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Solitary waves in Abelian Gauge Theories with strongly nonlinear potentials

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Abstract

We study the existence of radially symmetric solitary waves for a system of a nonlinear Klein–Gordon equation coupled with Maxwell's equation in presence of a positive mass. The nonlinear potential appearing in the system is assumed to be positive and with more than quadratical growth at infinity.

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1. Introduction, motivations and main result

In recent past years great attention was paid to some classes of systems of partial differential equations arising in Abelian Gauge Theories, i.e. theories consisting of field equations that provide a model for the interaction of matter with the electromagnetic field. In particular we recall the papers [1–3,5,8–10,12,14,17–19], where existence or nonexistence results are proved in the whole physical space.

Among all classes of solutions for these equations, we are interested in a special one, consisting of the so-called *solitary waves*, i.e. solutions of a field equation whose energy travels as a localized packet. Solutions of this type play a crucial rôle in these theories, in particular if such solutions exhibit some strong form of stability, and in this case they are called *solitons*. Solitons posses a particle-like behavior and they appear in a natural way in many situations of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics (for example see [11,13,20]). Therefore, the first step to prove the existence of solitons is to prove the existence of solitary waves and then, as a second step, one can try to prove that they are stable, as done in [17], where the author proves that solitary waves decaying at infinity are stable under some reasonable assumptions (see also [7] as a fundamental example of orbital stability in the case of only one nonlinear Schrödinger equation).

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In this paper we are interested in showing existence results for systems obtained by coupling a Klein–Gordon equation with Maxwell's ones. For the derivation of the general system and for a detailed description of the physical meaning of the unknowns we refer to the papers cited above and their references, passing to the formulation of the system itself, which is, therefore, a model for electrodynamics:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |\nabla S - q\mathbf{A}|^2 u - \left(\dfrac{\partial S}{\partial t} + q\phi\right)^2 u + W'(u) = 0, \\ \dfrac{\partial}{\partial t}\left[\left(\dfrac{\partial S}{\partial t} + q\phi\right)u^2\right] - \operatorname{div}\left[(\nabla S - q\mathbf{A})u^2\right] = 0, \\ \operatorname{div}\left(\dfrac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right) = q\left(\dfrac{\partial S}{\partial t} + q\phi\right)u^2, \\ \nabla \times (\nabla \times \mathbf{A}) + \dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{A}}{\partial t} + \nabla\phi\right) = q(\nabla S - q\mathbf{A})u^2, \end{cases} \tag{1.1}$$

where the equations are respectively the matter equation, the charge continuity equation, the Gauss equation and the Ampère equation.

We recall that the Klein–Gordon–Maxwell system is a special case of the Yang–Mills–Higgs equation when the Lie group G is the circle, and there is no potential energy term (see [23]). Moreover, also system (1.1) has been widely investigated. For example in [14] the authors prove that the system is well posed in $\mathbb{R}^3$ for initial data having total finite energy, while local well posedness in Sobolev spaces and local regularity in time of solutions are investigated in [18]. We also mention that the system was studied in higher dimensions, see [21], where global solutions are found in a suitable Sobolev space.

As we said above, we are interested in electrostatic standing waves, i.e. solutions having the special form

$$u = u(x), \qquad \mathbf{A} = 0, \qquad \phi = \phi(x), \qquad S = -\omega t, \quad \omega \in \mathbb{R}.$$

Once set

$$\Phi = \frac{\phi}{\omega},$$

the charge continuity and the Ampère equations are automatically satisfied, while the other two equations reduce to the following stationary system of Klein–Gordon–Maxwell type:

$$-\Delta u - \omega^2(1 - q\Phi)^2 u + W'(u) = 0 \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

$$-\Delta\Phi = q(1 - q\Phi)u^2 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

Here $q > 0$ represents the charge of the particle, ω is a real parameter, u represents the matter, while Φ is related to the electromagnetic field through Maxwell's equation (see [2]). Our attention is concentrated on W and in particular on the fact that it is assumed *nonnegative* and it possesses some good invariant (necessary to be considered in Abelian Gauge Theories), typically some conditions of the form

$$W\left(e^{i\theta}u\right) = W(u) \quad \text{and} \quad W'\left(e^{i\theta}u\right) = e^{i\theta}W'(u)$$

for any function u and any $\theta \in \mathbb{R}$.

In previous works (for example in [3,9,17]) the potential W was supposed to be

$$W(s) = \frac{m^2}{2}s^2 - \frac{|s|^\gamma}{\gamma} \tag{1.4}$$

for some $\gamma > 2$, and the authors showed existence results if $\gamma \in (4, 6)$ (in [3]) and then if $\gamma \in (2, 6)$ (in [9]), while nonexistence results for $\gamma \in (0, 2]$ or $\gamma \in [6, \infty)$ were proved in [8], where also more general potentials, behaving similarly to the one in (1.4), were considered.

However, the potential defined in (1.4) is not always positive, while for physical reasons a "good" potential should be. Indeed, the fact that W is assumed nonnegative implies that the energy density of a solution (u, Φ) of system (1.2)–(1.3) is nonnegative as well (for example, see [2]).

In this paper we are concerned with system (1.2)–(1.3), and we take [2] as starting point, where the authors assume that W satisfies the following assumptions:

W_1) $W \in C^2(\mathbb{R})$, $W \geqslant 0$ and $W(0) = W'(0) = 0$;
W_2) $W''(0) = m_0^2 > 0$;
W_3) $\exists C_1, C_2 > 0$ and $p \in (0,4)$ such that $|W''(s)| \leqslant C_1 + C_2|s|^p$ for every $s \in \mathbb{R}$;
W_4) $0 \leqslant sW'(s) \leqslant 2W(s)$ for every $s \in \mathbb{R}$;
W_5) $\exists m_1, c > 0$ with $m_1 < m_0^2/2$ such that $W(s) \leqslant m_1 s^2 + c$ for every $s \in \mathbb{R}$.

