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Optimal L^p Hardy-type inequalities

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Abstract

Let Ω be a domain in $\mathbb{R}^n$ or a noncompact Riemannian manifold of dimension $n \geq 2$, and $1 < p < \infty$. Consider the functional $\mathcal{Q}(\varphi) := \int_\Omega (|\nabla\varphi|^p + V|\varphi|^p)\, \mathrm{d}\nu$ defined on $C_0^\infty(\Omega)$, and assume that $\mathcal{Q} \geq 0$. The aim of the paper is to generalize to the quasilinear case ($p \neq 2$) some of the results obtained in [6] for the linear case ($p = 2$), and in particular, to obtain "as large as possible" nonnegative (optimal) Hardy-type weight W satisfying

$$\mathcal{Q}(\varphi) \geq \int_\Omega W|\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

Our main results deal with the case where $V = 0$, and Ω is a general punctured domain (for $V \neq 0$ we obtain only some partial results). In the case $1 < p \leq n$, an optimal Hardy-weight is given by

$$W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla G}{G}\right|^p,$$

where G is the associated positive minimal Green function with a pole at 0. On the other hand, for $p > n$, several cases should be considered, depending on the behavior of G at infinity in Ω. The results are extended to annular and exterior domains.

Keywords: Hardy inequality; Optimal; p-Laplacian

1. Introduction

In a recent paper [6], the authors studied a general second-order *linear* elliptic operator $P \geq 0$ in a general domain $\Omega \subset \mathbb{R}^n$ (or a noncompact smooth manifold of dimension n), where $n \geq 2$, and obtained an optimal improvement of the inequality $P \geq 0$. The improved inequality is of the form $P \geq W$, where W is "as large as possible" weight function, and (in the self-adjoint case) the inequality $P \geq W$ is meant in the quadratic form sense. The weight W is given explicitly using a simple construction called the *supersolution construction*; any two linearly independent positive (super)solutions u_0, u_1 of the equation $Pu = 0$ give rise to a one-parameter family of Hardy-type weights

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$\{W_\alpha\}_{\{0\le\alpha\le 1\}}$ satisfying the inequality $P \ge W_\alpha$ (for more details on this construction see Section 4). The optimal weight is obtained by a careful choice of u_0, u_1 and α.

In the case of a Schrödinger type operator P, the main result of [6] reads as follows.

Theorem 1.1. *Consider a* symmetric *second-order linear elliptic operator P of the form*

$$Pu := -\operatorname{div}\big(A(x)\nabla u\big) + V(x)u$$

which is subcritical in Ω. Let q be the associated quadratic form. Then there exists a nonzero, nonnegative weight W satisfying the following properties:

(a) *The following Hardy-type inequality holds true*

$$q(\varphi) \ge \lambda \int_\Omega W(x)\big|\varphi(x)\big|^2\,\mathrm{d}x \quad \forall\varphi\in C_0^\infty(\Omega), \tag{1.1}$$

with $\lambda > 0$. Denote by $\lambda_0 := \lambda_0(P, W, \Omega)$ the best constant satisfying (1.1).

(b) *The operator $P - \lambda_0 W$ is critical in Ω; that is, the inequality*

$$q(\varphi) \ge \int_\Omega W_1(x)\varphi^2(x)\,\mathrm{d}x \quad \forall\varphi\in C_0^\infty(\Omega)$$

is not valid for any $W_1 \gneqq \lambda_0 W$.

(c) *The constant λ_0 is also the best constant for (1.1) with test functions supported in the exterior of any fixed compact set in Ω.*

(d) *The operator $P - \lambda_0 W$ is null-critical in Ω; that is, the corresponding Rayleigh–Ritz variational problem*

$$\inf_{\varphi\in\mathcal{D}_P^{1,2}(\Omega)}\left\{\frac{q(\varphi)}{\int_\Omega W(x)|\varphi(x)|^2\,\mathrm{d}x}\right\} \tag{1.2}$$

admits no minimizer. Here $\mathcal{D}_P^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $u \mapsto \sqrt{q(u)}$.

(e) *If furthermore, $W > 0$, then the spectrum and the essential spectrum of the Friedrichs extension of the operator $W^{-1}P$ on $L^2(\Omega, W\,\mathrm{d}x)$ are both equal to $[\lambda_0, \infty)$.*

In the present paper we consider the quasilinear case. Let $1 < p < \infty$, and denote by $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ the p-Laplace operator. Throughout the paper, Ω is either a domain in $\mathbb{R}^n$, or a noncompact smooth Riemannian manifold of dimension n, $n \ge 2$, such that $0 \in \Omega$. Let $V \in L^\infty_{\mathrm{loc}}(\Omega)$ be a real valued potential, and let Q_V be the quasilinear operator

$$Q_V(u) = Q(u) := -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) + V(x)|u|^{p-2}u \tag{1.3}$$

defined on Ω. Denote by

$$\mathcal{Q}_V(\varphi) = \mathcal{Q}(\varphi) := \int_\Omega \big(|\nabla\varphi|^p + V|\varphi|^p\big)\,\mathrm{d}\nu$$

the associated energy defined on $C_0^\infty(\Omega)$. We say that $\mathcal{Q} \ge 0$ in Ω if $\mathcal{Q}(\varphi) \ge 0$ for all $\varphi \in C_0^\infty(\Omega)$.

Let $W \ge 0$ in Ω. We denote

$$\lambda_0(Q_V, W, \Omega) := \sup\{\lambda\in\mathbb{R} \mid \mathcal{Q}_{V-\lambda W} \ge 0 \text{ in } \Omega\},$$
$$\lambda_\infty(Q_V, W, \Omega) := \sup\{\lambda\in\mathbb{R} \mid \exists K \subset\subset \Omega \text{ s.t. } \mathcal{Q}_{V-\lambda W} \ge 0 \text{ in } \Omega\setminus K\},$$

respectively, the *best constant* and *best constant at infinity* in the Hardy-type inequality

$$\mathcal{Q}_V(\varphi) \ge \lambda \int_{\Omega^\star} W|\varphi|^p\,\mathrm{d}\nu \quad \forall\varphi\in C_0^\infty(\Omega).$$

Let us mention that in the linear case ($p=2$) on $\Omega=\mathbb{R}^n$, and if $V=V(|x|)$, $W(x)=W(|x|)$ are two radial functions on $\mathbb{R}^n$, then λ_∞ is the infimum of λ's such that the ODE

$$-\left(t^{n-1}u'\right)'+t^{n-1}\left(V(r)-\lambda W(r)\right)u=0$$

is oscillatory as $t\to\infty$.

The aim of the present article is to generalize Theorem 1.1 (obtained in the linear case), to the quasilinear case and to obtain "as large as possible" nonnegative (optimal) weight W satisfying

$$\mathcal{Q}(\varphi)\geq\lambda\int_\Omega W(x)|\varphi|^p\,\mathrm{d}\nu\quad\forall\varphi\in C_0^\infty(\Omega).$$

In particular, we answer affirmatively a problem posed by the authors in [6] (see Problem 13.12 therein).

The extension of Theorem 1.1 to the quasilinear case is not a straightforward task. First, due to the nonlinearity of the operator Q_V, the supersolution construction has to be modified, and in particular in the case $p>n$, the supersolution construction leading to optimal potentials is essentially different. In fact, we could not extend Theorem 1.1 to operators Q_V with $V\neq0$. Secondly, the proof of Theorem 1.1 given in [6] is mostly of linear nature, and therefore a new approach is needed for the quasilinear case. Moreover, the proof of Theorem 1.5 actually provides us with an alternative proof for parts (b) and (c) of Theorem 1.1. On the other hand, it seems that there is no analog to part (e) of Theorem 1.1 concerning the essential spectrum of the corresponding operator. We note that in the linear case, the proof of part (e) relies on a construction of a family of generalized eigenfunctions, and this construction does not apply to the quasilinear case.

Let us introduce first our definition of *optimal Hardy-weights* for Q_V in a punctured domain.

Definition 1.2. Suppose that $\mathcal{Q}_V\geq0$ in Ω, and denote $\Omega^\star:=\Omega\setminus\{0\}$. Assume that a nonzero nonnegative function W satisfies the following Hardy-type inequality

$$\mathcal{Q}_V(\varphi)\geq\lambda\int_{\Omega^\star}W|\varphi|^p\,\mathrm{d}\nu\quad\forall\varphi\in C_0^\infty\left(\Omega^\star\right),\tag{1.4}$$

where λ is a positive constant. Set $\lambda_0:=\lambda_0(Q_V,W,\Omega^\star)$.

We say that W is an *optimal Hardy-weight* for the operator Q_V in Ω if the following conditions hold true.

(1) The functional $\mathcal{Q}_{V-\lambda_0W}$ is *critical in* $\Omega^\star$, i.e. for any $W_1\gneqq\lambda_0W$, the Hardy-type inequality

$$\mathcal{Q}_V(\varphi)\geq\int_{\Omega^\star}W_1|\varphi|^p\,\mathrm{d}\nu\quad\forall\varphi\in C_0^\infty\left(\Omega^\star\right)$$

does not hold. In particular, the equation $Q_{V-\lambda_0W}(u)=0$ in $\Omega^\star$ admits, up to a multiplicative positive constant, a unique positive (super)solution v; such a v is called the *Agmon ground state*.

(2) λ_0 is also the best constant for inequality (1.4) restricted to functions φ that are compactly supported either in a fixed punctured neighborhood of the origin, or in a fixed neighborhood of infinity in Ω. In particular, $\lambda_\infty(Q_V,W,\Omega^\star)=\lambda_0$.

(3) Suppose further that $V\geq0$. For an open set $\tilde\Omega\subset\Omega$, let $\mathcal{D}^{1,p}_{\mathcal{Q}_V}(\tilde\Omega)$ be the completion of $C_0^\infty(\tilde\Omega)$ with respect to the norm $\mathcal{Q}_V(\cdot)^{1/p}$. Then the functional $\mathcal{Q}_{V-\lambda_0W}$ is *null-critical* at 0 and at infinity in the following sense: for any pre-compact open set O containing 0, the (Agmon) ground state v of $Q_{V-\lambda_0W}$ in $\Omega^\star$ satisfies

$$\int_{O\setminus\{0\}}\left(|\nabla v|^p+V|v|^p\right)\mathrm{d}\nu=\infty,\quad\text{and}\quad\int_{\Omega\setminus\bar O}\left(|\nabla v|^p+V|v|^p\right)\mathrm{d}\nu=\infty.$$

In particular, the variational problem

$$\inf_{v\in\mathcal{D}^{1,p}_{\mathcal{Q}_V}(\Omega^\star)}\left\{\frac{\mathcal{Q}_V(\varphi)}{\int_{\Omega^\star}|\varphi|^pW\,\mathrm{d}\nu}\right\}\tag{1.5}$$

does not admit a minimizer.

Remark 1.3. It is natural to ask whether all the above properties of an optimal Hardy-weight are independent. It is indeed the case; in fact, in [6] we gave the following example which shows that, in general, (3) is not a consequence of (1) and (2).

Let $0 \le V \in C_0^\infty(\mathbb{R}^n)$ be a potential such that the operator $-\Delta - V(x)$ is critical in $\mathbb{R}^n$. Consider the operator $Q := -\Delta + \mathbf{1} - V(x)$, and the potential $W(x) := \mathbf{1}$. Then $\lambda_0(Q, W, \mathbb{R}^n) = \lambda_\infty(Q, W, \mathbb{R}^n) = 1$. On the other hand, the operator $Q - W$ is null-critical in $\mathbb{R}^n$ for $n \le 4$, and positive-critical if $n > 4$.

Remark 1.4. If $p \ne 2$, the definition of $\mathcal{D}^{1,p}_{\mathcal{Q}_V}(\Omega)$ cannot be applied to the case where $V \not\geq 0$, since the positivity of the functional $\mathcal{Q}_V$ on $C_0^\infty(\Omega)$ does not necessarily imply its convexity, and thus it does not give rise to a norm (see the discussion in [14]).

Using a modified supersolution construction, we obtain the main result of our paper:

Theorem 1.5. *Let $\bar\infty$ denote the ideal point in the one-point compactification of Ω. Suppose that $-\Delta_p$ admits a positive p-harmonic function $\mathcal{G}$ in $\Omega^\star := \Omega \setminus \{0\}$ satisfying one of the following conditions (1.6) and (1.7):*

$$1 < p \le n, \quad \lim_{x\to 0} \mathcal{G}(x) = \infty, \quad \text{and} \quad \lim_{x\to\bar\infty} \mathcal{G}(x) = 0, \tag{1.6}$$

$$p > n, \quad \lim_{x\to 0} \mathcal{G}(x) = \gamma \ge 0, \quad \text{and} \quad \lim_{x\to\bar\infty} \mathcal{G}(x) = \begin{cases} \infty & \text{if } \gamma = 0, \\ 0 & \text{if } \gamma > 0. \end{cases} \tag{1.7}$$

Define a positive function v and a nonnegative weight W on $\Omega^\star$ as follows:

(1) *If either (1.6) is satisfied, or (1.7) is satisfied with $\gamma = 0$, then $v := \mathcal{G}^{(p-1)/p}$, and $W := (\frac{p-1}{p})^p |\frac{\nabla\mathcal{G}}{\mathcal{G}}|^p$.*
(2) *If (1.7) is satisfied with $\gamma > 0$, then $v := [\mathcal{G}(\gamma - \mathcal{G})]^{(p-1)/p}$, and*

$$W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla\mathcal{G}}{\mathcal{G}(\gamma - \mathcal{G})}\right|^p |\gamma - 2\mathcal{G}|^{p-2}\big[2(p-2)\mathcal{G}(\gamma - \mathcal{G}) + \gamma^2\big].$$

Then the following Hardy-type inequality holds in $\Omega^\star$:

$$\int_{\Omega^\star} |\nabla\varphi|^p \,\mathrm{d}\nu \ge \int_{\Omega^\star} W|\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty\big(\Omega^\star\big), \tag{1.8}$$

and W is an optimal Hardy-weight *for $-\Delta_p$ in Ω.*

Moreover, up to a multiplicative constant, v is the unique positive supersolution of the equation $Q_{-W}(w) = 0$ in $\Omega^\star$.

