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Controllability of a string submitted to unilateral constraint

Farid Ammar-Khodja [a,1], Sorin Micu [b,2], Arnaud Münch [c,*,3]

[a] *Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon cedex, France*
[b] *Facultatea de Matematica si Informatica, Universitatea din Craiova, 200585, Romania*
[c] *Laboratoire de Mathématiques de Clermont-Ferrand, Université Blaise Pascal, UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France*

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Abstract

This article studies the controllability property of a homogeneous linear string of length one, submitted to a time dependent obstacle (described by the function $\{\psi(t)\}_{0\leqslant t\leqslant T}$) located below the extremity $x=1$. The Dirichlet control acts on the other extremity $x=0$. The string is modelled by the wave equation $y''-y_{xx}=0$ in $(t,x)\in(0,T)\times(0,1)$, while the obstacle is represented by the Signorini's conditions $y(t,1)\geqslant\psi(t)$, $y_x(t,1)\geqslant 0$, $y_x(t,1)(y(t,1)-\psi(t))=0$ in $(0,T)$. The characteristic method and a fixed point argument allow to reduce the problem to the analysis of the solutions at $x=1$. We prove that, for any $T>2$ and initial data $(y^0,y^1)\in H^1(0,1)\times L^2(0,1)$ with $\psi(0)\leqslant y^0(1)$, the system is null controllable with controls in $H^1(0,T)$. Two distinct approaches are used. We first introduce a penalized system in y_ϵ, transforming the Signorini's condition into the simpler one $y_{\epsilon,x}(t,1)=\epsilon^{-1}[y_\epsilon(t,1)-\psi(t)]^-$, ϵ being a small positive parameter. We construct explicitly a family of controls of the penalized problem, uniformly bounded with respect to ϵ in $H^1(0,T)$. This enables us to pass to the limit and to obtain a control for the initial equation. A more direct approach, based on differential inequalities theory, leads to a similar positive conclusion. Numerical experiments complete the study.

Résumé

Cet article étudie les propriétés de contrôlabilité d'une corde homogène de longeur un, soumise à un obstacle dépendant du temps (décrit par la fonction $\{\psi(t)\}_{0\leqslant t\leqslant T}$) à l'extrémité $x=1$. Le contrôle Dirichlet agit à l'extrémité $x=0$. La corde est modélisée par l'équation des ondes $y''-y_{xx}=0$ dans $(t,x)\in(0,T)\times(0,1)$, tandis que l'obstacle est représenté par les conditions de Signorini $y(t,1)\geqslant\psi(t)$, $y_x(t,1)\geqslant 0$, $y_x(t,1)(y(t,1)-\psi(t))=0$ sur $(0,T)$. La méthode des caractéristiques et un argument de point fixe permettent de réduire le problème à l'analyse des solutions en $x=1$. Nous prouvons que, pour tout $T>2$ et donnée initiale $(y^0,y^1)\in H^1(0,1)\times L^2(0,1)$ avec $\psi(0)\leqslant y^0(1)$, le système est contrôlable à zéro avec des contrôles dans $H^1(0,T)$. Deux approches sont utilisées. On introduit tout d'abord un système pénalisé en y_ϵ, transformant les conditions de Signorini en l'égalité $y_{\epsilon,x}(t,1)=\epsilon^{-1}[y_\epsilon(t,1)-\psi(t)]^-$, ϵ étant un paramètre positif. On construit explicitement une famille de contrôle du problème pénalisé uniformément bornée par rapport à ϵ dans $H^1(0,T)$. Cela nous permet de passer à la limite et d'obtenir un contrôle pour

* Corresponding author.
E-mail addresses: farid.ammar-khodja@univ-fcomte.fr (F. Ammar-Khodja), sd_micu@yahoo.com (S. Micu), arnaud.munch@math.univ-bpclermont.fr (A. Münch).

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le système initial. Une approche plus directe, relevant de la théorie des inéquations différentielles, conduit à un résultat positif similaire. Quelques applications numériques complètent l'étude.

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1. Introduction

Let $T > 0$ and $Q_T = (0, T) \times (0, 1)$. We consider the following system

$$\begin{cases} y'' - y_{xx} = 0, & (t,x) \in Q_T, \\ y(t,0) = u(t), & t \in (0,T), \\ y(t,1) \geqslant \psi(t), \quad y_x(t,1) \geqslant 0, \quad \big(y(t,1) - \psi(t)\big) y_x(t,1) = 0, & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where the symbol $'$ denotes the derivative with respect to the variable t. System (1.1) models the vibration of a homogeneous and linear string of length one in the time interval $(0, T)$ submitted to an initial excitation (y^0, y^1) at time $t = 0$. On the left extremity $x = 0$ acts a control function $u(t)$, whereas on the right extremity $x = 1$, the string is limited by a lower, time dependent, obstacle so that $y(1,t) \geqslant \psi(t)$ for all $t > 0$. The function $\psi(t)$ represents the position of the obstacle at each moment $t \in [0, T]$. When the rod touches the obstacle, its reaction can be only upward, so that $y_x(1,t) \geqslant 0$ on the set $\{t\colon\ y(1,t) = \psi(t)\}$. When the rod does not touch the obstacle, the right end is free so that $y_x(1,t) = 0$ on the set $\{t\colon\ y(1,t) > \psi(t)\}$. These usual conditions, which permit to describe the presence of the obstacle, are called *unilateral Signorini conditions* (see for instance [4]).

Various papers have been devoted to the existence and uniqueness of a solution of the boundary obstacle problem for the wave equation. Among them, we mention [5,6] (whose idea is used in our present work) and [10,11].

We investigate in this work the null boundary controllability of the nonlinear system (1.1) stated as follows: for any T fixed large enough and any (y^0, y^1) in a given space, does there exist a Dirichlet control $u \in H^1(0, T)$ which drives the corresponding solution of (1.1) to rest, i.e.

$$y(T) = y'(T) = 0 \quad \text{in } (0,1)? \tag{1.2}$$

More precisely, we prove the following theorem.

Theorem 1.1. *Let $T > 2$ and $\psi \in H^1(0,T)$ with the property that there exists $\tilde{T} \in (2, \min\{3, T\})$ such that $\psi(t) \leqslant 0$ for any $t \in [\tilde{T}, T]$. For any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with*

$$\psi(0) \leqslant y^0(1) \tag{1.3}$$

there exists a control function $u \in H^1(0,T)$ such that (1.1) *admits a unique solution $y \in C([0,T], H^1(0,1)) \cap C^1([0,T], L^2(0,1))$ satisfying $y(T) = y'(T) = 0$ in* $(0,1)$.

To our knowledge, the exact controllability, when a unilateral constraint is involved, has not been studied so far. In the different context of stabilization, we mention the contribution [8] where the authors prove the exponential decay of the energy associated with the solution of a damped wave equation submitted to a boundary obstacle.

We approach this nonlinear controllability problem in a constructive way by using the characteristic method. This allows to compute the solution ϕ of the linear wave equation submitted to the initial condition (ϕ^0, ϕ^1) and the nonhomogeneous boundary conditions $\phi(t,0) = u(t)$, $\phi(t,1) = f(t)$, for any $u, f \in L^2(0,T)$. Consequently, the Dirichlet-to-Neumann map A defined by $A(\phi^0, \phi^0, u, f) = \phi_x(\cdot, 1)$ may also be computed explicitly (see Section 2). Now, the controllability conditions $\phi(T) = \phi'(T) = 0$ determine u as a function of f in the interval $(T-2, T)$. On the other hand, at the right extremity $x = 1$ of the string, the Signorini conditions are equivalent to the following inequations in $(0, T)$

$$\begin{cases} f-\psi \geqslant 0, & t \in (0,T),\\ A\big(\phi^0,\phi^1,u,f\big) \geqslant 0, & t \in (0,T),\\ (f-\psi)A\big(\phi^0,\phi^1,u,f\big) = 0, & t \in (0,T). \end{cases} \tag{1.4}$$

Hence, the problem is reduced to find u in $[0, T-2]$ such that f is a solution of (1.4). This may be regarded as a fixed point argument since we start with a nonhomogeneous problem and return to the nonlinear one by imposing conditions on (u, f). A control u of (1.1) is found as a solution of these restrictions.

In Section 4, by using general results for differential inequalities, we describe a class of such controls $u \in H^1(0,T)$, assuming that T is strictly greater than 2.

In Section 3, we obtain alternatively the controllability result by using a penalty method, classical in contact mechanics, which consists in relaxing the Signorini inequations by the equation $y_{\epsilon,x}(\cdot,1) = \epsilon^{-1}[y_\epsilon(\cdot,1)-\psi]^-$ in $(0,T)$ where $\epsilon > 0$ denotes the penalized parameter. From a mechanical point of view ϵ^{-1} may be interpreted as the stiffness of the obstacle which is not assumed to be perfectly rigid anymore. Therefore, a penetration of the obstacle is allowed at $x = 1$. In the uncontrolled case, i.e. when $y_\epsilon(t,0) = 0$, the convergence of y_ϵ as ϵ goes to zero toward a solution of (1.1) is discussed, for instance, in [5,11]. We remark that the penalty method described above is fundamental both for the theoretical study and the numerical approximation of solutions of any contact problem.

In Section 3, following the previous fixed point argument, we construct explicitly a class of couples (u_ϵ, f_ϵ), solutions of

$$A\big(\phi^0,\phi^1,u_\epsilon,f_\epsilon\big) = \epsilon^{-1}[f_\epsilon-\psi]^-, \quad t \in (0,T), \tag{1.5}$$

which are uniformly bounded with respect to ϵ (Sections 3.2 and 3.3). This property allows to pass to the limit and to obtain a control for (1.1) in Section 3.4. Relation (1.5) represents a fixed point type condition for the couple (u_ϵ, f_ϵ). Note that we were able to estimate the dependence of the control u_ϵ on ϵ due to the almost explicit formulas we have for it. With a more general approach, using for instance Schauder's theorem like in [13] (see also [2]), it would be more difficult to obtain such precise estimates on ϵ.

Section 5 presents some numerical applications in agreement with the theoretical part while Section 6 concludes with some related extensions and open problems.

2. The control Dirichlet-to-Neumann map of a linear system

Let $T > 0$, $Q_T = (0,T) \times (0,1)$ and consider the following system:

$$\begin{cases} \phi'' - \phi_{xx} = 0, & (t,x) \in Q_T,\\ \phi(t,0) = u(t), & t \in (0,T),\\ \phi(t,1) = f(t), & t \in (0,T),\\ \phi(0,x) = \phi^0(x), \quad \phi'(0,x) = \phi^1(x), & x \in (0,1), \end{cases} \tag{2.6}$$

where $u \in L^2(0,T)$ is a control function and $f \in L^2(0,T)$ is given. The following result is well known (see, for instance, [7]).

Proposition 2.1.

