Available online at www.sciencedirect.com
SciVerse ScienceDirect
ELSEVIER
Ann. I. H. Poincaré – AN 29 (2012) 377–399
ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations ☆

Songzhe Lian, Wenjie Gao *, Hongjun Yuan, Chunling Cao

Institute of Mathematics, Jilin University, Changchun, Jilin 130012, People's Republic of China

Received 1 September 2011; received in revised form 20 December 2011; accepted 10 January 2012
Available online 16 January 2012

Abstract

The existence and uniqueness of weak solutions are studied to the initial Dirichlet problem of the equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + f(x,t,u),$$

with $\inf p(x) > 2$. The problems describe the motion of generalized Newtonian fluids which were studied by some other authors in which the exponent p was required to satisfy a logarithmic Hölder continuity condition. The authors in this paper use a difference scheme to transform the parabolic problem to a sequence of elliptic problems and then obtain the existence of solutions with less constraint to $p(x)$. The uniqueness is also proved.

Keywords: Electrorheological fluids; $p(x)$-Laplace; Degenerate; Parabolic

1. Introduction

Let $\Omega \subset R^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Consider the following problem

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + f(x,t,u), \quad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

$$u|_{\Gamma_T} = 0, \qquad u|_{t=0} = u_0, \tag{1.2}$$

where $\Gamma_T = \partial\Omega \times [0,T]$ and $p(x)$ is a measurable function.

In the case when p is a constant, there have been many results about the existence, uniqueness and the regularity of the solutions. We refer the readers to the bibliography given in [5,11,12] and the references therein.

A new interesting kind of fluids of prominent technological interest has recently emerged: the so-called electrorheological fluids. This model includes parabolic equations which are nonlinear with respect to the gradient of the thought solution, and with variable exponents of nonlinearity. The typical case is the so-called evolution p-Laplace equation with exponent p as a function of the external electromagnetic field (see [1,2,10] and the references therein).

☆ The project is supported by NSFC (10771085, 10571072) and by the 985 program of Jilin University.
* Corresponding author.
E-mail addresses: liansz@jlu.edu.cn (S. Lian), gaowj@jlu.edu.cn (W. Gao), hjy@jlu.edu.cn (H. Yuan), caocl@jlu.edu.cn (C. Cao).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2012.01.001

In [13], Zhikov showed that

$$W_0^{1,p(x)}(\Omega) \neq \{v \in W^{1,p(x)}(\Omega) \mid v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega).$$

Hence, the property of the space is different from the case when p is a constant (see Section 2 for the definition of the function spaces).

As we have known, when p is a constant, the non-degenerate problems have classical solutions and hence the weak solutions exist. But to the case of $p(x)$-Laplace type, there is no results to the corresponding non-degenerate problems. These will bring us some new difficulties in studying the weak solutions.

For more general $p(x,t)$-Laplace equation, the authors of [3] established the existence and uniqueness results with the exponent $p(x,t)$ satisfying the so-called logarithmic Hölder continuity condition, i.e.

$$\left|p(x) - p(y)\right| \leqslant \omega\left(|x-y|\right), \quad \forall x, y \in Q_T,\ |x-y| < \frac{1}{2} \tag{1.3}$$

with

$$\overline{\lim_{s\to 0^+}}\, \omega(s)\ln\left(\frac{1}{s}\right) = C < \infty.$$

However if $p(x,t)$ satisfies (1.3), then (see [14])

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega).$$

Therefore, we can ask whether the logarithmic Hölder continuity to $p(x,t)$ is indispensable for the existence of solutions to the problem.

In the present work, we will study the existence of the solutions to problem (1.1)–(1.2) without the condition (1.3). Unlike [3] , we will, in this paper, adopt a method of difference in time. Note that the author in [9] considered the p-Laplace equation without the term $f(x,t,u)$ by using a similar method. To overcome the difficulties caused by $p(x)$, we will develop some new ideas and new techniques.

The outline of this paper is the following: In Section 2, we introduce some basic Lebesgue and Sobolev spaces and state our main theorems. In Section 3, we give the existence of weak solutions to a difference equation (approximating problem). In Section 4, we will prove the global existence of solutions to the problem (1.1)–(1.2). Section 5 will be devoted to the proof of the local existence and the existence of weak solutions under some weaker conditions to the initial function u_0.

2. Basic spaces and the main results

To study our problems, we need to introduce some new function spaces.

Denote

$$p^+ = \operatorname{ess}\sup_{\overline{\Omega}} p(x), \qquad p^- = \operatorname{ess}\inf_{\overline{\Omega}} p(x).$$

Throughout the paper we assume that

$$2 < p^- \leqslant p(x) \leqslant p^+ < \infty, \quad \forall x \in \Omega. \tag{2.1}$$

Set

$$L^{p(x)}(\Omega) = \left\{u \;\middle|\; u \text{ is a measurable real-valued function, } \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty\right\},$$

$$|u|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\},$$

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$.

In the following, we state some of the properties of the function spaces introduced as above (see [6] and [7]).

Proposition 2.1. (i) *The space* $(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), |\cdot|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *Let* $q_1(x)$ *and* $q_2(x)$ *be real functions with* $1/q_1(x) + 1/q_2(x) = 1$ *and* $q_1(x) > 1$. *Then, the conjugate space of* $L^{q_1(x)}(\Omega)$ *is* $L^{q_2(x)}(\Omega)$. *And for any* $u \in L^{q_1(x)}(\Omega)$ *and* $v \in L^{q_2(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right| \leqslant 2|u|_{L^{q_1(x)}(\Omega)}|v|_{L^{q_2(x)}(\Omega)}.$$

(iii)

$$|u|_{L^{p(x)}(\Omega)} = 1, \quad \text{then } \int_\Omega |u|^{p(x)}\,dx = 1,$$

$$|u|_{L^{p(x)}(\Omega)} > 1, \quad \text{then } |u|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \int_\Omega |u|^{p(x)}\,dx \leqslant |u|^{p^+}_{L^{p(x)}(\Omega)},$$

$$|u|_{L^{p(x)}(\Omega)} < 1, \quad \text{then } |u|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \int_\Omega |u|^{p(x)}\,dx \leqslant |u|^{p^-}_{L^{p(x)}(\Omega)}.$$

(iv) *If* $p_1(x) \leqslant p_2(x)$, *then* $L^{p_1(x)} \supset L^{p_2(x)}$.

Proposition 2.2. *If* $p(x) \in C(\overline{\Omega})$, *then there is a constant* $C > 0$, *such that*

$$|u|_{L^{p(x)}(\Omega)} \leqslant C|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $|\nabla u|_{L^{p(x)}(\Omega)}$ *and* $|u|_{W^{1,p(x)}(\Omega)}$ *are equivalent norms of* $W_0^{1,p(x)}$.

We now give the definition of the solutions to our problem.

Definition 2.1. A function u is said to be a weak solution of (1.1)–(1.2), if u satisfies the following:

$$u \in L^2(Q_T), \qquad f(x,t,u) \in L^1(Q_T), \qquad D_i u \in L^{p(x)}(Q_T),$$

$u = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces,

$$\int_0^T\int_\Omega \left(u\frac{\partial\varphi}{\partial t} - |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi + f\varphi\right)dx\,dt = 0, \tag{2.2}$$

for all $\varphi \in C_0^\infty(Q_T)$ and

$$\lim_{t\to 0}\int_\Omega \big(u(x,t) - u_0(x)\big)\psi(x)\,dx = 0, \tag{2.3}$$

for all $\psi \in C_0^\infty(\Omega)$, where $Q_T = \Omega \times (0,T)$.

In the study of the global existence of solutions, we need the following hypotheses to the function f:

$$f(x,t,z) \in C^1\big(\overline{\Omega}\times[0,T]\times R\big) \quad \text{and} \quad \big|f(x,t,z)\big| \leqslant C_0\big(\phi(x,t) + |z|^\alpha\big), \tag{A}$$

where $\phi \geqslant 0$, $\phi \in L^r(\Omega \times (0,T))$, $r > (N + p^-)/p^-$ and $C_0 > 0$, $\alpha \geqslant 0$ are constants.

Our main results are the following.

Theorem 2.1. *Let* $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ *and* (A) *hold.*

Assume that

$$\begin{aligned} &\alpha < p^- - 1 \quad \text{or} \\ &\alpha = p^- - 1 \quad \text{and} \quad |\Omega| \text{ is sufficiently small,} \end{aligned} \tag{B}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Then there exists a weak solution of (1.1)–(1.2) *such that*

$$u \in L^\infty(Q_T) \cap L^\infty\big(0,T; W_0^{1,p(x)}(\Omega)\big), \qquad u_t \in L^2\big(\Omega \times (0,T)\big).$$

Remark 2.1. In certain sense, the constrains to α in (B) is necessary even to the case when p is a constant (see [11]).

Theorem 2.2. *If $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and $f(x,t,z) \in C^1(\overline{\Omega} \times [0,T] \times R)$. Then there exists a $T^* > 0$ such that* (1.1)–(1.2) *has a solution u in Q_{T^*}.*

Theorem 2.3. *If $f(x,t,z) \in C^1(\overline{\Omega} \times [0,T] \times R)$, then the solution of* (1.1)–(1.2) *with*

$$u \in L^\infty(Q_T), \qquad u_t \in L^2\big(\Omega \times (0,T)\big),$$

is unique.

Remark 2.2. Combining Theorems 2.1 and 2.3, we can obtain the existence of global solutions.