In particular, W_4) is equivalent to saying that the function $W(s)/s^2$ is decreasing if $s > 0$ and increasing if $s < 0$. Therefore, if $s > \varepsilon > 0$, we get

$$W(s) \leqslant \frac{W(\varepsilon)}{\varepsilon^2} s^2,$$

and passing to the limit as $\varepsilon \to 0$ we obtain $W(s) \leqslant \frac{m_0^2}{2} s^2$. Of course, such an inequality can be proved in the same way also if $s < 0$. Then from W_5) we get

$$W(s) \leqslant \max\left\{ m_1 s^2 + c, \frac{m_0^2}{2} s^2 \right\}.$$

We want to relax this quadratic growth condition and show that an existence result still holds true if we replace W_4) and W_5) with the new conditions

$$W_4)' \qquad \exists k \geqslant 2 \quad \text{such that} \quad 0 \leqslant sW'(s) \leqslant kW(s) \quad \forall s \in \mathbb{R}$$

and

$$W_5)' \qquad \begin{cases} \exists m_1, c > 0 \text{ and } \vartheta > 2 \quad \text{such that} \quad W(s) \leqslant c|s| + m_1|s|^\vartheta \quad \forall s \in \mathbb{R}, \\ m_1 < \min\left\{ \dfrac{m_0^2}{k(\vartheta - 1)}, 5^{2-\vartheta} \dfrac{(\vartheta - 2)^{\vartheta - 2}}{(\vartheta - 1)^{\vartheta - 1}} \left(1 + \dfrac{c}{2}\right)^{2-\vartheta} k^{1-\vartheta} m_0^{2(\vartheta - 1)} \right\}. \end{cases}$$

Of course, if W satisfies W_4), it also satisfies $W_4)'$, but now we can include higher order functions. In fact, operating in a similar way as above, from $W_4)'$ we get that there exist $a, b > 0$ such that

$$W(s) \leqslant a|s|^k + b \quad \text{for every } s \in \mathbb{R},$$

so that W may have superquadratic growth at infinity, which is perfectly in agreement with W_3). On the other hand, in $W_4)'$ we exclude the case $k < 2$, since in this latter case we would get that $W(s)/s^k$ is decreasing if $s > 0$, so that $W(s) \leqslant \frac{W(\varepsilon)}{\varepsilon^k} s^k$ for $s > \varepsilon > 0$, and passing to the limit as $\varepsilon \to 0^+$, by W_1) we would get $W \leqslant 0$, so that $W \equiv 0$, which we exclude by W_2).

On the other hand, $W_5)'$ generalizes W_5), in the sense that we can include higher order functions. However, we do not claim that the bound on m_1 is the best possible, since it is just a sufficient condition to exclude the case $W(s) = \frac{m_0^2}{2} s^2$, which does not verify the fundamental Lemma 3.1 below, in accordance with the nonexistence result proved in [8] for such a potential.

Remark 1.1. Note that when $k = 2$ and $\vartheta \to 2$, the condition on m_1 stated in $W_5)'$ simply reduces to $m_1 < m_0^2/2$, as already expressed in W_5).

Remark 1.2. As in [2], we have that $s = 0$ is an absolute minimum point for W, so that $W''(0) \geqslant 0$, in agreement with W_2).

Remark 1.3. If we consider the electrostatic case, i.e. $-\Delta u + W'(u) = 0$, calling "rest mass" of the particle u the quantity

$$\int_{\mathbb{R}^3} W(u)\,dx,$$

see [4], our assumptions on W imply that we are dealing with systems for particles having *positive mass*, which is, of course, the physical interesting case.

As usual, for physical reasons, we look for solutions having finite energy, i.e. $(u, \Phi) \in H^1 \times D^1$, where $H^1 = H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the scalar product

$$\langle u, v\rangle_{H^1} := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\,dx$$

and norm $\|u\| = (\int |\nabla u|^2 + \int u^2)^{1/2}$, and $D^1 = D^1(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2\,dx,$$

induced by the scalar product $\langle u, v\rangle_{D^1} := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v\,dx$.

Before stating our main result, let us note that if $u = 0$ in (1.2), then $\Phi = 0$ in (1.3), and if $u \neq 0$, then also $\Phi \neq 0$. Therefore, we can say that a solution (u, Φ) of system (1.2)–(1.3) is *nontrivial* if both u and Φ are different from the trivial function, and a sufficient condition for this occurrence is that $u \neq 0$.

Our main result is the following.

Theorem 1.1. *Assume that W satisfies W_1), W_2), W_3), $W_4)'$ and $W_5)'$. Then there exists $q^* > 0$ such that for any $q < q^*$ there exist $\omega^2 > 0$ and nontrivial functions $(u, \Phi) \in H^1(\mathbb{R}^3) \times D^1(\mathbb{R}^3)$ which solve* (1.2)–(1.3).

For completeness, in the last section of this paper we also give the proof of the following existence result for (1.2)–(1.3) when we assume W_4) and W_5) in place of the more general $W_4)'$ and $W_5)'$. A similar result was already presented in [2], but, eventually, another system was considered, though this fact is not explicitly stated.

Theorem 1.2. *Assume that W satisfies W_1)–W_5). Then there exists $q^* > 0$ such that for any $q < q^*$ there exist $\omega^2 > 0$ and nontrivial functions $(u, \Phi) \in H^1(\mathbb{R}^3) \times D^1(\mathbb{R}^3)$ which solve* (1.2)–(1.3).

2. Preliminary setting

First we recall the standard notation $L^p \equiv L^p(\mathbb{R}^3)$ $(1 \leqslant p < +\infty)$ for the usual Lebesgue space endowed with the norm

$$\|u\|_p^p := \int_{\mathbb{R}^3} |u|^p\,dx.$$

We also recall the continuous embeddings

$$H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3) \quad \forall p \in [2,6] \quad \text{and} \quad D^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3), \tag{2.5}$$

being 6 the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$.

Now, for any $\omega^2 > 0$ let us consider the functional $F : H^1 \times D^1 \to \mathbb{R}$ defined as

$$F_\omega(u, \Phi) = J(u) - \omega^2 \mathcal{A}(u, \Phi),$$

where

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} W(u)\,dx,$$

and

$$\mathcal{A}(u,\Phi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \Phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2(1-q\Phi)^2\,dx.$$

It is easily seen that under the growth assumptions on W we have the following:

Proposition 2.1. *The functional F_ω belongs to $C^1(H^1 \times D^1)$ and its critical points solve* (1.2)–(1.3).

Moreover, a by-now standard approach shows the following result, appearing in a similar way for the first time in [19], then in [9] and recently in [2].

Proposition 2.2. *For every $u \in H^1$, there exists a unique $\Phi = \Phi[u] \in D^1$ which solves* (1.3). *Furthermore*

(i) $0 \leqslant \Phi[u] \leqslant \frac{1}{q}$;
(ii) *if u is radially symmetric, then $\Phi[u]$ is radial, too.*

As a consequence of Proposition 2.2, we can define the map

$$\Phi : H^1(\mathbb{R}^3) \to D^1(\mathbb{R}^3)$$

which maps each $u \in H^1$ in the unique solution of (1.3). By the Implicit Function Theorem $\Phi \in C^1(H^1, D^1)$ and from the very definition of Φ we get

$$\frac{\partial F_\omega}{\partial \Phi}\big(u,\Phi[u]\big) = 0 \quad \forall u \in H^1, \quad \text{or equivalently} \quad \frac{\partial \mathcal{A}}{\partial \Phi}\big(u,\Phi[u]\big) = 0. \tag{2.6}$$

More precisely, we have the following result, which will be used later in the stronger context of radial functions.

