small

$$\left| v_R(x, z) - V_R(x_n, z) \right| = \left| V_R(R - \rho, z) - V_R(x_n, z) \right| \leqslant C|R - \rho - x_n|^{1/2} \leqslant Cs,$$

where we have used that (recall that $\rho := \sqrt{|x'|^2 + (x_n - R)^2}$)

$$R - \rho - x_n = -\frac{|x'|^2}{R - x_n + \rho}. \tag{A.5}$$

It follows that for $(x, z) \in B_s$

$$
\begin{aligned}
\left| v_R(x, z) - U(x_n, z) \right| &\leqslant \left| v_R(x, z) - V_R(x_n, z) \right| + \left| V_R(x_n, z) - U(x_n, z) \right| \\
&\leqslant Cs + \left| V_R(x_n, z) - U(x_n, z) \right|.
\end{aligned}
$$

Thus from the formula for V_R

$$
\left| v_R(x, z) - U(x_n, z) \right| \leqslant Cs + (n - 1) \frac{|x_n|}{R} U(x_n, z) \leqslant C's, \quad (x, z) \in B_s
$$

which gives the desired expansion (A.4) with $\alpha = 1$.

Now, we show that v_R is strictly monotone increasing in the e_n-direction in $B_2^+(v_R)$. Outside of its zero plate,

$$
\partial_{x_n} v_R(x) = -\frac{x_n - R}{\rho} \partial_t V_R(R - \rho, z).
$$

Thus we only need to show that $V_R(t, z)$ is strictly monotone increasing in t outside $\{(t, 0): \ t \leqslant 0\}$. This follows immediately from (A.3) and the formula for U.

Step 2. In this step we show the existence of $\tilde{v}_R$ satisfying (A.1) and (A.2). Since we have a precise formula for v_R in terms of U, this is only a matter of straightforward (though tedious) computations which we present here for completeness.

First we show that there exists a unique $\tilde{t}$ such that (here B_1 is the 2-dimensional ball)

$$
U(t, z) = V_R(t + \tilde{t}, z), \quad \text{in } B_1 \setminus \{(t, 0): \ t \leqslant 0\}, \tag{A.6}
$$

and

$$
\left| \tilde{t} + \frac{2(n - 1)tr}{R} \right| \leqslant \frac{\widetilde{C}}{R^2} r^3, \quad r^2 = t^2 + z^2, \tag{A.7}
$$

with $\widetilde{C}$ universal. Since V_R is strictly increasing in the t-direction in $B_1 \setminus \{(t, 0): \ t \leqslant 0\}$ it suffices to show that

$$
V_R\left(t - \frac{2(n - 1)tr}{R} - \frac{\widetilde{C}}{R^2} r^3 \right) < U(t, z) < V_R\left(t - \frac{2(n - 1)tr}{R} + \frac{\widetilde{C}}{R^2} r^3 \right). \tag{A.8}
$$

Let us prove the lower bound. We call

$$
\bar{t} = -\frac{2(n - 1)tr}{R} - \frac{\widetilde{C}}{R^2} r^3,
$$

and we use Taylor's formula to compute

$$
V_R(t + \bar{t}, z) = V_R(t, z) + \partial_t V_R(t, z) \bar{t} + \frac{1}{2} E \bar{t}^2, \qquad |E| \leqslant \left| \partial_{tt} V_R(s, z) \right|, \tag{A.9}
$$

with s between t and $t + \bar{t}$. We claim that

$$
\left| \partial_{tt} V_R(s, z) \right| \leqslant \frac{C'}{r^2} U(t, z). \tag{A.10}
$$

Indeed one can compute that

$$
\begin{aligned}
\partial_{tt} V_R(s, z) &= \left(1 + (n - 1) \frac{s}{R} \right) \partial_{tt} U(s, z) + \frac{2(n - 1)}{R} \partial_t U(s, z) \\
&= \left(1 + (n - 1) \frac{s}{R} \right) r^{-3/2} \partial_{tt} U\left(\frac{s}{r}, \frac{z}{r} \right) + r^{-1/2} \frac{2(n - 1)}{R} \partial_t U\left(\frac{s}{r}, \frac{z}{r} \right)
\end{aligned} \tag{A.11}
$$

where we have used that U is homogeneous of degree $1/2$.

