On the existence of multi-transition solutions for a class of elliptic systems

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Abstract

The existence of solutions undergoing multiple spatial transitions between isolated periodic solutions is studied for a class of systems of semilinear elliptic partial differential equations. A key tool is a new result on the possible behavior of the set of single transition solutions.
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1. Introduction

In a recent paper, [1], the family of elliptic systems

(PDE) $-\Delta u + F_u(x, u) = 0, x \in \mathbb{R}^n$,

was studied, where F satisfies

(F_1) $F \in C^2(\mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^m/\mathbb{Z}^m, \mathbb{R})$, i.e. F is 1-periodic in x_i, u_j for $1 \le i \le n$, and $1 \le j \le m$.

Here F_u denotes the gradient of F with respect to the u variables: $(\frac{\partial F}{\partial u_j})$. It was shown in [1] that (PDE) possesses a set of solutions, $\mathcal{M}_0$, that are 1-periodic in $x_1, \cdots, x_n$, and minimize the functional

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$$J_0(u) = \int_{\mathbb{T}^n} \left(\frac{1}{2} |\nabla u|^2 + F(x, u) \right) dx$$

over $E_0 = (W^{1,2}(\mathbb{T}^n))^m$. Note that if $u \in \mathcal{M}_0$, so is $u + j$ for any $j \in \mathbb{Z}^m$. Under the further assumptions that

(N_0) $\mathcal{M}_0/\mathbb{Z}^m$ is finite,

and

(F_2) F is even in $x_1, \cdots, x_n$, i.e. F is spatially reversible,

it was also shown in [1] that for any $v \in \mathcal{M}_0$, there is a $w \in \mathcal{M}_0 \setminus \{v\}$ and a solution of (PDE), $U \in C^2(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$, such that U is heteroclinic in x_1 from v to w. More generally, there is a solution of (PDE) that is heteroclinic in x_i from v to w and is periodic in the remaining variables for each i, $1 \le i \le n$. This result is obtained by minimizing a so-called renormalized functional associated with (PDE) that will be introduced in Section 2. The corresponding set of minimizing heteroclinics will be denoted by $\mathcal{M}_1(v, w)$.

In this paper, we are interested in the existence of more complex homoclinic and heteroclinic solutions of (PDE) that are obtained using the calculus of variations to "glue" together basic heteroclinics obtained in [1]. The simplest example is a 2-transition solution in $C^2(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$ that is homoclinic in x_1 to v and is also near w, i.e. in the language of dynamical systems, shadows w on a region of the form $[p, q] \times \mathbb{T}^{n-1}$. Similarly one can seek solutions of (PDE) that undergo $k \in \mathbb{N}$ or even infinitely many transitions in the x_1 direction between v and w. After some preliminaries in Section 2, such results will be obtained in Section 3. Then in Section 4, the more general question of whether there exist solutions of (PDE) on $\mathbb{R} \times \mathbb{T}^{n-1}$ that are heteroclinic between a given pair $v \ne w \in \mathcal{M}_0$ will be studied. In [1], it was shown that for such a pair, v, w there exists a heteroclinic chain (in x_1) of solutions joining v and w and possessing certain minimality properties that will be described later. A heteroclinic q-chain of solutions is a collection of $q + 1$ members of $\mathcal{M}_0$: $v_0 = v, \cdots, v_q = w$ and solutions $U_1, \cdots, U_q$ of (PDE) such that U_i is heteroclinic (in x_1) from v_{i-1} to v_i, for $1 \le i \le q$. In Section 4, it will be proved that there are actual solutions of (PDE) heteroclinic in x_1 from v to w that shadow the intermediate periodics, v_i, $1 \le i \le q - 1$.

When $m = 1$, (PDE) becomes a single equation. For that case in [16], Moser began a study of a much more general class of equations than (PDE). He was interested in developing a version of Aubry–Mather Theory for partial differential equations. He treated the quasilinear partial differential equation arising formally as the Euler equation of the functional

$$\int_{\mathbb{R}^n} \mathcal{F}(x, u, Du) \, dx$$

where $\mathcal{F}(x, z, p)$ is 1-periodic in the components, x_i of x and in z, is convex in p, and satisfies the sort of growth and coercivity conditions needed for the weak solution of the Euler equation to be a classical solution. In the spirit of Aubry–Mather Theory, Moser established various qualitative properties for a family of solutions he called (i) minimal and (ii) without self-intersection. In particular he proved that there is an ordered family of periodic solutions that satisfies (i) and (ii). For the same setting, Bangert [17] used Moser's results as a tool to help find several other classes of solutions of (PDE) satisfying (i) and (ii), such as heteroclinics between periodics, and heteroclinics in x_2 between simpler heteroclinics in x_1. Specializing to the simplest class of equations with the same essential properties as the family treated by Moser, in [18], Rabinowitz and Stredulinsky studied the Lagrangian corresponding to (PDE):

$$\mathcal{F}(x, z, p) = \frac{1}{2} |\nabla p|^2 + F(x, z),$$

where F satisfies (F_1) with $m = 1$. They first showed Bangert's heteroclinic solutions could be obtained by minimization arguments. This minimization characterization and variational gluing arguments were then employed to construct various kinds of homoclinic and heteroclinic solutions of (PDE) as local minima of a renormalized functional associated with the equation. Other extensions of Moser's work have been carried out in [19–23].

To see the new difficulties encountered when dealing with systems, recall that when $m = 1$, the set $\mathcal{M}_0$ of minimizing period solutions of (PDE) is ordered. Moreover whenever there is a gap pair, $v, w \in \mathcal{M}_0$, i.e. $v < w$ and there are

no other members of $\mathcal{M}_0$ between v and w, the set, $\mathcal{M}_1(v, w)$, of minimizing heteroclinics from v to w (and likewise $\mathcal{M}_1(w, v)$) is also ordered. The presence of these gap pairs of periodics and heteroclinics plays a key role in constructing multi-transition solutions of (PDE) when $m = 1$. Moreover (F_2) need not be required for this case. The order structure of minimizers is a consequence of the Maximum Principle which is no longer available when $m > 1$. Therefore in the present context, a replacement is needed for some of the properties that working with gap pairs provides.

Towards that end, some of the ideas developed to study multi-transition solutions for Hamiltonian systems (see e.g. [2–15]) will be employed. When $n = 1$ and $m > 1$, (PDE) reduces to the Hamiltonian system:

(HS) $-u'' + F_u(t, u) = 0.$

Here we have written t for x_1 and F now represents the potential energy of the system. The existence of multi-transition solutions to (HS) has been studied in [8,11–13]. Already for this simpler setting, a hypothesis like the reversibility condition, (F_2), seems to be needed to get the simplest heteroclinic solutions. Multi-transition solutions for (HS) were studied in [12] with the aid of what might be called the All or Nothing Lemma – see Lemma 2.12 of [12] – which characterized in a sense how degenerate $\mathcal{M}_1(v, w)$ could be in that setting. It enabled the construction of multi-transition solutions of the associated Hamiltonian system via a minimization argument involving functions satisfying certain pointwise constraints. Fortunately we are able to extend the lemma to the setting of (PDE), see Proposition 2.43 below. However the use of pointwise constraints as in [12] does not seem to be a good tool for $n > 1$. Nevertheless in conjunction with Proposition 2.43, we are able to extend the ideas of [12] to a $W^{1,2}$ setting leading in part to the results obtained here.

To be more precise, let $v, w \in \mathcal{M}_0$ and suppose that $\mathcal{M}_1(v, w)$ consists of minimal one transition heteroclinics. Consider the set

$$\mathcal{S}(v, w) = \left\{ U|_{[0,1] \times \mathbb{T}^{n-1}} \mid U \in \mathcal{M}_1(v, w) \right\}.$$

Then as Proposition 2.32 shows, $\bar{\mathcal{S}}(v, w)$ is compact in $(W^{1,2}([0, 1] \times \mathbb{T}^{n-1}))^m$. Letting $\mathcal{C}_v(v, w)$ and $\mathcal{C}_w(v, w)$ denote the components of $\bar{\mathcal{S}}(v, w)$ to which, respectively, v and w belong, Proposition 2.43 tells us either

i) $\mathcal{C}_v(v, w) = \mathcal{C}_w(v, w)$ or   ii) $\mathcal{C}_v(v, w) = \{v\}$ and $\mathcal{C}_w(v, w) = \{w\}$.

This is the result which for the current setting, we call the *All or Nothing Lemma*: Either the components to which v and w belong in $\bar{\mathcal{S}}(v, w)$ coincide or they reduce to the singletons $\{v\}$ and $\{w\}$. In the first case, the set $\mathcal{M}_1(v, w)$ contains a continuum of solutions which, when restricted to $[0, 1] \times \mathbb{T}^{n-1}$ via $\mathcal{S}(v, w)$, accumulate at both v and w. Note that case i) obtains when F does not depend on the x_1 variable. It is instructive to view the use of gap pairs when $m = 1$ in the light of the All or Nothing alternative. Indeed, for $m = 1$, a stronger alternative occurs: either the graphs of the functions in $\mathcal{M}_1(v, w)$ form a foliation of $\{(x, y) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, \ v(x) \leq y \leq w(x)\}$ or there is a gap pair of heteroclinic solutions between v and w. In the first case $\mathcal{M}_1(v, w)$ is a continuum connecting v and w in $W^{1,2}([0, 1] \times \mathbb{T}^{n-1})$ while, the presence of a gap pair is equivalent to the case ii) of the All or Nothing Lemma.

When ii) of Proposition 2.43 holds, $\bar{\mathcal{S}}(v, w)$ can be written as the disjoint union of two compact sets, one containing v and the other w, and they are used in the constrained minimization procedures to find multi-transition solutions of (PDE). No ordering structure is needed. Of course if ii) fails, (PDE) has a continuum of heteroclinic solutions joining v and w. Thus in either event, (PDE) has infinitely many heteroclinic or multi-transition solutions. Of course this is trivial when F is independent of x, but not so otherwise.

To conclude this section, consider the Allen–Cahn model equation

(AC) $-\Delta u + a(x)G'(u) = 0, \ x \in \mathbb{R}^n,$

where $G(u) = u^2(1 - u)^2$ or a similar double well potential and $a(x)$ is continuous, positive, and 1-periodic in the components of x. (AC) involves a rather different nonlinearity than that of (PDE). Such equations arise as models of phase transitions. As was shown e.g. in [18], the Maximum Principle can be used to reduce the study of the existence of solutions of (AC) lying between 0 and 1 to that of (PDE). There have been many papers which study (AC) or related models directly, mainly for $n = 2$, and prove the existence of multi-transition solutions. See e.g. [23–30]. Finally there is a series of papers on autonomous Allen Cahn systems [31–41] where related discreteness or finiteness assumptions (modulo translations) on the set of minimal heteroclinics are used to produce different classes of entire solutions.

2. Some preliminaries

In this section, several preliminaries that are needed to prove the existence of multi-transition solutions of (PDE) will be given. For convenience, we also state some results that were established in [1].

