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On the planar Schrödinger–Poisson system

Silvia Cingolani [a,*], Tobias Weth [b]

[a] *Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125 Bari, Italy*
[b] *Institut für Mathematik, Goethe-Universität Frankfurt, D-60054 Frankfurt am Main, Germany*

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Abstract

We develop a variational framework to detect high energy solutions of the planar Schrödinger–Poisson system

$$\begin{cases} -\Delta u + a(x)u + \gamma w u = 0, \\ \Delta w = u^2 \end{cases} \quad \text{in } \mathbb{R}^2$$

with a positive function $a \in L^\infty(\mathbb{R}^2)$ and $\gamma > 0$. In particular, we deal with the periodic setting where the corresponding functional is invariant under $\mathbb{Z}^2$-translations and therefore fails to satisfy a global Palais–Smale condition. The key tool is a surprisingly strong compactness condition for Cerami sequences which is not available for the corresponding problem in higher space dimensions. In the case where the external potential a is a positive constant, we also derive, as a special case of a more general result, the existence of nonradial solutions (u, w) such that u has arbitrarily many nodal domains. Finally, in the case where a is constant, we also show that solutions of the above problem with $u > 0$ in $\mathbb{R}^2$ and $w(x) \to -\infty$ as $|x| \to \infty$ are radially symmetric up to translation. Our results are also valid for a variant of the above system containing a local nonlinear term in u in the first equation.

MSC: 35J50; 35Q40

Keywords: Schrödinger–Poisson system; Logarithmic convolution potential; Standing wave solutions

1. Introduction

The present paper is concerned with standing (or solitary) wave solutions of Schrödinger–Poisson systems of the type

$$\begin{cases} i\psi_t - \Delta\psi + E(x)\psi + \gamma w \psi = 0, \\ \Delta w = |\psi|^2 \end{cases} \quad \text{in } \mathbb{R}^d \times \mathbb{R}. \tag{1.1}$$

* Corresponding author.
E-mail addresses: silvia.cingolani@poliba.it (S. Cingolani), weth@math.uni-frankfurt.de (T. Weth).

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Here $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is the (time-dependent) wave function, $x \mapsto E(x)$ is a real external potential, and $\gamma \in \mathbb{R}$ is a parameter. The function w represents an internal potential for a nonlocal self-interaction of the wave function ψ. The standing wave ansatz $\psi(x,t) = e^{-i\lambda t}u(x)$, $\lambda \in \mathbb{R}$ reduces (1.1) to the system

$$\begin{cases} -\Delta u + a(x)u + \gamma\, wu = 0, \\ \Delta w = u^2 \end{cases} \quad \text{for } u : \mathbb{R}^d \to \mathbb{R} \tag{1.2}$$

with $a(x) = E(x) + \lambda$. The second equation determines $w : \mathbb{R}^d \to \mathbb{R}$ only up to harmonic functions, but it is natural to choose w as the Newton potential of u^2, i.e., the convolution of u^2 with the fundamental solution Φ_d of the Laplacian. With this formal inversion of the second equation in (1.2), we obtain the integro-differential equation

$$-\Delta u + a(x)u + \gamma\big[\Phi_d * |u|^2\big]u = 0 \quad \text{in } \mathbb{R}^d, \tag{1.3}$$

where $\Phi_d(x) = \frac{1}{d(2-d)\omega_d}|x|^{2-d}$ in case $d \geq 3$ and $\Phi_d(x) = \frac{1}{2\pi}\log|x|$ in case $d = 2$. Here, as usual, ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

In the three dimensional case, (1.3) has been extensively studied. For $d = 3$, $a \equiv \lambda > 0$ and $\gamma > 0$, it was introduced by Pekar [27] in 1954 for describing the quantum mechanics of a polaron at rest and by Choquard in 1976 for describing an electron trapped in its hole. In [20], Lieb proved the existence of a unique ground state of (1.2) which is positive and spherically symmetric, using a minimization argument. In [22], Lions proved the existence of infinitely many distinct spherically symmetric solutions when $a(x)$ is a nonnegative, radially symmetric potential. In [28] Penrose derived (1.3) in his discussion about the self gravitational collapse of a quantum-mechanical system. Recently, existence and regularity results have also been obtained for $a(x) \not\equiv \lambda$ and for more general convolution potentials, see [1,13,18,23,25].

In some recent works, local nonlinear terms of the form $b|u|^{p-2}u$, $b \in \mathbb{R}$, $p > 2$ have been added to the right hand side of (1.3). Those nonlinear terms are frequently used in Schrödinger equations to model the interaction among particles [7,9]. A large amount of papers are devoted to the study of existence of solutions of the corresponding Schrödinger–Poisson system (also called Schrödinger–Maxwell system) in the three dimensional case. In particular for $\gamma < 0$, $b > 0$, $2 < p < 6$, we quote the results [3,4,16,30,6]. Conversely, for $\gamma > 0$ and $b < 0$, the corresponding system has been studied in [8] and it represents a Hartree model for crystals (see also [26]).

The literature is scantier for the planar case $d = 2$, which is the focus of the present paper. In this case, Masaki [24] proved global well-posedness of the Cauchy problem for (1.1) in a subspace of $H^1(\mathbb{R}^2)$. Moreover, Stubbe and Vuffray [12] established existence and uniqueness of positive, spherically symmetric solutions of (1.2) in the case $a \equiv \lambda$, $d \leq 6$, using shooting methods for the associated ODE system (see also [11] for the one-dimensional case). Unlike in the case $d = 3$, variational methods have rarely been used in the planar case in which (1.3) becomes

$$-\Delta u + au + \frac{\gamma}{2\pi}\big(\log(|\cdot|) * |u|^2\big)u = 0 \quad \text{in } \mathbb{R}^2. \tag{1.4}$$

Note that, at least formally, (1.3) has a variational structure related to the energy functional

$$u \mapsto \frac{1}{2}\int_{\mathbb{R}^2} \big(|\nabla u|^2 + a(x)u^2\big)\,dx + \frac{\gamma}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Phi_d\big(|x-y|^2\big)u^2(x)u^2(y)\,dx\,dy.$$

In case $d \geq 3$ and $a \in L^\infty(\mathbb{R}^d)$, this functional is well-defined and of class C^1 on $H^1(\mathbb{R}^d)$, but it is not well defined on $H^1(\mathbb{R}^2)$ in case $d = 2$. Considering the case $d = 2$, $a \equiv \lambda \in \mathbb{R}$ and $\gamma > 0$, Stubbe [32] set up a variational framework for (1.4) within a subspace of $H^1(\mathbb{R}^2)$. By using strict rearrangement inequalities, he proved that there exists, for any $\lambda \geq 0$, a unique ground state, which is a positive spherically symmetric decreasing function. In addition, he proved that there exists a negative number λ^* such that for any $\lambda \in (\lambda^*, 0)$ there exist two ground states with different L^2 norm. In the limiting case $\lambda = \lambda^*$, there is again a unique ground state.

In the present work we focus on (1.4) in the case $\gamma > 0$, and by rescaling we may assume $\gamma = 2\pi$. More precisely, we will consider a generalization of (1.4) given by

$$-\Delta u + a(x)u + \big(\log(|\cdot|) * |u|^2\big)u = b|u|^{p-2}u \quad \text{in } \mathbb{R}^2 \tag{1.5}$$

with $b \geq 0$, $p \geq 2$ and $a \in L^\infty(\mathbb{R}^2)$. We wish to emphasize that all of our results are new even for $b = 0$, i.e., for Eq. (1.4). As remarked before, the energy functional

$$u \mapsto I(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + a(x)u^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(|x-y|^2\right) u^2(x)u^2(y)\, dx dy - \frac{b}{p}\int_{\mathbb{R}^2} |u|^p\, dx$$

is not well defined on $H^1(\mathbb{R}^2)$. Inspired by Stubbe [32], we will consider the smaller Hilbert space

$$X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|) u^2(x)\, dx < \infty \right\}.$$

It is not difficult to see that I defines a C^1-functional on X. Moreover, critical points $u \in X$ of I are strong solutions of (1.5) in $W^{2,p}(\mathbb{R}^2)$ for all $p \geq 1$, and they are classical solutions in $C^2(\mathbb{R}^2)$ if a is Hölder continuous, see Proposition 2.3 below. So X provides a variational framework for (1.5), but some difficulties appear due to the following unpleasant facts. First, the norm of X is *not* translation invariant, whereas the functional I is invariant under every translation which leaves the external potential a invariant. Second, the quadratic part of I is not coercive on X even if $\inf_{\mathbb{R}^2} a > 0$. These difficulties enforce the implementation of new ideas and estimates within the variational framework. On the other hand, somewhat surprisingly, the specific form of I also allows to establish much better existence and multiplicity results than those available in the case $d \geq 3$ or for the simpler equation

$$-\Delta u + a(x)u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^2). \tag{1.6}$$

In fact, our main results suggest that the structure of the solution set of (1.5) is in general much richer than the one of (1.6) or related problems in higher dimensions.

Our first main result is concerned with the periodic setting.

Theorem 1.1. *Suppose that $p \geq 4$, $b \geq 0$, and that $a : \mathbb{R}^2 \to (0,\infty)$ is continuous and $\mathbb{Z}^2$-periodic. Then (1.5) admits a sequence of solution pairs $\pm u_n \in X$ such that $I(u_n) \to \infty$ as $n \to \infty$. Moreover, the restriction of I to the associated Nehari manifold $\mathcal{N} := \{u \in X \setminus \{0\} : I'(u)u = 0\}$ attains a global minimum, and every minimizer $u \in \mathcal{N}$ of $I|_{\mathcal{N}}$ is a solution of (1.5) which does not change sign and obeys the variational characterization*

$$I(u) = \inf_{u \in X} \sup_{t \in \mathbb{R}} I(tu).$$

We remark that Theorem 1.1 yields infinitely many geometrically distinct solutions (i.e., solutions generating different $\mathbb{Z}^2$-orbits) and also ground state solutions of (1.5), i.e., energy minimizers within the set of nontrivial solutions. This seems to be the first existence result for (1.5) in the periodic setting even in the special case $b = 0$. We also note that a result on the existence of solutions with arbitrarily high energy is not available for (1.6) without additional nondegeneracy assumptions (see e.g. [15]).

Our second main result is concerned with a different symmetric setting in the case where a is a positive constant. We need to fix some notation. Let G be a closed subgroup of the orthogonal group $O(2)$ such that

$$\mathrm{Fix}(G) = \{0\}, \quad \text{where } \mathrm{Fix}(G) := \left\{x \in \mathbb{R}^2 : Ax = x \text{ for all } A \in G\right\}. \tag{1.7}$$

Moreover, let $\tau : G \to \{-1, 1\}$ be a group homomorphism. The pair (G, τ) gives rise to a group action of G on X defined by

$$A * u(x) := \tau(A)u\left(A^{-1}x\right) \quad \text{for } A \in G,\ u \in X,\ \text{and } x \in \mathbb{R}^2. \tag{1.8}$$

The following result is concerned with solutions of (1.5) in the invariant subspace

$$X_G := \{u \in X : A * u = u \text{ for all } A \in G\}.$$

Theorem 1.2. *Suppose that $p \geq 4$, $b \geq 0$, and that a in (1.5) is a positive constant. Let G, τ be as above, and suppose that $\tau \equiv 1$ or that G is finite. Then (1.5) admits a sequence of solution pairs $\pm u_n \in X_G$ such that $I(u_n) \to \infty$ as $n \to \infty$. Moreover, the restriction of I to the associated Nehari manifold $\mathcal{N}_G := \{u \in X_G \setminus \{0\} : I'(u)u = 0\}$ attains a global minimum, and every minimizer $u \in \mathcal{N}_G$ of $I|_{\mathcal{N}_G}$ is a G-invariant solution of (1.5) which obeys the variational characterization*

$$I(u) = \inf_{u \in X_G} \sup_{t \in \mathbb{R}} I(tu),$$

and it does not change sign if $\tau \equiv 1$.

Some remarks seem appropriate. The assumptions on G and τ ensure that the closed subspace $X_G \subset X$ is infinite dimensional. If τ is nontrivial and $A \in G$ is given with $\tau(A) = -1$, then every $u \in X_G$ satisfies $u(A^{-1}x) = -u(x)$. In particular, u vanishes on the set $\{x \in \mathbb{R}^2 : Ax = x\}$ and changes sign in $\mathbb{R}^2$ if $u \neq 0$. Let us briefly discuss some examples. In case $G = O(2)$, $\tau \equiv 1$ the space X_G consists of radial functions in X. In this case Theorem 1.2 yields the existence of infinitely many radial solutions. Next, let $G = \{\mathrm{id}, A_1, A_2\}$, where A_i is the reflection at the coordinate hyperplane $\{x_i = 0\}$ for $i = 1, 2$. Moreover, let $\tau : G \to \{-1, 1\}$ be the homomorphism defined by $\tau(A_1) = -1$ and $\tau(A_2) = 1$ (this defines a homomorphism since A_1 and A_2 commute). Then $u \in X_G$ if and only if

$$u(-x_1, x_2) = -u(x_1, x_2) \quad \text{and} \quad u(x_1, -x_2) = u(x_1, x_2) \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \tag{1.9}$$

Hence Theorem 1.2 yields a sequence of solution pairs $\pm u_n$, $n \in \mathbb{N}$ of (1.5) with unbounded energy and such that u_n is odd with respect to the hyperplane $\{x_1 = 0\}$ and even with respect to the hyperplane $\{x_2 = 0\}$. We point out that such a result is not available for (1.6) in the case where $a > 0$ is constant; in fact, (1.6) does not admit any nontrivial solutions which vanish on a hyperplane, see e.g. [17]. As a third example we may consider, for given $k \in \mathbb{N}$, the subgroup G of $O(2)$ of order $2k$ generated by the (counter-clockwise) $\frac{\pi}{k}$-rotation

$$A \in O(2), \qquad A(x) = (x_1 \cos \pi/k - x_2 \sin \pi/k, x_1 \sin \pi/k + x_2 \cos \pi/k) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Let $\tau : G \to \{-1, 1\}$ be the homomorphism defined by $\tau(A^j) = (-1)^j$ for $j \in \mathbb{N}$. Then Theorem 1.2 applies and yields sign changing solutions which are invariant under the corresponding action defined by (1.8). Note that any such solution has at least $2k$ (conical) nodal domains in $\mathbb{R}^2$. We also point out that the assumption (1.7) is not essential for the existence of the sequence $(u_n)_n$ in Theorem 1.2, since it can always be achieved by enlargening G suitably. More precisely, if $F := \mathrm{Fix}(G) \neq \{0\}$, we may consider the orthogonal splitting $\mathbb{R}^2 := E \oplus F$ and define $B \in O(2) \setminus G$ by $Bx = -x$ for $x \in F$ and $Bx = x$ for $x \in E$. Then B is an involution which commutes with every element of G, so we may consider $\tilde{G} = G \cup \{B\}$ and $\tilde{\tau} : \tilde{G} \to \{-1, 1\}$ defined by $\tilde{\tau}|_G = \tau$ and $\tilde{\tau}(B) = -1$ in place of G and τ. Denoting by $X_{\tilde{G}}$ the invariant subspace of X with respect to the corresponding group action defined by (1.8) with $\tilde{G}$, $\tilde{\tau}$ in place of G, τ, we then have $\mathrm{Fix}(\tilde{G}) = 0$ and $X_{\tilde{G}} \subset X_G$.