Remark 1.6. Let us discuss hypotheses (1.6) and (1.7). Suppose first that Ω is a $C^{1,\alpha}$-bounded domain with $0 < \alpha \le 1$. Let $G^\Omega(x, 0)$ be the positive minimal p-Green function of the operator $-\Delta_p$ in Ω with a pole at 0. Then $\mathcal{G} := G^\Omega(\cdot, 0)$ satisfies either (1.6), or (1.7) with $\gamma > 0$. This assertion follows, for example, from the results in [8,9] and is valid more generally for any subcritical operator Q_V with $V \in L^\infty(\Omega)$.

Suppose further that Ω is a $C^{1,\alpha}$-subdomain of a noncompact Riemannian manifold M (where $\alpha \in (0, 1]$), with a positive p-Green function G^M that satisfies

$$\lim_{x\to\bar\infty} G^M(x, 0) = 0.$$

Using a standard exhaustion argument, the monotonicity of the Green functions as a function of the domain, and the above remark, it follows that $\mathcal{G} := G^\Omega(\cdot, 0)$ satisfies either (1.6), or (1.7) with $\gamma > 0$.

If $\Omega = \mathbb{R}^n$, $Q = -\Delta_p$, and $1 < p < n$ (resp., $p > n$), then $\mathcal{G}(x) := |x|^{\frac{p-n}{p-1}}$ satisfies assumption (1.6) (resp., assumption (1.7) with $\gamma = 0$). In this case, $\Omega^\star = \mathbb{R}^n \setminus \{0\}$ is the punctured space, and $W(x) = (\frac{p-1}{p})^p |x|^{-p}$ is the classical Hardy potential. We note that the criticality of the operator

$$Q_{-W}(u) = -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) - \left(\frac{p-1}{p}\right)^p \frac{|u|^{p-2}u}{|x|^p} \quad \text{in } \Omega^\star$$

follows also from the proof of [16, Theorem 1.3] given by Poliakovsky and Shafrir.

Remark 1.7. In our study, the domain $\Omega^\star$ should be viewed as a manifold with two ends: the origin and $\bar{\infty}$, the ideal point obtained by the one-point compactification of Ω. In particular, the notion of optimal Hardy-weight can be extended analogously to the case of any manifold with two ends (see Section 6, for an extension of Theorem 1.5 to annular or exterior domains).

The outline of the present paper is as follows. In Section 2 we review the theory of positive solutions for p-Laplacian type equations. Section 3 is devoted to a coarea formula which is a key result in our study (see Proposition 3.1). Section 4 explains the supersolution construction of Hardy-weights in various situations. Section 5 is devoted to the proof of Theorem 1.5. In Section 6 we present extensions of Theorem 1.5 to the case of annular and exterior domains. In Section 7 we present some L^p-Rellich-type inequalities and discuss the optimality of the obtained constants. Finally, in Section 8 we study the supersolution construction for general operators Q_V of the form (1.3), where the obtained weight is in general not optimal.

2. Preliminaries

Let Ω be a domain in $\mathbb{R}^n$ (or in a noncompact Riemannian manifold of dimension n), where $n \geq 2$. We equip Ω with the one-point compactification, and denote by $\bar{\infty}$ the added ideal point which we call the *infinity* in Ω. So, $x_n \to \bar{\infty}$ if and only if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ eventually exits any compact subset of Ω. For example, if $\Omega \subset \mathbb{R}^n$ is bounded, then the infinity in Ω is just $\partial\Omega$, and $x_n \to \bar{\infty}$ if and only if $\operatorname{dist}(x_n, \partial\Omega) \to 0$, where $\operatorname{dist}(\cdot, \partial\Omega)$ is the distance function to $\partial\Omega$.

Throughout the paper we assume that Ω is equipped with an absolutely continuous measure ν with respect to the Lebesgue measure in $\mathbb{R}^n$ (or with respect to the Riemannian measure in the case of a Riemannian manifold), and that the corresponding density is positive and smooth.

We write $\Omega_1 \Subset \Omega_2$ if Ω_2 is open, $\overline{\Omega_1}$ is compact and $\overline{\Omega_1} \subset \Omega_2$. Let $f, g \in C(D)$ be nonnegative functions, we denote $f \asymp g$ on D if there exists a positive constant C such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x) \quad \text{for all } x \in D.$$

For $1 < p < \infty$, we consider a quasilinear operator

$$Q_V(u) = Q(u) := -\Delta_p(u) + V|u|^{p-2}u, \tag{2.1}$$

where $V \in L^\infty_{\mathrm{loc}}(\Omega)$. Here, the p-Laplacian Δ_p is defined by

$$\Delta_p(u) := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),$$

where div is the divergence with respect to the measure ν, so, the integration by parts formula

$$\int_\Omega -\operatorname{div}(X)\varphi \, \mathrm{d}\nu = \int_\Omega X \cdot \nabla\varphi \, \mathrm{d}\nu$$

holds for any smooth vector field X and function φ that are compactly supported in Ω. Associated to Q_V there is the energy functional

$$\mathcal{Q}_V(\varphi) = \mathcal{Q}(\varphi) := \int_\Omega \big(|\nabla\varphi|^p + V|\varphi|^p\big) \, \mathrm{d}\nu \quad \varphi \in C_0^\infty(\Omega). \tag{2.2}$$

We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a (weak) *solution* of the equation $Q(u) = f$ in Ω if for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \big(|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + V|u|^{p-2}u\varphi\big) \, \mathrm{d}\nu = \int_\Omega f\varphi \, \mathrm{d}\nu. \tag{2.3}$$

We define in a similar way the notions of *subsolution* and *supersolution* of $Q(u) = f$. Weak solutions of the equation $Q(u) = 0$ admit Hölder continuous first derivatives, and nonnegative solutions of the equation $Q(u) = 0$ satisfy the Harnack inequality (see for example [10,17–20]). Therefore, in the definition (2.3) with $f = 0$, one can equivalently take test functions in $C_0^1(\Omega)$ instead of $C_0^\infty(\Omega)$.

The notions of criticality and subcriticality of Q_V have been studied in this context, and we refer to [13] for an account on this. For completeness, we recall the essential notions and results that we need throughout the present paper.

The operator Q is said to be *nonnegative in* Ω (and we denote it by $Q \geq 0$) if the equation $Q(u) = 0$ in Ω admits a positive (super)solution. As in the (selfadjoint) linear case, the following Allegretto–Piepenbrink type theorem holds:

Theorem 2.1. *(See [13, Theorem 2.3].) $Q \geq 0$ in Ω if and only if $\mathcal{Q}(\varphi) \geq 0$ for every $\varphi \in C_0^\infty(\Omega)$.*

Throughout the paper, we assume that Q is nonnegative in Ω. As in the linear case, there is a dichotomy for nonnegative operators: Q of the form (2.1) is either *critical* or *subcritical* in Ω. We note that in the case of $Q = -\Delta_p$ on a Riemannian manifold M equipped with its Riemannian measure, criticality (resp., subcriticality) is often called *p-parabolicity* (resp., *p-hyperbolicity*). Criticality/subcriticality has several equivalent definitions, which we recall below, but first we need to introduce some notions.

Definition 2.2. We say that a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of nonnegative functions belonging to $C_0^\infty(\Omega)$ is a *null-sequence* for $\mathcal{Q}$ in Ω if there exists an open set $B \Subset \Omega$ such that

$$\lim_{k\to\infty} \mathcal{Q}(\varphi_k) = \lim_{k\to\infty} \int_\Omega \left(|\nabla\varphi_k|^p + V|\varphi_k|^p\right) \mathrm{d}\nu = 0, \quad \text{and} \quad \int_B |\varphi_k|^p \,\mathrm{d}\nu \asymp 1.$$

Definition 2.3. Let K_0 be a compact set in Ω. A positive solution u of the equation $Q(w) = 0$ in $\Omega \setminus K_0$ is said to be a *positive solution of minimal growth in a neighborhood of infinity in* Ω (or $u \in \mathcal{M}_{\Omega,K_0}$ for brevity) if for any compact set K in Ω, with a smooth boundary, such that $K_0 \Subset \mathrm{int}(K)$, and any positive supersolution $v \in C((\Omega \setminus K) \cup \partial K)$ of the equation $Q(w) = 0$ in $\Omega \setminus K$, the inequality $u \leq v$ on ∂K implies that $u \leq v$ in $\Omega \setminus K$.

Similarly, for $x_0 \in \Omega$, we define the notion of a positive solution of the equation $Q(w) = 0$ in a punctured neighborhood of x_0 of minimal growth at x_0.

We have

Theorem 2.4. *(See [13,8].) Suppose that $\mathcal{Q}$ is nonnegative in Ω, and fix $x_0 \in \Omega$. Then the equation $Q(w) = 0$ has (up to a multiplicative constant) a unique positive solution $u \in \mathcal{M}_{\Omega,\{x_0\}}$ of minimal growth in a neighborhood of infinity in Ω.*

Moreover, u is either a global *positive solution of $Q(w) = 0$ in Ω (such a solution is called* Agmon's ground state*), or u has singularity at x_0 with the following asymptotic:*

$$u(x) \underset{x\to x_0}{\sim} \begin{cases} |x - x_0|^{\frac{p-n}{p-1}} & \text{if } 1 < p < n, \\ -\log|x - x_0| & \text{if } p = n, \\ 1 & \text{if } p > n. \end{cases}$$

In the latter case, the appropriately normalized solution is called the positive minimal Green function of Q in Ω *with a pole at x_0, and is denoted by $G_Q^\Omega(x, x_0) = G(x)$.*

Furthermore, any positive solution v of $Q(w) = 0$ in a punctured neighborhood of x_0 of minimal growth at x_0 has the following asymptotic near x_0:

$$v(x) \underset{x\to x_0}{\sim} \begin{cases} 1 & \text{if } 1 < p \leq n, \\ |x - x_0|^{\frac{p-n}{p-1}} & \text{if } p > n. \end{cases}$$

Definition 2.5. Suppose that $Q \geq 0$ in Ω. Then Q is said to be *critical* in Ω if the equation $Q(u) = 0$ in Ω admits a (Agmon) ground state, and *subcritical* in Ω otherwise.

Lemma 2.6. *(See [13].) Suppose that $Q \geq 0$ in Ω. Then the following assertions are equivalent:*

(1) *Q is* critical *in Ω.*
(2) *The equation $Q(w) = 0$ in Ω admits a unique positive supersolution (up to a multiplicative constant).*
(3) *The only nonnegative function W such that the inequality*

$$\mathcal{Q}(\varphi) \geq \int_{\Omega} W(x)|\varphi|^p \, \mathrm{d}\nu$$

holds for every $\varphi \in C_0^\infty(\Omega)$ is $W = 0$.
(4) *$\mathcal{Q}$ admits a null sequence in Ω.*

A nonnegative functional $\mathcal{Q}$ might contain an indefinite term (if the potential has indefinite sign). Although, by the Picone identity [2], such functional $\mathcal{Q}$ can be represented as the integral of a nonnegative Lagrangian L, this L still contains an indefinite term. It was proved in [15] that $\mathcal{Q}$ is equivalent to a *simplified energy* containing only nonnegative terms, as we explain now.

Definition 2.7. Let v be a positive solution of the equation $Q(u) = 0$ in Ω. The *simplified energy* is defined for nonnegative functions $w \in C_0^\infty(\Omega)$ by

$$\mathcal{Q}_{\mathrm{sim}}^v(w) := \begin{cases} \int_\Omega (v^2|\nabla w|^2(v|\nabla w| + w|\nabla v|)^{p-2}) \, \mathrm{d}\nu & \text{if } 1 < p \leq 2, \\ \int_\Omega (v^p|\nabla w|^p + v^2|\nabla v|^{p-2} w^{p-2}|\nabla w|^2) \, \mathrm{d}\nu & \text{if } p > 2. \end{cases} \tag{2.4}$$

Since the Picone identity holds also on manifolds (cf. [15, Section 2]), it follows that Lemma 2.2 in [15] is valid also on manifolds. Therefore, we obtain the following equivalence between the functional $\mathcal{Q}$ and the simplified energy $\mathcal{Q}_{\mathrm{sim}}^v$:

Lemma 2.8. *(See [15, Lemma 2.2].) Assume that $Q = Q_V \geq 0$ in Ω. Let $v \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ be a fixed positive solution of the equation $Q(u) = 0$ in Ω. Then for all $w \in C_0^\infty(\Omega)$ we have*

$$\mathcal{Q}(w) \asymp \mathcal{Q}_{\mathrm{sim}}^v\left(\frac{w}{v}\right).$$

Lemma 2.8 is a generalization of the *ground state transform* (see [6]) to the nonlinear case. In the nonlinear case, one obtains the equivalence (and not equality, as in the linear case) between $\mathcal{Q}$ and a functional containing only positive terms. As a corollary of Lemma 2.8, we state the following obvious upper estimate for the simplified energy, which will be of use later.