Let $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ and set $L(u) = \frac{1}{2}|\nabla u|^2 + F(x, u)$. Define

$$J_0(u) = \int_{\mathbb{T}^n} L(u)\, dx$$

and

$$c_0 = \inf_{u \in E_0} J_0(u),$$

where $E_0 = (W^{1,2}(\mathbb{T}^n))^m$. Set $T_p = [p, p+1] \times \mathbb{T}^{n-1}$ for $p \in \mathbb{Z}$. For $A \subset E_0$ and $\rho > 0$, let $N_\rho(A)$ denote a ρ neighborhood of A in whatever topology is being put on A. E.g. for $A \subset E_0$,

$$N_\rho(A) = \left\{ u \in E_0 \,\Big|\, \inf_{v \in A} \|u - v\|_{E_0} \leq \rho \right\}.$$

Then it was shown in [1]:

Proposition 2.1. *If F satisfies (F_1), then*

$$\mathcal{M}_0 = \left\{ u \in E_0 \,\big|\, J_0(u) = c_0 \right\} \neq \emptyset.$$

Setting

$$\hat{\mathcal{M}}_0 = \left\{ u \in \mathcal{M}_0 \,\big|\, [u] \in [0, 1]^m \right\} \quad \left([u] = \int_{[0,1]^n} u(x)\, dx \right),$$

then:

i) $\hat{\mathcal{M}}_0$ *is compact in E_0.*
ii) *If $(u_k) \subset E_0$, $[u_k] \in [0, 1]^m$, and $J_0(u_k) \to c_0$ as $k \to \infty$, then there is a $u \in \hat{\mathcal{M}}_0$ such that $u_k \to u$ in E_0 along a subsequence as $k \to \infty$.*
iii) *For any $\rho > 0$ there exists a $\beta(\rho) > 0$ such that if $u \in (W^{1,2}(T_0))^m \setminus N_\rho(\mathcal{M}_0)$, then $J_0(u) - c_0 \geq \beta(\rho)$.*

If (F) also satisfies (F_2), then

$$c_0 = \inf_{u \in (W^{1,2}([0,1]^n))^m} J_0(u)$$

and any $u \in \mathcal{M}_0$ is symmetric in x_i about $x_i = 0$ and about $x_i = 1/2$ for any $i \in \{1, \dots, n\}$.

Let $k \in \mathbb{N}^n$ and set

$$W^{1,2}\big(k \cdot \mathbb{T}^n\big)^m \equiv \left\{ u \in W^{1,2}_{loc}\big(\mathbb{R}^n, \mathbb{R}^m\big) \,\big|\, u \text{ is } k_1\text{-periodic in } x_1, \cdots, k_n\text{-periodic in } x_n \right\}.$$

For $u \in (W^{1,2}(k \cdot \mathbb{T}^n))^m$, set

$$I_k(u) = \int_0^{k_1} \cdots \int_0^{k_n} L(u)\, dx$$

and define

$$c_0(k) = \inf_{u \in (W^{1,2}(k \cdot \mathbb{T}^n))^m} I_k(u).$$

Then from [1], we have

Proposition 2.2. *If (F_1)–(F_2) hold, then for each $k \in \mathbb{N}^n$, $c_0(k) = (\Pi_1^n k_i)c_0$ and if $\mathcal{M}_0(k)$ is the corresponding set of minimizers of I_k, then $\mathcal{M}_0(k) = \mathcal{M}_0$.*

Remark 2.3. In Section 3, the following estimate will be needed. Take $k = (2, 1, \cdots, 1) \equiv k_0$. Then the proof of iii) of Proposition 2.1 shows for any $\rho > 0$, there is a $\beta_2 = \beta_2(\rho) > 0$ such that if $u \in (W^{1,2}(T_0 \cup T_1))^m \setminus \{N_\rho(\mathcal{M}_0(k_0))\}$, then

$$\int\limits_{T_0 \cup T_1} L(u)\, dx - 2c_0 \geq \beta_2(\rho)$$

Propositions 2.1 and 2.2 allow one to study solutions of (PDE) that are heteroclinic between members of $\mathcal{M}_0$. Henceforth it is convenient to assume

(N_0) $\mathcal{M}_0/\mathbb{Z}^m$ *is a finite set.*

Condition (N_0) can be considerably weakened, see e.g. [13], but only at the expense of additional technicalities. By Proposition 2.1 and (N_0),

$$5r_0 \equiv \inf_{u \neq v \in \mathcal{M}_0} \|v - u\|_{(W^{1,2}(T_0))^m} > 0. \tag{2.4}$$

Set

$$E_1 = W_{loc}^{1,2}\big(\mathbb{R} \times \mathbb{T}^{n-1}\big)^m$$

with associated norm

$$\|u\|_{E_1}^2 = \int\limits_{\mathbb{R} \times \mathbb{T}^{n-1}} |\nabla u|^2\, dx + \int\limits_{T_0} u^2\, dx.$$

Now, for $u \in E_1$ and $p \in \mathbb{Z}$, define

$$J_{1,p}(u) = \int\limits_{T_p} L(u)\, dx - c_0$$

and observe that by Proposition 2.1, $J_{1,p}(u) \geq 0$ for any $u \in E_1$ and $p \in \mathbb{Z}$. Finally define the renormalized functional

$$J_1(u) = \sum_{p \in \mathbb{Z}} J_{1,p}(u), \quad u \in E_1.$$

A useful property of this functional is – see [1]:

Proposition 2.5. *If $J_1(u) < \infty$, there exist functions, $v^\pm \in \mathcal{M}_0$, such that*

$$\left\|u - v^\pm\right\|_{(W^{1,2}(T_p))^m} \to 0 \quad as\ p \to \pm\infty.$$

Heteroclinic solutions of (PDE) can now be obtained by minimizing J_1 on appropriate classes of sets. Choose $v \in \mathcal{M}_0$ and set

$$\Gamma(v) = \big\{u \in E_1 \ \big| \ \|u - v\|_{(L^2(T_i))^m} \to 0,\ i \to -\infty;\ \big\|u - \mathcal{M}_0 \setminus \{v\}\big\|_{(L^2(T_i))^m} \to 0,\ i \to \infty\big\}.$$

Define

$$c_1(v) = \inf_{u \in \Gamma(v)} J_1(u). \tag{2.6}$$

Then from [1], we have:

Theorem 2.7. *Suppose F satisfies (F_1)–(F_2), (N_0) holds, and $v \in \mathcal{M}_0$. Then*

$$\mathcal{M}_1(v) \equiv \left\{ u \in \Gamma(v) \mid J_1(u) = c_1(v) \right\} \neq \emptyset$$

and there is a $w \in \mathcal{M}_0 \setminus \{v\}$ such that as $p \to \infty$, $\|U - w\|_{C^2(T_p, \mathbb{R}^m)} \to 0$. Moreover any $U \in \mathcal{M}_1(v)$ is a classical solution of (PDE) and is even in $x_2, \cdots, x_n$.

Remark 2.8. An important ingredient of the proof is a 'Palais–Smale' type property for J_1 on $\Gamma(v)$: any minimizing sequence for (2.6) is bounded in E_1, i.e. for each $0 < l \in \mathbb{R}$, is bounded in $W^{1,2}([-l, l] \times \mathbb{T}^{n-1})^m$, and with an appropriate normalization possesses a subsequence that converges in $W^{1,2}([-l, l] \times \mathbb{T}^{n-1})^m$.

Let $w \in \mathcal{M}_0 \setminus \{v\}$, set

$$\Gamma_1(v, w) = \left\{ u \in E_1 \mid \|u - v\|_{(L^2(T_i))^m} \to 0, \ i \to -\infty; \ \|u - w\|_{(L^2(T_i))^m} \to 0, \ i \to \infty \right\},$$

and define

$$C_1(v, w) = \inf_{u \in \Gamma_1(v, w)} J_1(u). \tag{2.9}$$

Let

$$\mathcal{M}_1(v, w) = \left\{ u \in \Gamma_1(v, w) \mid J_1(u) = C_1(v, w) \right\}.$$

It is possible that $\mathcal{M}_1(v, w) = \emptyset$, i.e. $C_1(v, w)$ may not be achieved, but as was shown in [1], there is a heteroclinic chain of solutions of (PDE) corresponding to $C_1(v, w)$. More precisely, by a heteroclinic chain of solutions of (PDE) joining v and w, we mean there is a $q \in \mathbb{N}$, functions $v_i \in \mathcal{M}_0$, $0 \leq i \leq q$, with $v_0 = v$, $v_q = w$ and functions $U_i \in \mathcal{M}_1(v_{i-1}, v_i)$, $1 \leq i \leq q$. In [1], it was shown that:

Theorem 2.10. *Let F satisfy (F_1)–(F_2) and (N_0) hold. Then for any pair $v, w \in \mathcal{M}_0$ with $v \neq w$, there exists a heteroclinic chain of solutions of (PDE) joining v and w with the property that*

$$C_1(v, w) = \Sigma_1^q C_i(v_{i-1}, v_i) = \Sigma_1^q J_1(U_i). \tag{2.11}$$

Remark 2.12. The chain need not be unique. Moreover it is possible that for some pair, v_{i-1} and v_i, there is a further heteroclinic chain, $u_0 = v_{i-1}, \cdots, u_s = v_i$ joining v_{i-1} and v_i with

$$C_1(v_{i-1}, v_i) = \Sigma_1^s C_i(u_{i-1}, u_i). \tag{2.13}$$

Replacing the $C_1(v_{i-1}, v_i)$ term in (2.11) by the right hand side of (2.13) gives a new heteroclinic chain joining v and w. We claim there is a constant, $\omega > 0$ such that $c_1(v) > \omega$ independently of $v \in \mathcal{M}_0$. If so, this replacement process can be repeated at most a finite number of times, arriving at a new heteroclinic chain joining v and w which cannot be further decomposed. We call such a chain a minimal heteroclinic chain. (This terminology was used in a slightly different way in [1].) To see the existence of ω, without loss of generality, it can be assumed that $[v] \in [0, 1]^m$, i.e. $v \in \hat{\mathcal{M}}_0$. But by (N_0), the set of such functions v is finite. Thus $\omega = \min\{c_1(v) \mid v \in \hat{\mathcal{M}}_0\}$. Note that a minimal heteroclinic chain has the property that for any adjacent pair v_{i-1}, v_i in it, if $z \in \mathcal{M}_0 \setminus \{v_{i-1}, v_i\}$ then

$$C_1(v_{i-1}, v_i) < C_1(v_{i-1}, z) + C_1(z, v_i). \tag{2.14}$$

These facts about heteroclinic chains will be useful later.

Next suppose that w is as given by Theorem 2.7. Then by that result, $C_1(v, w) = c_1(v)$ and $\emptyset \neq \mathcal{M}_1(v, w) \subset \mathcal{M}_1(v)$. With this choice of v and w, we will show in Section 3 that there are solutions of (PDE) that e.g. in the simplest case, are homoclinic in x_1 to v and shadow members of $\mathcal{M}_1(v, w)$ and $\mathcal{M}_1(w, v)$. Such results were obtained in [18] when $m = 1$ using the fact that $\mathcal{M}_0$ is an ordered set. The ordering property for $\mathcal{M}_0$ is a consequence of the Maximum Principle. This important tool is no longer available when $m > 1$. However for $n = 1$, such homoclinic solutions were found in [12] with the aid of a closer study of $\mathcal{M}_1(v, w)$. Fortunately some key arguments from [12] extend to the current setting and will be exploited in what follows.

For $i \in \mathbb{Z}$, set $X_i = \bigcup_{i-2}^{i+2} T_j$.