Since s lies between t and $t + \bar{t}$ we get that $(s/r, z/r) \in B_{3/2} \setminus B_{1/2}$, thus by boundary Harnack inequality in this annulus we get

$$
\left| \partial_{tt} U\left(\frac{s}{r}, \frac{z}{r} \right) \right| \leqslant K_1 U\left(\frac{s}{r}, \frac{z}{r} \right), \quad \partial_t U\left(\frac{s}{r}, \frac{z}{r} \right) \leqslant K_2 U\left(\frac{s}{r}, \frac{z}{r} \right),
$$

with K_1, K_2 universal.

Combining the above inequalities with (A.11) we obtain that

$$\left|\partial_{tt} V_R(s,z)\right| \leqslant \overline{C} r^{-3/2} U\left(\frac{s}{r}, \frac{z}{r}\right).$$

Thus the claim in (A.10) will follow if we show that for some K universal

$$U\left(\frac{s}{r}, \frac{z}{r}\right) \leqslant K U\left(\frac{t}{r}, \frac{z}{r}\right).$$

Again, this follows by the boundary Harnack inequality in the annulus $B_{3/2} \setminus B_{1/2}$ between $U(\frac{t}{r}, \frac{z}{r})$ and its translation $U(\frac{t+\tau}{r}, \frac{z}{r})$, for τ small. Our claim is thus proved.

Thus, using (A.9) together with this claim, the lower bound in (A.8) will be proved if we show that

$$U(t,z) > V_R(t,z) + \partial_t V_R(t,z)\bar{t} + \frac{C'}{2r^2} U(t,z)\bar{t}^2.$$

From the definition of V_R this is equivalent to showing that

$$(n-1)\frac{t}{R} U(t,z) + \left[\left(1 + (n-1)\frac{t}{R}\right)\partial_t U(t,z) + \frac{n-1}{R} U(t,z)\right]\bar{t} + \frac{C'}{2r^2} U(t,z)\bar{t}^2 < 0.$$

Dividing everything by $\partial_t U(t,z) = \frac{1}{2r} U(t,z)$ we get

$$\frac{2(n-1)rt}{R} + \left[\left(1 + (n-1)\frac{t}{R}\right) + \frac{2r(n-1)}{R}\right]\bar{t} + \frac{C'}{r}\bar{t}^2 < 0,$$

and using the definition of $\bar{t}$ we finally need to show that

$$(n-1)\frac{t+2r}{R}\bar{t} + \frac{C'}{r}\bar{t}^2 < \frac{\widetilde{C}}{R^2} r^3.$$

Using that for R large

$$|\bar{t}| \leqslant K r^2/R,$$

for K universal, we easily obtain that the inequality above holds for the appropriate $\widetilde{C}$ (universal) and R large.

To conclude our proof, we use (A.5) to write

$$v_R(X - \tilde{v}_R e_n) = V_R\big(R - \rho(\tilde{v}_R), z\big) = V_R\left(x_n - \tilde{v}_R - \frac{|x'|^2}{R - x_n + \tilde{v}_R + \rho(\tilde{v}_R)}, z\right),$$

with

$$\rho(\eta) := \sqrt{|x'|^2 + (x_n - \eta - R)^2}.$$

In view of (A.6) if there exists $\tilde{v}_R = \tilde{v}_R(X)$ such that

$$-\tilde{v}_R - \frac{|x'|^2}{R - x_n + \tilde{v}_R + \rho(\tilde{v}_R)} = \bar{t}, \tag{A.12}$$

then

$$v_R(X - \tilde{v}_R e_n) = U(X), \quad \text{in } B_1^+(U).$$

By the strict monotonicity of v_R in the e_n-direction in $B_1^+(v_R)$, in such set $\tilde{v}_R$ must be unique.