Lemma 2.9. *Denote*

$$X(w) := \int_{\Omega^\star} v^p|\nabla w|^p \, \mathrm{d}\nu, \qquad Y(w) := \int_{\Omega^\star} |w|^p|\nabla v|^p \, \mathrm{d}\nu.$$

Then there exists $C > 0$ such that for all $w \in C_0^\infty(\Omega)$ we have

$$\mathcal{Q}_{\mathrm{sim}}^v(w) \leq \begin{cases} C X(w) & \textit{if } 1 < p \leq 2, \\ C[X(w) + (\frac{X(w)}{Y(w)})^{2/p} Y(w)] & \textit{if } p > 2. \end{cases} \tag{2.5}$$

We conclude this section with the following useful lemma

Lemma 2.10. *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$, and $f \in C^2$. Then the following formula holds in the weak sense:*

$$-\Delta_p\big(f(u)\big) = -\big|f'(u)\big|^{p-2}\big[(p-1)f''(u)|\nabla u|^p + f'(u)\Delta_p(u)\big]. \tag{2.6}$$

Proof. Denote $g := -\Delta_p(u)$, and let $\varphi \in C_0^\infty(\Omega)$. Then, by Leibniz's product rule and the chain rule we have

$$\begin{aligned}&\int_\Omega \big|\nabla f(u)\big|^{p-2}\nabla f(u)\cdot\nabla\varphi\,\mathrm{d}\nu\\ &\quad= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\varphi\big)\,\mathrm{d}\nu - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\big)\varphi\,\mathrm{d}\nu.\end{aligned}$$

Note that for $p \ge 2$, the function $\psi(s) := |s|^{p-2}s$ is continuously differentiable, and $\psi'(s) := (p-1)|s|^{p-2}$. Moreover, for $1 < p < 2$ the function ψ is not differentiable at zero but its derivative near zero is integrable. Recall that by our assumptions $u \in C^{1,\alpha}(\Omega)$. Therefore if $p \ge 2$, then the function $|f'(u)|^{p-2}f'(u)\varphi$ belongs to $C_0^1(\Omega)$. On the other hand, for $1 < p < 2$, $\nabla(|f'(u)|^{p-2}f'(u)\varphi)$ is integrable. Hence in both cases, $|f'(u)|^{p-2}f'(u)\varphi$ is a legitimate test function. Consequently,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\varphi\big)\,\mathrm{d}\nu = \int_\Omega g\big|f'(u)\big|^{p-2}f'(u)\varphi\,\mathrm{d}\nu.$$

Therefore,

$$\begin{aligned}&\int_\Omega \big|\nabla f(u)\big|^{p-2}\nabla f(u)\cdot\nabla\varphi\,\mathrm{d}\nu\\ &\quad= \int_\Omega g\big|f'(u)\big|^{p-2}f'(u)\varphi\,\mathrm{d}\nu - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\big)\varphi\,\mathrm{d}\nu.\end{aligned}$$

Consequently, in the weak sense we have

$$-\Delta_p\big(f(u)\big) = -|\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\big) - \Delta_p(u)\big|f'(u)\big|^{p-2}f'(u).$$

But since $\psi'(s) := (p-1)|s|^{p-2}$ for $s \ne 0$, and ψ' is integrable at 0, we have that in the weak sense

$$|\nabla u|^{p-2}\nabla u\cdot\nabla\big(\big|f'(u)\big|^{p-2}f'(u)\big) = (p-1)\big|f'(u)\big|^{p-2}|\nabla u|^p f''(u). \tag{2.7}$$

This completes the proof of Lemma 2.10. □

3. The coarea formula

The present section is devoted to the proof of a coarea formula associated with the p-Laplacian. It seems that this key result in our study cannot be extended to the case of an operator Q_V of the form (1.3) with $V \ne 0$ and $p \ne 2$ (cf. [6, Lemma 9.2], where an analogue coarea formula is obtained for any linear symmetric operator).

Proposition 3.1. *Let $\mathcal{G}$ be a positive p-harmonic function in $\Omega^\star := \Omega \setminus \{0\}$. Define $v := \mathcal{G}^{(p-1)/p}$. Then there exists positive constants c and $\tilde{c}$ such that for every real functions f and g, defined on $(0,\infty)$ such that $f(v)$ and $g(v)$ have compact support in Ω, the following formulae hold:*

$$\int_{\Omega^\star} f(v)|\nabla v|^p\,\mathrm{d}\nu = c\int_{\inf v}^{\sup v}\frac{f(\tau)}{\tau}\,\mathrm{d}\tau, \tag{3.1}$$

and

$$\int_{\Omega^\star} g(\mathcal{G})|\nabla\mathcal{G}|^p\,\mathrm{d}\nu = \tilde{c}\int_{\inf\mathcal{G}}^{\sup\mathcal{G}} g(t)\,\mathrm{d}t. \tag{3.2}$$

Proof. The idea is the same as in [6, Lemma 9.2]. Setting $g(t) := f(t^{(p-1)/p})$ and performing the change of variable $\tau := t^{(p-1)/p}$, it follows that (3.1) is equivalent to (3.2). By the coarea formula, we have

$$\begin{aligned}\int_{\Omega^\star} g(\mathcal{G})|\nabla\mathcal{G}|^p\,\mathrm{d}\nu &= \int_{\inf\mathcal{G}}^{\sup\mathcal{G}}\Bigg(\int_{\{\mathcal{G}=t\}} g(\mathcal{G})\frac{|\nabla\mathcal{G}|^p}{|\nabla\mathcal{G}|}\mathrm{d}\sigma\Bigg)\mathrm{d}t\\ &= \int_{\inf\mathcal{G}}^{\sup\mathcal{G}} g(t)\Bigg(\int_{\{\mathcal{G}=t\}} |\nabla\mathcal{G}|^{p-1}\mathrm{d}\sigma\Bigg)\mathrm{d}t, \end{aligned} \tag{3.3}$$

where $\mathrm{d}\sigma$ denotes the Hausdorff measure of dimension $n-1$. Indeed, $\mathcal{G} \in C^{1,\alpha}_{\mathrm{loc}}$, in particular $|\nabla\mathcal{G}|^{p-1} \in L^1_{\mathrm{loc}}$ and the use of the coarea formula is licit. We claim that $\int_{\{\mathcal{G}=t\}} |\nabla\mathcal{G}|^{p-1}\mathrm{d}\sigma$ does not depend on t. This essentially follows from Green's formula, but since $\mathcal{G}$ is not smooth, we have to be careful. Let us fix t_1, t_2 such that $\inf\mathcal{G} < t_1 < t_2 < \sup\mathcal{G}$, and define $\mathcal{A}$ to be the "annulus"

$$\mathcal{A} := \big\{x \in \Omega^\star \mid t_1 < \mathcal{G} < t_2\big\}.$$

The boundary of $\mathcal{A}$ is the disjoint union of $\partial_- := \{\mathcal{G} = t_1\}$ and of $\partial_+ := \{\mathcal{G} = t_2\}$. We claim that $\mathcal{A}$ has *finite perimeter*, i.e., $\chi_{\mathcal{A}}$, the characteristic function of $\mathcal{A}$, has bounded variation. Indeed,

$$\chi_{\mathcal{A}} = \chi_{(t_1,t_2)} \circ \mathcal{G},$$

therefore,

$$\nabla\chi_{\mathcal{A}} = \big(\chi'_{(t_1,t_2)}(\mathcal{G})\big)\nabla\mathcal{G} = (\delta_{\mathcal{G}=t_1} - \delta_{\mathcal{G}=t_2})\nabla\mathcal{G}.$$

Since $\nabla\mathcal{G}$ is continuous, we obtain that $\chi_{\mathcal{A}} \in BV$, hence $\mathcal{A}$ has finite perimeter. Since $|\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}$ is continuous, and has divergence which vanishes in $\mathcal{A}$ in the weak sense, Theorems 5.2 and 7.2 in [4] imply that the Gauss–Green formula is valid on $\mathcal{A}$:

$$0 = -\int_{\mathcal{A}} \operatorname{div}\big(|\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\big)\,\mathrm{d}\nu = \int_{\partial_+^\star} |\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_{\partial_-^\star} |\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\cdot\mathbf{n}\,\mathrm{d}\sigma, \tag{3.4}$$

where $\partial_+^\star$ and $\partial_-^\star$ are the *reduced boundaries* (see [4]), $\mathbf{n}$ is the *measure theoretic exterior unit normal*, and σ is the $(n-1)$-dimensional Hausdorff measure. If $x \in \partial_+$ (resp., $x \in \partial_-$) is such that $|\nabla\mathcal{G}(x)| \neq 0$, then the boundary of $\mathcal{A}$ is C^1 in a neighborhood of x, and the vector field $\nabla\mathcal{G}/|\nabla\mathcal{G}|$ is well-defined near x; it is equal to $\mathbf{n}$ (resp., $-\mathbf{n}$) around x. Furthermore, we can write around x

$$|\nabla\mathcal{G}|^{p-1} = |\nabla\mathcal{G}|^{p-2}|\nabla\mathcal{G}| = \pm|\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\cdot\mathbf{n} \quad \text{if } x \in \partial_\pm. \tag{3.5}$$

Since $\mathcal{G}$ is $C^{1,\alpha}$, we may use a generalization of Sard's theorem due to Bojarski, Hajłasz and Strzelecki [3] to infer that for almost every $t \in (0,\infty)$

$$\sigma\big(\{\mathcal{G} = t\} \cap \operatorname{Crit}(\mathcal{G})\big) = 0,$$

where $\operatorname{Crit}(\mathcal{G})$ is the set of critical points of $\mathcal{G}$. This implies that for almost all t, (3.5) holds σ-almost everywhere on $\{\mathcal{G} = t\}$, and that for almost all t_1 and t_2, the reduced boundaries $\partial_+^\star$ and $\partial_-^\star$ coincide with $\partial_+ = \{\mathcal{G} = t_2\}$ and $\partial_- = \{\mathcal{G} = t_1\}$, respectively, up to a set of zero measure for σ. Since $|\nabla\mathcal{G}|^{p-1}$ and $|\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}$ are continuous, we obtain that for almost all t_1 and t_2 we have

$$\begin{aligned}&\int_{\partial_+^\star} |\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_{\partial_-^\star} |\nabla\mathcal{G}|^{p-2}\nabla\mathcal{G}\cdot\mathbf{n}\,\mathrm{d}\sigma\\ &\quad= \int_{\partial_+} |\nabla\mathcal{G}|^{p-1}\mathrm{d}\sigma - \int_{\partial_-} |\nabla\mathcal{G}|^{p-1}\mathrm{d}\sigma,\end{aligned}$$

and therefore by (3.4),

$$\int_{\{\mathcal{G}=t_2\}} |\nabla \mathcal{G}|^{p-1}\,\mathrm{d}\sigma = \int_{\{\mathcal{G}=t_1\}} |\nabla \mathcal{G}|^{p-1}\,\mathrm{d}\sigma.$$

Thus, $\int_{\{\mathcal{G}=t\}} |\nabla \mathcal{G}|^{p-1}\,\mathrm{d}\sigma$ is equal (almost everywhere) to a constant independent of t. □

4. The supersolution construction for the p-Laplacian

In this section, we show how to extend the *supersolution construction*, which was a primary tool in the study of the linear case in [6], to the p-Laplace operator. As in the linear case, in some cases this construction will give us *optimal* Hardy weights. We postpone the study of the supersolution construction for Q_V with $V \neq 0$ to Section 8, and here we present two particular supersolution constructions which apply to the p-Laplace operator. These constructions will lead us to the optimal weights of Theorems 1.5.

For completeness, we recall the supersolution construction for linear (not necessarily symmetric) elliptic operators:

Lemma 4.1. *(See [12, Theorem 3.1] and [6, Remark 5.4].) Let P be a second-order* linear *elliptic operator with real coefficients defined in Ω. For $j = 0, 1$, let V_j be real valued potentials, and suppose that v_j are positive (super)solutions of the equations $(P + V_j)u = 0$ in Ω. Then for $0 \le \alpha \le 1$ the function*

$$v_\alpha := (v_1)^\alpha (v_0)^{1-\alpha}$$

is a positive (super)solution of the linear equation

$$\big[P + (1-\alpha)V_0 + \alpha V_1 - \alpha(1-\alpha)W\big]u = 0 \quad \text{in } \Omega, \tag{4.1}$$

where

$$W := \left|\nabla \log\left(\frac{v_0}{v_1}\right)\right|_A^2, \tag{4.2}$$

$A = A(x)$ is the nonnegative definite matrix associated with the principal part of the operator P, and for $\xi \in \mathbb{R}^n$, $|\xi|_A^2 := \xi \cdot A\xi$.

We notice that since the proof of Lemma 4.1 is purely local and algebraic, we obtain in fact the following *pointwise* result.