Lemma 2.1. *The functional $\Lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as*

$$\Lambda(u) := \mathcal{A}\big(u,\Phi[u]\big) = \frac{1}{2}\int_{\mathbb{R}^3} \big|\nabla \Phi[u]\big|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2\big(1-q\Phi[u]\big)^2\,dx$$

is of class C^1 and

$$\Lambda'(u)(v) = \int_{\mathbb{R}^3} u\big(1-q\Phi[u]\big)^2 v\,dx \quad \textit{for any } u,\ v \in H^1\big(\mathbb{R}^3\big).$$

Proof. The regularity of all characters appearing in the definition of Λ implies that it is of class C^1, as claimed.
Moreover,

$$\Lambda(u) = \mathcal{A}\big(u,\Phi[u]\big),$$

so that

$$\Lambda'(u) = \frac{\partial \mathcal{A}}{\partial u}\big(u,\Phi[u]\big) + \frac{\partial \mathcal{A}}{\partial \Phi}\big(u,\Phi[u]\big)\Phi'[u] = \frac{\partial \mathcal{A}}{\partial u}\big(u,\Phi[u]\big)$$

by (2.6), and thus for any $u,\ v \in H^1(\mathbb{R}^3)$ there holds

$$\Lambda'(u)(v) = \int_{\mathbb{R}^3} u\big(1-q\Phi[u]\big)^2 v\,dx. \qquad \square$$

Now let us consider the functional

$$I : H^1(\mathbb{R}^3) \to \mathbb{R}, \qquad I(u) := F_\omega\big(u, \Phi[u]\big).$$

By definition of F_ω, we obtain

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} W(u)\,dx - \frac{\omega^2}{2}\int_{\mathbb{R}^3} \big|\nabla \Phi[u]\big|^2\,dx - \frac{\omega^2}{2}\int_{\mathbb{R}^3} u^2\big(1 - q\Phi[u]\big)^2\,dx.$$

By Proposition 2.1 and Lemma 2.1, $I \in C^1(H^1, \mathbb{R})$ and by (2.6) we get

$$I'(u) = \frac{\partial F_\omega}{\partial u}\big(u, \Phi[u]\big) - \omega^2 \frac{\partial \mathcal{A}}{\partial \Phi}\big(u, \Phi[u]\big)\Phi'[u] = \frac{\partial F_\omega}{\partial u}\big(u, \Phi[u]\big) \quad \forall u \in H^1\big(\mathbb{R}^3\big).$$

Standard calculations (for example, see [2]) give the following result.

Lemma 2.2. *The following statements are equivalent*:

(i) $(u, \Phi) \in H^1 \times D^1$ *is a critical point of* F,
(ii) u *is a critical point of* I *and* $\Phi = \Phi[u]$.

Then, in order to get solutions of (1.2)–(1.3), we could look for critical points of I.

It is readily seen that the functional I is strongly indefinite, in the sense that it is unbounded both from above and below. Moreover, a more delicate problem in getting the existence of critical points is the fact that I presents a lack of compactness due to its invariance under the translation group, given by the set of transformations having the form $u(x) = u(x + x_0)$ for any $x_0 \in \mathbb{R}^3$. In order to avoid the latter problem, it is standard to restrict ourselves to the set of radial functions, so that we consider

$$H^1_r = H^1_r\big(\mathbb{R}^3\big) := \big\{u \in H^1\big(\mathbb{R}^3\big):\ u(x) = u\big(|x|\big)\big\},$$

and the functional $I_{|H^1_r} : H^1_r \to \mathbb{R}$. In this way, if $u \in H^1_r$, then also $\Phi[u]$ is radial by (ii) of Proposition 2.2, and we write

$$\Phi[u] \in D^1_r = D^1_r\big(\mathbb{R}^3\big) := \big\{\phi \in D^1\big(\mathbb{R}^3\big):\ \phi(x) = \phi\big(|x|\big)\big\}.$$

We recall that $H^1_r(\mathbb{R}^3)$ is compactly embedded in the Lebesgue space of radial functions $L^s_r(\mathbb{R}^3)$ for any $s \in (2, 6)$ (see [6] and [22]).

The introduction of $I_{|H^1_r}$ is very useful, since H^1_r is a natural constraint for I, in the following standard sense (the proof can be found, for example, in [2]):

Lemma 2.3. *A function* $u \in H^1_r$ *is a critical point for* $I_{|H^1_r}$ *if and only if* u *is a critical point for* I.

Thus, from now on, one could look for critical points of $I_{|H^1_r}$ in $H^1_r(\mathbb{R}^3)$. But more precisely, we look for nontrivial triples $(u, \Phi[u], \omega)$ which solve system (1.2)–(1.3).

To this purpose, we follow an approach which is similar to the one in [2] (where actually a different system of partial differential equations was derived): for $\sigma > 0$ introduce the set

$$V_\sigma = \left\{u \in H^1_r\big(\mathbb{R}^3\big):\ \Lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big|\nabla \Phi[u]\big|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2\big(1 - q\Phi[u]\big)^2\,dx = \sigma^2\right\}.$$

Then we look for minimizers of J constrained on a suitable V_σ, and in this way ω^2 will be found as the Lagrange multiplier.

Theorem 1.1 will be proved thanks to the following:

Proposition 2.3. *Let* W *satisfy* $W_1)$, $W_2)$, $W_3)$, $W_4)'$ *and* $W_5)'$. *Then there exists* $q^* > 0$ *such that for any* $q \in (0, q^*)$ *there exists* $\sigma^2 > 0$ *such that* J *has a minimizer on* V_σ *with Lagrange multiplier* $\omega^2 > 0$.

For further use, we note that by (1.3) we get

$$\int_{\mathbb{R}^3} \left|\nabla\Phi[u]\right|^2 dx = q \int_{\mathbb{R}^3} u^2\big(1 - q\Phi[u]\big)\Phi[u]\, dx,$$

so that an easy computation gives

$$\Lambda(u) = \mathcal{A}\big(u, \Phi[u]\big) = \frac{1}{2}\int_{\mathbb{R}^3} u^2\big(1 - q\Phi[u]\big)\, dx. \tag{2.7}$$

3. The case $k > 2$

In this section we prove Proposition 2.3 when $k > 2$, leaving to the last section a few comments for the case $k = 2$. A fundamental tool in this section will be the result below, which we prove under more general assumptions on W, since we only require that W satisfies W_1) and $W_5)'$.

Lemma 3.1. *Assume that W satisfies W_1) and $W_5)'$. Then for any $m_0^2 > 0$ there exists $q^* > 0$ such that for any $q \in (0, q^*)$ there exists $\bar{u} = \bar{u}(m_0^2) \in H_r^1$, $\bar{u} \neq 0$, such that*

$$\frac{kJ(\bar{u})}{\int_{\mathbb{R}^3} \bar{u}^2(1 - q\Phi[\bar{u}])\, dx} < m_0^2.$$

Proof. First, let us prove the result for $q = 0$.

