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Multi-bump standing waves with critical frequency for nonlinear Schrödinger equations

Jaeyoung Byeon [a], Yoshihito Oshita [b,*]

[a] *Department of Mathematics, Pohang University of Science and Technology, Pohang, Kyungbuk 790-784, Republic of Korea*
[b] *Department of Mathematics, Okayama University, Tsushimanaka 3-1-1, Okayama 700-8530, Japan*

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Abstract

We glue together standing wave solutions concentrating around critical points of the potential V with different energy scales. We devise a hybrid method using simultaneously a Lyapunov–Schmidt reduction method and a variational method to glue together standing waves concentrating on local minimum points which possibly have no corresponding limiting equations and those concentrating on general critical points which converge to solutions of corresponding limiting problems satisfying a non-degeneracy condition.

Résumé

Nous recollons des ondes stationnaires d'ordres différents en énergie, se concentrant autour de points critiques d'un potentiel V. Nous introduisons une méthode hybride, utilisant à la fois une méthode de réduction de Lyapunov–Schmidt, et une méthode variationnelle pour recoller des ondes stationnaires, se concentrant en des minima locaux, éventuellement sans équation-limite correspondante, et d'autres se concentrant en des points critiques quelconques, convergeant vers des solutions de problèmes-limites correspondants, satisfaisant une condition de non-dégénérescence.

1. Introduction and statement of main results

We consider a standing wave solution for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2}\Delta\psi-V(x)\psi+f(\psi)=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

where $\hbar$ denotes the Plank constant, i is the imaginary unit and $f(e^{i\theta}\psi)=e^{i\theta}f(\psi)$. A solution of the form $\psi(x,t)=\exp(-iEt/\hbar)v(x)$ is called a standing wave solution of the nonlinear Schrödinger equation (1.1). Then, a function $\psi(x,t)\equiv\exp(-iEt/\hbar)v(x)$ is a standing wave solution of (1.1) if and only if the function v satisfies

* Corresponding author.
E-mail addresses: jbyeon@postech.ac.kr (J. Byeon), oshita@math.okayama-u.ac.jp (Y. Oshita).

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$$\frac{\hbar^2}{2}\Delta v - \big(V(x) - E\big) + f(v) = 0, \quad x \in \mathbb{R}^n. \tag{1.2}$$

For the physical background, refer to [8,29] and [30].

In this paper, we study standing waves of (1.1) for small $\hbar > 0$. For small $\hbar > 0$, these standing waves of the nonlinear Schrödinger equation (1.1) are referred to as semi-classical states. Thus we are concerned on the following equation

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.3}$$

In this paper, we are interested in the situation where E is a critical frequency in the sense that $\min_{x\in\mathbb{R}^n} V(x) = 0$. Since the pioneering work [21], there haven been many further papers for the case $\inf_{x\in\mathbb{R}^n} V(x) > E$ (refer to [1,2, 4,5,14–21,24,25,27,28,31–34,37–39] and references therein). When $\inf_{x\in\mathbb{R}^n} V(x) > 0$, we see via a transformation $v(x) \equiv u(\varepsilon x)$ that the following equations with constant $c > 0$ serve as limiting equations of (1.3)

$$\begin{cases} \Delta u - cu + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.4}$$

Thus, if $\inf_{x\in\mathbb{R}^n} V(x) > 0$, for any solution u_ε of (1.3), $\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty} > 0$. On the other hand, in the case of $\min_{x\in\mathbb{R}^n} V(x) = 0$, it was shown in [8] and [9] that there exists a locally minimal energy solution w_ε of (1.3) concentrating around an isolated component of global minimum points of V as $\varepsilon \to 0$. In contrast to the case of $\inf_{x\in\mathbb{R}^n} V(x) > 0$, the amplitude $\|w_\varepsilon\|_{L^\infty}$ and energy of the localized solution w_ε concentrating around global minimum points of V decay to 0 as $\varepsilon \to 0$, and their decay rates depend subtly upon how the potential V decays to 0 around the concentration points. Moreover, if the decaying behavior of potential V to 0 is sufficiently *irregular*, there will be no corresponding limiting problem; in such a case, an exact estimation of the amplitude and energy of the corresponding solution may not be possible. This makes the gluing of the localized solutions very difficult. If the decaying behavior of the potential V to 0 around global minimum points is *regular* and the corresponding limiting problems have good properties, a gluing of solutions concentrating around global minimum points has been worked out in [7,10,12,11]. Recently, without requiring the existence of limiting problems, Sato [36] were able to glue localized solutions concentrating around local minimum points when $f(u) = u^p$, $p \in (1, 2^*)$, where $2^* = (n+2)/(n-2)$ for $n \geqslant 3$ and $2^* = \infty$ for $n = 1, 2$. He glues the solution via a minimization on a torus of finite codimension in a Sobolev space which depends strongly on the homogeneity of an $f(u) = u^p$. In this paper we devise a new approach to glue together the localized solution for a more general type of nonlinearity, where the solution cannot be obtained via a minimization argument. In fact, we glue together the localized solution concentrating around local minimum points for quite general nonlinearities f without a monotonicity assumption for $f(t)/t$. Furthermore, we use both a variational method and a Lyapunov–Schmidt reduction method to glue together the solutions concentrating around global minimum points (without requiring any existence of limiting problems) and the solutions concentrating around stable critical points of potential V. We have never seen this approach using simultaneously both variational method and Lyapunov–Schmidt reduction method in the literature.

To begin, we list some conditions for V and f:

- (V1) $V \in C(\mathbb{R}^n)$; $\liminf_{|x|\to\infty} V(x) > 0 = \inf_{x\in\mathbb{R}^n} V(x)$;
- (V2) there exist disjoint bounded open sets Ω_i with smooth boundary $\partial\Omega_i$, $i = 1, \dots, k$, satisfying $0 \leqslant m_i = \inf_{x\in\Omega_i} V(x) < \min_{x\in\partial\Omega_i} V(x)$;
- (V3) there are $x_1, \dots, x_m \in \mathbb{R}^n$ and disjoint bounded open sets $\Omega_{k+1}, \dots, \Omega_{k+m}$ with smooth boundaries $\partial\Omega_{k+j}$ such that $x_j \in \Omega_{k+j}$, $V \in C^2(\Omega_{k+j})$, $\nabla V(x) \neq 0$ for $x \in \Omega_{k+j} \setminus \{x_j\}$, $\inf_{x\in\Omega_{k+j}} V(x) > 0$ and x_j is a non-degenerate critical point of V, for $j \in \{1, \dots, m\}$;
- (f1) $f \in C^1(\mathbb{R})$, $f(t) = 0$ for $t \leqslant 0$ and there exist some $\mu_1 > 1$ and $C > 0$ satisfying $|f(t)| \leqslant Ct^{\mu_1}$ for $t \in (0, 1)$;
- (f2-1) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geqslant 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\liminf_{t\to 0+} \frac{1}{t^{p+1}} \int_0^t f(s)\,ds > 0$;
- (f2-2) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geqslant 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\limsup_{t\to\infty} \frac{|f(t)|+|f'(t)t|}{t^p} < \infty$;
- (f3-1) there exists $\mu_2 > 1$ such that $(\mu_2 + 1)\int_0^t f(s)\,ds \leqslant f(t)t$ for $t > 0$;
- (f3-2) there exists $\mu_3 > 1$ and $t_1 > 0$ such that $\mu_3 f(s) \leqslant f'(t)t$ for $t \in [0, t_1]$;
- (f3-3) there exists $\mu_3 > 1$ such that $\mu_3 f(s) \leqslant f'(t)t$ for $t > 0$;

(f4) for any $a \in \{V(x_1), \dots, V(x_m)\}$, the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n,\ u \in H^{1,2}(\mathbb{R}^n) \tag{1.5}$$

has a radially symmetric solution U_a which is non-degenerate in $H_r^{1,2}(\mathbb{R}^n) \equiv \{w \in H^{1,2}(\mathbb{R}^n);\ w(x) = w(|x|)\}$, and $f \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R})$ for some $\gamma \in (0,1)$.

It is proved in [8] that if (f1), (f2-1) and (f3-1) hold, there exists a positive solution of (1.3) concentrating around an isolated component of zeros of V, and that if (f1), (f2-1), (f2-2) and (f3-1) hold, there exists a positive solution of (1.3) concentrating around an isolated component of local minimum points of V. In this paper, we will glue together the solutions found in [8] under the same conditions.

Throughout this paper we assume (f1), (f2-1) and (f3-1). Then we see that $f(t) > 0$ for $t > 0$. Note that (f1), (f2-1) and (f3-1) hold for $f(t) = t_+^p + t_+^q$ with $p \in (1, 2^*)$, $q > 1$.

Some remarks about the above conditions on f are in order. If we take any $\mu \in (1, \min\{\mu_1, \mu_2, \mu_3\})$, then the conditions (f1), (f3-1), (f3-2) hold with μ instead of μ_1, μ_2, μ_3, respectively. Note that (f1) or (f3-2) implies $\limsup_{t\to 0} \frac{f(t)}{t^\mu} < \infty$, that (f3-1) implies $\limsup_{t\to 0} \frac{1}{t^{\mu_2+1}} \int_0^t f(s)\,ds < \infty$, and that $\mu_1, \mu_2, \mu_3 \leqslant p$ if the conditions for μ_1, μ_2, μ_3 and p hold. For $\mu < \mu_1$, we can find $t_0 \in (0, t_1)$ such that $\mu f(t_0) \leqslant f'(t_0)t_0$. If not, there exists a constant $C > 0$ such that $f(t) \geqslant Ct^\mu$ for $t \in (0,1)$; this contradicts (f1). Now, we define

$$\tilde f(t) = \begin{cases} f(t) & t \leqslant t_0, \\ f(t_0) + \frac{f'(t_0)}{\mu t_0^{\mu-1}}(t^\mu - t_0^\mu) & t > t_0. \end{cases}$$

Then we see that $(\mu+1)\int_0^t \tilde f(s)\,ds \leqslant \tilde f(t)t$ for all $t > 0$ if (f3-1) holds, and that $\mu \tilde f(t) \leqslant \tilde f'(t)t$ for all $t > 0$ if (f3-2) holds. Refer to [26] for the result related to the non-degeneracy condition appearing in (f4).

To state our main results we give some definitions. We define $A_i \equiv \{x \in \Omega_i \mid V(x) = m_i\}$ for $i = 1, \dots, k$, $\mathcal{Z} \equiv \{x \in \mathbb{R}^n \mid V(x) = 0\}$, $F(t) \equiv \int_0^t f(s)\,ds$, $\tilde F(t) \equiv \int_0^t \tilde f(s)\,ds$ and

$$L_a(u) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + au^2\,dx - \int_{\mathbb{R}^n} F(u)\,dx, \quad u \in H^1(\mathbb{R}^n).$$

For any $a > 0$, let S_a be the set of all least energy solutions of the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \qquad u(0) = \max_{x\in\mathbb{R}^n} u(x). \tag{1.6}$$

It is known in [37] that for each $a > 0$, S_a is nonempty and if (f1), (f2-2) and (f3-1) are satisfied, and in [22] that any $u \in S_a$ is radially symmetric. Moreover, any solution $U \in S_a$ satisfies

$$U(x) \leqslant Ce^{-c|x|},$$

for some constants $C, c > 0$,

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla U|^2\,dx + \frac{na}{2}\int_{\mathbb{R}^n} U^2\,dx - n\int_{\mathbb{R}^n} F(U)\,dx = 0,$$

and

$$L_a(U) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla U|^2\,dx + \frac{a}{2}\int_{\mathbb{R}^n} U^2\,dx - \int_{\mathbb{R}^n} F(U)\,dx = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla U|^2\,dx.$$

For $i = 1, \dots, k$, we consider the following localized problem

$$\begin{cases} \varepsilon^2\Delta u - V(x)u + h(u) = 0, \quad u > 0 & \text{in } \Omega_i, \\ u(x) = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{1.7}$$

where $h = f$ if $m_i > 0$ and $h = \tilde f$ if $m_i = 0$. We define

$$J_\varepsilon(u; \Omega_i) \equiv \frac{1}{2}\int_{\Omega_i} \varepsilon^2|\nabla u|^2 + Vu^2\,dx - \int_{\Omega_i} H(u)\,dx,$$

where

$$H(t)=\int_0^t h(s)\,ds=\begin{cases}F(t) & \text{if } m_i>0,\\ \tilde F(t) & \text{if } m_i=0.\end{cases} \tag{1.8}$$

Then a solution u_ε of (1.7) corresponds to a critical point of the energy functional $J_\varepsilon(u;\Omega_i)$ on $H_0^1(\Omega_i)$.

If (f3-1) holds, for each nonnegative function $h_i^\varepsilon \in H_0^1(\Omega_i)\setminus\{0\}$, we can find $t(h_i^\varepsilon)>0$ such that for $t\geqslant t(h_i^\varepsilon)$, $J_\varepsilon(th_i^\varepsilon;\Omega_i)<0$. Then we define

$$C_\varepsilon^i \equiv \inf_{\gamma\in\Phi_\varepsilon^i}\max_{t\in[0,1]} J_\varepsilon\big(\gamma(t);\Omega_i\big),$$

where $\Phi_\varepsilon^i \equiv \{\gamma\in C([0,1],H_0^1(\Omega_i)) \mid \gamma(0)=0,\ \gamma(1)=t(h_i^\varepsilon)h_i^\varepsilon\}$. Then, it follows from the Mountain Pass Theorem (refer to [35]) that if (f1) and (f3-1) are satisfied when $m_i=0$, and (f1), (f2-2) and (f3-1) when $m_i>0$, there exists a mountain pass solution u_ε^i of (1.7) with $J_\varepsilon(u_\varepsilon^i,\Omega_i)=C_\varepsilon^i$. The main result in [9] implies that in case $m_i=0$, if we further assume (f2-1), $\lim_{\varepsilon\to0}\varepsilon^{-n}C_\varepsilon^i=0$,

$$\lim_{\varepsilon\to0}\|u_\varepsilon^i\|_{L^\infty(\Omega_i)}=0,\qquad \liminf_{\varepsilon\to0}\varepsilon^{-\frac{2}{\mu-1}}\|u_\varepsilon^i\|_{L^\infty(\Omega_i)}>0$$

and in case $m_i>0$, $\lim_{\varepsilon\to0}\varepsilon^{-n}C_\varepsilon^i=L_{m_i}(U)$ for $U\in S_{m_i}$ and there exists a maximum point x_ε^i of u_ε^i with $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon^i,A_i)=0$ such that for some $C,c>0$, $u(x)\leqslant C\exp(-\frac{c|x-x_\varepsilon^i|}{\varepsilon})$ and $u_\varepsilon^i(\cdot+x_\varepsilon^i)$ converges (up to a subsequence) uniformly to a function $U\in S_{m_i}$.

For any set $A\subset\mathbb{R}^n$ and $d>0$, we define $A^d\equiv\{x\in\mathbb{R}^n \mid \operatorname{dist}(x,A)<d\}$.

Theorem 1. *Suppose that* (V1) *and* (V2) *hold. Assume that* $m_1=\cdots=m_l=0<m_{l+1},\ldots,m_k$. *Suppose that* (f1), (f2-1), (f3-1) *hold if* $l=k$, *and that* (f1), (f2-1), (f2-2), (f3-1) *hold if* $l<k$. *Then, for sufficiently small* $\varepsilon>0$, *there exists a positive solution* u_ε *of* (1.3) *such that*

(i) *for any sufficiently small* $d>0$, *there exist* $C,c>0$ *satisfying*

$$u_\varepsilon(x)\leqslant C\exp\big(-c\operatorname{dist}\big(x,(A_1\cup\cdots\cup A_k)^d\big)/\varepsilon\big);$$

(ii) *for* $i=1,\ldots,l$,

$$\lim_{\varepsilon\to0}\|u_\varepsilon\|_{L^\infty(\Omega_i)}=0,\qquad \lim_{\varepsilon\to0}\varepsilon^{-2/(\mu-1)}\|u_\varepsilon\|_{L^\infty(\Omega_i)}>0;$$

and for $i=l+1,\ldots,k$, *a least energy solution* $U\in S_{m_i}$ *and some* $x_\varepsilon^i\in\mathbb{R}^n$ *with* $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon^i,A_i)=0$, *a transformed solution* $u_\varepsilon(\varepsilon x+x_\varepsilon^i)$ *converges* (*up to a subsequence*) *uniformly to* $U(x)$ *on each bounded set in* $\mathbb{R}^n$.

Moreover, if (f3-2) *and* (f3-3) *are also satisfied when* $l=k$ *and* $l<k$, *respectively, there exist some* $C,c>0$ *such that for* $i=1,\ldots,k$,

$$\big|J_\varepsilon(u_\varepsilon;\Omega_i)-C_i^\varepsilon\big|\leqslant Ce^{-c/\varepsilon}.$$

Theorem 2. *Assume that* (V1), (V2) *and* (V3) *hold. Suppose that* (f1), (f2-1), (f3-1) *and* (f4) *hold. Then for sufficiently small* $\varepsilon>0$, *there exists a positive solution* u_ε *of* (1.3) *such that*

(i) *for any sufficiently small* $d>0$, *there exist* $C,c>0$ *satisfying*

$$u_\varepsilon(x)\leqslant C\exp\big(-c\operatorname{dist}\big(x,\big(A_1\cup\cdots\cup A_k\cup\{x_1\}\cup\cdots\cup\{x_m\}\big)^d\big)/\varepsilon\big);$$

(ii) *for* $m_i=0$,

$$\lim_{\varepsilon\to0}\|u_\varepsilon\|_{L^\infty(\Omega_i)}=0,\qquad \lim_{\varepsilon\to0}\varepsilon^{-2/(\mu-1)}\|u_\varepsilon\|_{L^\infty(\Omega_i)}>0,$$

and for $m_i>0$, *there exists* $x_\varepsilon^i\in\mathbb{R}^n$ *such that* $\lim_{\varepsilon\to0}\operatorname{dist}(x_\varepsilon^i,A_i)=0$ *and that* $u_\varepsilon(\varepsilon x+x_\varepsilon^i)$ *converges uniformly* (*up to a subsequence*) *to some* $U_{m_i}\in S_{m_i}$ *on each bounded set in* $\mathbb{R}^n$;

(iii) *for each* $i = 1, \dots, m$, *there exists* $y_\varepsilon^i \in \mathbb{R}^n$ *such that* $\lim_{\varepsilon \to 0} y_\varepsilon^i = x_i$ *and that* $u_\varepsilon(\varepsilon x + y_\varepsilon^i)$ *converges uniformly* (*up to a subsequence*) *to* $U_{V(x_i)}$ *on each bounded set in* $\mathbb{R}^n$. *Here* $U_{V(x_i)}$ *is a function given in* (f4).

Moreover, if (f3-2) *and* (f3-3) *are also satisfied when* $l = k$ *and* $l < k$, *respectively, there exist some* $C, c > 0$ *such that for* $i = 1, \dots, k$,

$$\left| J_\varepsilon(u_\varepsilon; \Omega_i) - C_i^\varepsilon \right| \leqslant C e^{-c/\varepsilon}.$$

We conclude the introduction with an outline of our proof of the main results.

We look for a critical point of some energy functional Γ_ε. First we choose a bounded open set Ω_0 enclosing $\mathcal{Z} \setminus \bigcup_{i=1}^k A_i$ (see Section 2 for a precise condition for Ω_0). Then we consider a modified problem outside $\bigcup_{i=0}^{k+m} \Omega_i$ for each given function u_i on Ω_i, and show the existence of a solution $\varphi(u_0, \dots, u_{k+m})$ for the external problem solving by a minimization problem. We also show the existence of a solution $P_i(u_i)$ of the modified problem by a minimization such that $P_i(u_i)(x) = 0$ for $\mathrm{dist}(x, \Omega_i) \geqslant \delta > 0$ and $P_i(u_i) = u_i$ on Ω_i. Then, we will show that finding a solution is reduced to finding a critical point of a reduced functional $I_\varepsilon(u_0, \dots, u_{k+m})$. Finding a good estimate for

$$\tilde{\varphi}(u_0, \dots, u_{k+m}) \equiv \varphi(u_0, \dots, u_{k+m}) - \sum_{i=0}^{k+m} P_i(u_i)$$

is important in our proof. In fact, we show that $\tilde{\varphi}$ is exponentially small with respect to small $\varepsilon > 0$ (see Proposition 3). That enables us to regard the sum $\sum_{i=0}^{k+m} \Gamma_\varepsilon(P_i(u_i))$ of localized functionals depending only on each u_i as an exponentially small perturbation of the reduced functional I_ε. This is a novelty of our argument.