Corollary 4.2. *Let P be a second-order* linear *elliptic operator with real coefficients defined in Ω. For $j = 0, 1$, let V_j be real valued potentials, and suppose that v_j are positive functions satisfying the differential (in)equality*

$$(P + V_j)v_j \underset{(\ge)}{=} 0 \quad \text{at } x_0 \in \Omega.$$

Then for $0 \le \alpha \le 1$ the function $v_\alpha = (v_1)^\alpha (v_0)^{1-\alpha}$ satisfies the differential (in)equality

$$\big[P + (1-\alpha)V_0 + \alpha V_1 - \alpha(1-\alpha)W\big]u \underset{(\ge)}{=} 0 \quad \text{at } x_0 \in \Omega, \tag{4.3}$$

where W is the function defined by (4.2).

A related – but weaker – convexity result is known in the case of p-Laplacian type equations:

Lemma 4.3. *(See [13, Proposition 4.3].) Let $V_0, V_1 \in L^\infty_{\mathrm{loc}}(\Omega)$, $V_0 \neq V_1$. For $\alpha \in [0, 1]$ we denote*

$$Q_\alpha(u) := Q_{(1-\alpha)V_0+\alpha V_1}(u) = (1-\alpha)Q_{V_0}(u) + \alpha Q_{V_1}(u), \tag{4.4}$$

and suppose that $Q_{V_i} \ge 0$ in Ω for $i = 0, 1$.

Then $Q_\alpha \ge 0$ in Ω for all $\alpha \in [0, 1]$. Moreover, Q_α is subcritical in Ω for all $\alpha \in (0, 1)$.

Remark 4.4. Lemma 4.3 does not provide us with an explicit nonzero Hardy-weight W_α for Q_α, although the subcriticality of Q_α ensures the existence of a strictly positive weight.

The supersolution construction has been extended to the p-Laplacian itself by several authors, with $v_\alpha := (v_1)^\alpha (v_0)^{1-\alpha}$, in the particular case where v_0 is a positive p-harmonic function, and $v_1 = \mathbf{1}$ (see for example [1,5,6] and references therein). In particular, the following Caccioppoli-type inequality has been obtained in [6]:

Proposition 4.5. *(See [6, Proposition 13.11].) Assume that $\mathcal{G}$ is a positive supersolution (resp., solution) of the equation $-\Delta_p(w) = 0$ in Ω. Then for $\alpha \in (0,1)$, $\mathcal{G}^\alpha$ is a positive supersolution (resp., solution) of the equation $Q_{-W_\alpha}(w) = 0$ in Ω, where*

$$W_\alpha := \alpha^{p-1}(1-\alpha)(p-1)\left|\frac{\nabla \mathcal{G}}{\mathcal{G}}\right|^p .$$

In particular, by taking the optimal value $\alpha = \frac{p-1}{p}$ we obtain the following logarithmic Caccioppoli inequality:

$$\int_\Omega |\nabla\varphi|^p \,\mathrm{d}\nu \ge \left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{\nabla v}{v}\right|^p |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega), \tag{4.5}$$

where v is any positive p-superharmonic function in Ω.

Proof. The first assertion of the proposition follows from Lemma 2.10 and in particular from (2.6) with $f(s) := s^\alpha$. Hence using the Allegretto–Piepenbrink Theorem 2.1, we obtain (4.5). □

Remark 4.6. Inequality (4.5) has been independently proved in [5] by L. D'Ambrosio and S. Dipierro, using a different approach.

Example 4.7. Consider Proposition 4.5 in the particular case $\Omega = \mathbb{R}^n \setminus \{0\}$, $p \ne n$, and $\mathcal{G}(x) = |x|^{\frac{p-n}{p-1}}$. Then (4.5) clearly implies the classical Hardy inequality (with the best constant):

$$\int_{\mathbb{R}^n\setminus\{0\}} |\nabla\varphi|^p \,\mathrm{d}x \ge \left|\frac{p-n}{p}\right|^p \int_{\mathbb{R}^n\setminus\{0\}} \frac{|\varphi(x)|^p}{|x|^p} \,\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.6}$$

We will see later that Proposition 4.5 yields an optimal Hardy weight if $\mathcal{G}$ further satisfies either assumption (1.6) or (1.7) with $\gamma = 0$ (see Theorem 1.5). However, as we shall see in Section 8, this supersolution construction does not provide us with an optimal Hardy weight if Ω is a bounded, $C^{1,\alpha}$-domain if $\mathcal{G}$ satisfies (1.7) with $\gamma > 0$. In this case and also in other cases (see Section 6), an *optimal* Hardy weight will be obtained using a different supersolution construction given by the following proposition.

Proposition 4.8. *Suppose that $\mathcal{G}$ is a $C^{1,\beta}$-positive supersolution (resp., solution) of $-\Delta_p w = 0$ in Ω satisfying $0 \le m < \mathcal{G} < M < \infty$ in Ω, where $0 < \beta \le 1$.*

Set $v_\alpha := [(\mathcal{G}-m)(M-\mathcal{G})]^\alpha$, and define

$$W_\alpha := (p-1)\alpha^{p-1}\left|\frac{\nabla\mathcal{G}}{v_1}\right|^p |m+M-2\mathcal{G}|^{p-2}\big[2(2\alpha-1)v_1 + (1-\alpha)(M-m)^2\big] \ge 0. \tag{4.7}$$

Then for α satisfying

$$\alpha \in \begin{cases} [1/2, 1] & \text{if } m > 0, \\ [0, 1] & \text{if } m = 0, \end{cases}$$

the function v_α is a positive supersolution (resp., solution) of the equation $Q_{-W_\alpha}(w) = 0$ in Ω.

In particular, let $\alpha = (p-1)/p$, and assume that either $\alpha = (p-1)/p \geq 1/2$, or $m = 0$. Define

$$W := W_{\frac{p-1}{p}} = \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla \mathcal{G}}{v_1}\right|^p |m + M - 2\mathcal{G}|^{p-2}\big[2(p-2)v_1 + (M-m)^2\big]. \tag{4.8}$$

Then

$$v := v_{\frac{p-1}{p}} = \big[(\mathcal{G} - m)(M - \mathcal{G})\big]^{\frac{p-1}{p}}$$

is a positive solution (resp., supersolution) of $Q_{-W}(w) = 0$ in Ω, and the following L^p-Hardy type inequality holds:

$$\int_\Omega |\nabla \varphi|^p \,\mathrm{d}\nu \geq \int_\Omega W|\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.9}$$

Proof. Let $0 \leq \alpha \leq 1$. By our assumption, $\mathcal{G} \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1]$. Moreover, the function $f(s) = [(s-m)(M-s)]^\alpha$ belongs to $C^2((0,\gamma))$. Consequently, one may apply Lemma 2.10 with $\mathcal{G}$ and f to obtain that in the weak sense,

$$-\Delta_p(v_\alpha) \underset{(\geq)}{=} -(p-1)|f'(\mathcal{G})|^{p-2}|\nabla \mathcal{G}|^p f''(\mathcal{G}) = W_\alpha v_\alpha^{p-1} \quad \text{in } \Omega.$$

Therefore, $v_\alpha = f(\mathcal{G})$ is a positive (super)solution of the equation $Q_{-W_\alpha}(w) = 0$ in Ω, and the Allegretto–Piepenbrink type theorem (Theorem 2.1) implies

$$\int_\Omega |\nabla \varphi|^p \,\mathrm{d}\nu \geq \int_\Omega W_\alpha|\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

In particular, for $\alpha = (p-1)/p$ we have (4.9). $\square$

Remark 4.9. Let $\Omega_1 \Subset \Omega_2 \subset \mathbb{R}^n$ be two open sets. Suppose that $\Omega := \Omega_2 \setminus \Omega_1$ is a $C^{1,\beta}$-bounded annular-type domain such that $\partial\Omega$ is the union of $\Gamma_1 = \partial\Omega_1$, and $\Gamma_2 = \partial\Omega_2$. Let $\mathcal{G}$ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta_p(u) = 0 & \text{in } \Omega, \\ u = m & \text{on } \Gamma_1, \\ u = M & \text{on } \Gamma_2, \end{cases}$$

where $0 \leq m < M$. Then $\mathcal{G}$ satisfies the assumptions of Proposition 4.8.

Moreover, if $p > n$, Ω is a $C^{1,\beta}$-bounded domain with $0 < \beta \leq 1$, and $\mathcal{G} := G^\Omega(\cdot, 0)$ is the positive minimal p-Green function of the operator $-\Delta_p$ in Ω with a pole at 0. Then $\mathcal{G}$ satisfies the assumptions of Proposition 4.8 in $\Omega^\star$, with $m := \lim_{x\to\partial\Omega} \mathcal{G}(x) = 0$, and $M := \lim_{x\to 0} \mathcal{G}(x)$.

Remark 4.10. If in Proposition 4.8 the supersolution $\mathcal{G}$ is unbounded and satisfies $\mathcal{G} > m$ in Ω, then one should simply consider the supersolution construction with $v_\alpha := (\mathcal{G} - m)^\alpha$ with $0 \leq \alpha \leq 1$ to obtain the Hardy-type inequality

$$\int_\Omega |\nabla \varphi|^p \,\mathrm{d}\nu \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \left|\frac{\nabla \mathcal{G}}{\mathcal{G} - m}\right|^p |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega) \tag{4.10}$$

(cf. Proposition 4.5).

Remark 4.11. A new phenomenon appears in Proposition 4.8: if $p \neq 2$, then the weight W_α *necessarily vanishes* in Ω. Indeed, $W_\alpha = 0$ on the set

$$\left\{x \in \Omega \,\middle|\, \mathcal{G}(x) = \frac{m+M}{2}\right\}.$$

5. Proof of Theorem 1.5

The present section is devoted to the proof of the main result of the paper, namely Theorem 1.5, that deals with the case $V=0$, and claims the optimality of the supersolution construction for the p-Laplacian in $\Omega^\star$. We divide the proofs into three parts: the criticality of $\mathcal{Q}_{-W}$, the optimality of the constant near infinity and zero, and finally the null-criticality of $\mathcal{Q}_{-W}$.

5.1. Criticality

In the present subsection, we prove the criticality of $\mathcal{Q}_{-W}$. We divide the proof into two parts, according to which of the assumptions (1.6), (1.7) is satisfied. We start by showing the criticality of $\mathcal{Q}_{-W}$ if either (1.6) or (1.7) with $\gamma=0$ is satisfied. This is a consequence of the following proposition:

Proposition 5.1. *Assume that in Theorem 1.5 the positive p-harmonic function $\mathcal{G}$ satisfies*

$$\begin{cases} \lim_{x\to 0}\mathcal{G}=\infty \quad \text{and} \quad \lim_{x\to\bar{\infty}}\mathcal{G}=0 & \text{if } 1<p\le n, \\ \lim_{x\to 0}\mathcal{G}=0 \quad \text{and} \quad \lim_{x\to\bar{\infty}}\mathcal{G}=\infty & \text{if } p>n. \end{cases} \tag{5.1}$$

Then the functional $\mathcal{Q}_{-W}$ is critical in $\Omega^\star$.

Proof. Let $v:=\mathcal{G}^{\frac{p-1}{p}}$. Proposition 4.5 implies that v is a positive solution of the equation $-\Delta_p(w)-W|w|^{p-2}w=0$ in $\Omega^\star$. We construct a null-sequence for the functional $\mathcal{Q}_{-W}$ in a similar fashion as in the proof of [16, Theorem 1.3]. Let

$$\varphi_n(t):=\begin{cases} 0 & 0\le t\le \frac{1}{n^2}, \\ 2+\frac{\log t}{\log n} & \frac{1}{n^2}\le t\le \frac{1}{n}, \\ 1 & \frac{1}{n}\le t\le n, \\ 2-\frac{\log t}{\log n} & n\le t\le n^2, \\ 0 & t\ge n^2. \end{cases}$$

Set $w_n:=\varphi_n(v)$, and consider the sequence $\{vw_n\}_{n\in\mathbb{N}}$.

