Stochastically symplectic maps and their applications to the Navier–Stokes equation

Fraydoun Rezakhanlou [1]

UC Berkeley, Department of Mathematics, Berkeley, CA 94720-3840, United States

Abstract

Poincaré's invariance principle for Hamiltonian flows implies Kelvin's principle for solution to Incompressible Euler equation. Constantin–Iyer Circulation Theorem offers a stochastic analog of Kelvin's principle for Navier–Stokes equation. Weakly symplectic diffusions are defined to produce stochastically symplectic flows in a systematic way. With the aid of symplectic diffusions, we produce a family of martigales associated with solutions to Navier–Stokes equation that in turn can be used to prove Constantin–Iyer Circulation Theorem. We also review some basic facts in symplectic and contact geometry and their applications to Euler equation.
© 2014 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

1. Introduction

The question of global regularity for the incompressible fluid equations is a major and challenging open problem in PDE theory. In the case of an inviscid fluid, the fluid evolution is modeled by the Euler equation and has a geometric interpretation on account of *Kelvin Circulation Theorem* or *Cauchy Vorticity Formula*. In the language of differential geometry, the differential 2-form associated with the vorticity of the fluid remains the same in a coordinate system that is carried along the fluid. This means that those geometric properties of the vorticity that are preserved under a change of coordinates, are preserved with time. When this differential form is nondegenerate, namely symplectic in even dimensions and contact in odd dimensions, we may appeal to Symplectic/Contact geometry as a guideline for exploring those geometric properties that may provide us with useful information about the vorticity.

The *Incompressible Navier–Stokes equation* is the corresponding PDE for a viscous fluid. In Lagrangian coordinates a viscous fluid is described by a stochastic differential equation. This description allowed Constantin and Iyer [3] to discover a stochastic Kelvin Theorem for fluid circulation that in essence gives a more geometric insight into the dynamics of the viscous fluid. Once the initial vorticity, regarded as a differential 2-form, is evolved by the underlying

stochastic flow, we have a random path of geometrically equivalent differential forms. Averaging out the randomness yields a time-dependent family of differential forms that satisfy the vorticity equation associated with Navier–Stokes equation. Again under a nondegeneracy assumption of the initial velocity, we may try to understand the effect of averaging of those geometric properties that are preserved under the aforementioned stochastic flow.

The primary goal of this article is to explore the connection between Symplectic/Contact Geometry, Stochastic Calculus and Fluid Mechanics. To focus on the main ideas and avoid various nontrivial technical issues, we will be considering only classical solutions of the incompressible fluid equations in the present paper. Nonclassical solutions will be addressed in a subsequent paper [15].

Hamiltonian systems appear in conservative problems of mechanics governing the motion of particles in fluid. Such a mechanical system is modeled by a Hamiltonian function $H(x, t)$ where $x = (q, p) \in \mathbb{R}^d \times \mathbb{R}^d$, $q = (q_1, \ldots, q_d)$, $p = (p_1, \ldots, p_d)$ denote the positions and the momenta of particles. The Hamiltonian equations of motion are

$$\dot{q} = H_p(q, p, t), \qquad \dot{p} = -H_q(q, p, t) \tag{1.1}$$

which is of the form

$$\dot{x} = J \nabla_x H(x, t), \qquad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \tag{1.2}$$

where I_d denotes the $d \times d$ identity matrix. It was known to Poincaré that if ϕ_t is the flow of the ODE (1.2) and γ is a closed curve, then

$$\frac{d}{dt} \int_{\phi_t(\gamma)} \bar{\lambda} = 0, \tag{1.3}$$

where $\bar{\lambda} := p \cdot dq$. We may use Stokes' theorem to rewrite (1.3) as

$$\frac{d}{dt} \int_{\phi_t(\Gamma)} d\bar{\lambda} = 0 \tag{1.4}$$

for every two-dimensional surface Γ. In words, the 2-form

$$\bar{\omega} := \sum_{i=1}^{d} dp_i \wedge dq_i,$$

is invariant under the Hamiltonian flow ϕ_t. Equivalently,

$$\phi_t^* \bar{\omega} = \bar{\omega}. \tag{1.5}$$

A Hamiltonian system (1.2) simplifies if we can find a function $u(q, t)$ such that $p(t) = u(q(t), t)$. If such a function u exists, then $q(t)$ solves

$$\frac{dq}{dt} = H_p(q, u(q, t), t). \tag{1.6}$$

The equation for the time evolution of p gives us an equation for the evolution of the velocity function u; since

$$\dot{p} = (Du)\dot{q} + u_t = (Du)H_p(q, u, t) + u_t,$$
$$\dot{p} = -H_q(q, u, t),$$

the function $u(q, t)$ must solve,

$$u_t + (Du)H_p(q, u, t) + H_q(q, u, t) = 0. \tag{1.7}$$

For example, if $H(q, p, t) = \frac{1}{2}|p|^2 + P(q, t)$, then (1.7) becomes

$$u_t + (Du)u + \nabla P(q, t) = 0, \tag{1.8}$$

and Eq. (1.6) simplifies to

$$\frac{dq}{dt} = u(q, t). \tag{1.9}$$

(Here and below we write Du and ∇P for the q-derivatives of the vector field u and the scalar-valued function P respectively.) Eq. (1.8), coupled with the incompressibility requirement $\nabla \cdot u = 0$ is the Euler equation of fluid mechanics. If the flow of (1.9) is denoted by Q_t, then $\phi_t(q, u(q, 0)) = (Q_t(q), u(Q_t(q), t))$. Now (1.3) means that for any closed q-curve η,

$$\frac{d}{dt} \int_{Q_t(\eta)} u(q, t) \cdot dq = \frac{d}{dt} \int_\eta (DQ_t)^* u \circ Q_t(q, t) \cdot dq = 0, \tag{1.10}$$

or equivalently

$$d\left(Q_t^* \alpha_t\right) = d\alpha_0, \tag{1.11}$$

where $\alpha_t = u(q, t) \cdot dq$. This is the celebrated *Kelvin's Circulation Theorem*. In summary Poincaré's invariance principle (1.3) implies Kelvin's principle for Euler equation. (Note that the incompressibility condition $\nabla \cdot u = 0$ is not needed for (1.10).)

Since the pullback operation $\alpha \to Q^* \alpha$ commutes with the exterior differentiation (a straight forward consequence of Stokes' formula), we may rewrite (1.11) as

$$Q_t^*(d\alpha_t) = d\alpha_0, \tag{1.12}$$

and this is equivalent to Euler equation (Eq. (1.8) with the incompressibility condition $\nabla \cdot u = 0$). Moreover, when $d = 3$, (1.12) can be written as

$$\xi^t \circ Q_t = (DQ_t)\xi^0, \quad \text{or} \quad \xi^t = \left((DQ_t)\xi^0\right) \circ Q_t^{-1}, \tag{1.13}$$

where $\xi^t(\cdot) = \nabla \times u(\cdot, t)$. Eq. (1.13) is known as *Cauchy vorticity formulation* of Euler equation and is equivalent to the vorticity equation by differentiating both sides with respect to t:

$$\xi_t + (D\xi)u = (Du)\xi. \tag{1.14}$$

Geometrically speaking, (1.12) says that the forms $d\alpha_t$ and $d\alpha_0$ are isomorphic i.e. the former can be obtained from the latter by a change of coordinates. To explore the geometry of $d\alpha_t$, we assume that $d\alpha_0$ is nondegenerate and this property persists at later times by (1.12). However this nondegeneracy can be realized only when the spacial dimension d is even; such nondegenerate (closed) forms are called *symplectic forms*. When the dimension is odd we will require a different nondegeneracy assumption (but this time on the form α_t) that would allow us to explore some highly nontrivial geometric properties that are extensively studied in Contact Geometry. To rephrase what we stated earlier under nondegeneracy assumption, (1.12) implies that the forms $(d\alpha_t : t \geq 0)$ share the same properties from symplectic/contact geometrical point of view. How useful such an observation is, remains to be seen. As a first attempt we state some fundamental results in Symplectic Geometry in Section 2 and discuss some of their consequences for solutions of Incompressible Euler equation (see Propositions 2.1–2.2 and Remark 2.3).

Constantin and Iyer [3] discovered a circulation invariance principle for Navier–Stokes equation that is formulated in terms of a diffusion associated with the velocity field. Given a solution u to the Navier–Stokes equation

$$u_t + (Du)u + \nabla P(q, t) = v\Delta u, \quad \nabla \cdot u = 0, \tag{1.15}$$

let us write Q_t for the (stochastic) flow of the SDE

$$dq = u(q, t)dt + \sqrt{2v}dW, \tag{1.16}$$

with W denoting the standard Brownian motion. If we write $A = Q^{-1}$ and $\xi^t = \nabla \times u(\cdot, t)$, and assume that $d = 3$, then Constantin and Iyer's circulation formula reads as

$$\xi^t = \mathbb{E}\left((DQ_t)\xi^0\right) \circ A_t, \tag{1.17}$$

where $\mathbb{E}$ denotes the expected value.

We are now ready to state one of the main results of this article.

Theorem 1.1. *Write $\alpha_t = u(q, t) \cdot dq$ with u a classical solution of (1.15) and set $B_t = Q_{T-t} \circ Q_T^{-1}$ for $t \in [0, T]$.*

(i) *Then the process $\beta_t = B_t^* d\alpha_{T-t}$, $t \in [0, T]$ is a 2-form-valued martingale. When $d = 3$, this is equivalent to saying that the process*

$$M_t = \left((DB_t^{-1}) \xi^{T-t} \right) \circ B_t, \quad t \in [0, T],$$

is a martingale.

(ii) *The process*

$$\|\beta_t\|^2 - 2\nu \int_0^t \sum_{i=1}^d \left\| B_s^* \zeta_i^{T-s} \right\|^2 ds,$$

is a martingale, where

$$\zeta_i^\theta = \sum_{j,k=1}^d u_{q_i q_j}^k(\cdot, \theta) dq_j \wedge dq_k,$$

or equivalently, $\zeta_i^\theta(v_1, v_2) = \mathcal{C}(u_{q_i}(\cdot, \theta)) v_1 \cdot v_2$, with $\mathcal{C}(w) = Dw - (Dw)^*$. (The norm of a 2-form $\eta(v_1, v_2) = Cv_1 \cdot v_2$, $C^* = -C$, $C = [c_{ij}]$, is defined by $\|\eta\|^2 = \sum_{i,j} c_{ij}^2$.)