Define

$$v(x) := \begin{cases} 1 - |x| & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then $v \in H_r^1(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} v^2\, dx = \frac{2}{15}\pi,$$

while, for future need, we also compute

$$\int_{\mathbb{R}^3} |\nabla v|^2\, dx = \frac{4}{3}\pi \quad \text{and} \quad \int_{\mathbb{R}^3} v\, dx = \frac{\pi}{3}.$$

Now, for $\lambda \geqslant 1$ define

$$u_\lambda(x) = \lambda v\left(\frac{x}{\lambda}\right), \qquad \sigma_\lambda^2 = \frac{1}{2}\int_{\mathbb{R}^3} u_\lambda^2\, dx, \qquad B_\lambda = B(0, \lambda).$$

Then by $W_5)'$

$$\begin{aligned} \frac{k}{2}\frac{J(u_\lambda)}{\sigma_\lambda^2} &= \frac{k}{2}\frac{\frac{1}{2}\int_{B_\lambda} |\nabla u_\lambda|^2\, dx + \int_{B_\lambda} W(u_\lambda)\, dx}{\sigma_\lambda^2} \\ &\leqslant \frac{k}{2}\frac{\frac{1}{2}\int_{B_\lambda} |\nabla u_\lambda|^2\, dx + m_1\int_{B_\lambda} |u_\lambda|^\vartheta\, dx + c\int_{B_\lambda} |u_\lambda|\, dx}{\sigma_\lambda^2}. \end{aligned} \tag{3.8}$$

By the change of variable $y = x/\lambda$ we immediately get

$$\int_{B_\lambda} |\nabla u_\lambda|^2 = \lambda^3 \int_{B_1} |\nabla v|^2, \qquad \sigma_\lambda^2 = \frac{\lambda^5}{2}\int_{B_1} v^2, \qquad \int_{B_\lambda} |u_\lambda|^\vartheta = \lambda^{\vartheta+3}\int_{B_1} v^\vartheta,$$

and obviously $|B_\lambda| = \lambda^3|B_1|$. Therefore (3.8) gives

$$\frac{k}{2}\frac{J(u_\lambda)}{\sigma_\lambda^2} \leqslant \frac{k}{2}\left(\frac{\int_{B_1}|\nabla v|^2\,dx}{\int_{B_1}v^2\,dx}\frac{1}{\lambda^2} + 2c\frac{\int_{B_1}v\,dx}{\int_{B_1}v^2\,dx}\frac{1}{\lambda}\right) + \frac{km_1\int_{B_1}v^\vartheta\,dx}{\int_{B_1}v^2}\lambda^{\vartheta-2}. \tag{3.9}$$

Since $v \leqslant 1$ and $\vartheta > 2$, we have $\int v^\vartheta \leqslant \int v^2$. Moreover, since $\lambda \geqslant 1$, from (3.9) and the calculations above we get

$$\frac{k}{2}\frac{J(u_\lambda)}{\sigma_\lambda^2} \leqslant \frac{k}{2}(10+5c)\frac{1}{\lambda} + km_1\lambda^{\vartheta-2}.$$

Now, note that by assumption $W_5)'$ the function $h : [1,\infty) \to \mathbb{R}$ defined as

$$h(\lambda) := km_1\lambda^{\vartheta-1} - m_0^2\lambda + \frac{k}{2}(10+5c)$$

has a minimum point in $(1,\infty)$, since $m_1 < \frac{m_0^2}{k(\vartheta-1)}$, and that the minimum value is strictly less than 0, due to the other bound on m_1 assumed in $W_5)'$; thus there exists $\lambda > 1$ such that, setting $\bar{u} = u_\lambda$, there holds

$$\frac{kJ(\bar{u})}{\int_{\mathbb{R}^3}\bar{u}^2\,dx} < m_0^2, \tag{3.10}$$

and the claim for $q = 0$ holds true.

From the last inequality above we get the final statement of the lemma by a continuity argument. First, let us denote by Φ_q the mapping defined in Proposition 2.2, emphasizing the dependence of Φ on q, and now we show that

$$\int_{\mathbb{R}^3}\bar{u}^2\big(1 - q\Phi_q[\bar{u}]\big)\,dx \to \int_{\mathbb{R}^3}\bar{u}^2\,dx \quad \text{as } q \to 0,$$

so that the result will follow by (3.10). Note that the function $\bar{u}$ found for (3.10) in correspondence of $q = 0$ depends only on m_0^2 and *not* on q. Then, since $\Phi_q[\bar{u}]$ solves

$$-\Delta\Phi_q[\bar{u}] + q^2\bar{u}^2\Phi_q[\bar{u}] = q\bar{u}^2,$$

by the Hölder and Sobolev inequalities we get

$$\big\|\Phi_q[\bar{u}]\big\|^2 + q^2\int_{\mathbb{R}^3}\bar{u}^2\Phi_q[\bar{u}]^2\,dx \leqslant q\|\bar{u}\|_{12/5}^2\big\|\Phi_q[\bar{u}]\big\|_6 \leqslant qS\|\bar{u}\|_{12/5}^2\big\|\Phi_q[\bar{u}]\big\|,$$

so that

$$\big\|\Phi[\bar{u}]\big\| \leqslant qS\|\bar{u}\|_{12/5}^2. \tag{3.11}$$

Then

$$0 \leqslant q\int_{\mathbb{R}^3}\bar{u}^2\Phi_q[\bar{u}]\,dx \leqslant q^2S\|\bar{u}\|_{12/5}^2\big\|\Phi_q[\bar{u}]\big\|,$$

and the claim follows from (3.11). □

Now, following Lemma 3.1, fix $m_0^2 > 0$, take $q < q^* = q^*(m_0^2)$ and the associated $\bar{u}$, define

$$\sigma^2 = \frac{1}{2}\int_{\mathbb{R}^3}\big|\nabla\Phi[\bar{u}]\big|^2\,dx + \frac{1}{2}\int\bar{u}^2\big(1 - q\Phi[\bar{u}]\big)^2\,dx$$

and consider the corresponding set

$$V_\sigma = \left\{u \in H_r^1(\mathbb{R}^3):\ \frac{1}{2}\int_{\mathbb{R}^3}\big|\nabla\Phi[u]\big|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}u^2\big(1 - q\Phi[u]\big)^2\,dx = \sigma^2\right\}. \tag{3.12}$$

It turns out that this set is a good one:

Lemma 3.2. *The set V_σ defined in* (3.12) *is a nonempty manifold of codimension* 1.

Proof. By definition $\bar u \in V_\sigma$, which is thus nonempty. Moreover, $V_\sigma = \{u \in H_r^1\colon\ \Lambda(u) = \sigma^2\}$, and by Lemma 2.1 we have

$$\Lambda'(u) = 0 \quad \Leftrightarrow \quad u\bigl(1 - q\Phi[u]\bigr)^2 = 0.$$

Then also $u^2(1 - q\Phi[u]) \equiv 0$, so that

$$\int_{\mathbb{R}^3} u^2\bigl(1 - q\Phi[u]\bigr)\,dx = 0,$$

and thus u cannot belong to V_σ by (2.7). The claim follows. □

Now, let us prove Proposition 2.3, showing that $J_{|V_\sigma}$ has a minimizer with Lagrangian multiplier $\omega^2 > 0$.