Proposition 2.15. *Suppose* $v \in \mathcal{M}_0$, $w \in \mathcal{M}_0 \setminus \{v\}$, *and* $C_1(v, w)$ *satisfies* (2.14). *Then there is a constant*, $r_1 = r_1(v, w)$ *such that whenever* $z \in \mathcal{M}_0 \setminus \{v, w\}$ *and* $U \in \mathcal{M}_1(v, w)$,

$$\|z - U\|_{(W^{1,2}(T_0))^m} \geq 5r_1. \tag{2.16}$$

Proof. If the result is false, for any $\varepsilon > 0$, there is a $z \in \mathcal{M}_0$ and a $U_z \in \mathcal{M}_1(v, w)$ such that

$$\|z - U_z\|_{(W^{1,2}(T_0))^m} < \varepsilon. \tag{2.17}$$

Define $g \in \Gamma_1(v, z)$ and $h \in \Gamma_1(z, w)$ by

$$g = \begin{cases} U_z, & x_1 \leq 0, \\ x_1 z + (1 - x_1)U_z, & 0 \leq x_1 \leq 1, \\ z, & 1 \leq x_1 \end{cases} \tag{2.18}$$

and

$$h = \begin{cases} z, & x_1 \leq 0, \\ x_1 U_z + (1 - x_1)z, & 0 \leq x_1 \leq 1, \\ U_z, & 1 \leq x_1. \end{cases} \tag{2.19}$$

Due to the continuity of $J_{1,i}(\cdot)$, there is a $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$J_{1,0}(g), J_{1,0}(h) \leq \kappa(\varepsilon). \tag{2.20}$$

But then by (2.20),

$$C_1(v, w) = J_1(U_z) \geq J_1(g) + J_1(h) - 2\kappa(\varepsilon) \geq C_1(v, z) + C_1(z, w) - 2\kappa(\varepsilon). \tag{2.21}$$

Since ε is arbitrary, (2.21) is incompatible with (2.14) and the result follows. $\square$

If v and $w \in \mathcal{M}_0$ and $c_1(v) = C_1(v, w)$ or more generally if the heteroclinic chain of Theorem 2.10 consists of a single heteroclinic, then any $U \in \mathcal{M}_1(v, w)$ is a classical solution of (PDE). Therefore for any $\alpha \in (0, 1)$, $U \in C_{loc}^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$. The next result provides a uniform bound on the members of $\mathcal{M}_1(v, w)$.

Proposition 2.22. *There is an* $M = M(\alpha, v, w)$ *such that for any* $U \in \mathcal{M}_1(v, w)$, $\|U\|_{C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^m)} \leq M$.

Proof. Since $J_1(U) = C_1(v, w)$,

$$J_{1,i}(U) = \int_{T_i} \left(\frac{1}{2}|\nabla U|^2 + F(x, U) - c_0 \right) dx \leq C_1(v, w)$$

so for each $i \in \mathbb{Z}$,

$$\int_{T_i} |\nabla U|^2 \, dx \leq 2\left(c_0 + C_1(v, w) + \|F\|_{L^\infty(\mathbb{R}^{n+m}, \mathbb{R}^m)}\right) \equiv M_1. \tag{2.23}$$

Choose r_1 as given by Proposition 2.15. Suppose

$$\|U|_{T_i} - \mathcal{M}_0\|_{(W^{1,2}(T_i))^m} < r_1. \tag{2.24}$$

This is the case for all large $|i|$ so for such i,

$$\|U\|_{(W^{1,2}(T_i))^m} \leq \max\left(\|v\|_{(W^{1,2}(T_0))^m}, \|w\|_{(W^{1,2}(T_0))^m}\right) + r_1 \equiv M_2 \tag{2.25}$$

Let l be the number of sets, T_i, such that

$$\|U|_{T_i} - \mathcal{M}_0\|_{(W^{1,2}(T_i))^m} \geq r_1. \tag{2.26}$$

By iii) of Proposition 2.1,

$$l\beta(r_1) \leq \Sigma_{i \in \mathbb{Z}} J_{1,i}(U) = C_1(v, w). \tag{2.27}$$

Let p be the smallest value of $i \in \mathbb{Z}$ for which (2.24) fails. Set $\hat{x} = (x_2, \cdots, x_n) \in \mathbb{T}^{n-1}$. For $(s, \hat{x}) \in T_{p-1}$ and $(\sigma, \hat{x}) \in T_p$,

$$U(\sigma, \hat{x}) = U(s, \hat{x}) + \int_s^\sigma U_{x_1}(t, \hat{x}) \, dt$$

so

$$\left| U(\sigma, \hat{x}) \right|^2 \le 2 \left| U(s, \hat{x}) \right|^2 + 4 \int_{p-1}^{p+1} \left| U_{x_1}(t, \hat{x}) \right|^2 dt. \tag{2.28}$$

Integrating (2.28) over $s, t, \hat{x}$ gives

$$\|U\|^2_{(L^2(T_p))^m} \le 2 \|U\|^2_{(L^2(T_{p-1}))^m} + 4 \|U_{x_1}\|^2_{(L^2(T_{p-1} \cup T_p))^m}. \tag{2.29}$$

Therefore by (2.23), (2.25) and (2.29),

$$\|U\|^2_{(W^{1,2}(T_p))^m} \le 2M_2^2 + 6M_1. \tag{2.30}$$

Repeating the argument of (2.28)–(2.30) at most $l-1$ times, we find there is an $M_3 = M_3(l, M_1, \|v\|_{(W^{1,2}(T_0))^m})$ such that for all $i \in \mathbb{Z}$,

$$\|U\|_{(W^{1,2}(T_i))^m} \le M_3. \tag{2.31}$$

Then standard bootstrap arguments a la Moser – see e.g. [42] or [43] – yield the $C^{2,\alpha}$ bounds for U. □

Next for $v, w \in \mathcal{M}_0$ with $\mathcal{M}_1(v, w)$ consisting of minimal one transition heteroclinics, define

$$\mathcal{S}(v, w) = \left\{ U|_{T_0} \mid U \in \mathcal{M}_1(v, w) \right\}.$$

We consider $\mathcal{S}(v, w)$ under the $(W^{1,2}(T_0))^m$ topology. It is clear that $v, w \in \bar{\mathcal{S}}(v, w)$. The main properties of $\mathcal{S}(v, w)$ that we require are:

Proposition 2.32.

  i) $\mathcal{S}(v, w)$ *is bounded in* $C^{2,\alpha}(T_0, \mathbb{R}^m)$.
 ii) $\bar{\mathcal{S}}(v, w) = \mathcal{S}(v, w) \cup \{v\} \cup \{w\}$.
iii) $\bar{\mathcal{S}}(v, w)$ *is a compact metric space (in* $\| \cdot \|_{(W^{1,2}(T_0))^m}$*)*.

Proof. Proposition 2.22 implies i). For ii), let $(u_k) \subset \mathcal{S}(v, w)$ with $u_k \to u \in \bar{\mathcal{S}}(v, w)$ as $k \to \infty$. Let X_i be as earlier. Since (u_k) is in fact defined and, by i), bounded in $C^{2,\alpha}(X_0, \mathbb{R}^m)$, it can be assumed that $u_k \to u$ in $C^2(X_0, \mathbb{R}^m)$ as $k \to \infty$. Suppose that $u \not\equiv v$ or w. We claim $u \in \mathcal{S}(v, w)$ or more precisely, $u = U|_{T_0}$ for some $U \in \mathcal{S}(v, w)$. To see this, observe that $u_k = U_k|_{T_0}$, where $U_k \in \mathcal{M}_1(v, w)$. Proposition 2.22 implies that there is a $U \in C^2(\mathbb{R}^n, \mathbb{R}^m)$ such that, along a subsequence, $U_k \to U$ in $C^2_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ as $k \to \infty$. Thus U is a solution of (PDE) and $U|_{T_0} = u \not\equiv v, w$. We claim $u \notin \mathcal{M}_0 \setminus \{v, w\}$. To see this, let $\varepsilon > 0$. For large k, $\|U_k - u\|_{(W^{1,2}(X_0))^m} \le \varepsilon$. Define $f_k \in \Gamma_1(v, w)$ by

$$f_k = \begin{cases} U_k, & x_1 \le -1, \\ (x_1 + 1)u - x_1 U_k, & -1 \le x_1 \le 0, \\ u, & 0 \le x_1 \le 1, \\ (x_1 - 1)U_k + (2 - x_1)u, & 1 \le x_1 \le 2, \\ U_k, & 2 \le x_1. \end{cases} \tag{2.33}$$

Then

$$0 \le J_1(f_k) - J_1(U_k) \le J_{1,-1}(f_k) + J_{1,1}(f_k) \le 2\kappa(\varepsilon) \tag{2.34}$$

with $\kappa(\varepsilon)$ as earlier. Further define functions $g_k \in \Gamma_1(v, u)$ and $h_k \in \Gamma_1(u, w)$ via

$$g_k = \begin{cases} f_k, & x_1 \le 1, \\ u, & x_1 \ge 1, \end{cases} \tag{2.35}$$

$$h_k = \begin{cases} u, & x_1 \le 1, \\ f_k, & x_1 \ge 1. \end{cases} \tag{2.36}$$

By construction,

$$J_1(f_k) = J_1(g_k) + J_1(h_k). \tag{2.37}$$

Then by (2.34) and (2.37),

$$C_1(v, w) = J_1(U_k) \ge J_1(f_k) - 2\kappa(\varepsilon) \ge C_1(v, u) + C_1(u, w) - 2\kappa(\varepsilon). \tag{2.38}$$

Since ε is arbitrary,

$$C_1(v, w) \ge C_1(v, u) + C_1(u, w). \tag{2.39}$$

But by (2.14), (2.39) is not possible. Consequently $u \notin \mathcal{M}_0 \setminus \{v, w\}$ and $J_{1,0}(U) > 0$.

Note that

$$\Sigma_{-p}^{p} J_{1,i}(U) \le \liminf_{k \to \infty} \Sigma_{-p}^{p} J_{1,i}(U_k) \le \liminf_{k \to \infty} J_1(U_k) = C_1(v, w).$$

Therefore $J_1(U) \le C_1(v, w)$. By Proposition 2.5, there are functions, $U_{\pm} \in \mathcal{M}_0$ such that as $p \to \pm\infty$, $\|U - U_{\pm}\|_{(W^{1,2}(T_p))^m} \to 0$. We will show $U_- = v$ and $U_+ = w$. If so, $U \in \mathcal{M}_1(v, w)$, $u \in \mathcal{S}(v, w)$ and ii) is proved. If $U_- \ne v$, let $\varepsilon > 0$. Then there is an $i_0 = i_0(\varepsilon) \in -\mathbb{N}$ such that $\|U - U_-\|_{(W^{1,2}(T_i))^m} \le \frac{\varepsilon}{10}$ for all $i \le i_0$. Hence $\|U - U_-\|_{(W^{1,2}(X_i))^m} \le \frac{\varepsilon}{2}$ for $i \le i_0 - 2$. Take $i = i_0 - 2$. For large $k = k(i)$, $\|U_k - U_-\|_{(W^{1,2}(X_i))^m} \le \varepsilon$. Define $f_k \in \Gamma_1(v, w)$ as in (2.33) with u replaced by U_- and X_0 by X_i. Then (2.34) remains unchanged. Now define functions $g_k \in \Gamma_1(v, U_-)$ and $h_k \in \Gamma_1(U_-, w)$ by (2.35)–(2.36), and using (2.34) and (2.37) shows

$$C_1(v, w) = J_1(U_k) \ge J_1(f_k) - 2\kappa(\varepsilon) \ge C_1(v, U_-) + C_1(U_-, w) - 2\kappa(\varepsilon). \tag{2.40}$$

Following (2.38)–(2.39), this leads to

$$C_1(v, w) \ge C_1(v, U_-) + C_1(U_-, w) \tag{2.41}$$

which is only possible if $U_- = v$ or w. But $U_- = v$ has already been excluded, so $U_- = w$. By (2.40) again,

$$C_1(v, w) \ge J_1(f_k) - 2\kappa(\varepsilon) \ge C_1(v, w) + J_{1,0}(h_k) - 2\kappa(\varepsilon). \tag{2.42}$$

As $k \to \infty$, $J_{1,0}(h_k) \to J_{1,0}(u) > 0$. Thus choosing ε so small that $J_{1,0}(u) > 2\kappa(\varepsilon)$ shows (2.42) is not possible for large k. Hence it must be the case that $U_- = v$. A similar argument implies $U_+ = w$ and ii) is proved.