Claim. *$\{vw_n\}$ is a null-sequence for the functional $\mathcal{Q}_{-W}$.*

Set $B:=\{x\in\Omega^\star \mid 1<v<2\}$, then $\bar{B}$ is compact in $\Omega^\star$. By Lemma 2.8 we have

$$\mathcal{Q}_{-W}(vw)\asymp \mathcal{Q}^v_{\mathrm{sim}}(w),$$

where $\mathcal{Q}^v_{\mathrm{sim}}$ is the simplified energy for the functional $\mathcal{Q}_{-W}$, associated to v (see (2.4)). Thus, we need to prove that

$$\lim_{n\to\infty}\frac{\mathcal{Q}^v_{\mathrm{sim}}(w_n)}{\int_B (vw_n)^p\,\mathrm{d}\nu}=0. \tag{5.2}$$

Set

$$X_n:=X(w_n)=\int\limits_{\Omega^\star} v^p|\nabla w_n|^p\,\mathrm{d}\nu, \quad \text{and} \quad Y_n:=Y(w_n)=\int\limits_{\Omega^\star} w_n^p|\nabla v|^p\,\mathrm{d}\nu.$$

Using the coarea formula (3.1), we obtain

$$X_n = c_1 \int_{\Omega^\star} v^p \left|\varphi_n'(v)\right|^p |\nabla v|^p \,\mathrm{d}\nu = c \int_0^\infty \left(t\left|\varphi_n'(t)\right|\right)^p \frac{\mathrm{d}t}{t}$$
$$= c\left(\frac{1}{\log n}\right)^p \left(\int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{\mathrm{d}t}{t} + \int_n^{n^2} \frac{\mathrm{d}t}{t}\right) = 2c\left(\frac{1}{\log n}\right)^{p-1}.$$

Using again (3.1), we get

$$Y_n = \int_{\Omega^\star} w_n^p |\nabla v|^p \,\mathrm{d}\nu = c \int_0^\infty \left|\varphi_n(t)\right|^p \frac{\mathrm{d}t}{t} \asymp \int_{\frac{1}{n}}^{n} \frac{\mathrm{d}t}{t} \asymp \log n.$$

On the other hand, we clearly have

$$\int_B (v w_n)^p \,\mathrm{d}\nu \asymp 1.$$

Recall that by (2.5), the simplified energy can be estimated from above by

$$\mathcal{Q}^v_{\mathrm{sim}}(w_n) \le C \begin{cases} X_n & \text{if } 1 < p \le 2, \\ X_n + (\frac{X_n}{Y_n})^{2/p} Y_n & \text{if } p > 2. \end{cases}$$

Therefore, $\lim_{n\to\infty} \mathcal{Q}^v_{\mathrm{sim}}(w_n) = 0$, and (5.2) is proved. Thus, $\{v w_n : n \in \mathbb{N}\}$ is a null-sequence for the functional $\mathcal{Q}_{-W}$, and $\mathcal{Q}_{-W}$ is critical in $\Omega^\star$. □

Next, we prove the criticality of $\mathcal{Q}_{-W}$ if assumption (1.7) with $\gamma > 0$ is satisfied:

Proposition 5.2. *Assume that in Theorem 1.5 $p > n$, and the positive p-harmonic function $\mathcal{G}$ satisfies*

$$\lim_{x\to 0} \mathcal{G} = \gamma > 0 \quad \text{and} \quad \lim_{x\to \bar\infty} \mathcal{G} = 0. \tag{5.3}$$

Then the functional $\mathcal{Q}_{-W}$ is critical in $\Omega^\star$.

Proof. The proof follows closely the proof of Proposition 5.1. Assume for simplicity that $\gamma = \mathcal{G}(0) = 1$. Recall that $v := [\mathcal{G}(1-\mathcal{G})]^{\frac{p-1}{p}}$. Proposition 4.8 implies that v is a positive solution of the equation $-\Delta_p(w) - W|w|^{p-2}w = 0$ in $\Omega^\star$. We construct a null-sequence for the functional $\mathcal{Q}_{-W}$. This time, let

$$\varphi_n(t) := \begin{cases} 0 & 0 \le t \le \frac{1}{n^2}, \\ 2 + \frac{\log t}{\log n} & \frac{1}{n^2} \le t \le \frac{1}{n}, \\ 1 & \frac{1}{n} \le t, \end{cases}$$

and consider the sequence $\{w_n = \varphi_n(v)\}_{n\in\mathbb{N}}$. By hypothesis, $v(0) = 0$ and $\lim_{x\to\bar\infty} v(x) = 0$. Therefore, for every $n \in \mathbb{N}$, w_n is compactly supported in $\Omega^\star$.

Claim. *The sequence $\{v w_n\}_{n\in\mathbb{N}}$ is a null-sequence for the functional $\mathcal{Q}_{-W}$.*

Set $B := \{x \in \Omega^\star \mid \frac{1}{4} < v < \frac{3}{4}\}$, then $\bar{B}$ is compact in $\Omega^\star$. As in the proof of Proposition 5.1, we set

$$X_n := X(w_n) = \int_{\Omega^\star} v^p |\nabla w_n|^p \,\mathrm{d}\nu, \quad \text{and} \quad Y_n := Y(w_n) = \int_{\Omega^\star} w_n^p |\nabla v|^p \,\mathrm{d}\nu.$$

Let $f(s) := [s(1-s)]^{\frac{p-1}{p}}$. Using the coarea formula (3.2), we obtain

$$X_n = \int_{\Omega^\star} v^p |\nabla v|^p \left|\varphi_n'(v)\right|^p \mathrm{d}\nu$$
$$= C \int_{\Omega^\star} \left[\mathcal{G}(1-\mathcal{G})\right]^{p-2} |1-2\mathcal{G}|^p \left|\varphi_n' \circ f(\mathcal{G})\right|^p |\nabla \mathcal{G}|^p \mathrm{d}\nu$$
$$= \frac{C}{(\log n)^p} \int_{f(t)\in[1/n^2,1/n]} \frac{|1-2t|^p}{t(1-t)} \mathrm{d}t \asymp \frac{1}{(\log n)^{p-1}}.$$

Using again the coarea formula (3.2), we get

$$Y_n = \int_{\Omega^\star} \left(\varphi_n(v)\right)^p |\nabla v|^p \mathrm{d}\nu = \int_0^1 \varphi_n\left(f(t)\right)^p \frac{|1-2t|^p}{t(1-t)} \mathrm{d}t \asymp \log n.$$

In light of (2.5) we have $\lim_{n\to\infty} Q^v_{\mathrm{sim}}(w_n) = 0$. On the other hand, we clearly have

$$\int_B (v w_n)^p \mathrm{d}\nu \asymp 1.$$

Hence, $\{v w_n\}_{n\in\mathbb{N}}$ is a null-sequence for the functional Q_{-W}. □

5.2. *Optimality of the constant near infinity and zero*

In the present subsection we prove the optimality of the constant $C_p := (\frac{p-1}{p})^p$ near the ends of $\Omega^\star$. As in the previous subsection, we split the proof into two parts.

Proposition 5.3. *Assume that in Theorem 1.5 the positive p-harmonic function $\mathcal{G}$ satisfies*

$$\begin{cases} \lim\limits_{x\to 0} \mathcal{G} = \infty \quad \text{and} \quad \lim\limits_{x\to \bar\infty} \mathcal{G} = 0 & \text{if } 1 < p \le n, \\ \lim\limits_{x\to 0} \mathcal{G} = 0 \quad \text{and} \quad \lim\limits_{x\to \bar\infty} \mathcal{G} = \infty & \text{if } p > n. \end{cases} \tag{5.4}$$

Then the constant $\lambda = C_p$ in the Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^p \mathrm{d}\nu \ge \lambda \int_{\Omega^\star} \left|\frac{\nabla \mathcal{G}}{\mathcal{G}}\right|^p |\varphi|^p \mathrm{d}\nu \tag{5.5}$$

is also the best constant for functions φ compactly supported either in a fixed punctured neighborhood of the origin, or in a fixed neighborhood of infinity in Ω.

Proof. We assume that $1 < p \le n$, and present the proof of the optimality at infinity, the other cases being proved similarly. We proceed by contradiction.

Suppose that there exists a positive constant λ and a compact set $K \Subset \Omega$ containing zero such that

$$\int_{\Omega\setminus K} \left(|\nabla\psi|^p - W|\psi|^p\right) \mathrm{d}\nu \ge \lambda \int_{\Omega\setminus K} W|\psi|^p \mathrm{d}\nu \quad \forall \psi \in C_0^\infty(\Omega\setminus K). \tag{5.6}$$

We apply inequality (5.6) to $\psi = v\varphi$, where $v := \mathcal{G}^{(p-1)/p}$ is a positive solution of $Q_{-W}(w) = 0$, and $\varphi \in C_0^\infty(\Omega\setminus K)$. Now, use Lemmas 2.8 and 2.9, and (5.6) to obtain that for some positive constant β we have

$$\beta Y(\varphi) \le \begin{cases} X(\varphi) & \text{if } 1 < p \le 2, \\ X(\varphi) + (\frac{X(\varphi)}{Y(\varphi)})^{\frac{2}{p}} Y(\varphi) & \text{if } p > 2, \end{cases} \qquad \forall \varphi \in C_0^\infty(\Omega\setminus K), \tag{5.7}$$

where we recall that $X(\varphi) := \int_{\Omega^\star} v^p|\nabla\varphi|^p\,\mathrm{d}\nu$ and $Y(\varphi) := \int_{\Omega^\star} \varphi^p|\nabla v|^p\,\mathrm{d}\nu = \int_{\Omega^\star} v^p\varphi^p W\,\mathrm{d}\nu$. In the case $p > 2$, using the fact that for every $\varepsilon > 0$, there is a constant $C > 0$ such that for every $t > 0$, $t + t^{2/p} \le Ct + \varepsilon$, we have that

$$X(\varphi) + \left(\frac{X(\varphi)}{Y(\varphi)}\right)^{\frac{2}{p}} Y(\varphi) \le CX(\varphi) + \varepsilon Y(\varphi).$$

Taking $\varepsilon < \beta$, we get by (5.7) that for any $1 < p < \infty$, there is a constant $C > 0$ such that

$$CY(\varphi) \le X(\varphi) \quad \forall\varphi \in C_0^\infty(\Omega \setminus K). \tag{5.8}$$

Assume without loss of generality that $\{v \le 1\} \subset \Omega \setminus K$. Using the coarea formula (3.1), and applying inequality (5.8) to $\varphi = \phi(v)$, where $\phi \in C_0^\infty((0,1))$ we get that

$$\int_0^1 |\phi(t)|^p \frac{\mathrm{d}t}{t} \le C\int_0^1 \big(t|\phi'(t)|\big)^p \frac{\mathrm{d}t}{t} \quad \forall\phi \in C_0^\infty\big((0,1)\big). \tag{5.9}$$

But by [11, Theorem 1 of Sec. 1.3.2], this inequality cannot hold.

Alternatively, an easy way to see that (5.9) does not hold is to define a sequence $\{\phi_\varepsilon\}$ of compactly supported Lipschitz continuous functions in $(0, 1)$ of the form

$$\phi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon|\log\varepsilon|^\gamma} & t \in (0,\varepsilon), \\ \frac{1}{|\log t|^\gamma} & t \in (\varepsilon, \frac{1}{2}), \\ \psi(t) & t \in (\frac{1}{2}, 1), \end{cases}$$

where ψ is a smooth function, independent of ε such that $\psi(1) = 0$, and $\gamma > 0$ will be determined later. Apply inequality (5.9) to ϕ_ε to get

$$\int_\varepsilon^{\frac{1}{2}} |\phi_\varepsilon(t)|^p \frac{\mathrm{d}t}{t} \le C\left(\frac{1}{\varepsilon|\log\varepsilon|^\gamma}\int_0^\varepsilon t^p\frac{\mathrm{d}t}{t} + \int_\varepsilon^1 \big(t|\phi_\varepsilon'(t)|\big)^p\frac{\mathrm{d}t}{t}\right). \tag{5.10}$$

Since $p > 1$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon|\log\varepsilon|^\gamma}\int_0^\varepsilon t^p\frac{\mathrm{d}t}{t} = \lim_{\varepsilon\to 0}\frac{\varepsilon^{p-1}}{|\log\varepsilon|^\gamma} = 0,$$

therefore, letting $\varepsilon \to 0$ in (5.10), we get

$$\int_0^{\frac{1}{2}} \left(\frac{1}{|\log t|^\gamma}\right)^p\frac{\mathrm{d}t}{t} \le C\left(\int_0^{\frac{1}{2}}\left(\frac{1}{|\log t|^{\gamma+1}}\right)^p\frac{\mathrm{d}t}{t} + \int_{\frac{1}{2}}^1 \big(t\psi'(t)\big)^p\frac{\mathrm{d}t}{t}\right).$$

The right-hand side is finite for every positive value of γ, since $p(\gamma+1) > 1$. The left-hand side, on the contrary, is finite if and only if $p\gamma > 1$. Thus, taking γ such that $p\gamma \le 1$, we get a contradiction. As a consequence, inequality (5.9) cannot hold. □

Next, we prove the optimality of the constant $C_p = (\frac{p-1}{p})^p$ near the ends of $\Omega^\star$ if assumption (1.7) with $\gamma > 0$ is satisfied:

Proposition 5.4. *Assume that in Theorem 1.5 $p > n$, and the positive p-harmonic function $\mathcal{G}$ satisfies*

$$\lim_{x\to 0}\mathcal{G} = \gamma > 0 \quad \textit{and} \quad \lim_{x\to\bar{\infty}}\mathcal{G} = 0. \tag{5.11}$$

Denote

$$V := \left|\frac{\nabla \mathcal{G}}{\mathcal{G}(\gamma-\mathcal{G})}\right|^p |\gamma - 2\mathcal{G}|^{p-2}\big[2(p-2)\mathcal{G}(\gamma-\mathcal{G})+\gamma^2\big].$$

Then in the Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^p \,\mathrm{d}\nu \ge \lambda \int_{\Omega^\star} V|\varphi|^p \,\mathrm{d}\nu \tag{5.12}$$

the constant $\lambda = C_p$ is also the best constant for functions compactly supported either in a fixed punctured neighborhood of the origin, or in a fixed neighborhood of infinity in Ω.

