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From the Klein–Gordon–Zakharov system to a singular nonlinear Schrödinger system

Du système de Klein–Gordon–Zakharov vers un système de Schrödinger nonlinéaire singulier

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Abstract

In this paper, we continue our investigation of the high-frequency and subsonic limits of the Klein–Gordon–Zakharov system. Formally, the limit system is the nonlinear Schrödinger equation. However, for some special case of the parameters going to the limits, some new models arise. The main object of this paper is the derivation of those new models, together with convergence of the solutions along the limits.

Résumé

Dans cet article, on continue l'investigation des limites haute fréquence et subsonique du système de Klein–Gordon–Zakharov. Formellement, le système limite est le système de Schrödinger nonlinéaire. Cependant, pour un cas particulier des paramètres, on trouve un nouveau modèle qui contient un terme singulier. L'objet de ce papier est de donner une dérivation rigoureuse de ce modèle et de montrer la convergence dans l'espace d'énergie.

1. Introduction

In this paper we continue the investigation of the Klein–Gordon–Zakharov system started in [11–13].

The Klein–Gordon–Zakharov system in nondimensional variables reads

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$$\begin{cases} c^{-2}\ddot{E} - \Delta E + c^2 E = -nE, \\ \alpha^{-2}\ddot{n} - \Delta n = \Delta |E|^2, \end{cases} \tag{1.1}$$

where $E : \mathbb{R}^{1+3} \to \mathbb{R}^3$ is the electric field[2] and $n : \mathbb{R}^{1+3} \to \mathbb{R}$, is the density fluctuation of ions, c^2 is the plasma frequency and α the ion sound speed. This system describes the interaction between Langmuir waves [1,3,22] and ion sound waves in a plasma (see Dendy [7] and Bellan [2]). It can be derived from the two-fluid Euler–Maxwell system (see Sulem and Sulem [17], Colin and Colin [5] and Texier [6,18,19] for some rigorous derivations). We also refer to [12, Introduction] for the rescaling with physical constants.

The system (1.1) has the following conserved energy

$$\int c^2|E|^2 + |\nabla E|^2 + c^{-2}|\dot{E}|^2 + \frac{1}{2}\big||\alpha\nabla|^{-1}\dot{n}\big|^2 + \frac{1}{2}|n|^2 + n|E|^2\,dx. \tag{1.2}$$

Notice that this energy is at least $O(c^2)$ due to the first term when c goes to infinity, so it is not useful by itself to get uniform bounds when c goes to infinity and does not give a conserved quantity for the limit system.

To explain the main contribution of this paper, we start by some formal considerations. Taking $F = e^{ic^2 t}E$, system (1.1) becomes

$$\begin{cases} c^{-2}\ddot{F} + 2i\dot{F} - \Delta F = -nF, \\ \alpha^{-2}\ddot{n} - \Delta n = \Delta |F|^2. \end{cases} \tag{1.3}$$

Its formal limit as $c, \alpha \to \infty$ is given by the nonlinear Schrödinger equation:

$$2i\dot{F} - \Delta F = |F|^2 F, \quad n = -|F|^2. \tag{1.4}$$

If we take the limit $c \to \infty$ first, we get the usual Zakharov system:

$$\begin{cases} 2i\dot{F} - \Delta F = -nF, \\ \alpha^{-2}\ddot{n} - \Delta n = \Delta |F|^2. \end{cases} \tag{1.5}$$

If we take the limit $\alpha \to \infty$ first in (1.1), we get the nonlinear Klein–Gordon system:

$$c^{-2}\ddot{E} - \Delta E + c^2 E = -|E|^2 E. \tag{1.6}$$

It is classically known that the limit when α goes to infinity in the Zakharov system (1.5) leads to the cubic nonlinear Schrödinger equation (1.4) and that the limit when c goes to infinity in the cubic nonlinear Klein–Gordon system (1.6) also leads to the cubic nonlinear Schrödinger equation.

However a more precise analysis involving the two different modes of oscillations of (1.1), namely writing $E = \mathbb{E}_1 e^{-ic^2 t} + \overline{\mathbb{E}_2} e^{ic^2 t}$ shows that these two limits do not commute. Indeed, the non-relativistic limit of the nonlinear Klein–Gordon was studied in [9,10]. In [10] we proved that the limit system is a coupled nonlinear Schrödinger system

$$\begin{cases} 2i\dot{\mathbb{E}}_1 - \Delta\mathbb{E}_1 - \big(|\mathbb{E}_1|^2 + 2|\mathbb{E}_2|^2\big)\mathbb{E}_1 = 0, \\ 2i\dot{\mathbb{E}}_2 - \Delta\mathbb{E}_2 - \big(|\mathbb{E}_2|^2 + 2|\mathbb{E}_1|^2\big)\mathbb{E}_2 = 0 \end{cases} \tag{1.7}$$

which differs from the one we can derive from the Zakharov system or the one derived in [12] where we took a simultaneous limit requiring that $\alpha < c$ where the limit system was

$$\begin{cases} 2i\dot{\mathbb{E}}_1 - \Delta\mathbb{E}_1 - \big(|\mathbb{E}_1|^2 + |\mathbb{E}_2|^2\big)\mathbb{E}_1 = 0, \\ 2i\dot{\mathbb{E}}_2 - \Delta\mathbb{E}_2 - \big(|\mathbb{E}_2|^2 + |\mathbb{E}_1|^2\big)\mathbb{E}_2 = 0. \end{cases} \tag{1.8}$$

In this paper, we will study the case where $\gamma\alpha = 2c^2$ for some fixed constant γ. At the limit we will get a singular Schrödinger system (1.13). Formally, we see that when γ goes to infinity we recover the nonlinear Schrödinger system (1.8) derived in [12], and when γ goes to zero we recover the nonlinear Schrödinger system (1.7) derived in [10]. The rigorous justification of these two limits is given in the forthcoming paper [14].

In this paper, γ will be called the resonant frequency and we will need extra work to bound n around the frequency γ. Indeed, the general definition of resonance comes from the theory of ODEs. Resonant terms are those that

[2] In our results the range of E may be $\mathbb{R}^d$ or $\mathbb{C}^d$ with arbitrary d.

cannot be eliminated by a Poincaré–Dulac normal forms. For dispersive equations, this notion is less precise. One can reduce to the theory of ODEs by considering plane wave solutions of the linear equation. In our case, this means that the frequencies $(\eta, \xi - \eta, \xi)$ are resonant if

$$\pm c\sqrt{c^2 + |\eta|^2} \pm c\sqrt{c^2 + |\xi - \eta|^2} = \alpha|\xi|. \tag{1.9}$$

This equation has many solutions. For instance $(\eta, -\eta, 0)$ for any η. These resonances will not be important due to the presence of the Δ in front of the $|E|^2$ term and that Δ vanishes at the frequency $\xi = 0$. Eq. (1.9) has also solutions which grow with c, namely if $|\eta| \ll |\xi| \sim c$, then the equation reduces to

$$c^2 \pm c\sqrt{c^2 + |\xi|^2} = 2\frac{c^2}{\gamma}|\xi|. \tag{1.10}$$

Hence, (1.9) has some solutions with $|\xi| \sim c$. These resonances do not affect the solution so much, since they involve high frequencies, where the interaction is smaller due to the regularity of the solutions, and so can be handled in a relatively simple way. The main resonance will be the one where $\eta = 0$ and $|\xi| = \gamma_c$, γ_c solving $c^2 + c\sqrt{c^2 + \gamma_c^2} = \alpha\gamma_c$. Hence, γ_c goes to γ when c goes to infinity. There is no factor which cancels this resonant interaction, and in fact the limit equation becomes singular at this frequency due to the resonance, which can be formally observed by partial integration in time, as we will do in Section 2. This justifies the importance of the frequency γ.

To write our limit system, we need some notations. We define the following operators as functions of $|\nabla| = \sqrt{-\Delta}$ (by using the Fourier transform)

$$\begin{aligned}
\mathcal{A}_\gamma^+ &:= \frac{|\nabla|}{|\nabla| - \gamma + 0i} = PV\left(\frac{|\nabla|}{|\nabla| - \gamma}\right) - i\gamma\pi\delta\big(|\nabla| - \gamma\big), \\
\mathcal{A}_\gamma^- &:= \frac{|\nabla|}{|\nabla| + \gamma}, \\
\mathcal{A}_\gamma &:= \frac{1}{2}\big(\mathcal{A}_\gamma^+ + \mathcal{A}_\gamma^-\big) = PV\left(\frac{|\nabla|^2}{|\nabla|^2 - \gamma^2}\right) - \frac{i\gamma\pi}{2}\delta\big(|\nabla| - \gamma\big), \\
I_c &:= \big(1 + |\nabla/c|^2\big)^{-1/2},
\end{aligned} \tag{1.11}$$

where PV denotes the principal value. The dot product $a \cdot b$ will always denote the Euclidian dot product, namely $a \cdot b = \sum_{i=1}^3 a_i b_i$. In particular, we have $|\mathbb{E}_1|^2 = \mathbb{E}_1 \cdot \overline{\mathbb{E}_1}$. Our main result is as follows.

Theorem 1.1. *Let (E^c, n^c) be a sequence of solutions for* (1.1) *such that $c \to \infty$ with $\gamma = 2c^2/\alpha > 0$ fixed, and initial data satisfying*

$$\begin{aligned}
&\big(E^c(0), c^{-2} I_c \dot{E}^c(0)\big) \to (\varphi, \psi) \quad \text{in } H^1, \\
&\big(n^c(0), |\alpha\nabla|^{-1}\dot{n}^c(0)\big) \quad \text{bounded in } L^2.
\end{aligned} \tag{1.12}$$

Let T^c be its maximal existence time. Let $\mathbb{E}^\infty := (\mathbb{E}_1^\infty, \mathbb{E}_2^\infty)$ be the solution of the following nonlinear Schrödinger type system

$$\begin{aligned}
&2i\dot{\mathbb{E}} - \Delta\mathbb{E} = |\mathbb{E}|^2\mathbb{E} + \mathcal{A}_\gamma(\mathbb{E}_1 \cdot \mathbb{E}_2)\overline{\mathbb{E}}^\perp, \\
&\mathbb{E}(0) = \frac{1}{2}(\varphi - i\psi, \overline{\varphi} - i\overline{\psi}),
\end{aligned} \tag{1.13}$$

where $\mathbb{E}^\perp = (\mathbb{E}_2, \mathbb{E}_1)$. Let T^∞ be the maximal existence time for $\mathbb{E}^\infty$. Then in the limit $c \to \infty$ with $\gamma\alpha = 2c^2$, we have $\liminf T^c \geqslant T^\infty$*, and for any $T < T^\infty$,*

$$E^c - \big(e^{ic^2 t}\mathbb{E}_1^\infty + e^{-ic^2 t}\overline{\mathbb{E}_2^\infty}\big) \to 0 \quad \text{in } C\big([0, T]; H^1\big). \tag{1.14}$$

We have asymptotic formula also for $\dot{E}^c$, n^c and $\dot{n}^c$, which we will give in a more precise and general version of the above theorem (see Theorem 3.1). Here we just remark that the singular part in the equation for $\mathbb{E}^\infty$ actually comes from the singular behavior of n^c and $\dot{n}^c$.

Remark that in the limit system (1.13), the L^2 norm of the solution decreases in t by the nonlinear interaction of $\mathbb{E}_1 \cdot \mathbb{E}_2$ at the frequency of size γ, because of the dissipative part of $\mathcal{A}_\gamma^+$, i.e. $\Im \mathcal{A}_\gamma^+ = -\gamma\pi\delta(|\nabla| - \gamma)$:

$$\partial_t \|\mathbb{E}^\infty(t)\|_{L_x^2}^2 = -\frac{\gamma}{2(2\pi)^2} \int_{|\xi|=\gamma} \left| \int_{\mathbb{R}^3} \left(\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty\right)(t,\xi) e^{-ix\xi}\, dx \right|^2 d\xi. \tag{1.15}$$

This property is used in a forthcoming paper [14] to study the limit when γ goes to infinity in (1.13). A similar phenomenon is known in the context of stability of nonlinear bound states, to cause the radiation damping [16] in the nonlinear Klein–Gordon equation (the linear ground state decays by the nonlinear resonance), and the relaxation of excited states [21] in the nonlinear Schrödinger equation (the excited states decay by the nonlinear resonance). In those cases, the operator $\mathcal{A}_\gamma^+$ involving a potential gives decay in the ODE governing the amplitude of the bound states. But as far as the authors know, the above theorem seems the first observation with a rigorous proof for a nonlinear resonance leading to the decrease of the energy for the limit wave functions.

The rest of the paper is organized as follows: First, in the end of this introduction, we give some notations which are necessary to the statement of the main result in Section 3. In the next section, we will rewrite our equation into a first order system such that we can formally derive the limit system. Then we restate our main result in Section 3 in the new variables, allowing more general initial data, which can introduce some additional singular terms into the limit system. After preparing some notations and tools in Section 4, we prove first a set of uniform estimates in Section 5, and then prove the convergence in Section 6.