(iii) *We have the equality*

$$\|d\alpha_T\|^2 + 2\nu \mathbb{E} \int_0^T \sum_{i=1}^d \left\| B_s^* \zeta_i^{T-s} \right\|^2 ds = \mathbb{E} \left\| B_T^* d\alpha_0 \right\|^2.$$

Remark 1.1.

(i) In a subsequent paper [15], we will show how Theorem 1.1 can be extended to certain weak solutions. To make sense of martingales β_t and M_t, we need to make sure that DQ_t exists weakly and belongs to suitable L^ℓ spaces. As it turns out, a natural condition to guarantee $DQ_t \in L^\ell$ for all $\ell \in [1, \infty)$ is

$$\int_0^T \left[\int_{\mathbb{R}^d} |u(q, t)|^r dx \right]^{r'/r} dt < \infty,$$

for some $r, r' \geq 1$ such that $d/r + 2/r' < 1$.

(ii) Our result takes a simpler form if u is a solution to the backward Navier–Stokes equation. In other words if u is a divergence-free vector field such that

$$u_t + (Du)u + \nabla P(q, t) + \nu \Delta u = 0, \quad t < T,$$

then the process $(\beta_t = Q_t^* d\alpha_t : t \leq T)$ is a martingale. When $d = 3$, we deduce that $M_t = ((DA_t)\xi^t) \circ Q_t$ is a martingale. We note that a backward Navier–Stokes equation must be solved for a given final condition $u(\cdot, T) = u^0(\cdot)$.

(iii) In some sense, Theorem 1.1(i) is compatible with a conjecture that the circulation is preserved only in some statistical sense for a singular solution of Euler equation. We refer to [6] and [7] for some heuristic justification of this conjecture. We note that if for a surface Γ, the martingale $m_t^\nu = \int_\Gamma \beta_t$ has a limit m_t^0 in low ν limit, then m_t^0 remains a martingale. In other words, even if the circulation is not conserved for a singular solution of the Euler equation, it is still a martingale and conserved in a suitable averaged sense. We refer to Appendix A.2 for more details.

(iv) As an immediate consequence of Theorem 1.1(iii) we learn

$$\|d\alpha_T\|^2, \qquad 2\nu\mathbb{E}\int_0^T \sum_{i=1}^d \|B_s^*\zeta_i^{T-s}\|^2 ds \le \mathbb{E}\|A_T^* d\alpha_0\|^2, \tag{1.18}$$

where $A_T = Q_T^{-1}$. In particular, a bound on $\mathbb{E}\|A_T\|^2$ would yield a bound on the vorticity at time T provided that the initial vorticity is uniformly bounded.

(v) As a preparation for the proof of Theorem 1.1, we define *symplectic and semisymplectic diffusions* and give practical criteria for checking whether or not a diffusion is (semi)symplectic. These criteria appear in Propositions 3.2–3.4 and Corollary 3.1. □

The organization of the paper is as follows:

- In Section 2 we discuss Weber's formulation of Euler equation and show how (1.12) implies (1.13). We also discuss two fundamental results in Symplectic Geometry that are related to the so-called *Clebsch* variables.
- In Section 3 we address some geometric questions for stochastic flows of general diffusions and study *symplectic diffusions*.
- In Section 4 we use symplectic diffusions to establish Theorem 1.1.
- In Appendix A.1 we discuss *contact* diffusions.
- In Appendix A.2 we discuss the invariance of the martingale property in a vanishing viscosity limit.

2. The Euler equations

In this section we review some basic facts in differential geometry and their applications to Euler equation. Even though most of the discussion of this section is either well-known or part of folklore, a reader may find our discussion useful as we use similar ideas to prove Theorem 1.1. We also use this section as an excuse to demonstrate/advertise the potential use of symplectic/contact geometric ideas in fluid mechanics.

We start with giving an elementary proof of (1.4): By Cartan's formula

$$\frac{d}{dt}\phi_t^*\bar\lambda = \phi_t^*\mathcal{L}_{Z_H}\bar\lambda = \phi_t^* dK = d(K \circ \phi_t), \tag{2.1}$$

where $\mathcal{L}_Z$ denotes the Lie derivative with respect to the vector field Z, $Z_H = J\nabla_x H$ for $H(q,p,t) = |p|^2/2 + P(q,t)$, and

$$K(q,p,t) = p \cdot H_p(q,p,t) - H(q,p,t) = \frac{1}{2}|p|^2 - P(q,t). \tag{2.2}$$

If we integrate both sides of (2.1) over an arbitrary (non-closed) curve of the form $(\eta, u(\eta, t))$, or equivalently restrict the form $\bar\lambda$ to the graph of the function u, then we obtain

$$\frac{d}{dt}\big[(DQ_t)^* u \circ Q_t\big] = \nabla(L \circ Q_t), \tag{2.3}$$

where $L(q,t) = K(q, u(q,t), t) = |u(q,t)|^2/2 - P(q,t)$. Here by A^* we mean the transpose of the matrix A. Recall $A_t = Q_t^{-1}$, so that

$$(DQ_t)^{-1} = DA_t \circ Q_t.$$

As a consequence of (2.3),

$$u(\cdot, t) = (DA_t)^* u^0 \circ A_t + \nabla(R \circ A_t),$$

for $R = \int_0^t L \circ Q_s ds$. As a result,

$$u(\cdot, t) = \mathcal{P}\big[(DA_t)^* u^0 \circ A_t\big], \tag{2.4}$$

where u^0 is the initial data and $\mathcal{P}$ denotes the Leray–Hodge projection onto the space of divergence-free vector fields. Formula (2.4) is *Weber's formulation* and is equivalent to Euler's equation.

So far we have shown that the Kelvin principle (1.3) is equivalent to the Weber formulation of Euler equation. If we use (1.5) instead, we obtain a different equivalent formulation of Euler equation, namely the vorticity equation (1.12) or (1.13). Recall

$$\bar{\omega}(v_1, v_2) = J v_1 \cdot v_2.$$

If we choose v_1 and v_2 to be tangent to the graph of u, i.e. $v_i = (w_i, Du(q, t)w_i)$ for $i = 1, 2$, then

$$\bar{\omega}(v_1, v_2) = \mathcal{C}(u)w_1 \cdot w_2,$$

where $\mathcal{C}(u) = Du - (Du)^*$. Hence (1.12) really means

$$\begin{aligned}
&\left[\mathcal{C}\big(u(\cdot, t)\big) \circ Q_t\right](DQ_t)w_1 \cdot (DQ_t)w_2 = \mathcal{C}\big(u(\cdot, 0)\big)w_1 \cdot w_2, \\
&\mathcal{C}\big(u(\cdot, t)\big) = (DA_t)^*\left[\mathcal{C}\big(u(\cdot, 0)\big) \circ A_t\right](DQ_t).
\end{aligned} \tag{2.5}$$

Let us assume now that $d = 3$ so that, $\mathcal{C}(u)w = \xi \times w$, where $\xi = \nabla \times u$ denotes the vorticity. Hence

$$\bar{\omega}(v_1, v_2) = (\xi \times w_1) \cdot w_2 =: [\xi, w_1, w_2].$$

We note that the right-hand side is the volume form evaluated at the triple (ξ, w_1, w_2). Now the invariance (2.5) becomes

$$\left[\xi^t \circ Q_t, (DQ_t)w_1, (DQ_t)w_2\right] = \left[\xi^0, w_1, w_2\right], \tag{2.6}$$

where we have written ξ^t for $\xi(\cdot, t)$. Since u is divergence-free, the flow Q_t is volume preserving. As a result,

$$\left[\xi^0, w_1, w_2\right] = \left[(DQ_t)\xi^0, (DQ_t)w_1, (DQ_t)w_2\right].$$

From this and (2.6) we deduce

$$\left[\xi^t \circ Q_t, (DQ_t)w_1, (DQ_t)w_2\right] = \left[(DQ_t)\xi^0, (DQ_t)w_1, (DQ_t)w_2\right].$$

Since w_1 and w_2 are arbitrary, we conclude that (1.13) is true.

Example 2.1. When $d = 3$, we may use cylindrical coordinates $q_1 = r\cos\theta$, $q_2 = r\sin\theta$, $q_3 = z$ to write $u = ae(\theta) + cf(\theta) + be_3$, where

$$e(\theta) = (\cos\theta, \sin\theta, 0), \qquad f(\theta) = (-\sin\theta, \cos\theta, 0), \qquad e_3 = (0, 0, 1).$$

A solution is called *axisymmetric* if a, b, and c are functions of (r, z) only and do not depend on θ. Let us write

$$\eta = a_z - b_r, \qquad \hat{\eta} = r^{-1}\eta, \qquad \bar{c} = rc, \qquad \hat{c} = r^{-1}c.$$

If we write $\hat{Q}_t(r, z, \theta)$ for the flow $Q_t(q)$ in the cylindrical coordinates, then

$$\hat{Q}_t(r, z, \theta) = \left(\psi_t(r, z), \theta + \int_0^t \hat{c}\big(\psi_s(r, z), s\big)ds\right),$$

where ψ_t is the flow of the ODE

$$\dot{r} = a(r, z, t), \qquad \dot{z} = b(r, z, t).$$

One can easily check that the divergence free condition $\nabla \cdot u = 0$ means that $(ra)_r + (rb)_z = 0$. This condition is equivalent to $\psi_t^* \gamma = \gamma$ for the area form $\gamma = r\,dr \wedge dz$, simply because

$$\mathcal{L}_v \gamma = di_v \gamma = d(ra\,dz - rb\,dr) = \left[(ra)_r + (rb)_z\right]dr \wedge dz,$$

where $v = (a, b)$. We also have

$$\alpha_t = a(\cdot, t)dr + b(\cdot, t)dz + \bar{c}(\cdot, t)d\theta =: \hat{\alpha}_t + \bar{c}(\cdot, t)d\theta.$$