Proof. We prove the optimality of the constant C_p at infinity, the proof of the optimality at zero being similar (by replacing $\mathcal{G}$ with $(\gamma-\mathcal{G})$). Note that $W = C_p V$. Assume by contradiction that C_p is not optimal at infinity, then there is a positive constant λ and a compact subset K of Ω containing 0, such that

$$\int_{\Omega\setminus K} \big(|\nabla\psi|^p - W|\psi|^p\big)\,\mathrm{d}\nu \ge \lambda \int_{\Omega\setminus K} W|\psi|^p\,\mathrm{d}\nu \quad \forall\psi\in C_0^\infty(\Omega\setminus K). \tag{5.13}$$

Since by our assumption $\lim_{x\to\bar\infty}\mathcal{G}(x)=0$, we have

$$W \underset{x\to\bar\infty}{\sim} \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla\mathcal{G}}{\mathcal{G}}\right|^p.$$

Therefore, by enlarging K, we may assume that the following inequality is satisfied, for some $\mu>0$:

$$\int_{\Omega\setminus K} \big(|\nabla\psi|^p - W|\psi|^p\big)\,\mathrm{d}\nu \ge \mu \int_{\Omega\setminus K} \left|\frac{\nabla\mathcal{G}}{\mathcal{G}}\right|^p |\psi|^p\,\mathrm{d}\nu \quad \forall\psi\in C_0^\infty(\Omega\setminus K). \tag{5.14}$$

We apply this inequality to $\psi=\varphi v$, where $v=[\mathcal{G}(\gamma-\mathcal{G})]^{\frac{p-1}{p}}$ is a positive solution of $Q_{-W}(w)=0$, and $\varphi\in C_0^\infty(\Omega\setminus K)$. Define $\tilde v := \mathcal{G}^{\frac{p-1}{p}}$, and notice that at infinity,

$$v \underset{x\to\bar\infty}{\sim} \tilde v,$$

and

$$\left|\frac{\nabla v}{v}\right|^p \underset{x\to\bar\infty}{\sim} \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla\mathcal{G}}{\mathcal{G}}\right|^p = \left|\frac{\nabla\tilde v}{\tilde v}\right|^p.$$

Therefore, from Lemma 2.8, (2.5) with $p>2$, and (5.14), one gets that for some positive constant β,

$$\beta\tilde Y(\varphi) \le \tilde X(\varphi) + \left(\frac{\tilde X(\varphi)}{\tilde Y(\varphi)}\right)^{\frac{2}{p}} \tilde Y(\varphi) \quad \forall\varphi\in C_0^\infty(\Omega\setminus K), \tag{5.15}$$

where $\tilde X(\varphi) := \int_{\Omega^\star} \tilde v^p |\nabla\varphi|^p\,\mathrm{d}\nu$ and $\tilde Y(\varphi) := \int_{\Omega^\star} \varphi^p|\nabla\tilde v|^p\,\mathrm{d}\nu$. We are back to inequality (5.7) of the proof of Proposition 5.3, where we have shown that such an inequality cannot hold. Consequently, (5.13) does not hold, and the constant $(\frac{p-1}{p})^p$ in (5.12) is optimal at infinity. □

5.3. Null-criticality

The null-criticality of the operators Q_{-W} in $\Omega^\star$ follows from our coarea formula (3.1). First, we have:

Proposition 5.5. *Assume that in Theorem 1.5 the positive p-harmonic function $\mathcal{G}$ satisfies*

$$\begin{cases} \lim\limits_{x\to 0} \mathcal{G} = \infty \quad \text{and} \quad \lim\limits_{x\to\bar\infty} \mathcal{G} = 0 \quad \text{if } 1 < p \le n, \\ \lim\limits_{x\to 0} \mathcal{G} = 0 \quad \text{and} \quad \lim\limits_{x\to\bar\infty} \mathcal{G} = \infty \quad \text{if } p > n. \end{cases} \tag{5.16}$$

Then the functional Q_{-W} *is null-critical at* 0 *and at infinity in* Ω.

Proof. Let $v := \mathcal{G}^{\frac{p-1}{p}}$, and denote also $u := \mathcal{G}$. A minimizer of the variational problem (1.5) is necessarily a positive solution of the equation $Q_{-W} = 0$ in Ω^*. Since Q_{-W} is critical, a minimizer in $\mathcal{D}^{1,p}(\Omega)$ should be the ground state v. We claim that for any neighborhood O of 0, the ground state v satisfies

$$\int_{O\setminus\{0\}} |\nabla v|^p \,\mathrm{d}\nu = \infty, \quad \text{and} \quad \int_{\Omega\setminus\bar O} |\nabla v|^p \,\mathrm{d}\nu = \infty.$$

Indeed, the coarea formula (3.1) implies that

$$\int_{\{t_- < u(x) < t_+\}} |\nabla v|^p \,\mathrm{d}\nu = c_1 \int_{t_-}^{t_+} \frac{\mathrm{d}t}{t} \underset{t_\pm \to \varepsilon_\pm}{\longrightarrow} \infty,$$

with $\varepsilon_+ = \infty$ and $\varepsilon_- = 0$. Thus, the claim is proved. □

The corresponding result, under assumption (1.7) with $\gamma > 0$, reads as follows

Proposition 5.6. *Assume that in Theorem 1.5* $p > n$, *and the positive* p*-harmonic function* $\mathcal{G}$ *satisfies*

$$\lim_{x\to 0} \mathcal{G} = \gamma > 0 \quad \text{and} \quad \lim_{x\to\bar\infty} \mathcal{G} = 0. \tag{5.17}$$

Then the functional Q_{-W} *is null-critical at* 0 *and at infinity in* Ω.

Proof. The proof is similar to the proof of Proposition 5.5. Indeed, recall that $v := [\mathcal{G}(\gamma - \mathcal{G})]^{\frac{p-1}{p}}$. Let $\varepsilon_+ = \gamma$ and $\varepsilon_- = 0$. It is enough to prove that

$$\lim_{t_\pm \to \varepsilon_\pm} \int_{\{t_- < \mathcal{G} < t_+\}} |\nabla v|^p \,\mathrm{d}\nu = \infty.$$

We prove it when $t_- \to 0$, the other case is similar, (replace $\mathcal{G}$ with $(\gamma - \mathcal{G})$). Define $\tilde v = \mathcal{G}^{\frac{p-1}{p}}$. At infinity in Ω, we have

$$v \underset{x\to\bar\infty}{\sim} \tilde v,$$

and

$$\left|\frac{\nabla v}{v}\right|^p \underset{x\to\bar\infty}{\sim} \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla \mathcal{G}}{\mathcal{G}}\right|^p = \left|\frac{\nabla \tilde v}{\tilde v}\right|^p.$$

Therefore, by the coarea formula (3.1), one has as $t_- \to 0$,

$$\int_{\{t_- < \mathcal{G} < \gamma/2\}} |\nabla v|^p \,\mathrm{d}\nu \sim \int_{\{t_- < \mathcal{G} < \gamma/2\}} |\nabla \tilde v|^p \,\mathrm{d}\nu = \int_{t_-}^{\frac{\gamma}{2}} \frac{\mathrm{d}t}{t},$$

and consequently,

$$\lim_{t_- \to \varepsilon_-} \int_{\{t_- < \mathcal{G} < \gamma/2\}} |\nabla v|^p \,\mathrm{d}\nu = \infty. \qquad \square$$

We conclude the present section with a corollary concerning Caccioppoli inequality. Recall the logarithmic Caccioppoli inequality (4.5) which holds in particular in $\Omega^\star$:

$$\int_{\Omega^\star} |\nabla\varphi|^p \,\mathrm{d}\nu \geq \mu \int_{\Omega^\star} \left|\frac{\nabla v}{v}\right|^p |\varphi|^p \,\mathrm{d}\nu \quad \forall\varphi \in C_0^\infty(\Omega^\star), \tag{5.18}$$

where v is any positive p-superharmonic functions in $\Omega^\star$, and $\mu \geq C_P = (\frac{p-1}{p})^p$. By the results of [6] it follows that in the linear case (where $p=2$) the constant $C_2 = 1/4$ in (5.18) is optimal.

Now, Theorem 1.5 clearly implies the optimality of the constant C_p also for any $1 < p \leq n$. More precisely, we have.

Corollary 5.7. *Assume that $1 < p \leq n$, and suppose that Ω is a $C^{1,\alpha}$-domain of a noncompact Riemannian manifold M (where $\alpha \in (0,1]$), and $-\Delta_p$ is subcritical in M. Let G^M be the positive minimal Green function, and assume that* $\lim_{x\to\bar\infty} G^M(x,0) = 0$.

Then the best constant in the logarithmic Caccioppoli inequality (5.18) equals to $(\frac{p-1}{p})^p$.

6. Optimal weights for annular and exterior domains

In the present section we extend our main result (Theorem 1.5), obtained for punctured domains, to two additional types of domains: annular-type domains and exterior-type domains. As in the case of punctured domains, we view these two types of domains as manifolds with two ends. In particular, Definition 1.2 of optimal Hardy-type weight (which was given for a punctured domain) is extended naturally to handle annular-type and exterior-type domains.

We assume that the given positive p-harmonic function admits limits at the two ends (one limit might be infinity). We use the supersolution constructions obtained in Propositions 4.5 and 4.8, and the techniques used in the proof of Theorem 1.5 to obtain optimal Hardy-weights for these cases. We omit the proofs since they differ only slightly from the proof of Theorem 1.5.

Theorem 6.1. *Let Ω be a $C^{1,\alpha}$ domain for some $\alpha > 0$. Let $U \Subset \Omega$ be an open $C^{1,\alpha}$ subdomain of Ω, and consider $\tilde\Omega := \Omega \setminus U$. Denote by $\bar\infty$ the infinity in Ω, and assume that $-\Delta_p$ admits a positive p-harmonic function $\mathcal{G}$ in $\tilde\Omega$ satisfying the following conditions*

$$\lim_{x\to\partial U} \mathcal{G}(x) = \gamma_1, \qquad \lim_{x\to\bar\infty} \mathcal{G}(x) = \gamma_2, \tag{6.1}$$

where $\gamma_1 \neq \gamma_2$, and $0 \leq \gamma_1, \gamma_2 \leq \infty$. Denote

$$m := \min\{\gamma_1,\gamma_2\}, \qquad M := \max\{\gamma_1,\gamma_2\}.$$

Define positive functions v_1 and v, and a nonnegative weight W on $\tilde\Omega$ as follows:

(a) *If $M < \infty$, assume further that either $m = 0$ or $p \geq 2$, and let*

$$v_1 := (\mathcal{G}-m)(M-\mathcal{G}), \qquad v := v_1^{(p-1)/p} = \big[(\mathcal{G}-m)(M-\mathcal{G})\big]^{(p-1)/p},$$

and

$$W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla\mathcal{G}}{v_1}\right|^p |m+M-2\mathcal{G}|^{p-2}\big[2(p-2)v_1 + (M-m)^2\big]. \tag{6.2}$$

(b) *If $M = \infty$, define*

$$v_1 := (\mathcal{G}-m), \qquad v := v_1^{(p-1)/p} = (\mathcal{G}-m)^{(p-1)/p},$$

and

$$W := \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla\mathcal{G}}{v_1}\right|^p. \tag{6.3}$$

Then the following Hardy-type inequality holds true

$$\int_{\tilde{\Omega}} |\nabla\varphi|^p \,\mathrm{d}\nu \geq \int_{\tilde{\Omega}} W|\varphi|^p \,\mathrm{d}\nu \quad \forall\varphi \in C_0^\infty(\tilde{\Omega}), \tag{6.4}$$

and W is an optimal *Hardy-weight for $-\Delta_p$ in $\tilde{\Omega}$.*

Moreover, up to a multiplicative constant, v is the unique positive supersolution of the equation $Q_{-W}(w) = 0$ in $\tilde{\Omega}$.

7. Optimal L^p Rellich-type inequalities

Throughout the present section we consider a *linear* operator P. In [6] we proved the following L^2-Rellich-type inequality.