In view of the fact that, by the results above, (1.5) has a large solution set in general, it is a natural aim to classify different types of solutions via their geometric properties. The first step within this rather ambitious program is the study of the shape and possible uniqueness of positive solutions (1.5). Related to this, we have the following result.

Theorem 1.3. *Suppose that $p \geq 2$, and that a is a positive constant in (1.5). Then every positive solution $u \in X$ of (1.5) is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover, if $b = 0$, then u is unique up to translation.*

We note that Ma and Zhao [23] proved the corresponding statement for (1.3) in the case $d = 3$, $a \equiv 1$ and $\gamma = 1$. Their approach relies on a reformulation of (1.3) as an integral equation and a moving plane argument. The results of [23] apply to a general class of integral equations, but they do not apply to (1.5) since the logarithmic integral kernel is sign-changing. Our proof of Theorem 1.3 relies on a more direct and simpler variant of the moving plane method for the corresponding system, see Section 6 below. We also point out that the uniqueness claim in Theorem 1.3 is a simple consequence of the uniqueness result for positive, radial solutions in [12].

The paper is organized as follows. In Section 2 we set up the variational framework and establish useful preliminary estimates. We also address the notion and regularity of weak solutions, see Proposition 2.3 below. Section 3 contains the key compactness condition, Proposition 3.1 which is fundamental for our existence results. Roughly speaking, Proposition 3.1 asserts that the functional I satisfies the Cerami condition at arbitrary energy levels' up to translations' if a is periodic and positive. In Section 4 we then prove Theorem 1.1. The proof is based on deformation arguments which differ considerably from previous works. In particular, we need to implement new estimates on the Krasnoselski genus of neighborhoods of the *noncompact* set of critical points of I at a given energy level. Moreover, special care is needed at some points since the norm of X is not translation invariant. Section 5 is devoted to the proof of

Theorem 1.2. The key observation here is the fact that, in the G-invariant setting described above, the restriction of I to the invariant space X_G satisfies the Cerami condition. Finally, in Section 6 we will prove a slightly more general variant of Theorem 1.3, and we will deduce Theorem 1.3 in the end of this section.

2. Preliminaries

In the following, we assume that $a \in L^\infty(\mathbb{R}^2)$ satisfies $\inf_{\mathbb{R}} a > 0$. Then we may endow $H^1(\mathbb{R}^2)$ with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^2} \big(\nabla u \cdot \nabla v + a(x)uv\big)\, dx, \quad \text{for } u, v \in H^1\big(\mathbb{R}^2\big).$$

The corresponding norm given by

$$\|u\|^2 := \int_{\mathbb{R}^2} \big(|\nabla u|^2 + a(x)u^2\big)dx, \quad \text{for } u \in H^1\big(\mathbb{R}^2\big)$$

is equivalent to the standard $H^1(\mathbb{R}^2)$-norm. We define the symmetric bilinear forms

$$(u, v) \mapsto B_1(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\big(1 + |x-y|\big)u(x)v(y)\, dx dy,$$

$$(u, v) \mapsto B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right)u(x)v(y)\, dx dy,$$

$$(u, v) \mapsto B_0(u, v) = B_1(u, v) - B_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\big(|x-y|\big)u(x)v(y)\, dx dy.$$

Here the definition is restricted, in each case, to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Note that, since $0 \le \log(1 + r) \le r$ for $r > 0$, we have by the Hardy–Littlewood–Sobolev inequality [21]

$$\big|B_2(u, v)\big| \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}\big|u(x)v(y)\big|\, dx dy \le C_0 |u|_{\frac{4}{3}} |v|_{\frac{4}{3}} \quad \text{for } u, v \in L^{\frac{4}{3}}\big(\mathbb{R}^2\big) \tag{2.1}$$

with a constant $C_0 > 0$. Here and in the following, $|u|_p$ stands for the L^p-norm of a function $u \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$. We now define the functionals

$$V_1 : H^1\big(\mathbb{R}^2\big) \to [0, \infty], \qquad V_1(u) = B_1\big(u^2, u^2\big) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\big(1 + |x-y|\big)u^2(x)u^2(y)\, dx dy,$$

$$V_2 : L^{\frac{8}{3}}\big(\mathbb{R}^2\big) \to [0, \infty), \qquad V_2(u) = B_2\big(u^2, u^2\big) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right)u^2(x)u^2(y)\, dx dy,$$

$$V_0 : H^1\big(\mathbb{R}^2\big) \to \mathbb{R} \cup \{\infty\}, \qquad V_0(u) = B_0\big(u^2, u^2\big) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\big(|x-y|\big)u^2(x)u^2(y)\, dx dy.$$

Note that, as a consequence of (2.1), we have

$$\big|V_2(u)\big| \le C_0 |u|_{\frac{8}{3}}^4 \quad \text{for all } u \in L^{\frac{8}{3}}\big(\mathbb{R}^2\big), \tag{2.2}$$

so V_2 only takes finite values on $L^{\frac{8}{3}}(\mathbb{R}^2) \subset H^1(\mathbb{R}^2)$. Next we define, for any measurable function $u : \mathbb{R}^2 \to \mathbb{R}$,

$$|u|_*^2 := \int_{\mathbb{R}^2} \log\big(1 + |x|\big)u^2(x)\, dx \in [0, \infty].$$

We note that, since

$$\log\bigl(1+|x-y|\bigr) \le \log\bigl(1+|x|+|y|\bigr) \le \log\bigl(1+|x|\bigr) + \log\bigl(1+|y|\bigr) \quad \text{for } x, y \in \mathbb{R}^2, \tag{2.3}$$

we have the estimate

$$\begin{aligned} B_1(uv, wz) &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bigl[\log\bigl(1+|x|\bigr) + \log\bigl(1+|y|\bigr)\bigr] \bigl|u(x)v(x)\bigr|\bigl|w(y)z(y)\bigr|\,dx\,dy \\ &\le |u|_*|v|_*|w|_2|z|_2 + |u|_2|v|_2|w|_*|z|_* \quad \text{for } u, v, w, z \in L^2\bigl(\mathbb{R}^2\bigr) \end{aligned} \tag{2.4}$$

with the conventions $\infty \cdot 0 = 0$ and $\infty \cdot s = \infty$ for $s > 0$. The following lemma will also be useful in the sequel.

Lemma 2.1. *Let $(u_n)_n$ be a sequence in $L^2(\mathbb{R}^2)$ such that $u_n \to u \in L^2(\mathbb{R}^2) \setminus \{0\}$ pointwise a.e. on $\mathbb{R}^2$. Moreover, let $(v_n)_n$ be a bounded sequence in $L^2(\mathbb{R}^2)$ such that*

$$\sup_{n \in \mathbb{N}} B_1\bigl(u_n^2, v_n^2\bigr) < \infty. \tag{2.5}$$

Then there exists $n_0 \in \mathbb{N}$ and $C > 0$ such that $|v_n|_* < C$ for $n \ge n_0$.

If, moreover,

$$B_1\bigl(u_n^2, v_n^2\bigr) \to 0 \quad \text{and} \quad |v_n|_2 \to 0 \quad \text{as } n \to \infty, \tag{2.6}$$

then

$$|v_n|_* \to 0 \quad \text{as } n \to \infty,\ n \ge n_0. \tag{2.7}$$

Proof. By assumption and Egorov's Theorem, there exists $n_0 \in \mathbb{N}$, $R, \delta > 0$ and a measurable subset $A \subset B_R(0)$ such that $|A| > 0$ and $u_n^2(x) \ge \delta$ for every $n \ge n_0$. Since

$$1 + |x-y| \ge 1 + \frac{|y|}{2} \ge \sqrt{1+|y|} \quad \text{for every } x \in B_R(0),\ y \in \mathbb{R}^2 \setminus B_{2R}(0),$$

we may then estimate

$$\begin{aligned} B_1\bigl(u_n^2, v_n^2\bigr) &\ge \int_{\mathbb{R}^2 \setminus B_{2R}(0)} \int_A \log\bigl(1+|x-y|\bigr) u_n^2(x) v_n^2(y)\,dx\,dy \\ &\ge \frac{\delta|A|}{2} \int_{\mathbb{R}^2 \setminus B_{2R}(0)} \log\bigl(1+|y|\bigr) v_n^2(y)\,dy = \frac{\delta|A|}{2}\biggl(|v_n|_*^2 - \int_{B_{2R}(0)} \log\bigl(1+|y|\bigr) v_n^2(y)\,dy\biggr) \\ &\ge \frac{\delta|A|}{2}\bigl(|v_n|_*^2 - \log(1+2R)|v_n|_2^2\bigr). \end{aligned}$$

Since $|v_n|_2^2$ and $B_1(u_n^2, v_n^2)$ remain bounded in n by assumption, it follows that $|v_n|_*$ also remains bounded in n, as claimed. If moreover (2.6) holds, then the estimate above yields (2.7). □

In the following, we fix $b \ge 0$, $p \ge 4$ and consider the functional

$$I : H^1\bigl(\mathbb{R}^2\bigr) \to \mathbb{R} \cup \{\infty\}, \qquad I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}V_0(u) - \frac{b}{p}|u|_p^p.$$

We also define the Hilbert space

$$X := \bigl\{u \in H^1\bigl(\mathbb{R}^2\bigr) : |u|_*^2 < \infty\bigr\} \quad \text{with norm given by } u \mapsto \|u\|_X^2 := \|u\|^2 + |u|_*^2.$$

Note that, by (2.4), the restriction of I to X (also denoted by I in the following) only takes finite values in $\mathbb{R}$. We have the following properties.

Lemma 2.2.

(i) *The space X is compactly embedded in $L^s(\mathbb{R}^2)$ for all $s \in [2,\infty)$.*
(ii) *The functionals V_0, V_1, V_2 and I are of class C^1 on X.*

Moreover, $V_i'(u)v = 4B_i(u^2, uv)$ for $u, v \in X$ and $i = 0, 1, 2$.

(iii) *V_2 is continuous (in fact continuously differentiable) on $L^{\frac{8}{3}}(\mathbb{R}^2)$.*
(iv) *V_1 is weakly lower semicontinuous on $H^1(\mathbb{R}^2)$.*
(v) *I is weakly lower semicontinuous on X.*
(vi) *I is lower semicontinuous on $H^1(\mathbb{R}^2)$.*

Proof. (i) follows from Rellich's Theorem (see Theorem XIII.65 in [29]).

We prove (ii) and (iii). Let u_n be a sequence in X converging to some $u \in X$. It follows that u_n is bounded and

$$\begin{aligned}
\big|V_1(u_n) - V_1(u)\big| &= \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)u_n^2(x)\big(u_n^2(y)-u^2(y)\big)dxdy \\
&\quad + \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)\big(u_n^2(x)-u^2(x)\big)u^2(y)dxdy \\
&\le \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x|\big)u_n^2(x)\big|u_n(y)-u(y)\big|\big|u_n(y)+u(y)\big| \\
&\quad + \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|y|\big)u_n^2(x)\big|u_n(y)-u(y)\big|\big|u_n(y)+u(y)\big| \\
&\quad + \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x|\big)\big|u_n(x)-u(x)\big|\big|u_n(x)+u(x)\big|u^2(y)dxdy \\
&\quad + \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|y|\big)\big|u_n(x)-u(x)\big|\big|u_n(x)+u(x)\big|u^2(y)dxdy \\
&\le |u_n|_*^2|u_n-u|_2|u_n+u|_2 + |u_n|_2^2|u_n-u|_*|u_n+u|_* \\
&\quad + |u|_2^2|u_n-u|_*|u_n+u|_* + |u|_*^2|u_n-u|_2|u_n+u|_2 \\
&\le C\|u_n-u\|_X
\end{aligned} \tag{2.8}$$

for a suitable positive constant $C > 0$. Hence we derive that $V_1(u_n)$ tends to $V_1(u)$, as $n \to \infty$.

For any $v \in X$ the Gateaux derivative of V_1 at $u \in X$ is given by

$$V_1'(u)v = 4\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)u^2(x)u(y)v(y)dxdy.$$

Since

$$\begin{aligned}
\big|V_1'(u)v\big| &\le 4\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x|\big)u^2(x)u(y)v(y) + 4\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|y|\big)u^2(x)u(y)v(y) \\
&\le |u|_*^2|u|_2|v|_2 + |u|_2^2|u|_*|v|_* \le K\|u\|_X^3\|v\|_X
\end{aligned} \tag{2.9}$$

we deduce that $V_1'(u) \in X^*$ and $\|V_1'(u)\| \le K\|u\|_X^3$ for any $u \in X$.

Now we prove that $V'(u_n)$ tends to $V'(u)$ in X^* if $u_n \to u$ in X.