We conclude the introduction with some notations used throughout the paper. More notations will be given in Section 4.

$$\begin{aligned} &\langle a\rangle := \left(1+|a|^2\right)^{1/2}, \qquad \langle a,b\rangle := \Re(a\cdot\bar{b}), \\ &\langle f \mid g\rangle_x := \int_{\mathbb{R}^3} \left\langle f(x), g(x)\right\rangle dx, \qquad \langle u \mid v\rangle_{t,x} := \int_{\mathbb{R}} \left\langle u(t) \mid v(t)\right\rangle_x dt, \end{aligned} \tag{1.16}$$

where a, b, f, g, u and v may be scalar or vector valued. We denote by $\mathcal{F}_d$ the d-dimensional Fourier transform. In particular, the space and the space–time Fourier transform are denoted by

$$\mathcal{F}_3\varphi = \tilde{\varphi}(\xi) = \int_{\mathbb{R}^3} \varphi(x) e^{-ix\xi}\, dx, \qquad \mathcal{F}_4 u = \hat{u}(\tau,\xi) = \int_{\mathbb{R}^{1+3}} u(t,x) e^{-it\tau - ix\xi}\, dt\, dx \tag{1.17}$$

and their inverse are given by

$$\mathcal{F}_3^{-1}\tilde{\varphi} = \varphi(x) = \frac{1}{(2\pi)^3}(\mathcal{F}_3\tilde{\varphi})(-x), \qquad \mathcal{F}_4^{-1}\hat{u} = u(t,x) = \frac{1}{(2\pi)^4}(\mathcal{F}_4\hat{u})(-t,-x). \tag{1.18}$$

For any function φ, we define the Fourier multiplier $\varphi(\nabla) := \mathcal{F}_3^{-1}\varphi(\xi)\mathcal{F}_3$. We will use the following multipliers repeatedly:

$$I_c := \langle \nabla/c\rangle^{-1}, \qquad \Delta_c := -2\omega(\nabla), \qquad \omega(\xi) := c^2\left(\langle \xi/c\rangle - 1\right). \tag{1.19}$$

Finally we recall the definition of norms for intersection and sum of two compatible Banach spaces X and Y (see e.g. [4,20])

$$\|f\|_{X\cap Y} = \|f\|_X + \|f\|_Y, \qquad \|f\|_{X+Y} = \inf_{f=f_1+f_2} \|f_1\|_X + \|f_2\|_Y. \tag{1.20}$$

2. Reduction of equations

In this section we rewrite the Klein–Gordon–Zakharov system (1.1) into first order equations in time and also decompose n into different time oscillations, from which one can easily obtain the limit system. The reduced systems will be suited also to get uniform estimates as well as the convergence.

First we define (we will remove the c dependence from E^c and n^c)

$$E_+ := \frac{1}{2}\{E - ic^{-2}I_c\partial_t E\}, \qquad E_- := \frac{1}{2}\{\overline{E} - ic^{-2}I_c\partial_t \overline{E}\},$$
$$\mathbb{E} := e^{-ic^2t}(E_+, E_-), \qquad N := n - i|\alpha\nabla|^{-1}\dot{n}.$$

We also define $\mathbb{E}^\perp = (\mathbb{E}_2, \mathbb{E}_1)$ and $\mathbb{E}^* := e^{-2ic^2t}\overline{\mathbb{E}}^\perp$ for any $\mathbb{E} = (\mathbb{E}_1, \mathbb{E}_2)$. The original functions are given by

$$E = e^{ic^2t}\mathbb{E}_1 + e^{-ic^2t}\overline{\mathbb{E}_2}, \qquad \dot{E} = ic^2 I_c^{-1}\big(e^{ic^2t}\mathbb{E}_1 - e^{-ic^2t}\overline{\mathbb{E}_2}\big),$$
$$n = \Re N, \qquad \dot{n} = -\Im\big(|\alpha\nabla|N\big),$$

where $\Re$ and $\Im$ represent the real and imaginary parts. Hence, the system (1.1) is reduced to

$$\begin{cases} 2i\dot{\mathbb{E}} - \Delta_c\mathbb{E} = -I_c n\big(\mathbb{E} + \mathbb{E}^*\big), \\ i\dot{N} + |\alpha\nabla|N = -|\alpha\nabla|\big\langle\mathbb{E}, \mathbb{E} + \mathbb{E}^*\big\rangle. \end{cases} \tag{2.1}$$

From now on, we will concentrate on system (2.1). Further we rewrite it into integral form as

$$\mathbb{E} = e^{-i\Delta_c t/2}\mathbb{E}(0) - S_E I_c n\big(\mathbb{E} + \mathbb{E}^*\big), \tag{2.2}$$
$$N = e^{i|\alpha\nabla|t}N(0) - S_n|\alpha\nabla|\big\langle\mathbb{E}, \mathbb{E} + \mathbb{E}^*\big\rangle, \tag{2.3}$$

where the space–time operators S_E and S_n are defined by

$$S_E f := \frac{1}{2i}\int_0^t e^{-i\Delta_c(t-s)/2} f(s)\,ds, \qquad S_n f := \frac{1}{i}\int_0^t e^{i|\alpha\nabla|(t-s)} f(s)\,ds. \tag{2.4}$$

Next we decompose N into components with different time phases

$$\begin{aligned} &N = N^f + N^0 + N^+ + N^-, \\ &N^f := e^{i|\alpha\nabla|t}N(0), \\ &N^0 := -S_n|\alpha\nabla||\mathbb{E}|^2, \\ &N^+ := -S_n e^{i\alpha\gamma t}|\alpha\nabla|(\mathbb{E}_1\cdot\mathbb{E}_2), \\ &N^- := -S_n e^{-i\alpha\gamma t}|\alpha\nabla|(\overline{\mathbb{E}_1\cdot\mathbb{E}_2}), \end{aligned} \tag{2.5}$$

where the oscillation $\alpha\gamma t = 2c^2 t$ is coming from $\mathbb{E}^*$ in the equation for N. Integrating on the phase $e^{i\alpha(|\nabla|\pm\gamma)(t-s)}$ in s, we get

$$\begin{aligned} &N^0 = -|\mathbb{E}|^2(t) + e^{i|\alpha\nabla|t}|\mathbb{E}|^2(0) + iS_n\partial_t|\mathbb{E}|^2, \\ &N^+ = -e^{i\alpha\gamma t}\mathcal{A}_\gamma^+(\mathbb{E}_1\cdot\mathbb{E}_2)(t) + e^{i|\alpha\nabla|t}\mathcal{A}_\gamma^+(\mathbb{E}_1\cdot\mathbb{E}_2)(0) + i\mathcal{A}_\gamma^+ S_n e^{i\alpha\gamma t}\partial_t(\mathbb{E}_1\cdot\mathbb{E}_2), \\ &N^- = -e^{-i\alpha\gamma t}\mathcal{A}_\gamma^-(\overline{\mathbb{E}_1\cdot\mathbb{E}_2})(t) + e^{i|\alpha\nabla|t}\mathcal{A}_\gamma^-(\overline{\mathbb{E}_1\cdot\mathbb{E}_2})(0) + i\mathcal{A}_\gamma^- S_n e^{-i\alpha\gamma t}\partial_t(\overline{\mathbb{E}_1\cdot\mathbb{E}_2}). \end{aligned} \tag{2.6}$$

The second and the third terms on each line will go to zero in the limit due to dispersion of $e^{i|\alpha\nabla|t}$ (the decay for the singular operator $e^{i|\alpha\nabla|t}\mathcal{A}_\gamma^+$ is given in Lemma 4.3). Hence plugging each first term into the nonlinearity for $\mathbb{E}$, we get the leading terms

$$n\big(\mathbb{E} + \mathbb{E}^*\big) = -|\mathbb{E}|^2\mathbb{E} - \frac{1}{2}\big[\mathcal{A}_\gamma^+(\mathbb{E}_1\cdot\mathbb{E}_2) + \mathcal{A}_\gamma^-(\mathbb{E}_1\cdot\mathbb{E}_2)\big]\overline{\mathbb{E}}^\perp + osc. + o(1), \tag{2.7}$$

where $osc.$ represents those terms with rapid oscillation $e^{\pm i\alpha\gamma t}$ or $e^{-2i\alpha\gamma t}$, and hence goes to zero weakly in time. Thus we arrive at the limit system (1.13).

3. Main result

Now we restate our main result in terms of the new variables $(\mathbb{E}, N)$ introduced in Section 2, slightly extending the initial data space for N. For that purpose, we introduce the Banach space $\mathcal{W}^{k,p}$ on $\mathbb{R}^3$ for $p \geqslant 2$ and $k \in \mathbb{Z}$ by the norm

$$\|\varphi\|_{\mathcal{W}^{k,p}} := \sup_{t \geqslant 0} \left\| e^{it|\nabla|}\varphi \right\|_{W^{k,p}}. \tag{3.1}$$

Theorem 3.1. *Let $(\mathbb{E}^c, N^c)$ be a sequence of solutions to (2.1), such that $c \to \infty$ with $\gamma = 2c^2/\alpha > 0$ fixed, and $\|\mathbb{E}^c(0)\|_{H^1} + \|N^c(0)\|_{L^2+\mathcal{W}^{k,p}}$ bounded for some $p > 3$ and $k \geqslant 1$. Let T^c be the maximal existence time of $(\mathbb{E}^c, N^c)$. Then there is $T > 0$, depending only on the size of the above initial norm, such that $T^c \geqslant T$ and $\|\mathbb{E}^c(t)\|_{H^1} + \|N^c(t)\|_{L^2+\mathcal{W}^{k,p}}$ is uniformly bounded on $[0, T]$ for large c.*

Moreover, assume that the initial data satisfies for $\sigma = 0, +$

$$\begin{aligned} &\mathbb{E}^c(0) \to \Phi^\infty \quad \text{in } H^1, \\ &e^{-\sigma i\alpha\gamma t} e^{i|\alpha\nabla|t} N^c(0) \to \mu^{\sigma\infty} \quad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^3\big), \end{aligned} \tag{3.2}$$

as $c \to \infty$, for some Φ^∞ and some $\mu^{\sigma\infty}$. Let $\mathbb{E}^\infty$ be the solution of the following limit system

$$\begin{aligned} &2i\dot{\mathbb{E}}^\infty - \Delta\mathbb{E}^\infty = \big[|\mathbb{E}^\infty|^2 - \Re\mu^{0\infty}\big]\mathbb{E}^\infty + \big[\mathcal{A}_\gamma\big(\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty\big) - \mu^{+\infty}/2\big]\bar{\mathbb{E}}^{\infty\perp}, \\ &\mathbb{E}^\infty(0) = \Phi^\infty. \end{aligned} \tag{3.3}$$

Let $T^\infty > 0$ be the maximal existence time of $\mathbb{E}^\infty$. Then we have a lower bound $\liminf T^c \geqslant T^\infty$*, and for all $T < T^\infty$ we have uniform convergence*

$$\mathbb{E}^c - \mathbb{E}^\infty \to 0 \quad \text{in } C\big([0,T]; H^1\big), \tag{3.4}$$

and also, by decomposing $N^c = N^{fc} + N^{0c} + N^{+c} + N^{-c}$ according to (2.6),

$$\begin{aligned} &N^{0c} + |\mathbb{E}^\infty|^2 - N^{0Ic} \to 0 \quad \text{in } C\big([0,T]; L^2\big), \\ &N^{-c} + e^{i\alpha\gamma t}\mathcal{A}_\gamma^-\big(\overline{\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty}\big) - N^{-Ic} \to 0 \quad \text{in } C\big([0,T]; L^2\big), \\ &N^{+c} + e^{-i\alpha\gamma t}\mathcal{A}_\gamma^+\big(\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty\big) - N^{+Ic} \to 0 \quad \text{in } C\big([0,T]; L^2 + \mathcal{W}^{k,p}\big), \end{aligned} \tag{3.5}$$

for all $p > 3$, where

$$\begin{aligned} &N^{fc} = e^{i|\alpha\nabla|t} N^c(0), \qquad N^{0Ic} = e^{i|\alpha\nabla|t}|\Phi^\infty|^2, \\ &N^{+Ic} = e^{i|\alpha\nabla|t}\mathcal{A}_\gamma^+\big(\Phi_1^\infty \cdot \Phi_2^\infty\big), \qquad N^{-Ic} = e^{i|\alpha\nabla|t}\mathcal{A}_\gamma^-\big(\overline{\Phi_1^\infty \cdot \Phi_2^\infty}\big). \end{aligned} \tag{3.6}$$

Moreover, we have

$$\lim_{s\to\infty} \left\| e^{is|\nabla|}\big(N^c - N^{fc}\big)(t) \right\|_{L^2+\mathcal{W}^{k,p}} = 0 \tag{3.7}$$

uniformly for $t \in [0, T]$ and for large c.

Remark 3.2. The uniform bound of $N^c(0)$ implies that the convergence to $\mu^{\sigma\infty}$ in (3.2) actually holds $*$-weakly in $L^\infty(0,\infty; L^2 + W^{k,p})$, so that we can make sense of the products with $\mu^{\sigma\infty}$ in the limit system.

Remark 3.3. (3.7) implies that the singular parts $\mu^{\sigma\infty}$ are preserved for later time, namely

$$e^{-i\sigma\alpha\gamma t} e^{i\alpha(t-t_0)|\nabla|} N^c(t_0) \to \mu^{\sigma\infty}. \tag{3.8}$$

In other words, the singular initial layer N^{+Ic} does not affect these terms (neither do the regular ones N^{0Ic} and N^{-Ic}). This follows from the decay property of $e^{it|\nabla|}\mathcal{A}_\gamma^+$, see Lemma 4.3.

In particular, if we start with initial data $N^c(0)$ bounded in H^σ for some $\sigma \in \mathbb{R}$, then we will never encounter $\mu^{*\infty}$, because for any $\chi \in C_0^\infty(\mathbb{R}^3)$ we have

$$\left\|\chi(x/R)e^{i|\alpha\nabla|t}N^c(0)\right\|_{L^2H^\sigma} \lesssim \sqrt{R/\alpha}\left\|N^c(0)\right\|_{H^\sigma}, \tag{3.9}$$

see [12, Lemma 8.1].