We also consider the problem under a weaker condition for u_0.

Theorem 2.4. *Let $u_0 \in L^\infty(\Omega)$.*

(i) *If* (A) *and* (B) *hold, then there exists a weak solution u of* (1.1)–(1.2) *such that*

$$u \in L^\infty(Q_T) \cap L^\infty\big(\epsilon,T; W_0^{1,p(x)}(\Omega)\big), \qquad u_t \in L^2\big(\Omega \times (\epsilon,T)\big), \tag{2.4}$$

where $0 < \epsilon < T$ is a constant.

(ii) *If $f(x,t,z) \in C^1(\overline{\Omega} \times [0,T] \times R)$, then there exists a $T^* > 0$ such that* (1.1)–(1.2) *has a solution u in Q_{T^*} satisfying* (2.4).

3. Existence of weak solutions to a difference equation

Let

$$F^i(x,u) = \int_{u_{i-1}}^{u} \left(\frac{1}{h} \int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau \right) ds, \tag{3.1}$$

and

$$\psi^i(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx - \int_\Omega F^i(x,u)\,dx + \frac{1}{2h} \int_\Omega (u - u_{i-1})^2\,dx, \quad i = 1,2,\ldots. \tag{3.2}$$

Denote

$$p^* = \begin{cases} \frac{Np^-}{N-p^-}, & \text{if } N > p^-, \\ (\frac{N+p^-}{N})p^-, & \text{if } N \leqslant p^-. \end{cases} \tag{3.3}$$

Lemma 3.1. *Assume that $p(x) \in C(\overline{\Omega})$, $u_{i-1}(x) \in L^{p^*}(\Omega)$ and* (A), (B) *hold. Then the functional $\psi^i(u)$ achieves its minimum on the set*

$$S = \big\{u \in W_0^{1,p(x)}(\Omega)\big\}. \tag{3.4}$$

Proof. We will show, in three steps, that $\psi^i(u)$ satisfies the conditions which assure the existence of a minimum on the set.

Step 1. S is weakly closed.

By Proposition 2.1(i) we know that $W_0^{1,p(x)}(\Omega)$ is a reflexive Banach space and then by Mazur theorem it is weakly closed.

Step 2. $\psi^i(u)$ satisfies the coerciveness conditions.

By (A) we have

$$\psi^i(u) \geqslant \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - C_0\int_\Omega\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)|u-u_{i-1}|\,dx$$
$$- C_1\int_\Omega\left(|u|^{\alpha+1}+|u_{i-1}|^{\alpha+1}\right)dx + \frac{1}{2h}\int_\Omega (u-u_{i-1})^2\,dx, \tag{3.5}$$

where $C_1 > 0$ is a constant.

We first estimate the second term on the right-hand side of the inequality.

Denote $r_1 = (N+p^-)/p^-$ and $r_2 = (N+p^-)/N$.

By (A) and the Hölder inequality, we get

$$\begin{aligned}
I_1 &= C_0\int_\Omega\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)|u-u_{i-1}|\,dx\\
&\leqslant C_0\left(\int_\Omega\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)^{r_1}dx\right)^{1/r_1}\left(\int_\Omega|u-u_{i-1}|^{r_2}\,dx\right)^{1/r_2}\\
&\leqslant C\left(\frac{1}{h}\int_\Omega\int_{ih}^{(i+1)h}\phi^{r_1}(x,\tau)\,d\tau\,dx\right)^{1/r_1}\left(\int_\Omega|u-u_{i-1}|^{r_2}\,dx\right)^{1/r_2}\\
&\leqslant C\|u-u_{i-1}\|_{L^{r_2}(\Omega)} \leqslant C\left(\|u\|_{L^{r_2}(\Omega)}+\|u_{i-1}\|_{L^{r_2}(\Omega)}\right).
\end{aligned}$$

Notice that $r_2 = (N+p^-)/N < p^*$ for $N > p^-$. By the imbedding inequality and Young's inequality, for all $N \geqslant 1$, we have

$$I_1 \leqslant \frac{1}{4}\int_\Omega|\nabla u|^{p^-}\,dx + C \leqslant \frac{1}{4}\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}\,dx + C.$$

Now, we estimate $C_1\int_\Omega(|u|^{\alpha+1}+|u_{i-1}|^{\alpha+1})\,dx$ in the following two cases.

(i) $\alpha < p^- - 1$.

By Young's inequality and the Poincaré inequality, we get

$$\begin{aligned}
C_1\int_\Omega\left(|u|^{\alpha+1}+|u_{i-1}|^{\alpha+1}\right)dx &\leqslant \epsilon\int_\Omega|u|^{p^-}\,dx + C\\
&\leqslant \frac{1}{4}\int_\Omega|\nabla u|^{p^-}\,dx + C \leqslant \frac{1}{4}\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}\,dx + C.
\end{aligned}$$

(ii) $\alpha = p^- - 1$, but the Lebesgue measure of Ω is sufficiently small.

By the *Poincaré* inequality, we get

$$\begin{aligned}
C_1\int_\Omega\left(|u|^{p^-}+|u_{i-1}|^{p^-}\right)dx &\leqslant C_1\int_\Omega|u|^{p^-}\,dx + C\\
&\leqslant \frac{1}{4}\int_\Omega|\nabla u|^{p^-}\,dx + C \leqslant \frac{1}{4}\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}\,dx + C.
\end{aligned}$$

Summarizing up the above estimates and combining Proposition 2.2, we get

$$\psi^i(u) \geqslant \frac{1}{2p^+}|\nabla u|^{p^-}_{L^{p(x)}(\Omega)} - C$$
$$\geqslant \frac{1}{2Cp^+}|u|^{p^-}_{W^{1,p(x)}(\Omega)} - C \to \infty, \quad \text{as } |u|_{W^{1,p(x)}(\Omega)} \to \infty.$$

Step 3. $\psi^i(u)$ is weakly lower semicontinuous.

At first, by the convexity of the functional, we know that for $\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$, weakly lower semicontinuous is equivalent to lower semicontinuous (see [4]).

Let

$$v_l \to v, \quad \text{in } W_0^{1,p(x)} \quad \text{as } l \to \infty. \tag{3.6}$$

Then by Proposition 2.1(iii), we have $|v_l|_{L^{p(x)}(\Omega)}, \int_\Omega |\nabla v_l|^{p(x)}\,dx \leqslant C, l = 1, 2, \ldots.$

Now

$$\left|\int_\Omega \frac{1}{p(x)}|\nabla v_l|^{p(x)}\,dx - \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right|$$
$$\leqslant \int_\Omega \int_0^1 \left|s\nabla v_l + (1-s)\nabla v\right|^{p(x)-1} \cdot |\nabla v_l - \nabla v|\,ds\,dx$$
$$= \int_0^1 \int_\Omega \left|s\nabla v_l + (1-s)\nabla v\right|^{p(x)-1} \cdot |\nabla v_l - \nabla v|\,dx\,ds.$$

Then combining Proposition 2.1(ii), (iii), we know that $\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$ is a continuous functional. Therefore, $\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$ is weakly lower semicontinuous.

Now, consider the functional

$$I_2 = -\int_\Omega F^i(x,u)\,dx + \frac{1}{2h}\int_\Omega (u-u_{i-1})^2\,dx.$$

By (3.6), using (ii) and (iv) of Proposition 2.1, for any $0 < \epsilon < p^-$, we have

$$v_l \overset{\text{weak}}{\rightharpoonup} v, \quad \text{in } W_0^{1,p^- - \epsilon},$$

and then using the Sobolev compact imbedding theorem we get

$$v_l \to v, \quad \text{in } L^{p^*_\epsilon},$$

where

$$p^*_\epsilon = \begin{cases} \frac{N(p^- - \epsilon)}{N-(p^- - \epsilon)}, & \text{if } N > p^- - \epsilon, \\ \frac{N+(p^- - \epsilon)}{N}(p^- - \epsilon), & \text{if } N \leqslant p^- - \epsilon. \end{cases}$$

For small enough ϵ, we have $L^{p^*_\epsilon} > \max\{r/(r-1), 2\}$. Combining (A), we may prove that the functional I_2 is continuous in $L^{p^*_\epsilon}$. Hence I_2 is weakly lower semicontinuous.

Obviously, the sum of two weakly lower semicontinuous functionals is weakly lower semicontinuous functional and our conclusion follows.

By above results and a standard argument (see [4]), we know that the functional $\psi^i(u)$ achieves its minimum on the set S. □

Lemma 3.2. *Let* $u_+ = \max\{0, u\}$. *Assume that* u *is a minima obtained in Lemma* 3.1. *Then for any constant* $k \geqslant 1$, *both* u *and* $-u$ *satisfy*

$$\frac{1}{h}\int_\Omega (u-k)_+^2\,dx + \int_{D_k} |\nabla u|^{p(x)}\,dx \leqslant \frac{1}{h}\int_\Omega (u-k)_+\left(\int_{ih}^{(i+1)h} \left|f(x,\tau,u)\right|d\tau + |u_{i-1}|\right)dx, \tag{3.7}$$

where $D_k = \{x \in \Omega: u(x) > k\}$.

