Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 221–242

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Qualitative analysis of rupture solutions for a MEMS problem

Juan Dávila [a,*], Kelei Wang [b], Juncheng Wei [c,d]

[a] *Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*
[b] *Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, China*
[c] *Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong*
[d] *Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada*

Received 11 April 2013; received in revised form 2 October 2013; accepted 2 September 2014

Available online 16 October 2014

Abstract

We prove sharp Hölder continuity and an estimate of rupture sets for sequences of solutions of the following nonlinear problem with negative exponent

$$\Delta u = \frac{1}{u^p} \quad \text{in } \Omega, \; p > 1.$$

As a consequence, we prove the existence of rupture solutions with isolated ruptures in a bounded convex domain in $\mathbb{R}^2$.
© 2014 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

Keywords: Semilinear elliptic equations with negative power; Hölder continuity; Monotonicity formula

1. The setting and main results

Of concern is the following MEMS problem in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\Delta u = u^{-p} \quad \text{in } \Omega \tag{1.1}$$

where $p > 1$.

Problem (1.1) arises in modeling an electrostatic Micro-Electromechanical System (MEMS) device. We refer to the books by Pelesko and Bernstein [18] for physical derivations and Esposito, Ghoussoub and Guo [7] for mathematical analysis.

Of special interest are solutions that give rise to singularities in the equation, that is, such that $u \approx 0$ in some region, which in the physical model represents a **rupture** in the device. The main result of this paper is to give a sharp estimate on the Hölder continuity of solutions near the ruptures and estimates on Hausdorff dimensions of such rupture sets under natural energy assumptions.

* Corresponding author.
E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), wangkelei@wipm.ac.cn (K. Wang), wei@math.cuhk.edu.hk (J. Wei).

http://dx.doi.org/10.1016/j.anihpc.2014.09.009
0294-1449/© 2014 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

We now state our main results.

Theorem 1.1. *Let (u_i) be a sequence of positive smooth solutions to (1.1) in $B_2(0)$, satisfying*

$$\sup_i \int_{B_2(0)} |\nabla u_i|^2 + u_i^{1-p} + u_i^2 = M < +\infty, \tag{1.2}$$

where $B_2(0) \subset \mathbb{R}^n$ is the open ball of radius 2. Then:

- *The functions u_i are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{B_1})$;*
- *Up to subsequence, u_i converges uniformly to u_∞ in B_1, strongly in $H^1(B_1)$, and u_i^{-p} converges to u_∞^{-p} in $L^1(B_1)$;*
- *Outside $\{u_\infty = 0\}$, u_i converges to u_∞ in any C^k norm, for any k;*
- *u_∞ is a* stationary *solution of (1.1).*

By a solution we mean that $u \in H^1$, $u^{-p} \in L^1$ and it satisfies (1.1) in the sense of distributions. We say a solution $u \in H^1 \cap L^{1-p}$ is stationary if for any smooth vector field Y with compact support,

$$\int \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \operatorname{div} Y - DY(\nabla u, \nabla u) = 0. \tag{1.3}$$

For positive smooth solutions this condition follows from variations of the energy functional

$$E(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{u^{1-p}}{p-1}$$

with respect to perturbations of the parametrization of the domain, that is,

$$\frac{d}{dt} E\big(u\big(x + tY(x)\big)\big)\Big|_{t=0} = 0. \tag{1.4}$$

Formula (1.3) can also be obtained by multiplying (1.1) by $Y \cdot \nabla u$ and integrating by parts. Such condition is classical in many works dealing with partial regularity, for example in the work Evans [8] and Bethuel [1] in obtaining partial regularity for harmonic maps. It also appeared in the work of Pacard [16] on partial regularity results for weak solutions of semilinear supercritical equations.

In harmonic map and many other problems, it is not always true that the weak limit of stationary solutions is still stationary. For the problem (1.1), it is also not so direct to prove that the weak limit of distributional solutions is still a distributional solution. This is where the uniform Hölder continuity enters into our arguments. In particular, this uniform Hölder continuity is crucial for the establishment of those strong convergence in the above theorem.

Next we consider the partial regularity problem for stationary solutions.

Theorem 1.2. *Assume u is a $C^{\frac{2}{p+1}}$ continuous, stationary solution of (1.1). Then $\{u = 0\}$ is a closed set with Hausdorff dimension no more than $n - 2$. Moreover, if $n = 2$, $\{u = 0\}$ is a discrete set.*

Previous estimates on the zero set of solutions include Jiang, Lin, who prove that the dimension of $\{u = 0\}$ is at most $n - 2 + \frac{4}{p+2}$, for solutions $u \in H^1$, $u^{-p} \in L^1$. The exponent was later improved to $n - 2 + \frac{2}{p+1}$ by Dupaigne, Ponce and Porretta [5] for the same class of solutions. Jiang and Lin [13] and Guo and Wei [12] also considered *finite energy* solutions, and proved that the dimension of the zero set of solutions is at most $n - 2 + \frac{4}{p+1}$. Dávila and Ponce [4] improved the exponent for these solutions to $n - 2 + \frac{2}{p+1}$. In all these cases it is not known whether the best exponent obtained is optimal. The dimension estimate in Theorem 1.2 is the smallest compared to these previous results, although we assume a Hölder and stationary condition. However it is optimal because in 2 dimensions $u(r) = c_p r^{\frac{2}{p+1}}$ is a radial singular solution, which satisfies the conditions in Theorem 1.2.

We also would like to mention that, the sharp Hölder continuity corresponds to the classical $C^{1,1}$ regularity in the obstacle problem (see for example Caffarelli [2]),

$$\Delta u = \chi_{\{u>0\}}, \quad u \geq 0,$$

which can be viewed as the special case $p = 0$ of (1.1). When $p \in (0, 1)$, the $C^{1,\frac{1-p}{p+1}}$ regularity for minimizers was also studied in Phillips [17]. Since we will use the blow up analysis and Federer's dimension reduction principle to prove Theorem 1.2, which involve various steps of passing to the limit, this uniform continuity and the consequent strong convergence property will play an important role in this proof.

As an application of the preceding theorems, we consider the original MEMS problem in a bounded domain

$$-\Delta v = \frac{\lambda}{(1-v)^p} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Here rupture means $v = 1$.

It is known that there exists a critical parameter $\lambda_* > 0$ such that for $\lambda < \lambda_*$, problem (1.5) has a minimal solution and for $\lambda > \lambda_*$ there are no positive solutions. In [6], Esposito, Ghoussoub and Guo showed that when $n \leq 7$, the extremal solution at λ_* is smooth and hence there is a secondary bifurcation near λ_*. When the domain is convex, it is known that the only solutions for λ small is the minimal solution. Thus by Rabinowitz's bifurcation theorem [19], there exists a sequence of $\lambda_i \geq c_0 > 0$ and a sequence of solutions $\{u_i = 1 - v_i\}$ such that $\min u_i \to 0$. By convexity of Ω and the moving plane method, there is a neighborhood Ω_δ of $\partial\Omega$ such that u_i remains uniformly positive in Ω_δ (see Lemma 3.2 in [12]). As a consequence of Theorem 1.1, u_i are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{\Omega})$ and hence converges uniformly to a Hölder continuous function u_∞ with nonempty rupture set $\{u_\infty = 0\}$. Applying Theorem 1.2 we obtain the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a convex set. Then there exists a $\lambda^* > 0$ such that the following problem*

$$\Delta u = \frac{\lambda^*}{u^p} \quad \text{in } \Omega, \qquad u = 1 \quad \text{on } \partial\Omega \tag{1.6}$$

admits a weak solution u such that u is Hölder continuous and the rupture set of u consists a finite number of points.

Theorem 1.3 was proved by Guo and the third author [12] under the condition that $p < 3$ and that the domain has two axes of symmetries.

The proof of the uniform Hölder estimate for positive solutions in Theorem 1.1 is inspired by the work of Noris, Tavares, Terracini and Verzini [15], where uniform Hölder estimates are established for a strongly competitive Schrödinger system. A contradiction argument leads after scaling to a globally Hölder stationary nontrivial solution of

$$u\Delta u = 0, \quad u \geq 0 \text{ in } \mathbb{R}^n. \tag{1.7}$$

But a Liouville theorem of [15] says that u is trivial. The argument is carried out in Section 2 and we give the Liouville theorem in Appendix A for completeness. The proof of the remaining statements of Theorem 1.1 is given in Section 4, after some preliminaries in Section 3. The proof actually applies to a sequence of stationary solutions having a uniform Hölder bound. Section 5 contains the proof of Theorem 1.2.

2. The uniform Hölder continuity

In this section we prove

Theorem 2.1. *Let u_i be a sequence of positive solutions to (1.1) in B_4 with*

$$\sup_i \int_{B_4} u_i < +\infty.$$

Then

$$\sup_i \|u_i\|_{C^{\frac{2}{p+1}}(\bar{B}_1)} < +\infty.$$

An important result that we will use is the following Liouville type theorem obtained by Noris, Tavares, Terracini and Verzini [15]. By completeness we give a proof in Appendix A.

Theorem 2.2. *Let $\alpha \in (0,1)$. Assume $\bar{u}_\infty \geq 0$, $\bar{u}_\infty \in H^1_{loc}(\mathbb{R}^n)$, is a globally $C^\alpha(\mathbb{R}^n)$ function satisfying*

$$\bar{u}_\infty \Delta \bar{u}_\infty = 0 \quad \text{in } \mathbb{R}^n, \tag{2.1}$$

and that $\bar{u}_\infty$ is stationary, i.e.