(1) *If* $((\phi^0,\phi^1),(u,f)) \in (L^2(0,1) \times H^{-1}(0,1)) \times L^2(0,T)^2$ *there exists a unique solution* ϕ *of* (2.6) *such that* $\phi \in C([0,T], L^2(0,1)) \cap C^1([0,T], H^{-1}(0,1))$ *and a positive constant C such that*

$$\big\|\big(\phi(t),\phi'(t)\big)\big\|_{L^2(0,1)\times H^{-1}(0,1)} \leqslant C\big(\big\|\big(\phi^0,\phi^1\big)\big\|_{L^2(0,1)\times H^{-1}(0,1)} + \|(u,f)\|_{L^2(0,T)^2}\big).$$

(2) *If* $((\phi^0,\phi^1),(u,f)) \in H^1(0,1) \times L^2(0,1) \times H^1(0,T)^2$ *verifies the compatibility conditions*

$$u(0) = \phi^0(0), \qquad f(0) = \phi^0(1), \tag{2.7}$$

there exists a unique solution of (2.6) $\phi \in C([0,T], H^1(0,1)) \cap C^1([0,T], L^2(0,1))$ *and a positive constant C such that*

$$\big\|\big(\phi(t),\phi'(t)\big)\big\|_{H^1(0,1)\times L^2(0,1)} \leqslant C\big(\big\|\big(\phi^0,\phi^1\big)\big\|_{H^1(0,1)\times L^2(0,1)} + \|(u,f)\|_{H^1(0,T)^2}\big).$$

We define the space

$$\mathbb{H}=\big\{\big((\phi^0,\phi^1),(u,f)\big)\in H^1(0,1)\times L^2(0,1)\times H^1(0,T)^2,\ u(0)=\phi^0(0),\ f(0)=\phi^0(1)\big\}.$$

Given (ϕ^0,ϕ^1,f), our aim is to find a family of explicit controls u for which the solution ϕ of (2.6) satisfies $\phi(T)=\phi'(T)=0$ in $(0,1)$. Setting

$$p=\phi'-\phi_x,\qquad q=\phi'+\phi_x, \tag{2.8}$$

it follows that (2.6) is equivalent to

$$\begin{cases} p'+p_x=q'-q_x=0, & (t,x)\in Q_T,\\ (p+q)(\cdot,0)=2u', & t\in(0,T),\\ (p+q)(\cdot,1)=2f', & t\in(0,T),\\ p^0=\phi^1-\phi^0_x,\quad q^0=\phi^1+\phi^0_x, & x\in(0,1). \end{cases} \tag{2.9}$$

If $((p^0,q^0),(u,f))\in L^2(0,1)^2\times H^1(0,T)^2$ system (2.9) admits a unique generalized solution $(p,q)\in C([0,T],L^2(0,1)^2)$ (see for instance [9, Theorem 3.1, p. 650]). In view of (2.8), this solution corresponds to a solution ϕ of (2.6) satisfying

$$\phi\in C\big([0,T],H^1(0,1)\big)\cap C^1\big([0,T],L^2(0,1)\big)$$

associated with the data $((\phi^0,\phi^1),(u,f))\in\mathbb{H}$.

Proposition 2.2. *Let $T\in(2,3)$ and assume that $((\phi^0,\phi^1),(u,f))\in\mathbb{H}$. Then the solution (p,q) of (2.9) satisfies $(p,q)(T)=0$ in $(0,1)$ if and only if $((\phi^0,\phi^1),(u,f))$ satisfies*

$$\begin{cases} u'(t)=f'(t+1)+\dfrac{1}{2}q^0(t) & \text{if } T-2<t<1,\\ u'(t)=f'(t+1)+f'(t-1)-\dfrac{1}{2}p^0(2-t) & \text{if } 1<t<T-1,\\ u'(t)=f'(t-1)-\dfrac{1}{2}p^0(2-t) & \text{if } T-1<t<2,\\ u'(t)+u'(t-2)=f'(t-1)+\dfrac{1}{2}q^0(t-2) & \text{if } 2<t<T. \end{cases} \tag{2.10}$$

Proof. Solving system (2.9) using the characteristics method gives the expressions:

$$p(t,x)=\begin{cases} p^0(x-t) & \text{if } 0<t<x<1,\\ 2u'(t-x)-q^0(t-x) & \text{if } 0<x<t<1+x,\\ 2u'(t-x)-2f'(t-x-1)+p^0(2-t+x) & \text{if } 1+x<t<2+x,\\ 2u'(t-x)-2f'(t-x-1)+2u'(t-x-2)-q^0(t-x-2) & \text{if } 2+x<t<T \end{cases} \tag{2.11}$$

and

$$q(t,x)=\begin{cases} q^0(x+t) & \text{if } 0<t<1-x,\\ 2f'(t+x-1)-p^0(2-t-x) & \text{if } 1-x<t<2-x,\\ 2f'(x+t-1)-2u'(t+x-2)+q^0(t+x-2) & \text{if } 2-x<t<3-x,\\ 2f'(t+x-1)-2u'(t+x-2)+2f'(t+x-3)-p^0(4-t-x) & \text{if } 3-x<t<T. \end{cases} \tag{2.12}$$

It follows that

$$p(T,x)=\begin{cases} 2u'(T-x)-2f'(T-x-1)+p^0(x-T+2) & \text{if } 0<T-2<x<1,\\ 2u'(T-x)-2f'(T-x-1)+2u'(T-x-2)-q^0(T-x-2) & \text{if } 0<x<T-2<1 \end{cases}$$

and

$$q(T,x)=\begin{cases} -2u'(x+T-2)+2f'(x+T-1)+q^0(x+T-2) & \text{if } 0<x<3-T,\\ -2u'(T+x-2)+2f'(T+x-1)+2f'(T+x-3)-p^0(4-T-x) & \text{if } 0<3-T<x<1. \end{cases}$$

Consequently, $(p,q)(T)=0$ in (0, 1) if and only if u satisfies

$$\begin{cases} u'(T-x)=f'(T-x-1)-\frac{1}{2}p^0(x-T+2) & \text{if } T-2<x<1,\\ u'(T-x)=-u'(T-x-2)+f'(T-x-1)+\frac{1}{2}q^0(T-x-2) & \text{if } 0<x<T-2,\\ u'(x+T-2)=f'(x+T-1)+\frac{1}{2}q^0(x+T-2) & \text{if } 0<x<3-T,\\ u'(x+T-2)=f'(T+x-1)+f'(T+x-3)-\frac{1}{2}p^0(4-T-x) & \text{if } 3-T<x<1,\end{cases}$$

which is equivalent to (2.10). □

Remark 2.1. We point out that in Proposition 2.2, the values of the control functions u are not prescribed on $(0, T-2)$. Consequently, there exists an infinite number of such control functions. The fact that u is "free" in $(0, T-2)$ plays a crucial role in the sequel.

Let us define the space

$$\mathbb{H}_c=\{(\phi^0,\phi^1,u,f)\in\mathbb{H}\mid (2.10)\text{ is verified and } u(T)=f(T)=0\}. \tag{2.13}$$

Corollary 2.1. *Let $T\in(2,3)$ and $(\phi^0,\phi^1,u,f)\in\mathbb{H}$. Then the solution ϕ of* (2.6) *satisfies $\phi(T)=\phi'(T)=0$ in* (0, 1) *if and only if $(\phi^0,\phi^1,u,f)\in\mathbb{H}_c$.*

Proof. From Proposition 2.2 and (2.8), it follows that (2.10) is a necessary and sufficient condition for $\phi'(T)=\phi_x(T)=0$ on (0, 1). Thus, in order to get $\phi(T)=0$ on (0, 1), it is necessary and sufficient to choose $u(T)=f(T)=0$. □

Definition 2.1. The **Dirichlet-to-Neumann map** associated with the system (2.6) is the application $A:\mathbb{H}\to L^2(0,T)$ defined by

$$A(\phi^0,\phi^1,u,f)=\phi_x(.,1),$$

where ϕ is the solution of (2.6) associated with (ϕ^0,ϕ^1,u,f).

The **Control Dirichlet-to-Neumann map** is the application

$$A_c=A_{|\mathbb{H}_c}.$$

The following lemma gives a characterization of these two maps.

Lemma 2.1. *Let $T\in(2,3)$ and $(\phi^0,\phi^1,u,f)\in\mathbb{H}$. Then*

$$A(\phi^0,\phi^1,u,f)(t)=\begin{cases} f'(t)-p^0(1-t), & 0<t<1,\\ f'(t)-2u'(t-1)+q^0(t-1), & 1<t<2,\\ f'(t)+2f'(t-2)-2u'(t-1)-p^0(3-t), & 2<t<T.\end{cases} \tag{2.14}$$

As a consequence of (2.10), *if $(\phi^0,\phi^1,u,f)\in\mathbb{H}_c$, then*

$$A_c(\phi^0,\phi^1,u,f)(t)=\begin{cases} f'(t)-p^0(1-t), & 0<t<1,\\ f'(t)-2u'(t-1)+q^0(t-1), & 1<t<T-1,\\ -f'(t), & T-1<t<T,\end{cases} \tag{2.15}$$

where $p^0=\phi^1-\phi^0_x$ and $q^0=\phi^1+\phi^0_x$.

Proof. Taking into account the expressions of $p=\phi'-\phi_x$ and $q=\phi'+\phi_x$ derived in (2.11) and (2.12), we get:

$$\phi_x(t,1)=\frac{q(t,1)-p(t,1)}{2},$$

which leads to (2.14). The expression (2.15) is obtained using (2.10). □

Remark 2.2. Note that the expression of $A_c(\phi^0,\phi^1,u,f)$ in (2.15) involves only the part of u defined on $(0,T-2)$, i.e. the "free" part of u.

Remark 2.3. Clearly, $A_c\in\mathcal{L}(\mathbb{H}_c;L^2(0,T))$. If moreover

$$\left(u,f,\phi^0,\phi^1\right)\in H^2(0,T)\times H^2(0,T)\times H^2(0,1)\times H^1(0,1)$$

with the compatibility conditions

$$u'(0)=-\phi_x^0(0),\qquad f'(T-1)=u'(T-2)+\frac{1}{2}\left(\phi_x^0(T-2)+\phi^1(T-2)\right),$$

then $A_c(\phi^0,\phi^1,u,f)\in H^1(0,T)$. From (2.15) other regularity results can be easily derived.

3. A penalty method

We are going to study (1.1) by introducing the penalized problem:

$$\begin{cases} y_\epsilon''-y_{\epsilon,xx}=0, & (t,x)\in Q_T,\\ y_\epsilon(t,0)=u_\epsilon(t), & t\in(0,T),\\ y_{\epsilon,x}(t,1)=\epsilon^{-1}\left[y_\epsilon(t,1)-\psi(t)\right]^-, & t\in(0,T),\\ y_\epsilon(0,x)=y^0(x),\quad y_\epsilon'(0,x)=y^1(x), & x\in(0,1), \end{cases}\tag{3.16}$$

where $[\cdot]^-$ denotes the negative part so that $[y_\epsilon(t,1)-\psi(t)]^-=-\min\{0,y_\epsilon(t,1)-\psi(t)\}$ and $\epsilon>0$.

In the sequel we denote

$$p^0=y^1-y_x^0,\qquad q^0=y^1+y_x^0.\tag{3.17}$$

Let us first give a definition of the weak solutions of (3.16).

Definition 3.1. For any $(y^0,y^1)\in H^1(0,1)\times L^2(0,1)$ and $u_\epsilon\in H^1(0,T)$ with $u_\epsilon(0)=y^0(0)$, a **weak solution of (3.16)** is a function

$$y_\epsilon\in C\left([0,T],H^1(0,1)\right)\cap C^1\left([0,T],L^2(0,1)\right)$$

with the property that there exists $f_\epsilon\in H^1(0,T)$ such that

(1) $f_\epsilon(0)=y^0(1)$.
(2) y_ϵ verifies (2.6) with nonhomogeneous terms (u_ϵ,f_ϵ) and initial data (y^0,y^1).
(3) $A(y^0,y^1,u_\epsilon,f_\epsilon)=\epsilon^{-1}[f_\epsilon-\psi]^-$.