To prove Theorem 1, we consider an energy gradient flow in a product of an appropriate small ball in $H^1(\Omega_0)$ and appropriate annuli in $H^1(\Omega_i), i = 1, \dots, k$. To take appropriate radii of the ball and annuli is also important in our proof. If there exist no solutions in the product of the ball and the annuli, we show that via a gradient estimation near the boundary of the product of the ball and the annuli, we can deform a product of localized mountain paths into a surface where the maximum energy is less than a sum of independent local mountain pass levels by an algebraic order of $\varepsilon > 0$. Then, from the exponential smallness of $\tilde{\varphi}(u_0, \dots, u_{k+m})$, we will get a contradiction.

For the proof of Theorem 2, we use a Lyapunov–Schmidt reduction method in a region $\Omega_{k+1} \cup \dots \cup \Omega_{k+m}$ before we use the variational argument in a region $\Omega_0 \cup \dots \cup \Omega_k$ as in the proof of Theorem 1. For the typical case $m = 1$ we will find a critical point of the functional $u_{k+1} \mapsto I(u_0, \dots, u_{k+1})$ by the reduction method which depends smoothly on $(u_0, \dots, u_k)$, and then use the same argument as in the proof of Theorem 1. So we will skip the variational procedure in the proof of Theorem 2 since the required variational argument after the reduction is exactly the same as that of the proof of Theorem 1.

This paper is organized as follows. In Section 2, some preliminaries about the above reduction are given. Theorems 1 and 2 will be proved in Sections 3 and 4, respectively.

2. Preliminaries

We define $A_0 := \mathcal{Z} \setminus \bigcup_{i=1}^k A_i$ and $\Omega_0 \supset A_0$ a bounded open set with a smooth boundary such that $\overline{\Omega_0} \cap \overline{\Omega_i} = \emptyset$ for $i = 1, \dots, k+m$. For $\delta > 0$, let $\Omega_i^\delta = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \Omega_i) \leqslant \delta\}$, and $\Omega_i^{-\delta} = \{x \in \Omega_i \mid \mathrm{dist}(x, \partial\Omega_i) \geqslant \delta\}$. Taking sufficiently small $\delta > 0$, we may assume that $A_i \subset \Omega_i^{-2\delta}$, $\Omega_i^{2\delta} \cap \Omega_j^{2\delta} = \emptyset$ for each $0 \leqslant i \neq j \leqslant k+m$, and that $\partial\Omega_i^{\delta'}$ is smooth for each $0 \leqslant i \leqslant k+m$ and $\delta' \in [-2\delta, 2\delta]$. Reordering the index, we may assume that $m_1 = \dots = m_l = 0$ and $m_{l+1}, \dots, m_k > 0$ for some $l \in \{1, \dots, k\}$. In this section, we assume that (f1), (f2-1) and (f3-1) are satisfied and (f2-2) are also satisfied if $l < k$. As in the previous section, we can find $t_0 > 0$ such that $\mu f(t_0) \leqslant f'(t_0)t_0$. Then, we define

$$\tilde{f}(t) = \begin{cases} f(t) & t \leqslant t_0, \\ f(t_0) + \frac{f'(t_0)}{\mu t_0^{\mu-1}}(t^\mu - t_0^\mu) & t > t_0. \end{cases}$$

Then we see that $(\mu+1)\int_0^t \tilde{f}(s)\,ds \leqslant \tilde{f}(t)t$ for all $t > 0$ if (f3-1) holds, and that $\mu \tilde{f}(t) \leqslant \tilde{f}'(t)t$ for all $t > 0$ if (f3-2) holds. We define

$$b \equiv \inf\left\{ V(x) \,\middle|\, x \notin \Omega_0^{-\delta} \cup \dots \cup \Omega_k^{-\delta} \right\} > 0.$$

We define $V_\varepsilon(x) := V(\varepsilon x)$, and H_ε the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to a norm $\|u\|_\varepsilon = (\int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon u^2\, dx)^{1/2}$. Note that $H_\varepsilon \subset H^1(\mathbb{R}^n)$ by (V1). We also define $\Omega_{i,\varepsilon} = \{x \mid \varepsilon x \in \Omega_i\}$, $\Omega_{i,\varepsilon}^\delta = \{x \mid \varepsilon x \in \Omega_i^\delta\}$. Then we see that

$$\inf\{V_\varepsilon(x) \mid x \notin \Omega_{0,\varepsilon}^{-\delta} \cup \cdots \cup \Omega_{k,\varepsilon}^{-\delta}\} = b > 0.$$

From (f3-1), there exist some $C_1, C_2 > 0$ such that $f(t) \leqslant C_1 t^{\mu_2}$ for $t \in [0,1]$ and $f(t) \geqslant C_2 t^{\mu_2}$ for $t \in [1,\infty)$. Thus, taking sufficiently small $\lambda \in (0, \min\{b/10, b(1/2 - 1/(\mu+1))\})$, we can construct a function $f_\lambda \in C^1(\mathbb{R})$ for small λ_0 and large $\lambda_1 > 0$ such that

$$f_\lambda(t) = \begin{cases} f(t) & t \leqslant \lambda_0, \\ \lambda t & t \geqslant \lambda_1 \end{cases}$$

with $0 < \lambda_0 < \lambda_1$, and $f_\lambda(t) \leqslant \min\{f(t), \tilde{f}(t), \lambda t\}$, $|f_\lambda'(t)| \leqslant 2\lambda$ for $t \geqslant 0$. Note that $f_\lambda(t) = 0$ for $t \leqslant 0$ and

$$\frac{|f_\lambda(t_1) - f_\lambda(t_2)|}{|t_1 - t_2|} \leqslant 2\lambda \quad \text{for } t_1 \neq t_2.$$

We find a function $\chi \in C^1(\mathbb{R}^n)$ such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 1$ for $x \in \bigcup_{i=0}^{k+m} \Omega_i^{-\delta}$ and $\chi(x) = 0$ for $x \notin \bigcup_{i=0}^{k+m} \Omega_i$.

We define

$$g(x,t) = \begin{cases} \chi(\varepsilon x)\tilde{f}(t) + (1 - \chi(\varepsilon x)) f_\lambda(t) & x \in \bigcup_{i=0}^{l} \Omega_{i,\varepsilon}, \\ \chi(\varepsilon x) f(t) + (1 - \chi(\varepsilon x)) f_\lambda(t) & x \in \bigcup_{i=l+1}^{k+m} \Omega_{i,\varepsilon}, \\ f_\lambda(t) & x \notin \bigcup_{i=0}^{k+m} \Omega_{i,\varepsilon}, \end{cases}$$

$G(x,t) \equiv \int_0^t g(x,s)\, ds$, and for $u \in H_\varepsilon$,

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon(x) u^2\, dx - \int_{\mathbb{R}^n} G(x,u)\, dx.$$

Then $\Gamma_\varepsilon \in C^2(H_\varepsilon)$. For a measurable subset $U \subset \mathbb{R}^n$, we define

$$\Gamma_\varepsilon(u; U) = \frac{1}{2} \int_U |\nabla u|^2 + V_\varepsilon(x) u^2\, dx - \int_U G(x,u)\, dx,$$

$$\|u\|_{\varepsilon,U}^2 = \int_U |\nabla u|^2 + V_\varepsilon(x) u^2\, dx$$

for $u \in H_\varepsilon$. For $\vec{u} = (u_0, \ldots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+m,\varepsilon})$, we define a norm

$$\|\vec{u}\|_\varepsilon^2 = \sum_{i=0}^{k+m} \|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}^2.$$

For $u_i \in H^1(\Omega_{i,\varepsilon})$, let

$$X_i^\varepsilon(u_i) = \{u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \mid u = u_i \text{ on } \Omega_{i,\varepsilon}\}.$$

We regard $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ as an element in $H^1(\mathbb{R}^n)$ by defining $u(x) = 0$ for $x \notin \Omega_{i,\varepsilon}^\delta$.

Proposition 1. *For each $u_i \in H^1(\Omega_{i,\varepsilon})$, $i \in \{0, \ldots, k+m\}$, there exists a unique minimizer $P_i(u_i)$ of Γ_ε on $X_i^\varepsilon(u_i)$, which satisfies the following*:

(i) $w = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ *solves*

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(ii) $P_i : H^1(\Omega_{i,\varepsilon}) \to H_0^1(\Omega_{i,\varepsilon}^\delta)$ *is of class* C^1, *and for all* $h \in H^1(\Omega_{i,\varepsilon})$, $w = P_i'(u_i)h$ *solves*

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda'(P_i(u_i))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = h & \text{in } \Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(iii) *there exists a positive constant* C, *independent of small* $\varepsilon > 0$ *such that*

$$\|P_i(u_i)\|_\varepsilon \leqslant C\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}} \quad \text{for all } u_i \in H^1(\Omega_{i,\varepsilon}),\ i \in \{0, \dots, k+m\}.$$

This proposition can be proved in a similar way as in [36], so we omit the proof.
For $\vec{u} = (u_0, \dots, u_{k+m})$, $u_i \in H^1(\Omega_{i,\varepsilon})$, let

$$X_*^\varepsilon(\vec{u}) = \{u \in H_\varepsilon \mid u = u_i \text{ on } \Omega_{i,\varepsilon},\ i = 0, \dots, k+m\}.$$

By the same procedure as in the proof of Proposition 1, we can prove the following proposition.

Proposition 2. *For each* $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+m,\varepsilon})$, *there exists a unique minimizer* $\varphi(\vec{u})$ *of* Γ_ε *on* $X_*^\varepsilon(\vec{u})$, *which satisfies the following*:

(i) $w = \varphi(\vec{u})$ *solves*

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \cdots \cup \Omega_{k,\varepsilon})^c, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}\ (i = 0, \dots, k), \end{cases}$$

(ii) $\varphi : H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon}) \to H_\varepsilon$ *is of class* C^1, *and for all* $h \in H^1(\Omega_{i,\varepsilon})$, $v = \frac{\partial\varphi(\vec{u})}{\partial u_i}h$ *solves*

$$\begin{cases} \Delta v - V_\varepsilon v + f_\lambda'(\varphi(\vec{u}))v = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \cdots \cup \Omega_{k,\varepsilon})^c, \\ v = h & \text{in } \Omega_{i,\varepsilon}, \\ v = 0 & \text{in } \Omega_{j,\varepsilon},\ j \neq i, \end{cases}$$

(iii) *there exists a positive constant* C, *independent of small* $\varepsilon > 0$ *such that*

$$\|\varphi(\vec{u})\|_\varepsilon \leqslant C\|\vec{u}\|_\varepsilon.$$

Let $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u}) - \sum_{i=0}^{k+m} P_i(u_i)$. Then it follows that $\tilde{\varphi}(\vec{u}) \in X_*^\varepsilon(\vec{0})$. Now we obtain the following estimates for $\tilde{\varphi}(\vec{u})$.

Proposition 3. *For any* $R > 0$ *and* $\varepsilon_0 > 0$, *there exist constants* $C, c > 0$ *such that*

$$\|\tilde{\varphi}(\vec{u})\|_\varepsilon \leqslant Ce^{-c/\varepsilon}$$

for $\varepsilon \in (0, \varepsilon_0)$ *and* $\|\vec{u}\|_\varepsilon \leqslant R$.

Proof. For $u_i \in H^1(\Omega_{i,\varepsilon})$, $i = 0, \dots, k+m$, $w_i = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ solves

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta. \end{cases}$$

By Proposition 1, there exists a constant $C > 0$, independent of $\vec{u}$ with $\|\vec{u}\|_\varepsilon \leqslant R$, such that $\|w_i\|_{H^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})} \leqslant C$. Hence by elliptic estimates (refer to [23]), we deduce that for each $s \in (0,1)$, there exists a constant $c > 0$ such that $\|w_i\|_{L^\infty(\Omega_{i,\varepsilon}^{s\delta} \setminus \Omega_{i,\varepsilon})} \leqslant C$. By comparison principle, we conclude that for some $C > 0$, independent of small $\varepsilon > 0$,

$$w_i \leqslant Ce^{-c/\varepsilon} \quad \text{on } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}^{\delta/2}.$$

Then, it follows from boundary Schauder estimates [23, Corollary 8.36] that $|w_i|_{C^{1,\theta}(\partial\Omega^\delta_{i,\varepsilon})} \leqslant Ce^{-c/\varepsilon}$. Denoting $\varphi = \varphi(\vec{u}) = \sum_{i=0}^{k+m} P_i(u_i) + \tilde{\varphi}(\vec{u})$, we see that φ satisfies

$$\begin{cases} \Delta\varphi - V_\varepsilon\varphi + f_\lambda(\varphi) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \cdots \cup \Omega_{k+m,\varepsilon})^c, \\ \varphi = u_i & \text{on } \partial\Omega_{i,\varepsilon}. \end{cases}$$

If $\|\vec{u}\|_\varepsilon \leqslant R$, it follows from (iii) of Proposition 2 that $\|\varphi(\vec{u})\|_\varepsilon \leqslant C$. Hence by elliptic estimates, we see that for each $s \in (0, 1)$,

$$\|\varphi(\vec{u})\|_{L^\infty(\Omega^{s\delta}_{0,\varepsilon} \cup \cdots \cup \Omega^{s\delta}_{k+m,\varepsilon})^c} \leqslant C.$$

Then by comparison principle, we deduce that

$$\varphi \leqslant Ce^{-c/\varepsilon} \quad \text{in } \bigcup_{i=0}^{k+m} \Omega^{3\delta/2}_{i,\varepsilon} \setminus \Omega^{\delta/2}_{i,\varepsilon}.$$

Then it follows from interior Schauder estimates [23, Theorem 8.32] that

$$\left|\varphi(\vec{u})\right|_{C^{1,\theta}(\bigcup_{i=0}^{k+m} \partial\Omega^\delta_{i,\varepsilon})} \leqslant Ce^{-c/\varepsilon}.$$

Now a function $w_i = P_i(u_i)$ satisfies

$$\Delta(w_i - \varphi) - V_\varepsilon(w_i - \varphi) + \frac{f_\lambda(w_i) - f_\lambda(\varphi)}{w_i - \varphi}(w_i - \varphi) = 0 \quad \text{in } \Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}.$$

Multiplying by $w_i - \varphi$ and integrating by parts, we have

$$\int\limits_{\partial\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} (w_i - \varphi)\frac{\partial(w_i - \varphi)}{\partial n}\, dS - \int\limits_{\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} \left|\nabla(w_i - \varphi)\right|^2 dx$$
$$- \int\limits_{\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} V_\varepsilon(w_i - \varphi)^2\, dx + \int\limits_{\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} \frac{f_\lambda(w_i) - f_\lambda(\varphi)}{w_i - \varphi}(w_i - \varphi)^2\, dx = 0.$$

Since $w_i - \varphi = 0$ on $\partial\Omega_{i,\varepsilon}$ and $\|w_i - \varphi\|_{C^1(\partial\Omega^\delta_{i,\varepsilon})} = O(e^{-c/\varepsilon})$, we see that

$$\int\limits_{\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} \left|\nabla(w_i - \varphi)\right|^2 + \int\limits_{\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon}} (w_i - \varphi)^2 = O\left(e^{-c/\varepsilon}\right);$$

hence

$$\|\tilde{\varphi}\|_{H^1(\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon})} = \|\varphi - w_i\|_{H^1(\Omega^\delta_{i,\varepsilon} \setminus \Omega_{i,\varepsilon})} = O\left(e^{-c/\varepsilon}\right). \tag{2.1}$$

We note that $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u})$ on $\mathbb{R}^n \setminus (\Omega^\delta_{0,\varepsilon} \cup \cdots \cup \Omega^\delta_{k+m,\varepsilon})$. From a decay property of φ, we see that $\|\varphi\|_{H^1(\mathbb{R}^n \setminus \Omega^\delta_{0,\varepsilon} \cup \cdots \cup \Omega^\delta_{k+m,\varepsilon})} = O(e^{-c/\varepsilon})$. Thus combining this with (2.1), we get the required estimate. □

Let $I_\varepsilon(\vec{u}) = \Gamma_\varepsilon(\varphi(\vec{u}))$. Then from Proposition 3, we conclude that for any $R, \varepsilon_0 > 0$, there exist constants $C, c > 0$ such that

$$\left|I_\varepsilon(\vec{u}) - \sum_{i=0}^{k+m} \Gamma_\varepsilon\left(P_i(u_i)\right)\right| \leqslant Ce^{-c/\varepsilon} \quad \text{for } \|\vec{u}\|_\varepsilon \leqslant R \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{2.2}$$

Moreover, we have the following properties for $I_\varepsilon(\vec{u})$.

Proposition 4. *The following hold.*

(i) *A vector function $\vec{u} = (u_0, \ldots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+m,\varepsilon})$ is a critical point of I_ε if and only if $\varphi(\vec{u})$ is a critical point of Γ_ε.*

(ii) *The functional* $\vec{u} \mapsto I_\varepsilon(\vec{u})$ *satisfies* (PS) *condition if* Γ_ε *does.*
(iii) *For any* $R > 0$, $i = 0, \dots, k+m$ *and* $\varepsilon_0 > 0$, *there exist constants* $C, c > 0$ *such that*

$$\left| \frac{\partial I_\varepsilon}{\partial u_i}(u_0, \dots, u_{k+m}) - \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} \right| \leqslant Ce^{-c/\varepsilon} \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } \|\vec{u}\|_\varepsilon \leqslant R.$$

Proof. (i) For all $\vec{\zeta} = (\zeta_0, \dots, \zeta_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$, we have $\tilde{\varphi}'(\vec{u})\vec{\zeta} \in X_*^\varepsilon(\vec{0})$. Then, it follows from the definition of φ that for all $h \in X_*^\varepsilon(\vec{0})$,

$$\begin{aligned} I_\varepsilon'(\vec{u})\vec{\zeta} &= \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)\varphi'(\vec{u})\vec{\zeta} \\ &= \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)\left[\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i + \tilde{\varphi}'(\vec{u})\vec{\zeta}\right] \\ &= \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)\left[\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i\right] \\ &= \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)\left[\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i + h\right]. \end{aligned}$$

Then, from the fact $H_\varepsilon = \{\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i + h \mid \zeta_i \in H^1(\Omega_{i,\varepsilon}), h \in X_*^\varepsilon(\vec{0})\}$, the equivalence of (i) follows.

(ii) We see from Propositions 1 and 2 that for $\vec{\zeta} = (\zeta_0, \dots, \zeta_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$,

$$\begin{aligned} \left| \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)\left(\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i + h\right)\right| &\leqslant \big|I_\varepsilon'(\vec{u})\vec{\zeta}\big| \\ &\leqslant \big\|I_\varepsilon'(\vec{u})\big\| \|\vec{\zeta}\|_\varepsilon \\ &\leqslant \big\|I_\varepsilon'(\vec{u})\big\| \left\|\sum_{i=0}^{k+m} P_i'(u_i)\zeta_i + h\right\|_\varepsilon. \end{aligned}$$

Thus it follows that $\|\Gamma_\varepsilon'(\varphi(\vec{u}))\| \leqslant \|I_\varepsilon'(\vec{u})\|$. Note that $\Gamma_\varepsilon(\varphi(\vec{u})) = I_\varepsilon(\vec{u})$ and for some $C > 0$, $\|\vec{u}\|_\varepsilon \leqslant \|\varphi(u)\|_\varepsilon \leqslant C\|\vec{u}\|_\varepsilon$. Thus, we conclude that I_ε satisfies (PS) condition if Γ_ε does.