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla \bar{u}_\infty|^2 \operatorname{div} Y - DY(\nabla \bar{u}_\infty, \nabla \bar{u}_\infty) = 0,$$

for any vector field $Y \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then $\bar{u}_\infty$ is constant.

The remaining part of this section will be devoted to the proof of Theorem 2.1.

Proof of Theorem 2.1. Note that because u_i is subharmonic and positive,

$$\sup_i \|u_i\|_{L^\infty(B_2(0))} < +\infty.$$

Take $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $B_1(0)$, $\{\eta > 0\} = B_2(0)$, $\eta = 0$ in $\mathbb{R}^n \setminus B_2(0)$. Denote

$$\hat{u}_i = u_i \eta.$$

We will actually prove that

$$\sup_i \|\hat{u}_i\|_{C^{\frac{2}{p+1}}(\bar{B}_2(0))} < +\infty.$$

Assume this is not true. Because $\hat{u}_i$ are smooth in B_2, there exist $x_i, y_i \in B_2(0)$ such that as $i \to +\infty$,

$$L_i = \frac{|\hat{u}_i(x_i) - \hat{u}_i(y_i)|}{|x_i - y_i|^{\frac{2}{p+1}}} = \max_{x,y\in B_2(0), x\neq y} \frac{|\hat{u}_i(x) - \hat{u}_i(y)|}{|x-y|^{\frac{2}{p+1}}} \to +\infty. \tag{2.2}$$

Note that because $\hat{u}_i$ are uniformly bounded, as $i \to +\infty$, $|x_i - y_i| \to 0$.

Denote $r_i = |x_i - y_i|$ and $z_i = (y_i - x_i)/r_i$. Since $|z_i| = 1$, we can assume that $z_i \to z_\infty \in \mathbb{S}^{n-1}$. Define

$$\widetilde{u}_i(x) := L_i^{-1} r_i^{-\frac{2}{p+1}} \hat{u}_i(x_i + r_i x) = L_i^{-1} r_i^{-\frac{2}{p+1}} u_i(x_i + r_i x)\eta(x_i + r_i x),$$

and

$$\bar{u}_i(x) := L_i^{-1} r_i^{-\frac{2}{p+1}} u_i(x_i + r_i x)\eta(x_i).$$

These functions are defined in $\Omega_i = \frac{1}{r_i}(B_2(0) - x_i)$. Note that Ω_i converges to Ω_∞, which may be the entire space or a half space.

We first present some facts about these rescaled functions, which will be used below. By definition we have

$$\widetilde{u}_i(x) = \frac{\eta(x_i + r_i x)}{\eta(x_i)} \bar{u}_i(x),$$

and

$$\begin{aligned}
\nabla \widetilde{u}_i(x) &= \frac{r_i \nabla \eta(x_i + r_i x)}{\eta(x_i)} \bar{u}_i(x) + \frac{\eta(x_i + r_i x)}{\eta(x_i)} \nabla \bar{u}_i(x) \\
&= L_i^{-1} r_i^{\frac{p-1}{p+1}} u_i(x_i + r_i x) \nabla \eta(x_i + r_i x) + \frac{\eta(x_i + r_i x)}{\eta(x_i)} \nabla \bar{u}_i(x) \\
&= \frac{\eta(x_i + r_i x)}{\eta(x_i)} \nabla \bar{u}_i(x) + O\big(L_i^{-1} r_i^{\frac{p-1}{p+1}}\big).
\end{aligned}$$

By (2.2) and noting that $|z_i| = 1$, we have

$$1 = \left|\widetilde{u}_i(0) - \widetilde{u}_i(z_i)\right| = \max_{x,y\in\Omega_i, x\neq y} \frac{|\widetilde{u}_i(x) - \widetilde{u}_i(y)|}{|x-y|^{\frac{2}{p+1}}}. \tag{2.3}$$

Next, because η is Lipschitz continuous in $\overline{B_2(0)}$, for $x \in \Omega_i$, we have a constant C which depends only on $\sup_{B_2(0)} u_i$ and the Lipschitz constant of η, such that

$$\begin{aligned}\left|\widetilde{u}_i(x) - \bar{u}_i(x)\right| &\le \frac{C}{L_i r_i^{\frac{2}{p+1}}} \left|\eta(x_i + r_i x) - \eta(x_i)\right| \\ &\le C L_i^{-1} r_i^{\frac{p-1}{p+1}} |x|. \end{aligned} \tag{2.4}$$

This converges to 0 uniformly on any compact set of Ω_∞ as $i \to +\infty$. By the Lipschitz continuity of η, we also have

$$\widetilde{u}_i(x) \le C L_i^{-1} r_i^{\frac{p-1}{p+1}} \operatorname{dist}(x, \partial\Omega_i). \tag{2.5}$$

Finally, we note that $\bar{u}_i$ satisfies

$$\Delta \bar{u}_i = \varepsilon_i \bar{u}_i^{-p}. \tag{2.6}$$

Here $\varepsilon_i = L_i^{-p-1} \eta(x_i)^{p+1} \to 0$ as $i \to +\infty$.

We divide the proof into two cases.

Case 1. $A_i := \widetilde{u}_i(0) \to +\infty$.

By (2.5),

$$\operatorname{dist}(0, \partial\Omega_i) \ge c L_i r_i^{-\frac{p-1}{p+1}} A_i \to +\infty.$$

Hence Ω_i converges to $\mathbb{R}^n$. By (2.3), we can assume that (after passing to a subsequence of i) $\widetilde{u}_i - A_i$ converges to $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By (2.4), $\bar{u}_i - A_i$ converges to the same $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$.

For any $R > 0$, if i large, (2.3) and (2.4) imply that

$$\inf_{B_R(0)} \bar{u}_i \ge \inf_{B_R(0)} \widetilde{u}_i - C L_i^{-1} r_i^{\frac{p-1}{p+1}} R \ge A_i - R^{\frac{2}{p+1}} - C L_i^{-1} r_i^{\frac{p-1}{p+1}} R \ge \frac{A_i}{2}.$$

So

$$0 \le \Delta(\bar{u}_i - A_i) \le 2^p \varepsilon_i A_i^{-p} \to 0.$$

By standard $W^{2,q}$ estimates, for any $q \in (1, +\infty)$, $\bar{u}_i - A_i$ are uniformly bounded in $W^{2,q}_{loc}(\mathbb{R}^n)$. Then by the Sobolev embedding theorem, for any $\alpha \in (0,1)$, $\bar{u}_i - A_i$ are uniformly bounded in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$. By letting $i \to +\infty$ in (2.6), we see $\bar{u}_\infty$ is a harmonic function on $\mathbb{R}^n$.

By the uniform convergence of $\bar{u}_i - A_i$, we can take the limit in (2.3) to get

$$1 = \left|\bar{u}_\infty(0) - \bar{u}_\infty(z_\infty)\right| = \max_{x,y\in\Omega_i} \frac{|\bar{u}_\infty(x) - \bar{u}_\infty(y)|}{|x-y|^{\frac{2}{p+1}}}.$$

The first equality implies that $\bar{u}_\infty$ is non-constant, while the second one implies that $\bar{u}_\infty$ is globally $2/(p+1)$-Hölder continuous, hence a constant by the Liouville theorem for harmonic functions. This is a contradiction.

Case 2. $A_i := \widetilde{u}_i(0) \to A_\infty \in [0, +\infty)$.

By the first equality in (2.3),

$$1 \le \widetilde{u}_i(0) + \widetilde{u}_i(z_i). \tag{2.7}$$

Then by (2.5),

$$cL_i r_i^{-\frac{p-1}{p+1}} \le \mathrm{dist}(0, \partial\Omega_i) + \mathrm{dist}(z_i, \partial\Omega_i) \le 2\mathrm{dist}(0, \partial\Omega_i) + 1.$$

So we still have $\mathrm{dist}(0, \partial\Omega_i) \to +\infty$, and $\Omega_\infty = \mathbb{R}^n$.

By (2.3), we can assume that (by passing to a subsequence of i) $\widetilde{u}_i$ converges to $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By (2.4), $\bar{u}_i$ converges to the same $\bar{u}_\infty$ uniformly on any compact set of $\mathbb{R}^n$. By this uniform convergence, we can take the limit in (2.7) to get

$$1 \le \bar{u}_\infty(0) + \bar{u}_\infty(z_\infty).$$

So the open set $D := \{\bar{u}_\infty > 0\}$ is non-empty.

Let $D' \Subset D$ be a compact set, and $\delta = \frac{1}{2}\inf_{D'} \bar{u}_\infty$, so that $\delta > 0$. Then if i is large,

$$\inf_{D'} \bar{u}_i \ge \delta.$$

By the same argument as in Case 1, we see

$$\Delta \bar{u}_\infty = 0 \quad \text{in } D.$$

Hence $\bar{u}_\infty$ is smooth in D. In particular, if $\{\bar{u}_\infty = 0\} = \emptyset$, we can use the same argument as in Case 1 to get a contradiction.

In the following we assume $\{\bar{u}_\infty = 0\} \ne \emptyset$. Without loss of generality, assume that $\bar{u}_\infty(0) = 0$.