To understand the meaning of (1.12) in the axisymmetric case, observe

$$Q_t^* \alpha_t = \psi_t^* \hat{\alpha}_t + (\bar{c}_t \circ \psi_t)d\theta + (\bar{c}_t \circ \psi_t) \int_0^t d(\hat{c}_s \circ \psi_s)ds,$$

where we have written $\bar{c}_t$ and $\hat{c}_t$ for $\bar{c}(\cdot, t)$ and $\hat{c}(\cdot, t)$ respectively. As a result, the identity $d(Q_t^* \alpha_t - \alpha_0) = 0$, is equivalent to $\bar{c}(\psi_t(r, z), t) = \bar{c}(r, z, 0)$, and

$$d\left[\psi_t^* \hat{\alpha}_t + (\bar{c}_t \circ \psi_t) \int_0^t d(\hat{c}_s \circ \psi_s)ds \right] = d\hat{\alpha}_0. \tag{2.7}$$

To simplify (2.7), recall that $\bar{c}_t \circ \psi_t = c_0$, and

$$d\left(\psi_t^* \hat{\alpha}_t\right) = \psi_t^*(d\hat{\alpha}_t) = -\psi_t^*(\hat{\eta}_t \gamma) = -(\hat{\eta}_t \circ \psi_t)\gamma,$$

where $\hat{\eta}_t = \hat{\eta}(\cdot, t)$ and we have used $\psi_t^* \gamma = \gamma$. On the other hand

$$d\left[(\bar{c}_t \circ \psi_t) \int_0^t d(\hat{c} \circ \psi_s)ds \right] = d\left[\int_0^t \bar{c}_0 d(\hat{c} \circ \psi_s)ds \right] = d\left[\int_0^t (\bar{c}_s \circ \psi_s)d(\hat{c} \circ \psi_s)ds \right]$$

$$= \int_0^t \psi_s^* d[\bar{c}_s d\hat{c}_s]ds = \int_0^t \psi_s^*[d\bar{c}_s \wedge d\hat{c}_s]ds = \int_0^t \psi_s^*\left[\{\bar{c}_s, \hat{c}_s\}dz \wedge dr\right]ds$$

$$= -2\int_0^t \psi_s^*\left[r^{-1}bb_z dz \wedge dr\right]ds = 2\int_0^t \left[(r^{-2}bb_z) \circ \psi_s \gamma\right]ds,$$

because $\psi_s^* \gamma = \gamma$. Here we are writing $\{\cdot, \cdot\}$ for the Poisson bracket. From this and (2.7) we deduce that (1.12) is equivalent to the equality

$$\hat{\eta}_t \circ \psi_t - 2\int_0^t \left(r^{-2}bb_z\right) \circ \psi_s ds = \hat{\eta}_0. \qquad \square$$

To apply ideas from Symplectic Geometry to solutions of Euler equation, we need to assume some non-degeneracy of the initial data. However such a non-degeneracy is possible only in even dimensions. When the dimension is odd, we instead assume the best non-degeneracy that is possible for the initial data. This would take us to the realm of contact geometry.

Definition 2.1.

(i) A closed 2-form ω is symplectic if it is non-degenerate. We say that symplectic forms ω^1 and ω^2 are *isomorphic* if there exists a diffeomorphism Ψ such that $\Psi^* \omega^1 = \omega^2$.

(ii) A 1-form α is contact if $l_x = \{v : d\alpha(x; v, w) = 0 \text{ for every } w\}$ is a line and for every $v \in l_x$, we have that $\alpha(x; v) \neq 0$. We say that contact forms α^1 and α^2 are *isomorphic* if there exists a diffeomorphism Ψ such that $\Psi^* \alpha^1 = \alpha^2$. We say that contact forms α^1 and α^2 are *conformally isomorphic* if there exist a diffeomorphism Ψ and a scaler-valued continuous function $f > 0$ such that $\Psi^* \alpha^1 = f\alpha^2$.

(iii) A solution u of Euler equation is called *symplectic* if $\omega_0 = d\alpha_0$ is symplectic.

(iv) A solution u of Euler equation is *contact* if there exists a scalar-valued C^1 function f_0 such that $\alpha_0 + df_0$ is contact. (Recall $\alpha_t = u(\cdot, t) \cdot dx$.) $\square$

Remark 2.1.

(i) As it is well-known, the non-degeneracy of a 2-form can only happen when the dimension d is even. Recall $\alpha_t = u(\cdot, t) \cdot dx$. If u is a symplectic solution, then $\omega_t = d\alpha_t$ is symplectic for all t because by (1.12), the form ω_t is isomorphic to ω_0.

(ii) When u is a contact solution of Euler equation, then $\tilde{\alpha}_t = Q_t^* \alpha_0 + df_t$ is contact for all t where $f_t = f_0 \circ Q_t$. In general $\tilde{\alpha}_t \neq \alpha_t$. However, by Eq. (1.12), we have $d\alpha_t = d\tilde{\alpha}_t$. Hence there exists a scalar-valued function g_t such that $\alpha_t + dg_t = \tilde{\alpha}_t$ is contact.   □

We continue with some general properties of symplectic and contact solutions of Euler equation.

As for symplectic solutions, assume that the dimension $d = 2k$ is even and write

$$(q_1, \ldots, q_d) = (x_1, y_1, \ldots, x_k, y_k).$$

A classical theorem of Darboux asserts that all symplectic forms are isomorphic to the standard form $\bar{\omega} = d\bar{\lambda} = \sum_{i=1}^k dy_i \wedge dx_i$. A natural question is whether such an isomorphism exists globally.

Definition 2.2. Let u be a symplectic solution of Euler equation. We say that *Clebsch* variables exist for u in the interval $[0, T]$, if we can find C^1 functions

$$X_1, \ldots, X_k, Y_1, \ldots, Y_k : \mathbb{R}^d \times [0, T] \to \mathbb{R}, \qquad F : \mathbb{R}^d \times [0, T] \to \mathbb{R}$$

such that $\Psi_t = (X_1, Y_1, \ldots, X_k, Y_k)(\cdot, t)$ is a diffeomorphism, and

$$u(x, t) = \left(\sum_{i=1}^k Y_i \nabla X_i \right)(x, t) + \nabla F(x, t),$$

for every $t \in [0, T]$. Alternatively, we may write $\alpha_t = \Psi_t^* \bar{\lambda} + dF$, which implies that $d\alpha_t = \Psi_t^* \bar{\omega}$.   □

Proposition 2.1. *Let u be a symplectic solution to Euler equation in the interval $[0, T]$.*

(i) *If Clebsch variables exist for $t = 0$, then they exist in the interval $[0, T]$.*

(ii) *If $d = 4$ and Clebsch variables exist for $t = 0$ outside some ball $B_r = \{x : |x| \leq r\}$, then they exist globally in the interval $[0, T]$.*

Proof. **(i)** This is an immediate consequence of (1.12): If $\Psi_0^* \bar{\omega} = \omega_0 = d\alpha_0$, then

$$\left(Q_t \circ \Psi_0^{-1} \right)^* d\alpha_t = \Psi_0^{-1*} Q_t^* d\alpha_t = \Psi_0^{-1*} d\alpha_0 = \bar{\omega},$$

which means that we can choose $\Psi_t = \Psi_0 \circ A_t$ for the Clebsch change of variables.

(ii) This is a consequence of a deep theorem of Gromov [9]: When $d = 4$, a symplectic form is isomorphic to the standard form $\bar{\omega}$, if this is the case outside a ball B_r.   □

Remark 2.2. Part (i) of Proposition 2.1 is well-known and can be found in Section 2 of [4]. We also refer to [14] for various historical comments and numerical results on Clebsch variables. For a detailed study of generalized Clebsch variables and their connection to Eq. (1.13), see Constantin [2].   □

Observe that Euler equation can be rewritten as

$$\frac{d}{dt} \alpha_t + i_u(d\alpha_t) = -dH, \tag{2.8}$$

where $H(q, t) = P(q, t) + |u(q, t)|^2 / 2$ is the Hamiltonian function. For a steady solution, α_t is independent of t and we simply get

$$i_u(d\alpha) = -dH.$$

If u is a symplectic steady solution of Euler equation, then $i_u(d\alpha) = -dH$ means that u is a *Hamiltonian vector field* with respect to the symplectic form $d\alpha$. Of course the associated the Hamiltonian function is H. Alternatively, we may write

$$u = -\mathcal{C}(u)^{-1}\nabla H. \tag{2.9}$$

Proposition 2.2. *Let u be a steady symplectic solution to Euler equation, and let c be a regular level set of $H(q,t) = P(q,t) + \frac{1}{2}|u(q,t)|^2$ i.e. $\nabla H(q) \neq 0$ whenever $H(q) = c$. Then the restriction of the form α to the submanifold $H = c$ is contact. In words, regular level sets of H are contact submanifolds.*

Proof. We say a vector field X is *Liouville* with respect to the form $d\alpha$ if $\mathcal{L}_X d\alpha = d\alpha$. By a standard fact in Symplectic Geometry (see for example [16] or [10]), the level set $H = c$ is contact if and only if we can find a Liouville vector field X that is transversal to $M_c = \{H = c\}$. More precisely,

$$\mathcal{L}_X d\alpha = d\alpha, \qquad X(q) \notin T_q M_c,$$

for every $q \in M_c$. Here $T_q M_c$ denotes the tangent fiber to M_c at q. The first condition means that $di_X d\alpha = d\alpha$. This is satisfied if $i_X d\alpha = \alpha$. This really means that $\mathcal{C}(u)X = u$ and as a result, we need to choose $X = \mathcal{C}(u)^{-1}u$. It remains to show that X is never tangent to M_c. For this, it suffices to check that $X \cdot \nabla H \neq 0$. Indeed, when $H = c$,

$$X \cdot \nabla H = \mathcal{C}(u)^{-1}u \cdot \nabla H = -u \cdot \mathcal{C}(u)^{-1}\nabla H = \left|\mathcal{C}(u)^{-1}\nabla H\right|^2 \neq 0,$$

by (2.9) because by assumption $\nabla H \neq 0$. We are done. $\quad\square$

Remark 2.3. According to the Weinstein Conjecture, every compact contact manifold carries a period *Reeb* orbit. This conjecture has been established by Viterbo in the Euclidean setting (see for example [16] or [10]) and is applicable to the setting of Proposition 2.2. In the language of fluid mechanics, Viterbo's theorem asserts that every compact nondegenerate level set of the energy function associated with a symplectic steady solution carries at least one periodic orbit of the fluid. $\quad\square$