Let $(u_n)_n$ be a minimizing sequence for $J_{|V_\sigma}$, so that

$$J(u_n) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} W(u_n)\,dx \to \inf_{V_\sigma} J. \tag{3.13}$$

By Ekeland's Variational Principle (see for example [24, Theorem 8.5]) we can also assume that $J'_{|V_\sigma}(u_n) \to 0$, i.e. there exists a sequence $(\lambda_n)_n$ of real numbers such that

$$J'(u_n) - \lambda_n \Lambda'(u_n) \to 0 \quad \text{in } H^{-1} := \bigl(H_r^1\bigr)'. \tag{3.14}$$

By Lemma 2.1, taken $v \in H_r^1$, we have $\Lambda'(u_n)(v) = \int u_n(1 - q\Phi[u_n])^2 v$ for any $n \in \mathbb{N}$, so that, setting $\Phi_n := \Phi[u_n]$, we have

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla v\,dx + \int_{\mathbb{R}^3} W'(u_n)v\,dx - \lambda_n \int_{\mathbb{R}^3} u_n(1 - q\Phi_n)^2 v \to 0 \quad \forall v \in H_r^1.$$

Our first essential result is the following.

Lemma 3.3. *The sequence $(u_n)_n$ is bounded in $H_r^1(\mathbb{R}^3)$ and the sequence $(\Phi_n)_n$ is bounded in $D_r^1(\mathbb{R}^3)$.*

Proof. Since $W \geqslant 0$, by (3.13) it is immediately seen that the sequence $(\nabla u_n)_n$ is bounded in $(L^2(\mathbb{R}^3))^3$. Moreover, since $u_n \in V_\sigma$, from (3.12) we immediately get that $(\Phi_n)_n$ is bounded in D_r^1.

Since Φ_n is radial, by the Radial Lemma (see [6] and [22]) there exists $c > 0$ such that

$$\bigl|\Phi_n(x)\bigr| \leqslant c\frac{\|\Phi_n\|}{|x|^{1/2}} \quad \text{if } |x| \geqslant 1,$$

and then

$$\bigl|\Phi_n(x)\bigr| \leqslant \frac{c'}{|x|^{1/2}} \quad \text{if } |x| \geqslant 1,$$

since $(\Phi_n)_n$ is bounded in D_r^1. This implies that there exists $R > 0$ such that $1 - q\Phi_n(x) \geqslant 1/2$ if $|x| \geqslant R$ for any $n \in \mathbb{N}$. In this way (i) of Proposition 2.2 and (2.7) imply

$$\begin{aligned} 2\sigma^2 = \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)\,dx &\geqslant \int_{\{x\in\mathbb{R}^3\colon\ |x|\geqslant R\}} u_n^2(1 - q\Phi_n)\,dx \\ &\geqslant \frac{1}{2}\int_{\{x\in\mathbb{R}^3\colon\ |x|\geqslant R\}} u_n^2\,dx. \end{aligned}$$

But we have already proved that $(u_n)_n$ is bounded in $D^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, so that it is also bounded in $L^6(B_R)$, and thus in $L^2(B_R)$, and the claim follows. □

Note that the following estimate holds true by an easy integration of (1.3), using the fact that $0 \leqslant \Phi_n \leqslant 1/q$ in $\mathbb{R}^3$ for any $n \in \mathbb{N}$:

$$\exists C > 0 \quad \text{such that} \quad \|\Phi_n\| \leqslant C\|u_n\|_{12/5}^2 \quad \forall n \in \mathbb{N}.$$

Now, $(u_n)_n$ and $(\Phi_n)_n$ are bounded in $H_r^1(\mathbb{R}^3)$ and $D_r^1(\mathbb{R}^3)$ respectively, so that there exists a subsequence (still labelled $(u_n)_n$) such that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$; the associated (sub)sequence $(\Phi_n)_n$ admits a weakly convergent subsequence in $D_r^1(\mathbb{R}^3)$. Therefore we can assume, after relabelling, that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$ and $\Phi_n \rightharpoonup \Phi$ in $D_r^1(\mathbb{R}^3)$. Moreover, we can assume that $(u_n)_n$ converges strongly to u in $L^p(\mathbb{R}^3)$ for any $p \in (2,6)$.

Let us first show that the weak limits u and Φ are related by the fact that $\Phi = \Phi[u]$: indeed for any $\psi \in D^1(\mathbb{R}^3)$ there holds

$$\int_{\mathbb{R}^3} \nabla\Phi_n \cdot \nabla\psi \, dx = q \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)\psi \, dx;$$

but $u_n^2 \to u^2$ in $L^{6/5}(\mathbb{R}^3)$, while, as for $\int u_n^2 \Phi_n \psi$, we have that $\Phi_n \rightharpoonup \Phi$ in $L^6(\mathbb{R}^3)$, $\psi \in L^6(\mathbb{R}^3)$ and $u_n^2 \to u^2$ in $L^{3/2}(\mathbb{R}^3)$. Thus, passing to the limit,

$$\int_{\mathbb{R}^3} \nabla\Phi \cdot \nabla\psi \, dx = q \int_{\mathbb{R}^3} u^2(1 - q\Phi)\psi \, dx,$$

i.e. $\Phi = \Phi[u]$, as claimed.

We now want to prove that the (sub)sequence $(u_n)_n$, which converges weakly, actually converges strongly in $H_r^1(\mathbb{R}^3)$. First we need the following crucial result:

Lemma 3.4. *There exists $\omega^2 \in [0, \infty)$ such that $\lambda_n \to \omega^2$ as $n \to \infty$ (up to subsequences).*

Remark 3.1. Actually at the end we will prove that $\omega^2 > 0$ (see Lemma 3.7 below), but for the moment this preliminary result is enough.

Proof of Lemma 3.4. The proof is quite long and will be divided in several steps. Let us start noting that by (3.14), and since $(u_n)_n$ is bounded, we have

$$\langle J'(u_n), u_n \rangle - \lambda_n \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)^2 \, dx := \varepsilon_n \to 0 \quad \text{in } \mathbb{R}. \tag{3.15}$$

First step. Assume by contradiction that along a subsequence

$$\int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)^2 \, dx \to 0, \tag{3.16}$$

that is

$$\int_{\mathbb{R}^3} u_n^2 \, dx = 2q \int_{\mathbb{R}^3} u_n^2 \Phi_n \, dx - q^2 \int_{\mathbb{R}^3} u_n^2 \Phi_n^2 \, dx + o(1), \tag{3.17}$$

where here and in the following we write $o(1)$ for any real sequence approaching 0 as $n \to \infty$.