We claim that

$$\bar{u}_i \to \bar{u}_\infty \quad \text{in } H^1_{loc}(\mathbb{R}^n) \tag{2.8}$$

and

$$\varepsilon_i \bar{u}_i^{1-p} \to 0 \quad \text{in } L^1_{loc}(\mathbb{R}^n) \tag{2.9}$$

Indeed, take a function $\eta \in C_0^\infty(\mathbb{R}^n)$. Testing the equation of $\bar{u}_i$ with $\bar{u}_i\eta^2$, we get

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2\eta^2 + \varepsilon_i \bar{u}_i^{1-p}\eta^2 + 2\bar{u}_i\eta\nabla\bar{u}_i\nabla\eta = 0. \tag{2.10}$$

First, by applying the Cauchy inequality to the last term, we have

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2\eta^2 + \varepsilon_i \bar{u}_i^{1-p}\eta^2 \le 4\int_{\mathbb{R}^n} \bar{u}_i^2|\nabla\eta|^2.$$

Because $\bar{u}_i$ are uniformly bounded in any compact set of $\mathbb{R}^n$, $\bar{u}_i$ are uniformly bounded in $H^1_{loc}(\mathbb{R}^n)$. By the uniform convergence of $\bar{u}_i$, they must converge weakly to $\bar{u}_\infty$ in $H^1_{loc}(\mathbb{R}^n)$.

By taking limit in (2.10), we obtain

$$\lim_{i\to+\infty}\int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2\eta^2 - |\nabla\bar{u}_\infty|^2\eta^2 + \varepsilon_i \bar{u}_i^{1-p}\eta^2 = -\int_{\mathbb{R}^n} |\nabla\bar{u}_\infty|^2\eta^2 + 2\bar{u}_\infty\eta\nabla\bar{u}_\infty\nabla\eta.$$

On the other hand, take a $\sigma > 0$ small so that $\{\bar{u}_\infty = \sigma\}$ is a smooth hypersurface. Then because $\bar{u}_\infty$ is harmonic in $\{\bar{u}_\infty > \sigma\}$,

$$\begin{aligned}\int_{\{\bar{u}_\infty > \sigma\}} |\nabla\bar{u}_\infty|^2\eta^2 + 2\bar{u}_\infty\eta\nabla\bar{u}_\infty\nabla\eta &= \int_{\{\bar{u}_\infty = \sigma\}} \frac{\partial\bar{u}_\infty}{\partial\nu}\bar{u}_\infty\eta^2 \\ &= \sigma\int_{\{\bar{u}_\infty = \sigma\}} \frac{\partial\bar{u}_\infty}{\partial\nu}\eta^2 \\ &= \sigma\int_{\{\bar{u}_\infty > \sigma\}} \nabla\bar{u}_\infty\nabla\eta^2 \\ &= O(\sigma).\end{aligned}$$

Here ν is the outward unit normal vector to $\partial\{\bar{u}_\infty > \sigma\}$. By letting $\sigma \to 0$, we see

$$\int_{\mathbb{R}^n} |\nabla \bar{u}_\infty|^2 \eta^2 + 2\bar{u}_\infty \eta \nabla \bar{u}_\infty \nabla \eta = 0.$$

Hence

$$\lim_{i\to+\infty} \int_{\mathbb{R}^n} |\nabla \bar{u}_i|^2 \eta^2 - |\nabla \bar{u}_\infty|^2 \eta^2 + \varepsilon_i \bar{u}_i^{1-p} \eta^2 = 0.$$

This proves both (2.8) and (2.9).

Because $\bar{u}_i > 0$ in Ω_i, it is smooth.

Let $Y \in C_0^\infty(\Omega_i, \mathbb{R}^n)$. Then by standard domain variation calculation, i.e. (1.4),

$$\int_{\Omega_i} \left(\frac{1}{2}|\nabla \bar{u}_i|^2 - \frac{\varepsilon_i}{p-1}\bar{u}_i^{1-p}\right) \operatorname{div} Y - DY(\nabla \bar{u}_i, \nabla \bar{u}_i) = 0.$$

By the previous lemma, we can take the limit to get

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla \bar{u}_\infty|^2 \operatorname{div} Y - DY(\nabla \bar{u}_\infty, \nabla \bar{u}_\infty) = 0.$$

Now we can apply Theorem 2.2, which implies $\bar{u}_\infty$ is a constant. This is a contradiction because both $\{\bar{u}_\infty > 0\}$ and $\{\bar{u}_\infty = 0\}$ are nonempty.

In conclusion, the assumption (2.2) does not hold. So $\hat{u}_i$ are uniformly bounded in $C^{\frac{2}{p+1}}(\overline{B_2})$. Since $\hat{u}_i = u_i$ in B_1, this finishes the proof of Theorem 2.1. □

Remark 2.3. An essential point in this proof is the fact that

$$\bar{u}_\infty \Delta \bar{u}_\infty = 0.$$

This is well defined, because $\Delta \bar{u}_\infty$ is a Radon measure and $\bar{u}_\infty$ is continuous. From this we also get, in the distributional sense

$$\Delta \bar{u}_\infty^2 = 2|\nabla \bar{u}_\infty|^2.$$

3. Some tools

In this section we first present some consequences of the uniform Hölder continuity, which we will use to prove Theorems 1.1 and 1.2. Therefore, throughout this section we assume that (u_i) is a sequence of stationary solutions of (1.1) in $B_2(0)$ satisfying

$$\sup_i \|u_i\|_{C^{\frac{2}{p+1}}(\bar{B}_{3/2}(0))} < +\infty. \tag{3.1}$$

By Theorem 2.1, this includes the case that u_i are positive solutions of (1.1) in $B_2(0)$ satisfying (1.2).

Lemma 3.1. *There exists a constant C such that for any i, $x \in B_1$ and $r \in (0, 1/2)$,*

$$\int_{B_r(x)} u_i^{-p} \le C r^{n-2\frac{p}{p+1}}.$$

Proof. Take a nonnegative function $\eta \in C_0^\infty(B_{2r}(x))$ such that $\eta \equiv 1$ in $B_r(x)$ and $|\Delta\eta| \le Cr^{-2}$. Then

$$\int u_i^{-p}\eta = \int \big(u_i - u_i(x)\big)\Delta\eta \le Cr^{n-2+\frac{2}{p+1}}.$$

Here we have used the uniform $2/(p+1)$-Hölder continuity of u_i, which implies that

$$\sup_{B_r(x)} \left|u_i - u_i(x)\right| \le C r^{\frac{2}{p+1}}. \qquad \square \tag{3.2}$$

Lemma 3.2. *There exists a constant C depending only on M, such that for any i, $x \in B_1$ and $r \in (0, 1/2)$,*

$$\int_{B_r(x)} |\nabla u_i|^2 + u_i^{1-p} \le C r^{n-2\frac{p-1}{p+1}}.$$

Proof. First by the previous lemma and the Hölder inequality,

$$\int_{B_r(x)} u_i^{1-p} \le \left(\int_{B_r(x)} u_i^{-p}\right)^{\frac{p-1}{p}} \left|B_r(x)\right|^{\frac{1}{p}} \le C r^{n-2\frac{p-1}{p+1}}.$$

Take a nonnegative function $\eta \in C_0^\infty(B_{2r}(x))$ such that $\eta \equiv 1$ in $B_r(x)$ and $|\nabla\eta| \le 2r^{-1}$. Testing the equation of u_i with $(u_i - u_i(x))\eta^2$, we get

$$\int |\nabla u_i|^2\eta^2 + u_i^{-p}\big(u_i - u_i(x)\big)\eta^2 = -2\int \nabla u_i \nabla\eta\big(u_i - u_i(x)\big)\eta.$$

The Cauchy inequality gives

$$\int |\nabla u_i|^2\eta^2 \le \int u_i^{-p}\left|u_i - u_i(x)\right|\eta^2 + 8\int |\nabla\eta|^2\big(u_i - u_i(x)\big)^2.$$

Then using the previous lemma and (3.2) we have

$$\begin{aligned}\int |\nabla u_i|^2\eta^2 &\le \sup_{B_r(x)}\left|u_i - u_i(x)\right| \int u_i^{-p}\eta^2 + 8\sup_{B_r(x)}\left|u_i - u_i(x)\right|^2 \int |\nabla\eta|^2 \\ &\le C r^{n-2\frac{p-1}{p+1}}. \qquad \square\end{aligned}$$

The following result holds for any $2/(p+1)$-Hölder continuous solutions.

Lemma 3.3. *If $x \in \{u_i > 0\}$,*

$$\left|\nabla u_i(x)\right| \le C u_i(x)^{-\frac{p-1}{2}}.$$

Proof. Denote $h^{\frac{2}{p+1}} = u_i(x) > 0$. By the Hölder continuity, $u_i \ge \frac{h^{\frac{2}{p+1}}}{2}$ in $B_{\delta h}(x)$, where δ depends only the $C^{2/(p+1)}$ norm of u_i. Note that we also have $u_i \le 2h^{\frac{2}{p+1}}$ in $B_{\delta h}(x)$.