To obtain iii), suppose $(u_k) \subset \bar{\mathcal{S}}(v, w)$. Then as for ii), there is a $z \in \bar{\mathcal{S}}(v, w)$ such that, along a subsequence, $u_k \to z$ in $(W^{1,2}(T_0))^m$. Hence $\bar{\mathcal{S}}(v, w)$ is a compact metric space. $\square$

To continue, let $\mathcal{C}_v(v, w)$ denote the component of $\bar{\mathcal{S}}(v, w)$ to which v belongs and define $\mathcal{C}_w(v, w)$ similarly. Then we have:

Proposition 2.43. *Either*

 i) $\mathcal{C}_v(v, w) = \mathcal{C}_w(v, w)$ *or*
 ii) $\mathcal{C}_v(v, w) = \{v\}$ *(and $\mathcal{C}_w(v, w) = \{w\}$).*

If ii) holds, there exist nonempty disjoint compact sets, $K_v^- \equiv K_v(v, w)$, $K_w^- \equiv K_w(v, w) \subset \bar{\mathcal{S}}(v, w)$ such that

 iii) $v \in K_v^-$, $w \in K_w^-$,
 iv) $\bar{\mathcal{S}}(v, w) = K_v^- \cup K_w^-$,
 v) $\mathrm{dist}(K_v^-, K_w^-) \equiv 5r_2 > 0$.

Proof. To establish the first statement, suppose that i) and ii) fail to hold. Then $\{v\} \neq \mathcal{C}_v(v, w)$ and $\mathcal{C}_v(v, w)$ is a connected set which does not meet w. For $u \in \bar{\mathcal{S}}(v, w)$ and $k \in \mathbb{N}$, set $f_k(u)(x) = u(x + ke_1)$ where e_1 is a unit vector in the x_1 direction. Then $f_k : \bar{\mathcal{S}}(v, w) \to \bar{\mathcal{S}}(v, w)$ and is continuous. Therefore $f_k(\mathcal{C}_v(v, w))$ is compact and connected with $v \in f_k(\mathcal{C}_v(v, w))$. Hence $f_k(\mathcal{C}_v(v, w)) \subset \mathcal{C}_v(v, w)$ for each $k \in \mathbb{N}$. But as $k \to \infty$, $f_k(u) \to w$ for each $u \in \mathcal{S}(v, w)$. Since $w \notin \mathcal{C}_v(v, w)$, we have a contradiction.

Next suppose that ii) of the proposition holds. Since there does not exist a subcontinuum of $\bar{\mathcal{S}}(v, w)$ that joins v and w, by a standard separation theorem in point set topology, [44], there are nonempty disjoint compact sets, K_v^-, K_w^- that satisfy iii) and iv) and hence v). $\quad\square$

Remark 2.44. Note that i) of Proposition 2.43 occurs if e.g. F is independent of x_1.

Remark 2.45. With v and w as above, since $\mathcal{M}_1(w, v)$ also consists of minimal one transition heteroclinics, $\mathcal{S}(w, v)$ is well defined. Thus there is a version of Proposition 2.43 in this setting for, using the obvious notation, the sets $\mathcal{C}_v(w, v)$, $\mathcal{C}_w(w, v)$, $K_v^+ \equiv K_v(w, v)$, $K_w^+ \equiv K_w(w, v)$. Due to (F_2), $\text{dist}(K_v^+, K_w^+) = 5r_2$. Note also that $\text{dist}(N_{r_1}(K_v^\pm), N_{r_1}(K_w^\pm)) \geq r_2$.

For constructing multi-transition solutions of (PDE), the following technical result is needed. Set $r = \min(r_0, r_1, r_2)$ and define the set

$$\Lambda^-(v, w) = \left\{ u \in \Gamma_1(v, w) \mid \left\| u - K_v^- \right\|_{(W^{1,2}(T_0))^m} = r \right\}$$

and associated minimization value

$$d^-(v, w) = \inf_{u \in \Lambda^-(v, w)} J_1(u). \tag{2.46}$$

Then we have

Proposition 2.47. *If $\mathcal{S}(v, w)$ satisfies ii) of Proposition 2.43, $d^-(v, w) > C_1(v, w)$.*

Proof. The argument of Theorem 3.4 in [1] – see also [18] – shows that a minimizing sequence for (2.46) has a subsequence converging in $W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ to a $V \in W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ with $J_1(V) \leq d^-(v, w)$. Therefore V satisfies the constraint of $\Lambda^-(v, w)$. Moreover by Proposition 2.5, there are functions, $V_\pm \in \mathcal{M}_0$ which are the limits in $(W^{1,2}(T_0))^m$ of $V|_{T_p}$ as $p \to \pm\infty$. The argument in Proposition 2.32 involving $U_\pm$ shows $V_- = v$ and $V_+ = w$. Hence $V \in \Lambda^-(v, w)$ and $J_1(V) \geq d^-(v, w)$ so $J_1(V) = d^-(v, w)$. Since $\Lambda^-(v, w) \subset \Gamma_1(v, w)$, $C_1(v, w) \leq d^-(v, w)$. If $C_1(v, w) = d^-(v, w)$, $V \in \mathcal{M}_1(v, w)$ and $V|_{T_0} \in \mathcal{S}(v, w)$. In particular, either $V|_{T_0} \in K_v^-$ or $V|_{T_0} \in K_w^-$. But due to the constraint, $\|V - K_v^-\|_{(W^{1,2}(T_0))^m} = r$, and v) of Proposition 2.43, neither alternative is possible. Thus $d^-(v, w) > C_1(v, w)$. $\quad\square$

Remark 2.48. By the same argument as that of the proof of Proposition 2.47, there is a version of that result with the constraint replaced by $\|u - K_w^-\|_{(W^{1,2}(T_0))^m} = r$, the new set of functions being called $\Lambda^+(v, w)$ and $d^-(v, w)$ replaced by $d^+(v, w)$. Setting $d(v, w) = \min(d^-(v, w), d^+(v, w))$ and $\Lambda(v, w) = \Lambda^-(v, w) \cup \Lambda^+(v, w)$, we have

Corollary 2.49. *If $\mathcal{S}(v, w)$ satisfies ii) of Proposition 2.43, $d(v, w) > C_1(v, w)$.*

Remark 2.50. Likewise with the aid of Remark 2.45, there are variants of Proposition 2.47 and Corollary 2.49 for $\mathcal{S}(w, v)$ with the natural changes in notation: $d^\pm(w, v)$, etc.

3. The simplest multi-transition solutions

Let $v \in \mathcal{M}_0$ and let $w \in \mathcal{M}_0 \setminus \{v\}$. In this section, the existence of solutions of (PDE) that as a function of x_1 go back and forth between v and w will be established. We begin with the simplest case of homoclinic solutions to v that also shadow w on a region of the form $[p, q] \times \mathbb{T}^{n-1}$. These solutions undergo two transitions. Then we consider solutions with any finite number of transitions. For both of these cases, the solutions will be found as local minima

of J_1 on an appropriate class of functions. Lastly, by a limit process, solutions having an infinite number of transitions will be treated.

To formulate the variational problem for the simplest case, let $\mathbf{m} = (m_1, \cdots, m_4) \in \mathbb{Z}^4$ and $l \in \mathbb{N}$ with $m_1 + 2l < m_2 - 2l < m_2 + 2l < m_3 - 2l < m_3 + 2l < m_4 - 2l$. Let $K_v^\pm$, $K_w^\pm$ be as in Proposition 2.43. Define

$$\mathcal{A}_2 = \mathcal{A}_2(\mathbf{m}, l; v, w) = \{u \in E_1 \mid u \text{ satisfies } (3.1)\}$$

where

$$
u(\cdot + je_1)|_{T_0} \in
\begin{cases}
N_r(K_v^-), & j < m_1 + l, \\
N_r(K_w^-), & m_2 - l \leq j < m_2 + l, \\
N_r(w), & m_2 + 2l \leq j < m_3 - 2l, \\
N_r(K_w^+), & m_3 - l \leq j < m_3 + l, \\
N_r(K_v^+), & m_4 - l \leq j.
\end{cases}
\tag{3.1}
$$

The members of $\mathcal{A}_2$ are candidates for the type of 2-transition solutions that we seek. It is possible to remove the $N_r(w)$ constraint at the expense of some additional work in the following existence result.

Define

$$b_2 = b_2(\mathbf{m}, l; v, w) = \inf_{u \in \mathcal{A}_2} J_1(u) \tag{3.2}$$

and set

$$\mathcal{M}(b_2) = \{u \in \mathcal{A}_2 \mid J_1(u) = b_2\}.$$

Then we have

Theorem 3.3. *Suppose (F_1)–(F_2) and (N_0) are satisfied. Let v and $w \in \mathcal{M}_0$ with $c_1(v) = C_1(v, w)$ and assume ii) of Proposition 2.43 holds for $\mathcal{S}(v, w)$ and $\mathcal{S}(w, v)$. There exists an $m_0 = m_0(v, w) \in \mathbb{N}$ for which whenever $l \geq m_0$ and $m_{i+1} - m_i - 6l \geq m_0$ for $i = 1, 2, 3$, then $\mathcal{M}(b_2) \neq \emptyset$. Moreover any $U \in \mathcal{M}(b_2)$ is a classical solution of (PDE).*

Remark 3.4. Varying l and m in Theorem 3.3 shows that there are infinitely many such 2-transition homoclinic solutions of (PDE). These solutions shadow members of $\mathcal{M}_1(v, w)$ and $\mathcal{M}_1(w, v)$ in subregions of $\mathbb{R} \times T^{n-1}$. The constant, m_0, works equally well if the roles of v and w are reversed. Hence we also obtain a theorem giving solutions homoclinic to w.

Before proving Theorem 3.3, one final preliminary providing an upper bound for b_2 is needed.