Hence nontrivial $\mu^{*\infty}$ can be created only from singular (in the Fourier space) initial data. For example, if μ_0^0 and μ_0^+ are bounded complex-valued measures on $[0,b)$ and $(a,b)\times S^2$ respectively, then

$$\begin{aligned} N^c(0) &:= \mathcal{F}_3^{-1}\int_0^b |\xi|^{-2}\delta\big(|\xi|-\tau/\alpha\big)\mu_0^0(\tau)\,d\tau + \mathcal{F}_3^{-1}\int_a^b \delta\big(|\xi|-\gamma-\tau/\alpha\big)\mu_0^+\big(\tau,\xi/|\xi|\big)\,d\tau \\ &= \mathcal{F}_3^{-1}\alpha|\xi|^{-2}\mu_0^0\big(\alpha|\xi|\big) + \mathcal{F}_3^{-1}\alpha\mu_0^+\big(\alpha\big(|\xi|-\gamma\big),\xi/|\xi|\big) \end{aligned} \tag{3.10}$$

is bounded in $\mathcal{W}^{k,\infty}$ for any $k \in \mathbb{N}$, and the limit profiles are given by

$$\begin{aligned} \mu^{0\infty}(t) &= (2\pi)^{-3}\int_0^b e^{i\tau t}\mu_0^0(\tau)\,d\tau = (2\pi)^{-2}\mathcal{F}_1^{-1}\mu_0^0(\tau), \\ \mu^{+\infty}(t) &= \mathcal{F}_3^{-1}\delta\big(|\xi|-\gamma\big)\int_a^b e^{i\tau t}\mu_0^+\big(\tau,\xi/|\xi|\big)\,d\tau = \mathcal{F}_4^{-1}\delta\big(|\xi|-\gamma\big)\mu_0^+\big(\tau,\xi/|\xi|\big). \end{aligned} \tag{3.11}$$

Note that $\mu^{0\infty}$ and $\mu^{+\infty}$ do not see each other because of the rapid oscillation $e^{-i\alpha\gamma t}$. If we choose $N^c(0) = N(0)$ independent of c, then the convergence (3.2) implies that $\mu^{0\infty}$ is a constant, and $\mu^{+\infty}$ is time-independent with Fourier support on $\{|\xi| = \gamma\}$. We remark that $\sigma = -1$ in (3.2) would give always 0 in the limit because of the oscillation $e^{i\alpha(|\nabla|+\gamma)t}$, which is uniformly rapid for all frequency ξ.

Remark 3.4. For the uniform bounds, we can sharpen the $\mathcal{W}^{k,p}$ norm by replacing L^p with the Lorentz space $L^{3,\infty}$.

Remark 3.5. Theorem 1.1 easily follows from the above theorem by transforming the variables back to the original (E,n), in the case $N^c(0)$ is bounded in L^2 and hence $\mu^{0\infty} = \mu^{+\infty} = 0$. However the singular part $\mathcal{W}^{k,p}$ is needed even for the proof in this case. Indeed, to prove the above result, we will work on some small time interval $(0,T_1)$ on which we can prove some uniform estimates, then we will pass to the limit. Then, to extend the convergence to the maximal existence interval $(0,T^\infty)$, we need to iterate the same argument on some interval (T_1,T_2). We notice that at the time T_1, $N^{+c}(T_1)$ contains the singular part

$$N^{+c}(T_1) \to -\mathcal{A}_\gamma^+\big(\mathbb{E}_1^\infty(T_1)\cdot\mathbb{E}_2^\infty(T_1)\big), \tag{3.12}$$

which is bounded in $\mathcal{W}^{k,p} + L^2$ for all $p > 3$ and $k \in \mathbb{N}$ by Lemma 4.3, but does not belong to L^2 in general.

Our first order system (2.1) is not exactly invariant for time shift, because of the oscillation factors $e^{\pm ic^2t}$, but for the modulated translation

$$(\mathbb{E},N) \mapsto \big(e^{ic^2t_0}\mathbb{E}(t+t_0), N(t+t_0)\big), \tag{3.13}$$

for any $t_0 \in \mathbb{R}$. Correspondingly, we have an immediate

Corollary 3.6. *In the above theorem, assume instead of* (3.2)

$$\begin{aligned} &e^{-ic^2t_0}\mathbb{E}^c(0) \to \Phi^\infty \quad \text{in } H^1, \\ &e^{-\sigma i\alpha\gamma(t+t_0)}e^{i|\alpha\nabla|t}N^c(0) \to \mu^{\sigma\infty} \quad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^3\big), \end{aligned} \tag{3.14}$$

for some $t_0 \in \mathbb{R}$. Then we have the convergence

$$e^{-ic^2t_0}\mathbb{E}^c - \mathbb{E}^\infty \to 0, \qquad N^{0c} + \left|\mathbb{E}^\infty\right|^2 - N^{0Ic} \to 0,$$
$$N^{-c} + e^{i\alpha\gamma(t+t_0)}\mathcal{A}_\gamma^-\left(\overline{\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty}\right) - N^{-Ic} \to 0,$$
$$N^{+c} + e^{-i\alpha\gamma(t+t_0)}\mathcal{A}_\gamma^+\left(\mathbb{E}_1^\infty \cdot \mathbb{E}_2^\infty\right) - N^{+Ic} \to 0, \tag{3.15}$$

in the same topologies and with the same $\mathbb{E}^\infty$ *and* N^{*Ic} *as above.*

Proof. Assume by contradiction that one of the convergences fails. Extracting a subsequence of c, we may assume in addition that $e^{ic^2t} \to e^{i\theta}$ for some $\theta \in \mathbb{R}$. Then we can apply the above theorem replacing Φ^∞ with $e^{i\theta}\Phi^\infty$ and $\mu^{\sigma\infty}$ with $e^{2\sigma i\theta}\mu^{\sigma\infty}$. Since the limit system is invariant with respect to the "Gauge transform"

$$\left(\mathbb{E}^\infty, \mu^\sigma\right) \mapsto \left(e^{i\theta}\mathbb{E}^\infty, e^{2\sigma i\theta}\mu^\sigma\right), \tag{3.16}$$

the theorem gives all the desired convergences for this subsequence, a contradiction. □

Strictly speaking, we will be using the above logic implicitly in the proof of the above theorem when extending the convergence from the first time step T_1 to the maximal existence time T^∞. Namely, we should apply the above argument to the modulated translation $(e^{ic^2T_1}\mathbb{E}(t+T_1), N(t+T_1))$ to get the convergence in the next time step (cf. (3.8) for the persistence of (3.14)). We will not repeat this in the proof given below.

4. Preliminaries and notations

Before starting the proof, we prepare basic settings and estimates together with some notations.

4.1. Frequency decomposition

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfy $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ for $|\xi| \leqslant 4/3$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 5/3$. For any $a > 0$ and any function φ, we denote

$$f_{\leqslant a} := \chi\left(|\nabla/a|\right)f, \qquad f_{>a} := f - f_{\leqslant a}, \qquad f_a := \begin{cases} f_{\leqslant a} - f_{\leqslant a/2} & (a > 1), \\ f_{\leqslant 1} & (a \leqslant 1). \end{cases} \tag{4.1}$$

Hence we have the inhomogeneous Littlewood–Paley decomposition

$$f = \sum_{j\in\mathbb{D}} f_j, \quad \mathbb{D} := \left\{1, 2, 2^2, 2^3, 2^4, \ldots\right\}. \tag{4.2}$$

In addition, we denote the nonresonant frequency part by

$$N_X := N - N_\gamma. \tag{4.3}$$

We note that the singularity of $\mathcal{A}_\gamma^+$ is only around $|\xi| = \gamma$ in the Fourier space, and so it is regular in the physical space.

For bilinear interactions, we denote frequency trichotomy by

$$\begin{aligned} fg &= (fg)_{LH} + (fg)_{HL} + (fg)_{HH} \\ &:= \sum_{l<h/4} f_l g_h + \sum_{h>4l} f_h g_l + \sum_{4i\geqslant j\geqslant i/4} f_i g_j, \end{aligned} \tag{4.4}$$

where i, j, k, l, h run over the dyadic numbers $\mathbb{D}$, and *LH*, *HL* and *HH* respectively indicate low-high, high-low and high-high frequency interactions.

If no ambiguity can occur, we often abbreviate such as $(fg)_{Yl} := ((fg)_Y)_l$ and $(fg)_{Y+Z} := (fg)_Y + (fg)_Z$ where $Y, Z = HH, HL$ or LH and $l = a, > a, \leqslant a$. For example, $(EF)_{HLX} = ((EF)_{HL})_X$, $(EF)_{HH>a} = ((EF)_{HH})_{>a}$, etc.

4.2. Strichartz norms

We briefly recall the Strichartz estimate for $e^{-it\Delta_c/2}$ and $e^{it|\alpha\nabla|}$ on $\mathbb{R}^3$ (see [8,9]). We note that all the linear estimates can be reduced to the case $c = 1$ and $\alpha = 1$ by rescaling.

For the Klein–Gordon equation, we have the L^1–L^∞ decay estimate

$$\left\| e^{-it\Delta_c/2}\varphi \right\|_{I_c^{-5\theta/2} B^0_\infty(\mathbb{R}^3)} \lesssim |t|^{-(2+\theta)/2} \|\varphi\|_{B_1^{2(1-\theta)}(\mathbb{R}^3)}, \tag{4.5}$$

for $\theta \in [0,1]$, where $B^s_p := B^s_{p,2}$ denotes the inhomogeneous Besov space (cf. [4]). The case $\theta = 0$ corresponds to the wave equation, and $\theta = 1$ (without I_c) is for the Schrödinger equation (we will not use the intermediate case $0 < \theta < 1$). From the decay estimate, the standard argument derives the Strichartz estimate

$$\left\| e^{-it\Delta_c/2}\varphi \right\|_{I_c^{-\mu} L^p(\mathbb{R}; B^\sigma_q(\mathbb{R}^3))} \lesssim \|\varphi\|_{H^1}, \tag{4.6}$$

for the exponents $0 \leqslant \theta \leqslant 1$, $2 \leqslant p, q \leqslant \infty$ satisfying $(p,q) \neq (2,\infty)$,

$$\frac{2}{p} = (2+\theta)\left(\frac{1}{2} - \frac{1}{q}\right), \qquad \sigma = 1 - (1-\theta)\left(1 - \frac{2}{q}\right), \qquad \mu = \frac{5\theta}{2}\left(\frac{1}{2} - \frac{1}{q}\right). \tag{4.7}$$

Moreover, for the wave equation, we have

$$\left\| e^{it|\alpha\nabla|}\varphi \right\|_{\alpha^{-1/p} L^p(\mathbb{R}; \dot B^\sigma_q(\mathbb{R}^3))} \lesssim \|\varphi\|_{L^2}, \tag{4.8}$$

where $\dot B^s_q := \dot B^s_{q,2}$ denotes the homogeneous Besov space, provided that

$$2 < p \leqslant \infty, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \qquad \sigma = -\frac{2}{p}. \tag{4.9}$$

For the Duhamel terms we have similarly

$$\|S_E f\|_{ST(E)_1} \lesssim \|f\|_{ST(E)_2^*}, \qquad \|S_n f\|_{ST(N)_3} \lesssim \|f\|_{ST(N)_4^*}, \tag{4.10}$$

where for each $ST(E)_j$ (resp. $ST(N)_j$) we could choose any space in (4.6) (resp. in (4.8)), but for the sake of concreteness we choose the following specific exponents:

$$\begin{aligned} ST(E)_1 &= L^\infty H^1 \cap I_c^{-5/9} L^3 B^1_{18/5} \cap I_c^{-1/3} L^3 B^{2/3}_6, \\ ST(E)_2^* &= L^1 H^1 + I_c^{25/36} L^{12/7} B^1_{9/7} + I_c^{1/2} L^{10/7} B^1_{10/7}, \end{aligned} \tag{4.11}$$

where we chose $\theta = 1$ for the second, the fifth, and the sixth spaces, and $\theta = 0$ for the third one, in view of (4.7). Here the sum and the intersection imply in practice that we can choose any member of the intersections to estimate the solution and any member of the sum for the nonlinearity. The power of I_c indicates loss or gain of regularity for higher frequencies $|\xi| \gtrsim c$ ($\gg 1$), which will be often exchanged with power of c (e.g. $c^{-\theta} I_c^{-1} \lesssim \langle\nabla\rangle^{-\theta}$ on $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$ and $0 \leqslant \theta \leqslant 1$). For the wave component we define similarly

$$\begin{aligned} ST(N)_3 &= L^\infty L^2 \cap \alpha^{-1/3} L^3 B^{-2/3}_6, \\ ST(N)_4^* &= L^1 L^2 + \alpha^{1/3} L^{3/2} B^{2/3}_{6/5} + \alpha^{1/6} L^{6/5} B^{1/3}_{3/2}. \end{aligned} \tag{4.12}$$

In applying the Strichartz estimates, we will write these exponents explicitly.

4.3. Fourier restriction norms

For any $s \in \mathbb{R}$ and any interval $I \subset \mathbb{R}$, we define

$$\begin{aligned} X^{s,1} &:= \left\{ e^{-i\Delta_c t/2} u(t) \;\middle|\; u \in H^1_t\left(I; H^s_x\right) \right\}, \\ Y^{s,1} &:= \left\{ e^{i|\alpha\nabla|t} u(t) \;\middle|\; u \in H^1_t\left(I; H^s_x\right) \right\}, \end{aligned} \tag{4.13}$$

with the norms

$$\|u\|_{X^{s,1}(I)} = \|e^{i\Delta_c t/2}u(t)\|_{H_t^1(I;H_x^s)}, \qquad \|v\|_{Y^{s,1}(\mathbb{R})} = \|e^{-i|\alpha\nabla|t}v(t)\|_{H_t^1(I;H_x^s)}. \tag{4.14}$$

Those norms on the whole line $t \in \mathbb{R}$ can be represented by the Fourier transform

$$\|u\|_{X^{s,1}(\mathbb{R})} = \|\langle\tau - \omega(\xi)\rangle\langle\xi\rangle^s \hat{u}\|_{L^2_{\tau,\xi}}, \qquad \|v\|_{Y^{s,1}(\mathbb{R})} = \|\langle\tau - \alpha|\xi|\rangle\langle\xi\rangle^s \hat{v}\|_{L^2_{\tau,\xi}}. \tag{4.15}$$

The distance from the characteristic surface, such as $|\tau - \omega(\xi)|$ for $X^{s,1}$, plays an essential role in using those norms. So, we consider an explicit extension from $(0, T)$ to $\mathbb{R}$. We define an extension operator ρ_T for any $T \in (0, 1)$ by

$$\rho_T u(t) = \chi(t)u(\mu_T(t)), \tag{4.16}$$

where $\mu_T(t) := \max(\min(t, 2T - t), 0)$ and $\chi \in C_0^\infty(\mathbb{R})$ satisfies $\chi(t) = 1$ for $|t| \leqslant 2$ and $\chi(t) = 0$ for $|t| \geqslant 3$. It is clear that $\rho_T u(t) = u(t)$ for $t \in (0, T)$, and ρ_T is bounded on $H_t^1(0, T; H^s) \to H_t^1 H^s(\mathbb{R}^{1+3})$ uniformly for $s \in \mathbb{R}$ and $0 < T \leqslant 1$.