(iii) For $\vec{h} = (h_0, \dots, h_{k+m})$, $\vec{u} = (u_0, \dots, u_{k+m})$ with $h_i, u_i \in H^1(\Omega_{i,\varepsilon})$, a function $v = \sum_{i=0}^{k+m} \frac{\partial\varphi(\vec{u})}{\partial u_i} h_i$ solves

$$\begin{cases} \Delta v - V_\varepsilon v + f_\lambda'(\varphi(\vec{u}))v = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k+m,\varepsilon})^c, \\ v = h_i & \text{in } \Omega_{i,\varepsilon}. \end{cases}$$

By the minimization characterization of v, we have $\|v\|_\varepsilon \leqslant C\|\vec{h}\|_\varepsilon \leqslant C$ for a constant C independent of $\vec{h}$ with $\|\vec{h}\|_\varepsilon \leqslant 1$.

Similarly, a function $w_i = P_i'(u_i)h_i$ solves

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda'(P_i(u_i))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\ w = h_i & \text{in } \Omega_{i,\varepsilon}, \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta. \end{cases}$$

By the minimization characterization of w_i, we have $\|w_i\|_\varepsilon \leqslant C\|h_i\|_\varepsilon \leqslant C$ for a constant C independent of $\vec{h}$ with $\|\vec{h}\|_\varepsilon \leqslant 1$.

Since

$$\begin{aligned} &\left| \sum_{i=0}^{k+m} \Gamma_\varepsilon'\big(\varphi(\vec{u})\big)w_i - \sum_{i=0}^{k+m} \Gamma_\varepsilon'\big(P_i(u_i)\big)w_i \right| \\ &\quad \leqslant \left| \sum_{i=0}^{k+m} \int_{\Omega_{i,\varepsilon}^\delta} \nabla\big(\varphi(\vec{u}) - P_i(u_i)\big) \cdot \nabla w_i + V_\varepsilon\big(\varphi(\vec{u}) - P_i(u_i)\big)w_i \, dx \right| \end{aligned}$$

$$+\sum_{i=0}^{k+m}\int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}}\big|f_\lambda\big(\varphi(\vec u)\big)-f_\lambda\big(P_i(u_i)\big)\big||w_i|\,dx$$

$$\leqslant\sum_{i=0}^{k+m}\big\|\varphi(\vec u)-P_i(u_i)\big\|_{\varepsilon,\Omega^\delta_{i,\varepsilon}}\|w_i\|_{\varepsilon,\Omega^\delta_{i,\varepsilon}}+2\lambda\sum_{i=0}^{k+m}\big\|\varphi(\vec u)-P_i(u_i)\big\|_{L^2(\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon})}\|w_i\|_{L^2(\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon})},$$

we see from Proposition 3 that for any $\varepsilon_0, R>0$, there exist $C,c>0$ satisfying

$$\left|\sum_{i=0}^{k+m}\Gamma'_\varepsilon\big(\varphi(\vec u)\big)w_i-\sum_{i=0}^{k+m}\Gamma'_\varepsilon\big(P_i(u_i)\big)w_i\right|\leqslant Ce^{-c/\varepsilon}\quad\text{for }\varepsilon\in(0,\varepsilon_0)\text{ and }\|\vec u\|_\varepsilon\leqslant R.$$

Then, it follows that

$$\begin{aligned}\left|I'_\varepsilon(\vec u)\vec h-\sum_{i=0}^{k+m}\frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i}h_i\right|&=\left|\Gamma'_\varepsilon\big(\varphi(\vec u)\big)v-\sum_{i=0}^{k+m}\Gamma'_\varepsilon\big(P_i(u_i)\big)w_i\right|\\&\leqslant\left|\Gamma'_\varepsilon\big(\varphi(\vec u)\big)\left(v-\sum_{i=0}^{k+m}w_i\right)\right|+Ce^{-c/\varepsilon}\\&=Ce^{-c/\varepsilon}\end{aligned}$$

since $v-\sum w_i\in X^\varepsilon_*(\vec 0)$. This completes the proof. □

We define $\Gamma^j_\varepsilon(u)=\Gamma_\varepsilon(P_j(u))$ for $u\in H^1(\Omega_{j,\varepsilon})$ and $j\in\{1,\dots,k+m\}$.

Proposition 5. *For each $j\in\{1,\dots,k+m\}$, the following hold.*

(i) *A function u_j is a critical point of Γ^j_ε if and only if $P_j(u_j)\in H^1_0(\Omega^\delta_{j,\varepsilon})$ is a critical point of Γ_ε on $H^1_0(\Omega^\delta_{j,\varepsilon})$.*
(ii) *The functional Γ^j_ε on $H^1(\Omega_{j,\varepsilon})$ satisfies* (PS) *condition if Γ_ε on $H^1_0(\Omega^\delta_{j,\varepsilon})$ does.*

Proof. We can prove these following the same argument as in the proof of (i) and (ii) in Proposition 4. □

Let $\alpha=2\frac{\mu+1}{\mu-1}\geqslant 2\max\{\frac{p+1}{p-1},\frac{\mu+1}{\mu-1}\}$. By Proposition 1(iii), we can choose a constant $M>1$, independent of small $\varepsilon>0$, such that

$$\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}\leqslant\big\|P_i(u_i)\big\|_\varepsilon\leqslant M\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}\quad\text{for all }u_i\in H^1(\Omega_{i,\varepsilon}),\ i\in\{0,\dots,k+m\}.$$

Proposition 6. *Let $i\in\{0,\dots,l\}$. For sufficiently small $\varepsilon>0$, it holds that $\Gamma_\varepsilon(u)\geqslant 6M^2\varepsilon^{2\alpha}$ for all $u\in H^1_0(\Omega^\delta_{i,\varepsilon})$ with $\|u\|_\varepsilon=4M\varepsilon^\alpha$.*

Proof. Note that from conditions (f2-1) and (f3-1), we have $\mu\leqslant p$. Also it follows from (f3-1) and (f2-2) that $G(x,t)\leqslant Ct^{p+1}+Ct^{\mu+1}$ for some $C>0$. Then, for $u\in H^1_0(\Omega^\delta_{i,\varepsilon})\subset H_\varepsilon$, denoting $v(x)\equiv u(x/\varepsilon)\in H^1_0(\Omega^\delta_i)$, we see that for sufficiently small $\varepsilon>0$,

$$\begin{aligned}\Gamma_\varepsilon(u)&\geqslant\int_{\Omega^\delta_{i,\varepsilon}}\frac12\big(|\nabla u|^2+V_\varepsilon u^2\big)-Cu^{p+1}-Cu^{\mu+1}\,dx\\&=\frac12\|u\|^2_\varepsilon-\varepsilon^{-n}\left\{\int_{\Omega^\delta_i}Cv^{p+1}+Cv^{\mu+1}\,dx\right\}\\&\geqslant\frac12\|u\|^2_\varepsilon-C\varepsilon^{-n}\left\{\left(\int_{\Omega^\delta_i}|\nabla v|^2\right)^{(p+1)/2}+\left(\int_{\Omega^\delta_i}|\nabla v|^2\right)^{(\mu+1)/2}\right\}\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{1}{2}\|u\|_\varepsilon^2 - C\varepsilon^{n(p-1)/2-(p+1)}\|u\|_\varepsilon^{p+1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|u\|_\varepsilon^{\mu+1} \\
&\geqslant \left(\frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)}\|u\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|u\|_\varepsilon^{\mu-1}\right)\|u\|_\varepsilon^2 \\
&\geqslant 6M^2\varepsilon^{2\alpha}.
\end{aligned}$$

This completes the proof. □

Proposition 7. *Let $i \in \{0,\dots,l\}$. For sufficiently small $\varepsilon > 0$, it holds that $|\Gamma_\varepsilon(u)| \leqslant 5M^2\varepsilon^{2\alpha}$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon \leqslant 3M\varepsilon^\alpha$, and $\Gamma_\varepsilon(u) \geqslant 0$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon \leqslant 5M\varepsilon^\alpha$.*

Proof. As in the proof of Proposition 6, we note that $\mu \leqslant p$ and $G(x,t) \leqslant Ct^{p+1} + Ct^{\mu+1}$ for some $C > 0$. Then, for $u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \subset H_\varepsilon$, we deduce as in Proposition 6 that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\left|\Gamma_\varepsilon(u)\right| &\leqslant \int_{\Omega_{i,\varepsilon}^\delta} \frac{1}{2}\left(|\nabla u|^2 + V_\varepsilon u^2\right) + Cu^{p+1} + Cu^{\mu+1}\,dx \\
&\leqslant \frac{1}{2}\|u\|_\varepsilon^2 + C\varepsilon^{n(p-1)/2-(p+1)}\|u\|_{H_0^1(\Omega_{i,\varepsilon}^\delta)}^{p+1} + C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|u\|_{H_0^1(\Omega_{i,\varepsilon}^\delta)}^{\mu+1} \\
&= \left\{\frac{1}{2} + C\varepsilon^{n(p-1)/2-(p+1)}\|u\|_\varepsilon^{p-1} + C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|u\|_\varepsilon^{\mu-1}\right\}\|u\|_\varepsilon^2 \\
&\leqslant 5M^2\varepsilon^{2\alpha}.
\end{aligned}$$

Similarly we see that for $u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \subset H_\varepsilon$ with $\|u\|_\varepsilon \leqslant 5M\varepsilon^\alpha$,

$$\begin{aligned}
\Gamma_\varepsilon(u) &\geqslant \int_{\Omega_{i,\varepsilon}^\delta} \frac{1}{2}\left(|\nabla u|^2 + V_\varepsilon u^2\right) - Cu^{p+1} - Cu^{\mu+1}\,dx \\
&\geqslant \left\{\frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)}\|u\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|u\|_\varepsilon^{\mu-1}\right\}\|u\|_\varepsilon^2 \\
&\geqslant \left\{\frac{1}{2} - C\varepsilon^{n(p-1)/2}(5M)^{p-1} - C\varepsilon^{n(\mu-1)/2}(5M)^{\mu-1}\right\}\|u\|_\varepsilon^2.
\end{aligned}$$

Thus we see that if $\varepsilon > 0$ is sufficiently small, $\Gamma_\varepsilon(u) \geqslant 0$ for $u \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ with $\|u\|_\varepsilon \leqslant 5M\varepsilon^\alpha$. This completes the proof. □

Proposition 8. *For all $u \in H^1(\Omega_{i,\varepsilon})$, $i \in \{0,\dots,k+m\}$, it holds that*

$$\begin{aligned}
\Gamma_\varepsilon'\left(P_i(u)\right)P_i'(u)u = {}&\int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2\,dx - \int_{\Omega_{i,\varepsilon}} g(x,u)u\,dx + \int_{\Omega_{i,\varepsilon}^\delta\setminus\Omega_{i,\varepsilon}} \left|\nabla P_i(u)\right|^2 + V_\varepsilon\left(P_i(u)\right)^2 dx \\
&- \int_{\Omega_{i,\varepsilon}^\delta\setminus\Omega_{i,\varepsilon}} f_\lambda\left(P_i(u)\right)P_i(u)\,dx.
\end{aligned}$$

Proof. Note that $w = P_i'(u)u$ solves

$$\begin{cases}
\Delta w - V_\varepsilon w + f_\lambda'(P_i(u_i))w = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}, \\
w = u & \text{in } \Omega_{i,\varepsilon}, \\
w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta.
\end{cases}$$

Since $w - P_i(u) \in H_0^1(\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon})$, we deduce that $\Gamma_\varepsilon'(P_i(u))P_i'(u)u = \Gamma_\varepsilon'(P_i(u))P_i(u)$. Then, the claim follows. □

Note that we take $\alpha = 2\frac{\mu+1}{\mu-1} \geqslant 2\max\{\frac{p+1}{p-1}, \frac{\mu+1}{\mu-1}\}$.

Proposition 9. *For sufficiently small $\varepsilon > 0$,*

$$\left\| I'_\varepsilon(\vec{u}) \right\| \geqslant \frac{1}{2}\varepsilon^\alpha \quad \text{if } 2\varepsilon^\alpha \leqslant \left\| P_i(u_i) \right\|_\varepsilon \leqslant 3M\varepsilon^\alpha \quad \text{for some } i \in \{0, \dots, k\}.$$

Proof. We denote $w_i = P_i(u_i)$. Note that $\lambda < b/10$. Then, from Proposition 8, conditions (f2-1) and (f3-2), we see that

$$\begin{aligned}
\Gamma'_\varepsilon\bigl(P_i(u_i)\bigr)P'_i(u_i)u_i &= \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2\,dx - \int_{\Omega_{i,\varepsilon}} g(x,u_i)u_i\,dx \\
&\quad + \int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2\,dx - \int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}} f_\lambda(w_i)w_i\,dx \\
&\geqslant \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2\,dx - C\int_{\Omega_{i,\varepsilon}} u_i^{\mu+1} + u_i^{p+1}\,dx \\
&\quad + \int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2\,dx - \lambda\int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}} (w_i)^2\,dx \\
&\geqslant \int_{\Omega_{i,\varepsilon}} |\nabla u_i|^2 + V_\varepsilon u_i^2\,dx - C\int_{\Omega_{i,\varepsilon}} u_i^{\mu+1} + u_i^{p+1}\,dx + \frac{1}{2}\int_{\Omega^\delta_{i,\varepsilon}\setminus\Omega_{i,\varepsilon}} |\nabla w_i|^2 + V_\varepsilon(w_i)^2\,dx \\
&\geqslant \frac{1}{2}\|w_i\|_\varepsilon^2 - C\varepsilon^{n(p-1)/2-(p+1)}\|w_i\|_{H_0^1}^{p+1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|w_i\|_{H_0^1}^{\mu+1} \\
&\geqslant \left\{\frac{1}{2} - C\varepsilon^{n(p-1)/2-(p+1)}\|w_i\|_\varepsilon^{p-1} - C\varepsilon^{n(\mu-1)/2-(\mu+1)}\|w_i\|_\varepsilon^{\mu-1}\right\}\|w_i\|_\varepsilon^2.
\end{aligned}$$

Hence for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\left\| \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} \right\| &\geqslant \frac{\Gamma'_\varepsilon(P_i(u_i))P'_i(u_i)u_i}{\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}} \\
&\geqslant \left\{\frac{1}{2} - C\varepsilon^{\frac{n(p-1)}{2}-(p+1)}\left\|P_i(u_i)\right\|_\varepsilon^{p-1} - C\varepsilon^{\frac{n(\mu-1)}{2}-(\mu+1)}\left\|P_i(u)\right\|_\varepsilon^{\mu-1}\right\}\frac{\|P_i(u_i)\|_\varepsilon^2}{\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}} \\
&\geqslant \left\{\frac{1}{2} - C\varepsilon^{\frac{n(p-1)}{2}-(p+1)}\left\|P_i(u_i)\right\|_\varepsilon^{p-1} - C\varepsilon^{\frac{n(\mu-1)}{2}-(\mu+1)}\left\|P_i(u)\right\|_\varepsilon^{\mu-1}\right\}\left\|P_i(u_i)\right\|_\varepsilon \\
&\geqslant \frac{1}{2}\varepsilon^\alpha.
\end{aligned}$$

Then, the claim follows from Proposition 4. □

Proposition 10. *Let $E > 0$ be a given constant. Then for sufficiently large $R_1 > 0$, there exists a constant ε_0 such that if $I_\varepsilon(\vec{u}) \leqslant E$, $\varepsilon \in (0, \varepsilon_0)$, $\frac{R_1}{4M} \leqslant \|P_i(u_i)\|_\varepsilon \leqslant R_1$ for some $i = 1, \dots, k$, then $\|I'_\varepsilon(\vec{u})\| \geqslant 1$.*

Proof. Denoting $w_i = P_i(u_i)$ and $F_\lambda(t) = \int_0^t f_\lambda(s)\,ds$, we see from Proposition 8 and the condition (f3-1) that

$$\begin{aligned}
&(\mu+1)\Gamma_\varepsilon\bigl(P_i(u)\bigr) - \Gamma'_\varepsilon\bigl(P_i(u)\bigr)P'_i(u)u \\
&\quad = \frac{\mu-1}{2}\int_{\Omega_{i,\varepsilon}} |\nabla u|^2 + V_\varepsilon u^2\,dx + \int_{\Omega_{i,\varepsilon}} g(x,u)u - (\mu+1)G(x,u)\,dx
\end{aligned}$$

$$+\int_{\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}} f_\lambda(w_i)w_i-(\mu+1)F_\lambda(w_i)\,dx+\frac{\mu-1}{2}\int_{\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}}|\nabla w_i|^2+V_\varepsilon(w_i)^2\,dx$$

$$\geqslant\frac{\mu-1}{2}\int_{\Omega_{i,\varepsilon}}|\nabla u|^2+V_\varepsilon u^2\,dx+\frac{\mu-1}{2}\int_{\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}}|\nabla w_i|^2+V_\varepsilon(w_i)^2\,dx-\int_{\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}^{-\delta}}(\mu+1)\frac{\lambda}{2}(w_i)^2\,dx$$

$$\geqslant\frac{\mu-1}{4}\|w_i\|_\varepsilon^2.$$

By Proposition 4, we see that

$$I_\varepsilon'(\vec{u})(0,\dots,0,u_i,0,\dots,0)=\frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i}u_i+O\big(e^{-c/\varepsilon}\big).$$

Then it follows that

$$\begin{aligned}\big\|I_\varepsilon'(\vec{u})\big\|&\geqslant\frac{-\Gamma_\varepsilon'(P_i(u_i))P_i'(u_i)u_i+O(e^{-c/\varepsilon})}{\|(0,\dots,0,u_i,0,\dots,0)\|_\varepsilon}\\&\geqslant\frac{\frac{\mu-1}{4}\|P_iu_i\|_\varepsilon^2-(\mu+1)\Gamma_\varepsilon(P_i(u_i))+O(e^{-c/\varepsilon})}{\|(0,\dots,0,u_i,0,\dots,0)\|_\varepsilon}\\&\geqslant\frac{\frac{\mu-1}{4^3M^2}(R_1)^2-(\mu+1)I_\varepsilon(\vec{u})+O(e^{-c/\varepsilon})}{R_1}\\&\geqslant\frac{\frac{\mu-1}{4^3M^2}(R_1)^2-(\mu+1)E+O(e^{-c/\varepsilon})}{R_1}.\end{aligned}$$

Thus we conclude that for sufficiently large $R_1>0$, there exists $\varepsilon_0>0$ satisfying $\|I_\varepsilon'(\vec{u})\|\geqslant 1$, $\varepsilon\in(0,\varepsilon_0)$. □

Proposition 11. *Suppose that* (f3-1) *holds. Then, there exists $h_i\in H_0^1(\Omega_{i,\varepsilon}^{-\delta})\setminus\{0\}$ and $t_i>0$ for each $i\in\{1,\dots,k\}$ such that $\Gamma_\varepsilon(th_i)<0$ for $t>t_i$ and small $\varepsilon>0$.*

Proof. Note that for $u\in H_0^1(\Omega_{i,\varepsilon}^{-\delta})$,

$$\Gamma_\varepsilon\big(P_i(tu)\big)=\Gamma_\varepsilon(tu)\leqslant\frac{M^2t^2}{2}\int_{\Omega_{i,\varepsilon}}|\nabla u|^2+V_\varepsilon u^2\,dx-\int_{\Omega_{i,\varepsilon}}H(tu)\,dx.$$