Note that relation (3) from Definition 3.1 represents a fixed point type condition ensuring that

$$y_{\epsilon,x}(t,1)=\epsilon^{-1}\left[f_\epsilon(t)-\psi(t)\right]^-=\epsilon^{-1}\left[y_\epsilon(t,1)-\psi(t)\right]^-,$$

i.e. exactly the boundary condition on $x=1$ from (3.16). We have the following result of existence and uniqueness of solutions.

Proposition 3.1. *Let $T\in(2,3)$. For any $(y^0,y^1)\in H^1(0,1)\times L^2(0,1)$ and $u_\epsilon\in H^1(0,T)$ with $u_\epsilon(0)=y^0(0)$, there exists a unique weak solution of* (3.16).

Proof. From Proposition 2.1, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$, $u_\epsilon \in H^1(0,T)$ with $u_\epsilon(0) = y^0(0)$ and $f_\epsilon \in H^1(0,T)$ with $f_\epsilon(0) = y^0(1)$ there exists a unique solution $y_\epsilon \in C([0,T], H^1(0,1)) \cap C^1([0,T], L^2(0,1))$ of (2.6). Hence, the problem is reduced to show the existence and uniqueness of $f_\epsilon \in H^1(0,T)$ with $f_\epsilon(0) = y^0(1)$ and such that $A(y^0, y^1, u_\epsilon, f_\epsilon) = \epsilon^{-1}[f_\epsilon - \psi]^-$. By taking into account (2.14) this is equivalent to prove that the following differential equation has a unique solution

$$\begin{cases} f_\epsilon'(t) = \begin{cases} \epsilon^{-1}[f_\epsilon(t) - \psi(t)]^- + p^0(1-t), & 0 < t < 1, \\ \epsilon^{-1}[f_\epsilon(t) - \psi(t)]^- + 2u_\epsilon'(t-1) - q^0(t-1), & 1 < t < 2, \\ \epsilon^{-1}[f_\epsilon(t) - \psi(t)]^- - 2f_\epsilon'(t-2) + 2u_\epsilon'(t-1) + p^0(3-t), & 2 < t < T, \end{cases} \\ f_\epsilon(0) = y^0(1). \end{cases} \tag{3.18}$$

In the interval $(0,2)$ the right-hand side of Eq. (3.18) is Lipschitz with respect to f_ϵ and the existence and uniqueness of solutions of (3.18) are a consequence of the classical results for the ordinary differential equations. In the interval $(2,T)$ the function $f_\epsilon'(t-2)$ is completely known, it belongs to $L^2(2,T)$ and may be seen as a nonhomogeneous term. The existence and uniqueness of solution in this interval follow as before. □

3.1. Controllability of the penalized problem

In this section we pass to study the controllability properties of (3.16).

Definition 3.2. Problem (3.16) is **null controllable in time *T*** if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$, there exists a control $u_\epsilon \in H^1(0,T)$ with $u_\epsilon(0) = y^0(0)$ such that the corresponding weak solution of (3.16) verifies $y_\epsilon(T) = y_\epsilon'(T) = 0$.

The following characterization of the controllability property of (3.16) is a direct consequence of Corollary 2.1.

Proposition 3.2. *Let $T \in (2,3)$ and $\psi \in H^1(0,T)$ with $\psi(T) \leqslant 0$. Problem* (3.16) *is null controllable in time T if and only if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with $y^0(1) \geqslant \psi(0)$, there exist a control $u_\epsilon \in H^1(0,T)$ and a function $f_\epsilon \in H^1(0,T)$ such that*

(1) $(y^0, y^1, u_\epsilon, f_\epsilon) \in \mathbb{H}_c$.
(2) *y_ϵ is the solution of* (2.6) *with nonhomogeneous terms (u_ϵ, f_ϵ) and initial data (y^0, y^1).*
(3) $A_c(y^0, y^1, u_\epsilon, f_\epsilon) = \epsilon^{-1}[f - \psi]^-$.

Proof. Indeed, if conditions (1)–(3) are fulfilled there exists a unique weak solution y_ϵ of (3.16) with initial data (y^0, y^1) and control u_ϵ. Since y_ϵ verifies (2.6), it follows from Corollary 2.1 that $(y^0, y^1, u_\epsilon, f_\epsilon) \in \mathbb{H}_c$ implies $y_\epsilon(T) = y_\epsilon'(T) = 0$ in $(0,1)$.

Reciprocally, if (3.16) is null controllable, from the existence of a weak solution of (3.16) we deduce the existence of $f_\epsilon \in H^1(0,T)$ such that $(y^0, y^1, u_\epsilon, f_\epsilon) \in \mathbb{H}$ and conditions (2)–(3) are verified. Since $y_\epsilon(T) = y_\epsilon'(T) = 0$ in $(0,1)$ and y_ϵ verifies (2.6), from Corollary 2.1, we deduce that $(y^0, y^1, u_\epsilon, f_\epsilon) \in \mathbb{H}_c$. □

Using (2.15), the controllability of (3.16) becomes equivalent to the following nonlinear control problem: given $T \in (2,3)$,

$$\begin{cases} \texttt{find } f_\epsilon \in H^1(0,T) \texttt{ with } f_\epsilon(T) = 0 \texttt{ and } u_\epsilon \in H^1(0,T-2) \texttt{ with } u_\epsilon(0) = y^0(0) \texttt{ such that} \\ f_\epsilon'(t) = G\big(t, f_\epsilon(t), u_\epsilon(t)\big), \quad t \in (0,T), \\ f_\epsilon(0) = y^0(1), \end{cases} \tag{3.19}$$

where

$$G\big(t, f_\epsilon(t), u_\epsilon(t)\big) = \begin{cases} \frac{1}{\epsilon}[f_\epsilon(t) - \psi(t)]^- + p^0(1-t), & t \in (0,1), \\ \frac{1}{\epsilon}[f_\epsilon(t) - \psi(t)]^- + 2u_\epsilon'(t-1) - q^0(t-1), & t \in (1, T-1), \\ -\frac{1}{\epsilon}[f_\epsilon(t) - \psi(t)]^-, & t \in (T-1, T). \end{cases} \tag{3.20}$$

The study of the null controllability property of (3.16) will be carried out in two steps:

(1) In a first step, we will prove that, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ and $\psi \in H^1(0,T)$, satisfying (1.3), there exists a family of controls $u_\epsilon \in H^1(0,T)$ such that the solution of (3.16) satisfies $y_\epsilon(T) = y'_\epsilon(T) = 0$ on $(0,1)$. At this level, we again make use of the Dirichlet-to-Neumann map and the characterization (3.19) of the controllability property.
(2) In a second step, we provide some estimates on u_ϵ and y_ϵ which will allow to pass to the limit in ϵ in order to obtain a solution of (1.1) satisfying $y(T) = y'(T) = 0$ on $(0,1)$.

3.2. Existence of solutions of the controlled penalized problem

Lemma 3.1. *For any $\epsilon > 0$, problem* (3.19) *admits an infinite number of solutions (f_ϵ, u_ϵ).*

Proof. On $[0,1]$, let $l_\epsilon \in H^1(0,1)$ be the unique solution of

$$\begin{cases} f'_\epsilon(t) = \epsilon^{-1}\big[f_\epsilon(t) - \psi(t)\big]^- + p^0(1-t), & t \in (0,1), \\ f_\epsilon(0) = y^0(1) \end{cases} \tag{3.21}$$

and set $f_{1,\epsilon} = l_\epsilon(1)$. Similarly, on $(T-1,T)$, let r_ϵ be the unique solution of the backward problem

$$\begin{cases} f'_\epsilon(t) = -\epsilon^{-1}\big[f_\epsilon(t) - \psi(t)\big]^-, & t \in (T-1,T), \\ f_\epsilon(T) = 0 \end{cases} \tag{3.22}$$

and set $f_{T-1,\epsilon} = r_\epsilon(T-1)$.

We then consider the following nonlinear control problem:

$$\begin{cases} \texttt{find}\ u_\epsilon \in H^1(0,T-2)\ \texttt{such that}\ u_\epsilon(0) = y^0(0)\ \texttt{and}\ f_\epsilon \in H^1(1,T-1)\ \texttt{such that} \\ f'_\epsilon(t) = \epsilon^{-1}\big[f_\epsilon(t) - \psi(t)\big]^- + 2u'_\epsilon(t-1) - q^0(t-1), & t \in (1,T-1), \\ f_\epsilon(1) = f_{1,\epsilon}, \quad f_\epsilon(T-1) = f_{T-1,\epsilon} \end{cases} \tag{3.23}$$

which consists to find a control u_ϵ steering the solution of the differential equation

$$f'_\epsilon(t) = \frac{1}{\epsilon}\big[f_\epsilon(t) - \psi(t)\big]^- + 2u'_\epsilon(t-1) - q^0(t-1), \quad t \in (1,T-1),$$

from the initial data $f_{1,\epsilon}$ to the final data $f_{T-1,\epsilon}$. We proceed as follows: we first consider the linear control problem

$$\begin{cases} \texttt{find}\ v_\epsilon \in H^1(0,T-2)\ \texttt{and}\ \theta_\epsilon \in H^1(1,T-1)\ \texttt{such that} \\ \theta'_\epsilon(t) = 2v'_\epsilon(t-1), & t \in (1,T-1), \\ \theta_\epsilon(1) = f_{1,\epsilon}, \quad \theta_\epsilon(T-1) = f_{T-1,\epsilon}. \end{cases} \tag{3.24}$$

Clearly, any $(v_\epsilon, \theta_\epsilon) \in H^1(0,T-2) \times H^1(0,T-1)$ satisfying

$$\begin{cases} v_\epsilon(T-2) = \dfrac{1}{2}(f_{T-1,\epsilon} - f_{1,\epsilon}), \\ v_\epsilon(0) = 0, \\ \theta_\epsilon(t) = 2v_\epsilon(t-1) + f_{1,\epsilon} \end{cases} \tag{3.25}$$

is a solution of (3.24). Now let us choose

$$\begin{cases} \theta_\epsilon(t) = 2v_\epsilon(t-1) + f_{1,\epsilon}, \\ 2u'_\epsilon(t-1) = 2v'_\epsilon(t-1) - \dfrac{1}{\epsilon}\big[2v_\epsilon(t-1) + f_{1,\epsilon} - \psi(t)\big]^- + q^0(t-1), & t \in (1,T-1), \\ u_\epsilon(0) = y^0(0), \end{cases} \tag{3.26}$$

where v_ϵ is any function from $H^1(0,T-2)$ such that $v_\epsilon(T-2) = \frac{1}{2}(f_{T-1,\epsilon} - f_{1,\epsilon})$ and $v_\epsilon(0) = 0$.

It is straightforward that the couple $(\theta_\epsilon, u_\epsilon)$ defined by formulas (3.26) satisfies (3.19). Thus, a family of solutions (f_ϵ, u_ϵ) to problem (3.20) is constructed if we take

$$f_\epsilon = \begin{cases} l_\epsilon, & [0,1], \\ \theta_\epsilon, & [1, T-1], \\ r_\epsilon, & [T-1, T], \end{cases} \tag{3.27}$$

where l_ϵ, r_ϵ and $(\theta_\epsilon, u_\epsilon)$ are given by (3.21), (3.22) and (3.26) respectively. □

3.3. Boundedness of solutions of the controlled penalized problem

Now, we prove that the sequence (u_ϵ, f_ϵ) may be chosen uniformly bounded with respect to ϵ in $H^1(0,T)$. From now on, C denotes a strictly positive constant that may vary from line to line but is independent on ϵ.