Suppose that u_n tends to u in X. It follows that u_n is bounded and for any $v \in X$ we have

$$\begin{aligned}
\frac{|V_1'(u_n)v - V_1'(u)v|}{4} &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)\big(u_n^2(x)u_n(y) - u^2(x)u(y)\big)v(y)dxdy \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)\big(u_n^2(x) - u^2(x)\big)u_n(y)v(y)dxdy \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x-y|\big)\big(u_n(y) - u(y)\big)u^2(x)v(y)dxdy \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x|\big)\big|u_n(x)-u(x)\big|\big|u_n(x)+u(x)\big|\big|u_n(y)\big|\big|v(y)\big|dxdy \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|y|\big)\big|u_n(x)-u(x)\big|\big|u_n(x)+u(x)\big|\big|u_n(y)\big|\big|v(y)\big|dxdy \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|x|\big)\big|u_n(y)-u(y)\big|u^2(x)\big|v(y)\big|dxdy \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\big(1+|y|\big)\big|u_n(y)-u(y)\big|u^2(x)\big|v(y)\big|dxdy \\
&\le |u_n-u|_*|u_n+u|_*|u_n|_2|v|_2 + |u_n-u|_2|u_n+u|_2|u_n|_*|v|_* \\
&\quad + |u_n-u|_2|v|_2|u|_*^2 + |u_n-u|_*|v|_*|u|_2^2 \\
&\le K\|u_n-u\|_X\|v\|_X
\end{aligned} \tag{2.10}$$

for a suitable some $K > 0$, independent of v. Hence $\|V_1'(u_n) - V_1'(u)\| \le K\|u_n - u\|_X$ and thus V_1 is continuously differentiable on X. Moreover $V_1'(u)v = 4B_1(u^2, uv)$ holds.

Let u_n be a sequence on $L^{8/3}(\mathbb{R}^2)$ converging to some $u \in L^{8/3}(\mathbb{R}^2)$ in $L^{8/3}(\mathbb{R}^2)$. Now we evaluate

$$\begin{aligned}
\big|V_2(u_n) - V_2(u)\big| &\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u_n^2(x)\big|u_n^2(y) - u^2(y)\big|dxdy \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)\big|u_n^2(x) - u^2(x)\big|u^2(y)dxdy \\
&\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}u_n^2(x)\big|u_n(y)-u(y)\big|\big|u_n(y)+u(y)\big| \\
&\quad + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}\big|u_n(x)-u(x)\big|\big|u_n(x)+u(x)\big|u^2(y)dxdy \\
&\le C|u_n-u|_{8/3}|u_n+u|_{8/3}|u_n|_{8/3}^2 + |u_n-u|_{8/3}|u_n+u|_{8/3}|u|_{8/3}^2 \\
&\le C|u_n-u|_{8/3}\big[\big(|u_n|_{8/3}+|u|_{8/3}\big)|u_n|_{8/3}^2 + \big(|u|_{8/3}+|u|_{8/3}\big)|u|_{8/3}^2\big] \\
&\le C|u_n-u|_{8/3}\big(|u_n|_{8/3}+|u|_{8/3}\big)\big(|u_n|_{8/3}^2+|u|_{8/3}^2\big).
\end{aligned} \tag{2.11}$$

From (2.11) we derive that if u_n tends to u in $L^{8/3}$ as $n \to \infty$, then $V_2(u_n)$ tends to $V_2(u)$ as $n \to \infty$. Hence V_2 is continuous on $L^{\frac{8}{3}}(\mathbb{R}^2)$. For any $v \in L^{8/3}(\mathbb{R}^2)$ the Gateaux derivative of V_2 at $u \in L^{8/3}(\mathbb{R}^2)$ along v is given by

$$V_2'(u)v = 4\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u^2(x)u(y)v(y)dxdy.$$

Since

$$\begin{aligned}\left|V_2'(u)v\right| &\le 4 \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|} u^2(x)\left|u(y)\right|\left|v(y)\right| \\ &\le 4C_0|u^2|_{4/3}|uv|_{4/3} \le 4C_0|u|_{8/3}^3|v|_{8/3} \end{aligned} \tag{2.12}$$

we deduce $\|V_2'(u)\| \le K|u|_{8/3}^3$ for any $u \in L^{8/3}(\mathbb{R}^2)$.

Now we prove that V_2 is continuously differentiable on $L^{8/3}(\mathbb{R}^2)$. Suppose that u_n tends to u in $L^{8/3}(\mathbb{R}^2)$. It follows that u_n is bounded and for any $v \in L^{8/3}(\mathbb{R}^2)$

$$\begin{aligned}\frac{|V_2'(u_n)v - V_2'(u)v|}{4} &= \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)\left(u_n^2(x)u_n(y)-u^2(x)u(y)\right)v(y)\,dxdy \\ &= \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)\left(u_n^2(x)-u^2(x)\right)u_n(y)v(y)\,dxdy \\ &\quad + \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}\left|u_n(y)-u(y)\right|u^2(x)\left|v(y)\right|dxdy \\ &\le \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}\left|u_n(x)-u(x)\right|\left|u_n(x)+u(x)\right|\left|u_n(y)\right|\left|v(y)\right|dxdy \\ &\le C|u_n-u|_{8/3}|v|_{8/3}|u|_{8/3}^2 + |u_n-u|_{8/3}|u_n+u|_{8/3}|u_n|_{8/3}|v|_{8/3} \\ &= C|u_n-u|_{8/3}|v|_{8/3}\left(|u|_{8/3}^2 + |u_n+u|_{8/3}|u_n|_{8/3}\right) \\ &\le C_1|u_n-u|_{8/3}|v|_{8/3} \end{aligned} \tag{2.13}$$

for some suitable some positive constants C, C_1, independent of v. Hence $\|V_2'(u_n) - V_2'(u)\| \le C_1|u_n - u|_{8/3}$ and thus V_2 is continuous differentiable on $L^{8/3}(\mathbb{R}^2)$.

Furthermore we deduce that V_2 is C^1 on X and $V_2'(u)v = 4B_2(u^2, uv)$ holds.

It follows that $V_0 = V_1 - V_2$ is C^1 on X and $V_0'(u)v = 4B_0(u^2, uv)$ holds. We also infer that I is C^1 on X.

We prove (iv). Let u_n be a sequence in $H^1(\mathbb{R}^2)$ and $u \in H^1(\mathbb{R}^2)$. Suppose that u_n weakly converges to u in $H^1(\mathbb{R}^2)$. For $R > 0$ fixed, u_n strongly converges to u in $L^2(B_R)$ and

$$\lim_{n\to\infty} \iint\limits_{B_R(0)\times B_R(0)} \log\left(1+|x-y|\right)u_n^2(x)u_n^2(y)\,dxdy = \iint\limits_{B_R(0)\times B_R(0)} \log\left(1+|x-y|\right)u^2(x)u^2(y)\,dxdy.$$

It follows that

$$\liminf_{n\to\infty} V_1(u_n) \ge \iint\limits_{B_R(0)\times B_R(0)} \log\left(1+|x-y|\right)u^2(x)u^2(y)\,dxdy. \tag{2.14}$$

By Monotone Convergence theorem, we derive

$$\lim_{R\to\infty} \iint\limits_{B_R(0)\times B_R(0)} \log\left(1+|x-y|\right)u^2(x)u^2(y)\,dxdy = V_1(u)$$

and then

$$\liminf_{n\to\infty} V_1(u_n) \ge V_1(u).$$

We prove (v). Now let u_n be a sequence in X and $u \in X$. Assume that u_n weakly converges to u in X. It follows that u_n strongly converges to u in $L^{8/3}(\mathbb{R}^2)$ and thus from (ii) we have $\lim_{n\to\infty} V_2(u_n) = V_2(u)$.

Moreover u_n weakly converges to u in $H^1(\mathbb{R}^2)$ and by (iv), we have $\liminf_{n\to\infty} V_1(u_n) \ge V_1(u)$. Taking also into account (i), we conclude that

$$\liminf_{n\to\infty} I(u_n) \ge I(u).$$

Finally, (vi) follows from (iv) and the continuity of the functional

$$u \mapsto I(u) - \frac{1}{4}V_1(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}V_2(u) - \frac{b}{p}|u|_p^p$$

with respect to the H^1-norm. □

As a consequence of 2.2(ii), we find that

$$I'(u)v = \langle u, v\rangle + B_0\big(u^2, uv\big) - b\int_{\mathbb{R}^2} |u|^{p-2}uv\,dx \quad \text{for } u, v \in X, \tag{2.15}$$

so $u \in X$ is a critical point of I if and only if it is a weak solution of (1.5) in the sense that the RHS of (2.15) vanishes for every $v \in X$. Moreover, we have the following regularity result.

Proposition 2.3. *Let $u \in X$ be a weak solution of (1.5), i.e., a critical point of I. Then $u \in W^{2,p}(\mathbb{R}^2)$ for every $p \ge 1$, and u is a strong solution of (1.5). Moreover, we have:*

(i) $|u(x)|e^{\alpha|x|} \to 0$ *as* $|x| \to \infty$ *for every* $\alpha > 0$.
(ii) *The function* $w : \mathbb{R}^2 \to \mathbb{R}$, $w(x) = \int_{\mathbb{R}^2} \log|x-y|u^2(y)\,dy$ *is of class* C^3 *on* $\mathbb{R}^2$ *and satisfies*

$$\Delta w = u^2 \quad \text{in } \mathbb{R}^2, \qquad w(x) - |u|_2^2 \log|x| \to 0 \quad \text{as } |x| \to \infty.$$

Furthermore, if a is Hölder continuous, then $u \in C^2(\mathbb{R}^2)$ is a classical solution of (1.5).

Proof. We first show that the function w defined in (ii) is locally bounded and has the asserted asymptotics. If $|x| \le 1$, we have

$$\big|w(x)\big| \le \int_{B_2(x)} \big|\log|x-y|\big|u^2(y)\,dy + \int_{\mathbb{R}^2\setminus B_2(x)} \log|x-y|u^2(y)\,dy, \tag{2.16}$$

Since $1 \le |x-y| \le 1+|y|$ for $y \in \mathbb{R}^2 \setminus B_2(x)$, we find that

$$\int_{\mathbb{R}^2\setminus B_2(x)} \log|x-y|u^2(y)\,dy \le \|u\|_X^2.$$

Moreover, by Young's inequality and Sobolev embeddings, we have

$$\int_{B_2(x)} \big|\log|x-y|\big|u^2(y)\,dy \le \left(\int_{B_2(0)} \big|\log|y|\big|^2\,dy\right)^{1/2} \|u\|_{L^4(B_2(x))}^2 \le c_0\|u\|_X^2, \tag{2.17}$$

where $c_0 > 0$ is a constant. Hence (2.16) implies that $w \in L^\infty(B_1(0))$. Next, we consider $x \in \mathbb{R}^2$ with $|x| \ge 1$, and we note that

$$w(x) - |u|_2^2\log|x| = \int_{\mathbb{R}^2} h(x,y)u^2(y)\,dy \quad \text{with } h(x,y) = \log|x-y| - \log|x| = \log\frac{|x-y|}{|x|}.$$

Note that $h(x,y) \to 0$ as $|x| \to \infty$ for every $y \in \mathbb{R}^2$. Moreover,

$$\log\frac{1}{2} \le h(x,y)1_{\{|y-x|\ge\frac{1}{2}\}}(y) \le \log\big(1+|y|\big) \quad \text{for all } x, y \in \mathbb{R}^2 \text{ with } |x| \ge 1.$$

Since the functions $[\log \frac{1}{2}]u^2$ and $\log(1+|\cdot|)u^2$ are in $L^1(\mathbb{R}^2)$, Lebesgue's Theorem implies that

$$\int\limits_{|y-x|\geq\frac{1}{2}} h(x,y)u^2(y)\,dy \to 0 \quad \text{as } |x|\to\infty. \tag{2.18}$$

Moreover, since $u\in X$, we have

$$0\leq \log|x| \int\limits_{|y-x|\leq\frac{1}{2}} u^2(y)\,dy \leq \int\limits_{|y|\geq\frac{|x|}{2}} \log\bigl(2\bigl[1+|y|\bigr]\bigr)u^2(y)\,dy \to 0 \quad \text{as } |x|\to\infty,$$

and, similarly as in (2.17),

$$\int\limits_{|y-x|\leq\frac{1}{2}} \bigl|\log|x-y|\bigr|u^2(y)\,dy \leq c_1\|u\|^2_{L^4(B_{\frac{1}{2}}(x))} \to 0 \quad \text{as } |x|\to\infty$$

with a constant $c_1>0$. Combining these estimates with (2.18), we conclude that

$$\int\limits_{\mathbb{R}^2} h(x,y)u^2(y)\,dy \to 0 \quad \text{as } |x|\to\infty.$$

and thus the asymptotics in (ii) are proved. By Agmon's Theorem (see [2]), it follows that u satisfies (i), and elliptic regularity theory then yields that $u\in W^{2,p}(\mathbb{R}^2)$ for every $p\in[1,\infty)$, and that u is a strong solution of (1.5). Moreover, $u\in C^{1,\beta}_{loc}(\mathbb{R}^2)$ for all $\beta\in(0,1)$ by Sobolev embeddings. Hence elliptic regularity implies that $w\in C^{3,\beta}_{loc}(\mathbb{R}^2)$ satisfies $\Delta w = u^2$.

Finally, if a is Hölder continuous, then u satisfies an equation of the form $-\Delta u = f$ with (locally) Hölder continuous $f:\mathbb{R}^2\to\mathbb{R}$, so $u\in C^2(\mathbb{R}^2)$ by elliptic regularity. □

We add some observations on the functional geometry of I.

Lemma 2.4. *There exists $\alpha>0$ such that*

$$\inf\bigl\{I(u) : u\in X : \|u\|=\beta\bigr\} > 0 \quad \textit{for } 0<\beta\leq\alpha \tag{2.19}$$

and

$$\inf\bigl\{I'(u)u : u\in X : \|u\|=\beta\bigr\} > 0 \quad \textit{for } 0<\beta\leq\alpha. \tag{2.20}$$

Proof. By (2.2) and Sobolev embeddings we have

$$\begin{aligned} I(u) &\geq \frac{\|u\|^2}{2} - \frac{V_2(u)}{4} - \frac{b}{p}|u|_p^p \geq \frac{\|u\|^2}{2} - \frac{C_0}{4}|u|^4_{\frac{8}{3}} - \frac{b}{p}|u|_p^p \\ &\geq \frac{1}{2}\bigl(\|u\|^2 - C_1\|u\|^4 - C_2\|u\|^p\bigr) = \frac{\|u\|^2}{2}\bigl(1 - C_1\|u\|^2 - C_2\|u\|^{p-2}\bigr) \end{aligned}$$

for $u\in X$ with constants $C_1, C_2>0$. This readily implies that (2.19) holds for $\alpha>0$ sufficiently small.