Lemma 3.5. *There is a constant, $M > 0$ such that $b_2 \leq M$ independently of l and $\mathbf{m}$.*

Proof. Take $\bar{u} \in E_1$ such that

$$
\bar{u} =
\begin{cases}
v, & x_1 \leq m_1 + l, \\
(x_1 - (m_1 + l))w + (m_1 + l + 1 - x_1)v, & m_1 + l \leq x_1 \leq m_1 + l + 1, \\
w, & m_1 + l + 1 \leq x_1 \leq m_3 + l, \\
(x_1 - (m_3 + l))v + (m_3 + l + 1 - x_1)w, & m_3 + l \leq x_1 \leq m_3 + l + 1, \\
v, & x_1 \geq m_3 + l + 1.
\end{cases}
\tag{3.6}
$$

Then $\bar{u} \in \mathcal{A}_2$ so

$$b_2 \leq J_1(\bar{u}) \equiv M. \qquad \square$$

Proof of Theorem 3.3. Let (u_k) be a minimizing sequence for (3.2). Then by Lemma 3.5, $J_1(u_k) \leq M$ for all $k \in \mathbb{N}$. As e.g. in Proposition 2.22, this shows that (u_k) is bounded in E_1. Hence there is a $U \in E_1$ such that, along a subsequence, $u_k \to U$ weakly in E_1 and strongly in $L^2_{loc}(\mathbb{R}^n, \mathbb{R}^m)$. In fact as in the proof of Theorem 2.7, $u_k \to U$ in E_1. Therefore $U \in \mathcal{A}_2$, so $J_1(U) \geq b_2$. But for any $p \in \mathbb{N}$, Σ_{-p}^p, $J_{1,i}(U) \leq \liminf_{k \to \infty} J_{1,i}(u_k) \leq$

$\liminf_{k\to\infty} J_1(u_k) = b_2$. Thus letting $p \to \infty$, it follows that $J_1(U) \le b_2$ and hence $J_1(U) = b_2$. By Proposition 2.5, there are functions, $U_\pm$, such that $\|U - U_\pm\|_{(W^{1,2}(T_p))^m} \to 0$ as $p \to \pm\infty$. The form of the constraints in $\mathcal{A}_2$ implies $U_- = v$ and $U_+ = w$.

It remains to show that U is a solution of (PDE). Standard arguments in elliptic regularity theory imply this is the case for the regions, T_i, where there are no constraints. The same arguments work for any T_i where there is strict inequality in the associated constraint. Thus due to the x_1 asymptotics, there is a $p_0 \in \mathbb{N}$ such that U is a solution for $|x_1| \ge p_0$.

To show that U is a solution in the remaining regions, the characterization of U as a minimum in (3.2) and the freedom we have in the choice of the parameters $\mathbf{m}$, l will be exploited. First we will show that for large l and each of the constraint regions aside from the middle one involving w directly, there is a region, X_i, where X_i is as in the proof of Proposition 2.32, such that U is very close to v or w in X_i. This information will then imply that U is near v or w in a much larger region. Then a comparison argument can be used to prove that there is strict inequality for all of the constraints of $\mathcal{A}_2$ and therefore U is globally a solution of (PDE).

To carry out this program, let $\varepsilon \in (0, r)$. Further conditions will be imposed on ε in the course of the proof. The parameters l, $\mathbf{m}$ will depend on ε. Suppose that for every $X_i \subset [m_2 - l, m_2 + l + 1] \times \mathbb{T}^{n-1}$, there is some $T_j \subset X_i$, such that $\|U - \mathcal{M}_0\|_{W^{1,2}(T_j)^m} \ge \varepsilon$. Then by iii) of Proposition 2.1,

$$b_2 = J_1(U) \ge (2l + 1)\beta(\varepsilon)/5 \tag{3.7}$$

which is contrary to Lemma 3.5 for large $l = l(\varepsilon)$. Thus there is some such X_{i_2} for which $\|U - \mathcal{M}_0\|_{W^{1,2}(T_j)^m} < \varepsilon$ for $j \in [i_2 - 2, i_2 + 2] \cap \mathbb{Z}$. But by (3.1) and the choice of r, the only member of $\mathcal{M}_0$ for which this is possible is w.

Remark 3.8. The same argument gives an $X_{i_3} \subset [m_3 - l, m_3 + l + 1] \times \mathbb{T}^{n-1}$ with $\|U - w\|_{W^{1,2}(T_j)^m} < \varepsilon$ for $j \in [i_3 - 2, i_3 + 2] \cap \mathbb{Z}$ as well as $X_{i_1} \subset [m_1 - l, m_1 + l + 1] \times \mathbb{T}^{n-1}$ with $\|U - v\|_{W^{1,2}(T_j)^m} < \varepsilon$ for $j \in [i_1 - 2, i_1 + 2] \cap \mathbb{Z}$ and $X_{i_4} \subset [m_4 - l, m_4 + l + 1] \times \mathbb{T}^{n-1}$ with $\|U - v\|_{W^{1,2}(T_j)^m} < \varepsilon$ for $j \in [i_4 - 2, i_4 + 2] \cap \mathbb{Z}$.

Next we will show that U is near w in T_i for $i_2 \le i \le i_3$. First suppose that there is a $p \in [i_2, i_3] \cap \mathbb{Z}$ such that $\|U - \mathcal{M}_0\|_{W^{1,2}(T_p)^m} \ge r$. Thus $i_2 + 2 \le p \le i_3 - 2$ and by Proposition 2.1,

$$\Sigma_{i_2-1}^{i_3+1} J_{i,k}(U) \ge J_{1,p}(U) \ge \beta(r). \tag{3.9}$$

On the other hand, define

$$V = \begin{cases} U, & x_1 \le i_2 - 1, \\ (x_1 - (i_2 - 1))w + (i_2 - x_1)U, & i_2 - 1 \le x_1 \le i_2, \\ w, & i_2 \le x_1 \le i_3 + 1, \\ (x_1 - (i_3 + 1))U + (i_3 + 2 - x_1)w, & i_3 + 1 \le x_1 \le i_3 + 2, \\ U, & x_1 \ge i_3 + 2. \end{cases} \tag{3.10}$$

Then for ε sufficiently small, $V \in \mathcal{A}_2$ and with $\kappa(\varepsilon)$ as earlier,

$$J_1(U) - J_1(V) \ge \beta(r) - 2\kappa(\varepsilon) > 0. \tag{3.11}$$

But (3.11) is contrary to the minimality of U in $\mathcal{A}_2$, so for each p with $i_2 + 2 \le p \le i_3 - 2$, $\|U - \mathcal{M}_0\|_{W^{1,2}(T_p)^m} < r$. We claim that in fact

$$\|U - \mathcal{M}_0\|_{W^{1,2}(T_p)^m} = \|U - w\|_{W^{1,2}(T_p)^m} \tag{3.12}$$

so U is interior to this part of the constraint set and hence satisfies (PDE) here. To verify (3.12), take k_0 as in Remark 2.3. Using Proposition 2.2 and Remark 2.3, suppose there is a pair $T_p \cup T_{p+1}$ with $p \in [i_2, i_3]$ such that

$$\left\| U - \mathcal{M}_0(k_0) \right\|_{W^{1,2}(T_p \cup T_{p+1})^m} \ge 2r.$$

Then as in (3.11),

$$\Sigma_{i_2-1}^{i_3+1} J_{i,k}(U) \ge J_{1,p}(U) + J_{1,p+1}(U) \ge \beta_2(2r) \tag{3.13}$$

so for ε possibly still smaller,

$$J_1(U) - J_1(V) \geq \beta_2(2r) - 2\kappa(\varepsilon) > 0 \tag{3.14}$$

again a contradiction. Hence for all $p \in [i_2, i_3]$,

$$\left\| U - \mathcal{M}_0(k_0) \right\|_{W^{1,2}(T_p \cup T_{p+1})^m} < 2r.$$

Thus for each such p, there is a $z_p \in \mathcal{M}_0$ such that

$$\| U - z_p \|_{W^{1,2}(T_p \cup T_{p+1})^m} < 2r. \tag{3.15}$$

Due to the choice of r, (3.15) implies that z_p is independent of p. But for $p = i_2$, $z_p = w$. Hence

$$\| U - w \|_{W^{1,2}(T_p \cup T_{p+1})^m} < 2r \tag{3.16}$$

for each $p \in [i_2, i_3]$ and (3.12) follows.

Since $\| U - w \|_{W^{1,2}(T_p)^m} < r$ for each $p \in [i_2, i_3]$, by the construction of i_2 and i_3, U satisfies the middle set of constraints of (3.1) with strict inequality. Moreover the same is true of the remaining constraints. To see this, observe first that if $m_3 - m_2 - 4l > 2l$, exactly as in (3.7), there is an $X_{i_5} \subset [m_2 + l, m_3 - l]$ such that $\| U - w \|_{W^{1,2}(T_p)^m} < \varepsilon$ for $p \in [i_5 - 2, i_5 + 2] \cap \mathbb{Z}$. Next, arguing as in (2.33)–(2.38), define Ψ, g, h by

$$\Psi = \begin{cases} U, & x_1 \leq i_5 - 1, \\ (x_1 - (i_5 - 1))w + (i_5 - x_1)U, & i_5 - 1 \leq x_1 \leq i_5, \\ w, & i_5 \leq x_1 \leq i_5 + 1, \\ (x_1 - (i_5 + 1))U + (i_5 + 2 - x_1)w, & i_5 + 1 \leq x_1 \leq i_5 + 2, \\ U, & x_1 \geq i_5 + 2, \end{cases} \tag{3.17}$$

and

$$g = \begin{cases} \Psi, & x_1 \leq i_5 + 1, \\ w, & x_1 \geq i_5 + 1, \end{cases} \tag{3.18}$$

$$h = \begin{cases} w, & x_1 \leq i_5 + 1, \\ \Psi, & x_1 \geq i_5 + 1. \end{cases} \tag{3.19}$$

Note that $g \in \Gamma_1(v, w)$, $h \in \Gamma_1(w, v)$, and

$$J_1(\Psi) = J_1(g) + J_1(h). \tag{3.20}$$

Suppose that there is not strict inequality for the constraint for $U|_{T_p}$ for some T_p with $p < i_5$. Then $g \in \Lambda^-(v, w) \cup \Lambda^+(v, w)$. Therefore by (3.20) and the choice of Ψ,

$$b_2 = J_1(U) \geq J_1(\Psi) - J_{1, i_5 - 1}(\Psi) - J_{1, i_5 + 1}(\Psi)$$
$$\geq d(v, w) + \Sigma_{i \geq i_5 + 2} J_{1,i}(U) - 2\kappa(\varepsilon) \tag{3.21}$$

with $\kappa(\varepsilon)$ as earlier.

To continue, a better upper bound than that of Lemma 3.5 is needed for b_2. Choose $u^- \in \mathcal{M}_1(v, w)$. Then replacing u^- by $u^-(\cdot + pe_1)$ for appropriate $p \in \mathbb{Z}$ if need be, we can assume that u^- satisfies the first constraint in (3.1). If $m_2 - m_1 - 2l$ is large enough (depending on r), it also satisfies the second constraint. It can also be assumed that for l sufficiently large, $\| u^- - w \|_{W^{1,2}(X_{i_5})^n} < \varepsilon$. Therefore choosing Φ so that

$$\Phi = \begin{cases} u^-, & x_1 \leq i_5 - 1, \\ w, & i_5 \leq x_1 \leq i_5 + 1, \\ U, & x_1 \geq i_5 + 2, \end{cases} \tag{3.22}$$

and interpolating as e.g. in (2.33) in the remaining regions, it follows that $\Phi \in \mathcal{A}_2$. Now as in (3.18)–(3.21),

$$b_2 \leq J_1(\Phi) \leq c_1(v) + 2\kappa(\varepsilon) + \Sigma_{i \geq i_5 + 2} J_{1,i}(U). \tag{3.23}$$

Combining (3.21) and (3.23) yields

$$c_1(v) + 2\kappa(\varepsilon) \geq J_1(U) \geq d_1(v, w) - 2\kappa(\varepsilon)$$

or

$$c_1(v) + 4\kappa(\varepsilon) \geq d_1(v, w) \tag{3.24}$$

Since $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$, choosing ε so small that

$$\kappa(\varepsilon) < \frac{1}{4}\big(d_1(v, w) - c_1(v)\big), \tag{3.25}$$

(3.24)–(3.25), Proposition 2.47 and Remark 2.48 yield a contradiction. Thus U is a solution of (PDE) in $(-\infty, i_5 + 2] \times \mathbb{T}^{n-1}$.