Lemma 3.2. *There exists a constant $C = C(y^0, y^1, \psi) > 0$ such that for any $\epsilon > 0$ the function f_ϵ given by* (3.27) *satisfies the estimate*:

$$\big(f_\epsilon(t)\big)^2 \leqslant C, \quad t \in [0,1] \cup [T-1, T]. \tag{3.28}$$

If moreover $y^0(1) - \psi(0) \geqslant 0$, then:

$$\frac{1}{\epsilon}\big(\big[f_\epsilon(t) - \psi(t)\big]^-\big)^2 \leqslant C, \quad t \in [0,1] \cup [T-1, T], \tag{3.29}$$

$$\int_0^t \big(f_\epsilon'\big)^2(s)\,ds \leqslant C, \quad t \in [0,1], \tag{3.30}$$

$$\int_t^T \big(f_\epsilon'\big)^2(s)\,ds \leqslant C, \quad t \in [T-1, T]. \tag{3.31}$$

Proof. We set $h_\epsilon = f_\epsilon - \psi$ so that problem (3.21) can be written as

$$\begin{cases} h_\epsilon'(t) = \dfrac{1}{\epsilon} h_\epsilon^-(t) + p^0(1-t) - \psi'(t), & t \in (0,1), \\ h_\epsilon(0) = y^0(1) - \psi(0). \end{cases} \tag{3.32}$$

Multiplying this equation by h_ϵ and integrating over $(0,t)$ for $t < 1$, we get

$$\begin{aligned} h_\epsilon^2(t) &= h_\epsilon^2(0) - \frac{2}{\epsilon}\int_0^t \big[h_\epsilon^-(s)\big]^2\,ds + 2\int_0^t \big(p^0(1-s) - \psi'(s)\big)h_\epsilon(s)\,ds \\ &\leqslant \big(h_\epsilon^2(0) + \|p^0\|^2_{L^2(0,1)} + \|\psi'\|^2_{L^2(0,1)}\big) + \int_0^t h_\epsilon^2(s)\,ds. \end{aligned}$$

From Gronwall's lemma, we deduce that

$$h_\epsilon^2(t) \leqslant C\big(\big(y^0(1) - \psi(0)\big)^2 + \|p^0\|^2_{L^2(0,1)} + \|\psi'\|^2_{L^2(0,1)}\big).$$

Thus

$$f_\epsilon^2(t) \leqslant C\big(\big(y^0(1) - \psi(0)\big)^2 + \|p^0\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\big), \quad t \in (0,1). \tag{3.33}$$

Similarly, problem (3.22) can be written as

$$\begin{cases} h_\epsilon'(t) = -\dfrac{1}{\epsilon} h^-(t) - \psi'(t), & t \in (T-1, T), \\ h_\epsilon(T) = -\psi(T). \end{cases}$$

Multiplying as previously this equation by h_ϵ and integrating over (t,T) for $t\in(T-1,T)$, the same arguments lead to the estimate

$$h_\epsilon^2(t)\leqslant C\big(\psi^2(T)+\|\psi'\|^2_{L^2(T-1,T)}\big).$$

This implies

$$f_\epsilon^2(t)\leqslant C\big(\psi^2(T)+\|\psi\|^2_{H^1(T-1,T)}\big),\quad t\in(T-1,T). \tag{3.34}$$

Estimates (3.33) and (3.34) prove the first part of the lemma.

We now multiply the equation of h_ϵ in (3.32) by h'_ϵ and integrating over $(0,t)$ with $t\in(0,1)$, we get

$$\int_0^t \big(h'_\epsilon(s)\big)^2\,ds+\frac{1}{\epsilon}\big(h_\epsilon^-(t)\big)^2=\frac{1}{2\epsilon}\big(h_\epsilon^-(0)\big)^2+\int_0^t\big(p^0(1-s)-\psi'(s)\big)h'_\epsilon(s)\,ds.$$

If we assume that $h_\epsilon(0)=y^0(1)-\psi(0)\geqslant 0$, then $h_\epsilon^-(0)=0$ and Cauchy–Schwarz inequality imply

$$\int_0^t \big(h'_\epsilon(s)\big)^2\,ds+\frac{1}{\epsilon}\big(h_\epsilon^-(t)\big)^2\leqslant\int_0^t\big(p^0(1-s)-\psi'(s)\big)^2\,ds,\quad t\in(0,1).$$

From this last inequality, it follows that

$$\frac12\int_0^t\big(f'_\epsilon(s)\big)^2\,ds+\frac{1}{\epsilon}\big(\big[f_\epsilon(t)-\psi(t)\big]^-\big)^2\leqslant\int_0^t\big[\psi'(s)\big]^2\,ds+\int_0^t\big(p^0(1-s)-\psi'(s)\big)^2\,ds,\quad t\in(0,1).$$

With the same argument, we get on $(T-1,T)$:

$$\int_t^T\big(f'_\epsilon(s)\big)^2\,ds+\frac{1}{\epsilon}\big(\big[f_\epsilon(t)-\psi(t)\big]^-\big)^2\leqslant C\|\psi\|^2_{H^1(T-1,T)},\quad t\in(T-1,T).$$

This ends the proof. □

Remark 3.1. From (3.30) and (3.21) (resp. (3.31) and (3.22)) it can be deduced that

$$\frac{1}{\epsilon^2}\int_0^t\big(\big[f_\epsilon(s)-\psi(s)\big]^-\big)^2\,ds\leqslant C,\quad t\in[0,1]$$

(resp.

$$\frac{1}{\epsilon^2}\int_t^T\big(\big[f_\epsilon(s)-\psi(s)\big]^-\big)^2\,ds\leqslant C,\quad t\in[T-1,T]).$$

Note that, so far the only conditions imposed to f_ϵ in $[1,T-1]$ are $f_\epsilon\in H^1(1,T-1)$, $f_\epsilon(1)=f_{1,\epsilon}$ and $f_\epsilon(T-1)=f_{T-1,\epsilon}$. The next step is to prove that f_ϵ may be chosen such that the estimates (3.28)–(3.31) hold true on $(1,T-1)$ too.

Lemma 3.3. *There exists a sequence $(f_\epsilon)_{\epsilon>0}\subset H^1(1,T-1)$ such that $f_\epsilon(1)=f_{1,\epsilon}$, $f_\epsilon(T-1)=f_{T-1,\epsilon}$ and*

$$\|f_\epsilon\|_{H^1(1,T-1)}\leqslant C,\qquad \frac{1}{\epsilon^2}\int_1^{T-1}\big(\big[f_\epsilon(t)-\psi(t)\big]^-\big)^2\,dt\leqslant C,\qquad \|u_\epsilon\|_{H^1(0,T-2)}\leqslant C,$$

where u_ϵ is the solution of (3.26).

Proof. Fix $\epsilon > 0$ sufficiently small. The idea behind the following construction of f_ϵ is to take $f_\epsilon(t) \geqslant \psi(t)$ in order to have $[f_\epsilon(t) - \psi(t)]^- = 0$ for all $t \in (1, T-1)$. However, this is not possible if $f_{1,\epsilon} < \psi(1)$ or $f_{T-1,\epsilon} < \psi(T-1)$. Taking into account (3.29), we can still keep bounded the integral $\frac{1}{\epsilon^2}\int_1^{T-1}([f_\epsilon(s) - \psi(s)]^-)^2\,ds$ if $f_\epsilon(t) < \psi(t)$ in a neighborhood of length ϵ of 1 and $T-1$. Thus, if $f_{1,\epsilon} < \psi(1)$ we have to construct a function f_ϵ in $[1, 1+\epsilon]$ such that

$$f_\epsilon(1) = f_{1,\epsilon}, \qquad f_\epsilon(1+\epsilon) = \psi(1+\epsilon), \qquad \frac{1}{\epsilon^2}\int_1^{1+\epsilon}\left(\left[f_\epsilon(s) - \psi(s)\right]^-\right)^2 ds \leqslant C.$$

Analogously, if $f_{T-1,\epsilon} < \psi(T-1)$ we have to find a function f_ϵ in $[T-1-\epsilon, T-1]$ such that

$$f_\epsilon(T-1) = f_{T-1,\epsilon}, \qquad f_\epsilon(T-1-\epsilon) = \psi(T-1-\epsilon), \qquad \frac{1}{\epsilon^2}\int_{T-1-\epsilon}^{T-1}\left(\left[f_\epsilon(s) - \psi(s)\right]^-\right)^2 ds \leqslant C.$$

In both cases, $f_\epsilon - \psi$ will be an interpolation polynomial of degree one in t. More precisely:

- If $\psi(1) - f_{1,\epsilon} < 0$ and $\psi(T-1) - f_{T-1,\epsilon} < 0$, we choose any function $f_\epsilon \in H^1(1, T-1)$ satisfying $f_\epsilon(1) = f_{1,\epsilon}$ and $f_\epsilon(T-1) = f_{T-1,\epsilon}$. For instance:

$$f_\epsilon(t) = \max\left\{\frac{f_{T-1,\epsilon} - f_{1,\epsilon}}{T-2}(t-1) + f_{1,\epsilon}, \psi(t)\right\}, \quad t \in (1, T-1).$$

- If $\psi(1) - f_{1,\epsilon} \geqslant 0$ and $\psi(T-1) - f_{T-1,\epsilon} \geqslant 0$,

$$f_\epsilon(t) = \begin{cases} \psi(t) + (\psi(1) - f_{1,\epsilon})(\frac{1}{\epsilon}(t-1) - 1), & t \in [1, 1+\epsilon], \\ \psi(t), & t \in [1+\epsilon, T-1-\epsilon], \\ \psi(t) + (\psi(T-1) - f_{T-1,\epsilon})(-1 + \frac{T-1-t}{\epsilon}), & t \in [T-1-\epsilon, T-1]. \end{cases} \tag{3.35}$$

- If $\psi(1) - f_{1,\epsilon} \geqslant 0$ and $\psi(T-1) - f_{T-1,\epsilon} < 0$,

$$f_\epsilon(t) = \begin{cases} \psi(t) + (\psi(1) - f_{1,\epsilon})(\frac{1}{\epsilon}(t-1) - 1), & t \in [1, 1+\epsilon], \\ \psi(t), & t \in [1+\epsilon, \frac{T}{2}], \\ \max\{\frac{f_{T-1,\epsilon} - \psi(\frac{T}{2})}{\frac{T}{2}-1}(t - \frac{T}{2}) + \psi(\frac{T}{2}), \psi(t)\}, & t \in [\frac{T}{2}, T-1]. \end{cases}$$

- If $\psi(1) - f_{1,\epsilon} < 0$ and $\psi(T-1) - f_{T-1,\epsilon} \geqslant 0$,

$$f_\epsilon(t) = \begin{cases} \max\{\frac{\psi(\frac{T}{2}) - f_{1,\epsilon}}{\frac{T}{2}-1}(t-1) + f_{1,\epsilon}, \psi(t)\}, & t \in [1, \frac{T}{2}], \\ \psi(t), & t \in [\frac{T}{2}, T-1-\epsilon], \\ \psi(t) + (\psi(T-1) - f_{T-1,\epsilon})(-1 + \frac{T-1-t}{\epsilon}), & t \in [T-1-\epsilon, T-1]. \end{cases}$$

By construction $f_\epsilon \in H^1(1, T-1)$ and satisfies $f_\epsilon(1) = f_{1,\epsilon}$, $f_\epsilon(T-1) = f_{T-1,\epsilon}$ in all cases. Moreover, note that from (3.26) one has

$$2u_\epsilon'(t-1) = f_\epsilon'(t) - \frac{1}{\epsilon}\left[f_\epsilon(t) - \psi(t)\right]^- + q^0(t-1), \quad t \in (1, T-1), \qquad u_\epsilon(0) = y^0(0).$$

Consequently, the uniform boundedness of u_ϵ follows from the uniform estimates with respect to ϵ for f_ϵ and $\frac{1}{\epsilon}[f_\epsilon - \psi]^-$.