Lemma 7.1. *(See [6, Corollary 10.3].) Assume that P is a subcritical linear Schrödinger-type operator in Ω of the form*

$$P := -\operatorname{div}\big(A(x)\nabla\cdot\big) + V(x),$$

and let v_0 and v_1 be two linearly independent positive solutions of the equation $Pu = 0$ in Ω. Let $W := \frac{1}{4}|\nabla \log(\frac{v_0}{v_1})|_A^2$ be the Hardy-weight obtained by the supersolution construction with a pair (v_0, v_1) (see (4.2)). Suppose that W is strictly positive, and fix $0 \leq \lambda \leq 1$. Then

(a) *For a fixed $0 \leq \alpha < 1$ and all $\varphi \in C_0^\infty(\Omega)$ the following Rellich-type inequality holds true*

$$\int_\Omega \frac{|P\varphi|^2}{W(x)}\left(\frac{v_0}{v_1}\right)^\alpha \mathrm{d}\nu \geq \lambda\big(1-\alpha^2\big)^2 \int_\Omega |\varphi|^2 W(x)\left(\frac{v_0}{v_1}\right)^\alpha \mathrm{d}\nu. \tag{7.1}$$

(b) *If $P - W$ is critical in Ω, then $\lambda = 1$ is the best constant in (7.1).*

We are interested in generalizing the L^2-Rellich-type inequalities (7.1) to L^p-Rellich-type inequalities for the operator P. Our result hinges on the following L^p-Rellich-type inequality of E.B. Davies and A.M. Hinz:

Theorem 7.2. *(See [7, Theorem 4].) Let Ω be a domain in a Riemannian manifold of dimension $n \geq 2$, and let $1 \leq p < \infty$. If $0 < v \in C(\Omega)$ with $-\Delta v > 0$ and $-\Delta(v^\delta) \geq 0$ for some $\delta > 1$, then*

$$\int_\Omega \frac{v^p}{|\Delta v|^{p-1}}|\Delta\varphi|^p \,\mathrm{d}\nu \geq \frac{[(p-1)\delta+1]^p}{p^{2p}} \int_\Omega |\Delta v||\varphi|^p \,\mathrm{d}\nu \quad \forall\varphi \in C_0^\infty(\Omega).$$

If $P = -\operatorname{div}(A(x)\nabla\cdot)$ (i.e., $V = 0$), Theorem 7.2 implies the following L^p-Rellich-type inequality:

Theorem 7.3. *Let $P := -\operatorname{div}(A\nabla\cdot)$ be a subcritical operator in Ω, and let v_0 be a positive (super)solution of the equation $Pu = 0$ in Ω and $v_1 := \mathbf{1}$. Let $W := \frac{1}{4}|\nabla \log v_0|_A^2$ be the Hardy-weight obtained by the supersolution construction with a pair (v_0, v_1), and suppose that $W > 0$. Then for every $\alpha \in (0, 1)$ and $1 \leq p < \infty$ the following Rellich-type inequality holds:*

$$\int_\Omega \frac{|P\varphi|^p}{W^{p-1}}(v_0)^\alpha \,\mathrm{d}\nu \geq \frac{4^p(1-\alpha)^p(p-1+\alpha)^p}{p^{2p}} \int_\Omega |\varphi|^p W(v_0)^\alpha \,\mathrm{d}\nu \quad \forall\varphi \in C_0^\infty(\Omega). \tag{7.2}$$

Proof. Apply Theorem 7.2, with $v := (v_0)^\alpha$, and $\delta = 1/\alpha$. Since $-\Delta v \geq 4\alpha(1-\alpha)Wv > 0$, and $-\Delta v^\delta \geq 0$, we obtain (7.2). □

Using the ground state transform with a positive solution v_1, Theorem 7.3 implies:

Theorem 7.4. *Let $P := -\operatorname{div}(A\nabla\cdot) + V$ be a subcritical linear Schrödinger-type operator in Ω, and let v_0 and, v_1 be two positive solutions of the equation $Pu = 0$ in Ω. Let $W := \frac{1}{4}|\nabla \log(v_0/v_1)|_A^2$ be the Hardy-weight obtained by the supersolution construction with a pair (v_0, v_1), and suppose that $W > 0$. Then for every $\alpha \in (0,1)$ and $1 \le p < \infty$ the following L^p-Rellich-type inequality holds:*

$$\int_\Omega \frac{|P\varphi|^p}{W^{p-1}}\left(\frac{v_0}{v_1}\right)^\alpha v_1^{2-p}\,\mathrm{d}\nu \ge \frac{4^p(1-\alpha)^p(p-1+\alpha)^p}{p^{2p}} \int_\Omega |\varphi|^p W\left(\frac{v_0}{v_1}\right)^\alpha v_1^{2-p}\,\mathrm{d}\nu \tag{7.3}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Remark 7.5. In the case $p = 2$, we recover the best constant $(1-\alpha^2)^2$ obtained in Lemma 7.1. We note that for $p \ne 2$, the constant of the L^p-Rellich-type inequalities (7.2) and (7.3) is optimal at least in the classical case, where $\Omega = \mathbb{R}^n \setminus \{0\}$, $P = -\Delta$, $v_0 = |x|^{2-n}$ and $v_1 = \mathbf{1}$. The optimality of the constant in this case follows from the remark in [7, page 521].

8. The supersolution construction for Q_V

In the present section we study the supersolution construction for operators Q_V of the form (1.3) under the assumption that (roughly speaking) the supersolutions v_j have the same level sets. In Appendix A we present a proof of the particular case of radially symmetric potentials.

The following result generalizes Lemma 4.1 for $p \ne 2$.

Theorem 8.1. *Let v_j, $j = 0, 1$, be two positive, linearly independent, C^2-(super)solutions of the equation $Q_{V_j}(u) = 0$ in Ω. Assume that ∇v_0 does not vanish in Ω, and that $v_1 = \varphi_1(v_0)$ for some C^2-function φ_1 such that $\varphi_1'(u) \ne 0$. For $0 \le \alpha \le 1$, define the function*

$$v_\alpha := v_1^\alpha v_0^{1-\alpha},$$

and let

$$V_\alpha := \big((1-\alpha)V_0|\nabla \log v_0|^{2-p} + \alpha V_1|\nabla \log v_1|^{2-p}\big)|\nabla \log v_\alpha|^{p-2},$$

$$W_\alpha := \alpha(1-\alpha)(p-1)\left|\nabla \log\left(\frac{v_0}{v_1}\right)\right|^2 |\nabla \log v_\alpha|^{p-2}.$$

Then v_α is a positive (super)solution of the equation

$$Q_{V_\alpha - W_\alpha}(u) = 0 \quad \text{in } \Omega, \tag{8.1}$$

and the following improved inequality holds

$$\mathcal{Q}_{V_\alpha}(\varphi) \ge \int_\Omega W_\alpha |\varphi|^p\,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

Remark 8.2. If both v_0 and v_1 do not admit critical points, then the condition $v_1 = \varphi_1(v_0)$ is equivalent to the fact that ∇v_0 and ∇v_1 are collinear at every point, and also to the fact that the level sets of v_0 and v_1 coincide, that is, for every $t_0 > 0$, there is $t_1 > 0$ such that

$$\big\{x \in \Omega \mid v_0(x) = t_0\big\} = \big\{x \in \Omega \mid v_1(x) = t_1\big\},$$

and vice versa. A particular case appears when v_j are radially symmetric positive supersolutions (see Appendix A).

Proof. Fix $x \in \Omega$ and set $u := v_0(x)$. By Lemma 2.10 we have

$$\begin{aligned} Q_{V_1}(v_1) &= -\Delta_p\big(\varphi_1(v_0)\big) + V_1\big(\varphi_1(v_0)\big)^{p-1} \\ &= \big|\varphi_1'(u)\big|^{p-2}|\nabla v_0|^p\left(-(p-1)\varphi_1''(u) - \frac{\Delta_p(v_0)}{|\nabla v_0|^p}\varphi_1'(u) + V_1\frac{|(\log\varphi_1)'(u)|^{2-p}}{|\nabla v_0|^p}\varphi_1(u)\right) \end{aligned} \tag{8.2}$$

in the weak sense. On the other hand, with the identity map $\varphi_0(t) := t$ on $\mathbb{R}_+$ we have at x

$$\begin{aligned} Q_{V_0}(v_0) &= -\Delta_p\big(\varphi_0(v_0)\big) + V_0\big(\varphi_0(v_0)\big)^{p-1} \\ &= \big|\varphi_0'(u)\big|^{p-2}|\nabla v_0|^p\left(-(p-1)\varphi_0''(u) - \frac{\Delta_p(v_0)}{|\nabla v_0|^p}\varphi_0'(u) + V_0\frac{|(\log\varphi_0)'(u)|^{2-p}}{|\nabla v_0|^p}\varphi_0(u)\right). \end{aligned} \tag{8.3}$$

Therefore, for $j = 0, 1$, $\varphi_j(u)$ satisfies at the point u the following linear ordinary differential inequality

$$-(p-1)\varphi_j''(u) - \frac{\Delta_p(v_0)}{|\nabla v_0|^p}\varphi_j'(u) + V_j\frac{|(\log\varphi_j)'(u)|^{2-p}}{|\nabla v_0|^p}\varphi_j(u) \underset{(\geq)}{=} 0.$$

Denote $\varphi_\alpha(u) := \varphi_0(u)^{1-\alpha}\varphi_1(u)^\alpha$, and apply the one-dimensional version of Corollary 4.2. We obtain the following linear differential inequality at u

$$\begin{aligned} &-(p-1)\varphi_\alpha''(u) - \frac{\Delta_p(v_0)}{|\nabla v_0|^p}\varphi_\alpha'(u) + (1-\alpha)V_0\frac{|(\log\varphi_0)'(u)|^{2-p}}{|\nabla v_0|^p}\varphi_\alpha(u) \\ &\quad + \alpha V_1\frac{|(\log\varphi_1)'(u)|^{2-p}}{|\nabla v_0|^p}\varphi_\alpha(u) - (p-1)\alpha(1-\alpha)\left|\left[\log\left(\frac{\varphi_0(u)}{\varphi_1(u)}\right)\right]'\right|^2\varphi_\alpha(u) \underset{(\geq)}{=} 0. \end{aligned} \tag{8.4}$$

In view of Lemma 2.10 we have

$$-\Delta_p(\varphi_\alpha) = \big|\varphi_\alpha'\big|^{p-2}|\nabla v_0|^p\left(-(p-1)\varphi_\alpha'' - \frac{\Delta_p(v_0)}{|\nabla v_0|^p}\varphi_\alpha'\right).$$

On the other hand,

$$\begin{aligned} &\big|(\log\varphi_j)'\big|^{2-p}\big|(\log\varphi_\alpha)'\big|^{p-2} = |\nabla\log v_j|^{2-p}|\nabla\log v_\alpha|^{p-2} \quad j = 0, 1, \\ &\left|\left[\log\left(\frac{\varphi_0}{\varphi_1}\right)\right]'\right|^2\big|(\log\varphi_\alpha)'\big|^{p-2}|\nabla v_0|^p = \left|\nabla\log\left(\frac{v_0}{v_1}\right)\right|^2|\nabla\log v_\alpha|^{p-2}. \end{aligned}$$

Hence, (8.4) implies the result of the theorem. □

Remark 8.3. In particular, let $V = 0$ and $v_0 = G$ be the p-Laplacian's Green function with a pole at $0 \in \Omega$, and $v_1 = \mathbf{1}$. Then $V_\alpha = 0$, and a computation shows that $W_\alpha = (p-1)\alpha^{p-1}(1-\alpha)|\frac{\nabla G}{G}|^p$ (cf. Proposition 4.5).

Corollary 8.4. *Assume that $p > n$, $V = 0$, and $-\Delta_p$ is subcritical in Ω. Let G be (up to a constant) the p-Green function with a pole at $0 \in \Omega$. Suppose that*

$$\lim_{x\to 0} G(x) = \gamma > 0 \quad \text{and} \quad \lim_{x\to\bar\infty} G(x) = 0.$$

For $0 \leq \alpha \leq 1$, let

$$v_\alpha := G^{1-\alpha}(\gamma - G)^\alpha, \qquad W_\alpha := \alpha(1-\alpha)(p-1)\big|\gamma(1-\alpha) - G\big|^{p-2}\left|\frac{\nabla G}{G(\gamma - G)}\right|^p. \tag{8.5}$$

Then the following improved Hardy inequality holds in $\Omega^\star$:

$$\int_{\Omega^\star}|\nabla\varphi|^p\,\mathrm{d}\nu \geq \int_{\Omega^\star}W_\alpha|\varphi|^p\,\mathrm{d}\nu \quad \forall\varphi \in C_0^\infty\big(\Omega^\star\big). \tag{8.6}$$

Moreover, for any $0 \leq \alpha \leq 1$ the operator Q_{-W_α} is subcritical in Ω.

Proof. By our assumption, $\gamma - G$ is a positive p-harmonic function in $\Omega^\star$. Apply Theorem 8.1 with $v_0 = G$ and $v_1 = \gamma - G$ to obtain (8.6).

Assume to the contrary that Q_{-W_α} is critical in $\Omega^\star$. Two cases should be considered: either $\alpha < (p-1)/p$, or $1 - \alpha < (p-1)/p$.

Let us assume for example $\alpha < (p-1)/p$, the other case being similar (exchanging the roles of zero and infinity). Then $v_{\frac{p-1}{p}}$ is a positive supersolution of Q_{-W_α} in a neighborhood of zero, and v_α is a positive solution of Q_{-W_α} of minimal growth in $\Omega^\star$. Therefore, there exists $C > 0$ such that $v_\alpha \le C v_{\frac{p-1}{p}}$ in a neighborhood of zero. But since $\alpha < (p-1)/p$, this is impossible, and we get a contradiction. □

Remark 8.5. A priori it is clear that for W_α (given by (8.5)) to be optimal at the origin, it is needed that $\alpha = (p-1)/p$, but for the constant to be optimal at $\bar{\infty}$, we must choose $\alpha = 1/p$, and thus v_α cannot be a ground state (if $p \ne 2$). Thus, in the nontrivial cases ($v_j \ne$ constant), the supersolution construction of the form $v_\alpha = v_1^\alpha v_0^{1-\alpha}$, does not provide us with an optimal Hardy weight. On the other hand, let $\psi(G) := [G(\gamma - G)]^{(p-1)/p}$ and

$$\begin{aligned} W &:= \frac{-\Delta_p(\psi(G))}{\psi(G)^{p-1}} \\ &= \left(\frac{p-1}{p}\right)^p \left|\frac{\nabla G}{G(\gamma - G)}\right|^p |\gamma - 2G|^{p-2}\big[2(p-2)G(\gamma - G) + \gamma^2\big] \ge 0. \end{aligned} \tag{8.7}$$

Then under the conditions of Theorem 1.5, W is an optimal Hardy-weight for $-\Delta_p$, and $\psi(G)$ is the ground state of the critical operator Q_{-W} in $\Omega^\star$. Note that nevertheless, $W = 0$ on the set $\{x \in \Omega^\star \mid G(x) = \gamma/2\}$.