Proof. For $0 \leqslant \epsilon < 1$, we have $u - \epsilon(u-k)_+ \in S$ and then

$$g(-\epsilon) = \psi^i\big(u - \epsilon(u-k)_+\big) \geqslant \psi^i(u) = g(0).$$

Therefore,

$$\lim_{\epsilon\to 0,\,\epsilon>0} \frac{g(-\epsilon)-g(0)}{-\epsilon} \leqslant 0.$$

Plugging into the definition of g, we get

$$\frac{1}{h}\int_\Omega (u-k)_+(u-u_{i-1})\,dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(u-k)_+\,dx \leqslant \frac{1}{h}\int_\Omega (u-k)_+\left(\int_{ih}^{(i+1)h} f(x,\tau,u)\,d\tau\right)dx.$$

Notice that

$$(u-k)_+(u-u_{i-1}) = (u-k)_+^2 - (u-k)_+(u_{i-1}-k) \geqslant (u-k)_+^2 - (u-k)_+(u_{i-1}-k)_+,$$

the conclusion of the lemma can be proved easily.

Also, by

$$\psi^i\big(u + \epsilon(-u-k)_+\big) \geqslant \psi^i(u),$$

we know that the conclusion of the lemma holds for $-u$. □

Remark 3.1. Moreover, if $u \in L^\infty(\Omega)$, we have

$$\int_\Omega \frac{(u-k)_+^{(q+1)}}{h}\,dx + q\int_\Omega |\nabla u|^{p(x)}(u-k)_+^{(q-1)}\,dx \leqslant \int_\Omega \frac{(u_{i-1}-k)_+(u-k)_+^q}{h}\,dx + \frac{1}{h}\int_\Omega (u-k)_+^q \int_{ih}^{(i+1)h} f(x,\tau,u)\,d\tau\,dx, \tag{3.8}$$

where $q \geqslant 1$ is a constant.

Now we consider the following problem.

$$\frac{1}{h}(u_i - u_{i-1}) = \operatorname{div}\big(|\nabla u_i|^{p(x)-2}\nabla u_i\big) + \frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau, \quad x \in \Omega, \tag{3.9}$$

$$u_i|_{\partial\Omega} = 0, \quad i = 1,2,\ldots, \tag{3.10}$$

where $h > 0$ is a constant.

By Lemma 3.1, similarly to Lemma 3.2, we get

Lemma 3.3. *Let* (A), (B) *hold. Assume that* $p(x) \in C(\overline{\Omega})$ *and* $u_{i-1}(x) \in L^{p^*}(\Omega)$. *Then there exists a weak solution* u_i *of* (3.9)–(3.10) *such that* $u_i \in W_0^{1,p(x)}(\Omega)$.

4. Global existence of weak solutions

In the following we assume that

$$lh \leqslant T < (l+1)h,$$

where l is an integer.

Define $u^h : \Omega \times [0,\infty) \to R$ such that

$$u^{(h)}(\cdot,t) = u_i, \quad \text{for } t \in \big[ih,(i+1)h\big),\ i = 0,1,\dots,l \tag{4.1}$$

where u_i is a solution obtained in Lemma 3.3.

We will prove that a subsequence of $u^{(h)}$ converges and the limiting function is a solution of (1.1)–(1.2).

Denote

$$\begin{aligned}\partial^{(-h)}u^{(h)}(\cdot,t) &= \frac{1}{-h}\big(u^{(h)}(\cdot,t-h) - u^{(h)}(\cdot,t)\big)\\ &= \begin{cases}\frac{1}{h}(u_i - u_{i-1})(\cdot), & \text{for } t \in \big[ih,(i+1)h\big),\ 1 \leqslant i \leqslant l,\\ 0, & \text{for } t \in [0,h).\end{cases}\end{aligned} \tag{4.2}$$

Define the following new functions $f^{(h)}(x,t)$ and $\phi^{(h)}(x,t)$ as

$$f^{(h)}(x,t) = \frac{1}{h}\int_{ih}^{(i+1)h} f\big(x,\tau,u_i(x)\big)\,d\tau, \quad \text{for } t \in \big[ih,(i+1)h\big),\ i = 0,1,\dots,l \tag{4.3}$$

$$\phi^{(h)}(x,t) = \frac{1}{h}\int_{ih}^{(i+1)h} \phi(x,\tau)\,d\tau, \quad \text{for } t \in \big[ih,(i+1)h\big),\ i = 0,1,\dots,l. \tag{4.4}$$

By (A) we have

$$\big|f^{(h)}(x,t)\big| \leqslant C_0\big(\phi^{(h)} + \big|u^{(h)}\big|^{\alpha}\big). \tag{A$'$}$$

Lemma 4.1. *If $\phi \in L^r(Q_T)$, then $\phi^{(h)} \in L^r(Q_T)$ and*

$$\iint_{Q_T} \big(\phi^{(h)}\big)^r\,dx\,dt \leqslant \iint_{Q_T} \phi^r\,dx\,dt,$$

where r is given in (A).

Proof. By Hölder's inequality

$$\begin{aligned}\iint_{Q_T} \big(\phi^{(h)}\big)^r\,dx\,dt &= \sum_i \int_{ih}^{(i+1)h}\int_{\Omega}\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)^r dx\\ &= h\int_{\Omega}\sum_i\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)^r dx \leqslant h\int_{\Omega}\sum_i\left(\frac{1}{h}\int_{ih}^{(i+1)h}\phi^r(x,\tau)\,d\tau\right)dx\\ &= \int_{\Omega}\sum_i\left(\int_{ih}^{(i+1)h}\phi^r(x,\tau)\,d\tau\right)dx = \iint_{Q_T}\phi^r\,dx\,dt. \qquad \square\end{aligned}$$

In the following, we will give the estimate to the maximum norm of the solution by adopting the method in [11].

Lemma 4.2. *Let* (A), (B) *hold. Assume that* $p(x) \in C(\overline{\Omega})$ *and* $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$*. Then for any integer* $1 \leqslant q < \infty$*, there is a constant* $C(q) > 0$ *independent of* h *such that*

$$\|u^{(h)}\|_{L^{q+1}(Q_T)} \leqslant C(q), \quad \forall h > 0.$$

Proof. Let $u_+ = \max\{0, u\}$ and k be chosen so that $\|u_0\|_{L^\infty(\Omega)} \leqslant k$. Multiplying (3.9) by $(q+1)(u_i - k)_+^q$ and integrating over Ω we get

$$\begin{aligned}
&(q+1)\int_\Omega \frac{(u_i-k)_+^{(q+1)}}{h}\,dx + q(q+1)\int_\Omega |\nabla u_i|^{p(x)}(u_i-k)_+^{(q-1)}\,dx \\
&\quad = (q+1)\int_\Omega \frac{(u_{i-1}-k)(u_i-k)_+^q}{h}\,dx + (q+1)\int_\Omega (u_i-k)_+^q f^{(h)}(x,ih)\,dx \\
&\quad \leqslant (q+1)\int_\Omega \frac{(u_{i-1}-k)_+(u_i-k)_+^q}{h}\,dx + (q+1)\int_\Omega (u_i-k)_+^q f^{(h)}(x,ih)\,dx. \qquad (4.5)
\end{aligned}$$

By Young's inequality

$$(u_{i-1}-k)_+(u_i-k)_+^q \leqslant \frac{q}{q+1}(u_i-k)_+^{(q+1)} + \frac{1}{q+1}(u_{i-1}-k)_+^{(q+1)}.$$

Hence we have

$$\begin{aligned}
&\int_\Omega \frac{(u_i-k)_+^{(q+1)}}{h}\,dx + q(q+1)\int_\Omega |\nabla u_i|^{p(x)}(u_i-k)_+^{(q-1)}\,dx \\
&\quad \leqslant \int_\Omega \frac{(u_{i-1}-k)_+^{(q+1)}}{h}\,dx + (q+1)\int_\Omega (u_i-k)_+^q f^{(h)}(x,ih)\,dx, \quad i = 1,\dots,l. \qquad (4.6)
\end{aligned}$$

Summing over i in (4.6) and considering the definition of $u^{(h)}$, we have

$$\begin{aligned}
&\int_\Omega \big(u^{(h)}-k\big)_+^{(q+1)}(\cdot,t)\,dx + q(q+1)\int_h^{(l+1)h}\int_\Omega \big|\nabla u^{(h)}\big|^{p(x)}\big(u^{(h)}-k\big)_+^{(q-1)}\,dx\,dt \\
&\quad \leqslant \int_\Omega (u_0-k)_+^{(q+1)}\,dx + (q+1)\int_h^{(l+1)h}\int_\Omega \big(u^{(h)}-k\big)_+^q f^{(h)}\,dx\,dt, \qquad (4.7)
\end{aligned}$$

where $t \in [h, (l+1)h)$. By Young's inequality

$$\big|\nabla u^{(h)}\big|^{p^-} \leqslant \big|\nabla u^{(h)}\big|^{p(x)} + C.$$

Using $\|u_0\|_{L^\infty(\Omega)} \leqslant k$, we get

$$\begin{aligned}
&\sup_{t\in(h,(l+1)h)}\int_\Omega \big(u^{(h)}-k\big)_+^{(q+1)}(\cdot,t)\,dx + q(q+1)\int_h^{(l+1)h}\int_\Omega \big|\nabla u^{(h)}\big|^{p^-}\big(u^{(h)}-k\big)_+^{(q-1)}\,dx\,dt \\
&\quad \leqslant Cq(q+1)\int_h^{(l+1)h}\int_\Omega \big(u^{(h)}-k\big)_+^{(q-1)}\,dx\,dt + (q+1)\int_h^{(l+1)h}\int_\Omega \big(u^{(h)}-k\big)_+^q \big|f^{(h)}\big|\,dx\,dt \\
&\quad = Cq(q+1)I_1 + (q+1)I_2. \qquad (4.8)
\end{aligned}$$

Denote

$$\mu(k)=\left|\left\{(x,t)\in\Omega\times\big(0,(l+1)h\big)\colon u^{(h)}\geqslant k\right\}\right|.$$

By Young's inequality and the Poincaré inequality, we get

$$\begin{aligned}I_1&\leqslant\int_h^{(l+1)h}\int_\Omega\big(u^{(h)}-k\big)_+^{(q+p^--1)}\,dx\,dt+C(q)\mu(k)\\&\leqslant C\big(|\Omega|\big)\int_h^{(l+1)h}\int_\Omega\big|\nabla\big(u^{(h)}-k\big)^{(q+p^--1)/p^-}\big|^{p^-}\,dx\,dt+C(q)\mu(k).\end{aligned}\tag{4.9}$$

Now we estimate I_2.