For the first case, note that $f_\epsilon \geqslant \psi$ on $(1, T-1)$ and that from (3.28), $|f_{T-1,\epsilon}|$ and $|f_{1,\epsilon}|$ are uniformly bounded with respect to ϵ. It is straightforward that this implies uniform bounds with respect to ϵ for $\|f_\epsilon\|_{H^1(1,T-1)}$ and $\int_1^{T-1}(\frac{[f_\epsilon(t) - \psi(t)]^-}{\epsilon})^2\,dt = 0$.

Assume now that $\psi(1) - f_{1,\epsilon} \geqslant 0$. Then

$$\int_1^{1+\epsilon} |f_\epsilon(t)|^2\,dt = \int_1^{1+\epsilon} \left|\psi(t) + \big(\psi(1) - f_{1,\epsilon}\big)\left(\frac{1}{\epsilon}(t-1) - 1\right)\right|^2 dt$$
$$\leqslant C\left(\int_1^{1+\epsilon} |\psi(t)|^2\,dt + \big(\psi(1) - f_{1,\epsilon}\big)^2 \int_1^{1+\epsilon} \left|\left(\frac{1}{\epsilon}(t-1) - 1\right)\right|^2 dt\right)$$
$$\leqslant C\left(\int_1^{1+\epsilon} |\psi(t)|^2\,dt + \epsilon\big(\psi(1) - f_{1,\epsilon}\big)^2\right).$$

On the other hand

$$\int_1^{1+\epsilon} |f'_\epsilon(t)|^2\,dt = \int_1^{1+\epsilon} \left|\psi'(t) + \frac{1}{\epsilon}\big(\psi(1) - f_{1,\epsilon}\big)\right|^2 dt$$
$$\leqslant C\left(\int_1^{1+\epsilon} |\psi'(t)|^2\,dt + \frac{|(\psi(1) - f_{1,\epsilon})|^2}{\epsilon}\right).$$

These two last inequalities together with (3.29) give

$$\|f_\epsilon\|_{H^1(1,1+\epsilon)} \leqslant C.$$

To prove a similar estimate for u_ϵ on $(1, 1+\epsilon)$, we just need to estimate $\int_1^{1+\epsilon} (\frac{[f_\epsilon(t)-\psi(t)]^-}{\epsilon})^2\,dt$. But, from (3.35), we get

$$\int_1^{1+\epsilon} \left(\frac{[f(t) - \psi(t)]^-}{\epsilon}\right)^2 = \frac{(\psi(1) - f_{1,\epsilon})^2}{\epsilon^2} \int_1^{1+\epsilon} \left(1 - \frac{1}{\epsilon}(t-1)\right)^2 dt$$
$$\leqslant \frac{(\psi(1) - f_{1,\epsilon})^2}{\epsilon} \leqslant C$$

thanks again to (3.29). Thus

$$\int_1^{1+\epsilon} |u'_\epsilon(t)|^2\,dt \leqslant C.$$

The same arguments on $(T-1-\epsilon, T-1)$ with $\psi(T-1) - f_{T-1,\epsilon} \geqslant 0$ give the estimates

$$\|f_\epsilon\|_{H^1(T-1-\epsilon,T-1)} \leqslant C, \qquad \int_{T-1-\epsilon}^{T-1} |u'_\epsilon(t)|^2\,dt \leqslant C.$$

The other situations are easier to treat. This ends the proof of the lemma. □

As a summary, we have proved:

Corollary 3.1. *Let $T \in (2,3)$ and $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$, $\psi \in H^1(0,T)$ with $\psi(T) \leqslant 0$ and $y^0(1) - \psi(0) \geqslant 0$. Then problem* (3.19) *admits a sequence (u_ϵ, f_ϵ) of solutions such that*

$$f_\epsilon^2(t) \leqslant C, \quad t \in [0,T],$$
$$\|f_\epsilon\|_{H^1(0,T)} \leqslant C,$$
$$\|u_\epsilon\|_{H^1(0,T-2)} \leqslant C,$$
$$\int_0^T \left(\frac{[f_\epsilon(t) - \psi(t)]^-}{\epsilon}\right)^2 dt \leqslant C.$$

3.4. *Controllability of the obstacle problem*

The aim of this section is to prove Theorem 1.1. In fact we shall obtain a solution of (1.1) satisfying (1.2) by passing to the limit in the penalized problem (3.16). First of all, let us define the weak solutions of (1.1).

Definition 3.3. Given any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ and $u \in H^1(0,T)$ with $u(0) = y^0(0)$, a **weak solution of (1.1)** is a function

$$y \in C\big([0,T], H^1(0,1)\big) \cap C^1\big([0,T], L^2(0,1)\big)$$

with the property that there exists $f \in H^1(0,T)$ such that

(1) $f(0) = y^0(1)$.
(2) y is the solution of (2.6) with nonhomogeneous terms $(u, f) \in (H^1(0,T))^2$ and initial data (y^0, y^1).
(3) $f - \psi \geqslant 0$ in $(0,T)$.
(4) $A(y^0, y^1, u, f) \geqslant 0$ in $(0,T)$.
(5) $(f - \psi)A(y^0, y^1, u, f) = 0$ in $(0,T)$.

Also, we have the following definition.

Definition 3.4. Problem (1.1) is **null controllable in time *T*** if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with $y^0(1) \geqslant \psi(0)$, there exists a control $u \in H^1(0,T)$ with $u(0) = y^0(0)$ such that the corresponding weak solution of (1.1) verifies $y(T) = y'(T) = 0$.

The following characterization of the controllability property is a direct consequence of Corollary 2.1 and the definition of weak solutions of (1.1).

Proposition 3.3. *Problem* (1.1) *is null controllable in time T if and only if, for any $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ with $y^0(1) \geqslant \psi(0)$, there exist a control $u \in H^1(0,T)$ and a function $f \in H^1(0,T)$ such that*

(1) $(y^0, y^1, u, f) \in \mathbb{H}_c$.
(2) *y is the solution of* (2.6) *with nonhomogeneous terms $(u, f) \in (H^1(0,T))^2$ and initial data (y^0, y^1).*
(3) $f - \psi \geqslant 0$ *in* $(0,T)$.
(4) $A_c(y^0, y^1, u, f) \geqslant 0$ *in* $(0,T)$.
(5) $(f - \psi)A_c(y^0, y^1, u, f) = 0$ *in* $(0,T)$.

We can now pass to prove our main result.

Proof of Theorem 1.1. Let us first consider the case $T \in (2,3)$ and $\tilde{T} = T$. If (f_ϵ, u_ϵ) is the solution of (3.19) from Corollary 3.1, we may extract from the sequence (f_ϵ, u_ϵ) a subsequence, still denoted by (f_ϵ, u_ϵ), such that

$$\begin{aligned}
&\big(f_\epsilon(t), u_\epsilon(t)\big) \to \big(f(t), u(t)\big), \quad t \in [0, T],\\
&(f_\epsilon, u_\epsilon) \rightharpoonup (f, u) \quad \text{in } \big(H^1(0,T)\big)^2 \text{ weak},\\
&A_c\big(y^0, y^1, u_\epsilon, f_\epsilon\big) \rightharpoonup A_c\big(y^0, y^1, u, f\big) \quad \text{in } L^2(0,T) \text{ weak},\\
&\frac{[f_\epsilon - \psi]^-}{\epsilon} \rightharpoonup \mu \quad \text{in } L^2(0,T) \text{ weak}
\end{aligned}$$

with (f, u) satisfying (2.10) on $(T-2, T)$.

Since $A_c(y^0, y^1, u_\epsilon, f_\epsilon) = \epsilon^{-1}[f(t) - \psi(t)]^-$, it follows that

$$A_c\big(y^0, y^1, u, f\big) = \mu \geqslant 0. \tag{3.36}$$

On the other hand, since

$$\big[f_\epsilon(t) - \psi(t)\big]^- \to 0, \quad t \in [0, T],$$

it follows that

$$f - \psi \geqslant 0 \quad \text{on } [0, T]. \tag{3.37}$$

We now prove that $(f - \psi)A_c(y^0, y^1, u, f) = 0$ on $(0, T)$. We have that:

$$(f_\epsilon - \psi)A_c\big(y^0, y^1, u_\epsilon, f_\epsilon\big) = (f_\epsilon - \psi)\frac{[f_\epsilon - \psi]^-}{\epsilon} = -\frac{([f_\epsilon - \psi]^-)^2}{\epsilon}.$$

Using Corollary 3.1 we get

$$\frac{[f_\epsilon - \psi]^-}{\sqrt{\epsilon}} \to 0 \quad \text{in } L^2(0,T).$$

On the other hand,

$$\begin{aligned}
&f_\epsilon - \psi \to f - \psi \quad \text{in } L^2(0,T),\\
&A_c\big(y^0, y^1, u_\epsilon, f_\epsilon\big) \rightharpoonup A_c\big(y^0, y^1 u, f\big) \quad \text{in } L^2(0,T) \text{ weak}.
\end{aligned}$$

Thus

$$(f_\epsilon - \psi)A_c\big(y^0, y^1, u_\epsilon, f_\epsilon\big) \rightharpoonup (f - \psi)A_c\big(y^0, y^1, u, f\big) \quad \text{in } L^2(0,T) \text{ weak}$$

and

$$(f - \psi)A_c\big(y^0, y^1 u, f\big) = 0 \quad \text{in } (0, T).$$

It remains to prove that the solution y_ϵ of (2.6) associated with the data $(y^0, y^1, u_\epsilon, f_\epsilon)$ converges to the solution y of (2.6) associated with the data (y^0, y^1, u, f). By linearity, $y - y_\epsilon$ is the solution of (2.6) associated with $(0, 0, u - u_\epsilon, f - f_\epsilon)$. Thus by the first part of Proposition 2.1, we get that

$$\lim_{\epsilon \to 0} \big\| \big(y - y_\epsilon, y' - y'_\epsilon\big) \big\|_{L^2 \times H^{-1}(0,T)} = 0.$$

Thus y is the limit of the sequence $(y_\epsilon)_{\epsilon > 0}$ and consequently verifies $y(T) = y'(T) = 0$ in $(0, 1)$. The proof of the theorem in this case is finished.