Since

$$I'(u)(u) = \|u\|^2 + V_0(u) - b|u|_p^p \geq \|u\|^2 - V_2(u) - b|u|_p^p$$

for $u\in X$, a similar estimate shows that (2.20) holds for $\alpha>0$ sufficiently small. □

Lemma 2.5. *Let $u \in X \setminus \{0\}$. Then the function $\varphi_u : \mathbb{R} \to \mathbb{R}$, $\varphi_u(t) = I(tu)$ is even and has the following properties.*

(i) *If*

$$V_0(u) - b|u|_4^4 < 0 \quad \text{in case } p = 4 \quad \text{and} \quad V_0(u) < 0 \quad \text{or} \quad b > 0 \quad \text{in case } p > 4, \tag{2.21}$$

then there exists a unique $t_u \in (0, \infty)$ such that $\varphi_u' > 0$ on $(0, t_u)$ and $\varphi_u' < 0$ on (t_u, ∞). Moreover, $\varphi_u(t) \to -\infty$ as $t \to \infty$.

(ii) *If (2.21) does not hold, then $\varphi_u' > 0$ on $(0, \infty)$, and $\varphi_u(t) \to \infty$ as $t \to \infty$.*

Proof. The assertions follow easily from the fact that

$$\frac{\varphi_u'(t)}{t} = \|u\|^2 + t^2 V_0(u) - bt^{p-2}|u|_p^p \quad \text{for } t > 0. \qquad \square$$

We also need the following continuity property of the bilinear form B_1.

Lemma 2.6. *Let $(u_n)_n, (v_n)_n, (w_n)_n$ be bounded sequences in X such that $u_n \to u$ weakly in X. Then, for every $z \in X$, we have $B_1(v_n w_n, z(u_n - u)) \to 0$ as $n \to \infty$.*

Proof. We put $c_1 := \sup_{n\in\mathbb{N}} |v_n|_2|w_n|_2$ and $c_2 := \sup_{n\in\mathbb{N}} |v_n|_*|w_n|_*$. Then $c_1, c_2 < \infty$ by assumption. Using (2.3) again, we estimate

$$\begin{aligned} \big|B_1\big(v_n w_n, z(u_n - u)\big)\big| &\le |v_n|_2|w_n|_2 \int_{\mathbb{R}^2} \log\big(1 + |x|\big)\big|z(x)\big|\big|u_n(x) - u(x)\big|\,dx + |v_n|_*|w_n|_*|z|_2|u_n - u|_2 \\ &\le c_1 \int_{\mathbb{R}^2} \log\big(1 + |x|\big)\big|z(x)\big|\big|u_n(x) - u(x)\big|\,dx + o(1) \end{aligned} \tag{2.22}$$

as $n \to \infty$, since $u_n \to u$ strongly in $L^2(\mathbb{R}^2)$ by Lemma 2.2(i). For fixed $R > 0$, we have

$$\int_{\mathbb{R}^2} \log\big(1 + |x|\big)\big|z(x)\big|\big|u_n(x) - u(x)\big|\,dx = f_n(R) + g_n(R)$$

with

$$f_n(R) := \int_{B_R(0)} \log\big(1 + |x|\big)\big|z(x)\big|\big|u_n(x) - u(x)\big|\,dx \le \log(1 + R)|z|_2|u_n - u|_2 \to 0 \tag{2.23}$$

as $n \to \infty$ and

$$\begin{aligned} g_n(R) &:= \int_{\mathbb{R}^2\setminus B_R(0)} \log\big(1 + |x|\big)\big|z(x)\big|\big|u_n(x) - u(x)\big|\,dx \\ &\le \Bigg(\int_{\mathbb{R}^2\setminus B_R(0)} \log\big(1 + |x|\big)\big|z(x)\big|^2\,dx\Bigg)^{\frac{1}{2}} \Bigg(\int_{\mathbb{R}^2} \log\big(1 + |x|\big)|u_n - u|^2\,dx\Bigg)^{\frac{1}{2}} \le h(R), \end{aligned} \tag{2.24}$$

where

$$h(R) := \Bigg(\int_{\mathbb{R}^2\setminus B_R(0)} \log\big(1 + |x|\big)\big|z(x)\big|^2\,dx\Bigg)^{\frac{1}{2}} \sup_{n\in\mathbb{N}}\big(|u_n|_* + |u|_*\big) \to 0 \quad \text{as } R \to \infty. \tag{2.25}$$

Combining (2.22), (2.23) and (2.24), we find that

$$\limsup_{n\to\infty}\big|B_1\big(v_n w_n, z(u_n - u)\big)\big| \le c_1 h(R) \quad \text{for every } R > 0$$

and therefore $|B_1(v_n w_n, z(u_n - u))| \to 0$ as $n \to \infty$ by (2.25). $\square$

3. A compactness condition

In the following, we assume that

$$a : \mathbb{R}^2 \to (0, \infty) \quad \text{is continuous and } \mathbb{Z}^2\text{-periodic.} \tag{3.1}$$

Then the functional I is invariant under translations with respect to $\mathbb{Z}^2$. In the following, we fix some notation for the action of translation. For a function $u : \mathbb{R}^2 \to \mathbb{R}$ and $x \in \mathbb{R}^2$, we define

$$x * u : \mathbb{R}^2 \to \mathbb{R}, \qquad [x * u](y) = u(y - x) \quad \text{for } y \in \mathbb{R}^2.$$

The main result of this section is the following.

Proposition 3.1. *Let $(u_n)_n$ be a sequence in X such that*

$$I(u_n) \to d > 0 \quad \text{and} \quad \left\| I'(u_n) \right\|_{X'} \left(1 + \|u_n\|_X\right) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Then, after passing to a subsequence, there exist points $x_n \in \mathbb{Z}^2$, $n \in \mathbb{N}$ such that

$$x_n * u_n \to u \quad \text{strongly in } X \text{ as } n \to \infty$$

for some nonzero critical point $u \in X$ of I.

The remainder of this section will be occupied with the proof of Proposition 3.1. So we assume from now on that we are given a sequence $(u_n)_n \subset X$ satisfying (3.2).

Lemma 3.2. *If $(t_n)_n$ is a bounded sequence in $[0, \infty)$, then*

$$I(t_n u_n) \le I(u_n) + o(1) \quad \text{as } n \to \infty. \tag{3.3}$$

Moreover, if $t_n \to 0$ as $n \to \infty$, then

$$\liminf_{n\to\infty} I(t_n u_n) \ge 0. \tag{3.4}$$

Proof. We have

$$I(t_n u_n) - I(u_n) = \frac{t_n^2 - 1}{2}\|u_n\|^2 + \frac{t_n^4 - 1}{4} V_0(u_n) - b\frac{t_n^p - 1}{p}|u_n|_p^p$$

and

$$V_0(u_n) = I'(u_n)(u_n) + b|u_n|_p^p - \|u_n\|^2 = o(1) + b|u_n|_p^p - \|u_n\|^2 \quad \text{as } n \to \infty, \tag{3.5}$$

so that

$$\begin{aligned} I(t_n u_n) - I(u_n) &= -\left(\frac{t_n^4 - 1}{4} - \frac{t_n^2 - 1}{2}\right)\|u_n\|^2 - b\left(\frac{t_n^p - 1}{p} - \frac{t_n^4 - 1}{4}\right)|u_n|_p^p + o(1) \\ &= -\frac{(t_n^2 - 1)^2}{4}\|u_n\|^2 - \frac{b}{p}\left(t_n^p - \frac{p}{4}t_n^4 + \frac{p}{4} - 1\right)|u_n|_p^p + o(1) \\ &\le o(1) \quad \text{as } n \to \infty, \end{aligned}$$

where the last step follows from the fact that the map $t \mapsto t^p - \frac{p}{4}t^4 + \frac{p}{4} - 1$ is nonnegative on $[0, \infty)$ since $p \ge 4$ (with global minimum 0 attained at $t = 1$). This shows (3.3). If moreover $t_n \to 0$ as $n \to \infty$, then we have, using (3.5) again

$$I(t_n u_n) = \left(\frac{t_n^2}{2} - \frac{t_n^4}{4}\right)\|u_n\|^2 + b\left(\frac{t_n^4}{4} - \frac{t_n^p}{p}\right)|u_n|^p + o(1) \ge o(1)$$

as $n \to \infty$. Hence (3.4) holds. □

Lemma 3.3. *The sequence $(u_n)_n$ is bounded in $H^1(\mathbb{R}^2)$.*

Proof. Suppose by contradiction that, after passing to a subsequence,

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.6}$$

Put $v_n := \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$, so that $\|v_n\| = 1$ for all n. We claim that

$$\inf_{n\in\mathbb{N}} \sup_{x\in\mathbb{Z}^2} \int_{B_2(x)} v_n^2(y)\, dy > 0. \tag{3.7}$$

Suppose by contradiction that this is false. Then, after passing to a subsequence, $v_n \to 0$ in $L^s(\mathbb{R}^2)$ for every $s > 2$ by Lions' Lemma, see e.g. [33], and thus, by Lemma 2.2(iii),

$$V_2(tv_n) \to 0 \quad \text{and} \quad |tu_n|_p \to 0 \quad \text{as } n \to \infty \text{ for every } t \geq 0.$$

Hence

$$I(tv_n) = \frac{t^2}{2}\|v_n\|^2 + \frac{t^4}{4}V_1(tv_n) + o(1) \geq \frac{t^2}{2} + o(1) \quad \text{for every } t \geq 0$$

as $n \to \infty$, whereas, on the other hand,

$$I(tv_n) = I\left(\frac{t}{\|u_n\|}u_n\right) \leq I(u_n) + o(1) = d + o(1) \quad \text{as } n \to \infty \text{ for every } t \geq 0 \tag{3.8}$$

by Lemma 3.2. This is a contradiction for $t > \sqrt{2d}$, so we conclude that (3.7) holds. Thus there exists a sequence of points $x_n \in \mathbb{Z}^2$, $n \in \mathbb{N}$ such that, after passing to a subsequence, the sequence of the functions

$$w_n := x_n * v_n \in X, \quad n \in \mathbb{N}$$

converges weakly in $H^1(\mathbb{R}^2)$ to some nonzero function $w \in H^1(\mathbb{R}^2)$. Passing to a subsequence again if necessary, we may also assume that $w_n \to w$ pointwise a.e. on $\mathbb{R}^2$. Since, by (3.8), we have $I(w_n) = I(v_n) \leq d + o(1)$ as $n \to \infty$, we find that

$$V_1(w_n) = I(w_n) - \frac{1}{2} + V_2(w_n) + \frac{b}{p}|w_n|_p^p \leq C \quad \text{for all } n \in \mathbb{N} \tag{3.9}$$

with a constant $C > 0$. We now distinguish two cases:

Case 1: $b > 0$ and $p > 4$. In this case we estimate, using (3.9), that

$$\begin{aligned} I(tv_n) = I(tw_n) &= \frac{t^2}{2} + \frac{t^4}{4}\big(V_1(w_n) - V_2(w_n)\big) - b\frac{t^p}{p}|w_n|_p^p \\ &\leq \frac{t^2}{2} + \frac{t^4}{4}C - b\frac{t^p}{p}|w|_p^p + o(1), \end{aligned}$$

so that, since $b > 0$ and $p > 4$, there exists $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that

$$I\left(\frac{t_0}{\|u_n\|}u_n\right) = I(t_0 v_n) \leq -1 \quad \text{for } n \geq n_0. \tag{3.10}$$

Since $\frac{t_0}{\|u_n\|} \to 0$ as $n \to \infty$, (3.10) contradicts (3.4).

Case 2: $b = 0$ or $p = 4$. In this case we have

$$I(tv_n) = I(tw_n) = \frac{t^2}{2} + \frac{t^4}{4}\rho_n \quad \text{for } t \geq 0 \tag{3.11}$$

with $\rho_n := V_0(w_n) - \frac{b}{p}|w_n|_p^p$ for $n \in \mathbb{N}$. We claim that

$$\rho := \limsup_{n\to\infty} \rho_n < 0. \tag{3.12}$$

Indeed, if, by contradiction, $\liminf_{n\to\infty} \rho_n \geq 0$ after passing to a subsequence, then Lemma 3.2 and (3.11), applied with $t_1 = \sqrt{4d}$ imply that

$$d + o(1) = I(u_n) \geq I(t_1 v_n) + o(1) \geq \frac{t_1^2}{2} + o(1) = 2d + o(1) \quad \text{as } n \to \infty,$$

a contradiction. Hence (3.12) holds, and thus (3.11) implies that there exists $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that (3.10) also holds in this case. Again, we arrive at a contradiction to (3.4).