Similarly if there is not strict inequality for the constraint for $U|_{T_q}$ for some T_q with $q > i_5$, choose $u^+ \in \mathcal{M}_1(w, v)$. Then as for u^-, for an appropriate $p \in \mathbb{Z}$, $u^+(\cdot + pe_1)$ will satisfy the 3rd set of constraints in (3.1) and for $m_4 - m_3 - 2l$ sufficiently large, the 4th set also. Hence slightly modifying the argument of (3.22)–(3.25) shows that U is also a solution of (PDE) in $[i_5 - 2, \infty) \times \mathbb{T}^{n-1}$ and Theorem 3.3 is proved. $\square$

Essentially the same proof as for Theorem 3.3 can be used to show that there are k-transition solutions of (PDE) for any $k \in \mathbb{N}$ with $k > 2$. They are homoclinics if k is even and heteroclinics if k is odd. However, in carrying out the k-transition case, a priori the parameter ε may depend on k and go to 0 as $k \to \infty$, thus eliminating the possibility of finding ∞ – transition solutions in this fashion. We will show that this does not happen if m_0 is chosen more carefully. Towards that end, note that m_0 depends on l which in turn depends on ϵ. Thus m_0 will have to be adjusted slightly to satisfy some additional smallness conditions required of ϵ in the argument below.

To formulate the k-transition case, we proceed recursively. Suppose that $k > 2$ and $\mathbf{m} \in \mathbb{Z}^{2k}$. For $j \in \mathbb{Z}$, set $K_j = N_r(K_v^-)$ if $j \in 4\mathbb{Z} + 1$; $= N_r(K_w^-)$ if $j \in 4\mathbb{Z} + 2$; $= N_r(K_w^+)$ if $j \in 4\mathbb{Z} + 3$; $= N_r(K_v^+)$ if $j \in 4\mathbb{Z}$. Define $\mathcal{A}_k = \mathcal{A}_k(\mathbf{m}, l; v, w)$ to be the class of functions satisfying the constraints of $\mathcal{A}_{k-1}$ except for the last one which is replaced by

$$u(\cdot + je_1)|_{T_0} \in K_{2k-2} \quad \text{for } m_{2k-2} - l \leq j \leq m_{2k-2} + l$$

and the three additional constraints:

$$u(\cdot + je_1)|_{T_0} \in \begin{cases} N_r(\varphi), & m_{2k-2} + 2l \leq j < m_{2k-1} - 2l, \\ K_{2k-1}, & m_{2k-1} - l \leq j < m_{2k-1} + l, \\ K_{2k}, & m_{2k} - l \leq j. \end{cases} \tag{3.26}$$

where $\varphi = v$ if K_{2k-1} corresponds to v and $\varphi = w$ if K_{2k-1} corresponds to w. Define

$$b_k = b_k(\mathbf{m}, l; v, w) = \inf_{u \in \mathcal{A}_k} J_1(u)$$

and set

$$\mathcal{M}(b_k) = \big\{ u \in \mathcal{A}_k \,\big|\, J_1(u) = b_k \big\}.$$

Then we will show:

Theorem 3.27. *Assume* (F_1)–(F_2) *and* (N_0) *are satisfied. Suppose* v *and* $w \in \mathcal{M}_0$ *with* $c_1(v) = C_1(v, w)$ *and ii) of Proposition 2.43 holds for* $\mathcal{S}(v, w)$ *and* $\mathcal{S}(w, v)$. *If* $k \geq 3$ *and the inequalities*

$$l, m_{i+1} - m_i - 6l \geq m_0, \quad i = 1, \cdots, k, \tag{3.28}$$

are satisfied, $\mathcal{M}(b_k) \neq \emptyset$. *Moreover any* $U \in \mathcal{M}(b_k)$ *is a classical solution of (PDE).*

The next result provides the first step towards proving the theorem. Set $M = 1 + c_1(v)$, $\mu_0 = -\infty$, and $\mu_k = \infty$.

Proposition 3.29. *Let* $\varepsilon \in (0, r)$ *and let* l, $\mathbf{m}$ *satisfy* (3.28) *and* $k \geq 3$. *There exists an* ε_0 *such that whenever* $\varepsilon \leq \varepsilon_0$ *and* $U \in \mathcal{A}_k$ *with* $J_1(U) = b_k$, *then for* $1 \leq i \leq k - 1$, *there are integers* $\mu_i \in (m_{2i} + 2l + 2, m_{2i+1} - 2l - 2)$ *and corresponding sets* $X_{\mu_i} = \bigcup_{j=-2}^{j=2} T_{\mu_i + j}$ *such that*

$$\|U - \mathcal{M}_0\|_{W^{1,2}(X_{\mu_i})^m} < \varepsilon \tag{3.30}$$

and

$$\Sigma_{\mu_{i-1}}^{\mu_i} J_{1,i}(U) \le M + 1. \tag{3.31}$$

Proof. Let $V \in \mathcal{M}_1(v, w)$ and $W \in \mathcal{M}_1(w, v)$. Then e.g. in the spirit of (3.10), for l sufficiently large, by taking appropriate truncations V_l of V and W_l of W and gluing an appropriate phase shift of V_l to w to an appropriate phase shift of W_l to v to an appropriate phase shift of V_l and so on, we construct a member, u, of $\mathcal{A}_k$ such that $b_k \le J_1(u) \le k(c_1(v) + \frac{1}{2}) < kM$. Therefore $J_1(U) \le kM$. Suppose that for every $X_p \subset [m_{2i} + 2l, m_{2i+1} - 2l] \times T^{n-1}$, $1 \le i \le k - 1$, there is some $T_j \subset X_p$ such that $\|U - \mathcal{M}_0\|_{W^{1,2}(T_j)^m} \ge \frac{\varepsilon}{4}$. Then as in (3.7),

$$k(M + 1) \ge kM \ge J_1(U) \ge 2l\beta\left(\frac{\varepsilon}{4}\right)(k - 1)/5. \tag{3.32}$$

Thus for

$$l = l(\varepsilon) \ge \frac{5M}{\beta(\frac{\varepsilon}{4})}, \tag{3.33}$$

we have a contradiction to (3.32). Hence there is some q in $1 \le i \le k - 1$ and $\mu_q \in (m_{2q} + 2l + 2, m_{2q+1} - 2l - 2)$ such that $\|U - \mathcal{M}_0\|_{W^{1,2}(X_{q_\mu})^m} < \varepsilon$. But in this region, $\|U - \mathcal{M}_0\|_{W^{1,2}(X_i)^m} = \|U - \psi\|_{W^{1,2}(X_i)^m}$ where $\psi = v$ or w. The arguments being the same for either case suppose $\psi = w$. With $u \in \mathcal{A}_k$ as above, define

$$\Psi = \begin{cases} u, & x_1 \le \mu_q, \\ U, & x_1 \ge \mu_q + 1, \end{cases} \tag{3.34}$$

interpolating as usual. Then $\Psi \in \mathcal{A}_k$, and $0 \le J(\Psi) - J(U)$ so by the choice of u,

$$\sum_{-\infty}^{\mu_q} J_{1,p}(U) \le \sum_{-\infty}^{\mu_q} J_{1,p}(u) + \kappa(\varepsilon) \le qM + \kappa(\varepsilon) \le q(M + 1) \tag{3.35}$$

provided that ε is small enough. Similar reasoning shows

$$\sum_{\mu_q}^{\infty} J_{1,p}(U) \le (k - q)(M + 1). \tag{3.36}$$

If $q = 1$, (3.35) gives the proposition for $i = 1$. Thus suppose $q > 1$. Using (3.35) and the argument of (3.32)–(3.33) shows that there is a p in $1 \le i \le q - 1$ and $\mu_p \in (m_{2p} + 2l + 2, m_{2p+1} - 2l - 2)$ such that

$$\|U - \mathcal{M}_0\|_{W^{1,2}(X_{\mu_p})^m} = \|U - \varphi\|_{W^{1,2}(X_{\mu_p})^m} < \varepsilon$$

where φ is the choice of v or w appropriate for X_{μ_p}. Thus continuing this process for a finite number of steps and doing the same for the contribution of (3.36) yields the proposition. $\square$

Proof of Theorem 3.27. Let $\mathbf{m}$ satisfy (3.28). As in the proof of Theorem 3.3,

$$b_k(\mathbf{m}, l; v, w) = \inf_{u \in \mathcal{A}_k} J_1(u)$$

is finite and any minimizing sequence for b_k is bounded in E_1. Therefore as earlier, a subsequence converges in E_1 to $U_k \in \mathcal{A}_k$ where U_k is a solution of (PDE) except possibly for those sets, T_p, in a constraint region for which equality occurs in a constraint. It remains to show that such an equality cannot occur. Three types of equalities are possible: (a) the "interior" case where $U_k(\cdot + pe_1)|_{T_0} \in \partial K_i$ for some $p \in [m_i - l, m_i + l) \cap \mathbb{Z}$ where $i \ne 1, 2, 2k - 1, 2k$, (b) the "boundary" case where $U_k(\cdot + pe_1)|_{T_0} \in \partial K_i$ for some p in the constraint region associated with K_i where $i = 1, 2, 2k - 1, 2k$, and (c) $\|U_k - \varphi\|_{W^{1,2}(T_j)} = r$ where $\varphi \in \{v, w\}$. Excluding cases (a) and (b) involves modifying U_k to get contradictory upper and lower bounds for b_k. Once cases (a) and (b) are eliminated, it is straightforward to

exclude case (c). We will show case (a), which requires more cutting and pasting is not possible. Case (b) requires a slightly simpler argument similar to the proof of Theorem 3.3 and will be omitted.