If $T \geqslant 3$ or $\tilde{T} < T < 3$, we work in the interval $[0, \tilde{T}]$ where the hypotheses of the previous case are verified. Consequently, we may find a control $\tilde{u} \in H^1(0, \tilde{T})$ of (1.1) in $[0, \tilde{T}]$. If we extend $\tilde{u}$ by zero in $[\tilde{T}, T]$, we obtain a control $u \in H^1(0, T)$ for (1.1) in $[0, T]$. The proof of the theorem is now complete. □

4. A direct solution for the control problem

We now proceed to give a direct proof of Theorem 1.1. Let $T \in (2,3)$, $(y^0, y^1) \in H^1(0,1) \times L^2(0,T)$, $\Psi \in H^1(0,T)$ with the conditions

$$\psi(0) \leqslant y^0(1), \qquad \psi(T) \leqslant 0.$$

Given (y^0, y^1), we know from Proposition 3.3 that the solution of (1.1) is controlled if and only if we can find (u,f) such that $(y^0, y^1, u, f) \in \mathbb{H}_c$ and solves the problem

$$\begin{cases} f - \psi \geqslant 0, & (0,T), \\ A_c\big(y^0, y^1, u, f\big) \geqslant 0, & (0,T), \\ (f-\psi) A_c\big(y^0, y^1, u, f\big) = 0, & (0,T). \end{cases} \tag{4.38}$$

Taking into account (2.15), problem (4.38) is decomposed into two parts.

- On $(0, T-1)$, problem (4.38) can be written as

$$\begin{cases} f - \psi \geqslant 0, \\ f' - v \geqslant 0, \\ (f-\psi)\big(f' - v\big) = 0, \\ f(0) = y^0(1), \end{cases} \quad (0, T-1), \tag{4.39}$$

  where

$$v(t) = \begin{cases} p^0(1-t), & 0 < t < 1, \\ 2u'(t-1) - q^0(t-1), & 1 < t < T-1. \end{cases} \tag{4.40}$$

- On $(T-1, T)$:

$$\begin{cases} f - \psi \geqslant 0, \\ f' \leqslant 0, \\ (f-\psi) f' = 0, \\ f(T) = 0, \end{cases} \quad (T-1, T). \tag{4.41}$$

We solve separately the problems (4.39) and (4.41) using the following result which is for instance a consequence of [1]:

Lemma 4.1. *Let $h \in H^1(0,T)$ and $\theta_0 \geqslant h(0)$. Then the function*

$$\theta(t) = \max\Big(\theta_0, \sup_{0 \leqslant s \leqslant t} h(s)\Big), \quad t \in [0, T[,$$

belongs to $H^1(0,T)$ and is the unique solution of the problem

$$\begin{cases} \theta \geqslant h & \text{in } (0,T), \\ \theta' \geqslant 0 & \text{in } (0,T), \\ \theta'(\theta - h) = 0 & \text{in } (0,T), \\ \theta(0) = \theta_0. \end{cases} \tag{4.42}$$

Using this lemma and the notation $[f]^+ = \max(0, f)$, we get:

Proposition 4.1. *Let $v \in L^2(0, T-1)$ defined by* (4.40) *and $V(t) = \int_0^t v(s)\,ds$. Then the unique solution of* (4.39) *is given by*

$$f(t) = V(t) + \max\Big(y^0(1), \sup_{0 \leqslant s \leqslant t} \big(\psi(s) - V(s)\big)\Big), \quad t \in (0, T-1). \tag{4.43}$$

The unique solution of (4.41) *is given by*

$$f(t)=\Big[\sup_{t\leqslant s\leqslant T}\psi(s)\Big]^{+},\quad t\in(T-1,T). \tag{4.44}$$

Proof. In $(0,T-1)$, let us set

$$V(t)=\int_0^t v(s)\,ds,\qquad \theta(t)=f(t)-V(t),\qquad h(t)=\psi(t)-V(t) \tag{4.45}$$

so that system (4.39) transforms into (4.42) with $\theta_0=y^0(1)$. From Lemma 4.1, it follows that the unique solution of (4.39) in $H^1(0,T-1)$ is given by

$$f(t)=V(t)+\max\Big(y^0(1),\sup_{0\leqslant s\leqslant t}\big(\psi(s)-V(s)\big)\Big),\quad t\in(0,T-1).$$

Similarly, in $(T-1,T)$, let us set $\delta(t)=f(T-t)$ and $g(t)=\psi(T-t)$ for $t\in(0,1)$ so that (4.41) transforms into the following system:

$$\begin{cases}\delta\geqslant g & \text{in }(0,1),\\ \delta'\geqslant 0 & \text{in }(0,1),\\ \delta'(\delta-g)=0 & \text{in }(0,1),\end{cases}$$

which (again as a consequence of Lemma 4.1), since by assumption $\delta(0)=f(T)=0$ and $g(0)=\psi(T)\leqslant 0$, has a unique solution in $H^1(0,1)$ given by

$$\delta(t)=\max\Big(\delta(0),\sup_{0\leqslant s\leqslant t}g(s)\Big).$$

In other words,

$$f(T-t)=\max\Big(0,\sup_{0\leqslant s\leqslant t}\psi(T-s)\Big),\quad 0<t<1,$$

or equivalently (4.44). □

Proposition 4.2. *There exists u such that the function f given by* (4.43) *and* (4.44) *belongs to $H^1(0,T)$.*

Proof. To get a function $f\in H^1(0,T)$, we have to ensure the continuity of f at $t=T-1$:

$$\lim_{t\to(T-1)^-}f(t)=\lim_{t\to(T-1)^+}f(t). \tag{4.46}$$

But from

$$\lim_{t\to(T-1)^-}f(t)=V(T-1)+\max\Big(y^0(1),\sup_{0\leqslant s\leqslant T-1}\big(\psi(s)-V(s)\big)\Big)$$

and

$$\lim_{t\to(T-1)^+}f(t)=\Big[\sup_{T-1\leqslant s\leqslant T}\psi(s)\Big]^{+},$$

we are led to solve the following problem: find $u\in H^1(0,T-2)$ such that

$$V(T-1)+\max\Big(y^0(1),\sup_{0\leqslant s\leqslant T-1}\big(\psi(s)-V(s)\big)\Big)=\Big[\sup_{T-1\leqslant s\leqslant T}\psi(s)\Big]^{+}. \tag{4.47}$$

Note that the number $\Lambda=[\sup_{T-1\leqslant s\leqslant T}\psi(s)]^+$ does not depend on u and that from (4.40)

$$V(t)=\begin{cases}\int_{1-t}^1 p^0(s)\,ds, & 0\leqslant t\leqslant 1,\\ \int_0^1 p^0(s)\,ds+2(u(t-1)-u(0))-\int_0^{t-1}q^0(s)\,ds, & 1\leqslant t\leqslant T-1,\end{cases}$$

and in particular

$$V(T-1)=2\big(u(T-2)-u(0)\big)+\int_0^1 p^0(s)\,ds-\int_0^{T-2} q^0(s)\,ds.$$

Let $A=\max(y^0(1),\sup_{0\leqslant s\leqslant 1}(\psi(s)-V(s)))$ and look for a control u such that $\sup_{1\leqslant s\leqslant T-1}(\psi(s)-V(s))\geqslant A$, i.e. such that, for all $s\in(1,T-1)$,

$$2\big(u(s-1)-u(0)\big)\leqslant\psi(s)-\int_0^1 p^0(y)\,dy-A+\int_0^{s-1} q^0(y)\,dy\equiv g(s). \tag{4.48}$$

We check that $g(1)\leqslant 0$ from the definition of A and Λ. The continuity condition (4.47) then becomes

$$2\big(u(T-2)-u(0)\big)=\Lambda-\int_0^1 p^0(y)\,dy-A+\int_0^{T-2} q^0(y)\,dy\equiv B,$$

compatible with (4.48) since we compute $B-g(T-1)\geqslant 0$. We then choose $u(s-1)$ and $u(0)$ such that $2(u(s-1)-u(0))=g(s)+G(s)$ where $G(s)$ is a corrector function – linear positive – with $g(1)+G(1)=0$ and $g(T-1)+G(T-1)=B$. From the condition $u(0)=y^0(0)$, this permits to fix the control u in $(0,T-2)$ as follows:

$$u(s)=y^0(0)+\frac{1}{2}\big(g(s+1)+G(s+1)\big),\quad 0\leqslant s\leqslant T-2. \tag{4.49}$$

This ends the proof. □

Propositions 4.1 and 4.2 then prove Theorem 1.1.

5. Numerical illustrations

We illustrate our controllability results with some simple applications corresponding to the numerical value $T=2.2$ and the initial data

$$\big(y^0(x),y^1(x)\big)=\left(x\left(1-\frac{x}{2}\right),-3x\right),\quad x\in(0,1), \tag{5.50}$$

which ensure that the string touches the obstacle at the right extremity $x=1$ for some $t\in(0,T)$. We consider the constant case $\psi(t)=L\leqslant 0$ and the time dependent case with $\psi(t)=\sin(n\pi t/T)/5$ for some $n\in\mathbb{N}$.

5.1. The penalty method

If ϵ, T, y^0, y^1 and the obstacle function ψ are given, the numerical process associated with the penalized approach is as follows: the function $f(t)=y_\epsilon(t,1)$ is first computed on $(0,T)$ by solving the nonlinear ordinary differential equations (3.21) and (3.22) using the explicit Euler scheme. On $[1,T-1]$ we construct f_ϵ as in Lemma 3.3. This allows us to find the control function $u_\epsilon=y_\epsilon(t,0)$ on the interval $[0,T-2]$ from (3.26). In the rest of the time interval u is computed by solving system (2.10). Once the displacement y_ϵ is known at both extremities, the solution of the partial differential equation (3.16) on Q_T is finally obtained using a P_1 (finite element) approximation in space and the leapfrog scheme for the time derivative. In the case where the obstacle behavior is not known a priori, specific approximations are necessary (we refer to [3,12] where accurate and consistent schemes preserving the energy are proposed). The set $Q_T=(0,T)\times(0,1)$ is discretized with a uniform grid with $h=dt=1/1000$.

Figs. 1, 2 and Table 1 report some results obtained in the constant case $\psi(t)=-1/10$ on $(0,T)$ with $\epsilon=1/200$. As expected, the penalty approach ensures a small penetration of the obstacle. Thus, for some time, the quantity $y_\epsilon(1,t)-\psi(t)$ is strictly negative, but remains of order $-\epsilon$. We also check that the control u_ϵ remains uniformly bounded with respect to ϵ. Figs. 3, 4 and Table 2 reports similar results in the time dependent case $\psi(t)=\sin(2\pi t/T)/5$.

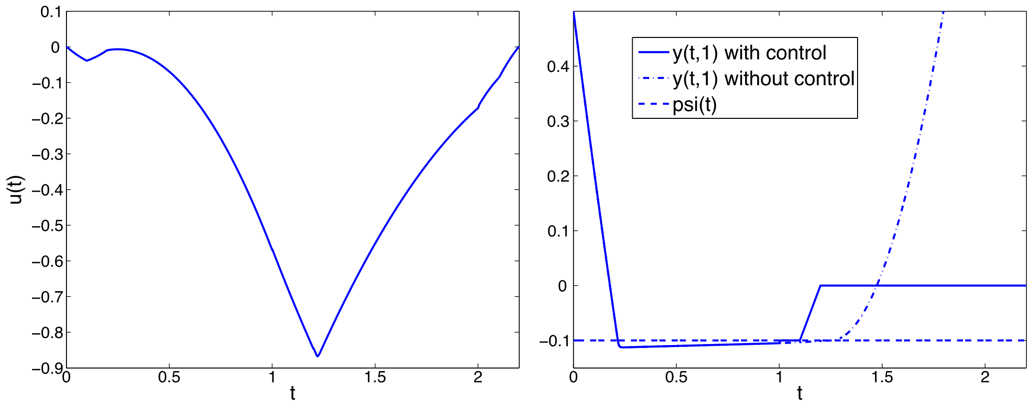

Fig. 1. Penalty method, $\epsilon = 1/200$, $\psi(t) = L = -1/10$. Evolution of the control u_ϵ (**left**) and corresponding displacement $y_\epsilon(\cdot, 1)$ (**right**) vs $t \in [0, T]$, $\|u_\epsilon\|_{L^2(0,T)} \approx 6.131 \times 10^{-1}$.