Here H is defined as in (1.8). By condition (f3-1) and the definition of $\tilde{f}$, there exist constants $C>0$, $\mu>1$ such that $H(t)\geqslant Ct^{\mu+1}$ for all $t\geqslant 1$. This implies that for any nonnegative function $u_i\in H_0^1(\Omega_{i,\varepsilon})\setminus\{0\}$, there exists $t_i>0$ satisfying

$$\Gamma_\varepsilon(tu_i)<0,\quad t\geqslant t_i.$$

Then, the claim follows. □

Proposition 12. *For each $\varepsilon>0$, Γ_ε satisfies the* (PS) *condition.*

Proof. Note from the definition of g that $(\mu+1)G(x,t)\leqslant g(x,t)t$ for $x\in\bigcup_{i=0}^{k+m}\Omega_{i,\varepsilon}^{-\delta}$, $g(x,t)t-(\mu+1)G(x,t)\geqslant -\frac{(\mu+1)\lambda}{2}t^2$ for $x\notin\bigcup_{i=0}^{k+m}\Omega_{i,\varepsilon}^{-\delta}$. Then, we can follow the same scheme as in the proof of Lemma 1.1 in [16]. □

3. Proof of Theorem 1

We use the same notations as in Section 2 with $m=0$. By the assumptions, $m_1=\cdots=m_l=0$ and $m_{l+1},\dots,m_k>0$. Recall that $M>1$ is a constant, independent of small $\varepsilon>0$, such that

$$\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}\leqslant\big\|P_i(u_i)\big\|_\varepsilon\leqslant M\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}}\quad\text{for all } u_i\in H^1(\Omega_{i,\varepsilon}),\ i\in\{0,\dots,k\}.$$

Now we take $\alpha \equiv 2\frac{\mu+1}{\mu-1} \geqslant 2\max\{\frac{p+1}{p-1}, \frac{\mu+1}{\mu-1}\}$. By Proposition 6, for each $i \in \{0, \dots, k\}$, it holds that for small $\varepsilon > 0$,

$$\inf\{\Gamma_\varepsilon(u);\ u \in H_0^1(\Omega_{i,\varepsilon}^\delta),\ \|u\|_\varepsilon = 4M\varepsilon^\alpha\} \geqslant 6M^2\varepsilon^{2\alpha}.$$

By Proposition 11, we can take a nonnegative function $h_i \in H_0^1(\Omega_{i,\varepsilon})$ such that for some $t_i(h_i) > 0$, $\Gamma_\varepsilon(P_i(th_i)) = \Gamma_\varepsilon(th_i) < 0$ for $t \geqslant t_i(h_i)$ and each $i \in \{1, \dots, k\}$. Let $G_\varepsilon^i = \{\gamma \in C([0,1], H_0^1(\Omega_{i,\varepsilon}^\delta)) \mid \gamma(0) = 0,\ \gamma(1) = h_i\}$ and

$$c_\varepsilon^i := \inf_{\gamma \in G_\varepsilon^i} \max_{t \in [0,1]} \Gamma_\varepsilon(P_i(\gamma(t))) \geqslant 6M^2\varepsilon^{2\alpha}, \quad i \in \{1, \dots, k\}. \tag{3.1}$$

Some functions $h_i^\varepsilon, i = 1, \dots, k$ were introduced in Section 1 when we define C_ε^i. Now we take $h_i^\varepsilon(x) = h_i(x/\varepsilon)$. Then, it is obvious that $C_\varepsilon^i \geqslant \varepsilon^n c_\varepsilon^i$ for each $i = 1, \dots, k$. By Proposition 5, there exists a critical point $u_i \in H^1(\Omega_{i,\varepsilon})$ of $\Gamma_\varepsilon^i = \Gamma_\varepsilon \circ P_i$ which satisfying

$$\Delta P_i(u_i^\varepsilon) - V_\varepsilon P_i(u_i^\varepsilon) + g(x, P_i(u_i^\varepsilon)) = 0 \quad \text{in } \Omega_{i,\varepsilon}^\delta.$$

For each $i \in \{1, \dots, k\}$, let S_ε^i the set of critical points $u \in H^1(\Omega_{i,\varepsilon})$ of $\Gamma_\varepsilon^i \equiv \Gamma_\varepsilon \circ P_i$ with $\Gamma_\varepsilon^i(u_\varepsilon^i) = c_\varepsilon^i$. We define Π_ε^i the set of solution $u \in H^1(\Omega_{i,\varepsilon})$ of Γ_ε^i with $\Gamma_\varepsilon^i(u) \leqslant c_\varepsilon^i$.

Then we see the following result.

Proposition 13. *For each $i \in \{1, \dots, k\}$, Π_ε^i is compact, and for $i \in \{1, \dots, l\}$, $\limsup_{\varepsilon \to 0} \max\{\|u\|_\varepsilon \mid u \in \Pi_\varepsilon^i\} = 0$, and for $i \in \{l+1, \dots, k\}$, $\limsup_{\varepsilon \to 0} \max\{\|u\|_\varepsilon \mid u \in \Pi_\varepsilon^i\} < \infty$. Furthermore, for each $d > 0$, there exist constants $C, c > 0$ such that $P_i(u_\varepsilon^i)(x) \leqslant C\exp(-\frac{c}{\varepsilon}\operatorname{dist}(\varepsilon x, A_i^d))$ for $i \in \{1, \dots, k\}$ and $u_\varepsilon^i \in \Pi_\varepsilon^i$.*

Proof. The compactness follows from Proposition 5.

It follows by the same argument as in the proof of Proposition 4 in [9] that for $j = 1, \dots, l$, $\limsup_{\varepsilon \to 0} \Gamma_\varepsilon^j(u_\varepsilon^j) = 0$ and for $j = l+1, \dots, k$, $\limsup_{\varepsilon \to 0} \Gamma_\varepsilon^j(u_\varepsilon^j) \leqslant L_{m_j}(U)$ for $U \in S_{m_j}$. We see from (f3-1) and the definition of $\tilde{f}$ that

$$\begin{aligned}
\Gamma_\varepsilon^j(u_\varepsilon^j) &\geqslant \frac{1}{2}\int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx - \frac{1}{\mu+1}\int_{\Omega_{j,\varepsilon}} g(x, P_j(u_\varepsilon^j))P_j(u_\varepsilon^j)\,dx \\
&\quad - \frac{\lambda}{2}\int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2\,dx \\
&= \left(\frac{1}{2} - \frac{1}{\mu+1}\right)\int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx \\
&\quad + \frac{1}{\mu+1}\int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}} g(x, P_j(u_\varepsilon^j))P_j(u_\varepsilon^j)\,dx - \frac{\lambda}{2}\int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2\,dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu+1}\right)\int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx - \frac{\lambda}{2}\int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} (P_j(u_\varepsilon^j))^2\,dx.
\end{aligned}$$

Thus, we see from the condition $\lambda < \min\{\frac{b}{10}, b(\frac{1}{2} - \frac{1}{\mu+1})\}$ that

$$\begin{aligned}
\Gamma_\varepsilon^j(u_\varepsilon^j) &\geqslant \left(\frac{1}{2} - \frac{1}{\mu+1}\right)\int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx - \frac{\lambda}{2b}\int_{\Omega_{j,\varepsilon}^\delta \setminus \Omega_{j,\varepsilon}^{-\delta}} V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx \\
&\geqslant \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\mu+1}\right)\int_{\Omega_{j,\varepsilon}^\delta} |\nabla P_j(u_\varepsilon^j)|^2 + V_\varepsilon(P_j(u_\varepsilon^j))^2\,dx.
\end{aligned}$$

Then the second claim follows.

From the elliptic estimates (see Proposition 3.5 in [3]), we see that for $i \in \{1, \dots, l\}$,

$$\limsup_{\varepsilon\to 0} \max\bigl\{ \|u_\varepsilon^i\|_{L^\infty} \bigm| u \in \Pi_\varepsilon^i \bigr\} = 0,$$

and for $i \in \{l+1, \dots, k\}$, $\limsup_{\varepsilon\to 0} \max\{\|u\|_{L^\infty} \mid u \in \Pi_\varepsilon^i\} < \infty$. Then, we can show as in [9] that for any small $d > 0$, $\lim_{\varepsilon\to 0} \|u_\varepsilon^i\|_{L^\infty(\Omega_{i,\varepsilon}^\delta \setminus (A_i^d)_\varepsilon)} = 0$ uniformly with respect to $u_\varepsilon^i \in S_\varepsilon^i$. Then, we can show also as in [9] that for some $c, C > 0$ that $P_i(u_\varepsilon^i)(x) \leqslant C \exp(-\frac{c}{\varepsilon}\operatorname{dist}(\varepsilon x, A_i^d))$ for $i \in \{1, \dots, k\}$ and $u_\varepsilon^i \in \Pi_\varepsilon^i$. This completes the proof. □

Proposition 14. *For each $i \in \{1, \dots, k\}$ and small $\varepsilon > 0$, there exist $\gamma_\varepsilon^i \in G_\varepsilon^i$, $d_\varepsilon^i \in (c_\varepsilon^i, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon})]$ and $R_2 > 0$ such that $\max_{t\in[0,1]} \Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) \leqslant d_\varepsilon^i$, and $\|\gamma_\varepsilon^i(t)\|_{\varepsilon,\Omega_{i,\varepsilon}} \leqslant R_2$ if $\Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) \in [c_\varepsilon^i - \varepsilon^{2\alpha}, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon})]$ for some $t \in [0, 1]$.*

Proof. From Proposition 13, we see that $R' \equiv \sup_{0<\varepsilon<1} \max_{u\in\Pi_\varepsilon^i} \|u\|_\varepsilon < \infty$. Then, we can find $d_\varepsilon^i \in (c_\varepsilon^i, c_\varepsilon^i + \exp(-\frac{1}{\varepsilon})]$ and $a(\varepsilon) > 0$ such that when $\Gamma_\varepsilon^i(u) \leqslant d_\varepsilon^i$ and $\|u\|_\varepsilon \geqslant 2R'$, $|(\Gamma_\varepsilon^i)'(u)| \geqslant a(\varepsilon)$. From the definition of c_ε^i, we can find a path ζ_ε^i such that $\Gamma_\varepsilon^i(\zeta_\varepsilon^i(t)) \leqslant d_\varepsilon^i$ for $t \in [0, 1]$. Suppose that $\Gamma_\varepsilon^i(\zeta_\varepsilon^i(t)) \in [c_\varepsilon^i - \varepsilon^\alpha, d_\varepsilon^i]$ and $\|\zeta_\varepsilon^i(t)\|_\varepsilon > 2R'$ for some $t \in (0, 1)$. Then, by a deformation lemma, we can deform ζ_ε^i to $\gamma_\varepsilon^i \in G_\varepsilon^i$ such that if $\|\gamma_\varepsilon^i(t)\|_\varepsilon \geqslant 2R'$ for $t \in (0, 1)$, $\Gamma_\varepsilon^i(\gamma_\varepsilon^i(t)) < c_\varepsilon^i - \varepsilon^\alpha$. Thus, if we take $R_2 = 2R'$, our conclusion holds. □

For $i \in \{l+1, \dots, k\}$, we denote

$$E_i = L_{m_i}(U), \quad U \in S_{m_i}.$$

Here we state a result proved in [6].

Proposition 15. *For $n = 1, 2$ and $m_i > 0$, there exist $T > 0$ and a path $\gamma_i : [0, T] \to H^1(\mathbb{R}^n)$, such that*

(i) $$L_{m_i}\bigl(\gamma_i(T)\bigr) < -1, \qquad \max_{s\in[0,T]} L_{m_i}\bigl(\gamma_i(s)\bigr) = E_i;$$
(ii) *there exists $T_0 \in (0, T)$ such that $\gamma(T_0) \in S_{m_i}$ and $L_{m_i}(\gamma_i(T_0)) = E_i$ and $L_{m_i}(\gamma_i(t)) < E_i$ for $\|\gamma_i(t) - \gamma_i(T_0)\| \geqslant \delta$;*
(iii) *there exist $C, c > 0$ such that for any $t \in [0, T]$,*

$$\bigl|\gamma_i(t)(x)\bigr| + \bigl|\nabla_x \gamma_i(t)(x)\bigr| \leqslant Ce^{-c|x|}.$$

Proposition 16. *For each $i \in \{1, \dots, l\}$, $\lim_{\varepsilon\to 0} c_\varepsilon^i = 0$, and for each $i \in \{l+1, \dots, k\}$, $\lim_{\varepsilon\to 0} c_\varepsilon^i \leqslant L_{m_i}(U)$ with $U \in S_{m_i}$.*

Proof. The behavior $\lim_{\varepsilon\to 0} c_\varepsilon^i = 0$, $i \in \{1, \dots, l\}$ can be proved by the same method as in the proof of Proposition 4 in [9].

For the proof of remained cases, we find a nonnegative function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 0$ for $|x| \leqslant \delta/2$ and $\phi(x) = 0$ for $|x| \geqslant \delta$. For $x_i \in A_i$ and $U \in S_{m_i}$, we define

$$\gamma_\varepsilon(t)(x) = \begin{cases} \phi(\varepsilon(x - x_i))U(\frac{x-x_i}{t}) & n \geqslant 3, \\ \phi(\varepsilon(x - x_i))\gamma_i(t)(x) & n = 1, 2, \end{cases}$$

where γ_i is the curve satisfying the properties of Proposition 15. Then, we see that

$$\lim_{\varepsilon\to 0} \Gamma_\varepsilon^j\bigl(\gamma_\varepsilon(t)\bigr) = \begin{cases} \frac{t^{n-2}}{2}\int_{\mathbb{R}^n} |\nabla U|^2\,dx + \frac{t^n}{2}\int_{\mathbb{R}^n} m_i U^2\,dx - t^n \int_{\mathbb{R}^n} F(U)\,dx & \text{for } n \geqslant 3, \\ L_{m_i}(\gamma_i(t)) & \text{for } n = 1, 2. \end{cases}$$

Thus,

$$\lim_{\varepsilon\to 0} \max_{t\in(0,\infty)} \Gamma_\varepsilon^j\bigl(\gamma_\varepsilon(t)\bigr) \leqslant L_{m_i}(U).$$

This completes the proof. □

Now we define $\vec{\gamma}(s) = (0, \gamma_\varepsilon^1(s_1), \ldots, \gamma_\varepsilon^k(s_k))$ with $s = (s_1, \ldots, s_k) \in [0,1]^k := \overbrace{[0,1] \times \cdots \times [0,1]}^{k\text{-times}}$. We define

$$d_\varepsilon := \max_{s \in [0,1]^k} I_\varepsilon\big(\vec{\gamma}(s)\big).$$

It follows from $\max_{s\in[0,1]^k} I_\varepsilon(\vec{\gamma}(s)) = \sum_{i=1}^k \max_{s_i \in [0,1]} \Gamma_\varepsilon^i(\gamma_\varepsilon^i(s_i)) + O(\exp(-\frac{c}{\varepsilon}))$ for some $c > 0$, the definition of c_ε^i, and Proposition 14 that for some $c > 0$,

$$\sum_{i=1}^k c_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right) \leqslant \max_{s\in[0,1]^k} I_\varepsilon\big(\vec{\gamma}(s)\big) \leqslant \sum_{i=1}^k d_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right)$$
$$\leqslant \sum_{i=1}^k c_\varepsilon^i + O\left(\exp\left(-\frac{c}{\varepsilon}\right)\right).$$

Thus we see that $d_\varepsilon = c_\varepsilon^1 + \cdots + c_\varepsilon^k + O(e^{-c/\varepsilon})$ for some $c > 0$. From Proposition 16, we find that $d_\varepsilon \leqslant E$ for some constant E, independent of small $\varepsilon > 0$.

Let $R_1, R_2 > 0$ be constants given in Proposition 10 and Proposition 14, respectively. We define $R = 4M\max\{R_1, R_2\}$. In what follows, for the sake of simplicity, we write $P_i(u)$ for $P_i(u_i) = P_i(u|_{\Omega_{i,\varepsilon}})$, $u \in H_\varepsilon$. For each $c, d \in \mathbb{R}$, we define $\Gamma_\varepsilon^c = \{u \in H_\varepsilon \mid \Gamma_\varepsilon(u) \leqslant c\}$ and

$$I_\varepsilon^d = \big\{\vec{u} \in H_\varepsilon\big(\Omega_0^\varepsilon\big) \times \cdots \times H_\varepsilon\big(\Omega_k^\varepsilon\big) \bigm| I_\varepsilon(\vec{u}) \leqslant d\big\}.$$

Now we have the following existence result.

Proposition 17. *For sufficiently small $\varepsilon > 0$, there exists a critical point $u_\varepsilon \in H_\varepsilon$ of Γ_ε such that $u_\varepsilon \in (\Gamma_\varepsilon^{d_\varepsilon} \setminus \Gamma_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}})$ and*

$$u_\varepsilon \in \big\{u \in H_\varepsilon \bigm| \|P_0(u)\|_\varepsilon \leqslant 5M\varepsilon^\alpha \text{ and } \varepsilon^\alpha \leqslant \|P_i(u)\|_\varepsilon \leqslant 2R,\ i = 1, \ldots, k\big\}.$$

Proof. Recall from Proposition 4 that $\vec{u}$ is a critical point of I_ε if and only if $\varphi(\vec{u}) \in H_\varepsilon$ is a critical point of Γ_ε.

To the contrary, suppose that there are not such critical points of Γ_ε. Then, there are no critical points $\vec{u}$ of I_ε on the set

$$Y_\varepsilon \equiv \big(I_\varepsilon^{d_\varepsilon} \setminus I_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}}\big) \cap \big\{\vec{u} \bigm| \|P_0(u_0)\|_\varepsilon \leqslant 5M\varepsilon^\alpha,\ \ \varepsilon^\alpha \leqslant \|P_i(u)\|_\varepsilon \leqslant 2R \text{ for } i = 1, \ldots, k\big\}.$$

We define

$$A_\varepsilon \equiv \big(I_\varepsilon^{d_\varepsilon} \setminus I_\varepsilon^{d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}}\big) \cap \big\{\vec{u} \bigm| \|P_0(u_0)\|_\varepsilon \leqslant 3M\varepsilon^\alpha,\ 2\varepsilon^\alpha \leqslant \|P_i(u)\|_\varepsilon \leqslant R \text{ for } i = 1, \ldots, k\big\}.$$

Let χ be a Lipschitz continuous function such that $\chi(\vec{u}) = 1$ if $\vec{u} \in A_\varepsilon$, and $\chi(\vec{u}) = 0$ if $I_\varepsilon(\vec{u}) \leqslant d_\varepsilon - \varepsilon^{2\alpha}$ or

$$\vec{u} \notin \big\{\vec{u} \bigm| \|P_0(u_0)\|_\varepsilon \leqslant 5M\varepsilon^\alpha,\ \varepsilon^\alpha \leqslant \|P_i(u)\|_\varepsilon \leqslant 2R \text{ for } i = 1, \ldots, k\big\}.$$

Let $\vec{\gamma}(s,t)$, $t \geqslant 0$ be the solution of

$$\frac{\partial}{\partial t}\vec{\gamma}(s,t) = -\chi\big(\vec{\gamma}(s,t)\big)\nabla I_\varepsilon\big(\vec{\gamma}(s,t)\big)/\big\|\nabla I_\varepsilon\big(\vec{\gamma}(s,t)\big)\big\|, \quad \vec{\gamma}(s,0) = \vec{\gamma}(s),\ s \in [0,1]^k.$$

Note that $\vec{\gamma}$ is defined for all $t \geqslant 0$, and also that $I_\varepsilon(\vec{\gamma}(s,t))$ is nonincreasing in $t \geqslant 0$. We shall prove that there exits $t^* = t^*(\varepsilon) > 0$ such that

$$\max_{s\in[0,1]^k} I_\varepsilon\big(\vec{\gamma}\big(s, t^*\big)\big) \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.$$

If $I_\varepsilon(\vec{\gamma}(s)) < d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$ for some $s \in [0,1]^k$, then $I_\varepsilon(\vec{\gamma}(s,t)) \leqslant I_\varepsilon(\vec{\gamma}(s)) \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$ for any $t \geqslant 0$ by the monotonicity. Now let s be a point in $[0,1]^k$ such that $I_\varepsilon(\vec{\gamma}(s)) \geqslant d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$. Then we claim that $\|P_i(\gamma_i(s))\|_\varepsilon > 3M\varepsilon^\alpha$ for all $i \in \{1, \ldots, k\}$. Indeed, if it is not true, it follows from Proposition 7 that $|\Gamma_\varepsilon(P_i(\gamma_i(s)))| \leqslant 5M^2\varepsilon^{2\alpha}$ for some $i \in \{1, \ldots, k\}$. Then, we see from (3.1) that for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
I_\varepsilon\big(\vec\gamma(s)\big) &= \sum_{i=1}^{k} \Gamma_\varepsilon\big(P_i\big(\gamma_i(s_i)\big)\big) + O\big(e^{-c/\varepsilon}\big)\\
&\leqslant \big(c_\varepsilon^1 + \cdots + \check{c}_\varepsilon^i + \cdots + c_\varepsilon^k\big) + 5M^2\varepsilon^{2\alpha} + O\big(e^{-c/\varepsilon}\big)\\
&\leqslant d_\varepsilon - c_\varepsilon^i + 5M^2\varepsilon^{2\alpha} + O\big(e^{-c/\varepsilon}\big)\\
&< d_\varepsilon - 6M^2\varepsilon^{2\alpha} + 5M^2\varepsilon^{2\alpha} + O\big(e^{-c/\varepsilon}\big)\\
&< d_\varepsilon - \varepsilon^{2\alpha}.
\end{aligned}
$$

This contradicts the above condition $I_\varepsilon(\vec\gamma(s)) \geqslant d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$, and the claim follows. Now from the definition of R, we see that $\vec\gamma(s) \in A_\varepsilon$ if $I_\varepsilon(\vec\gamma(s)) \geqslant d_\varepsilon - \frac{1}{2}\varepsilon^{2\alpha}$.