Since in both cases we reached a contradiction, we conclude that the sequence $(u_n)_n$ is bounded in $H^1(\mathbb{R}^2)$, as claimed. □

Proof of Proposition 3.1 (completed). We first show that

$$\liminf_{n\in\mathbb{N}} \sup_{x\in\mathbb{Z}^2} \int_{B_2(x)} u_n^2(y)\,dy > 0. \tag{3.13}$$

Suppose by contradiction that (3.13) is false. Since the sequence $(u_n)_n$ is bounded in $H^1(\mathbb{R}^2)$, it then follows from Lions' Lemma (see e.g. [33]) that, after passing to a subsequence, $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for every $s > 2$, and thus, by (2.2) and (3.2),

$$\|u_n\|^2 + V_1(u_n) = I'(u_n)u_n + V_2(u_n) + b|u_n|_p^p \to 0 \quad \text{as } n \to \infty,$$

so that $\|u_n\|^2 \to 0$, $V_1(u_n) \to 0$ and therefore

$$I(u_n) = \frac{\|u_n\|^2}{2} + \frac{1}{4}\big(V_1(u_n) - V_2(u_n)\big) - \frac{b}{p}|u_n|_p^p \to 0 \quad \text{as } n \to \infty,$$

contradicting the assumption that $I(u_n) \to d \neq 0$. Hence (3.13) holds. Thus there exists a sequence of points $x_n \in \mathbb{Z}^2$, $n \in \mathbb{N}$ such that, after passing to a subsequence, the sequence of the functions

$$\tilde{u}_n := x_n * u_n \in X, \quad n \in \mathbb{N},$$

converges weakly in $H^1(\mathbb{R}^2)$ to some function $u \in H^1(\mathbb{R}^2) \setminus \{0\}$. We may also assume that $\tilde{u}_n \to u$ pointwise almost everywhere in $\mathbb{R}^2$. Moreover, we have, invoking (3.5) again, that

$$B_1\big(\tilde{u}_n^2, \tilde{u}_n^2\big) = V_1(\tilde{u}_n) = V_1(u_n) = o(1) + V_2(u_n) + b|u_n|_p^p - \|u_n\|^2,$$

and the RHS of this inequality remains bounded in n. Consequently, Lemma 2.1 implies that $|\tilde{u}_n|_*$ remains bounded in n, so that the sequence $(\tilde{u}_n)_n$ is bounded in X. We may thus assume, passing to a subsequence again if necessary, that $\tilde{u}_n \rightharpoonup u$ weakly in X, so that $u \in X$. It then follows by Lemma 2.2(i) that $\tilde{u}_n \to u$ strongly in $L^s(\mathbb{R}^2)$ for $s \in [2, \infty)$. Next we claim that

$$I'(\tilde{u}_n)(\tilde{u}_n - u) \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Indeed, we have

$$\big|I'(\tilde{u}_n)(\tilde{u}_n - u)\big| = \big|I'(u_n)\big(u_n - (-x_n) * u\big)\big| \leq \big\|I'(u_n)\big\|_{X'}\big(\|u_n\|_X + \big\|(-x_n) * u\big\|_X\big) \tag{3.15}$$

for every n. Moreover, it is easy to see that

$$|u_n|_* = \int_{\mathbb{R}^2} \log\big(1 + |x - x_n|\big)\tilde{u}_n^2\,dx \geq c_1 \log\big(1 + |x_n|\big) \quad \text{for all } n$$

with a constant $c_1 > 0$ and

$$\big|(-x_n) * u\big|_* = \int_{\mathbb{R}^2} \log\big(1 + |x - x_n|\big)u^2\,dx \leq c_2 \log\big(1 + |x_n|\big) \quad \text{for all } n$$

with a constant $c_2 > 0$. Since moreover $\|u_n\|$ and $\|(-x_n) * u\|$ are bounded in n, there exists a constant $c_3 > 0$ such that

$$\big\|(-x_n) * u\big\|_X \le c_3 \|u_n\|_X \quad \text{for all } n. \tag{3.16}$$

Hence (3.15) implies that

$$\big|I'(\tilde u_n)(\tilde u_n - u)\big| \le (1 + c_3)\big\|I'(u_n)\big\|_{X'} \|u_n\|_X \to 0 \quad \text{as } n \to \infty,$$

as claimed in (3.14). Using (3.14), we find that

$$\begin{aligned} o(1) &= I'(\tilde u_n)(\tilde u_n - u) \\ &= o(1) + \|\tilde u_n\|^2 - \|u\|^2 + \frac{1}{4} V'(\tilde u_n)(\tilde u_n - u) - b \int_{\mathbb{R}^2} |\tilde u_n|^{p-2} \tilde u_n (\tilde u_n - u)\, dx \\ &= o(1) + \|\tilde u_n\|^2 - \|u\|^2 + \frac{1}{4}\big[V_1'(\tilde u_n)(\tilde u_n - u) - V_2'(\tilde u_n)(\tilde u_n - u)\big], \end{aligned}$$

where

$$\left|\frac{1}{4} V_2'(\tilde u_n)(\tilde u_n - u)\right| = \big|B_2\big(\tilde u_n^2, \tilde u_n(\tilde u_n - u)\big)\big| \le |\tilde u_n|_{\frac{8}{3}}^3 |\tilde u_n - u|_{\frac{8}{3}} \to 0$$

as $n \to \infty$ and

$$\frac{1}{4} V_1'(\tilde u_n)(\tilde u_n - u) = B_1\big(\tilde u_n^2, \tilde u_n(\tilde u_n - u)\big) = B_1\big(\tilde u_n^2, (\tilde u_n - u)^2\big) + B_1\big(\tilde u_n^2, u(\tilde u_n - u)\big)$$

with

$$B_1\big(\tilde u_n^2, u(\tilde u_n - u)\big) \to 0 \quad \text{as } n \to \infty$$

by Lemma 2.6. Combining these estimates, we obtain that

$$o(1) = \|\tilde u_n\|^2 - \|u\|^2 + B_1\big(\tilde u_n^2, (\tilde u_n - u)^2\big) + o(1) \ge \|\tilde u_n\|^2 - \|u\|^2 + o(1) \ge o(1) \quad \text{as } n \to \infty,$$

which implies that $\|\tilde u_n\| \to \|u\|$ and $B_1(\tilde u_n^2, (\tilde u_n - u)^2) \to 0$ as $n \to \infty$. Hence $\|\tilde u_n - \tilde u\| \to 0$ and, by Lemma 2.1, $|\tilde u_n - \tilde u|_* \to 0$ as $n \to \infty$. We thus conclude that $\|\tilde u_n - u\|_X \to 0$ as $n \to \infty$, as claimed.

We still need to show that $I'(u) = 0$. Let $v \in X$. Then we have

$$\big\|(-x_n) * v\big\|_X \le c_4 \|u_n\|_X \quad \text{for all } n \text{ with a constant } c_4 > 0$$

by the same argument which leads to (3.16). Moreover, $I'(u)v = \lim_{n\to\infty} I'(\tilde u_n)v$, where

$$\big|I'(\tilde u_n)v\big| = \big|I'(u_n)(-x_n) * v\big| \le \big\|I'(u_n)\big\|_X \big\|(-x_n) * v\big\|_X \le c_4 \big\|I'(u_n)\big\|_X \|u_n\|_X \to 0$$

as $n \to \infty$. Hence $I'(u)v = 0$, and the proof is finished. □

4. The periodic setting

In this section, we will complete the proof of Theorem 1.1. As in the last section, we assume that $a : \mathbb{R}^2 \to (0, \infty)$ is continuous and $\mathbb{Z}^2$-periodic. For $c > 0$ we put

$$K_c := \big\{u \in X \,:\, I'(u) = 0,\ I(u) = c\big\}.$$

Moreover, for $\rho > 0$ we also consider the set

$$A_{c,\rho} := \big\{u \in X \,:\, \|u - v\| \le \rho \text{ for some } v \in K_c\big\}. \tag{4.1}$$

We note that $A_{c,\rho}$ is closed in X and in $H^1(\mathbb{R}^2)$. Indeed, the closedness in $H^1(\mathbb{R}^2)$ follows easily from Proposition 3.1. We also note that $A_{c,\rho}$ is symmetric with respect to the reflection $u \mapsto -u$ and invariant under $\mathbb{Z}^2$-translations for every $c \in \mathbb{R}$, $\rho > 0$. In the following, $\gamma(A) \in \mathbb{N} \cup \{0, \infty\}$ denotes the Krasnoselski genus of a closed and symmetric subset $A \subset X$ (defined with respect to continuity in $\|\cdot\|_X$), see e.g. [31, Chapter II.5] for the definition of γ. The following finiteness property is a key step in the proof of Theorem 1.1.

Proposition 4.1. *Let $c > 0$. Then there exists $\rho_0 = \rho_0(c) > 0$ such that $\gamma(A_{c,\rho}) < \infty$ for $\rho \in (0, \rho_0)$.*

Proof. We fix a continuous map

$$\beta : L^2(\mathbb{R}^2) \setminus \{0\} \to \mathbb{R}^2$$

which is equivariant under $\mathbb{Z}^2$-translations, i.e.,

$$\beta(x * u) = x + \beta(u) \quad \text{for } x \in \mathbb{R}^2 \text{ and } u \in L^2(\mathbb{R}^2) \setminus \{0\}.$$

We also require that $\beta(-u) = \beta(u)$ for every $u \in L^2(\mathbb{R}^2) \setminus \{0\}$. A map with these properties has been constructed independently in [10] and [5], and in [5] it is called *generalized barycenter map*. We also put

$$\tilde{K}_c := \{u \in K_c : \beta(u) \in [-4, 4]^2\}.$$

We first show that

$$\tilde{K}_c \text{ is compact in } X. \tag{4.2}$$

To prove this, let $u_n \in \tilde{K}_c$, $n \in \mathbb{N}$ be arbitrary. By Proposition 3.1, there exists a sequence $(x_n)_n \subset \mathbb{R}^2$ such that, after passing to a subsequence,

$$w_n := x_n * u_n \to u \in K_c \quad \text{with respect to } \|\cdot\|_X \text{ as } n \to \infty.$$

Consequently,

$$x_n + \beta(u_n) = \beta(w_n) \to \beta(u) \quad \text{as } n \to \infty.$$

Since $\beta(u_n) \in [-4, 4]^2$ for all n, this implies that the sequence $(x_n)_n$ is bounded in $\mathbb{R}^2$, so that $x_n \to x \in \mathbb{R}^2$ after passing to a subsequence. Consequently,

$$u_n \to u_0 := (-x) * u \quad \text{with respect to } \|\cdot\|_X \text{ as } n \to \infty.$$

Indeed, although $\|\cdot\|_X$ is not invariant under $\mathbb{Z}^2$-translations, we still have

$$\|u_n - (-x) * u\|_X \le \|u_n - (-x_n) * u\|_X + \|(-x_n) * u - (-x) * u\|_X \le C\|x_n * u_n - u\|_X + o(1) = o(1)$$

as $n \to \infty$ with a constant $C > 0$, since the sequence $(x_n)_n \subset \mathbb{R}^2$ is bounded. Moreover,

$$\beta(u_0) = \lim_{n\to\infty} \beta(u_n) \in [-4, 4]^2,$$

so that $u_0 \in \tilde{K}_c$. Hence (4.2) is proved.

By (4.2) and a standard result (see e.g. [31, Chapter II.5]), we have $k := \gamma(\tilde{K}_c) < \infty$, hence there exists an odd and continuous map $g_0 : \tilde{K}_c \to \mathbb{R}^k \setminus \{0\}$. A priori, g_0 is only continuous with respect to $\|\cdot\|_X$, but it easily follows from the compactness of $\tilde{K}_c$ in X that g_0 is also continuous with respect to $\|\cdot\|$. By Tietze's extension theorem, we may extend g_0 to an odd and continuous map $g : H^1(\mathbb{R}^2) \to \mathbb{R}^k$ (continuous with respect to $\|\cdot\|$). Since $\tilde{K}_c$ is also compact in $H^1(\mathbb{R}^2)$, there exists $\rho > 0$ such that

$$g(u) \neq 0 \quad \text{for every } u \in U_\rho, \tag{4.3}$$

where

$$U_\rho := \{u \in H^1(\mathbb{R}^2) : \|u - v\| \le \rho \quad \text{for some } v \in \tilde{K}_c\}.$$

In the following, we let $[\cdot]$ denote the Gauss bracket, i.e.

$$[\cdot] : \mathbb{R} \to \mathbb{Z}, \qquad [s] := \max\{n \in \mathbb{Z} : n \le s\}.$$

We also consider $[\cdot]$ as a map from $\mathbb{R}^2 \to \mathbb{Z}^2$, defining

$$[(x, y)] = ([x], [y]) \in \mathbb{Z}^2 \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Let

$$L_1 := \left\{ u \in L^2(\mathbb{R}^2) \setminus \{0\} : \left|\beta(u) - [\beta(u)]\right|_\infty \le \frac{1}{2} \right\}$$

where $|x|_\infty := \max\{|x_1|, |x_2|\}$ for $x \in \mathbb{R}^2$. Then L_1 is invariant under $\mathbb{Z}^2$-translations, i.e.

$$q * L_1 = L_1 \quad \text{for every } q \in \mathbb{Z}^2.$$

Moreover, it is easy to see that the map

$$h_1 : L_1 \to L_1, \qquad h_1(u) = \left(-[\beta(u)]\right) * u$$

is an isometry with respect to $\|\cdot\|$ which is odd and invariant under $\mathbb{Z}^2$-translations, i.e.

$$h_1(q * u) = h_1(u) = -h_1(-u) \quad \text{for every } u \in L_1,\ q \in \mathbb{Z}^2.$$

We also put

$$a_2 := \left(\frac{1}{2}, 0\right), \qquad a_3 := \left(0, \frac{1}{2}\right), \qquad a_4 := \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$$

and define the sets

$$L_i = a_i * L_1 \subset L^2(\mathbb{R}^2) \setminus \{0\} \quad \text{for } i = 2, 3, 4.$$

Then, by construction,

$$L^2(\mathbb{R}^2) \setminus \{0\} \subset \bigcup_{i=1}^{4} L_i.$$

We define the maps $h_i : L_i \to L_i$, $i = 2, 3, 4$ by

$$h_i(u) = a_i * \left[h_1\left((-a_i) * u\right)\right] \quad \text{for } u \in L_i.$$

These maps are also isometries with respect to $\|\cdot\|$, and they are odd and invariant under translations. Moreover, for every $i \in 1, \dots, 4$ and $u \in L_i$ we have $\beta(h_i(u)) \in [0,1]^2$. We now consider the sets

$$A_i := h_i^{-1}(U_\rho) \subset L_i \quad \text{for } i = 1, \dots, 4.$$

Then the sets A_i are closed in $H^1(\mathbb{R}^2)$, symmetric and invariant under $\mathbb{Z}^2$-translations for $i = 1, \dots, 4$. Let $A := \bigcup_{i=1}^4 A_i$. Using (4.3) and standard arguments in the context of the Krasnoselski genus, one may construct an odd map

$$\hat{g} : A \to \mathbb{R}^{4k} \setminus \{0\}$$

which is continuous with respect to $\|\cdot\|$. It remains to prove that

$$A_{c,\rho} \subset A \quad \text{for } \rho > 0 \text{ sufficiently small.} \tag{4.4}$$

Once this is established, the restriction of the map $\hat{g}$ to $A_{c,\rho}$ is odd and continuous with respect to $\|\cdot\|_X$, and thus it follows that $\gamma(A_{c,\rho}) \le 4k$.

To prove (4.4), we argue by contradiction, assuming that $A_{c,\rho} \not\subset A$ for every $\rho > 0$. Then there exists

$$u_n \in A_{c,\frac{1}{n}} \setminus A \quad \text{for every } n \in \mathbb{N}. \tag{4.5}$$

Moreover, there exist $v_n \in K_c$ such that $\|u_n - v_n\| \le \frac{1}{n}$ for every n. Without loss, we may assume that $u_n \in L_1$ for every $n \in \mathbb{N}$. Moreover, by the translation invariance of L_1, K_c and statement (4.5), we may assume that $\beta(v_n) \in [0,1]^2$ for every $n \in \mathbb{N}$, so in particular $v_n \in \tilde{K}_c$ for every n. By compactness of $\tilde{K}_c$, we may pass to a subsequence such that $\|v_n - v\| \to 0$ as $n \to \infty$ for some $v \in \tilde{K}_c$ with $\beta(v) \in [0,1]^2$. Since also $\|u_n - v\| \to 0$ as $n \to \infty$, it follows that $\beta(u_n) \in [-2,2]^2$ for n sufficiently large. Consequently,

$$w_n := \left(-[\beta(u_n)]\right) * v_n \in \tilde{K}_c \quad \text{for } n \text{ sufficiently large,}$$

whereas $\|h_1(u_n) - w_n\| = \|u_n - v_n\| \to 0$ as $n \to \infty$. Hence $h_1(u_n) \in U_\rho$ for n sufficiently large, which implies that $u_n \in A_1 \subset A$ for n sufficiently large. This contradicts (4.5). Hence (4.4) follows, and the proof is finished. □

Next we recall the notion of relative genus.