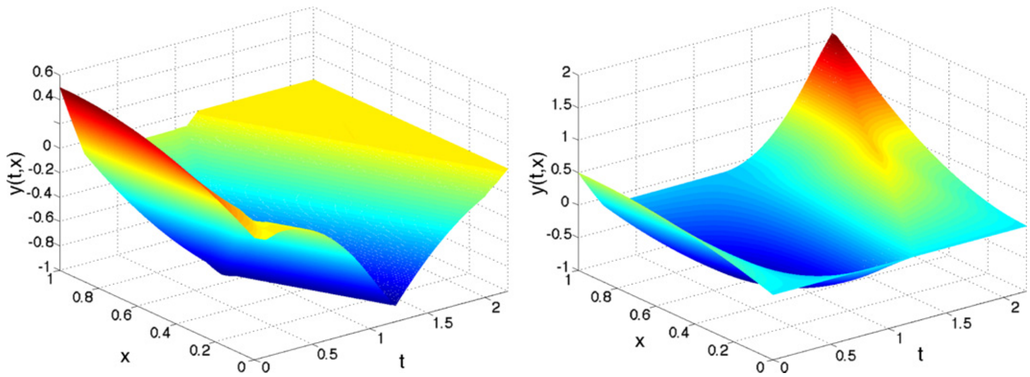

Fig. 2. Penalty method, $\epsilon = 1/200$, $\psi(t) = L = -1/10$. Evolution of y_ϵ on Q_T in the controlled (**left**) and uncontrolled case (**right**).

Table 1
Penalty approach, $\psi(t) = L = -1/10$.

	$\epsilon = 1/100$	$\epsilon = 1/200$	$\epsilon = 1/400$	$\epsilon = 1/800$
$\|u_\epsilon\|_{L^2(0,T)}$	6.175×10^{-1}	6.131×10^{-1}	6.108×10^{-1}	6.097×10^{-1}
$\|\epsilon^{-1}[y_\epsilon(\cdot,1) - \psi]^-\|_{L^2(0,T)}$	1.617	1.624	1.627	1.628
$\min_{t\in[0,T]}(y_\epsilon(t,1) - \psi(t))$	-2.47×10^{-2}	-1.25×10^{-2}	-6.34×10^{-3}	-3.19×10^{-3}

5.2. *Direct method*

For the direct method, the process is as follows: the control u is first computed on $(0, T-2)$ with the formula (4.49) which permits to compute the function v on $(0, T-1)$ defined by (4.40), then $V(s) = \int_0^s v(t)\,dt$ and finally the function $f(t) = y(t, 1)$ on $(0, T)$ with the formulas (4.43) and (4.44). The control u on $(T-2, T)$ is then given by (2.10). In the simple case $\psi(t) = L \in (-3/2, 0]$ in $[0, T]$, we obtain the following expressions. From (4.49), we deduce that

$$u(t) = -\frac{t}{2}\left(2t - 1 + \frac{L}{T-2}\right), \quad t \in (0, T-2), \tag{5.51}$$

leading to the function $V \in L^2(0, T-1)$ given by

$$V(t) = \begin{cases} t(-3+t), & 0 \leqslant t \leqslant 1, \\ \frac{4-2T-tL+L}{T-2}, & 1 \leqslant t \leqslant T-1, \end{cases} \tag{5.52}$$

and to the function f given by

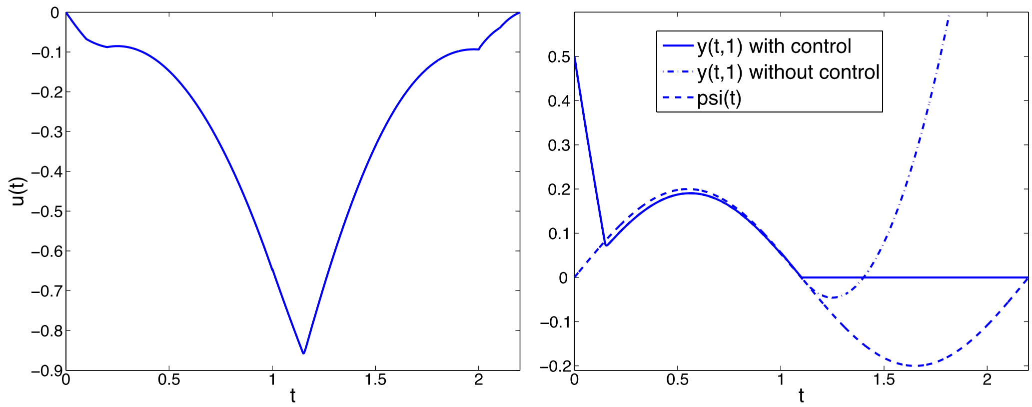

Fig. 3. Penalty method, $\epsilon = 1/200$, $\psi(t) = \sin(2\pi t/T)/5$. Evolution of the control u_ϵ (**left**) and corresponding displacement $y_\epsilon(\cdot, 1)$ (**right**) vs $t \in [0, T]$, $\|u_\epsilon\|_{L^2(0,T)} \approx 5.533 \times 10^{-1}$.

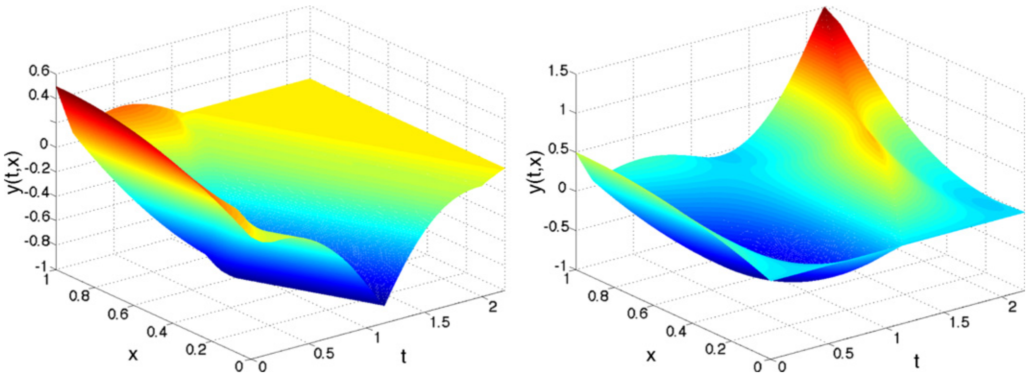

Fig. 4. Penalty method, $\epsilon = 1/200$, $\psi(t) = \sin(2\pi t/T)/5$. Evolution of y_ϵ on Q_T in the controlled (**left**) and uncontrolled case (**right**).

Table 2
Penalty approach, $\psi(t) = \sin(2\pi t/T)/5$.

	$\epsilon = 1/100$	$\epsilon = 1/200$	$\epsilon = 1/400$	$\epsilon = 1/800$
$\|u_\epsilon\|_{L^2(0,T)}$	5.586×10^{-1}	5.533×10^{-1}	5.506×10^{-1}	5.492×10^{-1}
$\|\epsilon^{-1}[y_\epsilon(\cdot,1) - \psi]^-\|_{L^2(0,T)}$	1.837	1.844	1.848	1.850
$\min_{t\in[0,T]}(y_\epsilon(t,1) - \psi(t))$	-3.09×10^{-2}	-1.57×10^{-2}	-7.97×10^{-3}	-4.01×10^{-3}

$$f(t) = \begin{cases} t(-3+t) + \frac{1}{2}, & 0 \leqslant t \leqslant t_L, \\ L, & t_L \leqslant t \leqslant 1, \\ \frac{L(-t+T-1)}{T-2}, & 1 \leqslant t \leqslant T-1, \\ 0, & T-1 \leqslant t \leqslant T, \end{cases} \tag{5.53}$$

with $t_L = (3 - \sqrt{7+4L})/2 \in (0,1)$. From (2.10), the function f then provides the control u in $(T-2, T)$

$$\begin{cases} u(t) = -\dfrac{L}{2} + \dfrac{t}{2} - t^2, & T-2 < t < 1, \\ u(t) = \dfrac{3}{2} - \dfrac{L}{2} + \dfrac{t^2}{2} - \dfrac{5t}{2}, & 1 < t < t_L + 1, \\ u(t) = -3 + \dfrac{L}{2} + \dfrac{5t}{2} - \dfrac{t^2}{2}, & t_L + 1 < t < 2, \\ u(t) = -\dfrac{1}{2}\dfrac{L(t-T)}{T-2}, & 2 < t < T, \end{cases} \tag{5.54}$$

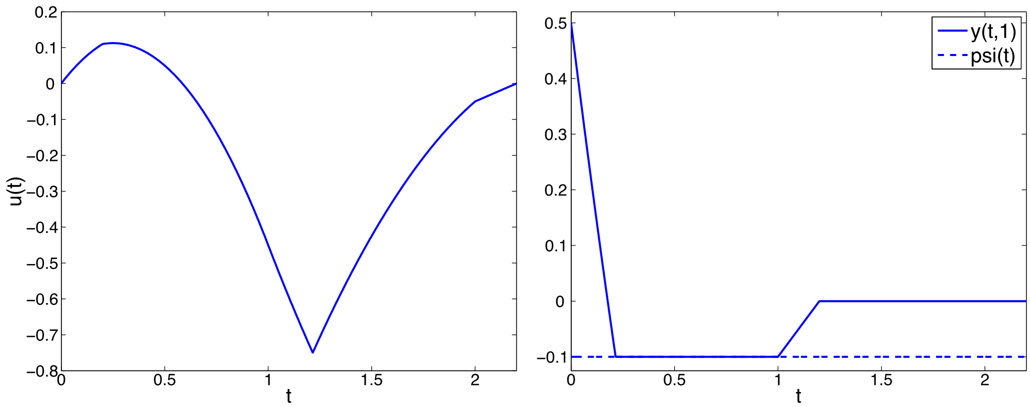

Fig. 5. $\psi(t) = L = -1/10$. Evolution of the control u (**left**) and corresponding displacement $y(\cdot, 1)$ (**right**) vs $t \in [0, T]$, $\|u\|_{L^2(0,T)} \approx 4.84 \times 10^{-1}$.