Define $\bar{u}(y) = h^{-\frac{2}{p+1}} u_i(x + hy)$. Then in $B_\delta(0)$, $1/2 \le \bar{u} \le 2$, and $\bar{u}$ satisfies Eq. (1.1). By standard elliptic estimates, there exists a constant C depending only on δ and n so that

$$\left|\nabla\bar{u}(0)\right| \le C.$$

Rescaling back we get the required claim. $\square$

This estimate implies that $|\nabla u_i^{\frac{p+1}{2}}| \le C$ in $\{u_i > 0\}$. Thus we get

Corollary 3.4. *The function $u_i^{\frac{p+1}{2}}$ is uniformly Lipschitz continuous.*

The next result is taken from [14], and it can be viewed as a non-degeneracy result.

Lemma 3.5. *There exists a constant c depending only on M, such that for any i, $x \in B_1$ and $r \in (0, 1/2)$,*

$$\int_{B_r(x)} u_i \geq cr^{n+\frac{2}{p+1}}.$$

Proof. By the Hölder inequality,

$$\int_{B_r(x)} 1 = \int_{B_r(x)} u_i^{-\frac{p}{p+1}} u_i^{\frac{p}{p+1}} \leq \Bigl(\int_{B_r(x)} u_i^{-p}\Bigr)^{\frac{1}{p+1}} \Bigl(\int_{B_r(x)} u_i\Bigr)^{\frac{p}{p+1}}.$$

Substituting Lemma 3.1 into this we get the estimate. □

Finally let us recall the monotonicity formula for stationary solutions.

Theorem 3.6. *Let u be a stationary solution of (1.1) in B_1. Then for any $B_R(x) \subset B_1$ and $r \in (0, R)$,*

$$E(r; x, u) = r^{-n+2\frac{p-1}{p+1}} \int_{B_r(x)} \Bigl(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\Bigr) - \frac{r^{-n+2\frac{p-1}{p+1}-1}}{p+1} \int_{\partial B_r(x)} u^2$$

is nondecreasing in r. Moreover, if $E(r; x, u) \equiv const.$, then u is homogeneous with respect to x:

$$u(x + \lambda y) = \lambda^{\frac{2}{p+1}} u(x + y), \quad y \in B_R(x),\ \lambda \in (0, 1).$$

Proof. Following the same proof of a monotonicity formula in [11, Lemma 2.2], we have

$$\frac{d}{dr}E(r; x, u) = c(n, p)r^{2\frac{p-1}{p+1}-n} \int_{\partial B_r(x)} \Bigl(\frac{\partial u}{\partial r} - \frac{2}{p+1}r^{-1}u\Bigr)^2 \geq 0. \tag{3.3}$$

The proof uses the stationary assumption on the solution to obtain a Pohozaev type identity used in the calculation.

Formula (3.3) also characterizes the case of equality. □

By the equation we have

$$\int_{B_r(x)} |\nabla u|^2 + u^{1-p} - \int_{\partial B_r(x)} uu_r = 0.$$

Multiplying this with $\frac{2}{p-3}r^{2\frac{p-1}{p+1}-n}$, and adding it into $E(r; x, u)$, we get another form for $E(r; x, u)$

$$E(r; x, u) = r^{-n+2\frac{p-1}{p+1}} \int_{B_r(x)} \Bigl(\frac{1}{2} + \frac{2}{p-3}\Bigr)|\nabla u|^2 + \Bigl(\frac{2}{p-3} - \frac{1}{p-1}\Bigr)u^{1-p} - \frac{1}{p-3}\frac{d}{dr}\Bigl[r^{-n+2\frac{p-1}{p+1}} \int_{\partial B_r(x)} u^2\Bigr].$$

4. The convergence

In this section we prove Theorem 1.1. We can prove actually a stronger statement, so we assume in this section that (u_i) is a sequence of stationary $C^{\frac{2}{p+1}}$ Hölder solutions of (1.1) in $B_2(0)$ satisfying the uniform estimate (3.1). By Theorem 2.1, this includes the case that u_i are positive solutions of (1.1) in $B_2(0)$ satisfying (1.2).

Let us list the results we obtained in the previous sections. There exists a constant C independent of i, such that:

(1) For any $x \in B_1$ and $r \in (0, 1/2)$,

$$\int_{B_r(x)} |\nabla u_i|^2 + u_i^{1-p} \leq Cr^{n-2\frac{p-1}{p+1}}. \tag{4.1}$$

(2) For any $x \in B_1$ and $r \in (0, 1/2)$,

$$\int_{B_r(x)} u_i^{-p} \le C r^{n-2\frac{p}{p+1}}. \tag{4.2}$$

(3) For any $x, y \in B_1$,

$$\left|u_i(x) - u_i(y)\right| \le C|x - y|^{\frac{2}{p+1}}. \tag{4.3}$$

(4) For any $x \in B_1$ and $r \in (0, 1/2)$,

$$\int_{B_r(x)} u_i \ge \frac{1}{C} r^{\frac{2}{p+1}}. \tag{4.4}$$

By (4.3), we can assume that, up to a subsequence, u_i converges uniformly to a function u_∞ in B_1. Then with (4.1), u_i are also uniformly bounded in $H^1(B_1)$, and we can assume that it converges to u_∞ weakly in $H^1(B_1)$. By the uniform convergence, we see u_∞ also satisfies the estimates (4.3) and (4.4).

By standard elliptic estimates, for any domain $\Omega \Subset \{u_\infty > 0\} \cap B_1$ and k, u_i converges to u_∞ in $C^k(\Omega)$.

Lemma 4.1. $H^{n-2+\frac{2}{p+1}}(\{u_\infty = 0\} \cap B_1) = 0$.

Remark 4.2. This statement can be obtained from [5, Theorem 12] where it is proved that if $u \in L^1(B_1)$, $u \ge 0$ a.e., is such that Δu is a bounded measure and $u^{-p} \in L^1(B_1)$, then $H^{n-2+\frac{2}{p+1}}(\{u = 0\}) = 0$. Note that such u need not be continuous, but is well-defined outside some set of zero Newtonian capacity, so this formula makes sense, since for any Borel set $E \subset B_1$ with zero capacity we have $H^{N-2+\theta}(E) = 0$, if $\theta > 0$. However, note that at this stage we do not know if u_∞ is a weak solution, even in the distributional sense. This fact will be proved after establishing Lemma 4.3.

In our context we can give a short proof.

Proof of Lemma 4.1. First by (4.4), for any $x \in \{u_\infty = 0\} \cap B_1$ and $r \in (0, 1/2)$,

$$\sup_{B_r(x)} u_\infty \ge c r^{\frac{2}{p+1}}.$$

Then by the Hölder continuity (4.3) for u_∞, there exists a ball $B_{\delta r}(y) \subset B_r(x)$ (δ depends on the Hölder constant of u_∞) such that

$$u_\infty \ge c r^{\frac{2}{p+1}} \quad \text{in } B_{\delta r}(y).$$

In particular, $B_{\delta r}(y) \subset \{u_\infty > 0\}$. This means for any $x \in \{u_\infty = 0\} \cap B_1$ and $r \in (0, 1/2)$,

$$\frac{|\{u_\infty = 0\} \cap B_r(x)|}{|B_r(x)|} \le 1 - c\delta.$$

By the Lebesgue differentiation theorem, $|\{u_\infty = 0\} \cap B_1| = 0$.

Then because u_i^{-p} converges to u^{-p} uniformly in any compact set of $\{u_\infty > 0\} \cap B_1$, u_i^{-p} converges to u^{-p} a.e. in B_1. By the Fatou lemma,

$$\int_{B_1} u_\infty^{-p} \le \liminf_{i \to +\infty} \int_{B_1} u_\infty^{-p} \le C.$$

For any $\varepsilon > 0$, take a maximal ε-separated set $\{x_i, 1 \le i \le N\}$ of $\{u_\infty = 0\} \cap B_1$. By definition, $B_{\varepsilon/2}(x_i)$ are disjoint, and

$$\{u_\infty = 0\} \cap B_1 \subset \bigcup_i^N B_\varepsilon(x_i).$$

Note that every $B_\varepsilon(x_i)$ belongs to the ε-neighborhood $\mathcal{N}_\varepsilon$ of $\{u_\infty = 0\} \cap B_1$. Hence

$$\sum_{i=1}^{N} \int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \le \int_{\mathcal{N}_\varepsilon} u_\infty^{-p}, \tag{4.5}$$

which goes to 0 as $\varepsilon \to 0$, by the monotone convergence theorem. Because $x_i \in \{u_\infty = 0\}$, by (4.3),

$$\sup_{B_{\varepsilon/2}(x_i)} u_\infty \le C\varepsilon^{\frac{2}{p+1}}.$$

Thus

$$\int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \ge C\varepsilon^{n-2\frac{p}{p+1}}.$$

Substituting this into (4.5), we see

$$\begin{aligned}\sum_{i=1}^{N} \left(\operatorname{diam}\left(B_\varepsilon(x_i)\right)\right)^{n-2\frac{p}{p+1}} &\le C \sum_{i=1}^{N} \int_{B_{\varepsilon/2}(x_i)} u_\infty^{-p} \\ &\le C \int_{\mathcal{N}_\varepsilon} u_\infty^{-p}.\end{aligned}$$

By letting $\varepsilon \to 0$, we get $H^{n-2\frac{p}{p+1}}(\{u_\infty = 0\} \cap B_1) = 0$. $\square$

Since u_i^{-1} converges to u_∞^{-1} a.e. in B_1, by passing limit in (4.1) and (4.2) and using the Fatou lemma, we see u_∞ also satisfies (4.1) and (4.2). (The estimate of $|\nabla u_\infty|$ is a direct consequence of weak convergence in $H^1(B_1)$.)