Example 2.2. In this example we describe some simple solutions when the dimension is even. We use polar coordinates to write $x_i = r_i \cos\theta_i$, $y_i = r_i \sin\theta_i$, and let e_i (respectively e_i') denote the vector for which the x_i-th coordinate (respectively y_i-th coordinate) is 1 and any other coordinate is 0. Set

$$e_i(\theta_i) = (\cos\theta_i)e_i + (\sin\theta_i)e_i', \qquad f_i(\theta_i) = -(\sin\theta_i)e_i + (\cos\theta_i)e_i'.$$

We may write

$$u = \sum_{i=1}^{k}\left(a^i e_i(\theta_i) + b^i f_i(\theta_i)\right).$$

The form $\alpha = u \cdot dx$ can be written as

$$\alpha = \sum_{i=1}^{k}\left(a^i dr_i + r_i b^i d\theta_i\right) =: \sum_{i=1}^{k}\left(a^i dr_i + B^i d\theta_i\right).$$

For a simple solution, let us assume that all a^i's and b^i's depend on $r = (r_1, \ldots, r_k)$ only. We then have

$$d\alpha = \sum_{i<j}\left(a_{r_j}^i - a_{r_i}^j\right)dr_j \wedge dr_i + \sum_{i,j} r_i^{-1} B_{r_j}^i dr_j \wedge (r_i d\theta_i). \tag{2.10}$$

Note

$$\nabla = \sum_{i=1}^{d} e_i(\theta_i)\frac{\partial}{\partial r_i} + r_i^{-1}f_i(\theta_i)\frac{\partial}{\partial\theta_i}, \qquad u \cdot \nabla = \sum_{i=1}^{d} a^i\frac{\partial}{\partial r_i} + r_i^{-1}b^i\frac{\partial}{\partial\theta_i}.$$

From this we learn that if u solves Euler equation, then the vector fields $a = (a^1, \ldots, a^k)$ and $b = (b^1, \ldots, b^k)$ satisfy

$$a_t + \mathcal{C}(a)a - \left[r_i^{-1} (b^i)^2 \right]_i + \nabla_r K = 0,$$
$$b_t + (Db)a + \left[r_i^{-1} (a^i b^i) \right]_i = 0,$$
$$\sum_{i=1}^k (r_i a^i)_{r_i} / r_i = 0, \tag{2.11}$$

for some scalar function $K(r) = |a|^2/2 + p(r)$. Alternatively, if we introduce the matrix $E(b) = [r_i^{-1} B_{r_j}^i]$, the first and second equations in (2.11) may be written as

$$a_t + \mathcal{C}(a)a - E(b)^* b + \nabla_r H = 0,$$
$$b_t + E(b)a = 0, \tag{2.12}$$

where $H(r) = |b|^2/2 + K(r) = |u|^2/2 + p(r)$. (This can be derived directly from (2.8) and (2.10).) In view of (2.10), u is a symplectic solution if and only if the matrix $E(b)$ is invertible. When u is a steady solution, (2.12) simplifies to

$$\mathcal{C}(a)a - E(b)^* b + \nabla_r H = 0, \quad E(b)a = 0. \tag{2.13}$$

Moreover, by taking the dot product of both sides of the second (respectively first) equation by b (respectively a), we learn

$$a \cdot \nabla_r H = 0. \tag{2.14}$$

Also, the second equation in (2.13) really means

$$a \cdot \nabla_r B^i = 0 \quad \text{for } i = 1, \ldots, k. \tag{2.15}$$

When $d = 4$ and u is independent of time, it is straight forward to solve (2.11): From the last equation in (2.11) we learn that there exists a function $\psi(r_1, r_2)$ such that

$$a^1 = \psi_{r_2}/(r_1 r_2), \qquad a^2 = -\psi_{r_1}/(r_1 r_2).$$

From this, (2.14) and (2.15) we learn that $\nabla_r H$, $\nabla_r B^1$, $\nabla_r B^2$ and $\nabla \psi$ are all parallel. So we may write

$$H = \mu(\psi), \qquad B^1 = \mu_1(\psi), \qquad B^2 = \mu_2(\psi),$$

for some C^1 functions $\mu, \mu_1, \mu_2 : \mathbb{R} \to \mathbb{R}$. Finally we go back to the first equation in (2.13) to write

$$a^2 (a_{r_2}^1 - a_{r_1}^2) - \frac{B_{r_1}^1 B^1}{r_1^2} - \frac{B_{r_1}^2 B^2}{r_2^2} + H_{r_1} = 0.$$

Expressing this equation in terms of ψ yields the elliptic PDE

$$(r_1 r_2)^{-1} \left[\left(\frac{\psi_{r_1}}{r_1 r_2} \right)_{r_1} + \left(\frac{\psi_{r_2}}{r_1 r_2} \right)_{r_2} \right] = \left(\frac{\mu_1 \mu_1'}{r_1^2} + \frac{\mu_2 \mu_2'}{r_2^2} - \mu' \right) (\psi).$$

This equation may be compared to the Bragg–Hawthorne equation [1] that is solved to obtain axisymmetric steady solutions in dimension three. $\quad \square$

We now turn to the odd dimensions. Assume that $d = 2k + 1$ for $k \in \mathbb{N}$. We write $(q_1, \ldots, q_n) = (x_1, y_1, \ldots, x_k, y_k, z)$ and when $k = 1$ we simply write $(q_1, q_2, q_3) = (x, y, z)$. In this case, the standard contact form is $\bar{\lambda} = \sum_{i=1}^k y_i dx_i + dz$. Again, locally all contact forms are isomorphic to $\bar{\lambda}$.

Definition 2.3. Let u be a solution of Euler equation. We say that *Clebsch* variables exist for u in the interval $[0, T]$, if we can find C^1 functions

$$X_1, \ldots, X_k, Y_1, \ldots, Y_k : \mathbb{R}^d \times [0, T] \to \mathbb{R}, \qquad f, Z : \mathbb{R}^d \times [0, T] \to \mathbb{R}$$

such that $\Psi_t = (X_1, Y_1, \ldots, X_k, Y_k, Z)(\cdot, t)$ is a diffeomorphism, $f > 0$, and

$$(fu)(x,t) = \left(\sum_{i=1}^k Y_i \nabla X_i\right)(x,t) + \nabla Z(x,t),$$

for every $t \in [0.T]$. Alternatively, we may write $f\alpha_t = \Psi_t^* \bar{\lambda}$. $\square$

As we recalled in the proof of Proposition 2.1(ii), if $d = 4$ and a symplectic form is isomorphic to the standard form at infinity, then the isomorphism can be extended to the whole R^d. This is no longer true when $d = 3$; in fact there is a countable collection of pairwise non-conformally-isomorphic forms λ^n in $\mathbb{R}^3$ such that each λ^n is conformally isomorphic to $\bar{\lambda}$ at infinity but not globally. A fundamental result of Eliashberg gives a complete classification of contact forms. According to Eliashberg's Theorem [5], any contact form in $\mathbb{R}^3$ is conformally isomorphic to one of the following forms

(i) The standard form $\bar{\lambda}$.
(ii) The form $\hat{\lambda} = \frac{\sin r}{r}(xdy - ydx) + \cos r\, dz$, where $r^2 = x^2 + y^2$.
(iii) A countable collection of pairwise non-conformally-isomorphic forms $\{\lambda^n : n \in \mathbb{Z}\}$, where each λ^n is conformally-isomorphic to $\bar{\lambda}$ outside the ball B_1 but not globally in $\mathbb{R}^3$.

The above classification is related to the important notion of *overtwisted* contact forms. In fact $\hat{\lambda}$ is globally over-twisted whereas λ^ns are overtwisted only in a neighborhood of the origin. (We refer to [5] or [8] for the definition of overtwisted forms.)

Example 2.3. Let us assume that $d = 3$, and u is an axisymmetric steady solution to Euler equation. Using the notation of Example 2.1, u is a contact solution if and only if

$$u \cdot \xi = r^{-1}\big(b(r)\bar{c}'(r) - b'(r)\bar{c}(r)\big) \neq 0.$$

For example, if $b(r) = 1, c(r) = r$, then we get

$$\alpha = r^2 d\theta + dz = xdy - ydx + dz,$$

which is isomorphic to $\bar{\lambda}$. On the other hand, choosing $c(r) = r\sin r, b(r) = \cos r$ would yield exactly $\hat{\lambda}$. $\square$

Remark 2.4. In view of Eliashberg's classification in dimension three, Clebsch variables would exist only if we are searching for a solution that is conformally isomorphic to λ. However, if we search for a solution that is conformally isomorphic to $\hat{\lambda}$ for example, then we need to find scalar-valued functions

$$R, \Theta, Z, f : \mathbb{R}^3 \times [0, T] \to \mathbb{R},$$

such that

$$fu = R(\sin R)\nabla\Theta + (\cos R)\nabla Z.$$ $\square$

3. Symplectic diffusions

In this section we examine the question of *symplectic invariance* for the flows of diffusions. After a prominent work of Itô in 1951, diffusions may be defined as solutions of *stochastic differential equations (SDE)*. To make sense of solutions to SDEs, we need to make sense of stochastic integrals and these integrals come in three flavors: Itô (or forward), backward and Stratonovich. The main advantage of Stratonovich Integral is that the chain rule of ordinary calculus holds. This makes Stratonovich Calculus more attractive when we are interested in geometric questions for flows associated with diffusions. However Stratonovich integration does not yield martingale property that is very essential for bounding various functionals of a diffusion. Fortunately it is often straightforward to convert a Stratonovich Integral to an Itô Integral in the end and take advantage of the martingale property of the latter.