By the Hölder inequality and the Poincaré inequality

$$\begin{aligned}&\int_h^{(l+1)h}\int_\Omega\big(u^{(h)}-k\big)_+^{q+\alpha}\,dx\,dt\\&\quad\leqslant C\int_h^{(l+1)h}\left(\int_\Omega\big(u^{(h)}-k\big)_+^{q+p^--1}\,dx\right)^{(q+\alpha)/(q+p^--1)}dt\\&\quad\leqslant C\big(|\Omega|\big)\int_h^{(l+1)h}\left(\int_\Omega\big|\nabla\big(u^{(h)}-k\big)_+^{(q+p^--1)/p^-}\big|^{p^-}\,dx\right)^{(q+\alpha)/(q+p^--1)}dt.\end{aligned}\tag{4.10}$$

Similarly to the above, using the imbedding theorem, we may prove (see Lemma 3.1 in [11]) that

$$\begin{aligned}&\int_h^{(l+1)h}\int_\Omega\big(u^{(h)}-k\big)_+^{q}\phi^{(h)}\,dx\,dt\\&\quad\leqslant C\Bigg(\sup_{t\in(h,(l+1)h)}\int_\Omega\big(u^{(h)}-k\big)_+^{(q+1)}(\cdot,t)\,dx\\&\qquad+\int_h^{(l+1)h}\int_\Omega\big|\nabla\big(u^{(h)}-k\big)_+^{(q+p^--1)/p^-}\big|^{p^-}\,dx\,dt\Bigg)^{q_1},\end{aligned}\tag{4.11}$$

where $q_1=q(N+p^-)/(q(N+p^-)+N(p^--1+p^-/N))<1$.

Combining (4.10), (4.11), Lemma 4.1 and (A′), we can obtain the estimate for I_2.

Substituting it into (4.8), by $|\mu(k)|\leqslant 2|Q_T|$ and Young's inequality, we get

$$\sup_{t\in(0,(l+1)h)}\int_\Omega\big(u^{(h)}-k\big)_+^{(q+1)}(\cdot,t)\,dx\leqslant C.\tag{4.12}$$

Here we used the fact that $(u^{(h)}-k)_+(\cdot,t)=0$, for $t\in[0,h)$.

If $\alpha=p^--1$ and $|\Omega|$ is sufficiently small, by the Poincaré inequality

$$C\big(|\Omega|\big)\to 0,\quad \text{as } |\Omega|\to 0,$$

in (4.9) and (4.10). Thus we can also obtain the estimate for I_2. Substituting it into (4.8), we may prove (4.12).

Similarly, we may prove

$$\sup_{t\in(0,(l+1)h)}\int_\Omega\big(-u^{(h)}-k\big)_+^{(q+1)}(\cdot,t)\,dx\leqslant C.$$

Thus

$$\left\|u^{(h)}\right\|_{L^{q+1}(Q_T)} \leqslant C(q). \qquad \square$$

Remark 4.1. If we take $k = 0$ and $q = 1$ in Lemma 4.2, then

$$\int_h^{(l+1)h} \int_\Omega \left|\nabla u^{(h)}\right|^{p(x)} dx\,dt \leqslant C. \tag{4.13}$$

Remark 4.2. For studying above problems, we have to study the terms like $\int |\nabla u_i|^{p(x)} |u_i|^q\,dx$ $(q \geqslant 0)$ for $|\nabla u_i| \in L^{p(x)}$ (see (4.5)). To insure that integrals to be well-defined, we need $u_i \in L^\infty$. Actually, the solution that we get in Lemma 3.1 can be considered as a bounded function (see Lemma 4.4).

Now, we give a uniform estimate to the maximum norm of the solution.

We shall need the following proposition.

Proposition 4.1. *(See [5, p. 12].) Let $\{Y_n\}$, $n = 0, 1, 2, \ldots,$ be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \leqslant B b^n Y_n^{1+\beta}$$

where $B, b > 1$ and $\beta > 0$ are given numbers. If

$$Y_0 \leqslant B^{-1/\beta} b^{-1/\beta^2}, \tag{4.14}$$

then $\{Y_n\}$ converges to zero as $n \to \infty$.

Lemma 4.3. *Let the assumptions of Lemma 4.2 hold. Then there is a constant $M_1 > 0$ depending only on T, $|\Omega|$, N, p^-, r, $\|u_0\|_{L^\infty(Q_T)}$ such that*

$$\left\|u^{(h)}\right\|_{L^\infty(Q_T)} \leqslant M_1, \quad \forall h > 0.$$

Proof. Let k be chosen so that $\|u_0\|_{L^\infty(\Omega)} \leqslant k$ and denote

$$J_k = \sup_{t \in (0,(l+1)h)} \int_\Omega \left(u^{(h)} - k\right)_+^2(\cdot, t)\,dx + \int_0^{(l+1)h} \int_\Omega \left|\nabla\left(u^{(h)} - k\right)_+\right|^{p^-} dx\,dt.$$

Take $q = 1$ in (4.8), then by (A′)

$$J_k \leqslant C_1\left(\int_0^{(l+1)h} \int_\Omega \left(\phi^{(h)} + \left|u^{(h)}\right|^\alpha\right)\left(u^{(h)} - k\right)_+ dx\,dt + \mu(k)\right). \tag{4.15}$$

Now we estimate the integral of the right-hand side of the inequality.

By Lemma 4.1 and Hölder's inequality, we get

$$\begin{aligned}
&\int_0^{(l+1)h} \int_\Omega \phi^{(h)}\left(u^{(h)} - k\right)_+ dx\,dt \\
&\quad\leqslant C_2\left(\int_0^{(l+1)h} \int_\Omega \left(u^{(h)} - k\right)_+^{r/(r-1)} dx\,dt\right)^{(r-1)/r} \\
&\quad\leqslant C_3\left(\int_0^{(l+1)h} \int_\Omega \left(u^{(h)} - k\right)_+^{p^- + 2p^-/N} dx\,dt\right)^{N/(p^-N+2p^-)} \mu(k)^{(r-1)/r - N/(p^-N+2p^-)}.
\end{aligned}$$

Hence, by imbedding inequality (see [5, p. 7] or [8, p. 62]) we have

$$\int_0^{(l+1)h}\int_\Omega \left(u^{(h)}-k\right)_+\phi^{(h)}\,dx\,dt \leqslant C_4 J_k^{(N+p^-)/(p^-N+2p^-)}\mu(k)^{(r-1)/r-N/(p^-N+2p^-)}. \tag{4.16}$$

Also, by Lemma 4.2 we have

$$\begin{aligned}
&\int_0^{(l+1)h}\int_\Omega \left|u^{(h)}\right|^\alpha \left(u^{(h)}-k\right)_+\,dx\,dt \\
&\quad\leqslant \left(\int_0^{(l+1)h}\int_\Omega \left(u^{(h)}\right)^{\alpha r}dx\,dt\right)^{1/r}\left(\int_0^{(l+1)h}\int_\Omega \left(u^{(h)}-k\right)_+^{r/(r-1)}dx\,dt\right)^{(r-1)/r} \\
&\quad\leqslant C_5\left(\int_0^{(l+1)h}\int_\Omega \left(u^{(h)}-k\right)_+^{r/(r-1)}dx\,dt\right)^{(r-1)/r} \\
&\quad\leqslant C_6 J_k^{(N+p^-)/(p^-N+2p^-)}\mu(k)^{(r-1)/r-N/(p^-N+2p^-)}.
\end{aligned} \tag{4.17}$$

Substituting (4.16), (4.17) into (4.15), we get

$$J_k \leqslant C_7 J_k^{(p^-+N)/(p^-N+2p^-)}\mu(k)^{(r-1)/r-N/(p^-N+2p^-)} + C_7\mu(k).$$

By Young's inequality

$$J_k \leqslant C_8\left(\mu(k)^{1+(r-N-2)p^-/(rN(p^--1)+rp^-)} + \mu(k)\right).$$

Hence, for all $k^{(2)} \geqslant k^{(1)}$, we have

$$\begin{aligned}
&\left(k^{(2)}-k^{(1)}\right)\left(\mu\left(k^{(2)}\right)\right)^{N/(p^-N+2p^-)} \\
&\quad\leqslant C_9\left(\int_0^{(l+1)h}\int_\Omega \left(u^{(h)}-k^{(1)}\right)_+^{(p^-+2p^-/N)}\right)^{N/(p^-N+2p^-)} \\
&\quad\leqslant C_9\gamma^{N/(p^-N+2p^-)} J_{k^{(1)}}^{(N+p^-)/(p^-N+2p^-)} \\
&\quad\leqslant C_{10}\left(\mu\left(k^{(1)}\right)^{1+(r-N-2)p^-/(rN(p^--1)+rp^-)} + \mu\left(k^{(1)}\right)\right)^{(N+p^-)/(p^-N+2p^-)},
\end{aligned} \tag{4.18}$$

where γ is a constant, depending only on N, p^-, T, comes from imbedding inequality (see [5, p. 7] or [8, p. 62]).