By our assumption and (PS) condition proved in Proposition 4(ii), we see that for given $\varepsilon > 0$, there exists a positive constant $c(\varepsilon) > 0$ such that $\|I_\varepsilon'(\vec u)\| \geqslant c(\varepsilon) > 0$ if $\vec u \in Y_\varepsilon$. Hence there exits a number $t^* = t^*(\varepsilon) > 0$ such that if $\gamma(s,t) \in A_\varepsilon$ for all $0 \leqslant t \leqslant t^*$, then

$$
I_\varepsilon\big(\vec\gamma\big(s,t^*\big)\big) \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.
$$

Hence we need only prove that if $\vec\gamma(s,t_1) \notin A_\varepsilon$ for some $t_1 \in (0,t^*]$, then

$$
I_\varepsilon\big(\vec\gamma(s,t_1)\big) \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}. \tag{3.2}
$$

Assume that $\vec\gamma(s,t_1) \notin A_\varepsilon$ for some $t_1 \in (0,t^*]$. Then there holds one of the following

(C0) $\|P_0(\gamma_0(s,t_1))\|_\varepsilon \geqslant 3M\varepsilon^\alpha$;
(C1) $\|P_i(\gamma_i(s,t_1))\|_\varepsilon < 2\varepsilon^\alpha$ for some $i \in \{1,\dots,k\}$;
(C2) $\|P_i(\gamma_i(s,t_1))\|_\varepsilon > R/2$ for some $i \in \{1,\dots,k\}$;
(C3) $I_\varepsilon(\vec\gamma(s,t_1)) < d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}$.

The case (C3) implies (3.2).

In case (C1), it follows from $\|P_i(\gamma_i(s))\|_\varepsilon > 3M\varepsilon^\alpha$ and $\|P_i(\gamma_i(s,t_1))\|_\varepsilon < 2\varepsilon^\alpha$ that $\|\gamma_i(s)\|_{\varepsilon,\Omega_{i,\varepsilon}} > 3\varepsilon^\alpha$, $\|\gamma_i(s,t_1)\|_{\varepsilon,\Omega_{i,\varepsilon}} < 2\varepsilon^\alpha$ for sufficiently small $\varepsilon > 0$. Thus there exists an interval $[t_2,t_3] \subset [0,t_1]$ such that $\|\gamma_i(s,t_2)\|_{\varepsilon,\Omega_{i,\varepsilon}} = 3\varepsilon^\alpha$, $\|\gamma_i(s,t_3)\|_{\varepsilon,\Omega_{i,\varepsilon}} = 2\varepsilon^\alpha$ and $2\varepsilon^\alpha \leqslant \|\gamma_i(s,t)\|_{\varepsilon,\Omega_{i,\varepsilon}} \leqslant 3\varepsilon^\alpha$ for all $t \in [t_2,t_3]$. Then we see that

$$
\begin{aligned}
\varepsilon^\alpha &\leqslant \big\|\vec\gamma(s,t_3) - \vec\gamma(s,t_2)\big\|_{\varepsilon,\Omega_{i,\varepsilon}} = \left\| \int_{t_2}^{t_3} \frac{\partial\vec\gamma}{\partial\tau}(s,\tau)\,d\tau \right\|_{\varepsilon,\Omega_{i,\varepsilon}}\\
&= \left\| \int_{t_2}^{t_3} \chi\big(\vec\gamma(s,\tau)\big)\nabla I_\varepsilon\big(\vec\gamma(s,\tau)\big)/\big\|\nabla I_\varepsilon\big(\vec\gamma(s,\tau)\big)\big\|\,d\tau \right\|_{\varepsilon,\Omega_{i,\varepsilon}} \leqslant \int_{t_2}^{t_3} \chi\big(\vec\gamma(s,\tau)\big)\,d\tau.
\end{aligned}
$$

Moreover since $\|I_\varepsilon'(\vec u)\|_\varepsilon \geqslant \frac{1}{2}\varepsilon^\alpha$ for $2\varepsilon^\alpha \leqslant \|P_i(u_i)\|_\varepsilon \leqslant 3M\varepsilon^\alpha$ by (i) of Proposition 9, we see that for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
I_\varepsilon\big(\vec\gamma(s,t_1)\big) &= I_\varepsilon\big(\vec\gamma(s)\big) + \int_0^{t_1} I_\varepsilon'\big(\vec\gamma(s,\tau)\big)\frac{\partial\vec\gamma}{\partial\tau}(s,\tau)\,d\tau\\
&= I_\varepsilon\big(\vec\gamma(s)\big) - \int_0^{t_1} \chi\big(\vec\gamma(s,\tau)\big)\big\|I_\varepsilon'\big(\vec\gamma(s,\tau)\big)\big\|\,d\tau\\
&\leqslant d_\varepsilon - \frac{1}{2}\varepsilon^\alpha \int_{t_2}^{t_3} \chi\big(\vec\gamma(s,\tau)\big)\,d\tau \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}.
\end{aligned}
$$

We get (3.2) as desired.

In case (C0), it follows from $\|P_0(\gamma_0(s))\|_\varepsilon = 0$ and $\|P_0(\gamma_0(s,t_1))\|_\varepsilon \geqslant 3M\varepsilon^\alpha$ that there exists an interval $[t_2,t_3] \subset [0,t_1]$ such that $\|\gamma_0(s,t_2)\|_{\varepsilon,\Omega_0^\varepsilon} = 3\varepsilon^\alpha$, $\|\gamma_0(s,t_3)\|_{\varepsilon,\Omega_0^\varepsilon} = 2\varepsilon^\alpha$ and $2\varepsilon^\alpha \leqslant \|\gamma_0(s,t)\|_{\varepsilon,\Omega_0^\varepsilon} \leqslant 3\varepsilon^\alpha$ for all $t \in [t_2,t_3]$. Then, by the same procedure as in the proof of case (C1), we get (3.2).

In case (C2), we assume that $\|P_i(\gamma_i(s,t_1))\|_\varepsilon > \frac{R}{2}$ for some $i \in \{1,\dots,k\}$. From the choice of R, it follows that for sufficiently small $\varepsilon > 0$, $\|\gamma_i(s)\|_{\varepsilon,\Omega_i^\varepsilon} \leqslant \frac{R}{4M}$ and $\|\gamma_i(s,t_1)\|_{\varepsilon,\Omega_i^\varepsilon} > \frac{R}{2M}$. Thus there exists an interval $[t_2,t_3] \subset [0,t_1]$ such that $\|\gamma_i(s,t_2)\|_{\varepsilon,\Omega_{i,\varepsilon}} = \frac{R}{4M}$, $\|\gamma_i(s,t_3)\|_{\varepsilon,\Omega_{i,\varepsilon}} = \frac{R}{2M}$ and $\frac{R}{4M} \leqslant \|\gamma_i(s,t)\|_{\varepsilon,\Omega_{i,\varepsilon}} \leqslant \frac{R}{2M}$ for all $t \in [t_2,t_3]$. We then have $\frac{R}{4M} \leqslant \|P_i(\gamma_i(s,t))\|_\varepsilon \leqslant \frac{R}{2}$ for all $t \in [t_2,t_3]$. We also have $\|P_i(\gamma_i(s,t))\|_\varepsilon \leqslant R$ for all $i \in \{1,\dots,k\}$ and $t \geqslant 0$ due to the cut-off function χ. Note from Proposition 10 that $\|I'_\varepsilon(\vec{u})\| \geqslant 1$ if $\frac{R}{4M} \leqslant \|P_i(u_i)\|_\varepsilon \leqslant R$ for some $i \in \{1,\dots,k\}$ and $\|P_i(u_i)\|_\varepsilon \leqslant R$ for all $i \in \{0,\dots,k\}$. Then, by a similar procedure as in the case (C1), we get (3.2) for sufficiently small $\varepsilon > 0$. Therefore we conclude that

$$\max_{s\in[0,1]^k} I_\varepsilon\big(\vec{\gamma}\big(s,t^*\big)\big) \leqslant d_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}. \tag{3.3}$$

We note that

$$I_\varepsilon\big(\vec{\gamma}\big(s,t^*\big)\big) = \sum_{i=0}^{k} \Gamma_\varepsilon\big(P_i\big(\gamma_i\big(s,t^*\big)\big)\big) + O\big(e^{-c/\varepsilon}\big),$$

and from Proposition 7 that

$$\Gamma_\varepsilon\big(P_0\big(\gamma_0\big(s,t^*\big)\big)\big) \geqslant 0.$$

Thus, it follows that

$$I_\varepsilon\big(\vec{\gamma}\big(s,t^*\big)\big) \geqslant \sum_{i=1}^{k} \Gamma_\varepsilon\big(P_i\big(\gamma_i\big(s,t^*\big)\big)\big) + O\big(e^{-c/\varepsilon}\big). \tag{3.4}$$

Furthermore, if $s \in \partial[0,1]^k$, we have $\vec{\gamma}(s,t) = \vec{\gamma}(s)$ for all $t \geqslant 0$ since it follows from (3.1) that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} I_\varepsilon\big(\vec{\gamma}(s)\big) &= \sum_{i=1}^{k} \Gamma_\varepsilon\big(P_i\big(\gamma_i(s_i)\big)\big) + O\big(e^{-c/\varepsilon}\big) \\ &\leqslant \max_i\big(c_\varepsilon^1 + \cdots + \check{c}_\varepsilon^i + \cdots + c_\varepsilon^k\big) + O\big(e^{-c/\varepsilon}\big) \\ &\leqslant d_\varepsilon - \varepsilon^{2\alpha}. \end{aligned}$$

The result [13, Proposition 3.4] of Coti Zelati and Rabinowitz says that

$$\max_{s\in[0,1]^k} \sum_{i=1}^{k} \Gamma_\varepsilon\big(P_i\big(\gamma_i\big(s,t^*\big)\big)\big) \geqslant c_\varepsilon^1 + \cdots + c_\varepsilon^k = d_\varepsilon + O\big(e^{-c/\varepsilon}\big).$$

This contradicts (3.3) and (3.4), and hence completes the proof. □

Completion of Proof of Theorem 1. From Proposition 17, there exists a solution u_ε satisfying $\Delta u_\varepsilon - V_\varepsilon(x)u_\varepsilon + g(x,u_\varepsilon) = 0$ for $x \in \mathbb{R}^n$. It also follows from the strong maximum principle that $u_\varepsilon > 0$. By elliptic estimates (see Proposition 3.5 in [3]), we see that $\{\|u_\varepsilon\|_{L^\infty}\}_{\varepsilon<1}$ is bounded. Then by elliptic estimates, we see that for some $\alpha \in (0,1)$, $\{\|u_\varepsilon\|_{C^{1,\alpha}}\}_{\varepsilon<1}$ is bounded.

From the definition of φ on $\mathbb{R}^n \setminus \bigcup_{i=0}^k \Omega_{i,\varepsilon}$, we see that $u_\varepsilon(x) \to 0$ uniformly with respect to small $\varepsilon > 0$ as $\mathrm{dist}(x, \bigcup_{i=0}^k \Omega_{i,\varepsilon}) \to \infty$. Then we can show by a comparison principle that for some $C, c > 0$, $u_\varepsilon(x) \leqslant C\exp(-c\,\mathrm{dist}(x, \bigcup_{i=0}^k \Omega_{i,\varepsilon}))$.

Note that $\{\|u_\varepsilon\|_\varepsilon, \Gamma_\varepsilon(u_\varepsilon)\}_{\varepsilon<1}$ is bounded. Then, it is standard to see (refer [24, Proposition 2.2]) that there exists $\{y_1^\varepsilon,\dots,y_e^\varepsilon\} \subset \bigcup_{i=l+1}^k \overline{\Omega_{i,\varepsilon}}$ satisfying

$$\liminf_{\varepsilon\to 0} u_\varepsilon\big(y_a^\varepsilon\big) > 0 \quad (a = 1, \dots, e),$$
$$\liminf_{\varepsilon\to 0} \min\big\{\big|y_a^\varepsilon - y_b^\varepsilon\big|,\ 1 \leqslant a \neq b \leqslant e\big\} = \infty,$$

and

$$\lim_{\operatorname{dist}(x,\{\Omega_{0,\varepsilon},\dots,\Omega_{l,\varepsilon},y_1^\varepsilon,\dots,y_e^\varepsilon\})\to\infty} u_\varepsilon(x) = 0.$$

Then we deduce that for some $C, c > 0$,

$$u_\varepsilon(x) \leqslant C\exp\big(-c\operatorname{dist}\big(x, \big\{\Omega_{0,\varepsilon}, \dots, \Omega_{l,\varepsilon}, y_1^\varepsilon, \dots, y_e^\varepsilon\big\}\big)\big). \tag{3.5}$$

Taking a subsequence if it is necessary, we can assume that $\lim_{\varepsilon\to 0}\varepsilon y_a^\varepsilon = y_a \in \overline{\bigcup_{i=l+1}^k \Omega_i}$ for each $a = 1, \dots, e$. Suppose that for some $i \in \{l+1, \dots, k\}$, $\liminf_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} = 0$. Then, from the definition of $\varphi(\vec{u}_\varepsilon)$, it is standard to see that for some $C, c > 0$, $\|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} \leqslant C\exp(-c/\varepsilon)$. This implies that some $C, c > 0$, $\Gamma_\varepsilon(P_i(u_\varepsilon)) \leqslant C\exp(-c/\varepsilon)$. This contradicts to the fact $\|P_i(u_\varepsilon)\|_\varepsilon \geqslant \varepsilon^\alpha$. Thus we see that for each $i \in \{l+1, \dots, k\}$, there exists $a \in \{1, \dots, e\}$ with $y_a \in \overline{\Omega_i}$. Then, defining $v_\varepsilon^i(x) \equiv u_\varepsilon(x + y_a^\varepsilon)$, we see that v_ε converges locally uniformly to some v in $C^1(\mathbb{R}^n)$ which satisfies

$$\Delta v - V(y_a)v + g(y_a, v) = 0, \quad v > 0 \quad \text{in } \mathbb{R}^n, \qquad \lim_{|x|\to\infty} v(x) = 0.$$

We define

$$L(v; y_a) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 + V(y_a)v^2\,dx - \int_{\mathbb{R}^n} G(y_a, v)\,dx,$$

where $G(y_a, v) = \int_0^v g(y_a, s)\,ds \leqslant F(t)$. Then, from Proposition 15 for $n = 1, 2$ and by defining $v_t(x) = v(x/t)$, $t > 0$ for $n \geqslant 3$, we see that there exists a continuous path v_t, $t \geqslant 0$ satisfying $v_0 = 0$, $v_1 = v$ and $\limsup_{t\to\infty} L(v_t; y_a) < 0$, $\max_{t\in(0,\infty)} L(v_t; y_a) = L(v; y_a)$. Then, we see that

$$\begin{aligned} L(v; y_a) &= \max_{t\in[0,\infty)} L(v_t; y_a) \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2\,dx - \int_{\mathbb{R}^n} G(y_a, v_t)\,dx \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2\,dx - \int_{\mathbb{R}^n} F(v_t)\,dx. \end{aligned}$$

Then it follows from Pohozaev identity that for $V \in S_{V(y_a)}$,

$$\begin{aligned} L_{V(y_a)}(V) &\leqslant \max_{t\in(0,\infty)}\left[\frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_t|^2 + V(y_a)(v_t)^2\,dx - \int_{\mathbb{R}^n} F(v_t)\,dx\right] \\ &\leqslant L(v; y_a). \end{aligned}$$

We note that if $c < d$, $L_c(V) < L_d(W)$ for $V \in S_c$ and $W \in S_d$ (refer to [9, Proposition 5]). Furthermore, from condition (f3-1), the decaying property (3.5) and the fact $f_\lambda(t) \leqslant \lambda t$ with $\lambda < \min\{b/10, b(1/2 - 1/(\mu+1))\}$, we deduce that for some $C, c > 0$,

$$\begin{aligned} &\frac{1}{2}\int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2\,dx - \int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} G(x, u_\varepsilon)\,dx \\ &\quad\geqslant \frac{1}{2}\int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2\,dx - \frac{1}{\mu+1}\int_{\bigcup_{i=0}^l \Omega_{i,\varepsilon}^\delta} g(x, u_\varepsilon)u_\varepsilon\,dx \end{aligned}$$

$$- \int_{\bigcup_{i=0}^{l}(\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}^{-\delta})} \big(1-\chi(\varepsilon x)\big)\left(F_\lambda(u_\varepsilon) - \frac{1}{\mu+1} f_\lambda(u_\varepsilon)u_\varepsilon\right) dx$$

$$\geqslant \frac{1}{2} \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 \, dx - \frac{1}{\mu+1} \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} g(x,u_\varepsilon)u_\varepsilon \, dx - \frac{\lambda}{2} \int_{\bigcup_{i=0}^{l}(\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}^{-\delta})} (u_\varepsilon)^2 \, dx$$

$$\geqslant \left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 \, dx - \frac{\lambda}{2} \int_{\bigcup_{i=0}^{l}(\Omega_{i,\varepsilon}^{\delta}\setminus\Omega_{i,\varepsilon}^{-\delta})} (u_\varepsilon)^2 \, dx + O\big(e^{-c/\varepsilon}\big).$$

Then we get

$$\frac{1}{2} \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 \, dx - \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} G(x,u_\varepsilon)\, dx \geqslant \frac{1}{3}\left(\frac{1}{2} - \frac{1}{\mu+1}\right) \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 \, dx.$$

Thus, reordering the index $\{a\}$ in $\{y_a^\varepsilon\}$ if it is necessary, we conclude from Proposition 16 that $e = k - l$, $y_a \in A_{l+a}$ and

$$\lim_{\varepsilon\to 0} \int_{\bigcup_{i=0}^{l}\Omega_{i,\varepsilon}^{\delta}} |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2 \, dx = 0.$$

This implies that for each $d > 0$,

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\{x \mid \varepsilon x \notin (\bigcup_{i=l+1}^{k} A_i)^d\})} = 0. \tag{3.6}$$

Then we see that $v_\varepsilon(x) \equiv u_\varepsilon(x/\varepsilon)$ satisfies (1.3). From comparison principle, we deduce that for each $\sigma > 0$, there exists constants $C, c > 0$ satisfying

$$v_\varepsilon(x) \leqslant C \exp\big(-c \operatorname{dist}\big(x, (A_0 \cup \cdots \cup A_k)^\sigma\big)/\varepsilon\big). \tag{3.7}$$

From (3.7), the fact $\|P_0(u_\varepsilon)\|_\varepsilon \leqslant 5M\varepsilon^\alpha$ and an L^∞-elliptic estimate [23, Theorem 8.25], we see that for some $C > 0$, $\|u_\varepsilon\|_{L^\infty(\Omega_0^\varepsilon)} \leqslant C\varepsilon^\alpha$. Then we see that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_0^\varepsilon)} \leqslant \lim_{\varepsilon\to 0} \varepsilon^{-2/(\mu-1)} \varepsilon^{2\frac{\mu+1}{\mu-1}} = \lim_{\varepsilon\to 0} \varepsilon^{2\mu/(\mu-1)} = 0.$$

By the same method as in the proof of [9, Proposition 10], we see that since $\lim_{\varepsilon\to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} = 0$, there exists $C', c' > 0$ satisfying

$$\|u_\varepsilon\|_{L^\infty(\Omega_{0,\varepsilon})} \leqslant C' \exp\big(-c'/\varepsilon\big) \quad \text{for sufficiently small } \varepsilon > 0.$$

By the same method as in the proof of [9, Proposition 10], we see that if $\liminf_{\varepsilon\to 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} = 0$, $i \in \{1, \dots, k\}$, there exists $C', c' > 0$ satisfying $\|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} \leqslant C' \exp(-c'/\varepsilon)$ for some sufficiently small $\varepsilon > 0$. This means that for some $C, c > 0$, $\|P_i(u_\varepsilon)\|_\varepsilon \leqslant C \exp(-c/\varepsilon)$; this contradicts that $\|P_i(u_\varepsilon)\|_\varepsilon \geqslant \varepsilon^\alpha$ for each $i \in \{1, \dots, k\}$. Thus we see that $\liminf_{\varepsilon\to 0} \varepsilon^{-2/(1-\mu)} \|u_\varepsilon\|_{L^\infty(\Omega_{i,\varepsilon})} > 0$ for each $i \in \{1, \dots, k\}$. This proves (i) and (ii) of Theorem 1.