Lemma 4.3. *The sequence (u_i^{-p}) converges to u_∞^{-p} in $L^1(B_1)$.*

Proof. By the Fatou lemma, we always have

$$\int_{B_1} u_\infty^{-p} \le \liminf_{i\to+\infty} \int_{B_1} u_i^{-p}.$$

Thus we only need to prove the reverse inequality

$$\int_{B_1} u_\infty^{-p} \ge \limsup_{i\to+\infty} \int_{B_1} u_i^{-p}.$$

By the previous lemma, for any $\varepsilon > 0$, there exists a covering of $\{u_\infty = 0\} \cap B_1$ by $\cap_k C_k$, with $\operatorname{diam} C_k \le \varepsilon$, and

$$\sum_i (\operatorname{diam} C_k)^{n-2\frac{p}{p+1}} \le \varepsilon. \tag{4.6}$$

For each k, take an $x_k \in \{u_\infty = 0\} \cap B_1 \cap C_k$. Denote the open set

$$U := \bigcup_k B_{\operatorname{diam} C_k}(x_k).$$

U is an open neighborhood of $\{u_\infty = 0\} \cap B_1$. So in $(\{u_\infty > 0\} \cap B_1) \setminus U$, for all i large, u_i^{-p} have a uniformly positive lower bound and they converge to u_∞^{-p} uniformly. Hence

$$\lim_{i\to+\infty} \int_{(\{u_\infty>0\}\cap B_1)\setminus U} u_i^{-p} = \int_{(\{u_\infty>0\}\cap B_1)\setminus U} u_\infty^{-p}. \tag{4.7}$$

For each i and k, by (4.2),

$$\int_{B_{\operatorname{diam} C_k}(x_k)} u_i^{-p} \le C(\operatorname{diam} C_k)^{n-2\frac{p}{p+1}}.$$

Summing in k and noting (4.6), we see

$$\int_U u_i^{-p} \le \sum_k \int_{B_{\operatorname{diam} C_k}(x_k)} u_i^{-p} \le C\varepsilon.$$

Combined with (4.7), we obtain

$$\int_{B_1} u_\infty^{-p} \ge \limsup_{i\to+\infty} \int_{B_1} u_i^{-p} - C\varepsilon.$$

Taking $\varepsilon \to 0$, we complete the proof. □

Corollary 4.4. *The function u_∞ is a solution to (1.1) in the distributional sense.*

Lemma 4.5. *We have that u_i^{1-p} converges to u_∞^{1-p} in $L^1(B_1)$ and u_i converges to u_∞ strongly in $H^1(B_1)$.*

Proof. Note that for any $t, s \ge 0$, $|t^{1-p} - s^{1-p}| \le C(p)|s-t|(s^{-p} + t^{-p})$. Thus, by the previous lemma

$$\int_{B_1} \left|u_i^{1-p} - u_\infty^{1-p}\right| \le C(p) \sup_{B_1} |u_i - u_\infty| \left(\int_{B_1} u_i^{-p} + u_\infty^{-p} \right) \le C \sup_{B_1} |u_i - u_\infty|.$$

This converges to 0 by the uniform convergence of u_i to u_∞.

By testing the equation of u_i with $u_i\eta^2$, where $\eta \in C_0^\infty(B_2)$, we have

$$\int_{B_2} |\nabla u_i|^2\eta^2 + u_i^{1-p}\eta^2 = \int_{B_2} u_i^2 \Delta \frac{\eta^2}{2}.$$

By the strong convergence of u_i in $L^2_{loc}(B_2)$, and the convergence of u_i^{1-p} proved above, we have

$$\lim_{i\to+\infty} \int_{B_2} |\nabla u_i|^2\eta^2 + \int_{B_2} u_\infty^{1-p}\eta^2 = \int_{B_2} u_\infty^2 \Delta \frac{\eta^2}{2}.$$

Since $u_\infty \in H^1(B_2)$ is a weak solution of (1.1), and $u_\infty^{1-p} \in L^1_{loc}$, we also have

$$\int_{B_2} |\nabla u_\infty|^2\eta^2 + u_\infty^{1-p}\eta^2 = \int_{B_2} u_\infty^2 \Delta \frac{\eta^2}{2}.$$

This gives

$$\lim_{i\to+\infty} \int_{B_2} |\nabla u_i|^2 = \int_{B_2} |\nabla u_\infty|^2,$$

and the strong convergence of u_i in $H^1(B_1)$. □

By this convergence, we can take limit in (1.3) for u_i to get the corresponding stationary condition for u_∞. This finishes the proof of Theorem 1.1.

5. Dimension reduction for stationary solutions

In this section we assume that u is a $2/(p+1)$-Hölder continuous, stationary solution of (1.1) in B_2, with

$$\int_{B_2} |\nabla u|^2 + u^{1-p} + u^2 = M < +\infty.$$

By the results in Section 3, u satisfies all of the estimates (4.1)–(4.4). In particular, $\{u = 0\}$ is a closed set satisfying (by Lemma 4.1)

$$H^{n-2+\frac{2}{p+1}}\left(\{u = 0\}\right) = 0.$$

We will use the Federer dimension reduction principle to prove Theorem 1.2. We will mainly follow the treatment in Giusti [10, Chapter 11]. For an account of this argument, see also [20, Appendix A].

First let us consider the blow up procedure. Assume that $u(0) = 0$, for $\lambda \to 0$, define the blow up sequence

$$u^\lambda(x) = \lambda^{-\frac{2}{p+1}} u(\lambda x).$$

By a rescaling, we see u^λ satisfies (4.1)–(4.4), for all ball $B_r(x) \subset B_{\lambda^{-1}}(0)$. By the results established in Section 4, we can get a subsequence of $\lambda_i \to 0$, so that $u_i := u^{\lambda_i}$ converges uniformly to a u_∞ on any compact set of $\mathbb{R}^n$. (This limit may depend on the choice of the sequence λ_i and thus not unique.)

We also have

(1) For each R, u_i^{-p} converges to u_∞^{-p} in $L^1(B_R)$;
(2) For each R, u_i^{1-p} converges to u_∞^{1-p} in $L^1(B_R)$;
(3) For each R, u_i converges to u_∞ in $H^1(B_R)$;
(4) u_∞ is a stationary weak solution of (1.1) in the distributional sense;
(5) u_∞ is nonzero.

To continue, we first note the following result.

Lemma 5.1. *For any $\varepsilon > 0$, if i large, $\{u_i = 0\} \cap B_1$ lies in an ε-neighborhood of $\{u_\infty = 0\} \cap B_1$.*

Proof. This is because u_i converges to u_∞ uniformly in any compact set $\Omega' \Subset \{u_\infty > 0\} \cap B_1$. Thus for i large, $u_i > 0$ in Ω'. □

Next we would like to use the monotonicity formula to explore the information of the limit u_∞.

Lemma 5.2. *The limit* $\lim_{r\to 0} E(r; 0, u)$ *exists and is finite.*

Proof. In view of the monotonicity of $E(r; 0, u)$, we only need to show that as $r \to 0$, $E(r; 0, u)$ has a uniform lower bound.

By Lemma 3.2, for each $r \in (0, 1)$,

$$r^{2\frac{p-1}{p+1}-n} \int_{B_r} |\nabla u|^2 + u^{1-p} \le C.$$

Next, by Theorem 2.1, $\sup_{B_r} u \le Cr^{\frac{2}{p+1}}$. Thus

$$r^{2\frac{p-1}{p+1}-n-1} \int_{\partial B_r} u^2 \le Cr^{2\frac{p-1}{p+1}-n-1+n-1+\frac{4}{p+1}} = C.$$

Substituting these into the first formulation of $E(r; 0, u)$, we get

$$E(r; 0, u) \ge -C. \qquad \square$$

By (3.3), for any $r \in (0, 1)$,

$$E(1; 0, u) - E(r; 0, u) = c \int_{B_1 \setminus B_r} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} r^{-1} u \right)^2 dx.$$

Corollary 5.3. *We have*

$$\int_{B_1} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} |x|^{-1} u \right)^2 dx < +\infty.$$

Lemma 5.4. *The blow up limit u_∞ is a homogeneous solution of (1.1) on $\mathbb{R}^n$.*

Proof. By the strong convergence of u_i in $H^1_{loc}(\mathbb{R}^n)$, for any $\eta \in (0, 1)$,

$$\begin{aligned} &\int_{B_{\eta^{-1}} \setminus B_\eta} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u_\infty}{\partial r} - \frac{2}{p+1} r^{-1} u_\infty \right)^2 dx \\ &= \lim_{i \to +\infty} \int_{B_{\eta^{-1}} \setminus B_\eta} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u_i}{\partial r} - \frac{2}{p+1} |x|^{-1} u_i \right)^2 dx \\ &= \lim_{i \to +\infty} \int_{B_{\eta^{-1}\lambda_i} \setminus B_{\eta\lambda_i}} |x|^{2\frac{p-1}{p+1}-n} \left(\frac{\partial u}{\partial r} - \frac{2}{p+1} |x|^{-1} u \right)^2 dx \\ &= 0. \end{aligned}$$

The last one is guaranteed by the previous corollary.