Thus our claim will follow if we show that there exists $\tilde{v}_R$ satisfying (A.12) and such that

$$\left|\tilde{v}_R(X) - \gamma_R(X)\right| \leqslant C \frac{|X|^2}{R^2}.$$

To do so, we call

$$f(\eta) = -\eta - \frac{|x'|^2}{R - x_n + \eta + \rho(\eta)}, \quad -1 \leqslant \eta \leqslant 1,$$

and we show that

$$f\left(\gamma_R(X) + C\frac{|X|^2}{R^2}\right) \leqslant \tilde{t} \leqslant f\left(\gamma_R(X) - C\frac{|x|^2}{R^2}\right).$$

In view of (A.7) we need to prove that

$$f\left(\gamma_R(X) + C\frac{|X|^2}{R^2}\right) \leqslant -\frac{2(n-1)x_n r}{R} - \widetilde{C}\frac{r^3}{R^2},$$

and

$$f\left(\gamma_R(X) - C\frac{|X|^2}{R^2}\right) \geqslant -\frac{2(n-1)x_n r}{R} + \widetilde{C}\frac{r^3}{R^2}.$$

Let us prove the first inequality (the second one follows similarly). Call

$$\bar{\eta} = \gamma_R(X) + C\frac{|X|^2}{R^2}.$$

From the definition of f and γ_R the desired inequality is equivalent to

$$\frac{|x'|^2}{2R} - C\frac{|X|^2}{R^2} - \frac{|x'|^2}{R - x_n + \bar{\eta} + \rho(\bar{\eta})} \leqslant -\widetilde{C}\frac{r^3}{R^2}.$$

Clearly $-1 \leqslant \bar{\eta} \leqslant 1$, and one can easily verify that

$$R - x_n + \bar{\eta} + \rho(\bar{\eta}) \leqslant 2R + 5.$$

Thus

$$\frac{|x'|^2}{2R} - \frac{|x'|^2}{R - x_n + \bar{\eta} + \rho(\bar{\eta})} \leqslant |x'|^2\left(\frac{1}{2R} - \frac{1}{2R+5}\right) \leqslant \frac{|x'|^2}{R^2},$$

and the desired inequality follows if we show that

$$\frac{|x'|^2}{R^2} - C\frac{|X|^2}{R^2} \leqslant -\widetilde{C}\frac{r^3}{R^2}.$$

This inequality is trivially satisfied as long as $C - \widetilde{C} \geqslant 1$. $\square$

We now recall the statement of Corollary 6.5 and sketch its proof.

Corollary 6.5. *There exist δ, c_0, C_0, C_1 universal constants, such that*

$$v_R\left(X + \frac{c_0}{R}e_n\right) \leqslant \left(1 + \frac{C_0}{R}\right)U(X), \quad in\ \overline{B}_1 \setminus B_{1/4}, \tag{A.13}$$

with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n)) \cap \overline{B}_1 \setminus B_{1/4}$,

$$v_R\left(X + \frac{c_0}{R}e_n\right) \geqslant U\left(X + \frac{c_0}{2R}e_n\right), \quad in\ B_\delta, \tag{A.14}$$

$$v_R\left(X - \frac{C_1}{R}e_n\right) \leqslant U(X), \quad in\ \overline{B}_1. \tag{A.15}$$

Proof. Estimates (A.14) and (A.15) are immediate consequences of (A.2) and Lemma 3.1.

To obtain (A.13), notice that in view of (A.2) and Lemma 3.1,

$$v_R\left(X + \frac{c_0}{R}e_n\right) \leqslant U(X) \quad in\ \{X \in B_1 : |x'| \geqslant 1/8,\ |x_n| \leqslant \bar{\delta}\},$$

for some $c_0, \bar{\delta}$ small universal and R large (with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n))$). Hence the estimate (A.13) holds on the set $\{X \in B_1 \setminus B_{1/4} : \sqrt{x_n^2 + z^2} \leqslant \bar{\delta}\}$ and we only need to prove it on the complement of this set.

Again, from (A.2) and Lemma 3.1 we get that

$$v_R\left(X + \frac{c_0}{R}e_n\right) \leqslant U\left(X + \frac{\overline{C}}{R}e_n\right) \quad \text{in } B_1, \tag{A.16}$$

for $\overline{C}$ large universal. From Lemma 5.2 we know that

$$U\left(x_n + \frac{\overline{C}}{R}, z\right) \leqslant \left(1 + C\frac{\overline{C}}{R}\right)U(x_n, z),$$

as long as $\sqrt{x_n^2 + z^2} > \overline{\delta}$, with $C = C(\overline{\delta})$ (and R large). Combining this fact with (A.16) we get

$$v_R\left(X + \frac{c_0}{R}e_n\right) \leqslant \left(1 + \frac{C_0}{R}\right)U(x_n, z),$$

on the desired set. $\square$

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