Now we assume that $l = k$ and (f3-2) is also satisfied. We now estimate $\Gamma_\varepsilon(u_\varepsilon; \Omega_{i,\varepsilon})$. From Proposition 17 and (2.2), we have

$$\sum_{i=0}^{k} \Gamma_\varepsilon\big(P_i(u_\varepsilon)\big) \leqslant d_\varepsilon + O\big(e^{-\varepsilon/c}\big). \tag{3.8}$$

Since $\Gamma_\varepsilon(P_0(u_\varepsilon)) \geqslant 0$, we see that

$$\sum_{i=1}^{k} \Gamma_\varepsilon\big(P_i(u_\varepsilon)\big) \leqslant d_\varepsilon + O\big(e^{-\varepsilon/c}\big). \tag{3.9}$$

We define

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + V_\varepsilon(x)u^2\,dx - \int_{\mathbb{R}^n} F(u)\,dx.$$

Then we see from (3.6) and (3.7) that $\Gamma_\varepsilon(u_\varepsilon) = \Gamma_\varepsilon(u_\varepsilon)$ and $\Gamma_\varepsilon(P_i(u_\varepsilon)) = \Gamma_\varepsilon(P_i(u_\varepsilon))$. By the definition of c_i^ε, we see that for each $i \in \{1,\ldots,k\}$,

$$c_i^\varepsilon \leqslant \max_{t\geqslant 0} \Gamma_\varepsilon\big(P_i(tu_\varepsilon)\big) \leqslant \max_{t\geqslant 0} \Gamma_\varepsilon\big(tP_i(u_\varepsilon)\big).$$

By the monotonicity of $\tilde{f}(t)/t$, we see that $g_\varepsilon(t) := \Gamma_\varepsilon(tP_i(u_\varepsilon))$, $t \geqslant 0$ has a unique maximum point. Let $g_\varepsilon(t_\varepsilon) = \max_{t\geqslant 0} g_\varepsilon(t)$.

Now we show $t_\varepsilon = 1 + O(e^{-c/\varepsilon})$. Indeed,

$$\begin{aligned} g_\varepsilon'(1) &= \Gamma_\varepsilon'\big(P_i(u_\varepsilon)\big)P_i(u_\varepsilon) \\ &= \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - f(u_\varepsilon)u_\varepsilon\,dx + O\big(e^{-c/\varepsilon}\big) \\ &= O\big(e^{-c/\varepsilon}\big). \end{aligned}$$

Here we used the fact that $P_i(u_\varepsilon) = O(e^{-c/\varepsilon})$ in $\Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon}$.

Moreover we estimate for $i = 1,\ldots,k$,

$$\begin{aligned} g_\varepsilon''(1) &= \Gamma_\varepsilon''\big(P_i(u_\varepsilon)\big)\big[P_i(u_\varepsilon), P_i(u_\varepsilon)\big] \\ &= \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - f'(u_\varepsilon)(u_\varepsilon)^2\,dx \\ &\leqslant \int_{\Omega_{i,\varepsilon}} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 - \mu_3 f(u_\varepsilon)u_\varepsilon\,dx + O\big(e^{-c/\varepsilon}\big) \\ &= (1-\mu_3)\int_{\Omega_{i,\varepsilon}^\delta} |\nabla u_\varepsilon|^2 + V_\varepsilon(u_\varepsilon)^2 + O\big(e^{-c/\varepsilon}\big) \\ &\leqslant -(\mu_3 - 1)\varepsilon^{2\alpha}, \end{aligned}$$

and so $|g_\varepsilon''(1)| \geqslant (\mu_3 - 1)\varepsilon^{2\alpha}$. Therefore we have $t_\varepsilon = 1 + O(e^{-c/\varepsilon})$ as claimed.

Hence

$$c_i^\varepsilon \leqslant \max_{t\geqslant 0} \Gamma_\varepsilon\big(tP_i(u_\varepsilon)\big) = g_\varepsilon(t_\varepsilon) = g_\varepsilon(1) + O\big(e^{-c/\varepsilon}\big) = \Gamma_\varepsilon\big(P_i(u_\varepsilon)\big) + O\big(e^{-c/\varepsilon}\big). \tag{3.10}$$

Then we have

$$d_\varepsilon = \sum_{i=1}^k c_i^\varepsilon \geqslant \Gamma_\varepsilon(u_\varepsilon) \geqslant \sum_{i=1}^k \Gamma_\varepsilon\big(P_i(u_\varepsilon)\big) + O\big(e^{-c/\varepsilon}\big) \geqslant \sum_{i=1}^k c_i^\varepsilon + O\big(e^{-c/\varepsilon}\big).$$

This implies that for each $i \in \{1,\ldots,l\}$,

$$\big|c_i^\varepsilon - \Gamma_\varepsilon\big(P_i(u_\varepsilon)\big)\big| \leqslant O\big(e^{-c/\varepsilon}\big).$$

From the decay property (3.7), we deduce that $|c_i^\varepsilon - \Gamma_\varepsilon(u_\varepsilon;\Omega_{i,\varepsilon})| \leqslant O(e^{-c/\varepsilon})$ for $i \in \{1,\ldots,l\}$. When (f3-3) holds for $l < k$, by the same argument, we conclude that $|c_i^\varepsilon - \Gamma_\varepsilon(u_\varepsilon;\Omega_{i,\varepsilon})| \leqslant O(e^{-c/\varepsilon})$ for $i \in \{1,\ldots,k\}$.

Finally, noting that $C_\varepsilon^i = \varepsilon^n c_i^\varepsilon + O(e^{-c/\varepsilon})$, $J_\varepsilon(v_\varepsilon;\Omega_i) = \varepsilon^n\Gamma_\varepsilon(u_\varepsilon;\Omega_{i,\varepsilon})$ for $i \in \{1,\ldots,k\}$, we see that v_ε satisfies the desired properties.

The proof of Theorem 1 is completed. □

4. Proof of Theorem 2

For the notational simplicity, we only prove the case $m = 1$. We may assume that $x_1 = 0$ and $\Omega_{k+1} = B_\sigma(0)$, $\sigma > 0$. We use the same notations as in Section 2 with $m = 1$. In addition, we use the following notations: $(\mathbf{u}, u_{k+1}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+1,\varepsilon})$, $\mathbf{u} = (u_0, \ldots, u_k) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$. We write $\Omega_\varepsilon = \Omega_{k+1,\varepsilon}$, $P = P_{k+1}$. For a given $\mathbf{u}$, we define $Q_\varepsilon(u_{k+1}) = I_\varepsilon(\mathbf{u}, u_{k+1})$, $\psi(u_{k+1}) = \varphi(\mathbf{u}, u_{k+1})$, $K_\varepsilon(u_{k+1}) = \Gamma_\varepsilon(P(u_{k+1}))$.

In this section, for a given $\mathbf{u}$, we solve $Q'_\varepsilon(u_{k+1}) = 0$ by the Lyapunov–Schmidt reduction method. In order to construct an approximate solution, we first find a solution Ψ_ε of the localized problem $K'_\varepsilon(u) = 0$. We then find a solution $u_{k+1} = \Phi_\varepsilon(\mathbf{u})$ of $Q'_\varepsilon(u_{k+1}) = 0$ in an exponentially small neighborhood of the solution Ψ_ε to the localized problem. The localized problem will be solved by the Lyapunov–Schmidt reduction method as well. To find a critical point of $I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$, we consider the following functional

$$\bar{I}_\varepsilon(\mathbf{u}) \equiv I_\varepsilon\big(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})\big) - K_\varepsilon(\Psi_\varepsilon).$$

Then we apply the same method as in the proof in Section 3 to find a critical point of $\bar{I}_\varepsilon$ and finish the proof of Theorem 2. We need the non-degeneracy condition of $D^2V(0)$ to ensure C^1-dependency of $\Phi_\varepsilon(\mathbf{u})$.

Let U be the radially symmetric solution of (1.5) with $a = V(0)$ which is non-degenerate in $H^{1,2}_{\mathrm{r}}(\mathbb{R}^n)$. Then we see that if $\Delta\phi - V(0)\phi + f'(U)\phi \equiv 0$ for some $\phi \in H^{1,2}(\mathbb{R}^n)$, then $\phi \in \operatorname{span}\{\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_n}\}$. Moreover U satisfies the exponential decay estimate $U(x) \leqslant Ce^{-c|x|}$ for some $C, c > 0$. We define $U_{\varepsilon,y}(x) := U(x - y)$, $x \in \Omega_\varepsilon = B_{\sigma/\varepsilon}(0)$,

$$E_{\varepsilon,y} := \left\{ w \in H^1(\Omega_\varepsilon);\ \left\langle w, \frac{\partial U_{\varepsilon,y}}{\partial x_i} \right\rangle_{\varepsilon,\Omega_\varepsilon} = 0,\ i \in \{1, \ldots, n\} \right\}$$

for $y \in B_1(0)$. Here $\langle u, v\rangle_{\varepsilon,\Omega_\varepsilon} \equiv \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v + V_\varepsilon(x)uv\,dx$.

Lemma 1. *$Q_\varepsilon, K_\varepsilon$ is of class C^2 and it holds that for $u, h, k \in H^1(\Omega_\varepsilon)$,*

(i) $Q''_\varepsilon(u)[h,k] = \Gamma''_\varepsilon(\psi(u))[\psi'(u)h, \psi'(u)k]$,
(ii) $K''_\varepsilon(u)[h,k] = \Gamma''_\varepsilon(P(u))[P'(u)h, P'(u)k]$.

Proof. We prove only (i) since the proof of (ii) is similar. Since $\psi'(u)h - \psi(h) \in X^\varepsilon_*(\vec{0})$, there holds

$$\begin{aligned} Q'_\varepsilon(u)h &= \Gamma'_\varepsilon\big(\psi(u)\big)\psi'(u)h \\ &= \Gamma'_\varepsilon\big(\psi(u)\big)\psi(h). \end{aligned}$$

Hence we have

$$\begin{aligned} Q''_\varepsilon(u)[h,k] &= \Gamma''_\varepsilon\big(\psi(u)\big)\big[\psi'(u)k, \psi(h)\big] \\ &= \Gamma''_\varepsilon\big(\psi(u)\big)\big[\psi'(u)k, \psi'(u)h\big]. \end{aligned}$$

This completes the proof. □

We identify $K'_\varepsilon(U_{\varepsilon,y} + \omega)$ with an element of $H^1(\Omega_\varepsilon)$, and $K''_\varepsilon(U_{\varepsilon,y})$ with a linear operator on $H^1(\Omega_\varepsilon)$ by Riesz representation theorem with respect to $\langle\cdot,\cdot\rangle_{\varepsilon,\Omega_\varepsilon}$. Let Π_y be the orthogonal projection from $H^1(\Omega_\varepsilon)$ to $E_{\varepsilon,y}$ with respect to $\langle\cdot,\cdot\rangle_{\varepsilon,\Omega_\varepsilon}$. Then we regard $\Pi_y K''_\varepsilon(U_{\varepsilon,y})$ as a linear operator on $E_{\varepsilon,y}$.

Lemma 2. *There exists a constant $c > 0$, independent of small $\varepsilon > 0$ and $y \in B_1(0)$, such that for all $h \in E_{\varepsilon,y}$, there holds*

$$\big\|\Pi_y K''_\varepsilon(U_{\varepsilon,y})[h]\big\|_{\varepsilon,\Omega_\varepsilon} \geqslant c\|h\|_{\varepsilon,\Omega_\varepsilon}.$$

Proof. Assume by contrary that there exist sequences $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$, $\{y_i\}$ and $h_i \in E_{\varepsilon_i,y_i}$ such that $\|h_i\|_{\varepsilon,\Omega_\varepsilon} = 1$ and

$$\lim_{i\to\infty} \big\|\Pi_y K''_{\varepsilon_i}(U_{\varepsilon_i,y_i})[h_i]\big\|_{\varepsilon,\Omega_\varepsilon} = 0.$$

For the sake of simplicity, we write $\varepsilon = \varepsilon_i$, $y = y_i$, $h_\varepsilon = h_i$. Set $v_\varepsilon = P'(U_{\varepsilon,y})h_\varepsilon \in H_0^1(\Omega_\varepsilon^\delta) \subset H^1(\mathbb{R}^n)$. By Lemma 1, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2 - g'\big(x, P(U_{\varepsilon,y})\big)v_\varepsilon^2\,dx = 0.$$

Since $\|v_\varepsilon\|_\varepsilon \in [1, C]$ for some constant $C > 1$, we may assume that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2\,dx \in [1, \infty). \tag{4.1}$$

We deduce that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} g'\big(x, P(U_{\varepsilon,y})\big)v_\varepsilon^2\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} g'\big(x, P(U_{\varepsilon,y})\big)(h_\varepsilon)^2\,dx.$$

We may assume that $v_\varepsilon(\cdot + y)$ converges weakly in $H^1(\mathbb{R}^n)$ to some function $v \in H^1(\mathbb{R}^n)$. Using $\lim_{\varepsilon\to 0}\langle K_\varepsilon''(U_{\varepsilon,y})[h_\varepsilon], \Pi_y\eta(\cdot - y)\rangle_{\varepsilon,\Omega_\varepsilon} = 0$ for $\eta \in C_0^\infty(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \nabla v\cdot\nabla\frac{\partial U}{\partial x_i} + V(0)v\frac{\partial U}{\partial x_i} - f'(U)v\frac{\partial U}{\partial x_i}\,dx = 0,$$

we then see that

$$\int_{\mathbb{R}^n} \nabla v\cdot\nabla\eta + V(0)v\eta - f'(U)v\eta\,dx = 0,$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$. However, since $h_\varepsilon \in E_{\varepsilon,y}$, we obtain that

$$\int_{\mathbb{R}^n} \nabla\frac{\partial U}{\partial x_i}\cdot\nabla v + V(0)\frac{\partial U}{\partial x_i}v\,dx = 0, \quad i \in \{1, \dots, n\}.$$

By the non-degeneracy condition (f4), we see that $v = 0$. It follows from $v_\varepsilon(\cdot + y)$ converges weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ to 0 that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} g'\big(x, P(U_{\varepsilon,y})\big)(h_\varepsilon)^2\,dx = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f'(U_{\varepsilon,y})(h_\varepsilon)^2\,dx = 0.$$

Then we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} |\nabla v_\varepsilon|^2 + V_\varepsilon(x)v_\varepsilon^2\,dx = 0.$$

This contradicts (4.1) and completes the proof of lemma. □

Proposition 18. *For any $\theta \in (0, 1)$, there exists a constant $C, \varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $y \in B_1(0)$, there exist constants $C_{i,\varepsilon,y}$, $i \in \{1, \dots, n\}$ and $w_{\varepsilon,y} \in E_{\varepsilon,y} \cap H^2(\Omega_\varepsilon)$ satisfying $K_\varepsilon'(U_{\varepsilon,y} + w_{\varepsilon,y}) = \sum_{i=1}^n C_{i,\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i}$, $\|w_{\varepsilon,y}\|_{\varepsilon,\Omega_\varepsilon} \leqslant C\varepsilon^{2-\theta}$, and $|C_{i,\varepsilon,y}| \leqslant C\varepsilon^{2-\theta}$. In addition, a function $y \mapsto (w_{\varepsilon,y}, C_{1,\varepsilon,y}, \dots, C_{n,\varepsilon,y}) \in H^1(\Omega_\varepsilon) \times \mathbb{R}^n$ is in C^1.*

Proof. A linear operator $\Pi_y K_\varepsilon''(U_{\varepsilon,y}) : E_{\varepsilon,y} \to E_{\varepsilon,y}$ has a bounded inverse $[\Pi_y K_\varepsilon''(U_{\varepsilon,y})]^{-1}$ by Lemma 2. We define

$$G_\varepsilon(\omega) := -\big[\Pi_y K_\varepsilon''(U_{\varepsilon,y})\big]^{-1}\big[\Pi_y K_\varepsilon'(U_{\varepsilon,y} + \omega) - \Pi_y K_\varepsilon''(U_{\varepsilon,y})[\omega]\big]$$

for $\omega \in E_{\varepsilon,y}$. We shall show that there is a constant $C' > 0$ such that G_ε has a fixed point $\omega_{\varepsilon,y}$ on the set

$$S_\varepsilon := \big\{\omega \in E_{\varepsilon,y};\ \|\omega\|_{\varepsilon,\Omega_\varepsilon} \leqslant C'\varepsilon^{2-\theta},\ \|\omega\|_{L^\infty(\Omega_\varepsilon)} \leqslant C'\varepsilon\big\}$$

for sufficiently small $\varepsilon > 0$. Note that ω is a fixed point of G_ε if and only if $\Pi_y K_\varepsilon'(U_{\varepsilon,y}+\omega) = 0$. We claim that if we choose a sufficiently large $C' > 0$ and small $\varepsilon_0 > 0$, then we have $G_\varepsilon(\omega) \in S_\varepsilon$ for all $\omega \in S_\varepsilon$, $\varepsilon \in (0, \varepsilon_0)$ and $\|G_\varepsilon(\omega) - G_\varepsilon(\tilde\omega)\|_{\varepsilon,\Omega_\varepsilon} \leqslant \frac{1}{2}\|\omega - \tilde\omega\|_{\varepsilon,\Omega_\varepsilon}$ for all $\omega, \tilde\omega \in S_\varepsilon$, $\varepsilon \in (0, \varepsilon_0)$.