If we take $k^{(2)} = \|u_0\|_{L^\infty(\Omega)} + j$ $(j > 1)$ and $k^{(1)} = \|u_0\|_{L^\infty(\Omega)} + 1$, then

$$\begin{aligned}
&\mu\left(k^{(2)}\right)^{N/(p^-N+2p^-)} \\
&\quad\leqslant \frac{C_{10}}{j-1}\left(\left((T+1)|\Omega|\right)^{1+(r-N-2)p^-/(rN(p^--1)+rp^-)} + (T+1)|\Omega|\right)^{(N+p^-)/(p^-N+2p^-)}.
\end{aligned}$$

Hence, there exists a constant $j_0 > 1$ depending only on T, $|\Omega|$, N, p^-, r such that

$$\mu\left(k^{(2)}\right) \leqslant 1, \quad \text{as } j \geqslant j_0.$$

We take $k_m = \tilde M(2-2^{-m})$, $m = 0, 1, 2, \ldots$, where $\tilde M \geqslant \|u_0\|_{L^\infty(\Omega)} + j_0$ is a constant.

Then it is easy to see that

$$\mu(k_m) \leqslant 1, \quad \text{for } m = 0, 1, \ldots. \tag{4.19}$$

Now we consider the following two cases.

If

(i) $1+(r-N-2)p^-/\big(rN\big(p^- -1\big)+rp^-\big)\geqslant 1$,

then by (4.18) and (4.19)

$$(k_{m+1}-k_m)\big(\mu(k_{m+1})\big)^{N/(p^-N+2p^-)}\leqslant C_{11}\big(\mu(k_m)\big)^{(N+p^-)/(p^-N+2p^-)},$$

i.e.

$$\mu(k_{m+1})\leqslant\left(\frac{2C_{11}}{\tilde M}\right)^{(p^-N+2p^-)/N}2^{m(p^-N+2p^-)/N}\mu(k_m)^{1+p^-/N}.$$

If

(ii) $1+(r-N-2)p^-/\big(rN\big(p^- -1\big)+rp^-\big)<1$,

then

$$(k_{m+1}-k_m)\big(\mu(k_{m+1})\big)^{N/(p^-N+2p^-)}$$
$$\leqslant C_{12}\big(\mu(k_m)^{1+(r-N-2)p^-/(rN(p^--1)+rp^-)}\big)^{(N+p^-)/(p^-N+2p^-)},$$

i.e.

$$\mu(k_{m+1})\leqslant\left(\frac{2C_{12}}{\tilde M}\right)^{(p^-N+2p^-)/N}2^{m(p^-N+2p^-)/N}\mu(k_m)^{1+\delta_1},$$

where

$$\delta_1=\frac{p^-}{N}+\frac{(r-N-2)p^-(N+p^-)}{(rN(p^- -1)+rp^-)N}>0.$$

Now, take

$$\tilde M=\max\big\{C_{11}2^{N/p^-+1},C_{12}2^{N/p^-\delta_1(N+2)+1},\|u_0\|_{L^\infty(\Omega)}+j_0\big\},$$

then in both cases (4.14) hold. Hence by Proposition 4.1 we have that

$$u^{(h)}\leqslant 2\tilde M.$$

Similarly, we may derive a lower bound and this completes the proof of Lemma 4.3. □

Up to now we required that $p(x)\in C(\overline{\Omega})$. This is in fact not necessary. We have

Lemma 4.4. *Assume that* $u_0\in L^\infty(\Omega)\cap W_0^{1,p(x)}(\Omega)$ *and* (A), (B) *hold. Then the conclusions in Lemmas* 3.1–3.3, 4.1–4.3 *still hold.*

Proof. Note that we need the condition $p(x)\in C(\overline{\Omega})$ only in Proposition 2.2. But, if the functions mentioned in the proofs are uniformly bounded, then Proposition 2.2 will still holds without continuity condition.

Now, replace S in (3.4) by

$$\tilde S=\big\{u\in W_0^{1,p(x)}\big\}\cap\big\{\|u\|_{L^\infty(\Omega)}\leqslant M_1+1\big\},\tag{4.20}$$

and still consider $\psi^i(u)$, where $M_1>\|u_0\|_{L^\infty(\Omega)}$ is a constant depending only on T, $|\Omega|$, N, p^-, r, $\|u_0\|_{L^\infty(Q_T)}$ (the same as that in Lemma 4.3).

Assume that

$$v_l\in\tilde S\quad\text{and}\quad v_l\overset{\text{weak}}{\rightharpoonup}v\quad\text{in }W_0^{1,p(x)}.$$

Then by (ii) and (iv) of Proposition 2.1, we have $v_l\rightharpoonup^{weak}v$ in W^{1,p^-}. Hence, $\|v\|_{L^\infty(\Omega)}\leqslant M_1+1$ and $\tilde S$ is weakly closed. So we can complete the proof of Lemma 3.1 without the condition $p(x)\in C(\overline{\Omega})$. On the other hand, the proof of Lemma 3.2 does not need any change. Note that by the definition of $\tilde{S}$, it is not clear whether the minima of $\psi^i(u)$ satisfies (3.9)or not. Hence, we need modify the proofs of Lemmas 4.2 and 4.3. Let u_i be a minima of $\psi^i(u)$ in $\tilde{S}$. Then for $0 \leqslant \epsilon < 1/(\|u_i\|_{L^\infty(\Omega)}+1)^{q-1}$, we have $\|u_i - \epsilon(u_i-k)_+^q\|_{L^\infty(\Omega)} \leqslant M_1 + 1$ and $u_i - \epsilon(u_i-k)_+^q \in \tilde{S}$. Similarly to Lemma 3.2, we get the estimate (3.8). This estimate is exactly the same as that in (4.5) in Lemma 4.2. Then without any change in Lemmas 4.2 and 4.3, we get

$$\|u_i\|_{L^\infty(\Omega)} \leqslant M_1.$$

Thus for any $\phi \in C_0^\infty(\Omega)$, $|\phi| \leqslant 1$ and $-1 \leqslant \epsilon \leqslant 1$ we have that $u_i + \epsilon\phi \in \tilde{S}$ and then $\psi^i(u_i+\epsilon\phi) \geqslant \psi^i(u_i)$. Similarly to Lemma 3.2, we may prove that u_i is a weak solution of (3.9)–(3.10). We must point out that the test function must satisfy $|\phi| \leqslant 1$. Considering the form the test function appeared in the equality, the constrain to ϕ may be removed. □

Lemma 4.5. *Let the assumptions of Lemma* 4.4 *hold. Then for any integer* $1 \leqslant \tilde{l} \leqslant l$, *we have*

$$\frac{1}{2}\int_0^{(\tilde{l}+1)h}\int_\Omega \left|\partial^{(-h)}u^{(h)}\right|^2 dx\,dt + \int_\Omega \frac{1}{p(x)}\left|\nabla u^{(h)}(x,\tilde{l}h)\right|^{p(x)} dx \leqslant \int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx. \tag{4.21}$$

Proof. From Lemma 4.4, we know that u_i is the minima of $\psi^i(u)$. Hence for $u_{i-1} \in \tilde{S}$,

$$\psi^i(u_i) \leqslant \psi^i(u_{i-1}),$$

and then

$$\int_\Omega \frac{1}{p(x)}|\nabla u_i|^{p(x)}\,dx + \int_\Omega \frac{1}{2h}|u_i - u_{i-1}|^2\,dx$$
$$\leqslant \int_\Omega \frac{1}{p(x)}|\nabla u_{i-1}|^{p(x)}\,dx + \int_\Omega\int_{u_{i-1}}^{u_i}\left(\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau\right) ds\,dx, \quad i = 1,2,\dots,\tilde{l}.$$

Summing over i, we have

$$\int_\Omega \frac{1}{p(x)}|\nabla u_{\tilde{l}}|^{p(x)}\,dx + \sum_{i=1}^{\tilde{l}}\frac{1}{2h}\int_\Omega |u_i - u_{i-1}|^2\,dx$$
$$\leqslant \int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx + \sum_{i=1}^{\tilde{l}}\int_\Omega\int_{u_{i-1}}^{u_i}\left(\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau\right) ds\,dx. \tag{4.22}$$

We estimate the second term in the inequality in the following.