To exclude case (a), note that by its definition, there are 4 possible choices for K_i. Suppose $K_i = N_r(K_\varphi^s)$ where $\varphi \in \{v, w\}$ and $s \in \{+, -\}$. Define $\psi = \{v, w\} \setminus \{\varphi\}$. Note that going from K_i to one, but not both of, (d) K_{i-1} or (e) K_{i+1} involves a transition from a neighborhood of φ to one of ψ. Essentially the same reasoning is used for each of these cases so we carry out the details for (e) taking $\varphi = v$ and $\psi = w$. Thus suppose

$$U_k(\cdot + je_1) \in \begin{cases} K_{i-1} = N_r(K_v^+), & m_{i-1} - l \leq j < m_{i-1} + l \\ K_i = N_r(K_v^-), & m_i - l \leq j < m_i + l \\ K_{i+1} = N_r(K_w^-), & m_{i+1} - l \leq j < m_{i+1} + l. \end{cases}$$

By Proposition 3.29, there is an $i_v \in (m_{i-1} + 2l + 2, m_i - 2l - 2) \times T^{n-1}$ and an $i_w \in (m_{i+1} + 2l + 2, m_{i+2} - 2l - 2) \times T^{n-1}$ such that

$$\|U_k - v\|_{W^{1,2}(X_{i_v})^m}, \|U_k - w\|_{W^{1,2}(X_{i_w})^m} < \varepsilon. \tag{3.37}$$

Consider the function

$$f = \begin{cases} v, & x_1 \leq i_v, \\ U_k, & i_v + 1 \leq x_1 \leq i_w - 1, \\ w, & i_w \leq x_1, \end{cases} \tag{3.38}$$

interpolated as in (2.33) in the remaining regions. Observe that $f(\cdot + se_1) \in \Lambda^-(v, w)$ for some $s \in \mathbb{Z}$ so $d^-(v, w) \leq J_1(f)$. Therefore

$$d^-(v, w) \leq \sum_{i=i_v}^{i_w-2} J_{1,i}(U_k) + J_{1,i_v}(f) + J_{1,i_w-1}(f) \leq \sum_{i=i_v}^{i_w-1} J_{1,i}(U_k) + 2\kappa(\varepsilon). \tag{3.39}$$

By an earlier argument, for l sufficiently large, a $u \in \mathcal{M}_1(v, w)$ can be chosen such that $\|u - v\|_{W^{1,2}(T_p)^m} \leq \epsilon$ for any $p \leq i_v + 1$ and $\|u - w\|_{W^{1,2}(T_p)^m} \leq \epsilon$ for any $p \geq i_w - 1$. Define

$$\Phi = \begin{cases} U_k, & x_1 \leq i_v - 2, \\ v, & i_v - 1 \leq x_1 \leq i_v, \\ u, & i_v + 1 \leq x_1 \leq i_w - 1, \\ w, & i_w \leq x_1 \leq i_w + 1, \\ U_k, & i_w + 2 \leq x_1 \end{cases} \tag{3.40}$$

(with the usual interpolations). Since $\Phi \in \mathcal{A}_k$, by (3.39) we obtain

$$0 \leq J_1(\Phi) - J_1(U_k) = \sum_{i=i_v-2}^{i_w+1} \left(J_{1,i}(\Phi) - J_{1,i}(U_k) \right)$$

$$\leq \sum_{i=i_v}^{i_w-1} J_{1,i}(u) - \sum_{i=i_v}^{i_w-1} J_1(U_k) + 4\kappa(\epsilon) \leq c_1(v) - d^-(v, w) + 6\kappa(\epsilon). \tag{3.41}$$

Since ε is small, it can be assumed that $6\kappa(\epsilon) < d^-(v, w) - c_1(v)$. Thus (3.41), case (a) and similarly (b) are not possible.

To exclude (c), a familiar argument will be employed. Note that by (3.31), for $1 \leq i \leq k - 1$,

$$\Sigma_{\mu_{i-1}}^{\mu_i} J_{1,i}(U) \leq M + 1$$

so as in (3.32)–(3.33) again, there is an $X_{i_-} \subset [m_{2i} - l - 2, m_{2i} + l + 2] \times T^{n-1}$ and $X_{i_+} \subset [m_{2i+1} - l - 2, m_{2i+1} + l + 2] \times T^{n-1}$ such that $\|U_k - \varphi\|_{W^{1,2}(X_{i_-})}, \|U_k - \varphi\|_{W^{1,2}(X_{i_+})} < \varepsilon$ where φ is v or w as dictated by $i_\pm$. Define

$$\Xi = \begin{cases} U_k & x_1 \leq i_- - 1, \\ \varphi & i_- \leq x_1 \leq i_+, \\ U_k & i_+ + 1 \leq x_1 \end{cases} \tag{3.42}$$

Then $\varXi \in \mathcal{A}_k$ so if $\|U_k - \varphi\|_{W^{1,2}(T_j)} = r$ for any $j \in [m_{2k-2} + 2l, m_{2k-1} - 2l)$,

$$\beta(r) - 2\kappa(\varepsilon) \le J_1(U_k) - J_1(\varXi) \le 0,$$

which is not possible by our choice of ε. Consequently we have inequality for the constraints of type (c) and Theorem 3.27 is proved. $\quad\square$

The choice of m_0 in Theorem 3.27 is independent of k. Therefore we further obtain the existence of infinite transition solutions of (PDE):

Corollary 3.43. *Let $S = \mathbb{N}, -\mathbb{N},$ or $\mathbb{Z}$. Under the hypotheses of Theorem 3.27, if $(m_i)_{i \in S}$ satisfies (3.28) for $i \in S$, then there is a solution, U of (PDE) in $\mathcal{A}_S$ where*

$$\mathcal{A}_S = \left\{ u \in E_1 \mid u(\cdot + je_1)|_{T_0} \in K_i, \ m_i - l \le j < m_i + l, \ i \in S \right\}$$

(with the understanding that if $S = \mathbb{N}$ and $i = 1, m_1 - l$ is replaced by $-\infty$ and if $S = -\mathbb{N}$ and $i = -1, m_{-1} + l$ is replaced by ∞).

Proof. Make a finite truncation of S at say k and/or $-k$, obtaining a solution of (PDE), $U_k \in \mathcal{A}_k$, with $\mathcal{A}_k$ corresponding to the truncation. Then get bounds for U_k as in Proposition 2.22 and find U by passing to a limit along a subsequence. $\quad\square$

Remark 3.44. Since U_k has a minimality property in $\mathcal{A}_k$ and U_k converges to U in e.g. C^1_{loc}, U has a minimality property in $\mathcal{A}_S$, namely for any $u \in \mathcal{A}_S$ such that $u = U$ outside of a compact region, $\mathcal{D}$,

$$\int_{\mathcal{D}} \left(L(U) - L(u) \right) dx \le 0.$$

4. More complex multi-transition solutions

In this section, Theorem 3.27 will be generalized and the existence of multi-transition solutions that shadow heteroclinic chains will be established. Recall from Theorem 2.10 and Remark 2.12, for any distinct pair, $v, w \in \mathcal{M}_0$, there is a minimal heteroclinic chain of solutions of (PDE) joining v and w. More precisely, there is a $q \in \mathbb{N}$, periodics $v_i \in \mathcal{M}_0, 0 \le i \le q$, and heteroclinics $U_i \in \mathcal{M}_1(v_{i-1}, v_i), 1 \le i \le q$, such that the formal chain, $\mathcal{U} = \{U_1, \cdots, U_q\}$, joins v and w. Moreover $\mathcal{U}$ is a minimal chain, i.e.

$$C_1(v, w) = \Sigma_1^q C_i(v_{i-1}, v_i) = \Sigma_1^q J_1(U_i) \tag{4.1}$$

and for any adjacent pair $v_{i-1}, v_i, 1 \le i \le q$, if $z \in \mathcal{M}_0 \setminus \{v_{i-1}, v_i\}$, then

$$C_1(v_{i-1}, v_i) < C_1(v_{i-1}, z) + C_1(z, v_i). \tag{4.2}$$

Now we ask whether there exists an actual heteroclinic solution of (PDE) joining v and w and shadowing the minimal chain. This will be shown to be the case provided the non-degeneracy condition, ii) of Proposition 2.43 holds for $\mathcal{M}_1(v_{i-1}, v_i)$ for $1 \le i \le q$. In fact there exist infinitely many distinct such solutions. Then using this result, Theorem 4.7 below, some more general results that contain both Theorem 3.27 and Theorem 4.7 will be described.

To set the stage for Theorem 4.7, for $1 \le i \le q$, let $\mathcal{S}(v_{i-1}, v_i)$ be as in Proposition 2.32 and $\mathcal{C}_{v_{i-1}}(v_{i-1}, v_i)$ and $\mathcal{C}_{v_i}(v_{i-1}, v_i)$ be as in Proposition 2.43. Assume that:

(∗) For any $i = 1, \ldots, q, \mathcal{C}_{v_{i-1}}(v_{i-1}, v_i) = \{v_{i-1}\}$ and $\mathcal{C}_{v_i}(v_{i-1}, v_i) = \{v_i\}$

i.e. alternative ii) of Proposition 2.43 holds for any $i = 1, \ldots, q$. By (∗), there exist nonempty disjoint compact sets,

$$K_{v_{i-1}}(v_{i-1}, v_i), K_{v_i}(v_{i-1}, v_i) \subset \bar{\mathcal{S}}(v_{i-1}, v_i),$$

such that

$$v_{i-1} \in K_{v_{i-1}}(v_{i-1}, v_i), \qquad v_i \in K_{v_i}(v_{i-1}, v_i),$$
$$\bar{\mathcal{S}}(v_{i-1}, v_i) = K_{v_{i-1}}(v_{i-1}, v_i) \cup K_{v_i}(v_{i-1}, v_i).$$

By v) of Proposition 2.43, there is an $r_2 = r_2(v_{i-1}, v_i) > 0$ such that

$$\mathrm{dist}\big(K_{v_{i-1}}(v_{i-1}, v_i), K_{v_i}(v_{i-1}, v_i)\big) \geq 5r_2.$$

Recalling the parameters r_0, r_1 as defined in Section 2, set

$$\bar{r}_2 = \min_{1 \leq i \leq q} r_2(v_{i-1}, v_i) \quad \text{and} \quad \bar{r}_1 = \min_{1 \leq i \leq q} r_1(v_{i-1}, v_i)$$

where $r_1(v_{i-1}, v_i)$ is given by Proposition 2.15. Set $\bar{r} = \min\{r_0, \bar{r}_1, \bar{r}_2\}$. For each pair, $v_{i-1}, v_i, i \leq i \leq q$, define

$$\Lambda_i = \Lambda_i(v_{i-1}, v_i) = \big\{u \in \Gamma_1(v_{i-1}, v_i) \, \big| \, \|u - K_{v_{i-1}}(v_{i-1}, v_i)\|_{W^{1,2}(T_0)^m} = \bar{r} \text{ or }$$
$$\|u - K_{v_i}(v_{i-1}, v_i)\|_{W^{1,2}(T_0)^m} = \bar{r}\big\}$$

and

$$d_i = \inf_{u \in \Lambda_i} J_1(u). \tag{4.3}$$

Then the proof of Proposition 2.47 shows whenever (∗) holds,

$$d_i > C_1(v_{i-1}, v_i) \quad \text{for any } i \in \{1, \dots, q\}. \tag{4.4}$$

With the aid of these preliminaries, we can introduce a class of candidates for solutions of (PDE) that shadow the formal chain, $\mathcal{U}$. Let $l \in \mathbb{N}$ and $\mathbf{m} = (m_1, \dots, m_{2q}) \in \mathbb{Z}^{2q}$ with $m_j - m_{j-1} > 2l$ for $j = 2, \dots, 2q$. Define

$$\mathcal{A} = \mathcal{A}(q, \mathbf{m}, l) = \big\{u \in E_1 \, \big| \, u \text{ satisfies } (4.5)\big\}$$

where for $p \in \mathbb{Z}$,

$$u(\cdot + pe_1)|_{T_0} \in \begin{cases} N_{\bar{r}}(K_{v_0}(v_0, v_1)) & p \leq m_1 + l, \\ N_{\bar{r}}(K_{v_i}(v_{i-1}, v_i)) & m_{2i} - l \leq p < m_{2i} + l, \\ N_{\bar{r}}(v_i) & m_{2i} + 2l \leq p < m_{2i+1} - 2l, \\ N_{\bar{r}}(K_{v_i}(v_i, v_{i+1})) & m_{2i+1} - l \leq p < m_{2i+1} + l, \\ N_{\bar{r}}(K_{v_q}(v_{q-1}, v_q)) & m_{2q} - l \leq p \end{cases} \quad 1 \leq i \leq q - 1. \tag{4.5}$$