For the bilinear estimates using those norms, we introduce decomposition with respect to the distance from characteristic surface. For any $\beta : \mathbb{R}^3 \to \mathbb{R}$ and $\delta > 1$ and any function $u(t, x)$ on $\mathbb{R} \times \mathbb{R}^3$, we define

$$\begin{aligned} &P_{|\tau-\beta(\xi)|\leqslant\delta} u := \mathcal{F}_4^{-1}\chi\big((\tau - \beta(\xi))/\delta\big)\mathcal{F}_4 u, \\ &P_{|\tau-\beta(\xi)|>\delta} u := u - P_{|\tau-\beta(\xi)|\leqslant\delta} u. \end{aligned} \tag{4.17}$$

Estimating in the Fourier space, we easily obtain

$$\|P_{|\tau-\omega(\xi)|>\delta} u\|_{L^2 H^s} \lesssim \delta^{-1}\|u\|_{X^{s,1}}, \tag{4.18}$$

$$\|P_{|\tau-\alpha|\xi||>\delta} u\|_{L^2 H^s} \lesssim \delta^{-1}\|u\|_{Y^{s,1}}. \tag{4.19}$$

We can derive similar estimates in L_t^∞ setting without using $X^{s,b}$ spaces:

Lemma 4.1. *We have*

$$\left\|\rho_T \int_0^t (P_{|\tau|>\delta} f)(s)\,ds\right\|_{L_t^\infty(\mathbb{R};X)} \lesssim \min(\delta^{-1}, T)\|f\|_{L_t^\infty(\mathbb{R};X)}, \tag{4.20}$$

uniformly for any $\delta > 1$, any $T > 0$ and any Banach space X.

We note that $P_{|\tau|>\delta}$ is a Fourier multiplier in the time variable which cuts the low frequencies, and hence there is nothing to do with the x variable in the above lemma. We also note that the proof is simpler than that of Lemma 2.3 of [12] due to the different order of the integration and the extension ρ_T.

Proof. The left-hand side is bounded by

$$\begin{aligned} \left\|\int_0^t (P_{|\tau|>\delta} f)(s)\,ds\right\|_{L_t^\infty(0,T;X)} &\leqslant \|P_{|\tau|>\delta} f\|_{L^1(0,T;X)} \\ &\lesssim T\big\|f - \mathcal{F}_1^{-1}\big(\chi(\tau/\delta)\big) * f\big\|_{L_t^\infty(\mathbb{R};X)} \lesssim \|f\|_{L_t^\infty(\mathbb{R};X)}, \end{aligned} \tag{4.21}$$

where we used the triangle inequality for the Bochner integral in X, and that $\mathcal{F}_1^{-1}\chi(\tau/\delta) \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$. Similarly we have

$$\left\|\int_0^t (P_{|\tau|>\delta} f)(s)\,ds\right\|_{L_t^\infty(0,T;X)} \leqslant \big\|\big[\psi(\delta t) * f\big]_0^t\big\|_{L^\infty(0,T;X)} \lesssim \delta^{-1}\|f\|_{L_t^\infty(\mathbb{R};X)}, \tag{4.22}$$

where we denoted $\psi(t) := \mathcal{F}^{-1}\tau^{-1}(1 - \chi(\tau)) \in L^1(\mathbb{R})$. $\square$

4.4. Singular decay estimate

Here we derive some estimates on the singular operator $\mathcal{A}_\gamma^+$ together with the wave propagator. First, we have a pointwise decay estimate:

Lemma 4.2. *For any $\varphi \in \mathcal{S}(\mathbb{R}^3)$ with symmetry $\varphi(x) = \varphi(|x|)$, we have*

$$\left|e^{it|\nabla|}\mathcal{A}_\gamma^+\varphi(x)\right| \lesssim \begin{cases} \langle x\rangle^{-1}\langle t-|x|\rangle^{-1} & (|x|<t), \\ \langle x\rangle^{-1} & (|x|>t), \end{cases} \tag{4.23}$$

uniformly for $t>0$ and $x \in \mathbb{R}^3$.

Proof. By the Laplace transform, we have

$$\begin{aligned} e^{it|\nabla|}\mathcal{A}_\gamma^+\varphi(x) &= -ie^{it|\nabla|}\lim_{\varepsilon\to+0}\int_0^\infty |\nabla|e^{is(|\nabla|-\gamma+i\varepsilon)}\varphi\,ds \\ &= -ie^{it\gamma}\lim_{\varepsilon\to+0}\int_0^\infty |\nabla|e^{i(|\nabla|-\gamma+i\varepsilon)(t+s)}\varphi\,ds. \end{aligned} \tag{4.24}$$

Let $|x| = r$. By the Fourier transform, the expression before the limit is equal to

$$(2\pi)^{-2}\int_0^\infty\int_0^\infty \rho e^{i(\rho-\gamma+i\varepsilon)(t+s)}\mathcal{F}_3\varphi(\rho)\frac{\sin(\rho r)}{\rho r}\rho^2\,d\rho\,ds. \tag{4.25}$$

Define $f(t)$ by $\mathcal{F}_1 f(\rho) = \rho^2\mathcal{F}_3\varphi(\rho)$ for $\rho>0$, and $\mathcal{F}_1 f(\rho) = 0$ for $\rho<0$. By the inverse Fourier transform, we have

$$\begin{aligned} (4.25) &= \int_0^\infty \frac{f(t+r+s)-f(t-r+s)}{2\pi ir}e^{-(i\gamma+\varepsilon)(t+s)}\,ds \\ &= \frac{1}{i\pi}\int_{-1}^1\int_0^\infty f'(t+\theta r+s)e^{-(i\gamma+\varepsilon)(t+s)}\,ds\,d\theta. \end{aligned} \tag{4.26}$$

Since $\mathcal{F}_1 f \in W^{2,1}$ and $\mathcal{F}_1 f' = i\rho\mathcal{F}_1 f \in W^{3,1}$, we have

$$\left|f(t)\right| \lesssim \langle t\rangle^{-2}, \qquad \left|f'(t)\right| \lesssim \langle t\rangle^{-3}. \tag{4.27}$$

Hence we have for $r<t$,

$$\left|(4.25)\right| \lesssim \langle r\rangle^{-1}\int_0^\infty \langle t-r+s\rangle^{-2}\,ds \lesssim \langle r\rangle^{-1}\langle t-r\rangle^{-1}, \tag{4.28}$$

and for any $r>0$,

$$\left|(4.25)\right| \lesssim \langle r\rangle^{-1}\sup_{|\theta|\leqslant 1}\int_{\mathbb{R}} \langle t+\theta r+s\rangle^{-2}\,ds \lesssim \langle r\rangle^{-1}, \tag{4.29}$$

both uniformly in $\varepsilon>0$. Thus we get the desired bound by $\varepsilon\to+0$. □

Applied to the Littlewood–Paley decomposition, the above estimate immediately implies the following L^p decay.

Lemma 4.3. *If $q \geqslant 1$, $p \leqslant \infty$, $1/q - 1/p \geqslant 2/3$ and $(p,q) \neq (3,1), (\infty, 3/2)$, then we have*

$$\left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi_\gamma \right\|_{L^p(\mathbb{R}^3)} \lesssim t^{-3(1/q-1/p-2/3)} \|\varphi_\gamma\|_{L^q(\mathbb{R}^3)}. \tag{4.30}$$

In addition, we have

$$\left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi_\gamma \right\|_{L^{3,\infty}(\mathbb{R}^3)} \lesssim \|\varphi_\gamma\|_{L^1(\mathbb{R}^3)}, \qquad \left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi_\gamma \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\varphi_\gamma\|_{L^{3/2,1}(\mathbb{R}^3)}, \tag{4.31}$$

where $L^{p,q}$ denotes the Lorentz space.

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ be radially symmetric and $\mathcal{F}_3\psi(\xi) = 1$ for $|\xi| \lesssim \gamma + 1$ including $\operatorname{supp}\mathcal{F}_3\varphi_\gamma$, so that we have

$$e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi_\gamma = \varphi_\gamma * e^{it|\nabla|}\mathcal{A}_\gamma^+ \psi. \tag{4.32}$$

Hence by the Young inequality for the Lorentz space, we have for the first case,

$$\left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi_\gamma \right\|_{L^p} \lesssim \|\varphi_\gamma\|_{L^q} \left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \psi \right\|_{L^{r,\infty}}, \tag{4.33}$$

where $1/r = 1/p - 1/q + 1 \in [0, 1/3]$, and applying the above lemma to ψ,

$$\begin{aligned} \left\| e^{it|\nabla|}\mathcal{A}_\gamma^+ \psi \right\|_{L^{r,\infty}} &\lesssim \left\| |x|^{3/r} e^{it|\nabla|}\mathcal{A}_\gamma^+ \psi \right\|_{L^\infty} \\ &\lesssim \sup_{0<|x|<t} |x|^{3/r} \langle x\rangle^{-1} \langle t - |x| \rangle^{-1} + \sup_{|x|>t} |x|^{3/r} \langle x\rangle^{-1} \\ &\lesssim t^{3/r-1} = t^{-3(1/q-1/p-3/2)}, \end{aligned} \tag{4.34}$$

where we used that $|x|^{-3/r} \in L^{r,\infty}$.

The second case is just the critical case for the Young inequality. □

We will mainly use the above L^p decay with $q = 1$. From (4.26), it is clear that the pointwise estimate for $r > t$ cannot be improved, and hence $e^{it|\nabla|}\mathcal{A}_\gamma^+ \varphi$ does not belong to $L^3(\mathbb{R}^3)$ in general.

5. Uniform estimates

In this section and the next one, we prove the main Theorem 3.1. The main part of the proof consists in estimating the following norms uniformly in c (and α) and for small $T \in (0,1)$.

$$\begin{aligned} \|\mathbb{E}\| &:= \|\mathbb{E}\|_{\mathrm{Str}^E(0,T)} + \|\mathbb{E}\|_{\mathcal{X}(0,T)}, \\ \|N\| &:= \|N\|_{[\mathrm{Str}^n(0,T) + L_t^\infty(0,t;B^{1/2}_{2,\infty})] \cap \mathcal{Y}(0,T) + L_t^\infty(0,T;\mathcal{W}^{k,p})}, \end{aligned} \tag{5.1}$$

for arbitrarily fixed $k \geqslant 1$ and $p > 3$, where the spaces Str^E, Str^n, $\mathcal{X}$ and $\mathcal{Y}$ are defined by

$$\begin{aligned} \mathrm{Str}^E &:= \left\{ u \in L_t^\infty(H^1) \;\middle|\; u_{\leqslant c} \in L_t^3(B^1_{18/5}),\ u_{>c} \in c^{-1/3} L_t^3(B_6^{1/3}) \right\}, \\ \mathrm{Str}^n &:= L_t^\infty(L^2) \cap \alpha^{-1/3} L_t^3(B_6^{-2/3}), \\ \mathcal{X} &= I_c^{5/6} X^{0,1}, \qquad \mathcal{Y} = I_c^{-1/6} \alpha Y^{0,1}, \end{aligned} \tag{5.2}$$

and the norm for Str^E is given by

$$\|E\|_{\mathrm{Str}^E} := \|E\|_{L^\infty(H^1)} + \|E_{\leqslant c}\|_{L^3(B^1_{18/5})} + \|E_{>c}\|_{c^{-1/3}L^3(B_6^{1/3})}. \tag{5.3}$$

We recall that the existence of solutions was already proved in [15]. Note that in the Str^E norm, the frequencies lower than c are estimated in the second space in (4.11) and the higher part in the third space. The norm for N consists of three different components; we will use Str^n for the part similar to a free solution, $L^\infty(B^{1/2}_{2,\infty})$ for the part which is smoother but far from free solutions in the space–time frequency, and $\mathcal{W}^{k,p}$ for the singular part at the resonant frequency $|\xi| = \gamma$.

The uniform estimate will be done in this section, while Section 6 will be devoted to the convergence proof. Let us outline the proof for the uniform bounds. First in Section 5.1, we derive the estimates in the space–time Fourier spaces $\mathcal{X}$ and $\mathcal{Y}$ by simple product estimates, from the Strichartz and energy bounds.

To estimate the Strichartz norm of $\mathbb{E}$, we decompose

$$\mathbb{E} = e^{-i\Delta_c t/2}\mathbb{E}(0) - S_E I_c\Bigg[n_\gamma F + (n_X F_{\leqslant c})_{HH+LH} + \sum_{k>c}(n_{X(\leqslant\tilde\gamma k/c)}F_k)\Bigg] \\ - S_E I_c\Bigg[\sum_{k>c}(n_{X(>\tilde\gamma k/c)}F_k)_{LH+HH} + (n_X F)_{HL}\Bigg] \tag{5.4}$$

where $F = \mathbb{E} + \mathbb{E}^*$ and $\tilde\gamma = \gamma/\varepsilon$ with $\varepsilon > 0$ given in Lemma 5.4. The terms appearing on the first line of (5.4) will be treated in Proposition 5.2 using only Strichartz bounds. The terms on the second line of (5.4) require the use of the nonresonant property and are treated in Proposition 5.5.