This means for a.a. $x \in \mathbb{R}^n$,

$$\frac{\partial u_\infty}{\partial r} - \frac{2}{p+1} r^{-1} u_\infty = 0.$$

Integrating this in r, we get

$$u_\infty(x) = |x|^{\frac{2}{p+1}} u_\infty\left(\frac{x}{|x|} \right). \qquad \square$$

Define the density function (it may take value $-\infty$)

$$\Theta(x; u) := \lim_{r \to 0} E(r; x, u).$$

We have the following characterization of rupture points.

Lemma 5.5. *We have: $x \in \{u > 0\}$ if and only if $\Theta(x) = -\infty$.*

Proof. If $u(x) = 2h > 0$, by the continuity of u, $u > h$ in a ball $B_{r_0}(x)$ and it is smooth here. Hence for $r < r_0$,

$$r^{2\frac{p-1}{p+1}-n} \int_{B_r(x)} |\nabla u|^2 + u^{1-p} \le C r^{2\frac{p-1}{p+1}},$$

which goes to 0 as $r \to 0$.

On the other hand,

$$r^{2\frac{p-1}{p+1}-n-1} \int_{\partial B_r(x)} u^2 \ge h^2 r^{2\frac{p-1}{p+1}-2},$$

which goes to $+\infty$ as $r \to 0$. Substituting these into the first formulation of $E(r; x, u)$ we get

$$\lim_{r\to 0} E(r; x, u) = -\infty.$$

If $u(x) = 0$, the same proof of Lemma 5.2 gives

$$\Theta(x; u) = \lim_{r\to 0} E(r; x, u) \geq -C. \qquad \square$$

We record the following continuity property of this density function.

Lemma 5.6. *$\Theta(x; u)$ is upper semi-continuous in x and u (under the convergence specified as in Theorem 1.1).*

Proof. Because $u \in H^1(B_2)$ and $u^{1-p} \in L^1(B_2)$, by the first formulation of $E(r; x, u)$, $E(r; x, u)$ is a continuous function of x. Then since $\Theta(x)$ is the decreasing limit of this family of continuous functions, it is upper semi-continuous in x.

If we have a sequence of stationary weak solutions u_i that converges to u_∞ strongly in $H^1(B_r(x))$ and $L^{1-p}(B_r(x))$ as in Theorem 1.1, then by the trace theorem we also have

$$\int_{\partial B_r(x)} u_i^2 \to \int_{\partial B_r(x)} u_\infty^2.$$

This implies directly that

$$E(r; x, u_i) \to E(r; x, u_\infty). \tag{5.1}$$

For any $\varepsilon > 0$, by definition we can find an $r > 0$ so that

$$E(r; x, u_\infty) \leq \Theta(x; u_\infty) + \varepsilon.$$

On the other hand, by the monotonicity we always have

$$\Theta(x; u_i) \leq E(r; x, u_i).$$

Thus

$$\limsup_{i\to+\infty} \Theta(x; u_i) \leq \lim_{i\to+\infty} E(r; x, u_i) = E(r; x, u_\infty) \leq \Theta(x; u_\infty) + \varepsilon.$$

By taking $\varepsilon \to 0$ we can finish the proof. $\square$

Combining the above proof with Lemma 5.4 we can deduce that

Corollary 5.7. *Let $x_0 \in \{u = 0\}$ and u_∞ be a blow up limit of u at x_0, then for any $r > 0$,*

$$E(r; 0, u_\infty) = \Theta(0; u_\infty) = \Theta(x_0; u).$$

The first equality comes from the homogeneity of u_∞, and the second one can be obtained by combining (5.1) and the definition of Θ.

To prove Theorem 1.2, we argue by contradiction. So assume that the Hausdorff dimension of $\{u = 0\}$ is strictly larger than $n - 2$. Then by definition, there exists a $\delta > 0$ such that

$$H^{n-2+\delta}\big(\{u = 0\} \cap B_1\big) > 0. \tag{5.2}$$

For a set $A \subset \mathbb{R}^n$, define

$$H_\infty^{n-2+\delta}(A) := \inf\left\{\sum_j (\operatorname{diam} S_j)^{n-2+\delta},\ A \subset \bigcup_j S_j\right\}.$$

Then by [10, Lemma 11.2 and Proposition 11.3], (5.2) implies the existence of a density point $x_0 \in \{u = 0\} \cap B_1$, that is,

$$\limsup_{r\to 0} \frac{H_\infty^{n-2+\delta}(\{u=0\}\cap B_r(x_0))}{r^{n-2+\delta}} > 0. \tag{5.3}$$

We can perform the blow up procedure at x_0 to obtain a homogeneous solution $u_{\infty,0}$ on $\mathbb{R}^n$. By noting Lemma 5.1, we can prove as in [10, Lemma 11.5] to get

$$H_\infty^{n-2+\delta}\big(\{u_{\infty,0}=0\}\cap B_1(0)\big) \geq \limsup_{r\to 0} \frac{H_\infty^{n-2+\delta}(\{u=0\}\cap B_r(x))}{r^{n-2+\delta}} > 0, \tag{5.4}$$

if we choose a suitable sequence $\lambda_i \to 0$ in the definition of $u_{\infty,0}$.

Since $n \geq 2$, (5.4) implies that $\{u_{\infty,0} = 0\}$ contains a point $x_1 \neq 0$, which can also be chosen to be a density point by [10, Proposition 11.3]. Note that the origin 0 always belongs to $\{u_\infty = 0\}$ because u_∞ is homogeneous. This homogeneity also implies that the ray $\{tx_1 : t \geq 0\} \subset \{u_{\infty,0} = 0\}$, and

$$\Theta(tx_1; u_{\infty,0}) \equiv \Theta(x_1; u_{\infty,0}) \quad \text{for } t > 0. \tag{5.5}$$

The main step in the dimension reduction procedure is to blow up once again at x_1. Assume that one limit function is $u_{\infty,1}$ and we have a sequence $\lambda_i \to 0$ so that

$$u_i := \lambda_i^{-\frac{2}{p+1}} u_{\infty,0}(x_1 + \lambda_i x) \to u_{\infty,1},$$

in the sense of Theorem 1.1.

We want to show that $u_{\infty,1}$ is in fact translation invariant in the direction x_1, thus can be viewed as a function defined on $\mathbb{R}^{n-1}$. This can be achieved by the following lemma, together with the fact that, for any $t \in \mathbb{R}$,

$$\begin{aligned}\Theta(tx_1; u_{\infty,1}) \geq \limsup_{i\to+\infty} \Theta(tx_1; u_i) &= \limsup_{i\to+\infty} \Theta\big((1+t\lambda_i)x_1; u_{\infty,0}\big) \\ &= \Theta(x_1; u_{\infty,0}) = \Theta(0; u_{\infty,0}),\end{aligned}$$

where we have used Lemma 5.6 and Corollary 5.7.

Lemma 5.8. *Let u be a homogeneous stationary solution of (1.1) on $\mathbb{R}^n$, satisfying estimates (4.1)–(4.4) for all balls $B_r(x)$. Then for any $x \neq 0$, $\Theta(x,u) \leq \Theta(0,u)$. Moreover, if $\Theta(x,u) = \Theta(0,u)$, u is translation invariant in the direction x, i.e. for all $t \in \mathbb{R}$,*

$$u(tx + \cdot) = u(\cdot) \quad \text{in } \mathbb{R}^n.$$

Proof. With the help of the estimates (4.1)–(4.4), similar to the proof of Lemma 5.2, for any $x_0 \in \mathbb{R}^n$, there exists a constant C such that

$$\lim_{r\to+\infty} E(r; x_0, u) \leq C.$$

And we can define the blowing down sequence with respect to the base point x_0,

$$u_\lambda(x) = \lambda^{-\frac{2}{p+1}} u(x_0 + \lambda x), \qquad \lambda \to +\infty.$$

Since u is homogeneous with respect to 0,

$$u_\lambda(x) = u\big(\lambda^{-1}x_0 + x\big),$$

which converges to $u(x)$ as $\lambda \to +\infty$ uniformly in any compact set of $\mathbb{R}^n$. u_λ also converges strongly in $H^1_{loc}(\mathbb{R}^n)$, u_λ^{1-p} and u_λ^{-p} converge in $L^1_{loc}(\mathbb{R}^n)$. Then by the homogeneity of u and these convergence, we see

$$\begin{aligned}\Theta(0; u) = E(1; 0, u) &= \lim_{\lambda\to+\infty} E(1; 0, u_\lambda) \\ &= \lim_{\lambda\to+\infty} E(\lambda; x_0, u) \\ &\geq \Theta(x_0; u).\end{aligned}$$

Moreover, if $\Theta(x_0; u) = \Theta(0, u)$, the above inequality becomes an equality:

$$\lim_{\lambda \to +\infty} E(\lambda; x_0, u) = \Theta(x_0; u).$$

This then implies that $E(\lambda; x_0, u) \equiv \Theta(x_0; u)$ for all $\lambda > 0$. By (3.3), u is homogeneous with respect to x_0. Then for all $\lambda > 0$,

$$u(x_0 + x) = \lambda^{-\frac{2}{p+1}} u(x_0 + \lambda x) = u\big(\lambda^{-1} x_0 + x\big).$$

By letting $\lambda \to +\infty$ and noting that $u(\lambda^{-1}x_0 + \cdot)$ are uniformly bounded in $C^{\frac{2}{p+1}}(\mathbb{R}^n)$, we see

$$u(x_0 + \cdot) = u(\cdot) \quad \text{on } \mathbb{R}^n.$$

Because u is homogeneous with respect to 0, a direct scaling shows that $\Theta(tx_0; u) = \Theta(x_0; u)$ for all $t > 0$, so the above equality still holds if we replace x_0 by tx_0 for any $t > 0$. A change of variable shows this also holds for $t < 0$. □

We have shown that $u_{\infty,1}$ can be viewed as a weak solution of (1.1) in $\mathbb{R}^{n-1}$. Note that the $u_{\infty,1}$ is still in $H^1_{loc}(\mathbb{R}^{n-1})$ and $2/(p+1)$-Hölder continuous. The following result shows that the stationary condition is also preserved under this operation.