To study stochastic flows associated with diffusions, let us consider SDE

$$dx(t) = V_0\big(x(t), t\big)dt + \sum_{i=1}^{k} V_i\big(x(t), t\big) \circ dW^i(t),$$
(3.1)

where $(W^i : i = 1, \ldots, k)$ are standard one dimensional Brownian motions on some filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$, and $V_0, \ldots, V_k$ are C^r-vector fields in $\mathbb{R}^n$. Here we are using Stratonovich stochastic differentials for the second term on the right-hand of (3.1) and a solution to the SDE (3.1) is a diffusion with the infinitesimal generator

$$L = V_0 \cdot \nabla + \frac{1}{2}\sum_{i=1}^{k}(V_i \cdot \nabla)^2,$$

or in short $L = V_0 + \frac{1}{2}\sum_{i=1}^{k} V_i^2$, where we have simply written V for the V-directional derivative operator $V \cdot \nabla$. We assume that the random flow $\phi_{s,t}$ of (3.1) is well-defined almost surely. More precisely for $\mathbb{P}$-almost all realization of ω, we have a flow $\{\phi_{s,t}(\cdot, \omega) : 0 \le s \le t\}$ where $\phi_{s,t}(\cdot, \omega) : \mathbb{R}^n \to \mathbb{R}^n$ is a C^{r-1} diffeomorphism and $\phi_{s,t}(a, \omega) =: x(t)$ is a solution of (3.1) subject to the initial condition $x(s) = a$ (see Section 4.2 of Kunita [13] or Theorem 3.4 of [11]). We also write ϕ_t for $\phi_{0,t}$. For example a uniform bound on the C^r-norm of the coefficients $V_0, \ldots, V_k$ would guarantee the existence of such a stochastic flow provided that $r \ge 2$. We also remark that we can formally differentiate (3.1) with respect to the initial condition and derive a SDE for $\Lambda_{s,t}(x) = \Lambda_t(x) := D_x\phi_{s,t}(x)$:

$$d\Lambda_t(x) = D_x V_0\big(\phi_{s,t}(x), t\big)\Lambda_t(x)dt + \sum_{i=1}^{k} D_x V_i\big(\phi_{s,t}(x), t\big)\Lambda_t(x) \circ dW^i(t).$$
(3.2)

Given a differential ℓ-form $\alpha(x; v_1, \ldots, v_\ell)$, we define

$$\big(\phi_{s,t}^*\alpha\big)(x; v_1, \ldots, v_\ell) = \alpha\big(\phi_{s,t}(x); \Lambda_{s,t}(x)v_1, \ldots, \Lambda_{s,t}(x)v_\ell\big).$$

Given a vector field V, we write $\mathcal{L}_V$ for the Lie derivative in the direction V. More precisely, for every differential form α,

$$\mathcal{L}_V \alpha = (\hat{d} \circ i_V + i_V \circ \hat{d})\alpha,$$
(3.3)

where $\hat{d}$ and i_V denote the exterior derivative and V-contraction operator respectively. (To avoid a confusion between the stochastic differential and exterior derivative, we are using a hat for the latter.) We are now ready to state a formula that is the stochastic analog of Cartan's formula and it is a rather straight forward consequence of (3.2). We refer to Section 4.9 of [13] for a proof.

Proposition 3.1. *Set* $\mathbf{V} = (V_0, V_1, \ldots, V_m)$ *and*

$$\mathcal{A}_{\mathbf{V}} = \mathcal{L}_{V_0} + \frac{1}{2}\sum_{i=1}^{k} \mathcal{L}_{V_i}^2.$$

We also η^t *for* $\phi_{s,t}^*\eta$ *for any form* η. *We have*

$$d\alpha^t = (\mathcal{L}_{V_0}\alpha)^t dt + \sum_{i=1}^{k}(\mathcal{L}_{V_i}\alpha)^t \circ dW^i(t)$$

$$= (\mathcal{A}_{\mathbf{V}}\alpha)^t dt + \sum_{i=1}^{k}(\mathcal{L}_{V_i}\alpha)^t dW^i(t).$$
(3.4)

Example 3.1.

(i) If $\alpha = f$ is a 0-form, then $\mathcal{A}_{\mathbf{V}} f = Lf$ is simply the infinitesimal generator of the underlying diffusion.

(ii) If $\alpha = \rho dx_1 \wedge \cdots \wedge dx_n$, is a volume form, then

$$\mathcal{L}_V \alpha = \left(\nabla \cdot (\rho\alpha)\right) dx_1 \wedge \cdots \wedge dx_n, \quad \text{and} \quad \mathcal{A}_\mathbf{V}\alpha = \left(L^*\rho\right) dx_1 \wedge \cdots \wedge dx_n,$$

where L^* is the adjoint of the operator L.

(iii) If $\alpha = \rho dx_1 \wedge \cdots \wedge dx_n$, is a volume form, then $\alpha^t = \phi_t^*\alpha = \alpha$ if and only if $\nabla \cdot (\rho V_i) = 0$ for $i = 0, 1, \cdots, k$. For example, if $W = (W^1, \ldots, W^n)$ is an n-dimensional standard Brownian motion and

$$dx = V_0(x, t)dt + dW,$$

then the flow of this diffusion preserves the standard volume form $\bar\alpha = dx_1 \wedge \cdots \wedge dx_n$, if and only if $\nabla \cdot V_0 = 0$.

(iv) If $\alpha = \rho dx_1 \wedge \cdots \wedge dx_n$, is a volume form, and $L^*\rho = 0$, then α, regarded as a measure, is an invariant measure for the diffusion (3.1). However, $\alpha^t = \phi_t^*\alpha$ is a volume-form-valued martingale.　□

We now make two definitions:

Definition 3.1. Let α be a symplectic form.

(i) We say that the diffusion (3.1) is α-symplectic if its flow is symplectic with respect to α, almost surely. That is $\phi_t^*\alpha = \alpha$, a.s.

(ii) We say that the diffusion (3.1) is α-semisymplectic if $\alpha^t := \phi_t^*\alpha$, is a martingale.

Using Proposition 3.1 it is not hard to deduce

Proposition 3.2.

(i) *The diffusion* (3.1) *is α-symplectic if and only if the vector fields* $V_0, V_1, \ldots, V_k$ *are α-Hamiltonian, i.e.* $\mathcal{L}_{V_0}\alpha = \mathcal{L}_{V_1}\alpha = \cdots = \mathcal{L}_{V_k}\alpha = 0$.

(ii) *The diffusion* (3.1) *is α-semisymplectic if and only if* $\mathcal{A}_\mathbf{V}\alpha = 0$.

We discuss two systematic ways of producing semisymplectic diffusions.

Recipe (i) Given a symplectic form α, we write $X_H = X_H^\alpha$ for the Hamiltonian vector field associated with the Hamiltonian function H. Note that by non-degeneracy of α, there exists a unique vector field $X = \mathcal{X}^\alpha(\nu)$ such that $i_X\alpha = \nu$ for every 1-form ν and $X_H = -\mathcal{X}^\alpha(\hat{d}H)$. In the following proposition, we show that given $V_1, V_2, \ldots, V_k$, we can always find a unique $\hat{V}_0$ such that the diffusion associated with $\mathbf{V} = (X_H + \hat{V}_0, V_1, \ldots, V_k)$ is α-semisymplectic.

Proposition 3.3. *The diffusion* (3.1) *is α-semisymplectic if and only if there exists a Hamiltonian function H, such that*

$$V_0 = X_H - \frac{1}{2}\sum_{j=1}^k \mathcal{X}^\alpha(i_{V_j}\hat{d}i_{V_j}\alpha). \tag{3.5}$$

Proof. By definition,

$$\mathcal{A}_\mathbf{V}\alpha = \hat{d}\left[i_{V_0}\alpha + \frac{1}{2}\sum_{j=1}^k(i_{V_j}\hat{d}i_{V_j}\alpha)\right].$$

Hence $\mathcal{A}_\mathbf{V}\alpha = 0$ means that for some function H,

$$i_{V_0}\alpha + \frac{1}{2}\sum_{j=1}^k(i_{V_j}\hat{d}i_{V_j}\alpha) = -\hat{d}H.$$

From this we can readily deduce (3.5).　□

Recipe (ii) We now give a useful recipe for constructing $\bar{\omega}$-diffusions where $\bar{\omega}$ is the standard symplectic form and $n = 2d$.

Proposition 3.4. *Given a Hamiltonian function H, consider a diffusion $x(t) = (q(t), p(t))$ that solves*

$$dx(t) = J\nabla H\big(x(t), t\big)dt + \sum_{i=1}^{k} V_i\big(x(t), t\big)dW^i(t),$$

with $V_j = \begin{bmatrix} A_j \\ B_j \end{bmatrix}$, for $j = 1, \ldots, k$, where $A_j = (A_j^1, \ldots, A_j^d)$, and $B_j = (B_j^1, \ldots, B_j^d)$. Then $\mathcal{A}_\mathbf{V}\bar{\omega} = \frac{1}{2}\hat{d}\gamma$, where $\gamma = Z_1 \cdot dq + Z_2 \cdot dp$, with

$$Z_1^i = \sum_{r,j}\left(\frac{\partial A_j^r}{\partial q_i}B_j^r - \frac{\partial B_j^r}{\partial q_i}A_j^r\right),$$

$$Z_2^i = \sum_{r,j}\left(\frac{\partial A_j^r}{\partial p_i}B_j^r - \frac{\partial B_j^r}{\partial p_i}A_j^r\right). \tag{3.6}$$

Proof. The Stratonovich differential is related to Itô differential by

$$a \circ dW = adW + \frac{1}{2}[da, dW].$$

As a result, the diffusion $x(t)$ satisfies (3.1) for $V_0 = J\nabla H - \frac{1}{2}\hat{V}_0$ with $\hat{V}_0 = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$, where

$$A_0^i = \sum_{r,j}\left(\frac{\partial A_j^i}{\partial q_r}A_j^r + \frac{\partial A_j^i}{\partial p_r}B_j^r\right),$$

$$B_0^i = \sum_{r,j}\left(\frac{\partial B_j^i}{\partial q_r}A_j^r + \frac{\partial B_j^i}{\partial p_r}B_j^r\right). \tag{3.7}$$

By definition,

$$\mathcal{A}_\mathbf{V}\bar{\omega} = \frac{1}{2}\hat{d}\gamma := \frac{1}{2}\hat{d}(\eta - i_{\hat{V}_0}\bar{\omega}), \tag{3.8}$$

where

$$\eta = \sum_{j=1}^{k} i_{V_j}\hat{d}i_{V_j}\bar{\omega}.$$

To calculate γ and η, let us write $\beta(F)$ for the 1-form $F \cdot dx$ and observe

$$i_V\bar{\omega} = \beta(JV), \qquad \hat{d}\beta(F)(v, w) = \mathcal{C}(F)v \cdot w,$$

where $\mathcal{C}(F) = DF - (DF)^*$ with DF denoting the matrix of the partial derivatives of F with respect to x. From this we deduce

$$\eta = \sum_{j=1}^{k} \beta\big(\mathcal{C}(JV_j)V_j\big). \tag{3.9}$$

A straight forward calculation yields

$$\mathcal{C}(JV_j) = \begin{bmatrix} X_{11}^j & X_{12}^j \\ X_{21}^j & X_{22}^j \end{bmatrix}$$

where

$$X_{11}^j = \left[\frac{\partial B_j^i}{\partial q_r} - \frac{\partial B_j^r}{\partial q_i} \right]_{i,r=1}^n, \qquad X_{12}^j = \left[\frac{\partial B_j^i}{\partial p_r} + \frac{\partial A_j^r}{\partial q_i} \right]_{i,r=1}^n,$$

$$X_{21}^j = \left[-\frac{\partial A_j^i}{\partial q_r} - \frac{\partial B_j^r}{\partial p_i} \right]_{i,r=1}^n, \qquad X_{22}^j = \left[-\frac{\partial A_j^i}{\partial p_r} + \frac{\partial A_j^r}{\partial p_i} \right]_{i,r=1}^n.$$