It turns out that if V_j both have the same definite sign, then one can find potentials $\mathcal{V}_\alpha \ge V_\alpha$ (with the same definite sign) which does not depend on v_j, such that the corresponding Hardy inequality is satisfied with the same Hardy-weight W_α. We have

Corollary 8.6. *Let Ω, V_j, v_j (where $j = 0, 1$), v_α, and W_α be as in Theorem 8.1 (or as in Theorem A.1). Suppose further that $V_j \ge 0$ if $1 < p \le 2$ (resp., $V_j \le 0$ if $p \ge 2$), where $j = 0, 1$. Define*

$$\mathcal{V}_\alpha := \pm\big((1-\alpha)|V_0|^{1/(p-1)} + \alpha|V_1|^{1/(p-1)}\big)^{p-1},$$

where one should take the minus sign if $V_j \le 0$. Then v_α is a positive supersolution of the equation

$$Q_{\mathcal{V}_\alpha - W_\alpha}(u) = 0 \quad \text{in } \Omega, \tag{8.8}$$

and the following improved inequality holds

$$\mathcal{Q}_{\mathcal{V}_\alpha}(\varphi) \ge \int_\Omega W_\alpha |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

Moreover, if $p \ne 2$, and $|V_0| + |V_1| \ne 0$, then the functional $\mathcal{Q}_{\mathcal{V}_\alpha - W_\alpha}$ is subcritical *in Ω.*

Proof. Assume that the conditions of Theorem 8.1 are satisfied. Then v_α is a positive (super)solution of the equation $Q_{V_\alpha - W_\alpha}(u) = 0$.

We claim that the function $(\xi, \eta) \mapsto f(\xi, \eta) := \xi^{p-1}\eta^{2-p}$ on $\mathbb{R}_+^2$ is convex (resp., concave) if $p \ge 2$ (resp., $p \le 2$). Indeed,

$$\mathrm{Hess}(f) = (p-1)(p-2)\xi^{p-1}\eta^{2-p}\begin{bmatrix} \frac{1}{\xi^2} & -\frac{1}{\xi\eta} \\ -\frac{1}{\xi\eta} & \frac{1}{\eta^2} \end{bmatrix},$$

and it can be easily checked that $\mathrm{Hess}(f)$ is nonnegative (resp., nonpositive) on $\mathbb{R}_+^2$ if and only if $(p-1)(p-2) \ge 0$ (resp., $(p-1)(p-2) \le 0$). Hence,

$$\begin{aligned} &\big[(1-\alpha)|V_0|^{\frac{p-1}{p-1}}|\nabla \log v_0|^{2-p} + \alpha|V_1|^{\frac{p-1}{p-1}}|\nabla \log v_1|^{2-p}\big] \\ &\underset{(\text{respect. } \le)}{\ge} \big((1-\alpha)|V_0|^{1/(p-1)} + \alpha|V_1|^{1/(p-1)}\big)^{p-1}|(1-\alpha)\nabla \log v_0 + \alpha \nabla \log v_1|^{2-p}. \end{aligned}$$

So, $\mathcal{V}_\alpha \ge V_\alpha$, and hence v_α is a positive supersolution of the equation

$$Q_{\mathcal{V}_\alpha - W_\alpha}(u) = 0 \quad \text{in } \Omega,$$

and we have

$$\mathcal{Q}_{\mathcal{V}_\alpha}(\varphi) \geq \int_\Omega W_\alpha |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

If and $|V_0| + |V_1| \neq 0$, and $p \neq 2$, then the strict convexity (resp., concavity) of f implies that v_α is a positive supersolution of $Q_{\mathcal{V}_\alpha - W_\alpha}(u) = 0$ which is not a solution, and therefore by Lemma 2.6, the corresponding improved functional $\mathcal{Q}_{\mathcal{V}_\alpha - W_\alpha}$ is subcritical in Ω. □

Remark 8.7. 1. Suppose that $V_0 = V_1 \neq 0$ and V_0 has a definite sign, then $\mathcal{V}_\alpha = V_0$. By Corollary 8.6, the operator $Q_{V_0 - W_\alpha}$ is *subcritical* in Ω if $p \neq 2$. This is in contrast with the linear case where $p = 2$. Indeed, if v_0 is the Green function of the operator $Pu := -\operatorname{div}(A(x)\nabla\cdot) + V(x)$ in Ω with a pole 0, and if v_1 is a positive solution satisfying $\lim_{x\to\bar\infty} \frac{v_0(x)}{v_1(x)} = 0$, then $P - W_{1/2} = P - \frac{1}{4}|\nabla \log(\frac{v_0}{v_1})|^2$ is critical in $\Omega^\star$ (see [6, Theorem 2.2]).

2. In general, it is not clear how to optimize in α the potentials W_α in the case $V_0 = V_1 \neq 0$, and V_0 has a definite sign (so, $\mathcal{V}_\alpha = V_0$). But if we take $v_0 = \mathbf{1}$ (so, $V \geq 0$ and $1 < p \leq 2$), and $v_1 = v$ is a positive supersolution of the equation $Q_{V_0}(u) = 0$, then

$$W_\alpha = \alpha^{p-1}(1-\alpha)(p-1)\left|\frac{\nabla v}{v}\right|^p,$$

and by optimizing α one obtains

$$\mathcal{Q}_V(\varphi) \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \left(\frac{|\nabla v|}{v}\right)^p |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega), \tag{8.9}$$

which in particular reproves (2.12) in [1] if A is the identity matrix.

Conflict of interest statement

No conflict of interest.

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Appendix A. Radially symmetric potentials

In this Appendix we present a proof of a particular case of Theorem 8.1, where the two positive supersolutions are radially symmetric functions, and in particular, have the same level sets.

Theorem A.1. *Assume that for* $j = 0, 1$

$$\mathcal{Q}_{V_j}(\varphi) := \int_\Omega \left(|\nabla\varphi|^p + V_j|\varphi|^p\right) \mathrm{d}\nu \geq 0 \quad \varphi \in C_0^\infty(\Omega), \tag{A.1}$$

where Ω *is a domain in* $\mathbb{R}^n$ *not containing the origin, and the potentials* V_j *are two radially symmetric potentials. Let* v_j, $j = 0, 1$, *be two positive, linearly independent, radially symmetric,* C^2*-supersolutions of the equation* $Q_{V_j}(u) = 0$ *in* Ω. *For* $0 \leq \alpha \leq 1$, *define the function*

$$v_\alpha(r) := \left(v_1(r)\right)^\alpha \left(v_0(r)\right)^{1-\alpha},$$

where $r := |x|$. *Assume further that* $(v_0)'$, $(v_1)'$, *and* $(v_\alpha)'$ *do not vanish, and let*

$$V_\alpha(r) := \big((1-\alpha)V_0(r)\big|\big(\log v_0(r)\big)'\big|^{2-p} + \alpha V_1(r)\big|\big(\log v_1(r)\big)'\big|^{2-p}\big)\big|\big(\log v_\alpha(r)\big)'\big|^{p-2},$$

$$W_\alpha(r) := \alpha(1-\alpha)(p-1)\left|\left[\log\left(\frac{v_0(r)}{v_1(r)}\right)\right]'\right|^2 \big|\big(\log v_\alpha(r)\big)'\big|^{p-2}.$$

Then v_α is a positive supersolution of the equation

$$Q_{V_\alpha(|x|)-W_\alpha(|x|)}(u) = 0 \quad \text{in } \Omega, \tag{A.2}$$

and the following improved inequality holds

$$\mathcal{Q}_{V_\alpha}(\varphi) \geq \int_\Omega W_\alpha |\varphi|^p \,\mathrm{d}\nu \quad \forall \varphi \in C_0^\infty(\Omega).$$

Proof. Assume that v is a radially symmetric C^2-function, and denote $r := |x|$, $v' := \mathrm{d}v/\mathrm{d}r$. Then by Lemma 2.10 the p-Laplacian of v satisfies

$$-\Delta_p(v) = -\frac{1}{r^{n-1}}\big(r^{n-1}|v'|^{p-2}v'\big)' = -|v'|^{p-2}\left[(p-1)v'' + \frac{n-1}{r}v'\right] \tag{A.3}$$

in the weak sense. Denote the linear operator

$$Pu := -(p-1)u'' - \frac{n-1}{r}u'.$$

By our assumptions, v_j are positive radial (super)solutions of the equation $Q_{V_j}(u) = 0$ in Ω, where $j = 0, 1$. Hence,

$$Pv_j + \big(V_j\big|(\log v_j)'\big|^{2-p}\big)v_j \underset{(\geq)}{=} 0 \quad j = 0, 1.$$

Therefore, by Lemma 4.1, v_α is a positive (super)solution of the linear equation

$$\Bigg[P + (1-\alpha)V_0\big|(\log v_0)'\big|^{2-p} + \alpha V_1\big|(\log v_1)'\big|^{2-p}$$
$$- (p-1)\alpha(1-\alpha)\left|\left[\log\left(\frac{v_0}{v_1}\right)\right]'\right|^2\Bigg]u \underset{(\geq)}{=} 0. \tag{A.4}$$

Hence, v_α satisfies the quasilinear differential (in)equality

$$-\Delta_p(v_\alpha) + \big((1-\alpha)V_0\big|(\log v_0)'\big|^{2-p} + \alpha V_1\big|(\log v_1)'\big|^{2-p}\big)\big|(\log v_\alpha)'\big|^{p-2}v_\alpha^{p-1}$$
$$- (p-1)\alpha(1-\alpha)\left|\left[\log\left(\frac{v_0}{v_1}\right)\right]'\right|^2 \big|(\log v_\alpha)'\big|^{p-2}v_\alpha^{p-1} = Q_{V_\alpha - W_\alpha}(v_\alpha) \underset{(\geq)}{=} 0. \quad \square \tag{A.5}$$

Remark A.2. If $0 \in \Omega$, Ω is a radially symmetric domain, $V_0 = V_1$ is a radially symmetric potential, and Q_{V_0} is subcritical in Ω, then one can apply Theorem A.1 in $\Omega^\star = \Omega \setminus \{0\}$ with v_0 equals to the corresponding (unique) p-Green function of Q_{V_0} with a pole at the origin, and v_1 a global radial positive supersolution of the equation $Q_{V_0}(u) = 0$ in Ω.

References

[1] Adimurthi, A. Sekar, Role of the fundamental solution in Hardy–Sobolev-type inequalities, Proc. R. Soc. Edinb. A 136 (2006) 1111–1130.

[2] W. Allegretto, Y.X. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. 32 (1998) 819–830.

[3] B. Bojarski, P. Hajłasz, P. Strzelecki, Sard's theorem for mappings in Hölder and Sobolev spaces, Manuscr. Math. 118 (2005) 383–397.

[4] G.-Q. Chen, M. Torres, W.P. Ziemer, Gauss–Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws, Commun. Pure Appl. Math. 62 (2009) 242–304.

[5] L. D'Ambrosio, S. Dipierro, Hardy inequalities on Riemannian manifolds and applications, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 31 (2014) 449–475.

[6] B. Devyver, M. Fraas, Y. Pinchover, Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon, J. Funct. Anal. 266 (2014) 4422–4489.

[7] E.B. Davies, A.M. Hinz, Explicit constants for Rellich inequalities in $L^p(\Omega)$, Math. Z. 227 (1998) 511–523.
[8] M. Fraas, Y. Pinchover, Isolated singularities of positive solutions of p-Laplacian type equations in $\mathbb{R}^d$, J. Differ. Equ. 254 (2013) 1097–1119.
[9] J. García-Melián, J. Sabina de Lis, Maximum and comparison principles for operators involving the p-Laplacian, J. Math. Anal. Appl. 218 (1998) 49–65.
[10] O.A. Ladyženskaja, N.N. Ural'ceva, A variational problem for quasilinear elliptic equations with many independent variables, Sov. Math. Dokl. 1 (1960) 1390–1394.
[11] V. Maz'ya, Sobolev Spaces with Applications to Elliptic Partial Differential Equations, second, revised and augmented edition, Grundlehren der Mathematischen Wissenschaften, vol. 342, Springer, Heidelberg, 2011.
[12] Y. Pinchover, On criticality and ground states of second-order elliptic equations II, J. Differ. Equ. 87 (1990) 353–364.
[13] Y. Pinchover, K. Tintarev, Ground state alternative for p-Laplacian with potential term, Calc. Var. Partial Differ. Equ. 28 (2007) 179–201.
[14] Y. Pinchover, K. Tintarev, On the Hardy–Sobolev–Maz'ya inequality and its generalizations, in: V. Maz'ya (Ed.), Sobolev Spaces in Mathematics I: Sobolev Type Inequalities, in: International Mathematical Series, vol. 8, Springer, 2009, pp. 281–297.
[15] Y. Pinchover, A. Tertikas, K. Tintarev, A Liouville-type theorem for the p-Laplacian with potential term, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 25 (2008) 357–368.
[16] A. Poliakovsky, I. Shafrir, Uniqueness of positive solutions for singular problems involving the p-Laplacian, Proc. Am. Math. Soc. 133 (2005) 2549–2557.
[17] P. Pucci, J. Serrin, The Maximum Principle, Progress in Nonlinear Differential Equations and Their Applications, vol. 73, Birkhäuser Verlag, Basel, 2007.
[18] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.
[19] J. Serrin, Isolated singularities of solutions of quasi-linear equations, Acta Math. 113 (1965) 219–240.
[20] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differ. Equ. 51 (1984) 126–150.