We first estimate $\|G_\varepsilon(\omega)\|_{\varepsilon,\Omega_\varepsilon}$ for $\omega \in E_{\varepsilon,y}$. We have

$$\big\|G_\varepsilon(\omega)\big\|_{\varepsilon,\Omega_\varepsilon} \leqslant c_0 \sup_{h\in E_{\varepsilon,y}, \|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1} \big|K_\varepsilon'(U_{\varepsilon,y}+\omega)h - K_\varepsilon''(U_\varepsilon)[\omega, h]\big|$$

for some constant $c_0 > 0$. We denote $g'(x,t) = \frac{\partial g}{\partial t}(x,t)$. We note that

$$P'(U_{\varepsilon,y}+\omega)h - P'(U_{\varepsilon,y})h \in H_0^1\big(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon\big),$$
$$P(U_{\varepsilon,y}+\omega) - P'(U_{\varepsilon,y})\omega - P(U_{\varepsilon,y}) \in H_0^1\big(\Omega_\varepsilon^\delta \setminus \Omega_\varepsilon\big).$$

Then, for $h \in E_{\varepsilon,y}$, it follows from Proposition 1 that

$$\begin{aligned}
&K_\varepsilon'(U_{\varepsilon,y}+\omega)h - K_\varepsilon''(U_{\varepsilon,y})[\omega, h] \\
&\quad= \Gamma_\varepsilon'\big(P(U_{\varepsilon,y}+\omega)\big)P'(U_{\varepsilon,y}+\omega)h - \Gamma_\varepsilon''\big(P(U_{\varepsilon,y})\big)\big[P'(U_{\varepsilon,y})h, P'(U_{\varepsilon,y})\omega\big] \\
&\quad= \Gamma_\varepsilon'\big(P(U_{\varepsilon,y}+\omega)\big)P'(U_{\varepsilon,y})h - \Gamma_\varepsilon''\big(P(U_{\varepsilon,y})\big)\big[P'(U_{\varepsilon,y})h, P'(U_{\varepsilon,y})\omega\big] \\
&\quad= \Gamma_\varepsilon''\big(P(U_{\varepsilon,y})\big)\big[P'(U_{\varepsilon,y})h, P(U_{\varepsilon,y}+\omega) - P'(U_{\varepsilon,y})\omega\big] \\
&\qquad+ \int_{\mathbb{R}^n} \big\{g'\big(x, P(U_{\varepsilon,y})\big)P(U_{\varepsilon,y}+\omega) - g\big(x, P(U_{\varepsilon,y}+\omega)\big)\big\}P'(U_{\varepsilon,y})h \\
&\quad= \Gamma_\varepsilon''\big(P(U_{\varepsilon,y})\big)\big[P'(U_{\varepsilon,y})h, P(U_{\varepsilon,y})\big] \\
&\qquad+ \int_{\mathbb{R}^n} \big\{g'\big(x, P(U_{\varepsilon,y})\big)P(U_{\varepsilon,y}+\omega) - g\big(x, P(U_{\varepsilon,y}+\omega)\big)\big\}P'(U_{\varepsilon,y})h \\
&\quad= \Gamma_\varepsilon'\big(P(U_{\varepsilon,y})\big)P'(U_{\varepsilon,y})h - \int_{\mathbb{R}^n} \big[g\big(x, P(U_{\varepsilon,y}+\omega)\big) - g\big(x, P(U_{\varepsilon,y})\big) \\
&\qquad- g'\big(x, P(U_{\varepsilon,y})\big)\big\{P(U_{\varepsilon,y}+\omega) - P(U_{\varepsilon,y})\big\}\big]P'(U_{\varepsilon,y})h.
\end{aligned} \tag{4.2}$$

The first term in (4.2) denotes an error of the approximate solution $U_{\varepsilon,y}$. For any $\theta \in (0,1)$, there exist constants $C, c > 0$ such that

$$\begin{aligned}
\sup_{\substack{h\in E_{\varepsilon,y}\\ \|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1}} \big|\Gamma_\varepsilon'\big(P(U_{\varepsilon,y})\big)P'(U_{\varepsilon,y})h\big| &\leqslant \sup_{\substack{h\in E_{\varepsilon,y}\\ \|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1}} \int_{\Omega_\varepsilon} \nabla U_{\varepsilon,y}\cdot\nabla h + V_\varepsilon U_{\varepsilon,y}h - f(U_{\varepsilon,y})h\,dx + O\big(e^{-c/\varepsilon}\big) \\
&\leqslant \sup_{\substack{h\in E_{\varepsilon,y}\\ \|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1}} \int_{\Omega_\varepsilon} \big(V_\varepsilon - V(0)\big)U_{\varepsilon,y}h + O\big(e^{-c/\varepsilon}\big) \\
&\leqslant C\Big(e^{-c\varepsilon^{-\theta/2}} + \sup_{|\varepsilon\xi-\varepsilon y|\leqslant\varepsilon^{1-\theta/2}} \big|V(\varepsilon\xi) - V(0)\big|\Big) \\
&\leqslant C\varepsilon^{2-\theta}.
\end{aligned}$$

Here we divided the integral into the parts in $B_{\varepsilon^{-\theta/2}}(y)$ and $\Omega_\varepsilon \setminus B_{\varepsilon^{-\theta/2}}(y)$, and used the exponential decay of $U_{\varepsilon,y}$.

The remaining terms in (4.2) come from the nonlinearity of g. Using the uniform continuity of g' on compact sets, that is,

$$\lim_{\delta\to 0} \sup\big\{\big|g'(x,s) - g'(x,t)\big| \,\big|\, s,t \in [0,d],\ |s-t| \leqslant \delta\big\} = 0 \tag{4.3}$$

for any constant $d > 0$, we deduce that there exists a positive constant c_1, C_1, independent of small $\varepsilon > 0$ such that if $\|\omega\|_{L^\infty} \leqslant c_1$ then

$$\big\|G_\varepsilon(\omega)\big\|_{\varepsilon,\Omega_\varepsilon} \leqslant C_1\varepsilon^{2-\theta} + \frac{1}{2}\|\omega\|_{\varepsilon,\Omega_\varepsilon}. \tag{4.4}$$

Then, taking $C' \geqslant 2C_1$, we see that $\|G_\varepsilon(\omega)\|_{\varepsilon,\Omega_\varepsilon} \leqslant C'\varepsilon^{2-\theta}$.

Similarly we deduce that if $\|\omega\|_{L^\infty(\Omega_\varepsilon)} \leqslant c_1$ then

$$\big\|G_\varepsilon(\omega_1) - G_\varepsilon(\omega_2)\big\|_{\varepsilon,\Omega_\varepsilon} \leqslant \frac{1}{2}\|\omega_1 - \omega_2\|_{\varepsilon,\Omega_\varepsilon}.$$

Assume that $\|\omega\|_{\varepsilon,\Omega_\varepsilon} \leqslant C'\varepsilon$. We denote $\tilde\omega \equiv G_\varepsilon(\omega)$ and we estimate $\|\tilde\omega\|_{L^\infty(\Omega_\varepsilon)}$. Then there exist some constants a_i for $i \in \{1,\dots,n\}$ satisfying

$$-K''_\varepsilon(U_{\varepsilon,y})[\tilde\omega, h] = K'_\varepsilon(U_{\varepsilon,y} + \omega)h - K''_\varepsilon(U_{\varepsilon,y})[\omega, h] + \left\langle \sum_{i=1}^n a_i \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} \tag{4.5}$$

for all $h \in H^1(\Omega_\varepsilon)$. Substituting $h = \frac{\partial U_{\varepsilon,y}}{\partial x_j}$, and using that $\|\tilde\omega\|_{\varepsilon,\Omega_\varepsilon} = o(\varepsilon)$ holds by (4.4), we see that $a_i = o(\varepsilon)$. Since there hold

$$\begin{aligned} &K'_\varepsilon(U_{\varepsilon,y} + \omega)h - K''_\varepsilon(U_{\varepsilon,y})[\omega, h] \\ &\quad = \Gamma'_\varepsilon\big(P(U_{\varepsilon,y}+\omega)\big)P'(U_{\varepsilon,y}+\omega)h - \Gamma''_\varepsilon\big(P(U_{\varepsilon,y})\big)\big[P'(U_{\varepsilon,y})h, P'(U_{\varepsilon,y})\omega\big] \\ &\quad = \int_{\Omega_\varepsilon} \big\{-\Delta U_{\varepsilon,y} + V_\varepsilon U_{\varepsilon,y} - f(U_{\varepsilon,y}+\omega) + f'(U_{\varepsilon,y})\omega\big\} h\,dx \end{aligned}$$

and

$$-K''_\varepsilon(U_{\varepsilon,y})[\tilde\omega, h] = \int_{\Omega_\varepsilon} \big\{\Delta\tilde\omega - V_\varepsilon\tilde\omega + f'(U_{\varepsilon,y})\tilde\omega\big\} h\,dx$$

for any $h \in H^1_0(\Omega_\varepsilon)$, we see that $\tilde\omega$ solves

$$\Delta\tilde\omega - V_\varepsilon\tilde\omega + f'(U_{\varepsilon,y})\tilde\omega = f_{\varepsilon,y}(x) \quad \text{in } \Omega_\varepsilon,$$

where

$$f_{\varepsilon,y}(x) \equiv -\Delta U_{\varepsilon,y} + V_\varepsilon U_{\varepsilon,y} - f(U_{\varepsilon,y}+\omega) + f'(U_{\varepsilon,y})\omega + \sum_{i=1}^n a_i(-\Delta + V_\varepsilon)\frac{\partial U_{\varepsilon,y}}{\partial x_i}.$$

Hence $\xi = P'(U_{\varepsilon,y})\tilde\omega$ solves

$$\Delta\xi - V_\varepsilon\xi + g'\big(x, P(U_{\varepsilon,y})\big)\xi = \begin{cases} f_{\varepsilon,y}(x) & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega^\delta_\varepsilon \setminus \Omega_\varepsilon. \end{cases}$$

Since $a_i = o(\varepsilon)$, we have $|f_{\varepsilon,y}(x)| \leqslant o(\varepsilon) + o(\|\omega\|_{L^\infty(\Omega_\varepsilon)})$. Note that $\{\|V_\varepsilon - g'(x, P(U_{\varepsilon,y}))\|_{L^\infty(\Omega^\delta_\varepsilon)}\}_{\varepsilon<1}$ is bounded and $\|\xi\|_{L^2(\Omega^\delta_\varepsilon)} = o(\varepsilon)$. Then, by elliptic estimates, we see that

$$\|\xi\|_{L^\infty(B_1(z))} \leqslant C\big(\|\xi\|_{L^2(B_2(z))} + \|f_{\varepsilon,y}\|_{L^\infty(B_2(z))}\big).$$

Hence $\|\tilde\omega\|_{L^\infty(\Omega_\varepsilon)} \leqslant \|\xi\|_{L^\infty(\Omega_\varepsilon)} \leqslant C_2\varepsilon + \frac{1}{2}\|\omega\|_{L^\infty(\Omega_\varepsilon)}$ for some constant $C_2 > 0$, independent of C' for small $\varepsilon > 0$. If we take $C' = \max\{2C_1, 2C_2\}$, our claim follows.

By the contraction mapping theorem, there exists a unique fixed point $w_{\varepsilon,y}$ of G_ε on S_ε. By the elliptic estimates, we see that $w_{\varepsilon,y} \in H^2(\Omega_\varepsilon)$. We note that $(y, w) \mapsto \Pi_y K'_\varepsilon(U_{\varepsilon,y} + w)$ is of class C^1, and that

$$\lim_{\varepsilon\to 0}\big\|g'\big(x, P(U_{\varepsilon,y})\big)\big\|_{L^\infty(\Omega^\delta_\varepsilon\setminus\Omega_\varepsilon)} = \lim_{\varepsilon\to 0}\big\|g'\big(x, P(U_{\varepsilon,y} + w_{\varepsilon,y})\big)\big\|_{L^\infty(\Omega^\delta_\varepsilon\setminus\Omega_\varepsilon)} = 0.$$

Then, for $h, k \in E_{\varepsilon,y}$,

$$\begin{aligned} \big\|K''_\varepsilon(U_{\varepsilon,y} + w)[h,k] - K''_\varepsilon(U_{\varepsilon,y})[h,k]\big\|_{\varepsilon,\Omega_\varepsilon} &\leqslant \int_{\Omega_\varepsilon} \big|\big(g'\big(x, P(U_{\varepsilon,y})\big) - g'\big(x, P(U_{\varepsilon,y})\big)\big)hk\big| \\ &= o(1)\|h\|_{\varepsilon,\Omega_\varepsilon}\|k\|_{\varepsilon,\Omega_\varepsilon} \quad \text{as } \varepsilon \to 0. \end{aligned}$$

This and Lemma 2 imply $\Pi_y K''_\varepsilon(U_{\varepsilon,y}+w_{\varepsilon,y}) \neq 0$. Then, it follows from the implicit function theorem that $y \mapsto w_{\varepsilon,y}$ is of C^1.

Substituting $\omega = w_{\varepsilon,y}$, $h = \frac{\partial U_{\varepsilon,y}}{\partial x_i}$, and $a_i = -C_{i,\varepsilon,y}$ in (4.5), we see that $C_{i,\varepsilon,y} = O(\varepsilon^{2-\theta})$. This completes the proof. □

We define $\Psi_{\varepsilon,y} := U_{\varepsilon,y} + w_{\varepsilon,y}$.

Lemma 3. *There exists a constant $c > 0$, independent of small $\varepsilon > 0$ and $y \in B_1(0)$, such that for all $h \in E_{\varepsilon,y}$, there holds*

$$\left\| \Pi_y Q''_\varepsilon(\Psi_{\varepsilon,y})[h] \right\|_{\varepsilon,\Omega_\varepsilon} \geqslant c\|h\|_{\varepsilon,\Omega_\varepsilon}.$$

Proof. The proof is similar to one of Lemma 2, and so omitted. □

Proposition 19. *For any $R > 0$, there exist $C, c, \varepsilon_0 > 0$ such that for any $y \in B_1(0)$, $\varepsilon \in (0, \varepsilon_0)$ and $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$ with $\|\mathbf{u}\|_\varepsilon \leqslant R$, there exist a unique $z_{\varepsilon,y}(\mathbf{u}) \in E_{\varepsilon,y} \cap H^2(\Omega_\varepsilon)$ and constants $\tilde{C}_{i,\varepsilon,y}(\mathbf{u})$, $i \in \{1, \dots, n\}$ satisfying $Q'_\varepsilon(\Psi_{\varepsilon,y} + z_{\varepsilon,y}(\mathbf{u})) = \sum_{i=1}^n \tilde{C}_{i,\varepsilon,y}(\mathbf{u}) \frac{\partial U_{\varepsilon,y}}{\partial x_i}$ and $\|z_{\varepsilon,y}(\mathbf{u})\|_{\varepsilon,\Omega_\varepsilon} \leqslant Ce^{-c/\varepsilon}$. In addition, $(y, \mathbf{u}) \mapsto (z_{\varepsilon,y}(\mathbf{u}), \tilde{C}_{1,\varepsilon,y}(\mathbf{u}), \dots, \tilde{C}_{n,\varepsilon,y}(\mathbf{u})) \in H^1(\Omega_\varepsilon) \times \mathbb{R}^n$ is of C^1.*

Proof. The proof is almost similar to one of Proposition 18. We regard $\Psi_{\varepsilon,y}$ as an approximate solution and use Lemma 3. The only difference is the error estimate of the approximate solutions. Indeed, by Propositions 4(iii) and 18, we have

$$\sup_{h \in E_{\varepsilon,y}, \|h\|_{\varepsilon,\Omega_\varepsilon} \leqslant 1} \left| \Gamma'_\varepsilon\big(\psi(\Psi_{\varepsilon,y})\big)\psi'(\Psi_{\varepsilon,y})h \right| \leqslant Ce^{-c/\varepsilon}.$$

We omit the details. □

Proposition 20. *For sufficiently small $\varepsilon > 0$, there exists a $y_\varepsilon \in B_1(0)$ such that $K'_\varepsilon(U_{\varepsilon,y_\varepsilon} + w_{\varepsilon,y_\varepsilon}) = 0$.*

Proof. We need only to solve $C_{i,\varepsilon,y} = 0$ for $i = 1, \dots, n$. Note that $K'_\varepsilon(U_{\varepsilon,y} + w_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i} = 0$, $i = 1, \dots, n$ is equivalent to $C_{i,\varepsilon,y} = 0$, $i = 1, \dots, n$ since $\{\frac{\partial U_{\varepsilon,y}}{\partial x_i}\}_{i=1,\dots,n}$ is linearly independent.

Then, using $\frac{\partial U_{\varepsilon,y}}{\partial y_i} + \frac{\partial U_{\varepsilon,y}}{\partial x_i} = 0$ and the exponential decay property of U, we deduce that

$$\begin{aligned}
&K'_\varepsilon(U_{\varepsilon,y} + w_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i} \\
&\quad= \int_{\Omega_\varepsilon} \nabla U_{\varepsilon,y} \cdot \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} + V_\varepsilon(0)U_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i} - f(U_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V_\varepsilon(0)\big)U_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i} \\
&\qquad+ \int_{\Omega_\varepsilon} \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} \cdot \nabla w_{\varepsilon,y} + V(0)\frac{\partial U_{\varepsilon,y}}{\partial x_i}w_{\varepsilon,y} - f'(U_{\varepsilon,y})w_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V(0)\big)\frac{\partial U_{\varepsilon,y}}{\partial x_i}w_{\varepsilon,y} \\
&\qquad- \int_{\Omega_\varepsilon} \big\{f(U_{\varepsilon,y} + w_{\varepsilon,y}) - f(U_{\varepsilon,y}) - f'(U_{\varepsilon,y})w_{\varepsilon,y}\big\}\frac{\partial U_{\varepsilon,y}}{\partial x_i} + o\big(\varepsilon^2\big) \\
&\quad= \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V_\varepsilon(0)\big)U_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i} + \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V(0)\big)\frac{\partial U_{\varepsilon,y}}{\partial x_i}w_{\varepsilon,y} \\
&\qquad- \int_{\Omega_\varepsilon} \big\{f(U_{\varepsilon,y} + w_{\varepsilon,y}) - f(U_{\varepsilon,y}) - f'(U_{\varepsilon,y})w_{\varepsilon,y}\big\}\frac{\partial U_{\varepsilon,y}}{\partial x_i} + o\big(\varepsilon^2\big) \\
&\quad\equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + o\big(\varepsilon^2\big).
\end{aligned}$$

We note that $\|w_{\varepsilon,y}\|_{\varepsilon,\Omega_\varepsilon} \leqslant C\varepsilon^{2-\theta}$. Now, we can take $\theta \in (0,1/2)$ so that $(1+\gamma)(2-\theta) > 2$. From the property $f \in C^{1,\gamma}$ and an exponential decay of U, we deduce that $\mathrm{II} = o(\varepsilon^2)$ and $\mathrm{III} = o(\varepsilon^2)$. As for I, we see that

$$\mathrm{I} = -\frac{\varepsilon}{2} \int\limits_{B_{\sigma/\varepsilon}(-y)} \frac{\partial V}{\partial x_i}\big(\varepsilon(x+y)\big)\big(U(x)\big)^2\,dx + O\big(e^{-c/\varepsilon}\big).$$

We define

$$s_i^0(y) \equiv -\frac{1}{2}\sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0) y_j \int\limits_{\mathbb{R}^n} U^2.$$

Then since

$$s_i^0(y) = -\frac{1}{2}\int\limits_{\mathbb{R}^n} \sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0)(x_j+y_j)\big(U(x)\big)^2\,dx,$$

we have

$$\begin{aligned}
\big|\mathrm{I} - \varepsilon^2 s_i^0(y)\big| &= \frac{\varepsilon}{2} \int\limits_{B_{\sigma/\varepsilon}(-y)} \left| \sum_{j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(0)\varepsilon(x_j+y_j) - \frac{\partial V}{\partial x_i}\big(\varepsilon(x+y)\big)\right| \big(U(x)\big)^2 + O\big(e^{-c/\varepsilon}\big) \\
&\leqslant C\varepsilon \int\limits_{B_{\sigma/\varepsilon}(-y)} |\varepsilon x + \varepsilon y|^2 \big(U(x)\big)^2 + O\big(e^{-c/\varepsilon}\big) \\
&\leqslant C\varepsilon\big[\big(\varepsilon + \varepsilon^{1-\theta}\big)^2 + e^{-c\varepsilon^{-\theta}}\big] + O\big(e^{-c/\varepsilon}\big) = o\big(\varepsilon^2\big).
\end{aligned}$$

Setting $s^\varepsilon(y) = (s_1^\varepsilon(y), \ldots, s_n^\varepsilon(y))$, $s_i^\varepsilon(y) := \varepsilon^{-2} K_\varepsilon'(U_{\varepsilon,y} + w_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}$, $y \in B_1(0)$, we see that

$$s_i^\varepsilon(y) \to s_i^0(y)$$

uniformly in $y \in B_1(0)$. Since $D^2V(0)$ is non-degenerate, using the degree theory, we see that there exists a point y_ε such that $s^\varepsilon(y^\varepsilon) = 0$. Then we have $C_{i,\varepsilon,y_\varepsilon} = 0$. The proof is completed. □

Proposition 21. *For any $R > 0$, there exist $C, c, \varepsilon_0 > 0$ such that for any $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$, $\|\mathbf{u}\|_\varepsilon \leqslant R$ and $\varepsilon \in (0,\varepsilon_0)$, there exists a unique $\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(\mathbf{u}) \in B_1(0)$ satisfying $Q_\varepsilon'(\Psi_{\varepsilon,\tilde{y}_\varepsilon} + z_{\varepsilon,\tilde{y}_\varepsilon}(\mathbf{u})) = 0$ and $|\tilde{y}_\varepsilon(\mathbf{u}) - y_\varepsilon| \leqslant Ce^{-c/\varepsilon}$. In addition, $\mathbf{u} \mapsto \tilde{y}_\varepsilon(\mathbf{u})$ is of C^1.*

Proof. By Proposition 20, we have $Q_\varepsilon'(\Psi_{\varepsilon,y_\varepsilon}) = K_\varepsilon'(\Psi_{\varepsilon,y_\varepsilon}) + O(e^{-c/\varepsilon}) = O(e^{-c/\varepsilon})$. Hence it follows that $Q_\varepsilon'(\Psi_{\varepsilon,y_\varepsilon} + z_{\varepsilon,y_\varepsilon}) = O(e^{-c/\varepsilon})$. We define

$$t(y) := \big(t_1(y), \ldots, t_n(y)\big), \qquad t_i(y) := \varepsilon^{-2} Q_\varepsilon'(\Psi_{\varepsilon,y} + z_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}, \quad y \in B_1(0).$$

We have $t_i(y_\varepsilon) = O(e^{-c/\varepsilon})$, and we will find a solution of $t(y) = 0$ in an exponentially small neighborhood of y_ε.