Lemma 5.9. *Let $u = u(x_1, \cdots, x_{n-1}) \in H^1_{loc}(\mathbb{R}^{n-1}) \cap L^{-p}_{loc}(\mathbb{R}^{n-1})$ be a weak solution of (1.1) in $\mathbb{R}^{n-1}$. Take $\bar{u}$ to be the trivial extension of u to $\mathbb{R}^n$,*

$$\bar{u}(x_1, \cdots, x_n) = u(x_1, \cdots, x_{n-1}).$$

Then u is stationary if and only if $\bar{u}$ is stationary.

Proof. First assume $\bar{u}$ is stationary but u is not stationary. By definition there exists a vector field $Y \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$, such that

$$\int\limits_{\mathbb{R}^{n-1}} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \operatorname{div} Y - DY(\nabla u, \nabla u) = \delta > 0.$$

For any T, take a function $\eta_T \in C_0^\infty((-T-1, T+1))$ such that $\eta \equiv 1$ in $(-T, T)$, $|\eta'| \le 2$. Then

$$\bar{Y}(x_1, \cdots, x_{n-1}, x_n) = Y(x_1, \cdots, x_{n-1})\eta(x_n)$$

is a smooth vector field in $\mathbb{R}^n$ with compact support. So

$$\int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla \bar{u}|^2 - \frac{1}{p-1}\bar{u}^{1-p}\right) \operatorname{div} \bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u}) = 0.$$

However, direct calculation shows that this also equals

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n-1} \times \{-T < x_n < T\}} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \operatorname{div} Y - DY(\nabla u, \nabla u) \\
&\quad + \int\limits_{\mathbb{R}^{n-1} \times \{T < |x_n| < T+1\}} \left(\frac{1}{2}|\nabla \bar{u}|^2 - \frac{1}{p-1}\bar{u}^{1-p}\right) \operatorname{div} \bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u}) \\
&= 2T\delta + O(1).
\end{aligned}$$

Hence if we choose T large we get a contradiction with the stationary condition of $\bar{u}$. This proves the stationary condition for u.

Now assume u is stationary. For any vector field $\bar{Y} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, by noting that $\frac{\partial \bar{u}}{\partial x_n} = 0$ a.e., we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla \bar{u}|^2 - \frac{1}{p-1}\bar{u}^{1-p}\right) \operatorname{div} \bar{Y} - D\bar{Y}(\nabla \bar{u}, \nabla \bar{u}) \\
&= \int_{-\infty}^{+\infty} \left[\int_{\mathbb{R}^{n-1}} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \sum_{1\le i\le n-1} \frac{\partial \bar{Y}_i}{\partial x_i} - \sum_{1\le i,j\le n-1} \frac{\partial \bar{Y}_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right] \\
&\quad + \int_{\mathbb{R}^{n-1}} \left[\left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p-1}u^{1-p}\right) \int_{-\infty}^{+\infty} \frac{\partial \bar{Y}_n}{\partial x_n}\right] \\
&= 0.
\end{aligned}$$

This proves the stationary condition for $\bar{u}$. □

Similar to (5.4), when $u_{\infty,1}$ is viewed as a function defined on $\mathbb{R}^n$, we have

$$H_\infty^{n-2+\delta}\big(\{u_{\infty,1} = 0\} \cap B_1(0)\big) > 0.$$

Then if we view u_1 as a function defined on $\mathbb{R}^{n-1}$, this means

$$H_\infty^{n-3+\delta}\big(\{u_{\infty,1} = 0\} \cap B_1(0)\big) > 0.$$

We can repeat this reduction procedure until we get a stationary weak solution $u_{\infty,n-2}$ on $\mathbb{R}^2$, which satisfies

$$H_\infty^{\delta}\big(\{u_{\infty,n-2} = 0\} \cap B_1(0)\big) > 0.$$

In particular, $\{u_{\infty,n-2} = 0\}$ cannot be a singleton because $\delta > 0$. However, this contradicts the following lemma, and thus disproves our initial assumption (5.2).

Lemma 5.10. *Let u be a $2/(p+1)$-Hölder continuous, homogeneous solution of (1.1) in $\mathbb{R}^2$. Then $\{u = 0\} = \{0\}$.*

Here we only need the solution to be understood in the distributional sense, i.e. $u^{-p} \in L^1_{loc}(\mathbb{R}^2)$.

Proof. There exists a function $\varphi(\theta) \in C^{\frac{2}{p+1}}(\mathbb{S}^1)$ such that in the polar coordinates,

$$u(r,\theta) = r^{\frac{2}{p+1}}\varphi(\theta).$$

Then

$$\int_{B_1} u^{-p} = \int_0^1 \left(\int_{\mathbb{S}^1} \varphi(\theta)^{-p} d\theta\right) r^{-\frac{2p}{p+1}+1} dr < +\infty.$$

So

$$\int_{\mathbb{S}^1} \varphi(\theta)^{-p} d\theta < +\infty.$$

If there exists a $\theta_0 \in \mathbb{S}^1$ such that $\varphi(\theta_0) = 0$, then

$$\big|\varphi(\theta) - \varphi(\theta_0)\big| \le C|\theta - \theta_0|^{\frac{2}{p+1}}.$$

Hence near θ_0, φ^{-p} grows like $|\theta - \theta_0|^{-\frac{2p}{p+1}}$. Since $\frac{2p}{p+1} > 1$, φ^{-p} cannot be in $L^1(\mathbb{S}^1)$. This is a contradiction and we must have $\varphi > 0$ on $\mathbb{S}^1$. □

Remark 5.11. Similar arguments show that there does not exist homogeneous solutions in $\mathbb{R}^1$.

Finally, we prove the discreteness of $\{u = 0\}$ in the case of $n = 2$.

Assume there exists $x_i \in \{u = 0\} \cap B_1$, such that $x_i \to x_0$ but $x_i \neq x_0$. Take $r_i = |x - x_i|$ and define

$$u_i(x) = r_i^{-\frac{2}{p+1}} u(x_0 + r_i x).$$

After passing to a subsequence of i, we can assume that u_i converges uniformly to a $2/(p+1)$-Hölder continuous, homogeneous solution u_∞ in any compact set of $\mathbb{R}^2$. Since $z_i = (x_i - x_0)/r_i \in \mathbb{S}^1$, we can also assume that $z_i \to z_\infty \in \mathbb{S}^1$. By the uniform convergence of u_i,

$$u_\infty(z_\infty) = \lim_{i \to +\infty} u_i(z_i) = 0.$$

However, Lemma 5.10 says $u_\infty > 0$ outside the origin. This is a contradiction and $\{u = 0\} \cap B_1$ must be a discrete set.

Conflict of interest statement

The authors don't have conflicts of interest.

Acknowledgements

J. Dávila acknowledges support of Fondecyt 1130360, CAPDE-Anillo ACT-125 and Fondo Basal CMM. Kelei Wang is partially supported by the Joint Laboratory of CAS-Croucher in Nonlinear PDE. Juncheng Wei was supported by a GRF grant from RGC of Hong Kong. We thank Prof. L. Dupaigne for useful discussions.

Appendix A. A Liouville theorem

In this appendix we give a proof of Theorem 2.2, following the argument of [15].

Eq. (2.1) implies that

$$\Delta \bar{u}_\infty^2 = 2|\nabla \bar{u}_\infty|^2, \tag{A.1}$$

in the distributional sense. Moreover, $\bar{u}_\infty$ is harmonic in the open set $\{\bar{u}_\infty > 0\}$. So if $\bar{u}_\infty > 0$ everywhere, it is a harmonic globally Hölder function on $\mathbb{R}^n$ and we can use the standard arguments to deduce that is constant.

In the following we assume $\{\bar{u}_\infty = 0\} \neq \emptyset$. First we present some monotonicity formulas. It is here that the stationary condition on the solution is used.

Proposition A.1. *For $r > 0$ and $x \in \mathbb{R}^n$,*

$$D(r; x) := r^{2-n} \int_{B_r(x)} |\nabla \bar{u}_\infty|^2$$

is nondecreasing in r.