Definition 4.2. Let $D \subset Y$ be closed symmetric subsets of X. We define the genus of Y relative to D, denoted $\gamma_D(Y)$, as the smallest number k such that Y can be covered by closed and symmetric subsets U, V with the following properties:

(i) $D \subset U$, and there exists an odd continuous map $\chi : U \to D$ such that $\chi(u) = u$ for every $u \in D$.
(ii) $\gamma(V) \le k$.

If no such covering exists we set $\gamma_D(Y) := \infty$.

The following properties are easily verified (see e.g. [14, Proposition 3.4] for similar statements); we include the proof for the convenience of the reader.

Lemma 4.3. *Let D, Y and Z be closed symmetric subsets of X with $D \subset Y$. Then we have:*

(i) *(Subadditivity) $\gamma_D(Y \cup Z) \le \gamma_D(Y) + \gamma(Z)$.*
(ii) *If $D \subset Z$, and if there exists an odd continuous map $\varphi : Y \to Z$ with $\varphi(u) = u$ for every $u \in D$, then $\gamma_D(Y) \le \gamma_D(Z)$.*

Note in particular that (ii) implies the *monotonicity property* $\gamma_D(Y) \le \gamma_D(Z)$ if $D \subset Y \subset Z$.

Proof. (i) By definition of $\gamma_D(Y)$, there exist closed and symmetric sets $U, V \subset X$ with $D \subset U$, $Y \subset U \cup V$, $\gamma(V) = \gamma_D(Y)$ and an odd continuous map $\chi : U \to D$ with $\chi(u) = u$ for every $u \in D$. Since $V \cup Z$ is also closed and symmetric and $Y \cup Z \subset U \cup (V \cup Z)$, we conclude that $\gamma_D(Y \cup Z) \le \gamma(V \cup Z) \le \gamma(V) + \gamma(Z) = \gamma_D(Y) + \gamma(Z)$ by the subadditivity of γ (see [31, Chapter II.5]).

(ii) By definition of $\gamma_D(Z)$, there exist closed and symmetric sets $U, V \subset X$ with $D \subset U$, $Z \subset U \cup V$, $\gamma(V) = \gamma_D(Z)$ and an odd continuous map $\chi : U \to D$ with $\chi(u) = u$ for every $u \in D$. We then put $\tilde{U} = \varphi^{-1}(U \cap Z)$ and $\tilde{V} = \varphi^{-1}(V \cap Z)$, so that $Y \subset \tilde{U} \cup \tilde{V}$. Since the map $\tilde{\chi} = \chi \circ \varphi : \tilde{U} \to D$ is odd and continuous and satisfies $\tilde{\chi}(u) = u$ for every $u \in D$, it follows from the definition of γ that $\gamma_D(Y) \le \gamma(\tilde{V}) \le \gamma(V) = \gamma_D(Z)$. □

In the following, we put

$$I^c := \{u \in X : I(u) \le c\} \quad \text{for } c \in \mathbb{R} \quad \text{and} \quad D := I^0.$$

Moreover we define

$$c_k := \inf\{c \ge 0 : \gamma_D(I^c) \ge k\} \quad \text{for } k \in \mathbb{N}.$$

Lemma 4.4. *As in Theorem 1.1, let $\mathcal{N} := \{u \in X \setminus \{0\} : I'(u)u = 0\}$. Then we have*

$$c_1 = \inf_{\mathcal{N}} I = \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0, \tag{4.6}$$

Proof. The second equality in (4.6) is an easy consequence of Lemma 2.5, since $u \in \mathcal{N}$ if and only if $\varphi_u'(1) = 0$. The last inequality is an immediate consequence of Lemma 2.4.

Next we show that $c_1 \ge \inf_{\mathcal{N}} I$. We argue by contradiction, assuming that $c_1 < \inf_{\mathcal{N}} I$, and we fix $c \in (c_1, \inf_{\mathcal{N}} I)$. We also define $\mathcal{N}^\pm := \{u \in X : \pm I'(u)u > 0\}$ and the function

$$\tau : \mathcal{N}^- \to [1, \infty), \qquad \tau(u) := \inf\{t \ge 1 : I(tu) \le 0\}.$$

It follows from Lemma 2.5(i) that τ is a continuous (and even) function. Next we define the map

$$\chi : I^c \to X, \qquad \chi(u) = \begin{cases} 0 & u = 0, \\ 0 & \text{if } u \in \mathcal{N}^+, \\ \tau(u)u & \text{if } u \in \mathcal{N}^-. \end{cases}$$

Since $c < \inf_{\mathcal{N}} I$, χ is well-defined, and it is odd and continuous by (2.20) and the properties of τ. Since we also have $\chi|_D = \mathrm{id}|_D$, we conclude that $\gamma_D(I^c) = 0$. Since $c > c_1$, this contradicts the definition of c_1.

It remains to prove that $c_1 \le \inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu)$. For this let $u \in X \setminus \{0\}$, such that $\sup_W I < \infty$ for $W := \mathbb{R}u$. It then follows from Lemma 2.5 that there exists $R > 0$ such that $\{v \in W : \|v\| \ge R\} \subset D$. We need to show that $c_1 \le \sup_W I$, which follows immediately one we have shown that $\gamma_D(W \cup D) \ge 1$. Arguing by contradiction, we suppose that $\gamma_D(W \cup D) = 0$, which implies that there exists an odd continuous map $\chi : W \cup D \to D$ such that $\chi(v) = v$ for every $v \in D$. Since $\chi(0) = 0$ and $\chi(v) = v$ for $v \in W$ with $\|v\| \ge R$, there exists $v_\alpha \in W$ with $\|\chi(v_\alpha)\| = \alpha$, where $\alpha > 0$ is given in Lemma 2.4. Hence $\chi(v_\alpha) \notin D$, a contradiction. $\square$

Lemma 4.5. *We have $c_k < \infty$ for $k \in \mathbb{N}$.*

Proof. Let W be a k-dimensional subspace of functions in X with support contained in $B_{\frac{1}{4}}(0) \subset \mathbb{R}^2$. Then for any two points x, y contained in the support of function $u \in W$ we have $|x - y| < 1$, and this implies that $V_0(u) < 0$ for every $u \in W \setminus \{0\}$. As a consequence, it is easy to deduce from Lemma 2.5 that

$$\big\{u \in W : \|u\| \ge R\big\} \subset D \quad \text{for some } R > 0, \tag{4.7}$$

and that $c_W := \sup_{u \in W} I(u) < \infty$. We claim that

$$\gamma_D\big(I^{c_W}\big) \ge k \quad \text{and therefore} \quad c_k \le c_W. \tag{4.8}$$

Indeed, suppose the opposite; then there exist closed and symmetric subsets $U, V \subset X$ with $I^{c_W} \subset U \cup V$ and such that

(i) $D \subset U$, and there exists an odd continuous map $\chi : U \to D$ such that $\chi(u) = u$ for every $u \in D$;
(ii) $\gamma(V) \le k - 1$.

By Tietze's Theorem we may extend χ to an odd and continuous map $X \to X$ (still denoted by χ). We define

$$\mathcal{O} := \big\{u \in W : \big\|\chi(u)\big\| < \alpha\big\},$$

where α is given by Lemma 2.4. By (4.7) we see that $\mathcal{O}$ is a bounded and symmetric neighborhood of 0 in W. Hence $\gamma(\partial_W \mathcal{O}) = k$ by the Borsuk–Ulam Theorem, where $\partial_W \mathcal{O}$ denotes the relative boundary of $\mathcal{O}$ in W. We note that $\partial_W \mathcal{O} \cap U = \varnothing$, since $\|\chi(u)\| = \alpha$ for every $u \in \partial_W \mathcal{O}$ and $\chi(u) \in D$ for every $u \in U$. Consequently, $\partial_W \mathcal{O} \subset V$ and therefore

$$k = \gamma(\partial_W \mathcal{O}) \le \gamma(V) = k - 1,$$

a contradiction. This shows (4.8), as required. $\square$

We need the following deformation lemma.

Lemma 4.6. *Let $c > 0$. Then there exists $\rho_1 = \rho_1(c) > 0$ such that the following assertions hold for $\rho \in (0, \rho_1)$.*

(i) $A_{c,\rho} \cap D = \varnothing$.
(ii) *There exists $\varepsilon = \varepsilon(c, \rho) > 0$ and an odd and continuous map $\varphi : I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}$ such that $\varphi|_D = \mathrm{id}_D$.*

Proof. We first show that there exists $\rho_1 = \rho_1(c) > 0$ such that (i) holds for $\rho \in (0, \rho_1)$. Suppose by contradiction that this is false; then there exists numbers $\rho_n \to 0$ and $u_n \in A_{c,\rho_n} \cap D$ for every $n \in \mathbb{N}$. There also exist corresponding elements $v_n \in K_c$ such that $\|u_n - v_n\| \le \rho_n$. Moreover, by Proposition 3.1 there exists $x_n \in \mathbb{Z}^2$, $n \in \mathbb{N}$ such that $x_n * v_n \to v \in K_c$ with respect to $\|\cdot\|_X$. By $\mathbb{Z}^2$-translation invariance, we have $x_n * u_n \in A_{c,\rho_n} \cap D$ and

$$\|x_n * u_n - v\| \le \big\|x_n * (u_n - v_n)\big\| + \|x_n * v_n - v\| \le \|u_n - v_n\| + \|x_n * v_n - v\| \to 0 \quad \text{as } n \to \infty.$$

Since $x_n * u_n \in D$ for every n, we then infer from Lemma 2.2(vi) that

$$c = I(v) \le \liminf_{n\to\infty} I(x_n * u_n) \le 0,$$

contrary to our assumption. Hence there exists $\rho_1 = \rho_1(c) > 0$ such that (i) holds for $\rho \in (0, \rho_1)$, as claimed.

Next, we fix $\rho \in (0, \rho_1)$ and we consider

$$S := X \setminus A_{c,\rho} \quad \text{and} \quad \tilde{S}_\delta := \big\{u \in X : \|u - v\| \le \delta \text{ for some } v \in S\big\} \quad \text{for } \delta > 0.$$

We claim that for $\delta > 0$ sufficiently small we have

$$\big\|I'(u)\big\|_{X'}\big(1 + \|u\|_X\big) \ge 8\delta \quad \text{for all } u \in \tilde{S}_{2\delta} \text{ with } c - 2\delta^2 \le I(u) \le c + 2\delta^2. \tag{4.9}$$

Suppose by contradiction that this is false; then there exist sequences of numbers $\delta_n > 0$ and functions $u_n \in \tilde{S}_{2\delta_n}$ such that $\delta_n \to 0$ as $n \to \infty$, whereas

$$\big\|I'(u_n)\big\|_{X'}\big(1 + \|u_n\|_X\big) < 8\delta_n \quad \text{and} \quad c - 2\delta_n^2 \le I(u_n) \le c + 2\delta_n^2 \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 3.1, there exists, after passing to a subsequence, points $x_n \in \mathbb{Z}^2$, $n \in \mathbb{N}$ such that

$$x_n * u_n \to u \quad \text{strongly in } X \text{ as } n \to \infty$$

for some $u \in K_c$. Moreover, there exists $v_n \in S$ such that

$$\|v_n - u_n\| \le 2\delta_n \quad \text{for every } n \in \mathbb{N}.$$

It then follows that

$$\|x_n * v_n - u\| \le \|x_n * v_n - x_n * u_n\| + \|x_n * u_n - u\| \le \|v_n - u_n\| + \|x_n * u_n - u\| \to 0 \quad \text{as } n \to \infty.$$

Since, by $\mathbb{Z}^2$-translation invariance, $x_n * v_n \in \tilde{S}_{\delta_n}$ for every n, this implies that u is contained in the closure of S with respect to $\|\cdot\|$, so that

$$\|u - v\| \ge \rho \quad \text{for every } v \in K_c.$$

This contradicts the fact that u itself is contained in K_c. We conclude that (4.9) holds for $\rho > 0$ sufficiently small. Since $\|\cdot\| \le \|\cdot\|_X$, we may now fix $\delta > 0$ such that

$$\varepsilon := \delta^2 < \frac{c}{2} \tag{4.10}$$

and such that statement (4.9) holds with $\tilde{S}_{2\delta}$ replaced by the subset

$$S_{2\delta} := \big\{u \in X : \|u - v\|_X \le \delta \text{ for some } v \in S\big\}.$$

By [19, Lemma 2.6], there exists a continuous function $\eta : [0,1] \times X \to X$ such that

(i) $\eta(t,u) = u$ if $t = 0$ or $u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;
(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$;
(iii) $t \mapsto I(\eta(t,u))$ is nonincreasing for all $u \in X$.

An inspection of the proof of [19, Lemma 2.6] also shows that η can be constructed such that $\eta(t,-u) = -\eta(t,u)$ for $t \in [0,1]$, $u \in X$ since the underlying functional I is even in u. Since $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \varnothing$ by (4.10), the map

$$\varphi : I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}, \qquad \varphi(u) = \eta(1,u)$$

has the desired properties. $\square$

Corollary 4.7. *The values c_k are critical values of the functional I.*

Proof. Suppose by contradiction that $K_{c_k} = \varnothing$ for some k. Then also $A_{c_k,\rho} = \varnothing$ for every $\rho > 0$, and thus Lemma 4.6 yields $\varepsilon > 0$ and an odd and continuous map $\eta : I^{c_k+\varepsilon} \to I^{c_k-\varepsilon}$ such that $\eta|_D = \mathrm{id}_D$. This contradicts the definition of c_k. $\square$

Proposition 4.8. $c_k \to \infty$ *as* $k \to \infty$.