By Lemma 4.2, Young's inequality and the differentiability of f, we get

$$\int_\Omega\int_{u_{i-1}}^{u_i}\left(\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau\right) ds\,dx$$
$$\leqslant C\int_\Omega |u_i - u_{i-1}|\,dx \leqslant \frac{1}{4h}\int_\Omega |u_i - u_{i-1}|^2\,dx + 4Ch.$$

Plugging into (4.22), we get

$$\int_\Omega \frac{1}{p(x)}|\nabla u_l|^{p(x)}\,dx + \sum_{i=1}^{\tilde{l}} \frac{1}{4h}\int_\Omega |u_i - u_{i-1}|^2\,dx \leqslant \int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx + 4CT. \tag{4.23}$$

The conclusion of the lemma follows by noticing the definition of $u^{(h)}$. □

Define a new function:

$$w^{(h)}(\cdot,t) = \begin{cases} (\frac{t}{h} - i)(u_i - u_{i-1}) + u_{i-1}, & t \in [ih,(i+1)h),\ i = 1,2,\dots,l, \\ u_0, & t \in [0,h). \end{cases}$$

Then, we have

Lemma 4.6. *Let the assumptions of Lemma* 4.4 *hold. Then*

$$\int_0^T \int_\Omega \left|w^{(h)} - u^{(h)}\right|^2\,dx\,dt \to 0, \quad \textit{as } h \to 0.$$

Proof. By direct calculation,

$$\begin{aligned} &\int_\Omega \int_{ih}^{(i+1)h} \left|w^{(h)} - u^{(h)}\right|^2\,dx\,dt \\ &\quad = \int_\Omega (u_i - u_{i-1})^2\,dx \int_{ih}^{(i+1)h} \left(\frac{t}{h} - i - 1\right)^2 dt \\ &\quad = \frac{h}{3}\int_\Omega (u_i - u_{i-1})^2\,dx = \frac{h^3}{3}\int_\Omega \int_{ih}^{(i+1)h} \left(\partial^{-h} u^{(h)}\right)^2\,dx\,dt. \end{aligned}$$

Summing over i and using Lemma 4.5, we get

$$\int_h^T \int_\Omega \left|w^{(h)} - u^{(h)}\right|^2\,dx\,dt \to 0, \quad \text{as } h \to 0.$$

On the other hand,

$$\int_0^h \int_\Omega \left|w^{(h)} - u^{(h)}\right|^2\,dx\,dt = 0.$$

The conclusion follows. □

Lemma 4.7. *Let the assumptions of Lemma* 4.4 *hold. Then there exists a subsequence of* $\{u^{(h)}\}$ *denoted again by itself for the sake of simplicity, and a function* u *such that*

$$u^{(h)} \to u, \quad \textit{in } L^2(Q_T), \tag{4.24}$$

$$u^{(h)} \to u, \quad \textit{a.e. in } Q_T, \tag{4.25}$$

$$\partial^{(-h)} u^{(h)} \overset{\text{weak}}{\rightharpoonup} \partial_t u, \quad \textit{in } L^2(Q_T), \tag{4.26}$$

$$\nabla u^{(h)} \overset{\text{weak}}{\rightharpoonup} \nabla u, \quad \textit{in } L^{p(x)}(Q_T), \tag{4.27}$$

as $h \to 0$.

Proof. By Lemma 4.5 and Young's inequality,

$$\int_{\Omega} \left|\nabla u^{(h)}(\cdot,t)\right|^2 dx \leqslant C. \tag{4.28}$$

By the Poincaré inequality,

$$\int_{\Omega} \left|u^{(h)}(\cdot,t)\right|^2 dx \leqslant C,$$

and then

$$\int_0^T \int_{\Omega} \left|u^{(h)}(\cdot,t)\right|^2 dx \leqslant CT. \tag{4.29}$$

Therefore, there exists a subsequence of $u^{(h)}$ (denoted again by itself) and a function u such that

$$u^{(h)} \overset{\text{weak}}{\rightharpoonup} u, \quad \text{in } L^2(Q_T). \tag{4.30}$$

And then there exists a subsequence of $u^{(h)}$ such that (4.25) holds.

In the following, we consider $w^{(h)}$. Since

$$\nabla w^{(h)} = (\nabla u_i - \nabla u_{i-1})\left(\frac{t}{h} - i\right) + \nabla u_{i-1}, \quad \text{as } t \in \big[ih, (i+1)h\big),\ i = 1, 2, \dots, l$$

by (4.28), (4.29) and Lemma 4.6, we know that $w^{(h)}$ and $\nabla w^{(h)}$ are uniformly bounded in $L^2(Q_T)$. Since

$$w_t^{(h)} = \partial^{(-h)} u^{(h)} = \begin{cases} \frac{1}{h}(u_i - u_{i-1}), & \text{for } t \in [ih, (i+1)h),\ i = 1, 2, \dots, l, \\ 0, & \text{for } t \in [0, h), \end{cases}$$

by Lemma 4.5, we have $w_t^{(h)} \in L^2(Q_T)$. By the above estimates, we know that there exists a subsequence of $w^{(h)}$ (denoted again by itself) and a function u_* such that

$$w^{(h)} \to u_*, \quad \text{in } L^2(Q_T),$$
$$\nabla w^{(h)} \overset{\text{weak}}{\rightharpoonup} \nabla u_*, \quad \text{in } L^2(Q_T),$$
$$\partial^{(-h)} u^{(h)} = w_t^{(h)} \overset{\text{weak}}{\rightharpoonup} (u_*)_t, \quad \text{in } L^2(Q_T).$$

By Lemma 4.6, we get $u = u_*$. Using this, it is easy to prove (4.24) and (4.26).

By Proposition 2.1, $(L^{p(x)}, |\cdot|_{p(x)})$ is weakly compact and hence by Lemma 4.5, (4.27) can be easily obtained. □

Remark 4.3. By (4.21), we know that $u \in L^\infty(0,T; W_0^{1,p(x)}(\Omega))$.

Lemma 4.8. *Let the assumptions of Lemma* 4.4 *hold. Then*

$$f^{(h)} \to f(x,t,u), \quad \textit{in } L^1(Q_T), \quad \textit{as } h \to 0. \tag{4.31}$$

Proof. By Lemmas 4.3 and 4.7, we have

$$\left\|u^{(h)}\right\|_{L^\infty(Q_T)}, \|u\|_{L^\infty(Q_T)} \leqslant C, \quad \forall h > 0.$$

Since $f \in C^1$, by Hölder's inequality,

$$\iint_{Q_T} \left|f^{(h)} - f\right| dx\,dt$$
$$\leqslant \sum_i \int_{ih}^{(i+1)h} \int_\Omega \frac{1}{h} \int_{ih}^{(i+1)h} \left|f(x,\tau,u_i) - f(x,t,u)\right| d\tau\,dx\,dt$$
$$\leqslant C \sum_i \int_{ih}^{(i+1)h} \int_\Omega \frac{1}{h} \int_{ih}^{(i+1)h} \big(|\tau - t| + |u_i - u|\big)\,d\tau\,dx\,dt$$
$$\leqslant C \sum_i \int_{ih}^{(i+1)h} \int_\Omega \frac{1}{h} \int_{ih}^{(i+1)h} h\,d\tau\,dx\,dt + C \sum_i \int_{ih}^{(i+1)h} \int_\Omega |u_i - u|\,dx\,dt$$
$$\leqslant C \int_0^T \int_\Omega h\,dx\,dt + C \int_0^T \int_\Omega \left|u^{(h)} - u\right| dx\,dt \to 0, \quad as\ h \to 0. \qquad \square$$

Proof of Theorem 2.1. By Lemma 4.5 and the weak compactness of the space, there exists a subsequence such that

$$\left|\nabla u^{(h)}\right|^{p(x)-2} u^{(h)}_{x_i} \overset{\text{weak}}{\rightharpoonup} \chi_i, \quad \text{in } L^{p(x)/(p(x)-1)}(Q_T).$$

The same as that in [11], we may prove that $\chi_i = |\nabla u|^{p(x)-2} u_{x_i}$.

For test function $\phi(x,t) \in C_0^\infty(Q_T)$ and any constant $\tilde\tau \in [0,T]$, we have $\phi(x,\tilde\tau) \in C_0^\infty(\Omega)$.

Hence, by Lemma 3.3,

$$\int_\Omega \partial^{(-h)} u^{(h)} \phi(x,\tilde\tau)\,dx = \int_\Omega |\nabla u_i|^{p(x)-2} \nabla u_i \nabla\phi(x,\tilde\tau)\,dx + \int_\Omega f^{(h)} \phi(x,\tilde\tau)\,dx.$$

Integrating for $\tilde\tau$, combining Lemmas 4.7, 4.8 and Remark 4.1, we may prove that u is a weak solution of Eq. (1.1).

Now we prove that u satisfies the initial condition, i.e. (2.3) holds.

In the problem (3.9)–(3.10), taking a test function $\tilde\psi(x) \in C_0^\infty(\Omega)$, we get

$$\int_\Omega (u_i - u_{i-1})\tilde\psi\,dx + \int_{ih}^{(i+1)h} dt \int_\Omega |\nabla u_i|^{p(x)-2} \nabla u_i \cdot \nabla\tilde\psi\,dx$$
$$= \int_\Omega \left(\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau \right) \tilde\psi\,dx, \quad i = 1, 2, \ldots. \tag{4.32}$$

Summing over i, we get

$$\int_\Omega (u_{\tilde l} - u_0)\tilde\psi\,dx$$
$$= -\int_h^{(\tilde l+1)h} \int_\Omega |\nabla u_i|^{p(x)-2} \nabla u_i \cdot \nabla\tilde\psi\,dx\,dt + \sum_i h \int_\Omega \int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau\,\tilde\psi\,dx,$$

where $\tilde l > 0$ is an integer. Then by (4.13) and (ii), (iii) of Proposition 2.1, we have

$$\left| \int_h^{(\tilde l+1)h} \int_\Omega |\nabla u_i|^{p(x)-2} \nabla u_i \cdot \nabla\tilde\psi\,dx\,dt \right| \leqslant C(\sup \nabla\tilde\psi) |\nabla u_i|_{p(x)} |1|_{p(x)} \leqslant C(\tilde l h)^{\delta_1},$$

where $\delta_1 > 0$ is a constant depending only on p^+ and p^-.