From this we deduce

$$\mathcal{C}(JV_j)V_j = \begin{bmatrix} Y_1^j \\ Y_2^j \end{bmatrix},$$

where

$$Y_1^j = \left[\sum_r \left(\frac{\partial B_j^i}{\partial q_r} - \frac{\partial B_j^r}{\partial q_i} \right) A_j^r + \sum_r \left(\frac{\partial B_j^i}{\partial p_r} + \frac{\partial A_j^r}{\partial q_i} \right) B_j^r \right]_{i=1}^n,$$

$$Y_2^j = \left[\sum_r \left(\frac{\partial A_j^r}{\partial p_i} - \frac{\partial A_j^i}{\partial p_r} \right) B_j^r - \sum_r \left(\frac{\partial A_j^i}{\partial q_r} + \frac{\partial B_j^r}{\partial p_i} \right) A_j^r \right]_{i=1}^n.$$

Summing these expressions over j yields

$$\sum_j \mathcal{C}(JV_j)V_j = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} B_0 \\ -A_0 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + J\hat{V}_0,$$

where Z_1 and Z_2 are defined by (3.7). These (3.8) and (3.9) complete the proof. □

An immediate consequence of Proposition 3.4 is Corollary 3.1.

Corollary 3.1. *Let $x(t) = (q(t), p(t))$ be a diffusion satisfying*

$$dq = H_p(q, p)dt + \sqrt{2\nu}dW,$$
$$dp = -H_q(q, p)dt + \sqrt{2\nu}\Gamma(q, t)dW, \tag{3.10}$$

where Γ is a continuously differentiable $d \times d$-matrix valued function and $W = (W^1, \ldots, W^d)$ is a standard Brownian motion in $\mathbb{R}^d$. Then the process $x(t)$ is $\bar{\omega}$-semisymplectic.

Proof. Observe that $x(t)$ satisfies (3.6) for $A = \sqrt{2\nu}I_d$ and B that is independent of p. From this we deduce that $Z_2 = 0$ and $Z_1 = -\sqrt{2\nu}\nabla_q(\text{tr}\,\Gamma)$. We are done because $\gamma = -\sqrt{2\nu}\hat{d}(\text{tr}\,\Gamma)$, and $\mathcal{A}_{\mathbf{V}}\bar{\omega} = \frac{1}{2}\hat{d}\gamma = 0$. □

4. Martingale circulation

Proof of Theorem 1.1. Step 1. As in Section 1, we write D and ∇ for q-differentiation. For x-differentiation however, we write D_x and ∇_x instead. Let us write $x'(t) = (q'(t), p'(t))$ for a diffusion that satisfies

$$dq'(t) = p'(t)dt + \sqrt{2\nu}d\bar{W}$$
$$dp'(t) = -\nabla P(q'(t), t)dt + \sqrt{2\nu}Dw(q'(t), t)d\bar{W}, \tag{4.1}$$

for a time dependent C^1 vector field w in $\mathbb{R}^d$ and a standard Brownian motion $\bar{W}$. The flow of this diffusion is denoted by ϕ_t. We then apply Corollary 3.1 for $H(q, p, t) = \frac{1}{2}|p|^2 + P(q, t)$ and $\Gamma = Dw$, to assert that the diffusion x' is $\bar{\omega}$-semisymplectic. Let us now assume that w satisfies the backward Navier–Stokes equation

$$w_t + (Dw)w + \nabla P + \nu\Delta w = 0, \qquad \nabla \cdot w = 0, \quad t < T, \tag{4.2}$$

subject to a final condition $w(\cdot, T) = u^0$. We observe that if the process $q'(t)$ is a diffusion satisfying

$$dq'(t) = w\big(q'(t), t\big)dt + \sqrt{2\nu}d\bar{W}, \tag{4.3}$$

and $p'(t) = w(q'(t), t)$, then by Ito's formula,

$$
\begin{aligned}
dp'(t) &= \big[w_t + (Dw)w + \nu\Delta w\big]\big(q'(t), t\big)dt + \sqrt{2\nu}Dw\big(q'(t), t\big)d\bar{W} \\
&= -\nabla P\big(q'(t), t\big)dt + \sqrt{2\nu}Dw\big(q'(t), t\big)d\bar{W}.
\end{aligned} \tag{4.4}
$$

This means that if $\bar{Q}_t$ denotes the flow of the SDE (4.3), then

$$
\begin{aligned}
&\phi_t\big(q, w(q, 0)\big) = \big(\bar{Q}_t(q), w\big(\bar{Q}_t(q), t\big)\big), \\
&D_x\phi_t\big(q, w(q, 0)\big)\begin{bmatrix} a \\ Dw(q, 0)a \end{bmatrix} = \begin{bmatrix} (D\bar{Q}_t(q))a \\ (Dw(\bar{Q}_t(q), t))(D\bar{Q}_t(q))a \end{bmatrix}.
\end{aligned} \tag{4.5}
$$

By the conclusion of Corollary 3.1, the process

$$\hat{M}_t(x; v_1, v_2) = \big[J\big(D\phi_t(x)\big)v_1\big] \cdot \big[\big(D\phi_t(x)\big)v_2\big]$$

is a 2-form-valued martingale. By choosing

$$x = \big(q, w(q, 0)\big), \qquad v_1 = \begin{bmatrix} a_1 \\ Dw(q, 0)a_1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} a_2 \\ Dw(q, 0)a_2 \end{bmatrix}, \tag{4.6}$$

for the arguments of $\hat{M}_t$, we learn that $\bar{M}_t(q; a_1, a_2) := \hat{M}_t(x; v_1, v_2)$ is a 2-form-valued martingale in $\mathbb{R}^d$. Using (4.5) we have

$$
\begin{aligned}
\bar{M}_t(q; a_1, a_2) &= J\begin{bmatrix} (D\bar{Q}_t(q))a_1 \\ (Dw(\bar{Q}_t(q), t))(D\bar{Q}_t(q))a_1 \end{bmatrix} \cdot \begin{bmatrix} (D\bar{Q}_t(q))a_2 \\ (Dw(\bar{Q}_t(q), t))(D\bar{Q}_t(q))a_2 \end{bmatrix} \\
&= \big[(Dw - (Dw)^*)\big(\bar{Q}_t(q), t\big)\big]\big(D\bar{Q}_t(q)\big)a_1 \cdot \big(D\bar{Q}_t(q)\big)a_2 \\
&= \bar{Q}_t^*\hat{d}\bar{\alpha}_t(q; a_1, a_2),
\end{aligned} \tag{4.7}
$$

where $\bar{\alpha}_t = w(q, t) \cdot dq$. In summary, $\bar{M}_t = \bar{Q}_t^*\hat{d}\bar{\alpha}_t$ is a martingale (proving our first claim in Remark 1.1(ii)).

When $d = 3$,

$$
\begin{aligned}
\bar{Q}_t^*\hat{d}\bar{\alpha}_t(q; a_1, a_2) &= \big[\big(\eta^t \circ \bar{Q}_t(q)\big) \times \big(D\bar{Q}_t(q)\big)a_1\big] \cdot \big(D\bar{Q}_t(q)\big)a_2 \\
&= \big[\eta^t \circ \bar{Q}_t(q), \big(D\bar{Q}_t(q)\big)a_1, \big(D\bar{Q}_t(q)\big)a_2\big],
\end{aligned}
$$

where $\eta^t(\cdot) = \nabla \times w(\cdot, t)$ and $[a, b, c]$ is the determinant of a matrix with column vectors a, b and c. Since w is divergence-free, the flow $\bar{Q}_t$ is volume preserving (see Example 3.1(iii)). Hence

$$\bar{M}_t(q; a_1, a_2) = \big[\big(D\bar{A}_t \circ \bar{Q}_t(q)\big)\eta^t \circ \bar{Q}_t(q), a_1, a_2\big],$$

where $\bar{A}_t = \bar{Q}_t^{-1}$. Since $\bar{M}_t$ is a martingale, we deduce that the process

$$\tilde{M}_t(q) = \big(D\bar{A}^t \circ \bar{Q}_t(q)\big)\big(\eta^t \circ \bar{Q}_t(q)\big),$$

is a martingale (proving our second claim in Remark 1.1(ii)).

Step 2. Suppose that now u is a solution to the forward Navier–Stokes equation (1.15) and recall that when $d = 3$, we write $\xi = \nabla \times u$. We set $w(q, t) = -u(q, T - t)$ for $t \in [0, T]$. Then w satisfies (3.2) in the interval $t \in [0, T]$. Recall that $q(t)$ is the solution of SDE (1.16) with the flow Q_t. We choose $\bar{W}(t) = W(T - t) - W(T)$ in Eq. (4.3). According to a theorem of Kunita (see [12] or Theorem 13.15 in p. 139 of [17]), the flows Q and $\bar{Q}$ are related by the formula

$$\bar{Q}_t = Q_{T-t} \circ Q_T^{-1} = B_t.$$

Observe that $\bar{\alpha}_t = -\alpha_{T-t}$ and

$$\bar{M}_t = \bar{Q}_t^*\hat{d}\bar{\alpha}_t = -B_t^*\hat{d}\alpha_{T-t} = -\beta_t.$$

Hence $(\beta_t : t \in [0, T])$ is a martingale because $\bar{M}_t$ is a martingale by Step 1. Also, when $d = 3$,

$$\tilde{M}_t = \left((D\bar{A}_t)\eta^t\right) \circ \bar{Q}_t(q) = -\left((DB_t^{-1})\xi^{T-t}\right) \circ B_t.$$

This completes the proof of Part (i).