For a constant $\gamma \in (0,1)$ given in (f4), we choose a sufficiently small $\theta \in (0,1)$ in Proposition 18 to satisfy $(2-\theta)(1+\gamma) > 2$.

We write $\tilde{w}_{\varepsilon,y} = w_{\varepsilon,y} + z_{\varepsilon,y}$ and then have

$$\begin{aligned}
&\frac{\partial}{\partial y_j} Q_\varepsilon'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i} \\
&\quad= \left\langle \frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i}\right\rangle_{\varepsilon,\Omega_\varepsilon} - \int\limits_{\Omega_\varepsilon} f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right)\frac{\partial U_{\varepsilon,y}}{\partial x_i} \\
&\qquad+ \left\langle U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}, \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\right\rangle_{\varepsilon,\Omega_\varepsilon} - \int\limits_{\Omega_\varepsilon} f(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} + O\big(e^{-c/\varepsilon}\big)
\end{aligned}$$

$$= \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V_\varepsilon(0)\big)\left\{\frac{\partial U_{\varepsilon,y}}{\partial x_i}\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right) + (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\right\}$$

$$+ \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} \cdot \nabla \tilde{w}_{\varepsilon,y} + V_\varepsilon(0)\frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\tilde{w}_{\varepsilon,y} - f'(U_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\tilde{w}_{\varepsilon,y}$$

$$- \int_{\Omega_\varepsilon} \big\{f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) - f'(U_{\varepsilon,y})\big\}\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right)\frac{\partial U_{\varepsilon,y}}{\partial x_i} + o\big(\varepsilon^2\big)$$

$$\equiv E_1 + E_2 + E_3 + o\big(\varepsilon^2\big).$$

We estimate E_1 as follows:

$$E_1 = \int_{\Omega_\varepsilon} \big(V_\varepsilon(x) - V_\varepsilon(0)\big)\left\{\frac{\partial U_{\varepsilon,y}}{\partial x_i}\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j}\right) + (U_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\right\} + o\big(\varepsilon^2\big)$$

$$= \frac{1}{2}\int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j}\frac{\partial U^2_{\varepsilon,y}}{\partial x_i} + o\big(\varepsilon^2\big) = -\frac{1}{2}\int_{\Omega_\varepsilon} \frac{\partial^2 V_\varepsilon}{\partial x_i \partial x_j} U^2_{\varepsilon,y} + o\big(\varepsilon^2\big)$$

$$= -\frac{\varepsilon^2}{2}\frac{\partial^2 V}{\partial x_i \partial x_j}(0)\int_{\mathbb{R}^n} U^2 + o\big(\varepsilon^2\big).$$

Next we claim that $\|\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j}\|_{\varepsilon,\Omega_\varepsilon} = O(\varepsilon^{2-\theta})$.

We write $\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} = w_1 + w_2$, $w_1 \in \operatorname{span}\{\frac{\partial U_{\varepsilon,y}}{\partial x_1}, \ldots, \frac{\partial U_{\varepsilon,y}}{\partial x_n}\}$, $w_2 \in E_{\varepsilon,y}$.

Through by integration by parts, we see that

$$\left|\left\langle \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i}\right\rangle_{\varepsilon,\Omega_\varepsilon}\right| \leqslant \left|\left\langle \tilde{w}_{\varepsilon,y}, \frac{\partial U_{\varepsilon,y}}{\partial x_i \partial x_j}\right\rangle_{\varepsilon,\Omega_\varepsilon} + \int_{\Omega_\varepsilon} \varepsilon \frac{\partial V}{\partial x_j}(\varepsilon x)\tilde{w}_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial x_i}\right| + O\big(\varepsilon^{2-\theta}\big) \leqslant O\big(\varepsilon^{2-\theta}\big).$$

Since $\tilde{w}_{\varepsilon,y} \in E_{\varepsilon,y}$, we have $|\langle \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, \frac{\partial U_{\varepsilon,y}}{\partial x_i}\rangle_{\varepsilon,\Omega_\varepsilon}| = O(\varepsilon^{2-\theta})$. Hence $\|w_1\|_{\varepsilon,\Omega_\varepsilon} \leqslant O(\varepsilon^{2-\theta})$. On the other hand, we have

$$\|w_2\|_{\varepsilon,\Omega_\varepsilon} \leqslant C\big\|\Pi_y Q''_\varepsilon(\Psi_{\varepsilon,y})w_2\big\|_{\varepsilon,\Omega_\varepsilon}$$
$$\leqslant C\big\|\Pi_y Q''_\varepsilon(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})w_2\big\|_{\varepsilon,\Omega_\varepsilon} + o(1)\|w_2\|_{\varepsilon,\Omega_\varepsilon} \tag{4.6}$$

by Lemma 3. Differentiating the both sides of the following identity

$$Q'_\varepsilon(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})h = \sum_{i=1}^n \tilde{C}_i\left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon}, \quad h \in H^1(\Omega_\varepsilon),\ \tilde{C}_i = \tilde{C}_{i,\varepsilon,y}(\mathbf{u})$$

with respect to y_j, we get

$$\int_{\Omega_\varepsilon} \nabla\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right)\cdot \nabla h + \big(V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\big)\left(\frac{\partial U_{\varepsilon,y}}{\partial y_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right)h\,dx$$
$$= \sum_{i=1}^n \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i}\cdot \nabla h + V_\varepsilon \frac{\partial^2 U_{\varepsilon,y}}{\partial y_j \partial x_i} h\,dx + \sum_{i=1}^n \frac{\partial \tilde{C}_i}{\partial y_j}\left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} + O\big(e^{-c/\varepsilon}\big) \tag{4.7}$$

for $\|h\|_{\varepsilon,\Omega_\varepsilon} \leqslant 1$. Note that $\tilde{w}_{\varepsilon,y} \in H^2(\Omega_\varepsilon)$. The first integration in (4.7) equals to

$$\int_{\Omega_\varepsilon} \nabla\left(\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} - \frac{\partial(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})}{\partial x_j}\right)\cdot \nabla h$$

$$
\begin{aligned}
&+ \big(V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\big)\left(\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j} - \frac{\partial (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})}{\partial x_j}\right) h\,dx \\
&= \int_{\Omega_\varepsilon} \nabla (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \cdot \nabla \frac{\partial h}{\partial x_j} - f(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + V_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} \\
&\quad + \int_{\Omega_\varepsilon} \nabla \left(\frac{\partial w}{\partial x_j} + \frac{\partial w}{\partial y_j}\right) \cdot \nabla h + \big(V_\varepsilon - f'(U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y})\big)\left(\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right) h \\
&\quad + \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h\,dx \\
&= Q'_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h\right] \\
&\quad + \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h\,dx + O\big(e^{-c/\varepsilon}\big).
\end{aligned}
$$

The next term of the identity (4.7) equals to

$$
\begin{aligned}
&- \sum_{i=1}^{n} \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial^2 U_{\varepsilon,y}}{\partial x_j \partial x_i} \cdot \nabla h + V_\varepsilon \frac{\partial^2 U_{\varepsilon,y}}{\partial x_j \partial x_i} h\,dx + \sum_{i=1}^{n} \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} + O\big(e^{-c/\varepsilon}\big) \\
&= \sum_{i=1}^{n} \tilde{C}_i \int_{\Omega_\varepsilon} \nabla \frac{\partial U_{\varepsilon,y}}{\partial x_i} \cdot \nabla \frac{\partial h}{\partial x_j} + V_\varepsilon \frac{\partial U_{\varepsilon,y}}{\partial x_i} \frac{\partial h}{\partial x_j} + \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h\,dx \\
&\quad + \sum_{i=1}^{n} \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} + O\big(e^{-c/\varepsilon}\big) \\
&= Q'_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \frac{\partial h}{\partial x_j} + \sum_{i=1}^{n} \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h\,dx + \sum_{i=1}^{n} \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} + O\big(e^{-c/\varepsilon}\big).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}, h\right] \\
&= \sum_{i=1}^{n} \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} h\,dx + \sum_{i=1}^{n} \frac{\partial \tilde{C}_i}{\partial y_j} \left\langle \frac{\partial U_{\varepsilon,y}}{\partial x_i}, h\right\rangle_{\varepsilon,\Omega_\varepsilon} - \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) h\,dx + O\big(e^{-c/\varepsilon}\big)
\end{aligned}
$$

for $\|h\|_{\varepsilon,\Omega_\varepsilon} \leqslant 1$. We then obtain

$$
\begin{aligned}
&\left\langle \Pi_y Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right], h\right\rangle_{\varepsilon,\Omega_\varepsilon} \\
&= \left\langle Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right], \Pi_y h\right\rangle_{\varepsilon,\Omega_\varepsilon} \\
&= \sum_{i=1}^{n} \tilde{C}_i \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} \frac{\partial U_{\varepsilon,y}}{\partial x_i} \Pi_y h\,dx - \int_{\Omega_\varepsilon} \frac{\partial V_\varepsilon}{\partial x_j} (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \Pi_y h\,dx + O\big(e^{-c/\varepsilon}\big)
\end{aligned}
$$

for $\|h\|_{\varepsilon,\Omega_\varepsilon} \leqslant 1$,

$$
\left\| \Pi_y Q''_\varepsilon (U_{\varepsilon,y} + \tilde{w}_{\varepsilon,y}) \left[\frac{\partial \tilde{w}_{\varepsilon,y}}{\partial x_j} + \frac{\partial \tilde{w}_{\varepsilon,y}}{\partial y_j}\right] \right\|_{\varepsilon,\Omega_\varepsilon}
$$

$$= \sup_{\|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1}\left|\left\langle \Pi_y Q_\varepsilon''(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\left[\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+\frac{\partial \tilde w_{\varepsilon,y}}{\partial y_j}\right],h\right\rangle_{\varepsilon,\Omega_\varepsilon}\right|$$
$$\leqslant \sup_{\|h\|_{\varepsilon,\Omega_\varepsilon}\leqslant 1}\left|\int_{\Omega_\varepsilon}\varepsilon\frac{\partial V}{\partial x_j}(\varepsilon x)(U_{\varepsilon,y}+w_{\varepsilon,y})\Pi_y h\,dx\right|+O\big(\varepsilon^{2-\theta}\big)\leqslant O\big(\varepsilon^{2-\theta}\big),$$

and hence

$$\big\|\Pi_y Q_\varepsilon''(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})w_2\big\|_{\varepsilon,\Omega_\varepsilon}\leqslant\left\|\Pi_y Q_\varepsilon''(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\left[\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+\frac{\partial \tilde w_{\varepsilon,y}}{\partial y_j}\right]\right\|_{\varepsilon,\Omega_\varepsilon}+O\big(\varepsilon^{2-\theta}\big)$$
$$\leqslant O\big(\varepsilon^{2-\theta}\big).$$

Thus from (4.6), we get $\|w_2\|_{\varepsilon,\Omega_\varepsilon}=O(\varepsilon^{2-\theta})$. The claim follows.

Using this claim, we see that

$$E_3=-\int_{\Omega_\varepsilon}\big\{f'(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})-f'(U_{\varepsilon,y})\big\}\left(\frac{\partial U_{\varepsilon,y}}{\partial x_j}+\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}\right)\frac{\partial U_{\varepsilon,y}}{\partial x_i}+o\big(\varepsilon^2\big).$$

In addition, from the facts

$$\int_{\Omega_\varepsilon}f'(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\left(\frac{\partial U_{\varepsilon,y}}{\partial x_j}+\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}\right)\frac{\partial U_{\varepsilon,y}}{\partial x_i}=-\int_{\Omega_\varepsilon}f(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}+O\big(e^{-c/\varepsilon}\big)$$

and

$$\int_{\Omega_\varepsilon}f'(U_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_j}\frac{\partial U_{\varepsilon,y}}{\partial x_i}=-\int_{\Omega_\varepsilon}f(U_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}+O\big(e^{-c/\varepsilon}\big),$$

we get that

$$\int_{\Omega_\varepsilon}\big(f'(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})-f'(U_{\varepsilon,y})\big)\frac{\partial U_{\varepsilon,y}}{\partial x_j}\frac{\partial U_{\varepsilon,y}}{\partial x_i}$$
$$=-\int_{\Omega_\varepsilon}\big(f(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})-f(U_{\varepsilon,y})\big)\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}-\int_{\Omega_\varepsilon}f'(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+O\big(e^{-c/\varepsilon}\big)$$
$$=-\int_{\Omega_\varepsilon}f'(U_{\varepsilon,y})\tilde w_{\varepsilon,y}\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}-\int_{\Omega_\varepsilon}f'(U_{\varepsilon,y}+\tilde w_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+o\big(\varepsilon^2\big).$$

Since

$$E_2=\left\{\int_{\Omega_\varepsilon}\nabla\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}\cdot\nabla\tilde w_{\varepsilon,y}+V_\varepsilon(0)\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}\tilde w_{\varepsilon,y}-f'(U_{\varepsilon,y})\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}\tilde w_{\varepsilon,y}\right\},$$

we see that

$$E_2+E_3=\int_{\Omega_\varepsilon}\nabla\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}\cdot\nabla\tilde w_{\varepsilon,y}+V_\varepsilon(0)\frac{\partial^2 U_{\varepsilon,y}}{\partial x_j\partial x_i}\tilde w_{\varepsilon,y}+f'(U_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+o\big(\varepsilon^2\big)$$
$$=-\int_{\Omega_\varepsilon}\nabla\frac{\partial U_{\varepsilon,y}}{\partial x_i}\cdot\nabla\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+V_\varepsilon(0)\frac{\partial U_{\varepsilon,y}}{\partial x_i}\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}-f'(U_{\varepsilon,y})\frac{\partial U_{\varepsilon,y}}{\partial x_i}\frac{\partial \tilde w_{\varepsilon,y}}{\partial x_j}+o\big(\varepsilon^2\big)$$
$$=o\big(\varepsilon^2\big).$$

We conclude that $\frac{\partial}{\partial y_j} t_i(y) \to -\frac{1}{2} \frac{\partial^2 V}{\partial x_i \partial x_j}(0) \int_{\mathbb{R}^n} U^2\, dx$, and hence by the condition (f4), we have that $\nabla t^\varepsilon(y)$, $y \in B_1(0)$ is non-degenerate for small $\varepsilon > 0$. Therefore using a fixed point theorem, we see that there exists $\tilde{y}_\varepsilon$ such that $t(\tilde{y}_\varepsilon) = 0$ and $|\tilde{y}_\varepsilon - y_\varepsilon| = O(e^{-c/\varepsilon})$. Moreover by the implicit function theorem, $\mathbf{u} \mapsto \tilde{y}_\varepsilon$ is of C^1. □

We define a C^1 map $\Phi_\varepsilon(\mathbf{u}) = \Psi_{\varepsilon, \tilde{y}_\varepsilon(\mathbf{u})} + z_{\varepsilon, \tilde{y}_\varepsilon(\mathbf{u})}(\mathbf{u})$ and a C^1 functional

$$\bar{I}_\varepsilon(\mathbf{u}) = I_\varepsilon\big(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})\big) - K_\varepsilon(\Psi_{\varepsilon, y_\varepsilon}).$$

Proposition 22. *The following hold.*

(i) *For any $R > 0$, there exist constants $C, c, \varepsilon_0 > 0$ such that*

$$\left| \bar{I}_\varepsilon(\mathbf{u}) - \sum_{i=0}^{k} \Gamma_\varepsilon\big(P_i(u_i)\big) \right| \leqslant C e^{-c/\varepsilon}$$

for all $\mathbf{u} \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$ with $\|\mathbf{u}\|_\varepsilon \leqslant R$ and $\varepsilon \in (0, \varepsilon_0)$.

(ii) *The functional $\bar{I}_\varepsilon(\mathbf{u})$ satisfies* (PS) *condition.*

(iii) $\mathbf{u}$ *is a critical point of $\bar{I}_\varepsilon$ if and only if $(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$ is a critical point of I_ε.*

Proof. (i) This follows from (2.2) and the fact

$$\big\| \Phi_\varepsilon(\mathbf{u}) - \Psi_{\varepsilon, y_\varepsilon} \big\|_{\varepsilon, \Omega_\varepsilon} \leqslant \| z_{\varepsilon, \tilde{y}_\varepsilon(\mathbf{u})} \|_{\varepsilon, \Omega_\varepsilon} = O\big(e^{-c/\varepsilon}\big).$$

We can prove (ii) and (iii) following the same argument as in the proof of (i) and (ii) in Proposition 4. □

Finally we can find a critical point $\mathbf{u}_\varepsilon$ of $\bar{I}_\varepsilon$ similarly as in Section 3. Moreover it follows from the construction that $\limsup_{\varepsilon \to 0} \|\mathbf{u}_\varepsilon\|_\varepsilon < \infty$, and hence that $\Phi_\varepsilon(\mathbf{u}_\varepsilon)(\cdot + \tilde{y}_\varepsilon)$ converges uniformly to U on each bounded set in $\mathbb{R}^n$. This proves (iv) of Theorem 2. Other properties can be proved similarly as in Section 3. The proof of Theorem 2 is completed.

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