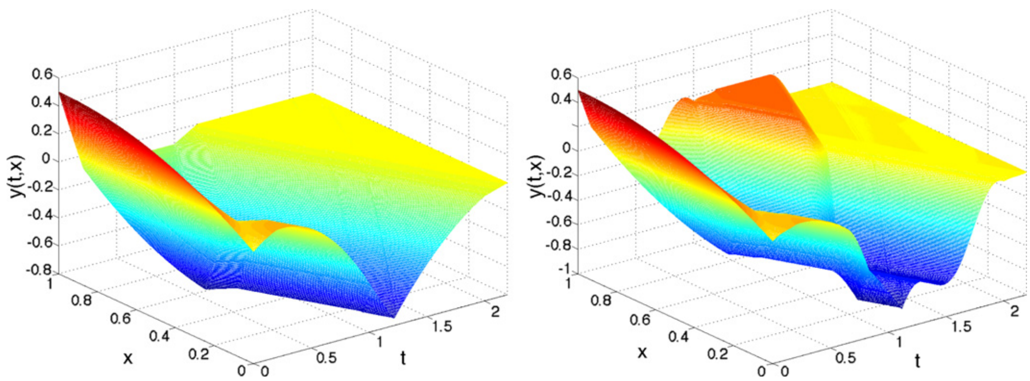

Fig. 6. Evolution of the controlled solution y in Q_T corresponding to $\psi(t) = -1/10$ (**left**) and $\psi(t) = \sin(6\pi t/T)/5$ (**right**).

assuming that $L \in (-3/2, 0)$ and $T \in (2, 3)$ are such that $t_L > T - 2$. The knowledge of $(y(t, 0), y(t, 1))$ in $(0, t)$ then permits to compute the entire solution y on Q_T by using the formulas given in Section 2. In practice, it is simpler to approximate y by a numerical discretization of the wave equation (1.1). Figs. 5 and 6 report the graph in the case $L = -1/10$. In particular, the set $\{t \in (0, T),\ f(t) = \psi(t) = L\}$ is reduced to one interval, corresponding to the contact period. Fig. 6(left) depicts the corresponding evolution of y on Q_T. The L^2-norm of the control is $\|u\|_{L^2(0,T)} \approx 4.84 \times 10^{-1}$. The other figures (Figs. 7 and 8) address the time dependent case $\psi(t) = \sin(n\pi t/T)/5$ for $n = 6$ and $n = 19$.

6. Comments and remarks

(1) It is clear from the proof that, instead of looking for controls such that the solution of (1.1) satisfies $y(T) = y'(T) = 0$ on $(0, 1)$, we may look for a control such that, given $(z^0, z^1) \in H^1(0, 1) \times L^2(0, 1)$, the solution satisfies $y(T) = z^0$, $y'(T) = z^1$? It suffices to suitably change the expression of u' in Proposition 2.2. The exact result is then

Theorem 6.1. *Let $T > 2$. For any $(y^0, y^1), (z^0, z^1) \in [H^1(0, 1) \times L^2(0, 1)]^2$, $\psi \in H^1(0, T)$ with*

$$y^0(1) \geqslant \psi(0), \qquad z^0(1) \geqslant \psi(T),$$

there exists $u \in H^1(0, T)$ such that (1.1) admits a unique solution $y \in C([0, T], H^1(0, 1)) \cap C^1([0, T], L^2(0, 1))$, satisfying $y(T) = z^0$, $y'(T) = z^1$ on $(0, 1)$.

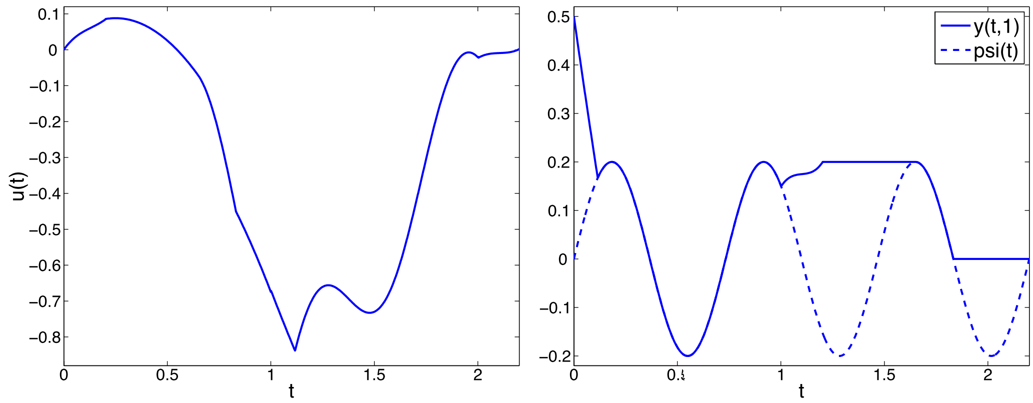

Fig. 7. $\psi(t) = \sin(6\pi t/T)/5-$. Evolution of the control u (**left**) and corresponding displacement $y(\cdot, 1)$ (**right**) vs $t \in [0, T]$, $\|u\|_{L^2(0,T)} \approx 6.44 \times 10^{-1}$.

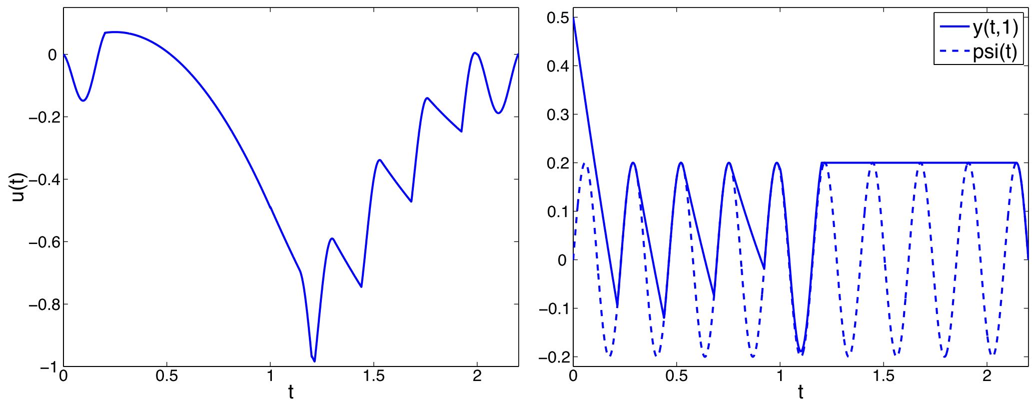

Fig. 8. $\psi(t) = \sin(19\pi t/T)/5-$. Evolution of the control u (**left**) and corresponding displacement $y(\cdot, 1)$ (**right**) vs $t \in [0, T]$.

(2) **The case $T = 2$.** Concerning the limit case $T = 2$, the control is then given by

$$\begin{cases} u'(t) = f'(t+1) + \dfrac{1}{2} q^0(t) & \text{if } 0 < t < 1, \\ u'(t) = f'(t-1) - \dfrac{1}{2} p^0(2-t) & \text{if } 1 < t < 2, \end{cases}$$

and the control Dirichlet-to-Neumann map is

$$A_c\big(y^0, y^1, u, f\big)(t) = \begin{cases} f'(t) - p^0(1-t) & \text{a.e. } 0 < t < 1, \\ -f'(t) & \text{a.e. } 1 < t < 2. \end{cases}$$

Note that A_c does not depend on u anymore and the differential equation corresponding to (3.20) becomes

$$\begin{cases} f'(t) = \begin{cases} \frac{1}{\epsilon}[f(t) - \psi(t)]^- + p^0(1-t), & t \in (0,1), \\ -\frac{1}{\epsilon}[f(t) - \psi(t)]^-, & t \in (1,2), \end{cases} \\ f(0) = y^0(1), \quad f(2) = 0. \end{cases} \tag{6.55}$$

For f to be an $H^1(0,2)$ function, we need the condition

$$f\big(1^-\big) = \lim_{t \to 1^-} f(t) = \lim_{t \to 1^+} f(t) = f\big(1^+\big). \tag{6.56}$$

If $v \in H^1(0,1)$ satisfies $v(0) = 0$ and $v(1) = f(1+) - y^0(1)$ we define the couple (f, p^0) on $(0,1)$ by

$$p^0(1-t) = v'(t) - \frac{[v(t) + y^0(1) - \psi(t)]^-}{\epsilon}; \qquad f(t) = v(t) + y^0(1).$$

With this choice, (6.55) and (6.56) are satisfied but the initial data depend on ϵ. We are able to pass to the limit with respect to ϵ only if we impose supplementary conditions on the initial data.

The conclusion is that, in general, even the penalized problem is not controllable in time $T = 2$ for any initial data. The same kind of problem occurs if one tries the direct method.

(3) Using this approach, we may also address the case of a lower and upper obstacles $\psi_l, \psi_u \in H^1(0,T)$ so that $\psi_l(t) \leqslant y(t,1) \leqslant \psi_u(t)$, $t \in (0,T)$ with the condition $\psi_l(T) \leqslant 0 \leqslant \psi_u(T)$ (see [3]).

(4) With the method used in Section 3, we can consider the nonlinear control problem

$$\begin{cases} y'' - y_{xx} = 0 & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = u(t) & t \in (0,T), \\ y_x(t,1) = f(t,y) & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1), \end{cases}$$

and prove the controllability for $(y^0, y^1) \in H^1(0,1) \times L^2(0,1)$ at any time $T > 2$ whenever f is continuous with respect to t and Lipschitz with respect to y.

If f is superlinear in y, there will be a problem to act on the blow-up time of a solution of (3.20) in (0, 1). But if we impose conditions on f that ensure the existence of the solution of (3.20) on (0, 1), the same technique will provide the controllability property.

(5) Also, we may address the analogous controllability problem for a nonhomogeneous string

$$\begin{cases} y'' - \big(a(x)y_x\big)_x = 0, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = u(t), & t \in (0,T), \\ y(t,1) \geqslant \psi(t), \quad y_x(t,1) \geqslant 0, \quad \big(y(t,1) - \psi(t)\big)y_x(t,1) = 0, & t \in (0,T), \\ y(0,x) = y^0(x), \quad y'(0,x) = y^1(x), & x \in (0,1), \end{cases} \tag{6.57}$$

where $a \in C^1[0,1]$ is such that there exists $a_0 > 0$ with $a_0 \leqslant a(x)$ for all $x \in (0,1)$.

As before, we may define the corresponding Dirichlet-to-Neumann map $A : \mathbb{H} \to L^2(0,T)$, $A(\phi^0, \phi^1, u, f) = \phi_x(\cdot, 1)$, where ϕ is the solution of

$$\begin{cases} \phi'' - \big(a(x)\phi_x\big)_x = 0, & (t,x) \in (0,T) \times (0,1), \\ \phi(t,0) = u(t), & t \in (0,T), \\ \phi(t,1) = f(t), & t \in (0,T), \\ \phi(0,x) = \phi^0(x), \quad \phi'(0,x) = \phi^1(x), & x \in (0,1). \end{cases} \tag{6.58}$$

If we define the space

$$\tilde{\mathbb{H}}_c = \big\{\big(\phi^0, \phi^1, u, f\big) \in \mathbb{H} \;\big|\; u(T) = f(T) = 0 \text{ and the solution of (6.58) verifies } \phi(T) = \phi'(T) = 0\big\},$$

the controllability of the solution of (6.57) in time T is equivalent to find $u, f \in H^1(0,T)$ such that

$$\begin{cases} \big(y^0, y^1, u, f\big) \in \tilde{\mathbb{H}}_c, \\ f - \psi \geqslant 0, & t \in (0,T), \\ A\big(y^0, y^1, u, f\big) \geqslant 0, & t \in (0,T), \\ (f - \psi)A\big(y^0, y^1, u, f\big) = 0, & t \in (0,T). \end{cases} \tag{6.59}$$

System (6.59) is similar to (4.38). However, in the constant coefficient case, we have essentially used the explicit computation of solution ϕ to translate (4.38) into the differential inequalities (4.39) and (4.41). This argument cannot be used for (6.59) and a different idea seems to be necessary.

(6) The same problem for the wave equation in higher dimension is an open problem.

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