Thus, by definition of V_σ and by (2.7), from (3.17) we get

$$2\sigma^2 = \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n) \, dx = q \int_{\mathbb{R}^3} u_n^2 \Phi_n \, dx - q^2 \int_{\mathbb{R}^3} u_n^2 \Phi_n^2 \, dx + o(1)$$

$$= q \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)\Phi_n \, dx + o(1). \tag{3.18}$$

Again by the Radial Lemma and recalling that Φ_n is bounded in $D_r^1(\mathbb{R}^3)$, there exists $M > 0$ such that

$$0 \leqslant \Phi_n(x) \leqslant c\frac{\|\Phi_n\|}{|x|^{1/2}} \leqslant \frac{1}{q} - \Phi_n(x) \quad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^3 \text{ with } |x| \geqslant M. \tag{3.19}$$

Thus, setting $B_M = \{x \in \mathbb{R}^3\colon |x| \leqslant M\}$ and $B_M^C = \mathbb{R}^3 \setminus B_M$, (3.18) and (3.19) imply

$$\begin{aligned}2\sigma^2 &\leqslant q\int_{B_M} u_n^2(1-q\Phi_n)\Phi_n\,dx + \int_{B_M^C} u_n^2(1-q\Phi_n)^2\,dx + o(1)\\ &\leqslant q\int_{B_M} u_n^2(1-q\Phi_n)\Phi_n\,dx + \int_{\mathbb{R}^3} u_n^2(1-q\Phi_n)^2\,dx + o(1)\\ &= q\int_{B_M} u_n^2(1-q\Phi_n)\Phi_n\,dx + o(1)\end{aligned} \tag{3.20}$$

by (3.16).

By Rellich's Theorem $u_n \to u$ and $\Phi_n \to \Phi$ in $L^q(B_M)$ for any $q \in [1,6)$; thus, writing

$$u_n^2(1-q\Phi_n)\Phi_n = u_n^2\Phi_n - qu_n^2\Phi_n^2,$$

in the first addendum $u_n^2 \to u^2$ and $\Phi_n \to \Phi$ in $L^2(B_M)$ and in the second addendum $u_n^2 \to u^2$ and $\Phi_n^2 \to \Phi^2$ in $L^2(B_M)$.

Since $\Phi = \Phi[u]$, as a consequence $0 \leqslant \Phi \leqslant 1/q$. In conclusion (3.20) gives

$$2\sigma^2 \leqslant q\int_{B_M} u^2(1-q\Phi)\Phi\,dx \leqslant q\int_{\mathbb{R}^3} u^2(1-q\Phi)\Phi\,dx = \int_{\mathbb{R}^3} |\nabla\Phi|^2\,dx. \tag{3.21}$$

But by definition of V_σ we have

$$2\sigma^2 = \int_{\mathbb{R}^3} |\nabla\Phi_n|^2\,dx + \int_{\mathbb{R}^3} u_n^2(1-q\Phi_n)^2\,dx \quad \forall n \in \mathbb{N},$$

while by the semicontinuity of the norm in $D_r^1(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} |\nabla\Phi|^2\,dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^3} |\nabla\Phi_n|^2\,dx,$$

and since $\int_{\mathbb{R}^3} u_n^2(1-q\Phi_n)^2\,dx \to 0$ by (3.21), we finally have

$$\Phi_n \to \Phi \quad \text{in } D^1\big(\mathbb{R}^3\big).$$

In addition, we note that $\Phi \neq 0$, since its norm equals $\sqrt{2}\sigma > 0$ by (3.12), and this also implies that

$$u \neq 0. \tag{3.22}$$

Second step. As a consequence of the convergence of Φ_n in $D_r^1(\mathbb{R}^3)$, and so in $L^6(\mathbb{R}^3)$ (the weak convergence here would be enough), and the strong convergence of u_n^2 to u^2 in $L^{6/5}(\mathbb{R}^3)$, from the equality $2\sigma^2 = \int u_n^2(1-q\Phi_n)$, we get

$$\int_{\mathbb{R}^3} u_n^2\,dx = 2\sigma^2 + q\int_{\mathbb{R}^3} u^2\Phi\,dx + o(1) \geqslant 2\sigma^2 \quad \forall n \in N,$$

so that

$$u_n \not\to 0 \quad \text{in } L^2\big(\mathbb{R}^3\big) \text{ as } n \to \infty.$$

In light of the three possibilities established by the Lions Concentration–Compactness Principle (see Lemma A.1 in Appendix A) we start by noting that dichotomy can never occur in the case of radial functions. If vanishing took place, by the final statement of Lemma A.1 we would have that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for any $r \in (2,6)$, which is in contradiction with (3.22). In conclusion we are in presence of compactness: there exists a sequence of points $(y_n)_n$ in $\mathbb{R}^3$ such that (possibly passing to a subsequence)

$$\forall \delta > 0\ \exists R = R(\delta) > 0 \quad \text{such that} \quad \int\limits_{B_R(y_n)} u_n^2\,dx \geqslant L^2 - \delta \quad \forall n \in \mathbb{N},$$

where, for shortness, we have set

$$L := \lim_{n\to\infty} \int\limits_{\mathbb{R}^3} u_n^2\,dx.$$

We now show that we can actually assume that

$$\forall \delta > 0\ \exists R = R(\delta) > 0 \quad \text{such that} \quad \int\limits_{B_{2R}} u_n^2\,dx \geqslant L^2 - \delta \quad \forall n \in \mathbb{N}, \tag{3.23}$$

i.e. concentration occurs at 0. Indeed, fix $\delta \in (0, L^2/2)$, find the corresponding R and assume by contradiction that for infinitely many n we have that $|y_n| > R$; then by the radial symmetry of u_n

$$L^2 - \delta \leqslant \int\limits_{B_R(y_n)} u_n^2\,dx = \frac{1}{2} \int\limits_{B_R(y_n)\cup B_R(-y_n)} u_n^2\,dx \leqslant \frac{1}{2}\int\limits_{\mathbb{R}^3} u_n^2\,dx \to \frac{L^2}{2},$$

which is absurd. Thus $|y_n| \leqslant R$ definitely, so, say, for any $n \in \mathbb{N}$. But in this way

$$\int\limits_{B_R(y_n)} u_n^2\,dx \leqslant \int\limits_{B_{2R}(0)} u_n^2\,dx,$$

and (3.23) follows.