Proof. Suppose by contradiction that

$$c_k \to c < \infty \quad \text{as } k \to \infty.$$

By Proposition 4.1 and Lemma 4.6, there exists $\rho, \varepsilon > 0$ such that $\gamma(A_{c,\rho}) < \infty$, $A_{c,\rho} \cap D = \varnothing$ and such that there exists an odd and continuous map $\varphi : I^{c+\varepsilon} \setminus A_{c,\rho} \to I^{c-\varepsilon}$ such that $\varphi|_D = \mathrm{id}_D$. Consequently,

$$\gamma_D\big(I^{c+\varepsilon} \setminus A_{c,\rho}\big) \le \gamma_D\big(I^{c-\varepsilon}\big) < \infty$$

by definition of c. Moreover, by the subadditivity property,

$$\gamma_D\big(I^{c+\varepsilon}\big) \le \gamma_D\big(I^{c+\varepsilon} \setminus A_{c,\rho}\big) + \gamma(A_{c,\rho}) < \infty,$$

which contradicts the fact that $c + \varepsilon > c_k$ for all k. The proof is finished. $\square$

Proof of Theorem 1.1 (completed). By Lemma 4.5, Corollary 4.7 and Proposition 4.8, there exists a sequence of critical points $\pm u_k \in X$ of I, $k \in \mathbb{N}$ such that $I(u_k) = c_k \to \infty$. Moreover, by Lemma 4.4, we have that

$$c_1 = \inf_{\mathcal{N}} I = \inf_{u \in X\setminus\{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0,$$

and by Corollary 4.7 this value is attained by a critical point of I (which is contained in $\mathcal{N}$).

Next, let $u \in \mathcal{N}$ be an arbitrary minimizer of $I|_{\mathcal{N}}$. We show that u is a critical point of I. Suppose by contradiction that $I'(u)v < 0$ for some $v \in X$. Since I' is continuous, there exists $\delta, \rho > 0$ such that

$$I'\big(t(u + sv)\big)v < 0 \quad \text{for } t \in [1-\delta, 1+\delta],\ s \in [-\rho, \rho]. \tag{4.11}$$

Moreover, since $I'((1-\delta)u)u > 0 > I'((1+\delta)u)u$, there exists $\hat{s} \in (0, \rho)$ with

$$I'\big((1-\delta)(u + \hat{s}v)\big)(u + \hat{s}v) > 0 > I'\big((1+\delta)(u + \hat{s}v)\big)(u + \hat{s}v),$$

which implies that $\hat{t}(u + \hat{s}v) \in \mathcal{N}$ for some $\hat{t} \in (1-\delta, 1+\delta)$. By Lemma 2.5 and (4.11), we find that

$$I\big(\hat{t}(u + \hat{s}v)\big) - I(u) \le I\big(\hat{t}(u + \hat{s}v)\big) - I(\hat{t}u) = -\int_0^{\hat{s}} I'\big(\hat{t}(u + sv)\big)v \, ds < 0,$$

contradicting the fact that u is a minimizer of $I|_{\mathcal{N}}$. Hence u is a critical point of I, as claimed.

Finally, if $u \in \mathcal{N}$ is a minimizer of $I|_{\mathcal{N}}$, it is easy to see that $|u| \in \mathcal{N}$ is a minimizer of $I|_{\mathcal{N}}$ as well, so it is a critical point of I by the considerations above. By Proposition 2.3, $|u| \in W^{2,p}(\mathbb{R}^2)$ for every $p \ge 1$, and $-\Delta|u| + q|u| = 0$ on $\mathbb{R}^2$ with some function $q \in L^\infty_{loc}(\mathbb{R}^2)$. The strong maximum principle and the fact that $u \not\equiv 0$ therefore imply that $|u| > 0$ on $\mathbb{R}^2$, which shows u does not change sign. $\square$

5. The G-invariant setting

This section is devoted to the proof of Theorem 1.2. Since the proof is similar but easier than the proof of Theorem 1.1, we skip some of the details. We thus assume from now on that

$$a \text{ is constant on } \mathbb{R}^2. \tag{5.1}$$

Moreover, we let $G \subset O(2)$ and $\tau : G \to \{-1, 1\}$ satisfy the assumptions of Theorem 1.2. Our aim is to detect critical points of the restriction of the function I to the invariant subspace

$$X_G := \{u \in X \,:\, A * u = u \text{ for all } A \in G\},$$

where $*$ denotes the group action defined in (1.8). By the principle of symmetric criticality, any critical point of the restriction of I to X_G (which we will denote by I as well in the following) is a critical point of I. We have the following compactness condition which improves Proposition 3.1 in the symmetric setting.

Proposition 5.1. *Suppose that G, τ satisfy the assumptions of Theorem 1.2. Then the restriction of I to X_G satisfies the Cerami condition at every energy level $d > 0$. More precisely, if $(u_n)_n$ is a sequence in X_G satisfying*

$$I(u_n) \to d > 0 \quad \text{and} \quad \left\| I'(u_n) \right\|_{X'_G} \big(1 + \|u_n\|_X\big) \to 0 \quad \text{as } n \to \infty,$$

then there exists a nonzero critical point $u \in X_G$ of I such that, after passing to a subsequence,

$$u_n \to u \quad \text{in } X \text{ as } n \to \infty.$$

Proof. The invariance of I under the action G implies that $I'(v)w = 0$ for all $v \in X_G$, $w \in X_G^\perp$, where $X_G^\perp$ denotes the orthogonal complement of X_G in X. Consequently, we have $\|I'(v)\|_{X'_G} = \|I'(v)\|_{X'}$ for all $v \in X_G$. Hence the sequence $(u_n)_n$ satisfies the assumption of Proposition 3.1, and thus we may pass to a subsequence such that

$$x_n * u_n \to u_0 \quad \text{strongly in } X \text{ as } n \to \infty \tag{5.2}$$

for some nonzero critical point $u_0 \in X$ of I and a suitable sequence $(x_n)_n \subset \mathbb{R}^2$. We show that the sequence $(x_n)_n$ is bounded. Suppose by contradiction that $|x_n| \to \infty$ after passing to a subsequence. We may assume that $\frac{x_n}{|x_n|} \to z_0$ as $n \to \infty$. By assumption (1.7), there exists $A \in G$ with $Az_0 \neq z_0$. Put $y_n := -x_n$ for $n \in \mathbb{N}$. Then we have, since $A * u_n = u_n$ for every $n \in \mathbb{N}$,

$$\begin{aligned} \left| A * (y_n * u_0) - y_n * u_0 \right|_2 &\le \left| A * (y_n * u_0 - u_n) \right|_2 + |u_n - y_n * u_0|_2 = 2|y_n * u_0 - u_n|_2 \\ &= 2|u_0 - x_n * u_n|_2 \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{5.3}$$

However,

$$\begin{aligned} \left| A * (y_n * u_0) - y_n * u_0 \right|_2^2 - 2|u_0|_2^2 &= -2\tau(A) \int_{\mathbb{R}^2} u_0\big(A^{-1}x - y_n\big) u_0(x - y_n)\, dx \\ &= -2\tau(A) \int_{\mathbb{R}^2} u_0\big(A^{-1}(z - z_n)\big) u_0(z)\, dz \\ &= -2 \int_{\mathbb{R}^2} (z_n * v) u_0 \, dz \end{aligned} \tag{5.4}$$

with $v := A * u_0$ and $z_n := Ay_n - y_n$. Moreover, since $Az_0 \neq z_0$, we find that

$$|z_n| = |x_n - Ax_n| = |x_n| \left| Az_0 - z_0 + o(1) \right| \to \infty \quad \text{as } n \to \infty,$$

and therefore

$$\int_{\mathbb{R}^2} (z_n * v) u_0 \, dx \to 0 \quad \text{as } n \to \infty.$$

Consequently, (5.4) implies that

$$\left| A * (y_n * u_0) - y_n * u_0 \right|_2 \to 2|u_0|_2 \neq 0,$$

contrary to (5.3). The contradiction shows that $(x_n)_n$ is bounded, so that $x_n \to x_0$ as $n \to \infty$ after passing to a subsequence. Hence (5.2) implies that $u_n \to u := (-x_0) * u_0$ in X, where u is also a nonzero critical point of I. Moreover, $u \in X_G$ since X_G is closed and $u_n \in X_G$ for every $n \in \mathbb{N}$. □

As a consequence of Proposition 5.1, the sets

$$K_c := \{u \in X_G \,:\, I'(u) = 0,\ I(u) = c\} \quad c > 0$$

are compact and therefore satisfy $\gamma(K_c) < \infty$, where, as before, the Krasnoselski genus γ is defined with respect to continuity in the norm $\|\cdot\|_X$. Setting

$$I^c := \{u \in X_G \,:\, I(u) \le c\} \quad \text{for } c \in \mathbb{R} \quad \text{and} \quad D := I^0,$$

we have the following deformation lemma.

Lemma 5.2. *Let $c > 0$. Then for every neighborhood U of K_c in X_G with $U \cap D = \varnothing$ there exists $\varepsilon > 0$ and an odd and continuous map $\varphi : I^{c+\varepsilon} \setminus U \to I^{c-\varepsilon}$ such that $\varphi|_D = \mathrm{id}_D$.*

Proof. Let

$$S := X_G \setminus U \quad \text{and} \quad S_\delta := \{u \in X_G \,:\, \|u - v\|_{X_G} \le \delta \text{ for some } v \in S\} \quad \text{for } \delta > 0.$$

It is standard to deduce from Proposition 5.1 that there exists $\delta \in (0, \sqrt{c/2})$ such that

$$\|I'(u)\|_{X'_G}(1 + \|u\|_{X_G}) \ge 8\delta \quad \text{for all } u \in S_{2\delta} \text{ with } c - 2\delta^2 \le I(u) \le c + 2\delta^2. \tag{5.5}$$

Let $\varepsilon := \delta^2$. By [19, Lemma 2.6], there exists a continuous function $\eta : [0,1] \times X_G \to X_G$ such that

(i) $\eta(t,u) = u$ if $t = 0$ or $u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;
(ii) $\eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}$;
(iii) $t \mapsto I(\eta(t,u))$ is nonincreasing for all $u \in X_G$.

Moreover, η can be constructed such that $\eta(t,-u) = -\eta(t,u)$ for $t \in [0,1]$, $u \in X_G$ since the underlying functional I is even in u. Since $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \varnothing$, the map

$$\varphi : I^{c+\varepsilon} \setminus U \to I^{c-\varepsilon}, \qquad \varphi(u) = \eta(1,u)$$

has the asserted properties. □

For closed symmetric subsets $D \subset Y \subset X_G$ we now define the relative genus $\gamma_D(Y)$ as in Definition 4.2, noting that Lemma 4.3 still holds. We then define the nondecreasing sequence of values

$$c_k := \inf\{c \ge 0 \,:\, \gamma_D(I^c) \ge k\}, \quad k \in \mathbb{N}.$$

Precisely as in the proof of Lemma 4.4, we see that

$$c_1 := \inf_{\mathcal{N}_G} I = \inf_{u \in X_G \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0, \tag{5.6}$$

where $\mathcal{N}_G := \{u \in X_G \setminus \{0\} \,:\, I'(u)u = 0\}$. Moreover we have

$$c_k < \infty \quad \text{for } k \in \mathbb{N}. \tag{5.7}$$

The proof of (5.7) is similar to the one of Lemma 4.5. The only difference is that we have to choose, for given $k \in \mathbb{N}$, a k-dimensional subspace W of functions in X_G with support contained in $B_{\frac{1}{4}}(0) \subset \mathbb{R}^2$. In the case where $\tau \equiv 1$, we may simply choose a space W consisting of radial C^1-functions supported in $B_{\frac{1}{4}}(0)$. In the case where $\tau \not\equiv 1$, G is assumed to be finite. This implies that there exists $x_0 \in \mathbb{R}^2 \setminus \{0\}$ with $Ax_0 \ne x_0$ for all $A \in G \setminus \{\mathrm{id}\}$, since the set

$$\{x \in \mathbb{R}^2 \,:\, Ax = x \text{ for some } A \in G,\ A \ne \mathrm{id}\}$$

is a finite union of subspaces of $\mathbb{R}^2$ of dimension less than or equal to one. As a consequence,

$$Ax_0 \ne Bx_0 \quad \text{for all } A, B \in G \text{ with } \tau(A) = 1,\ \tau(B) = -1. \tag{5.8}$$

Without loss, we may assume that $|x_0| < \frac{1}{4}$. It then follows from (5.8) that there exists an open neighborhood $U \subset B_{\frac{1}{4}}(0)$ of x_0 such that

$$A(U) \cap B(U) = \varnothing \quad \text{for all } A, B \in G \text{ with } \tau(A) = 1,\ \tau(B) = -1. \tag{5.9}$$

We may therefore pick a k-dimensional subspace $\tilde{W}$ of C^1-functions supported in U and define

$$W := \Bigg\{ \sum_{A \in \tau^{-1}(1)} \psi \circ A^{-1} - \sum_{B \in \tau^{-1}(-1)} \psi \circ B^{-1} : \psi \in \tilde{W} \Bigg\}.$$

By (5.9), W is a k-dimensional subspace of X_G of functions supported in $B_{\frac{1}{4}}(0)$. We may thus proceed as in the proof of Lemma 4.5 to obtain (5.7).

With the help of Lemma 5.2, we now deduce (similarly as in Corollary 4.7) that

$$\text{the values } c_k,\ k \in \mathbb{N},\ \text{are critical values of the functional } I \in C^1(X_G, \mathbb{R}). \tag{5.10}$$

We also observe that

$$c_k \to \infty \quad \text{as } k \to \infty. \tag{5.11}$$

This follows as in the proof of Proposition 4.8, with $A_{c,\rho}$ replaced by an arbitrary neighborhood U of K_c with $\gamma(U) < \infty$ and $U \cap D = \varnothing$. The existence of such a neighborhood follows from the compactness of K_c (see e.g. [31, Chapter II.5]).

By combining the statements (5.6), (5.7), (5.10) and (5.11), we may now finish the proof of Theorem 1.2 by the same arguments as in the proof of Theorem 1.1 (see the end of Section 4). The only difference is given by the fact that the implication

$$u \in X_G \quad \Longrightarrow \quad |u| \in X_G$$

is only true if $\tau \equiv 1$, and thus the last assertion in Theorem 1.2 requires this extra assumption.