By (A), combining Poincaré inequality and Hölder's inequality

$$\left|\int_\Omega (u_{\tilde{l}} - u_0)\tilde{\psi}\,dx\right| \leqslant C\big((\tilde{l}h)^{\delta_1} + \tilde{l}h + (\tilde{l}h)^{N/(N+p^-)} + (\tilde{l}h)^{1-\alpha/p^-}\big).$$

For $\tilde{l}h < 1$, there exists a constant $\delta_2 > 0$ depending only on α, p^-, N and δ_1 such that

$$\left|\int_\Omega (u_{\tilde{l}} - u_0)\tilde{\psi}\,dx\right| \leqslant C(\tilde{l}h)^{\delta_2}.$$

Noticing the definition of the $u^{(h)}$, we have

$$\sup_{t\in[h,(\tilde{l}+1)h)}\left|\int_\Omega \big(u^{(h)}(x,t) - u_0\big)\tilde{\psi}\,dx\right| \leqslant Ct^{\delta_2}. \tag{4.33}$$

For $t \in [0, h)$,

$$\int_\Omega \big(u^{(h)}(x,t) - u_0\big)\tilde{\psi}\,dx = 0.$$

Hence for all $0 \leqslant t < 1$, (4.33) holds. By (4.25), letting $h \to 0$ in (4.33), we may easily get (2.3). □

5. Local existence

As what we mentioned in Lemma 4.4, we will use the lemmas in Sections 3 and 4 without the assumption $p(x) \in C(\overline{\Omega})$.

Since $f \in C^1$, we know that there is a constant M such that for $|z| \leqslant \|u_0\|_{L^\infty(\Omega)} + 1$,

$$\big|f(x,t,z)\big| \leqslant M, \qquad \big|f_z(x,t,z)\big| \leqslant M. \tag{5.1}$$

Take $T^* > 0$ such that

$$T^*M < 1/2. \tag{5.2}$$

Without lose of generality, we may assume that $T^* \geqslant h$.

Consider the following problem

$$\frac{1}{h}(u_1 - u_0) = \operatorname{div}\big(|\nabla u_1|^{p(x)-2}\nabla u_1\big) + \frac{1}{h}\int_h^{2h} f(x,\tau,u_1)\,d\tau, \tag{5.3}$$

$$u_1|_{\partial\Omega} = 0. \tag{5.4}$$

We have

Lemma 5.1. *Suppose that f and u_0 satisfy the assumptions in Theorem* 2.2*, then* (5.3)–(5.4) *has a weak solution u_1 such that $\|u_1\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2hM$.*

Proof. We first consider the iteration problem,

$$\frac{1}{h}(v_m - u_0) = \operatorname{div}\big(|\nabla v_m|^{p(x)-2}\nabla v_m\big) + \frac{1}{h}\int_h^{2h} f(x,\tau,v_{m-1})\,d\tau, \tag{5.5}$$

$$v_m|_{\partial\Omega} = 0, \quad m = 1, 2, \ldots, \tag{5.6}$$

where $v_0 = u_0$.

Assume first $m = 1$. Since $|f(x,t,u_0)|$ is bounded, analogously to Lemmas 3.3 and 4.4, we may prove that (5.5)–(5.6) has a solution v_m in $L^\infty(\Omega) \cap W_0^{1,p(x)}$.

Now for any integer $q > 0$, we may take $(v_1 - Mh)_+^q$ as a test function in (5.5) to get

$$\int_\Omega \frac{1}{h}(v_1 - Mh)_+^{(q+1)}\,dx + q\int_\Omega \left|\nabla(v_1 - Mh)_+^{p(x)}\right|(v_1 - Mh)_+^{q-1}\,dx$$

$$= \int_\Omega \frac{1}{h}(v_1 - Mh)_+^q u_0\,dx + \frac{1}{h}\int_\Omega \int_h^{2h} f(x,\tau,v_0)\,d\tau\,(v_1 - Mh)_+^q\,dx.$$

By (5.1) and the Hölder inequality, we have

$$\int_\Omega (v_1 - Mh)_+^{(q+1)}\,dx \leqslant \int_\Omega (v_1 - Mh)_+^q(u_0 + hM)\,dx$$

$$\leqslant \left(\int_\Omega (v_1 - Mh)_+^{(q+1)}\,dx\right)^{q/(q+1)}\left(\int_\Omega (u_0 + hM)^{q+1}\,dx\right)^{1/(q+1)}.$$

Hence

$$\left\|(v_1 - Mh)_+\right\|_{L^{q+1}(\Omega)} \leqslant \|u_0 + hM\|_{L^{q+1}(\Omega)}.$$

Letting $q \to \infty$, we get $(v_1)_+ \leqslant \|u_0\|_{L^\infty(\Omega)} + 2hM$. Consider $-v_1$, we may get $(v_1)_- \geqslant -\|u_0\|_{L^\infty(\Omega)} - 2hM$, i.e. $\|v_1\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2hM$.

Since for $2hM < 1$, (5.1) holds, by induction, we may prove that there exist solutions $\{v_m\}$, $m = 1, 2, \ldots,$ of (5.5)–(5.6) satisfy that

$$\|v_m\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2hM. \tag{5.7}$$

Now, we prove that $\{v_m\}$, $m = 1, 2, \ldots,$ is a contracting sequence. Taking $v_m - v_{m-1}$ as a test function in (5.5)–(5.6) for m and $m-1$ respectively and then subtracting one from the other, we get

$$\int_\Omega \frac{1}{h}(v_m - v_{m-1})^2\,dx + \int_\Omega \left(|\nabla v_m|^{p(x)-2}\nabla v_m\,dx - |\nabla v_{m-1}|^{p(x)-2}\nabla v_{m-1}\right)\nabla(v_m - v_{m-1})\,dx$$

$$= \int_\Omega \frac{1}{h}\int_h^{2h}\left(f(x,\tau,v_{m-1})\,dx - f(x,\tau,v_{m-2})\right)d\tau\,(v_m - v_{m-1})\,dx$$

$$= \int_\Omega \frac{1}{h}\int_h^{2h} f_z(x,\tau,v_{m-1})\,d\tau\,(v_{m-1} - v_{m-2})(v_m - v_{m-1})\,dx.$$

It is easy to see that

$$\int_\Omega \left(|\nabla v_m|^{p(x)-2}\nabla v_m\,dx - |\nabla v_{m-1}|^{p(x)-2}\nabla v_{m-1}\right)\nabla(v_m - v_{m-1})\,dx \geqslant 0.$$

By (5.2) and Hölder inequality we have

$$\int_\Omega (v_m - v_{m-1})^2\,dx$$

$$\leqslant \frac{1}{2}\int_\Omega |v_{m-1} - v_{m-2}||v_m - v_{m-1}|\,dx$$

$$\leqslant \frac{1}{2}\|v_{m-1} - v_{m-2}\|_{L^2(\Omega)}\|v_m - v_{m-1}\|_{L^2(\Omega)},$$

i.e.

$$\|v_m - v_{m-1}\|_{L^2(\Omega)} \leqslant \frac{1}{2}\|v_{m-1} - v_{m-2}\|_{L^2(\Omega)}.$$

This proves that $\{v_m\}$, $m = 1, 2, \dots$, is a contracting sequence. Therefore, there is a function $u_1 \in L^2(\Omega)$ such that $v_m \to u_1$ in $L^2(\Omega)$ as $m \to \infty$.

Next, taking v_m as a test function in (5.5)–(5.6), we get

$$\int_\Omega \frac{1}{h} v_m^2\,dx + \int_\Omega |\nabla v_m|^{p(x)}\,dx = \int_\Omega \frac{1}{h} u_0 v_m\,dx + \int_\Omega \left(\frac{1}{h}\int_h^{2h} f(x,\tau,v_{m-1})\,d\tau\right) v_m\,dx.$$

Since $\{v_m\}$, $m = 1, 2, \dots$, is uniformly bounded, we get

$$\int_\Omega |\nabla v_m|^{p(x)}\,dx \leqslant C,$$

where the constant C is independent of m.

Therefore, there is a subsequence m_j such that ∇v_{m_j} converges weakly to ∇u_1 in $L^{p(x)}$ and $|\nabla v_{m_j}|^{p(x)-2}(v_{m_j})_i$ converges weakly to χ_i in $L^{p(x)/(p(x)-1)}$. Since $v_m \in W_0^{1,p(x)}$, it is easy to prove that $u_1 \in W_0^{1,p(x)}$.

Finally, we prove that u_1 is a solution of (5.3)–(5.4).

Taking a test function $\phi \in C_0^\infty$ in (5.5) and letting $i \to \infty$, we get

$$\int_\Omega \frac{1}{h}(u_1 - u_0)\phi\,dx + \int_\Omega \sum_i \chi_i \phi_i\,dx = \int_\Omega \left(\frac{1}{h}\int_h^{2h} f(x,\tau,u_1)\,d\tau\right)\phi\,dx.$$

The same as that in [11], we may prove that $|\nabla u_1|^{p(x)-2}(u_1)_i = \chi_i$ and hence u_1 is a solution. The proof is complete.

From (5.7), we know that $\|u_1\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2hM$. □

Proof of Theorem 2.2. Consider the following problem

$$\frac{1}{h}(u_i - u_{i-1}) = \operatorname{div}\left(|\nabla u_i|^{p(x)-2}\nabla u_i\right) + \frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau, \tag{5.8}$$

$$u_i|_{\partial\Omega} = 0, \tag{5.9}$$

where $i = 1, 2, \dots, l$ and $lh \leqslant T^*$.