To estimate N_X in $\mathrm{Str}^n + L^\infty B^{1/2}_{2,\infty}$, we write

$$N = e^{i|\alpha\nabla|t}N(0) - S_n|\alpha\nabla|\sum_k\sum_{j\leqslant\tilde\gamma k/c}\langle\mathbb{E}_k, \mathbb{E}+\mathbb{E}^*\rangle_j - S_n|\alpha\nabla|\sum_k\sum_{j>\tilde\gamma k/c}\langle\mathbb{E}_k, \mathbb{E}+\mathbb{E}^*\rangle_j. \tag{5.5}$$

For the part where $j \leqslant \tilde\gamma k/c$, we cannot use the nonresonant property but we can gain powers of c because j is much smaller than k. This part can be treated only by Strichartz in Proposition 5.3. The part $j > \tilde\gamma k/c$, is treated in Proposition 5.6 using the nonresonant property of the interaction.

Finally, the estimate on $N_\gamma \in L^\infty(\mathcal{W}^{k,p})$ is done in Section 5.4 by integrating by parts in time.

We emphasize that in the following estimates, the implicit constants are always independ of the parameters c, α, γ or T.

5.1. $\mathcal{X}\times\mathcal{Y}$ bounds from Strichartz bounds

Now we start the actual proof of Theorem 1.1, or the general version 3.1. Here we derive the $\mathcal{X}$ and $\mathcal{Y}$ type estimate from the Strichartz type bounds. We have the following proposition.

Proposition 5.1. *For any $T \in (0,1)$, any $p > 3$, and any functions n, E and F on $(0,T)\times\mathbb{R}^3$, we have*

$$\|S_E I_c n E\|_{\mathcal{X}(0,T)} \lesssim T^{1/6}\|n\|_{L^\infty(0,T;L^2+W^{1,p})}\|E\|_{\mathrm{Str}^E(0,T)}, \\ \big\|S_n|\alpha\nabla|\langle E,F\rangle\big\|_{\mathcal{Y}(0,T)} \lesssim T^{1/6}\|E\|_{\mathrm{Str}^E(0,T)}\|F\|_{\mathrm{Str}^E(0,T)}. \tag{5.6}$$

Proof. We decompose $n = n_1 + n_2$ such that $n_1 \in L^\infty L^2$ and $n_2 \in L^\infty W^{1,p}$. We use by Sobolev that $W^{1,p} \subset L^\infty$ and

$$\mathrm{Str}^E \subset I_c^{-1/3}L^3\big(B^{2/3}_6\big) \subset L^3\big(B^{2/3}_6\big) + c^{-1/3}L^3\big(B^{1/3}_6\big) \subset L^3L^\infty + c^{-1/3}L^3L^{18}, \tag{5.7}$$

where for the second embedding we separated the frequencies into $|\xi| \leqslant c$ and $|\xi| > c$. By Hölder and Sobolev, we have also that $L^{18}\times L^2$ (products of functions) $\subset L^{9/5} \subset H^{-1/6}$, where the exponents satisfy

$$\frac{1}{2} + \frac{1}{18} = \frac{5}{9} = \frac{1}{2} + \frac{1/6}{3}. \tag{5.8}$$

Thus we obtain

$$\|n_1E\|_{L^3L^2+c^{-1/3}L^3H^{-1/6}} \lesssim \|n_1\|_{L^\infty L^2}\|E\|_{\mathrm{Str}^E}, \\ \|n_2E\|_{L^\infty L^2} \lesssim \|n_2\|_{L^\infty L^\infty}\|E\|_{L^\infty L^2} \lesssim \|n_2\|_{L^\infty W^{1,p}}\|E\|_{L^\infty L^2}. \tag{5.9}$$

Summing these two, and using that $c^{-1/3}H^{-1/6} \subset c^{-1/6}H^{-1/6} \subset I_c^{-1/6}L^2$, we obtain the first estimate:

$$\|S_E I_c n E\|_{\mathcal{X}(0,T)} \lesssim \big\|(n_1+n_2)E\big\|_{L^2(I_c^{-1/6}L^2)} \lesssim T^{1/6}\|n\|_{L^\infty(L^2+W^{1,p})}\|E\|_{\mathrm{Str}^E}. \tag{5.10}$$

We get the second estimate exactly in the same way

$$\big\|\nabla\langle E,F\rangle\big\|_{L^3L^2+c^{-1/3}L^3H^{-1/6}} \lesssim \|\nabla E\|_{L^\infty L^2}\|F\|_{\mathrm{Str}^E} + \|\nabla F\|_{L^\infty L^2}\|E\|_{\mathrm{Str}^E}, \tag{5.11}$$

by putting the low frequency in the Strichartz space. □

5.2. Strichartz estimate for regular interactions

To derive $H^1 \times L^2$ and Strichartz bounds for $\mathbb{E}$ and N_X, we decompose the bilinear terms into frequencies as in (4.4). Those interactions where the less regular function has lower or similar frequency are relatively more regular. In [12], these term were treated only by the Strichartz estimate. Here, due to the low regularity, we have to treat some of those terms using their nonresonant property. We have the following estimates, which will be used with $E = \mathbb{E}$ or $\mathbb{E}^*$.

Proposition 5.2. *For any $T \in (0,1)$, any $p > 3$ and any functions E and n defined on $(0,T) \times \mathbb{R}^3$, we have*

$$\begin{aligned}
&\big\|S_E I_c(nE_{\leqslant c})_{HH+LH}\big\|_{\mathrm{Str}^E(0,T)} \lesssim T^{1/4}\|n\|_{L^\infty(0,T;L^2)}\|E\|_{\mathrm{Str}^E(0,T)},\\
&\big\|S_E I_c(nE)\big\|_{\mathrm{Str}^E(0,T)} \lesssim T\|n\|_{L^\infty(0,T;W^{1,p})}\|E\|_{L^\infty(0,T;H^1)},\\
&\Big\|S_E I_c \sum_{k>c}(n_{\leqslant\tilde{\gamma}k/c}E_k)\Big\|_{\mathrm{Str}^E(0,T)} \lesssim \big(T + T^{1/2}c^{-1/2}\big)\|n\|_{\mathrm{Str}^n(0,T)+L^\infty(0,T;B^{1/2}_{2,\infty})}\|E\|_{\mathrm{Str}^E}.
\end{aligned} \tag{5.12}$$

Proof. For the first estimate, we use the Strichartz estimate, hence it is bounded by

$$\begin{aligned}
\big\|(nE_{\leqslant c})_{HH+LH}\big\|_{\mathrm{Str}^{E^*}} &\lesssim \Big\|\sum_{l\in\mathbb{D}}\sum_{k\leqslant c}\sum_{j\lesssim k}(n_jE_k)_l\Big\|_{L^{12/7}B^1_{9/7}}\\
&\lesssim \big\|l/k\big\|n_j(t)\big\|_{L^2}\big\|E_k(t)\big\|_{B^1_{18/5}}\big\|_{L_t^{12/7}\ell^2_l\ell^1_{j,k}(j\lesssim l\sim k\leqslant c)}\\
&\lesssim T^{1/4}\|n\|_{L^\infty L^2}\|E_{\leqslant c}\|_{L^3B^1_{18/5}} \lesssim T^{1/4}\|n\|_{L^\infty L^2}\|E\|_{\mathrm{Str}^E},
\end{aligned} \tag{5.13}$$

where we used Hölder in $\mathbb{R}^3$ and $\mathbb{R}$ for x and t, and Young in $\mathbb{Z}$ for $l \in \mathbb{D} = 2^{\mathcal{N}}$ (cf. [12, Lemma 2.6]).

The second estimate easily follows from the fact that

$$\|nE\|_{L^\infty H^1} \lesssim \|n\|_{L_t^\infty(W^{1,p})}\|E\|_{L^\infty H^1}. \tag{5.14}$$

For the third estimate, we decompose $n = w + u$ such that $w \in \mathrm{Str}^n$ and $u \in L^\infty B^{1/2}_{2,\infty}$. For the part in Str^n, we have

$$\begin{aligned}
\Big\|\sum_{k>c}I_c(w_{\leqslant\tilde{\gamma}k/c}E_k)\Big\|_{H^1} &\lesssim \big\|(c/k)^{1-2/3}\|w\|_{B_6^{-2/3}}\|E_k\|_{B_3^{2/3}}k^{1/3}\big\|_{\ell^2_{k>c}}\\
&\lesssim c^{-1/2}\|w\|_{\alpha^{-1/3}B_6^{-2/3}}\|E_{>c}\|_{c^{-1/6}B_3^{2/3}}.
\end{aligned} \tag{5.15}$$

Integrating in time, we get

$$\Big\|S_E I_c\sum_{k>c}(w_{\leqslant\tilde{\gamma}k/c}E_k)\Big\|_{\mathrm{Str}^E} \lesssim T^{1/2}c^{-1/2}\|w\|_{\alpha^{-1/3}L^3B_6^{-2/3}}\|E_{>c}\|_{c^{-1/6}L^6B_3^{2/3}}. \tag{5.16}$$

For the part in $L^\infty B^{1/2}_{2,\infty}$, we have

$$\Big\|\sum_{k>c}I_c(u_{\leqslant\tilde{\gamma}k/c}E_k)\Big\|_{H^1} \lesssim \big\|\big\|(c/k)u_{\leqslant\tilde{\gamma}k/c}\big\|_{L^\infty}\|E_k\|_{H^1}\big\|_{\ell^2_{k>c}} \lesssim \|u\|_{B^{1/2}_{2,\infty}}\|E\|_{H^1}. \tag{5.17}$$

Integrating in time, we get

$$\Big\|S_E I_c\sum_{k>c}(u_{\leqslant\tilde{\gamma}k/c}E_k)\Big\|_{\mathrm{Str}^E} \lesssim T\|u\|_{L^\infty B^{1/2}_{2,\infty}}\|E\|_{L^\infty H^1}. \qquad \square \tag{5.18}$$

For the estimate of $S_n\alpha|\nabla|\langle E,F\rangle_X$ in $\mathrm{Str}^n + L^\infty B^{1/2}_{2,\infty}$ where $E, F = \mathbb{E}, \mathbb{E}^*$, we have to use the nonresonant property for almost all the interactions. However, there is a resonant case where we can only use the Strichartz estimate. The resonance we have here is actually less severe than the one at the frequency γ. This is the case when $cj \sim k \sim l$ and $E = \mathbb{E}$ and $F = \mathbb{E}^*$. For this case, we use the following proposition.

Proposition 5.3. *For any $T \in (0,1)$ and any functions E, F on $(0,T)\times\mathbb{R}^3$, we have*

$$\Big\| S_n \sum_{k\gtrsim c} \alpha|\nabla|(FE_k)_{\leqslant \tilde{\gamma}k/c}\Big\|_{\mathrm{Str}^n(0,T)} \lesssim T^{1/2}c^{-1/2}\|F\|_{\mathrm{Str}^E(0,T)}\|E\|_{\mathrm{Str}^E(0,T)}. \tag{5.19}$$

Proof. Here we use that they are *HH* interactions. Hence,

$$\begin{aligned}\big\|S_n\alpha|\nabla|(F_lE_k)_j\big\|_{\mathrm{Str}^n} &\lesssim \alpha j\big\|(F_lE_k)_j\big\|_{\alpha^{1/3}L^{3/2}B^{2/3}_{6/5}}\\ &\lesssim \alpha^{2/3}j^{5/3}\big\|(F_lE_k)_j\big\|_{L^{3/2}L^{6/5}}\\ &\lesssim \alpha^{2/3}j^{5/3}c^{-1/6}l^{-1}k^{-2/3}\|F_l\|_{L^2H^1}\|E_k\|_{c^{-1/6}L^6B^{2/3}_3}\\ &\lesssim c^{-1/2}(cj/l)^{5/3}\|F_l\|_{L^2H^1}\|E_{>c}\|_{c^{-1/6}L^6B^{2/3}_3},\end{aligned} \tag{5.20}$$

which can be summed in $\ell^2_j\ell^1_k\ell^1_l$ ($cj \lesssim k \sim l$), using the Young inequality for convolution in $\mathbb{Z}$, and yields a factor $c^{-1/2}$. □

5.3. Bilinear estimate for nonresonant interactions

The remaining terms cannot be estimated simply by using the Strichartz estimates. We need to take into account the nonresonance property and use the $X^{s,b}$ norms. Here nonresonance means the following simple trichotomy: one of three interacting functions (including the output) must be away from the characteristic surface in the space–time Fourier space. The $X^{s,b}$ spaces give a gain for functions away from the characteristics as in (4.18), (4.19) and (4.20).

Now we make the above statement into precise estimates. We estimate interactions of the form $\langle\Re(N)E \mid F\rangle_{t,x}$ for $N \in \mathcal{Y}(\mathbb{R})$ and $E, F \in \mathcal{X}(\mathbb{R})$, splitting each function with respect to the distance from the characteristic surfaces. Using (4.17), we define

$$\begin{aligned}&N^C = P_{|\tau-\alpha|\xi||\leqslant\delta}N, \qquad E^C = P_{|\tau-\omega(\xi)|\leqslant\delta}E, \qquad E^{*C} = P_{|\tau+\omega(\xi)+2c^2|\leqslant\delta}E^*,\\ &N^F = P_{|\tau-\alpha|\xi||>\delta}N, \qquad E^F = P_{|\tau-\omega(\xi)|>\delta}E, \qquad E^{*F} = P_{|\tau+\omega(\xi)+2c^2|>\delta}E^*,\end{aligned} \tag{5.21}$$

where $\delta > 0$ will be determined according to Lemma 5.4. We denote $n^F := \Re(N^F)$, $n^C := \Re(N^C)$. Notice also that $\mathbb{E}^{*C} = \mathbb{E}^{C*} = e^{-2ic^2t}(\overline{\mathbb{E}^C_2}, \overline{\mathbb{E}^C_1})$. Then the nonresonance property is expressed in the following way.