Finally, let us show that $u_n \to u$ in $L^2(\mathbb{R}^3)$. First, let us take δ and R as given in (3.23). Since $u \in L^2(\mathbb{R}^3)$, there exists $M \geqslant 2R$ such that

$$\int\limits_{B_M^C} u^2\,dx < \delta.$$

Since $u_n \to u$ in $L^2(B_M)$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we have

$$\|u_n - u\|_{L^2(B_M)} < \delta,$$

and by the monotonicity of the integral, since $M \geqslant 2R$, we also have that

$$\int\limits_{B_M} u_n^2\,dx \geqslant L^2 - \delta.$$

In conclusion

$$\begin{aligned}
\|u_n - u\|^2_{L^2(\mathbb{R}^3)} &= \|u_n - u\|^2_{L^2(B_M)} + \|u_n - u\|^2_{L^2(B_M^C)} \\
&< \delta + 2\|u_n\|^2_{L^2(B_M^C)} + 2\|u\|^2_{L^2(B_M^C)} \\
&< \delta + 2\|u_n\|^2_{L^2(\mathbb{R}^3)} - 2\|u_n\|^2_{L^2(B_M)} + 2\delta \\
&\leqslant 5\delta - 2L^2 + 2\|u_n\|^2_{L^2(\mathbb{R}^3)} \to 5\delta \quad \text{as } n \to \infty,
\end{aligned}$$

and the claim follows.

In this way by (3.16) and the strong convergence of u_n, Φ_n and Φ_n^2 in $L^2(\mathbb{R}^3)$, we get

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} u_n^2(1-q\Phi_n)^2\,dx \\ &= \int_{\mathbb{R}^3} u_n^2\,dx - 2q\int_{\mathbb{R}^3} u_n^2\Phi_n\,dx + q^2\int_{\mathbb{R}^3} u_n^2\Phi_n^2\,dx \\ &\to \int_{\mathbb{R}^3} u^2(1-q\Phi)^2\,dx. \end{aligned}$$

This implies that $u \equiv 0$ when $(1-q\Phi) \neq 0$, while from (3.17) we get

$$\int_{\mathbb{R}^3} u^2(1-q\Phi)\,dx = 2\sigma^2 > 0.$$

This is a contradiction and thus (3.16) cannot hold.

Third step. Since $(1-q\Phi_n)^2 \leqslant (1-q\Phi_n)$, we can assume that there exists $\Sigma^2 \in (0,\sigma^2]$ such that

$$\int_{\mathbb{R}^3} u_n^2(1-q\Phi_n)^2\,dx \to 2\Sigma^2.$$

By (3.15)

$$\begin{aligned} \lambda_n &= \frac{1}{2\Sigma^2+o(1)}\big(\langle J'(u_n),u_n\rangle - \varepsilon_n\big) \\ &= \frac{1}{2\Sigma^2+o(1)}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} W'(u_n)u_n\,dx\right) - \frac{\varepsilon_n}{2\Sigma^2+o(1)}. \end{aligned} \tag{3.24}$$

Thus by $W_4)'$, since $k \geqslant 2$,

$$\begin{aligned} \lambda_n &\leqslant \frac{1}{\Sigma^2+o(1)}\left(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{k}{2}\int_{\mathbb{R}^3} W(u_n)\,dx\right) - \frac{\varepsilon_n}{2\Sigma^2+o(1)} \\ &\leqslant \frac{k}{2\Sigma^2+o(1)}\left(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} W(u_n)\,dx\right) - \frac{\varepsilon_n}{2\Sigma^2+o(1)} \\ &= \frac{k}{2\Sigma^2+o(1)}J(u_n) - \frac{\varepsilon_n}{2\Sigma^2+o(1)}. \end{aligned}$$

On the other hand, again by $W_4)'$,

$$\langle J'(u_n),u_n\rangle = \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^3} W'(u_n)u_n\,dx \geqslant 0,$$

so that by (3.24) we finally get

$$-\frac{\varepsilon_n}{2\Sigma^2+o(1)} \leqslant \lambda_n \leqslant \frac{k}{2\Sigma^2+o(1)}J(u_n) - \frac{\varepsilon_n}{2\Sigma^2+o(1)}.$$

Thus, up to subsequences, $\lambda_n \to \omega^2$, where

$$0 \leqslant \omega^2 \leqslant \frac{k}{2\Sigma^2}\inf_{v\in V_\sigma} J(v). \qquad \square \tag{3.25}$$

By Lemma 3.3 we know that, up to a subsequence, $(u_n)_n$ converges weakly in H_r^1 and $\lambda_n \to \omega^2$. But, as said above, we now prove that the convergence of $(u_n)_n$ is strong. We start with the following result:

Lemma 3.5. *If $\omega^2 < m_0^2$, then $(u_n)_n$ is a Cauchy sequence in $H_r^1(\mathbb{R}^3)$.*

The proof of this lemma was given in [2] for $k = 2$, but it can be restated word by word also if $k > 2$, since the essential assumption in its proof was only W_3).

Thus, if we knew that $\omega^2 < m_0^2$, we could conclude by Lemma 3.5 that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$; in this way $u \in V_\sigma$ and it is a nontrivial solution of (1.2). Of course, if we knew that in any case $(u_n)_n$ converges strongly in $H_r^1(\mathbb{R}^3)$, we could conclude that u is a nontrivial solution of (1.2). For this reason we conclude with the following:

Lemma 3.6. *$(u_n)_n$ converges strongly in $H_r^1(\mathbb{R}^3)$.*

Proof. We already know that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, so that we need to show that $u_n \to u$ in $L^2(\mathbb{R}^3)$ and $\nabla u_n \to \nabla u$ in $(L^2(\mathbb{R}^3))^3$. Of course we have the following two possibilities: either $u_n \not\to u$ or $u_n \to u$ in $L^2(\mathbb{R}^3)$, and we want to exclude the first possibility.

By the Concentration–Compactness Principle, in the former case only vanishing can occur, since we are treating radial functions and then dichotomy cannot take place. If this is the case, by the final statement of Lemma A.1, $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for any $s \in (2, 6)$. We also recall that $\Phi_n \rightharpoonup \Phi = \Phi[u]$ in $D_r^1(\mathbb{R}^3)$. Then we would have that

$$2\sigma^2 = \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)\,dx = \int_{\mathbb{R}^3} u_n^2\,dx + o(1) = \int_{\mathbb{R}^3} u_n^2(1 - q\Phi_n)^2\,dx + o(1) \to 2\Sigma^2,$$

that is $\sigma^2 = \Sigma^2$. Then (3.25) and Lemma 3.1 imply

$$\omega^2 \leqslant \frac{k}{2\sigma^2} \inf_{v \in V_\sigma} J(v) < m_0^2,$$

so that Lemma 3.5 applies and in particular $u_n \to u$ in $L^2(\mathbb{R}^3)$, which is absurd under the assumptions of vanishing.