Proof. For a proof, see [3, Lemma 2.1]. In fact by the stationary condition, we have

$$(n-2) \int_{B_r(x)} |\nabla \bar{u}_\infty|^2 = r \int_{\partial B_r(x)} |\nabla \bar{u}_\infty|^2 - 2\left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2.$$

Then direct calculations give

$$\frac{d}{dr} D(r; x) = 2r^{2-n} \int_{\partial B_r(x)} \left(\frac{\partial \bar{u}_\infty}{\partial r}\right)^2 \geq 0. \qquad \square$$

Next let $H(r;x) := r^{1-n}\int_{\partial B_r}\bar{u}_\infty^2$. By (A.1), direct calculations give

$$\frac{dH}{dr} = 2r^{1-n}\int\limits_{\partial B_r}\bar{u}_\infty\frac{\partial\bar{u}_\infty}{\partial r} = 2r^{1-n}\int\limits_{B_r}\bar{u}_\infty\Delta\bar{u}_\infty = \frac{2}{r}D(r). \tag{A.2}$$

Then we get

Proposition A.2 *(Almgren monotonicity formula). For $r > 0$ and $x \in \mathbb{R}^n$,*

$$N(r;x) := \frac{D(r;x)}{H(r;x)}$$

is nondecreasing in r. Moreover, if $N(r;x) \equiv d$, then

$$\bar{u}_\infty(x+ry) = r^d\bar{u}_\infty(x+y).$$

Proof. Without loss of generality, take $x = 0$.

$$\begin{aligned}\frac{d}{dr}N(r) &= \frac{H(r)[2r^{2-n}\int_{\partial B_r}(\frac{\partial\bar{u}_\infty}{\partial r})^2] - D(r)(2r^{1-n}\int_{\partial B_r}\bar{u}_\infty\frac{\partial\bar{u}_\infty}{\partial r})}{H(r)^2}\\ &= 2r^{3-2n}\frac{\int_{\partial B_r}\bar{u}_\infty^2\int_{\partial B_r}(\frac{\partial\bar{u}_\infty}{\partial r})^2 - (\int_{\partial B_r}\bar{u}_\infty\frac{\partial\bar{u}_\infty}{\partial r})^2}{H(r)^2}\\ &\ge 0.\end{aligned}$$

If $N(r) \equiv d$, for any r,

$$\int\limits_{\partial B_r}\bar{u}_\infty^2\int\limits_{\partial B_r}\left(\frac{\partial\bar{u}_\infty}{\partial r}\right)^2 - \left(\int\limits_{\partial B_r}\bar{u}_\infty\frac{\partial\bar{u}_\infty}{\partial r}\right)^2 = 0.$$

By the characterization of the equality case of the Cauchy inequality, there exists a $\lambda(r)$ such that

$$\frac{\partial\bar{u}_\infty}{\partial r} = \lambda(r)\bar{u}_\infty.$$

Integrating in r we get a function $\varphi(r)$ such that

$$\bar{u}_\infty(y) = \varphi(|y|)\bar{u}_\infty\left(\frac{y}{|y|}\right).$$

Then a direct calculation shows that $\varphi(|y|) = |y|^d$. □

Proposition A.3. *If $N(r_0;x) \ge d$, then for $r > r_0$,*

$$r^{1-n-2d}\int\limits_{\partial B_r(x)}\bar{u}_\infty^2$$

is nondecreasing in r.

Proof. Direct calculation using (A.2) shows

$$\frac{d}{dr}\left(r^{1-n-2d}\int\limits_{\partial B_r(x)}\bar{u}_\infty^2\right) = -2dr^{-n-2d}\int\limits_{\partial B_r(x)}\bar{u}_\infty^2 + 2r^{1-n-2d}\int\limits_{B_r(x)}|\nabla\bar{u}_\infty|^2 \ge 0.$$

Here we have used Proposition A.2, in particular, the fact that $N(r) \ge d$ for every $r \ge r_0$. □

Because $\bar{u}_\infty$ is globally C^α,

$$\bar{u}_\infty(x) \le C\left(1+|x|^\alpha\right) \quad \text{in } \mathbb{R}^n.$$

Hence for any x and r large,

$$\int\limits_{\partial B_r(x)} \bar{u}_\infty^2 \le C r^{n-1+2\alpha}.$$

Combining this with the previous proposition we get

$$N(r;x) \le \alpha, \quad \text{for any } r>0,\ x \in \{\bar{u}_\infty = 0\}. \tag{A.3}$$

The next result is the so-called "doubling property".

Proposition A.4. *Let $x \in \{\bar{u}_\infty = 0\}$ and $R>0$ such that $N(R;x) \le d$, then for every $0<r\le R$*

$$H(r;x) \ge H(R;x)\frac{r^{2d}}{R^{2d}}. \tag{A.4}$$

Proof. By (A.2), if $H(r)>0$,

$$\frac{d}{dr}\log H(r) = \frac{2N(r)}{r} \le \frac{2d}{r}.$$

This means $r^{-2d}H(r)$ is non-increasing in r. Consequently, $H(r)>0$ for all $r \in (0,R)$, and (A.4) is a direct consequence of the monotonicity of $r^{-2d}H(r)$. □

Remark A.5. By this doubling property, we can prove that $\{\bar{u}_\infty = 0\}$ has zero Lebesgue measure. In fact, more properties such as the unique continuation property can be proved by this method, see [3, Lemma 3.3] and [9, Theorem 1.2].

By this doubling property, if $N(R;x) \le d < \alpha$, then for all $r \in (0,R)$,

$$H(r;x) \ge C r^{2d}.$$

However, if $\bar{u}_\infty(x) = 0$, because $\bar{u}_\infty$ is C^α continuous,

$$H(r;x) \le C r^{2\alpha}.$$

If r small, this is a contradiction. In other words, $N(r;x) \ge \alpha$ for any $r>0$.

Combining this fact with (A.3), we see for any $x \in \{\bar{u}_\infty = 0\}$ and $r>0$, $N(r;x) \equiv \alpha$. By Proposition A.2,

$$\bar{u}_\infty(x+y) = |y|^{\frac{2}{p+1}}\bar{u}_\infty\left(x + \frac{y}{|y|}\right).$$

In particular, $\{\bar{u}_\infty = 0\}$ is a cone with respect to any point in $\{\bar{u}_\infty = 0\}$. This then implies that $\{\bar{u}_\infty = 0\}$ is a linear subspace of $\mathbb{R}^n$. Assume $\{\bar{u}_\infty = 0\} = \mathbb{R}^k$ for some $k<n$. (Note that $\bar{u}_\infty$ is nontrivial, so $\{\bar{u}_\infty = 0\}$ cannot be the whole $\mathbb{R}^n$.) If $k \le n-2$, $\{\bar{u}_\infty = 0\}$ has zero capacity and then $\bar{u}_\infty$ is a harmonic function. Because $\bar{u}_\infty \ge 0$, by the strong maximum principle, either $\bar{u}_\infty > 0$ everywhere or $\bar{u}_\infty \equiv 0$. Both of these two lead to a contradiction.

If $k = n-1$, assume $\{\bar{u}_\infty = 0\} = \{x_1 = 0\}$. Then by the Schwarz reflection principle, $\bar{u}_\infty = c|x_1|$ for some constant $c>0$. This again contradicts the global α-Hölder continuity of $\bar{u}_\infty$ because $\alpha < 1$.

References

[1] F. Bethuel, On the singular set of stationary harmonic maps, Manuscr. Math. 78 (4) (1993) 417–443.

[2] L. Caffarelli, The obstacle problem revisited, J. Fourier Anal. Appl. 4 (4–5) (1998) 383–402.

[3] L.A. Caffarelli, F.H. Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, J. Am. Math. Soc. 21 (2008) 847–862.

[4] J. Dávila, A.C. Ponce, Hausdorff dimension of ruptures sets and removable singularities, C. R. Math. Acad. Sci. Paris 346 (1–2) (2008) 27–32.

[5] L. Dupaigne, A.C. Ponce, A. Porretta, Elliptic equations with vertical asymptotes in the nonlinear term, J. Anal. Math. 98 (2006) 349–396.
[6] P. Esposito, N. Ghoussoub, Y. Guo, Compactness along the branch of semistable and unstable solutions for an elliptic problem with a singular nonlinearity, Commun. Pure Appl. Math. 60 (2007) 1731–1768.
[7] P. Esposito, N. Ghoussoub, Y. Guo, Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS, Courant Lect. Notes Math., vol. 20, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 2010.
[8] L.C. Evans, Partial regularity for stationary harmonic maps into spheres, Arch. Ration. Mech. Anal. 116 (2) (1991) 101–113.
[9] N. Garofalo, F.H. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J. 35 (2) (1986) 245–268.
[10] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Springer Monogr. Math., vol. 80, Birkhäuser, 1984.
[11] Z. Guo, J. Wei, Hausdorff dimension of ruptures for solutions of a semilinear elliptic equation with singular nonlinearity, Manuscr. Math. 120 (2) (2006) 193–209.
[12] Z. Guo, J. Wei, Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity, Commun. Pure Appl. Anal. 7 (4) (2008) 765–786.
[13] H. Jiang, F.-H. Lin, Zero set of Sobolev functions with negative power of integrability, Chin. Ann. Math., Ser. B 25 (1) (2004) 65–72.
[14] L. Ma, J. Wei, Properties of positive solutions to an elliptic equation with negative exponent, J. Funct. Anal. 254 (4) (2008) 1058–1087.
[15] B. Noris, H. Tavares, S. Terracini, G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, Commun. Pure Appl. Math. 63 (3) (2010) 267–302.
[16] F. Pacard, Partial regularity for weak solutions of a nonlinear elliptic equation, Manuscr. Math. 79 (2) (1993) 161–172.
[17] D. Phillips, A minimization problem and the regularity of solutions in the presence of a free boundary, Indiana Univ. Math. J. 32 (1) (1983) 1–17.
[18] J.A. Pelesko, D.H. Bernstein, Modeling MEMS and NEMS, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[19] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.
[20] L. Simon, Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.