6. Symmetry and uniqueness of positive solutions

This section is concerned with the proof of Theorem 1.3. More precisely, we will prove a symmetry result in a slightly more general setting and deduce Theorem 1.3 afterwards. Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz with $f(0) = 0$. We study classical solutions (u, w) of the nonlinear system

$$\begin{cases} -\Delta u = wu + f(u) & \text{in } \mathbb{R}^2, \\ -\Delta w = 2\pi u^2 & \text{in } \mathbb{R}^2, \end{cases} \tag{6.1}$$

subject to the conditions

$$u \in L^\infty\big(\mathbb{R}^2\big) \quad \text{and} \quad w(x) \to -\infty \quad \text{as } |x| \to \infty. \tag{6.2}$$

By Agmon's Theorem (see [2]), (6.1) and (6.2) imply that

$$u(x) = o\big(e^{-\alpha|x|}\big) \quad \text{as } |x| \to \infty \text{ for every } \alpha > 0. \tag{6.3}$$

Moreover, since every semibounded harmonic function $\mathbb{R}^2 \to \mathbb{R}$ is constant, we have

$$w(x) = c - \int_{\mathbb{R}^2} \log|x-y| u^2(y)\, dy \quad \text{for } x \in \mathbb{R}^2 \text{ with a constant } c \in \mathbb{R}. \tag{6.4}$$

The main result of this section is the following.

Theorem 6.1. *Every classical solution (u, w) of (6.1), (6.2) with $u > 0$ in $\mathbb{R}^2$ is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover, if $f(u) = -au$ with a constant $a > 0$, then (u, w) is unique up to translation.*

The proof will be given by a variant of the moving plane method. For $\lambda \in \mathbb{R}$, we put

$$H_\lambda := \{x \in \mathbb{R}^2 : x_1 > \lambda\}, \qquad T_\lambda = \partial H_\lambda = \{x \in \mathbb{R}^2 : x_1 = \lambda\}.$$

Moreover, we let $\mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto x^\lambda$ denote the reflection of x at T_λ. From now on, we consider a fixed solution of (6.1), (6.2) with $u > 0$, and we set

$$u^\lambda(x) = u(x^\lambda), \qquad w^\lambda(x) = w(x^\lambda) \quad \text{for } x \in \mathbb{R}^2,\ \lambda \in \mathbb{R}.$$

On H_λ we define the difference functions

$$u_\lambda = u^\lambda - u, \qquad w_\lambda = w^\lambda - w$$

which satisfy the system of equations

$$\begin{cases} -\Delta u_\lambda + u_\lambda = w_\lambda u^\lambda + (w + h_\lambda)u_\lambda, \\ -\Delta w_\lambda = 2\pi\left[(u^\lambda)^2 - u^2\right] = 2\pi\left(u^\lambda + u\right)u_\lambda \end{cases} \quad \text{in } \mathcal{H}_\lambda \tag{6.5}$$

with

$$h_\lambda(x) := \begin{cases} \frac{f(u^\lambda(x)) - f(u(x))}{u(x)} + 1, & \text{if } u(x) \neq 0, \\ 1, & \text{if } u(x) = 0. \end{cases}$$

Since f is Lipschitz continuous on $[0, \|u\|_{L^\infty(\mathbb{R}^2)}]$, there exists a constant $C = C(u) > 0$ such that

$$\|h_\lambda\|_{L^\infty(H_\lambda)} \le C \quad \text{for every } \lambda \in \mathbb{R}. \tag{6.6}$$

It follows from (6.4) that

$$w_\lambda(x) = \int_{H_\lambda} \log\frac{|x - y^\lambda|}{|x - y|}\left(u^\lambda(y) + u(y)\right)u_\lambda(y)\,dy \quad \text{for } x \in H_\lambda. \tag{6.7}$$

Since $\log\frac{|x-y^\lambda|}{|x-y|} > 0$ for every $x, y \in H_\lambda$, we have the implication

$$u_\lambda \ge 0 \quad \text{in } H_\lambda \quad \Longrightarrow \quad w_\lambda \ge 0 \quad \text{in } H_\lambda \tag{6.8}$$

for every $\lambda \in \mathbb{R}$. In the following, we let $v^- := \min\{v, 0\}$ denote the negative part of a function defined on a subset of $\mathbb{R}^2$. Note that v^- is a nonpositive function with this convention. We need the following estimate:

Lemma 6.2. *There exists a constant $\kappa > 0$ such that*

$$\left\|w_\lambda^-\right\|_{L^2(H_\lambda)} \le \kappa c_\lambda \left\|u_\lambda^-\right\|_{L^2(H_\lambda)} \quad \textit{for every } \lambda \in \mathbb{R},$$

where

$$c_\lambda = \left(\int_{M_\lambda} (y_1 - \lambda)^2 u^2(y)\,dy\right)^{\frac{1}{2}} \quad \textit{and} \quad M_\lambda := \{x \in H_\lambda : u_\lambda(x) < 0\}.$$

Proof. We note that

$$0 \le \log\frac{|x - y^\lambda|}{|x - y|} \le \log\left(1 + \frac{|y - y^\lambda|}{|x - y|}\right) \le \frac{|y - y^\lambda|}{|x - y|} = \frac{2(y_1 - \lambda)}{|x - y|} \quad \text{for } x, y \in H_\lambda.$$

Since also $u_\lambda(y) < 0$ implies that $0 \le u^\lambda(y) \le u(y)$, we may use the integral representation (6.7) to conclude that

$$w_\lambda^-(x) \ge \int_{M_\lambda} \frac{2(y_1 - \lambda)}{|x - y|}\left(u^\lambda(y) + u(y)\right)u_\lambda^-(y)\,dy \ge 4\int_{M_\lambda} \frac{(y_1 - \lambda)}{|x - y|} u(y) u_\lambda^-(y)\,dy$$

for $x \in H_\lambda$, so that, by the Hardy–Littlewood–Sobolev inequality,

$$\left\|w_\lambda^-\right\|_{L^2(H_\lambda)} \le \kappa\, c_\lambda \left\|u_\lambda^-\right\|_{L^2(H_\lambda)}$$

with a constant $\kappa > 0$ independent of λ, as claimed. □

Lemma 6.3. *There exists $\bar{\lambda} > 0$ such that $u_\lambda \geq 0$ in H_λ for $\lambda \geq \bar{\lambda}$.*

Proof. By (6.2) and (6.6), we may choose $\lambda_1 > 0$ such that $w + h_\lambda \leq 0$ in H_λ for $\lambda \geq \lambda_1$. Multiplying the first equation in (6.5) by u_λ^- and integrating, we may estimate with the help of Lemma 6.2

$$\|u_\lambda^-\|_{L^2(H_\lambda)}^2 \leq \|u_\lambda^-\|_{H^1(H_\lambda)}^2 = \int_{H_\lambda} \big[w_\lambda u^\lambda u_\lambda^- + (w+h_\lambda)\big(u_\lambda^-\big)^2\big]\,dx \leq \int_{H_\lambda} w_\lambda^- u^\lambda u_\lambda^-\,dx$$
$$\leq \|w_\lambda^-\|_{L^2(H_\lambda)}\|u^\lambda\|_{L^\infty(H_\lambda)}\|u_\lambda^-\|_{L^2(H_\lambda)} \leq \kappa\, c_\lambda \|u\|_{L^\infty(\mathbb{R}^2)}\|u_\lambda^-\|_{L^2(H_\lambda)}^2.$$

Since $c_\lambda \to 0$ as $\lambda \to \infty$ by (6.3), there exists $\bar{\lambda} \geq \lambda_1$ such that

$$\kappa\, c_\lambda \|u\|_{L^\infty(\mathbb{R}^2)} < 1 \quad \text{for } \lambda \geq \bar{\lambda},$$

so that $u_\lambda^- \equiv 0$ on H_λ for $\lambda \geq \bar{\lambda}$, as claimed. □

Lemma 6.4. *If $\lambda \in \mathbb{R}$ is such that $u_\lambda \geq 0$ in H_λ, then also $w_\lambda \geq 0$ on H_λ. Moreover, either $u_\lambda \equiv 0 \equiv w_\lambda$ or*

$$u_\lambda > 0, \qquad w_\lambda > 0 \quad \textit{on } H_\lambda \tag{6.9}$$

and

$$\frac{\partial u}{\partial x_1} < 0, \qquad \frac{\partial w}{\partial x_1} < 0 \quad \textit{on } T_\lambda. \tag{6.10}$$

Proof. We already noted in (6.8) that $u_\lambda \geq 0$ in H_λ implies $w_\lambda \geq 0$ in H_λ. Moreover, if $u_\lambda \not\equiv 0$, then w_λ is strictly positive in H_λ and

$$\frac{\partial w_\lambda}{\partial x_1} = -2\frac{\partial w}{\partial x_1} > 0 \quad \text{on } T_\lambda$$

by the Hopf lemma. Conversely, if $w_\lambda \not\equiv 0$, then also $u_\lambda \not\equiv 0$ by (6.7), and u_λ satisfies

$$-\Delta u_\lambda + (1 - w - h_\lambda)u_\lambda \geq u^\lambda w_\lambda \geq 0 \quad \text{in } H_\lambda.$$

Hence $u_\lambda > 0$ in H_λ by the maximum principle, and

$$\frac{\partial u_\lambda}{\partial x_1} = -2\frac{\partial u}{\partial x_1} > 0 \quad \text{on } T_\lambda$$

by the Hopf lemma. □

Lemma 6.5. *Let $\lambda \in \mathbb{R}$. If $u_\lambda > 0$ in H_λ, then there exists $\varepsilon > 0$ such that $u_\mu \geq 0$ in H_μ for $\mu \in (\lambda - \varepsilon, \lambda)$.*

Proof. Let $B_R := B_R(0)$ for $R > 0$. By (6.2), (6.3) and (6.6), we may fix $R > 1$ large enough such that

$$w + h_\mu \leq 0 \quad \text{in } H_\mu \setminus B_R \text{ for every } \mu \in \mathbb{R} \tag{6.11}$$

and

$$\Bigg(\int_{\mathbb{R}^2 \setminus B_R} (y_1 - \mu)^2 u^2(y)\,dy\Bigg)^{\frac{1}{2}} < \frac{1}{\kappa \|u\|_{L^\infty(\mathbb{R}^2)}} \qquad \text{for every } \mu \in [\lambda - 1, \lambda], \tag{6.12}$$

where κ is as in Lemma 6.2. Moreover, by (6.9), (6.10) and the continuity of u, $\frac{\partial u}{\partial x_1}$, there exists $0 < \varepsilon < 1$ such that

$$u_\mu > 0 \quad \text{in } H_\mu \cap B_R \text{ for } \mu \in (\lambda - \varepsilon, \lambda]. \tag{6.13}$$

Arguing similarly as in the proof of Lemma 6.2 with $\mu \in (\lambda - \varepsilon, \lambda]$ in place of λ, we may multiply the first equation in (6.5) – with μ in place of λ – with u_μ^- and integrate, obtaining the estimate

$$\begin{aligned}\|u_\mu^-\|^2_{L^2(H_\mu)} \le \|u_\mu^-\|^2_{H^1(H_\mu)} &= \int\limits_{H_\mu\setminus B_R} \left[w_\mu u^\mu u_\mu^- + (w+h_\mu)(u_\mu^-)^2\right] dx \le \int\limits_{H_\mu\setminus B_R} w_\mu^- u^\mu u_\mu^- \, dx \\ &\le \|w_\mu^-\|_{L^2(H_\mu)} \|u^\mu\|_{L^\infty(H_\mu)} \|u_\mu^-\|_{L^2(H_\mu)} \le \kappa\, c_\mu \|u\|_{L^\infty(\mathbb{R}^2)} \|u_\mu^-\|^2_{L^2(H_\mu)}. \end{aligned} \tag{6.14}$$

Here we used (6.13) in the second step, (6.11) in the third step and Lemma 6.2 in the last step. We note that (6.12) and (6.13) also imply that

$$c_\mu \le \left(\int\limits_{\mathbb{R}^2\setminus B_R} (y_1-\mu)^2 u^2(y)\, dy \right)^{\frac{1}{2}} < \frac{1}{\kappa \|u\|_{L^\infty(\mathbb{R}^2)}} \quad \text{for } \mu \in (\lambda-\varepsilon, \lambda].$$

Consequently, (6.14) implies that $u_\mu^- \equiv 0$ for $\mu \in (\lambda-\varepsilon,\lambda]$. □

Proof of Theorem 6.1 (completed). Put

$$\lambda_1 := \inf\{\lambda \in \mathbb{R} : u_\lambda \ge 0 \text{ in } H_\lambda\}$$

By Lemma 6.3 we have $\lambda_1 < \infty$, whereas the positivity of u and (6.3) imply that $\lambda_1 > -\infty$. Moreover, as a consequence of Lemmas 6.4 and 6.5, we have that $u_{\lambda_1} \equiv 0$ and $w_{\lambda_1} \equiv 0$. Repeating the same argument with x_1 replaced by the second coordinate direction x_2, we also find $\lambda_2 \in \mathbb{R}$ such that u and w are symmetric with respect to the hyperplane $\{x \in \mathbb{R}^2 : x_2 = \lambda_2\}$. Now consider $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and the translated functions

$$\tilde u, \tilde w : \mathbb{R}^2 \to \mathbb{R}, \qquad \tilde u(x) = u(x-\lambda), \qquad \tilde w(x) = w(x-\lambda).$$

These functions also solve (6.1), (6.2) and satisfy

$$\tilde u(x) = \tilde u(-x), \qquad \tilde w(x) = \tilde w(-x) \quad \text{for } x \in \mathbb{R}^2.$$

It is then easy to see that every symmetry hyperplane of $\tilde u$ and $\tilde w$ must contain the origin. Consequently, repeating the moving plane procedure in an arbitrary direction in place of the x_1-coordinate direction, we obtain that $\tilde u$ and $\tilde w$ are symmetric with respect to any hyperplane containing zero, hence radially symmetric. The uniqueness of positive, radial solutions in the case where $f(u) = -au$ with $a > 0$ is proved in [12, Theorem 1.1.]. □

Proof of Theorem 1.3 (completed). Let $u \in X$ be a solution of (1.5) with constant a. By Proposition 2.3, u and the function $w : \mathbb{R}^2 \to \mathbb{R}$ defined by (6.4) satisfy condition (6.2), and (u, w) is a classical solution of (6.1) with the Lipschitz continuous function $f(u) = b|u|^{p-2}u - au$ (here we need the assumption $p \ge 2$). Consequently, the claim follows from Theorem 6.1. □

Conflict of interest statement

No conflict of interest is declared.

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