Lemma 5.4. *Let $\alpha\gamma = 2c^2$ for some fixed $\gamma > 0$. There exists $\varepsilon > 0$ (one can take $\varepsilon = 1/80$), such that we have the following* (i) *and* (ii) *for large c (say $c > 2(\gamma+1)$). Let $j,k,l \in \mathbb{D}$ be dyadic numbers.*

(i) *If $\delta \leqslant \varepsilon\alpha j$ and $j > 1$, then we have $\langle n^C_j E^C_k \mid F^C_l\rangle_{t,x} = 0$.*
(ii) *If $\delta \leqslant \varepsilon\alpha j$ and $\min(k,l) < \varepsilon\frac{c}{\gamma}j$, then we have $\langle n^C_{Xj} E^{*C}_k \mid F^C_l\rangle_{t,x} = 0$.*

Proof. By the Plancherel identity in space–time, we have

$$\begin{aligned}\big\langle n^C_j E^C_k \mid F^C_l\big\rangle_{t,x} &= \Re\int n^C_j E^C_k \overline{F^C_l}(t,x)\,dt\,dx\\ &= C\Re \iint_{(\tau_0,\xi_0)+(\tau_1,\xi_1)=(\tau,\xi)} \hat{n}^C_j(\tau_0,\xi_0)\hat{E}^C_k(\tau_1,\xi_1)\overline{\hat{F}^C_l}(\tau,\xi)\,d\xi_1\,d\tau_1\,d\xi\,d\tau.\end{aligned} \tag{5.22}$$

For the proof of the first point, we want to show that the set

$$A = \mathrm{Supp}\big(\hat{n}^C_j(\tau_0,\xi_0)\hat{E}^C_k(\tau_1,\xi_1)\hat{F}^C_l(\tau,\xi)\big) \cap \big\{(\tau_0,\xi_0)+(\tau_1,\xi_1) = (\tau,\xi)\big\} = \emptyset.$$

We denote the distance from each characteristic surface in the integrand on the right-hand side by

$$d_0 = \left|\tau_0 \mp \alpha|\xi_0|\right|, \qquad d_1 = \left|\tau_1 - \omega(\xi_1)\right|, \qquad d = \left|\tau - \omega(\xi)\right|, \tag{5.23}$$

where we recall that $\omega(\xi) = c^2(\langle|\xi|/c\rangle - 1)$ as defined in (1.19). Assume that $A \neq \emptyset$, and let $(\tau_0, \xi_0, \tau_1, \xi_1, \tau, \xi) \in A$. By the constraint $(\tau_0, \xi_0) + (\tau_1, \xi_1) = (\tau, \xi)$, we have

$$\begin{aligned} 6\delta > d_0 + d_1 + d &\geqslant \alpha|\xi_0| - \left|\omega(\xi) - \omega(\xi_1)\right| \\ &\geqslant \alpha|\xi_0| - c|\xi_0| \geqslant \frac{1}{2}\alpha|\xi_0|, \end{aligned} \tag{5.24}$$

since $\alpha = 2c^2/\gamma \gg c$ when c is large. Hence, by choosing ε small enough, we have $6\delta > d_0 + d_1 + d \geqslant \frac{1}{2}\alpha|\xi_0| \geqslant \frac{1}{3}\alpha j$ since $j > 1$, and we get a contradiction. Hence, $A = \emptyset$ and (i) is proved.

For the proof of the second point, we argue in a similar manner. We use that the characteristic surface for E^* is $\tau + c^2(\langle\xi/c\rangle + 1) = 0$, so the distance from the characteristic is given by

$$d_1 = \left|\tau_1 + c^2\left(\langle\xi_1/c\rangle + 1\right)\right|. \tag{5.25}$$

Hence we have

$$d_0 + d_1 + d \geqslant \left|\alpha|\xi_0| - 2c^2 - \omega(\xi_1) - \omega(\xi)\right|. \tag{5.26}$$

Since $|\xi_0| \not\sim \gamma$, we have

$$\left|\alpha|\xi_0| - 2c^2\right| \sim \alpha|\xi_0| + 2c^2 \gtrsim \alpha j, \tag{5.27}$$

where we used the fact that if $j = 1 > |\xi_0|$ then $\gamma > 1$ (otherwise $\mathcal{F}_3 n_X$ is supported away from $|\xi| \leqslant 1$), and hence $2c^2 = \alpha\gamma > \alpha = \alpha j$. The condition on k and l implies that

$$\omega(\xi_1) + \omega(\xi) < c\left(|\xi_1| + |\xi|\right) \leqslant c\left(2\min\left(|\xi_1|, |\xi|\right) + |\xi_0|\right) \leqslant (\varepsilon\alpha + 2c)j. \tag{5.28}$$

Hence we get a contradiction if ε is small enough and α, c are large. This ends the proof of (ii). □

Now we proceed to bilinear estimates. We start by looking at $S_E I_c(n\mathbb{E})$.

Proposition 5.5. *For any functions N and $\mathbb{E}$ on $(0,T) \times \mathbb{R}^3$, we have*

$$\left\|S_E I_c(n_X E)_{HL}\right\|_{\mathrm{Str}^E(0,T)} \lesssim \left(T^{1/6} + c^{-1/2}\right)\|N_X\|_{L^\infty L^2(0,T)\cap\mathcal{Y}(0,T)}\|\mathbb{E}\|_{\mathrm{Str}^E(0,T)\cap\mathcal{X}(0,T)}, \tag{5.29}$$

$$\begin{aligned} &\left\|S_E I_c\left[\sum_{k>c}(n_{X(>\tilde\gamma k/c)}E_k)_{LH+HH}\right]\right\|_{\mathrm{Str}^E(0,T)} \\ &\quad\lesssim \left(T^{1/5} + c^{-1/2}\right)\|N_X\|_{L^\infty L^2(0,T)\cap\mathcal{Y}(0,T)}\|\mathbb{E}\|_{\mathrm{Str}^E(0,T)\cap\mathcal{X}(0,T)} \end{aligned} \tag{5.30}$$

where $n := \Re N$ and $E = \mathbb{E}$ or $\mathbb{E}^*$.

Proof. In order to apply Lemma 5.4, we first extend those functions to $\mathbb{R}$ by using (4.16):

$$\mathbb{E}'(t) := e^{-it\Delta_c/2}\rho_T e^{it\Delta_c/2}\mathbb{E}(t), \qquad N'(t) = e^{i|\alpha\nabla|t}\rho_T e^{-i|\alpha\nabla|t}N(t), \tag{5.31}$$

which does not effect them nor the output on $(0,T)$, and we have

$$\begin{aligned} &\left\|\mathbb{E}'\right\|_{\mathcal{X}(\mathbb{R})} \lesssim \|\mathbb{E}\|_{\mathcal{X}(0,T)}, \qquad \left\|N'\right\|_{\mathcal{Y}(\mathbb{R})} \lesssim \|N\|_{\mathcal{Y}(0,T)}, \\ &\left\|\mathbb{E}'\right\|_{L^\infty H^1(\mathbb{R})} \lesssim \|\mathbb{E}\|_{L^\infty(0,T;H^1)}, \qquad \left\|N'\right\|_{L^\infty L^2(\mathbb{R})} \lesssim \|N\|_{L^\infty(0,T;L^2)}. \end{aligned} \tag{5.32}$$

In the following, we do not distinguish $(\mathbb{E}', N')$ and $(\mathbb{E}, N)$.

We decompose each function into dyadic pieces as $(n_j E_k)_l$, and let $\delta := \varepsilon\alpha j$ as in Lemma 5.4. Either by HL or by $j > \tilde\gamma k/c$, the condition of the lemma holds for both cases with $E = \mathbb{E}$ or $E = \mathbb{E}^*$, for sufficiently large c. Hence applying to nE the same decomposition as for E, we have

$$(n_j E_k)_l = \left(n_j^F E_k\right)_l + \left(n_j^C E_k^F\right)_l + \left(n_j^C E_k^C\right)_l^F. \tag{5.33}$$

Here, we have used Lemma 5.4 and the fact that the truncations P are self-adjoint.

Each term is estimated as follows, where we regard ε just as a constant.

First we prove (5.29), hence $k \lesssim j \sim l$. Using the Sobolev embedding $B^1_{18/5} + c^{-1/3} B^{1/3}_6 \subset I_c^{1/3} B^{1/6}_\infty$, we have

$$\begin{aligned} \big\| S_E I_c \big(n_j^F E_k\big)_l \big\|_{\mathrm{Str}^E} &\lesssim \langle l/c\rangle^{-1} \big\| \big(n_j^F E_k\big)_l \big\|_{L^1 H^1} \\ &\lesssim \langle l/c\rangle^{-1} l \big\| n_j^F \big\|_{L^2 L^2} \|E_k\|_{L^2 L^\infty} \\ &\lesssim \langle l/c\rangle^{-1} l \langle j/c\rangle^{1/6} \frac{\alpha}{\alpha j} \|N_j\|_{\mathcal{Y}} T^{1/6} \langle k/c\rangle^{1/3} k^{-1/6} \|E\|_{\mathrm{Str}^E(0,T)} \\ &\lesssim \langle l/c\rangle^{-1/2} T^{1/6} k^{-1/6} \|N_j\|_{\mathcal{Y}} \|E\|_{\mathrm{Str}^E}, \end{aligned} \tag{5.34}$$

where we have used (4.19) in the third line to estimate $\|n_j^F\|_{L^2L^2}$. Hence, this term can be summed in $\ell^2_l \ell^1_j \ell^1_k$ ($k \lesssim j \sim l$) and gives

$$\big\| S_E I_c \big(n^F E\big)_{HL} \big\|_{\mathrm{Str}^E} \lesssim T^{1/6} \|N\|_{\mathcal{Y}} \|\mathbb{E}\|_{\mathrm{Str}^E}. \tag{5.35}$$

Similarly, by using $L^2 \subset B^{-3/2}_\infty$ we have

$$\begin{aligned} \big\| S_E I_c \big(n_j^C E_k^F\big)_l \big\|_{\mathrm{Str}^E} &\lesssim \langle l/c\rangle^{-1} l \big\| N_j^C \big\|_{L^2L^2} k^{3/2} \langle k/c\rangle^{-5/6} (\alpha j)^{-1} \|\mathbb{E}_k\|_{\mathcal{X}} \\ &\lesssim \langle l/c\rangle^{-1} \langle k/c\rangle^{-5/6} (k/c)^{3/2} c^{-1/2} \|N\|_{L^2L^2} \|\mathbb{E}\|_{\mathcal{X}}. \end{aligned} \tag{5.36}$$

This can be summed in $\ell^1_l \ell^1_j \ell^1_k$ ($k \lesssim j \sim l$) and gives

$$\big\| S_E I_c \big(n^C E^F\big)^C_{HL} \big\|_{\mathrm{Str}^E} \lesssim c^{-1/2} T^{1/2} \|N\|_{L^\infty L^2(0,T)} \|\mathbb{E}\|_{\mathcal{X}}. \tag{5.37}$$

Now, using Lemma 4.1 and the Sobolev embedding, we have

$$\begin{aligned} \big\| S_E I_c \big(n_j^C E_k^C\big)^F_l \big\|_{c^{-1} L^\infty H^{3/2}} &\lesssim c\delta^{-1} \langle l/c\rangle^{-1} l^{3/2} \big\| N_j^C \big\|_{L^\infty L^2} k^{1/2} \big\| E_k^C \big\|_{L^\infty H^1} \\ &\lesssim (cl)^{-1} \langle l/c\rangle^{-1} l^{3/2} k^{1/2} \|N\|_{L^\infty L^2} \|\mathbb{E}\|_{L^\infty H^1}, \end{aligned} \tag{5.38}$$

which can be summed in $\ell^1_j \ell^1_k$ ($k \lesssim j \sim l$). We then use the fact that $c^{-1} L^\infty B^{3/2}_{2,\infty} \subset c^{-1/2} \mathrm{Str}^E$, deducing

$$\big\| S_E I_c \big(n^C E^C\big)^F_{HL} \big\|_{\mathrm{Str}^E} \lesssim c^{-1/2} \|N\|_{L^\infty L^2} \|\mathbb{E}\|_{L^\infty H^1}. \tag{5.39}$$

Next we concentrate on (5.30), hence we have $\tilde{\gamma} k/c \leqslant j$ and $k > c$. We will use the notation $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$. By Strichartz we have

$$\begin{aligned} \big\| S_E I_c \big(n_j^F E_k\big)_l \big\|_{\mathrm{Str}^E} &\lesssim \langle l/c\rangle^{-1} \big\| \big(n_j^F E_k\big)_l \big\|_{I_c^{1/2} L^{10/7} B^1_{10/7}(0,T)} \\ &\lesssim T^{1/5} \langle l/c\rangle^{-1/2} (l \wedge j)^{9/10} \big\| n_j^F \big\|_{L^2L^2} \|E_k\|_{L^\infty H^1} \\ &\lesssim T^{1/5} \langle l/c\rangle^{-1/2} (l \wedge j)^{9/10} \langle j/c\rangle^{1/6} \frac{\alpha}{\alpha j} \|N\|_{\mathcal{Y}} \|E\|_{\mathrm{Str}^E}, \end{aligned} \tag{5.40}$$

which can be bounded in $\ell^1_l \ell^1_j \ell^1_k$ ($j \vee l \sim k \gtrsim c$), and yields a factor $T^{1/5}$.