Similarly to Lemma 5.1, we may prove inductively that if $\nabla u_{i-1} \in L^{p(x)}$ and $\|u_{i-1}\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2(i-1)hM$ then there is a solution u_i of (5.8)–(5.9) such that $\|u_i\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 2ihM$.

Since $lh \leqslant T^*$, we have $\|u_i\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 1$ and then by (5.1) $\frac{1}{h}\int_{ih}^{(i+1)h} |f|\,d\tau \leqslant M$.

We should notice that the solutions u_i, $i = 1, 2, \dots$, may not be the minima of the functional mentioned in Theorem 2.1 and hence, comparing to the previous proof, we have to give the L^2 estimate to $\partial^{-h}u^{(h)}$.

Taking $(u_i - u_{i-1})/h$ as a test function in (5.8)–(5.9), we get

$$\int_\Omega \left(\frac{1}{h}(u_i - u_{i-1})\right)^2 dx + \frac{1}{h}\int_\Omega \left(|\nabla u_i|^{p(x)} - |\nabla u_i|^{p(x)-2}\nabla u_i \nabla u_{i-1}\right) dx = \frac{1}{h}\int_\Omega \left(\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau\right)(u_i - u_{i-1})\,dx. \tag{5.10}$$

By Young's inequality,

$$|a \cdot b| \leqslant |a||b| \leqslant \frac{p(x)-1}{p(x)}|a|^{p(x)/(p(x)-1)} + \frac{1}{p(x)}|b|^{p(x)},$$

and by (5.10), we get

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{h}(u_i - u_{i-1})\right)^2 dx + \frac{1}{h}\int_\Omega \frac{1}{p(x)}|\nabla u_i|^{p(x)}\,dx \\
&\quad \leqslant \frac{1}{h}\int_\Omega \frac{1}{p(x)}|\nabla u_{i-1}|^{p(x)}\,dx + \frac{1}{h}\int_\Omega M|u_i - u_{i-1}|\,dx \\
&\quad \leqslant \frac{1}{h}\int_\Omega \frac{1}{p(x)}|\nabla u_{i-1}|^{p(x)}\,dx + \frac{1}{2}\int_\Omega \left(\frac{1}{h^2}(u_i - u_{i-1})^2 + M^2\right) dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{ih}^{(i+1)h} d\tau \int_\Omega \left(\frac{1}{h}(u_i - u_{i-1})\right)^2 dx + 2\int_\Omega \frac{1}{p(x)}|\nabla u_i|^{p(x)}\,dx \\
&\quad \leqslant 2\int_\Omega \frac{1}{p(x)}|\nabla u_{i-1}|^{p(x)}\,dx + 2hM^2|\Omega|.
\end{aligned}$$

Summing over i and using the definition of $u^{(h)}$, we get

$$\begin{aligned}
&\int_h^{(l+1)h} \int_\Omega \left(\partial^{-h}u^{(h)}\right)^2 dx\,d\tau + 2\int_\Omega \frac{1}{p(x)}|\nabla u_l|^{p(x)}\,dx \\
&\quad \leqslant 2\int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)}\,dx + 2lhM^2|\Omega| \\
&\quad \leqslant C.
\end{aligned}$$

Since C is independent of h, we get a uniform L^2 estimate to $\partial^{-h}u^{(h)}$. Now, the same as the proof of Theorem 2.1, we may get the existence of solutions to (1.1)–(1.2). □

Proof of Theorem 2.3. Let u, v be two solutions of (1.1)–(1.2). Taking $u - v$ as a test function, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u-v)^2\,dx + \iint_{Q_t} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla(u-v)\,dx\,d\tau \\
&\quad = \iint_{Q_t} \left(f(x,t,u) - f(x,t,v)\right)(u-v)\,dx.
\end{aligned}$$

Since u, v are bounded and $f \in C^1$, we have

$$\int_\Omega (u-v)^2\,dx \leqslant C \iint_{Q_t} (u-v)^2\,dx\,d\tau.$$

Gronwall's inequality imples that $u = v$. The proof is complete. □

Proof of Theorem 2.4. (i) Assume that $u_{0,n} \in C_0^\infty(\Omega)$, such that $\|u_{0,n}\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + 1$ and

$$u_{0,n} \to u_0 \quad \text{in } L^2(\Omega).$$

Without any change in Lemmas 3.3 and 4.3, we may get the uniform boundedness of the solution $u_{i,n}$. More over we have (see Remark 4.1)

$$\int_h^T \int_\Omega \left|\nabla u_n^{(h)}\right|^{p(x)} dx\, dt \leqslant C. \tag{5.11}$$

For the L^2 estimate to u_t, take $(u_{i,n} - u_{i-1,n}) \int_{ih}^{(i+1)h} \xi(\tau)\, d\tau / h^2$ as a test function in (3.9)–(3.10), where

$$\xi(\tau) \in C_0^\infty(0, T), \qquad \xi(\tau) \geqslant 0, \quad \text{and} \quad \xi(\tau) = 1 \quad \text{for } \tau \geqslant \epsilon. \tag{5.12}$$

Without loss of generality, we may assume that $\epsilon > h$.

Similarly to the proof of Theorem 2.2, we have

$$\int_{ih}^{(i+1)h} \xi(\tau)\, d\tau \int_\Omega \left(\frac{1}{h}(u_{i,n} - u_{i-1,n})\right)^2 dx + \frac{2}{h} \int_\Omega \frac{1}{p(x)} |\nabla u_{i,n}|^{p(x)} dx \int_{ih}^{(i+1)h} \xi(\tau)\, d\tau$$
$$\leqslant \frac{2}{h} \int_\Omega \frac{1}{p(x)} |\nabla u_{i-1,n}|^{p(x)} dx \int_{ih}^{(i+1)h} \xi(\tau)\, d\tau + hC.$$

Summing over i, by (5.11) and (5.12), we get

$$\int_h^{(l+1)h} \int_\Omega \left(\partial^{-h} u_n^{(h)}\right)^2 \xi(\tau)\, dx\, d\tau + 2 \int_\Omega \frac{1}{p(x)} |\nabla u_{l,n}|^{p(x)} dx$$
$$\leqslant \sum_i \frac{2}{h} \int_\Omega \frac{1}{p(x)} |\nabla u_{i-1,n}|^{p(x)} dx \left(\int_{ih}^{(i+1)h} \xi(\tau)\, d\tau - \int_{(i-1)h}^{ih} \xi(\tau)\, d\tau \right) + C$$
$$\leqslant C(\xi_\tau) \sum_{i \geqslant 2} \int_\Omega \frac{1}{p(x)} |\nabla u_{i-1,n}|^{p(x)} dx + C$$
$$\leqslant C(\xi_\tau) \int_h^T \int_\Omega \left|\nabla u_n^{(h)}\right|^{p(x)} dx\, dt + C \leqslant C.$$

Then, the same as the proof of Theorem 2.1 (with diagonal process), we may prove the existence of solutions.

(ii) For $u_{0,n}$, by Lemma 5.1, there is a solution $u_{i,n}$ of (5.8)–(5.9) with uniform boundedness, and then similarly to (i), we can complete the proof. □

References

[1] E. Acerbi, G. Mingione, Regularity results for stationary electrorheological fluids, Arch. Ration. Mech. Anal. 164 (3) (2002) 213–259.
[2] E. Acerbi, G. Mingione, G.A. Seregin, Regularity results for parabolic systems related to a class of non Newtonian fluids, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (1) (2004) 25–60.
[3] S. Antontsev, S. Shmarev, Anisotropic parabolic equations with variable nonlinearity, Publ. Mat. 53 (2) (2009) 355–399.
[4] M. Berger, Nonlinearity and functional analysis, in: Lectures on Nonlinear Problems in Mathematical Analysis, Pure and Applied Mathematics, Academic Press, New York–London, 1977.
[5] E. Dibenedetto, Degenerate Parabolic Equations, Springer-Verlag, New York, 1993.
[6] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2) (2001) 424–446.
[7] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (116) (1991) 592–618.
[8] O.A. Ladyzhenskaja, V.A. Solonikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, American Mathematical Society, Providence, Rhode Island, 1968.
[9] L.Q. Peng, Non-uniqueness for the p-harmonic heat flow with potential into homogeneous spaces, Chinese Ann. Math. Ser. A 27 (4) (2006) 443–448 (in Chinese), translation in Chinese J. Contemp. Math. 27 (3) (2006) 231–238.

[10] M. Ružička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.
[11] J.N. Zhao, Existence and nonexistence of solutions for $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl. 172 (1) (1993) 130–146.
[12] J.N. Zhao, On the Cauchy problem and initial traces for the evolution p-laplacian equations with strongly nonlinear sources, J. Diff. Eqs. 121 (2) (1995) 329–383.
[13] V.V. Zhikov, Passage to the limit in nonlinear variational problems, Mat. Sb. 183 (8) (1992) 47–84 (in Russian), translation in Russian Acad. Sci. Sb. Math. 76 (2) (1993) 427–459.
[14] V.V. Zhikov, On the density of smooth functions in Sobolev–Orlicz spaces, Otdel. Mat. Inst. Steklov. (POMI) 310 (2004) 67–81, translation in J. Math. Sci. (N.Y.) 132 (3) (2006) 285–294.