Finally as in (3.2), define

$$b = b(q, \mathbf{m}, l) = \inf_{u \in \mathcal{A}(q, \mathbf{m}, l)} J_1(u) \tag{4.6}$$

and set

$$\mathcal{M}(b) = \big\{u \in \mathcal{A}(q, \mathbf{m}, l) \, \big| \, J_1(u) = b\big\}.$$

Then we have

Theorem 4.7. *Suppose* (F_1)–(F_2) *and* (N_0) *hold and there is a* $q \geq 2$, $\{v_0, v_1, \dots, v_q\} \subset \mathcal{M}_0$ *for which (4.1)–(4.2) and (∗) are satisfied. Let* $l \in \mathbb{N}$ *and* $\mathbf{m} = (m_1, \dots, m_{2q}) \in \mathbb{Z}^{2q}$. *Then there exists an* $m_0 = m_0(v_0, v_1, \dots, v_q) \in \mathbb{N}$ *such that* $l \geq m_0$ *and* $m_j - m_{j-1} - 6l \geq m_0$ *for* $j = 2, \dots, 2q$ *imply* $\mathcal{M}(b) \neq \emptyset$. *Moreover any* $U \in \mathcal{M}(b)$ *is a classical solution of (PDE).*

Proof. Let (u_k) be a minimizing sequence for (4.6). The argument of Theorem 3.3 and Theorem 3.27 again shows that along a subsequence, $u_k \to U$ in E_1 with $U \in \mathcal{A}$ and $J_1(U) = b$. Hence again, to show that U is a solution of (PDE), it suffices to prove that if m_0 is large enough, $U(x + pe_1)$ is in the interior of the corresponding set $N_{\bar{r}}(K_{v_i}(v_{i-1}, v_i))$ for the indices p and i specified in (4.5). If this is false, by (4.5), there exists $j \in \{1, \dots, 2q\}$ and

$$p_j \in \begin{cases} (-\infty, m_1 + l] \cap \mathbb{Z} & \text{if } j = 1, \\ [m_j - l, m_j + l) \cap \mathbb{Z} & \text{if } 1 < j < 2q, \\ [m_{2q} - l, +\infty) \cap \mathbb{Z} & \text{if } j = 2q, \end{cases}$$

for which

$$\bar{r} = \begin{cases} \text{dist}_{W^{1,2}(T_0)^m}(U(\cdot + p_j e_1)|_{T_0}, K_{v_{[j/2]}}(v_{[j/2]}, v_{[j/2]+1})), & \text{if } j \text{ is odd}, \\ \text{dist}_{W^{1,2}(T_0)^m}(U(\cdot + p_j e_1)|_{T_0}, K_{v_{[j/2]}}(v_{[j/2]-1}, v_{[j/2]})), & \text{if } j \text{ is even}. \end{cases}$$

Suppose that j is even so that

$$\bar{r} = \text{dist}_{W^{1,2}(T_0)^m}\big(U(\cdot + p_j e_1)|_{T_0}, K_{v_{j/2}}(v_{(j/2)-1}, v_{j/2})\big). \tag{4.8}$$

As in the proof of Theorem 3.27, we distinguish between (a) the "interior" case where $j \neq 2, 2q$ and (b) the "boundary" case where $j = 2$ or $j = 2q$. Consider the case (a). By the choice of constraints, the argument used to show (3.30) applies here providing an $i_- \in (m_{j-2} + l + 2, m_{j-1} - l - 2)$ and $i_+ \in (m_j + l + 2, m_{j+1} - l - 2)$ and corresponding regions X_{i_-} and X_{i_+} such that

$$\|U - v_{(j/2)-1}\|_{W^{1,2}(X_{i_-})^m}, \|U - v_{j/2}\|_{W^{1,2}(X_{i_+})^m} < \varepsilon. \tag{4.9}$$

Define

$$f = \begin{cases} v_{(j/2)-1}, & x_1 \leq i_-, \\ U, & i_- + 1 \leq x_1 \leq i_+ - 1, \\ v_{j/2}, & i_+ \leq x_1, \end{cases} \tag{4.10}$$

with the usual interpolations in the other regions. By (4.8), $f \in \Lambda_{j/2}$ so by (4.3) and (4.9)–(4.10),

$$d_{j/2} \leq \sum_{i=i_-}^{i_+ - 1} J_{1,i}(U) + J_{1,i_1}(f) + J_{1,i_2-1}(f) \leq \sum_{i=i_-}^{i_+ - 1} J_{1,i}(U) + 2\kappa(\epsilon). \tag{4.11}$$

As e.g. in the paragraph containing (3.40), if $l = l(\varepsilon, v_{i-1}, v_i)$ is sufficiently large, there exists $u \in \mathcal{M}_1(v_{\frac{j}{2}-1}, v_{j/2})$ such that $\|u - v_{\frac{j}{2}-1}\|_{W^{1,2}(T_p)^m} \leq \epsilon$ for any $p \leq i_- + 1$ and $\|u - v_{j/2}\|_{W^{1,2}(T_p)^m} \leq \epsilon$ for any $p \geq i_+ - 1$. Define

$$\Phi = \begin{cases} U, & x_1 \leq i_- - 2, \\ v_{\frac{j}{2}-1}, & i_- - 1 \leq x_1 \leq i_-, \\ u, & i_- + 1 \leq x_1 \leq i_+ - 1, \\ v_{j/2}, & i_+ \leq x_1 \leq i_+ + 1, \\ U, & x_1 \geq i_+ + 2 \end{cases} \tag{4.12}$$

making the usual interpolations. Observe that $\Phi \in \mathcal{A}(q, \mathbf{m}, l)$ and $\Phi|_{[i_- - 1, i_+ + 1] \times T^{n-1}}$ extended in the obvious way belongs to $\Gamma_1(v_{\frac{j}{2}-1}, v_{j/2})$. Consequently with the aid of (4.11), we obtain

$$0 \leq J_1(\Phi) - J_1(U) = \sum_{i=i_- - 2}^{i_+ + 1} \big(J_{1,i}(\Phi) - J_{1,i}(U)\big)$$

$$\leq \sum_{i=i_- - 1}^{i_+} J_{1,i}(\Phi) + 2\kappa(\epsilon) - \sum_{i=i_-}^{i_+ - 1} J_{1,i}(U) \leq C_1(v_{\frac{j}{2}-1}, v_{j/2}) - d_{j/2} + 4\kappa(\epsilon),$$

a contradiction to (4.4) if $4\kappa(\epsilon) < \min_{i=1,\dots,q}(C_1(v_{i-1}, v_i) - d_i)$. An analogous argument leads to a contradiction when $j = 2q$ or when j is odd and the theorem follows. $\square$

Next it will be shown how to extend Theorem 4.7 to more general settings via a few remarks. Since the ideas are quite similar to what has already been done, we will be sketchy.

Remark 4.13. Let $\mathcal{V} = (V_1, \cdots, V_q)$ and $\mathcal{W} = (W_1, \cdots, W_s)$ be respectively formal minimal heteroclinic q- and s-chains with associated periodic solutions $v_0, \cdots, v_q$ and $w_0, \cdots, w_s$ in $\mathcal{M}_0$. Suppose that $v_q = w_0$. Then we can
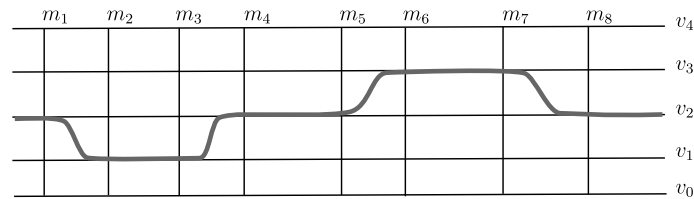
Fig. 1. A 4-transition solution obtained from a chain with periodics v_0, v_1, v_2, v_3, v_4.

formally glue $\mathcal{V}$ and $\mathcal{W}$ obtaining the heteroclinic $(q+s)$-chain, $\mathcal{U} = (V_1, \cdots, V_q, W_1, \cdots, W_s)$. It may not be minimal, but it does satisfy (4.2). We will show that there are infinitely many solutions of (PDE) that shadow $\mathcal{U}$ in the sense of Theorem 4.7, i.e. lie in the corresponding set $\mathcal{A}$. Indeed set $m_0(\mathcal{V}) \equiv m_0(v_0, \cdots, v_q)$ and $m_0(\mathcal{W}) \equiv m_0(w_0, \cdots, w_s)$ as given by Theorem 4.7. Choose $l \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^{q+s}$ such that $m_{i+1} - m_i - 6l > \max(m_0(\mathcal{V}), m_0(\mathcal{W}))$. Denoting the $\bar{r}$ (as in (4.5)) associated with $\mathcal{V}$ (resp. $\mathcal{W}$) by $\bar{r}(\mathcal{V})$ (resp. $\bar{r}(\mathcal{W})$), set $\bar{r}(\mathcal{U}) = \min(\bar{r}(\mathcal{V}), \bar{r}(\mathcal{W}))$. Now to find a solution of (PDE) associated with the chain, $\mathcal{U}$ and the current choice of l and $\mathbf{m}$, simply take $\mathcal{A}(q+s, \mathbf{m}, l)$ as the class of admissible functions. Moreover an examination of the proof of Theorem 4.7 shows that it carries over unchanged to the current setting.

Remark 4.14. In a similar fashion, any finite number of formal heteroclinic chains with matching endpoints can be glued to find shadowing heteroclinics or homoclinics, although m_0 and $\bar{r}$ will depend on the number of chains. Thus in general it will not be possible to do the same with an infinite number of such chains. However there are some important special cases where this can be done. The simplest is the setting of Section 3 where we go back and forth between neighborhoods of v and w, etc. Thus for example, Theorem 3.3 can be obtained from Theorem 4.7 and Remark 4.13 by gluing the 1-chain, $U(x_1, x_2, \cdots)$, to the 1-chain, $U(-x_1, x_2, \cdots)$ and similarly for Theorem 3.27. In the same fashion, one can go back and forth along a given heteroclinic q-chain or more generally go back and forth along some of the links of the chain. Fig. 1 provides an example.

For such situations, gluing infinite chains can be achieved by a limit process as in Corollary 3.43. All of these solutions are bounded in L^∞.

Remark 4.15. In conclusion, we mention one more generalization allowing for unbounded solutions. Remark 4.14 shows that there are infinite chain solutions whenever the chains are constructed from a finite number of basic heteroclinics. Observing that $\mathcal{M}_0$ is finite modulo $\mathbb{Z}^m$ and $\mathcal{M}_0$ is invariant under $\mathbb{Z}^m$ translations, one can find a finite number of heteroclinic solutions which together with their $\mathbb{Z}^m$ translates form chains joining any pair of members of $\mathcal{M}_0$. Thus by gluing such objects, shadowing heteroclinics of arbitrarily large L^∞ norm can be constructed as well as unbounded limits of such solutions using the arguments of Corollary 3.43.

Conflict of interest statement

The authors have no conflict of interest.

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