In the same way as (5.36), we have

$$\begin{aligned} \big\| S_E I_c \big(n_j^C E_k^F\big)_l \big\|_{\mathrm{Str}^E} &\lesssim \langle l/c\rangle^{-1} l (l \wedge j)^{3/2} \big\| N_j^C \big\|_{L^2L^2} \langle k/c\rangle^{-5/6} (\alpha j)^{-1} \|\mathbb{E}_k\|_{\mathcal{X}} \\ &\lesssim \langle l/c\rangle l (l \wedge j)^{3/2} k^{-5/6} c^{-7/6} j^{-1} \|N\|_{L^2L^2} \|\mathbb{E}\|_{\mathcal{X}}, \end{aligned} \tag{5.41}$$

where we used that $\alpha \sim c^2$. The last coefficient is bounded by

$$\begin{cases} \langle l/c\rangle^{-1} l^{1/6} j^{1/2} c^{-7/6} & (k \sim l), \\ \langle l/c\rangle^{-1} l^{5/2} j^{-11/6} c^{-7/6} & (k \sim j), \end{cases} \tag{5.42}$$

and hence we can bound (5.41) in $\ell^1_l \ell^1_k \ell^1_j$ ($j \vee l \sim k$), getting a factor $c^{-1/2} T^{1/2}$.

Finally, in the same way as (5.38), we have

$$\big\| S_E I_c \big(n_j^C E_k^C\big)^F_l \big\|_{c^{-1} L^\infty H^{3/2}} \lesssim \frac{c}{\alpha j} \langle l/c\rangle^{-1} l^{3/2} (l \wedge j)^{3/2} k^{-1} \|N\|_{L^\infty L^2} \|E\|_{L^\infty H^1}, \tag{5.43}$$

where the last coefficient is bounded by

$$\begin{cases} c^{-1}\langle l/c\rangle^{-1} j^{1/2} l^{1/2} & (k\sim l),\\ c^{-1}\langle l/c\rangle^{-1} j^{-2} l^{3} & (k\sim j),\end{cases} \tag{5.44}$$

and hence we can bound (5.43) in $\ell_l^\infty \ell_k^1 \ell_j^1$ $(j\vee l\sim k)$. We then use that $c^{-1}L^\infty B^{3/2}_{2,\infty}\subset c^{-1/2}\mathrm{Str}^E$, getting a factor $c^{-1/2}$. $\square$

Next we consider the nonresonant term in the equation for n.

Proposition 5.6. *For any functions* $\mathbb{E}$ *and* $\mathbb{F}$ *on* $(0,T)\times\mathbb{R}^3$*, we have*

$$\begin{aligned}&\Big\| S_n \sum_k \sum_{j>\tilde\gamma k/c} \alpha|\nabla|\langle \mathbb{E}_k, \mathbb{F}+\mathbb{F}^*\rangle_{Xj}\Big\|_{\mathrm{Str}^n(0,T)+L^\infty(0,T;B^{1/2}_{2,\infty})}\\ &\quad\lesssim \|\mathbb{E}\|_{\mathrm{Str}^E(0,T)\cap\mathcal{X}(0,T)}\|\mathbb{F}\|_{\mathrm{Str}^E(0,T)\cap\mathcal{X}(0,T)}.\end{aligned} \tag{5.45}$$

Proof. We will denote $E=\mathbb{E}$ and $F=\mathbb{F}$ or $\mathbb{F}^*$. Decomposing into dyadic pieces, we consider interactions of the form $\langle E_k, F_l\rangle_j$ for N with $j>\tilde\gamma k/c$. Hence,

$$\langle E_k, F_l\rangle_{Xj} = \langle E_k^F, F_l\rangle_{Xj} + \langle E_k, F_l^F\rangle_{Xj} - \langle E_k^F, F_l^F\rangle_{Xj} + \langle E_k^C, F_l^C\rangle_{Xj}^F \tag{5.46}$$

where we have used Lemma 5.4.

By using the Strichartz estimate, we have

$$\begin{aligned}\big\| S_n|\alpha\nabla|\langle E_k^F, F_l\rangle_j\big\|_{\mathrm{Str}^n} &\lesssim \alpha j\big\|\big(E_k^F F_l\big)_j\big\|_{\alpha^{1/6}L^{6/5}B^{1/3}_{3/2}}\\ &\lesssim \alpha j\alpha^{-1/6} j^{1/3}\big\|E_k^F\big\|_{L^2L^2}\langle l/c\rangle^{1/3} l^{-2/3}\|F_l\|_{I_c^{-1/3}L^3B_6^{2/3}}\\ &\lesssim \frac{\alpha j}{\delta}(j/c)^{1/3}\langle k/c\rangle^{-5/6}\langle l/c\rangle^{1/3} l^{-2/3}\|\mathbb{E}\|_{\mathcal{X}}\|F\|_{\mathrm{Str}^E}.\end{aligned} \tag{5.47}$$

This is summable in $\ell_j^1\ell_l^1\ell_k^1$ for $l\lesssim j\sim k$ and for $j\lesssim k\sim l$. The above term can be bounded also by

$$\begin{aligned}\alpha j\big\|\big(E_k^F F_l\big)_j\big\|_{L^1L^2} &\lesssim \alpha j T^{1/6}\big\|E_k^F\big\|_{L^2L^3}\langle l/c\rangle^{1/3} l^{-2/3}\|F_l\|_{I_c^{-1/3}L^3B_6^{2/3}}\\ &\lesssim T^{1/6}\frac{\alpha j}{\delta} k^{1/2}\langle k/c\rangle^{-5/6}\langle l/c\rangle^{1/3} l^{-2/3}\|\mathbb{E}\|_{\mathcal{X}}\|F\|_{\mathrm{Str}^E},\end{aligned} \tag{5.48}$$

which is summable in $\ell_j^1\ell_l^1\ell_k^1$ for $k\lesssim l\sim j$, yielding a factor $T^{1/6}$.

For $\langle E_k, F_l^F\rangle_j$, we have just to switch the roles of k and l. For $\langle E_k^F, F_l^F\rangle_j$, we have a better bound

$$\begin{aligned}\big\| S_n|\alpha\nabla|\langle E_k^F, F_l^F\rangle_j\big\|_{\mathrm{Str}^n} &\lesssim \alpha j\big\|\big(E_k^F F_l^F\big)_j\big\|_{L^1L^2}\\ &\lesssim (\alpha j)^{-1}(kl)^{3/4}\langle k/c\rangle^{-5/6}\langle l/c\rangle^{-5/6}\|\mathbb{E}_k\|_{\mathcal{X}}\|\mathbb{F}_k\|_{\mathcal{X}},\end{aligned} \tag{5.49}$$

which is summable for all j,k,l and gives a factor $c^{-1/2}$.

Finally, for the last term of (5.46), we have

$$\big\| S_n|\alpha\nabla|\langle E_k^C, F_l^C\rangle_j^F\big\|_{L^\infty H^{1/2}} \lesssim \frac{\alpha j}{\delta} s^{3/2} j^{1/2}(kl)^{-1}\|E_k\|_{L^\infty H^1}\|F_l\|_{L^\infty H^1} \tag{5.50}$$

where $s=\min(j,k,l)$. This can be summed in $\ell_l^1\ell_k^1$ for $k\lesssim l\sim j$, $l\lesssim k\sim j$ and $k\sim l\gtrsim j$ and gives a result in $L^\infty B^{1/2}_{2,\infty}$. $\square$

5.4. The resonant part of N

To estimate the resonant frequency part N_γ, we integrate by parts as in (2.6). Then the estimate on the boundary terms follows from Lemma 4.3. The estimate on the integral terms use the following proposition.

Proposition 5.7. *For any functions $\mathbb{E}$ and $\mathbb{F}$, we have*

$$\begin{aligned}
&\big\|S_n(\dot{\mathbb{E}}\cdot\mathbb{F})_\gamma\big\|_{L^\infty L^2}\lesssim T\|\dot{\mathbb{E}}\|_{L^\infty H^{-1}}\|\mathbb{F}\|_{L^\infty H^1},\\
&\big\|\mathcal{A}_\gamma^- S_n\big(\dot{\mathbb{E}}\cdot\mathbb{F}^*\big)_\gamma\big\|_{L^\infty L^2}\lesssim T\|\dot{\mathbb{E}}\|_{L^\infty H^{-1}}\|\mathbb{F}\|_{L^\infty H^1},\\
&\big\|\mathcal{A}_\gamma^+ S_n\big(\dot{\mathbb{E}}\cdot\mathbb{F}^*\big)_\gamma\big\|_{L^\infty \mathcal{W}^{k,p}}\lesssim T\|\dot{\mathbb{E}}\|_{L^\infty H^{-1}}\|\mathbb{F}\|_{L^\infty H^1}.
\end{aligned}\tag{5.51}$$

Proof. For the proof of the first two estimates, we have just to use that

$$\begin{aligned}
\big\|(\mathbb{F}\dot{\mathbb{E}})_\gamma\big\|_{L^2}&\lesssim\big\|(\mathbb{F}\dot{\mathbb{E}})_\gamma\big\|_{L^1}\\
&\lesssim\sum_{k\sim l}\|\mathbb{F}_k\|_{L^2}\|\dot{\mathbb{E}}_l\|_{L^2}\lesssim\|\mathbb{F}\|_{H^1}\|\dot{\mathbb{E}}\|_{H^{-1}},
\end{aligned}\tag{5.52}$$

and then apply the energy estimate. For the last term, we use the above L^1_x bound together with (4.31) to deal with $\mathcal{A}_\gamma^+$, where we may replace $L^{3,\infty}$ by $W^{k,p}$ $(p>3)$ thanks to the frequency restriction to γ. □

In the final argument, we have then to use that

$$\|\dot{\mathbb{E}}\|_{L^\infty H^{-1}}\leqslant\|\mathbb{E}\|_{L^\infty H^1}+\|\mathbb{E}\|_{L^\infty H^1}\|N\|_{L^\infty(L^2+\mathcal{W}^{1,p})},$$

which follows from the equation of $\mathbb{E}$ and that $\|\Delta_c\varphi\|_{H^{-1}}\lesssim\|\varphi\|_{H^1}$ uniformly in c.

5.5. Concluding the estimates

Applying the propositions of the previous subsections, we can estimate all the terms appearing in (5.4) and (5.5). Recall $\|\mathbb{E}\|$ and $\|N\|$ defined in (5.1).

Proposition 5.8. *If $(\mathbb{E},N)$ is a solution of (2.1) on $(0,T)$, then we have the following a priori bound*

$$\begin{aligned}
&\|\mathbb{E}\|\lesssim\big\|\mathbb{E}(0)\big\|_{H^1}+\big(T^{1/6}+c^{-1/2}\big)\|N\|\|\mathbb{E}\|,\\
&\|N\|\lesssim\big\|N(0)\big\|_{L^2+\mathcal{W}^{k,p}}+\|\mathbb{E}\|^2\big(1+T\|N\|\big).
\end{aligned}\tag{5.53}$$

Hence, it is clear that there exists a c_0 big enough and there exists a uniform time T such that the equation can be solved for $c>c_0$ on the time interval $(0,T)$.

6. Passage to the limit

In this section, we prove the convergence towards the limit system. We denote

$$\begin{aligned}
&N^{0\infty}=-\big|\mathbb{E}^\infty\big|^2,\\
&N^{+\infty}=-e^{i\alpha\gamma t}\mathcal{A}_\gamma^+\big(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\big),\qquad N^{-\infty}=-e^{-i\alpha\gamma t}\mathcal{A}_\gamma^-\big(\overline{\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty}\big),
\end{aligned}\tag{6.1}$$

$$\mathbb{E}^\omega:=e^{-i(\Delta_c-\Delta)t/2}\mathbb{E}^\infty,\qquad N^{\sigma\omega}:=N^{\sigma\infty}+N^{\sigma I}\quad\text{for }\sigma=0,\pm,\tag{6.2}$$

where $N^{\sigma I}=N^{\sigma Ic}$ were defined in Theorem 3.1. We also denote

$$\begin{aligned}
&N^\infty=N^{0\infty}+N^{+\infty}+N^{-\infty},\\
&N^I=N^f+N^{0I}+N^{+I}+N^{-I},\\
&N^\omega=N^\infty+N^I=N^{0\omega}+N^{+\omega}+N^{-\omega}.
\end{aligned}\tag{6.3}$$

Taking the real value, we define also $n^I = \Re N^I$,

$$n^\infty = \Re N^\infty = -\left|\mathbb{E}^\infty\right|^2 - \Re\left[e^{i\alpha\gamma t}\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right)\right], \tag{6.4}$$

and $n^\omega = \Re N^\omega = n^\infty + n^I$.

We will argue in a similar way as in [12] with the difference that here we have to estimate the whole Strichartz norm. For any Banach space Z for space–time functions on $(0,T)\times\mathbb{R}^3$ or space functions on $\mathbb{R}^3$, we will denote by

$$o(Z), \quad O(Z), \tag{6.5}$$

those sequence of functions which tends to 0 as $c\to\infty$ in the Z norm, and those sequence of functions bounded as $c\to\infty$ in the Z norm, respectively. We want to prove that

$$\begin{aligned}
&\mathbb{E}-\mathbb{E}^\omega \in o\left(\mathrm{Str}^E\right),\\
&\mathbb{E}-\mathbb{E}^\omega \in o(\mathcal{X}) + O\left(I_c X^{1,1}\right),\\
&N-N^\omega \in o\left(\mathrm{Str}^n + L^\infty\left(B^{1/2}_{2,\infty}+\mathcal{W}^{k,p}\right)\right),\\
&\left(N-N^\omega\right)_X \in o(\mathcal{Y}),
\end{aligned} \tag{6.6}$$

for any $p>3$, under the assumption that $\mathbb{E}(0) = \mathbb{E}^\infty(0) + o(H^1)$ and $N^f(0)\in O(L^2+\mathcal{W}^{k,p})$.