Step 3. The process $x'(t)$ is a diffusion of the form (3.1) with $k = d$ and

$$V_i(x', t) = V_i(q, t) = \sqrt{2\nu} \begin{bmatrix} e_i \\ w_{q_i} \end{bmatrix}, \quad i = 1, \ldots, d,$$

where $e_i = [\delta_i^j]_{j=1}^d$ is the unit vector in the i-th direction. A straight forward calculation yields that for the standard symplectic form $\bar{\omega} = \sum_j dp_j \wedge dq_j$,

$$i_{V_i(\cdot, t)}\bar{\omega} = \sqrt{2\nu}\left(w_{q_i}(\cdot, t) \cdot dq - dp_i\right) =: \sqrt{2\nu}\left(\gamma_i^t - dp_i\right),$$

$$\mathcal{L}_{V_i(\cdot, t)}\bar{\omega} = \sqrt{2\nu}\hat{d}\gamma_i^t = \sqrt{2\nu}\sum_{j,k} w_{q_iq_j}^k(\cdot, t)dq_j \wedge dq_k,$$

where $w = (w^1, \ldots, w^d)$. From this and (3.4) we deduce

$$\hat{M}_t = \bar{\omega} + \sqrt{2\nu}\sum_{i=1}^d \int_0^t \phi_s^*\left(\hat{d}\gamma_i^s\right)dW^i(s),$$

because by Step 1, we know that $\mathcal{A}_\mathbf{V}\omega = 0$. As in the calculation (4.7), we may choose as in (4.6) for the arguments of $\hat{d}\gamma^i$ to deduce

$$d\bar{M}_t = \sqrt{2\nu}\sum_{i=1}^d \bar{Q}_t^*\zeta_i^t dW^i(t),$$

where ζ_i^t is really $\hat{d}\gamma_i^t$ but now regarded as a 2-form in $\mathbb{R}^d$. Note

$$\zeta_i^t(a_1, a_2) = \mathcal{C}\left(w_{q_i}(\cdot, t)\right)a_1 \cdot a_2.$$

Let us write Z_t and Y_t^i for the antisymmetric matrices associated with the forms $\bar{Q}_t^*(\hat{d}\bar{\alpha}_t)$ and $\bar{Q}_t^*\zeta_i^t$, respectively. We then have

$$dZ_t = \sqrt{2\nu}\sum_{i=1}^d Y_t^i dW^i(t).$$

From this, we readily deduce that the quadratic variation of the process Z_t is given by

$$2\nu \int_0^t \sum_{i=1}^d \|Y_s^i\|^2 ds.$$

We now reverse time as in Step 2 to complete the proof of Part (ii). Part (iii) is an immediate consequence of the identity

$$\mathbb{E}\|Z_T\|^2 = \mathbb{E}\|Z_0\|^2 + 2\nu\mathbb{E}\int_0^T \sum_{i=1}^d \|Y_s^i\|^2 ds. \qquad \square$$

Example 4.1. In this example, we examine the consequences of Theorem 1.1(i) for the axisymmetric solutions of Navier–Stokes equation when $d = 3$. To ease the notation, we assume that w is a backward solution of Navier–Stokes equation (4.2). We follow the notation of Example 2.1 and write $w = ae(\theta) + cf(\theta) + be_3$, where a, b, and c are

functions of (r, z, t). As before, let us start from the SDE

$$dq = w(q, t)dt + v'dW, \tag{4.8}$$

with $W = (W^1, W^2, W^3)$ and $v' = \sqrt{2v}$. In cylindrical coordinates,

$$dr = \big[a(r, z, t) + vr^{-1}\big]dt + v'dW_r(t),$$
$$dz = b(r, z, t)dt + v'dW^3(t),$$
$$d\theta = \hat{c}(r, z, t)dt + v'r^{-1}dW_\theta(t), \tag{4.9}$$

where $W_r = e(\theta) \cdot W$ and $W_\theta = W \cdot f(\theta)$. In cylindrical coordinates, the SDE (4.9) has an infinitesimal generator of the form

$$\mathcal{A} = a\partial_r + b\partial_z + \hat{c}\partial_\theta + v\big(\partial_r^2 + r^{-1}\partial_r + \partial_z^2 + r^{-2}\partial_\theta^2\big) =: \hat{\mathcal{A}} + \hat{c}\partial_\theta + vr^{-2}\partial_\theta^2.$$

We note that the triplet (W_r, W_θ, W^3) consists of 3 (statistically) independent Brownian motions. However, we cannot apply Proposition 3.1 because the Brownian motions W_θ and W_r would depend on the initial choice of (r, θ). In order to apply Proposition 3.1, it is very essential that we use the same copy of Brownian motions for all starting $q(0)$. This is our assumption in (4.8) and would not be the case in (4.9). For a comparison, let us consider a modified system of the form

$$dr = \big[a(r, z, t) + vr^{-1}\big]dt + v'dB^1(t),$$
$$dz = b(r, z, t)dt + v'dB^2(t),$$
$$d\theta = \hat{c}(r, z, t)dt + v'r^{-1}dB^3(t), \tag{4.10}$$

where B^1, B^2 and B^3 are 3 independent standard Brownian motions. Here we use the same $B = (B^1, B^2, B^3)$ for all initial $q(0)$. If we write $\hat{Q}_t(r, z, \theta)$ for the flow of SDE (4.10), then

$$\hat{Q}_t(r, z, \theta) = \left(\psi_t(r, z), \theta + \int_0^t \hat{c}\big(\psi_s(r, z), s\big)ds + v' \int_0^t R_s^{-1}(r, z)dB^3(s) \right),$$

where $\psi_t(r, z) = (R_t(r, z), Z_t(r, z))$ is the flow of the first two equations of (4.10). As in Example 2.1, the divergence free condition $\nabla \cdot u = 0$ is equivalent to $\psi_t^* \gamma = \gamma$ for the area form $\gamma = r\hat{d}r \wedge \hat{d}z$. This can be shown with the aid of Proposition 3.1 as in Example 3.1(iii); this time we show that $\mathcal{L}_v \gamma = 0$, where $v = (a + vr^{-1}, b)$. As in Example 2.1, we write $\alpha_t = \hat{\alpha}_t + \bar{c}(\cdot, t)\hat{d}\theta$. We have

$$Q_t^*\alpha_t = \psi_t^*\hat{\alpha}_t + (\bar{c}_t \circ \psi_t)\hat{d}\theta + (\bar{c}_t \circ \psi_t) \int_0^t \hat{d}(\hat{c}_s \circ \psi_s)ds + \int_0^t \hat{d}R_s^{-1}dB^3(s).$$

Now, according to Theorem 1.1(i), the process $M_t := \hat{d}(Q_t^*\alpha_t)$, is a martingale. This is no longer true for the process $\hat{M}_t := \hat{d}(\hat{Q}_t^*\alpha_t)$. Indeed the component $\bar{c}(r, z, t)$ satisfies

$$\partial_t\bar{c} + \hat{\mathcal{A}}\bar{c} = 2r^{-1}\partial_r\bar{c},$$

which in turn implies that the process

$$\bar{c}_t \circ \psi_t - 2 \int_0^t R_s^{-1}\partial_r\bar{c}_s \circ \psi_s ds,$$

is a martingale.   □

Conflict of interest statement

The author does not know of any specific conflict of interest for this article.

Acknowledgements

Appendix A

A.1. Contact diffusions

Recall that contact forms are certain 1-forms that are non-degenerate in some rather strong sense. To explain this, recall that when α is a contact form in dimension $n = 2d + 1$, then the set $l_x = \{v : d\alpha_x(v, w) = 0 \text{ for all } w \in T_x M\}$ is a line. Also, if we define the kernel of α by

$$\eta_x^\alpha = \eta_x = \big\{v : \alpha(x; v) = 0\big\},$$

then the contact condition really means that l_x and η_x give a decomposition of $\mathbb{R}^n$ that depends solely on α:

$$\mathbb{R}^n = \eta_x \oplus l_x. \tag{A.1}$$

We also define the Reeb vector field $R(x) = R^\alpha(x)$ to be the unique vector such that

$$R(x) \in l_x, \qquad \alpha\big(x; R(x)\big) = 1.$$

The role of Hamiltonian vector fields in the contact geometry are played by contact vector fields.

Definition A.1. A vector field X is called an α-contact vector field if $\mathcal{L}_X \alpha = f\alpha$ for some scalar-valued positive continuous function f. $\quad \square$

It is known that for a given a function $H : M \to \mathbb{R}$, there exists a unique contact α-vector field $X_H = X_H^\alpha$ such that $i_{X_H} \alpha = \alpha(X_H) = H$. The vector field X_H is the analog of Hamiltonian vector field in contact geometry (with H playing the role of Hamiltonian function). The function f can be expressed in terms of H with the aid of the Reeb vector field $R = R^\alpha$; indeed, $f = dH(R^\alpha)$, and as a result,

$$\mathcal{L}_{X_H} \alpha = dH\big(R^\alpha\big)\alpha.$$

In our Euclidean setting, we consider a form $\alpha = u \cdot dx$ for a vector field u and

$$\beta(v_1, v_2) := d\alpha(v_1, v_2) = \mathcal{C}(u)v_1 \cdot v_2,$$

where $\mathcal{C}(u) = Du - (Du)^*$. (Recall that we are writing A^* for the transpose of A.) Since $C^* = -C$, we have that $\det \mathcal{C} = (-1)^n \det \mathcal{C}$. This implies that C cannot be invertible if the dimension is odd. Hence the null space l_x of $\mathcal{C}(u)(x)$ is never trivial and our assumption $\dim \ell_x = 1$ really means that this null space has the smallest possible dimension. Now (A.1) simply means that $u(x) \cdot R(x) \neq 0$. Of course R is chosen so that $u(x) \cdot R(x) \equiv 1$. Writing $u^\perp$ and $R^\perp$ for the space of vectors perpendicular to u and R respectively, then $\eta = u^\perp$, and we may define a matrix $\mathcal{C}'(u)$ which is not exactly the inverse of $\mathcal{C}(u)$ (because $\mathcal{C}(u)$ is not invertible), but it is specified uniquely by two requirements:

 (i) $\mathcal{C}'(u)$ restricted to $R^\perp$ is the inverse of $\mathcal{C}(u) : u^\perp \to R^\perp$.
(ii) $\mathcal{C}'(u)R = 0$.