Then we can conclude that $u_n \to u$ in $L^2(\mathbb{R}^3)$; from this very last information we obtain that $u_n \to u$ in H^1. Indeed,

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u)\,dx - \int_{\mathbb{R}^3} W'(u_n)(u_n - u)\,dx - \lambda_n \int_{\mathbb{R}^3} u_n(1 - q\Phi_n)^2(u_n - u)\,dx \to 0 \tag{3.26}$$

by (3.14), since $(u_n - u)_n$ is a bounded sequence. On the other hand, by W_3) we get

$$\left|W'(u_n)(u_n - u)\right| \leqslant C_1|u_n||u_n - u| + \frac{C_2}{p+1}|u_n|^{p+1}|u_n - u|.$$

By the Hölder inequality $\int |u_n||u_n - u| \leqslant \|u_n\|_2\|u_n - u\|_2 \to 0$. If $p \in (0, 2)$, then

$$\int_{\mathbb{R}^3} |u_n|^{p+1}|u_n - u|\,dx \leqslant \|u_n\|_{2(p+1)}\|u_n - u\|_2 \to 0,$$

since $2 < 2(p + 1) < 6$; if $p \in [2, 4)$ take $\varepsilon \in (0, 1)$ such that $\mu := (\frac{6-\varepsilon}{p+1})' \in (2, 6)$, so that

$$\int_{\mathbb{R}^3} |u_n|^{p+1}|u_n - u|\,dx \leqslant \|u_n\|_{6-\varepsilon}^{p+1}\|u_n - u\|_\mu \to 0,$$

since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 6)$ by assumption and the compact embedding of $H_r^1(\mathbb{R}^3)$.

Moreover,

$$\int_{\mathbb{R}^3} u_n(1 - q\Phi_n)^2(u_n - u)\,dx = \int_{\mathbb{R}^3} \left[\left(u_n^2 - u_n u\right) - 2q\left(u_n^2 - u_n u\right)\Phi_n + q^2\left(u_n^2 - u_n u\right)\Phi_n^2\right] dx;$$

now the first term in the r.h.s. of the previous identity goes to 0 since $u_n \to u$ in $L^2(\mathbb{R}^3)$, while $\Phi_n \rightharpoonup \Phi$ in $L^6(\mathbb{R}^3)$ and $u_n^2 - u_n u \to 0$ in $L^{6/5}(\mathbb{R}^3)$, Φ_n^2 is bounded in $L^3(\mathbb{R}^3)$ and $u_n^2 - u_n u \to 0$ in $L^{3/2}(\mathbb{R}^3)$, so that

$$\int_{\mathbb{R}^3} u_n(1-q\Phi_n)^2(u_n-u)\,dx \to 0.$$

In conclusion (3.26) gives $\|\nabla u_n\|_2 \to \|\nabla u\|_2$, so that, summing up, $u_n \to u$ in $H^1(\mathbb{R}^3)$, as claimed. □

At this point we have found that $u_n \to u$ strongly in $H^1_r(\mathbb{R}^3)$, so that u minimizes J on V_σ and hence it is a nontrivial solution of (1.2). This leads us to state the following final result, which lets us prove completely Proposition 2.3, and thus Theorem 1.1.

Lemma 3.7. $\omega^2 > 0$.

Proof. Assume by contradiction that $\omega^2 = 0$. Then $u \in V_\sigma$ would be a nontrivial solution of $J'(u) = 0$, i.e. u would solve

$$-\Delta u + W'(u) = 0 \quad \text{in } \mathbb{R}^3.$$

Multiplying by u and integrating by parts give

$$0 = \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} W'(u)u\,dx \geqslant \int_{\mathbb{R}^3} |\nabla u|^2\,dx,$$

which implies $u \equiv 0$, contradicting the fact that $u \in V_\sigma$. □

Theorem 1.1 is now a collection of the previous lemmas.

4. The case $k = 2$

In this short section we outline the proof of the existence result when $k = 2$, already claimed in [2], but not proved therein. First we need the following result, which is the counterpart of Lemma 3.1 when $k = \vartheta = 2$.

Lemma 4.1. *(See [2, Lemma 2.5].) Assume that W satisfies W_1) and W_5). Then for any $m_0^2 > 0$ there exists $q^* > 0$ such that for any $q \in (0, q^*)$ there exists $\bar u = \bar u(q) \in H^1_r$, $\bar u \neq 0$, such that*

$$\frac{2J(\bar u)}{\int_{\mathbb{R}^3} \bar u^2(1 - q\Phi[\bar u])\,dx} < m_0^2.$$

Once got this preliminary result, the rest of the proof of the existence of a nontrivial solution for problem (1.2)–(1.3) stated in Theorem 1.2 follows the same lines of the existence result given in Section 3, starting from Lemma 3.2 through Lemma 3.5, which hold true also if $k = 2$. Finally, in proving the corresponding result of Lemma 3.6, we need Lemma 4.1 above in place of Lemma 3.1.

Appendix A

We recall the properties of Lions's Concentration–Compactness Principle that we use in the form of the following lemma, which is the collection of [15, Lemma I.1] and [16, Lemma I.1]:

Lemma A.1 *(Concentration–Compactness Principle). Let $(\rho_n)_n$ be a sequence in $L^1(\mathbb{R}^N)$, $N \geqslant 1$, such that $\rho_n \geqslant 0$ and $\int \rho_n \to \lambda > 0$. Then there exists a subsequence $(\rho_{n_k})_k$ satisfying one of the three alternatives below*:

(compactness) *there exists a sequence of points $(y_k)_k$ in $\mathbb{R}^N$ such that $\forall \delta > 0$ $\exists R > 0$ with the property that*

$$\int_{B_R(y_k)} \rho_{n_k}\,dx \geqslant \lambda - \delta;$$

(vanishing)

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_{n_k}\,dx = 0 \quad \textit{for any } R > 0;$$

(dichotomy) *there exists* $\alpha \in (0,\lambda)$ *such that for every* $\delta > 0$ *there exist* $k_0 \geqslant 1$ *and two sequences of nonnegative functions* $(\rho_k^1)_k$, $(\rho_k^2)_k$ *in* $L^1(\mathbb{R}^N)$ *such that for any* $k \geqslant k_0$

$$\left\|\rho_{n_k} - \left(\rho_k^1 + \rho_k^2\right)\right\|_{L^1(\mathbb{R}^N)} < \delta, \qquad \left|\int_{\mathbb{R}^N} \rho_k^1\,dx - \alpha\right| \leqslant \delta,$$

$$\left|\int_{\mathbb{R}^N} \rho_k^2\,dx - (\lambda - \alpha)\right| \leqslant \delta,$$

and

$$\mathit{dist}\left(\mathrm{Supp}\left(\rho_k^1\right), \mathrm{Supp}\left(\rho_k^2\right)\right) \to \infty \quad \textit{as } k \to \infty.$$

Moreover, if $1 < p < N$, $1 \leqslant q < \frac{pN}{N-p}$, $(u_n)_n$ *is bounded in* $L^q(\mathbb{R}^N)$ *and* $(\nabla u_n)_n$ *is bounded in* $(L^p(\mathbb{R}^N))^N$, *and*

$$\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^q\,dx \to 0 \quad \textit{as } n \to \infty \textit{ for some } R > 0,$$

then $u_n \to 0$ *in* $L^s(\mathbb{R}^N)$ *for any* $s \in (q, \frac{pN}{N-p})$.

Of course the last statement is an improvement of the first part when $\rho_n = |u_n|^q$.

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