The contact vector field associated with H is given by

$$X_H = -\mathcal{C}'(u)\nabla H + HR.$$

In particular, when $n = 3$, the form $\alpha = u \cdot dx$ is contact if and only if $u \cdot \xi$ is never 0, where $\xi = \nabla \times u$ is the curl (vorticity) of u. In this case the Reeb vector field is given by $R = \xi/(u \cdot \xi)$, and

$$\mathcal{L}_Z u = \nabla(u \cdot Z) + \xi \times Z.$$

We also write $\bar{u} = u/(u \cdot \xi)$. The contact vector field associated with H is given by

$$X_H = \bar{u} \times \nabla H + HR.$$

Let $x(t)$ be a diffusion satisfying (3.1) and assume that this diffusion has a random flow ϕ_t. Given a contact form α, set $\alpha^t = \phi_t^* \alpha$ as before.

Definition A.2.

(i) We say that the diffusion (3.1) is α-contact, if for some scaler-valued semimartingale Z_t of the form,

$$dZ_t = g_0(x(t), t)dt + \sum_{i=1}^{k} g_i(x(t), t) \circ dW^i(t), \tag{A.2}$$

for $g_0, \ldots, g_k > 0$, we have

$$d\alpha^t = \alpha^t dZ_t.$$

(ii) We say that the diffusion (3.1) is α-semicontact, if there exists a continuous scalar-valued function positive $f(x, t)$ such that

$$M_t = \alpha^t - \int_0^t f(x(s), s)\alpha^s ds,$$

is a martingale. $\square$

We end this subsection with two propositions.

Proposition A.1. *The following statements are equivalent:*

(i) *The diffusion (3.1) is α-contact.*
(ii) *There exists a scaler-valued process A_t of the form*

$$dA_t = h_0(x(t), t)dt + \sum_{i=1}^{k} h_i(x(t), t) \circ dW^i(t),$$

with $h_1, \ldots, h_k > 0$ and $h_0 + \sum_{i=1}^{k} h_i^2 > 0$, such that $\alpha^t = e^{A_t}\alpha$.
(iii) *The vector fields $V_0, \ldots, V_k$ are α-contact.*

Proposition A.2. *The following statements are equivalent:*

(i) *The diffusion (3.1) is α-semicontact.*
(ii) *For some continuous scalar-valued positive function $f(x, t)$, we have $\mathcal{A}_{\mathbf{V}}\alpha = f\alpha$.*

The proof of Proposition A.2 is omitted because it is an immediate consequence of (3.4) and the definition.

Proof of Proposition A.1. Suppose that the vector fields $V_0, \ldots, V_k$ are α-contact. Then there exist scalar-valued functions $g_0(x, t), \ldots, g_k(x, t)$ such that $\mathcal{L}_{V_i}\alpha = g_i\alpha$. From this and Proposition 3.1 we learn that $d\alpha^t = \alpha^t dZ_t$ for Z_t as in (A.2). Hence (iii) implies (i).

Now assume (i) and set

$$Y_t = \exp\left(-Z_t + \frac{1}{2}[Z]_t\right),$$

where $[Z]$ denotes the quadratic variation of Z. We have

$$
\begin{aligned}
dY_t &= Y_t\left(-dZ_t + d[Z]_t\right), \\
d\left(Y_t\alpha^t\right) &= \alpha^t Y_t\left(-dZ_t + d[Z]_t\right) + Y_t d\alpha^t + d[Y,\alpha]_t \\
&= \alpha_t Y_t d[Z]_t + d[Y,\alpha]_t = \alpha_t Y_t d[Z]_t - \alpha_t Y_t d[Z]_t = 0,
\end{aligned}
$$

with $[Y,\alpha]_t$ denoting the quadratic co-variation process of Y_t and α^t. Hence $Y_t\alpha^t = \alpha$ and we have (ii) for $A_t = Z_t - \frac{1}{2}[Z]_t$.

We now assume (ii). We certainly have

$$
\begin{aligned}
d\alpha^t &= \alpha e^{A_t}\left(dA_t + \frac{1}{2}d[A]_t\right) = \alpha^t\left(dA_t + \frac{1}{2}d[A]_t\right) \\
&= \alpha^t\left(g_0\big(x(t),t\big)dt + \sum_{i=1}^{k} g_i\big(x(t),t\big) \circ dW^i(t)\right),
\end{aligned}
$$

for $g_0 = h_0 + \frac{1}{2}(\sum_i h_i^2)$ and $g_i = h_i$ for $i = 1,\ldots,k$. Comparing this to (3.4) yields $\mathcal{L}_{V_i}\alpha = g_i\alpha$ for $i = 0,\ldots,k$. Hence (iii) is true and this completes the proof. $\square$

A.2. Viscosity limit

In this subsection, we make some formal reasoning to support our comments in Remark 1.1(iii). For a weak solution of the Euler equation (1.15) it is not clear how to make sense of Circulation Formula (1.10) or (2.3). The main challenge is that for a rough vector field u that satisfies (1.15) in some weak sense, we do not have a satisfactory candidate for the flow of the ODE (1.9). As it turns out, it is much easier to construct a regular flow for SDE (1.16) (see Remark 1.1(i)). Motivated by this, let us start with a diffusion $x(t) = (q(t), p(t))$ satisfying

$$
\begin{aligned}
dq &= H_p(q, p, t)dt + \sqrt{2\nu}\,dW, \\
dp &= -H_q(q, p, t)dt + \sqrt{2\nu}\,\Gamma(q,t)dW,
\end{aligned}
\tag{A.3}
$$

and examine the Circulation Formula for it in low ν limit. Writing ω for the realization of the Brownian motion in (A.3) and $X_t^\nu(x; \omega) = X_t(x) = (Q_t(x), P_t(x))$ for the corresponding flow, the map $\omega \mapsto X^\nu(\cdot; \omega)$ induces a probability measure $\mathcal{P}_\nu$ on the space of measurable maps $X : \mathbb{R}^{2d} \times [0, T] \to \mathbb{R}^{2d}$. Let us write $\mathcal{F}_t$ for the σ-field generated by $(W(s) : 0 \le s \le t)$. From (A.3), it is not hard to see that $\mathcal{F}_t$ coincides with the σ-field generated by $(X_s(x) : 0 \le s \le t)$ for any given x. By Corollary 3.1,

$$
\int\left\{\int\left[\left(DQ_t(x)\right)^* P_t(x) - \left(DQ_s(x)\right)^* P_s(x)\right] \cdot J\big(x, X.(\cdot)\big)dx\right\} \mathcal{P}_\nu\big(dX.(\cdot)\big) = 0,
\tag{A.4}
$$

provided that $s \le t$, and $J(x, X.(\cdot))$ is a vector field that is divergence-free, smooth and of compact support in x-variable, and is $\mathcal{F}_s$ measurable in $X.(\cdot)$-variable. The question is what regularity of X is needed to guarantee a passage to the limit $\nu \to 0$. As was observed by Viterbo [18], we can make sense of the integral in the curly brackets if both P and Q possess half a derivative in L^2. If, for example we can control $H^{\frac{1}{2}}$-norm of X_t uniformly in t, then we can pass to the limit in (A.4) and assert that any limit point $\mathcal{P}$ of $\mathcal{P}_\nu$ in low ν limit is the law of the flow of a process $x(\cdot)$ that is $\bar{\omega}$-semisymplectic in some weak sense. We remark that in view of our proof in Section 4 and the system (4.1), the kind of Hamiltonian function $H(q, p, t)$ and $\Gamma(q, t)$ we have in mind depend on ν and are given by $|p|^2/2 + P(q, t)$ and $Du(q, t)$ respectively. For such choices of Hamiltonian function and Γ, we do not know how to

control the $H^{\frac{1}{2}}$-norm of the flow. We also note that in general the only known bound on Γ is

$$\nu \int_0^t |Du(q,s)|^2 dq ds < \infty,$$

which is not strong enough to imply that the martingale $\int_0^t \sqrt{2\nu} \Gamma(q(s),s) dW(s)$ goes to 0 as $\nu \to \infty$.

References

[1] S.L. Bragg, W.R. Hawthorne, Some exact solutions of the flow through annular cascade actuator discs, J. Aeronaut. Sci. 17 (1950) 243–249.
[2] P. Constantin, Euler equations, Navier–Stokes equations and turbulence, in: Mathematical Foundation of Turbulent Viscous Flows, in: Lect. Notes Math., vol. 1871, Springer, Berlin, 2006, pp. 1–43.
[3] P. Constantin, G. Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations, Commun. Pure Appl. Math. LXI (2008) 330–345.
[4] J. Deng, T.Y. Hou, X. Yu, A level set formulation for the 3D incompressible Euler equations, Methods Appl. Anal. 12 (4) (2005) 427–440.
[5] Y. Eliashberg, Classification of contact structures on $\mathbb{R}^3$, Int. Math. Res. Not. 3 (1993) 87–91.
[6] G.L. Eyink, Turbulent diffusion of lines and circulations, Phys. Lett. A 368 (2007) 486–490.
[7] G.L. Eyink, A stochastic least-action principle for the incompressible Navier–Stokes equation, Physica D 239 (2010) 1236–1240.
[8] H. Geiges, An Introduction to Contact Topology, Cambridge University Press, Cambridge, 2008.
[9] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82 (1985) 307–347.
[10] H. Hofer, E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser Verlag, Basel, 2011.
[11] H. Kunita, On the decomposition of solutions of stochastic differential equations, in: D. Williams (Ed.), Stochastic Integrals, in: Lect. Notes Math., vol. 851, Springer-Verlag, Berlin, Heidelberg, New York, 1981.
[12] H. Kunita, On backward stochastic differential equations, Stochastica 6 (1982) 293–313.
[13] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge university Press, Cambridge, 1990.
[14] K. Ohkitani, A geometrical study of 3D incompressible Euler flows with Clebsch potentials—a long-lived Euler flow and its power-law energy spectrum, Physica D 237 (14–17) (2008) 2020–2027.
[15] F. Rezakhanlou, Regular flows for diffusions with rough drifts, arXiv:1310.8353.
[16] F. Rezakhanlou, Lectures on symplectic geometry, www.math.berkeley.edu/rezakhan.
[17] L.C.G. Rogers, D. Williams, Diffusions, Markov Processes, and Martingales, vol. 2. Itô calculus, Cambridge University Press, Cambridge, 2000.
[18] C. Viterbo, Metric and isoperimetric problems in symplectic geometry, J. Am. Math. Soc. 13 (2) (2000) 411–431.