For the limit solution, we obtain $\mathbb{E}^\infty \in L^\infty H^1 \cap L^2 B^1_6$ in the same way as for the usual NLS, using the Strichartz estimate together with the following nonlinear estimates

$$\begin{aligned}
\left\|\left[\mathcal{A}_\gamma(EF)_\gamma + \mu^\infty\right]G\right\|_{L^\infty H^1} &\lesssim \left[\left\|(EF)_\gamma\right\|_{L^\infty L^1} + \left\|\mu^\infty\right\|_{L^\infty W^{1,p}}\right]\|G\|_{L^\infty H^1}\\
&\lesssim \left[\|E\|_{L^\infty H^1}\|F\|_{L^\infty H^1} + \left\|\mu^\infty\right\|_{L^\infty W^{1,p}}\right]\|G\|_{L^\infty H^1},\\
\left\|\mathcal{A}_\gamma(EF)_X G\right\|_{L^2 B^1_{6/5}} &\lesssim T^{1/2}\|EF\|_{L^\infty B^1_{3/2}}\|G\|_{L^\infty H^1}\\
&\lesssim T^{1/2}\|E\|_{L^\infty H^1}\|F\|_{L^\infty H^1}\|G\|_{L^\infty H^1},\\
\left\|\mathcal{A}_\gamma(EF)_X G\right\|_{L^1 H^1} &\lesssim T^{1/2}\|EF\|_{L^4 H^1}\|G\|_{L^4(B^1_3\cap L^\infty)}\\
&\lesssim T^{1/2}\|E\|_{L^\infty H^1}\|F\|_{L^4(B^1_3\cap L^\infty)}\|G\|_{L^4(B^1_3\cap L^\infty)}\\
&\lesssim T^{1/2}\|E\|_{L^\infty H^1}\|F\|^{1/2}_{L^\infty H^1}\|F\|^{1/2}_{L^2 B^1_6}\|G\|^{1/2}_{L^\infty H^1}\|G\|^{1/2}_{L^2 B^1_6},
\end{aligned} \tag{6.7}$$

where in the last step we used the real interpolation $(L^6, B^1_6)_{1/2,1} = B^{1/2}_{6,1} \subset L^\infty$, and in all cases we used Lemma 4.3 to treat $\mathcal{A}^+_\gamma$. Since $e^{i(\Delta_c-\Delta)t}$ is strongly convergent to 1 on H^1 uniformly for $t\in[0,T]$, we deduce that

$$\mathbb{E}^\omega = \mathbb{E}^\infty + o\left(\mathrm{Str}^E\right), \qquad \mathbb{E}^\omega \in O\left(X^{1,1}\right), \qquad N^I \in O\left(L^\infty\mathcal{W}^{k,p} + Y^{0,1}\right). \tag{6.8}$$

Denote $[\mathbb{E}] := \mathbb{E}+\mathbb{E}^*$ and $\mathbb{E}^\perp = (\mathbb{E}_2,\mathbb{E}_1)$. We decompose

$$\mathbb{E}-\mathbb{E}^\omega = E^1+E^2+E^3+E^4+E^5, \tag{6.9}$$

where each E^j is defined by the following

$$\begin{aligned}
E^1 &:= e^{-i\Delta_c t/2}\left(\mathbb{E}(0)-\mathbb{E}^\infty(0)\right),\\
E^2 &:= S_E I_c\left\{\left[\left|\mathbb{E}^\infty\right|^2 - \Re\mu^{0\infty}\right]\mathbb{E}^\omega + \left[\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right) - \mu^{+\infty}/2\right]\overline{\mathbb{E}}^{\omega\perp}\right\}\\
&\quad - S_E e^{i(\Delta-\Delta_c)t/2}\left\{\left[\left|\mathbb{E}^\infty\right|^2 - \Re\mu^{0\infty}\right]\mathbb{E}^\infty + \left[\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right) - \mu^{+\infty}/2\right]\overline{\mathbb{E}}^{\infty\perp}\right\},\\
E^3 &:= -S_E I_c n\left[\mathbb{E}-\mathbb{E}^\omega\right] - S_E I_c\left(n-n^\omega\right)\left[\mathbb{E}^\omega\right],\\
E^4 &:= -S_E I_c\left(n^I\left[\mathbb{E}^\omega\right] - \Re\mu^{0\infty}\mathbb{E}^\omega - \mu^{+\infty}\overline{\mathbb{E}}^{\omega\perp}/2\right),\\
E^5 &:= S_E I_c e^{-i\alpha\gamma t}\left|\mathbb{E}^\infty\right|^2\overline{\mathbb{E}}^{\omega\perp} - \frac{1}{2}S_E I_c e^{i\alpha\gamma t}\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right)\mathbb{E}^\omega\\
&\quad - \frac{1}{2}S_E I_c e^{-i\alpha\gamma t}\overline{\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right)}\mathbb{E}^\omega - \frac{1}{2}S_E I_c e^{-2i\alpha\gamma t}\overline{\mathcal{A}_\gamma\left(\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\right)}\overline{\mathbb{E}}^{\omega\perp}.
\end{aligned} \tag{6.10}$$

We treat each E^j one by one. We denote the small factor by

$$\kappa = T^{1/6}. \tag{6.11}$$

First we have immediately from the initial convergence,

$$\|E^1\|_{X^{1,1}} = o(1). \tag{6.12}$$

Next using the estimates (6.7) together with the Strichartz estimate as well as the strong convergence of $e^{i(\Delta-\Delta_c)t}$, we get

$$\|E^2\|_{\mathrm{Str}^E\cap\mathcal{X}} = o(1), \qquad \|E^5\|_{\mathrm{Str}^E\cap I_c X^{1,1}} = O(1). \tag{6.13}$$

Using Propositions 5.2 and 5.5, we get

$$\begin{aligned}\|E^3\|_{\mathrm{Str}^E} &\lesssim \kappa\|\mathbb{E}-\mathbb{E}^\omega\|_{\mathrm{Str}^E} + \kappa\|N-N^\omega\|_{Str^n+L^\infty(\mathcal{W}^{1,p}+B^{1/2}_{2,\infty})}\\ &\quad + \kappa\|(N-N^\omega)_X\|_{\mathcal{Y}} + o(1),\end{aligned} \tag{6.14}$$

and using Proposition 5.1,

$$\|E^3\|_{\mathcal{X}} \lesssim \kappa\|\mathbb{E}-\mathbb{E}^\omega\|_{\mathrm{Str}^E} + \kappa\|N-N^\omega\|_{L^\infty(\mathcal{W}^{1,p}+L^2)}. \tag{6.15}$$

For E^4 and E^5, we use their rapid oscillation in time. For the higher frequency $N^I_{>2\gamma+1}$, we integrate by parts after cutting the high frequency of $\mathbb{E}$. For any $\varepsilon > 0$, there exists $K > 1$ such that $\|\mathbb{E}^\omega_{>K}\|_{\mathrm{Str}^E\cap\mathcal{X}} \leqslant \varepsilon$ and so by Propositions 5.1, 5.2 and 5.5, as well as (6.7), we have

$$\big\|S_E I_c\big(n^I[\mathbb{E}^\omega_{>K}] - \Re\mu^{0\infty}\mathbb{E}^\omega_{>K} - \mu^{+\infty}\overline{\mathbb{E}}^{\omega\perp}_{>K}/2\big)\big\|_{\mathrm{Str}^E\cap\mathcal{X}} \lesssim \varepsilon. \tag{6.16}$$

Hence we may replace $\mathbb{E}^\omega$ by $\mathbb{E}^\omega_{\leqslant K}$ in E^4. Integrating by parts on $e^{i|\alpha\nabla|t}$, we get

$$\begin{aligned}2S_E I_c N^I_{>2\gamma+1}\mathbb{E}^\omega_{\leqslant K} &= \big[e^{i\Delta_c(s-t)/2} I_c\big\{\big(|\alpha\nabla|^{-1}N^I_{>2\gamma+1}\big)\mathbb{E}^\omega_{\leqslant K}\big\}\big]_0^t\\ &\quad - 2iS_E I_c\big\{\big(|\alpha\nabla|^{-1}N^I_{>2\gamma+1}\big)\partial_t\mathbb{E}^\omega_{\leqslant K}\big\}\\ &\quad + S_E\Delta_c I_c\big\{\big(|\alpha\nabla|^{-1}N^I_{>2\gamma+1}\big)\mathbb{E}^\omega_{\leqslant K}\big\}.\end{aligned} \tag{6.17}$$

Since $|\alpha\nabla|^{-1}N^I_{>2\gamma+1}$ is bounded in $\alpha^{-1}L^\infty(H^1 + W^{2,p})$, the first two lines are bounded in $I_c X^{1,1}$. For the last term, we need to integrate once more, which yields similar terms but with one more $|\alpha\nabla|^{-1}$ and Δ_c. Since $\Delta_c^2 I_c \leqslant O(\alpha^{3/2}|\nabla|)$ and $\Delta_c^2 \leqslant O(\alpha|\nabla|^2)$ on H^s, and $|\alpha\nabla|^{-2}N^I_{>2\gamma+1}$ is bounded in $\alpha^{-2}L^\infty(H^2 + W^{3,p})$, those terms after the second integration are bounded in $c^{-1}\mathrm{Str}^E \cap \alpha^{-1}\mathcal{X}$.

The other terms including $\overline{N}^I_{>2\gamma+1}$ or $\mathbb{E}^{\omega*}_{\leqslant K}$ are treated in the same way, integrating on the phase $e^{\pm i|\alpha\nabla|t}$ or $e^{i\alpha(\pm|\nabla|-\gamma)}$. The denominators are never singular thanks to the frequency restriction $> 2\gamma + 1$. Actually it is not needed for the term $\overline{N}\mathbb{E}^*$. Also, the same argument applies to E^5 without any low frequency cut-off, just by integrating on the phase $e^{\pm i\alpha\gamma t}$ or $e^{-2i\alpha\gamma t}$. Thus we obtain

$$\|E^5\|_{\mathrm{Str}^E} = o(1). \tag{6.18}$$

It remains to estimate the part in E^4 with $N^I_{\leqslant 2\gamma+1}$ and $\mathbb{E}^\omega_{\leqslant K}$, which we further cut-off in the physical space. Fix $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi(x) = 1$ for $|x| \leqslant 1$ and $\chi(x) = 0$ for $|x| \geqslant 2$. There exists $R > 1$ such that $\|(1-\chi(x/R))\mathbb{E}^\omega_{\leqslant K}\|_{L^\infty H^1} \leqslant \varepsilon$ and so

$$\big\|n^I_{\leqslant 2\gamma+1}\big[\big(1-\chi(x/R)\big)\mathbb{E}^\omega_{\leqslant K}\big]\big\|_{L^\infty H^1} + \big\|\mu^{*\infty}\big(1-\chi(x/R)\big)\overline{\mathbb{E}}^{\omega\perp}_{\leqslant K}\big\|_{L^\infty H^1} \lesssim \varepsilon. \tag{6.19}$$

Hence its contribution to E^4 is $O(\varepsilon)$ in $I_c X^{1,1}$.

Thus we may replace $\mathbb{E}_{\leqslant K}$ further by $\chi(x/R)\mathbb{E}_{\leqslant K}$. For the singular part, we may replace N^I by N^f because $(N^I - N^f)_{\leqslant 2\gamma+1} \to 0$ in $L^\infty W^{k,p}$ by L^p decay for $e^{it|\nabla|}$ and Lemma 4.3. Then the Fourier and physical cut-offs together with the time integration provide compactness for the convergence in (3.2) such that

$$\begin{aligned}&\left\|S_E I_c\chi(x/R)\left(n^I_{\leqslant 2\gamma+1}-\Re\mu^{0\infty}\right)\mathbb{E}\right\|_{L^\infty H^1}\to 0,\\&\left\|S_E I_c\chi(x/R)\left(e^{-i\alpha\gamma t}N^I_{\leqslant 2\gamma+1}-\mu^{+\infty}\right)\overline{\mathbb{E}}^\perp\right\|_{L^\infty H^1}\to 0,\end{aligned}\tag{6.20}$$

which implies also the decay in Str^E by interpolation. These terms might not go to 0 in $\mathcal{X}$, but are bounded in $I_c X^{1,1}$. Thus we conclude that

$$E^4\in o\left(\mathrm{Str}^E\right)\cap\left[o(\mathcal{X})+O\left(I_c X^{1,1}\right)\right].\tag{6.21}$$

Gathering the above estimates, we obtain

$$\begin{aligned}&\left\|\mathbb{E}-\mathbb{E}^\omega-E^4-E^5\right\|_{\mathcal{X}}\lesssim\kappa\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}+\kappa\left\|N-N^\omega\right\|_{L^\infty(L^2+\mathcal{W}^{1,p})}+o(1),\\&\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}\lesssim\kappa\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}+\kappa\left\|N-N^\omega\right\|_{\mathrm{Str}^n+L^\infty(\mathcal{W}^{1,p}+B^{1/2}_{2,\infty})}+\kappa\left\|\left(N-N^\omega\right)_X\right\|_{\mathcal{Y}}+o(1).\end{aligned}\tag{6.22}$$

Next, we estimate $N-N^\omega$. We will only concentrate on $N^+-N^{+\omega}$ since the other terms are easier. Integrating by parts for the lower frequency part

$$(\mathbb{E}\mathbb{E})_{ll}:=(\mathbb{E}_1)_{\leqslant c^{1/3}}\cdot(\mathbb{E}_2)_{\leqslant c^{1/3}},\tag{6.23}$$

as in (2.6), we get

$$\begin{aligned}N^+-N^{+\omega}&=-S_n|\alpha\nabla|e^{i\alpha\gamma t}\left[\mathbb{E}_1\cdot\mathbb{E}_2-(\mathbb{E}\mathbb{E})_{ll}\right]-e^{i\alpha\gamma t}\mathcal{A}^+_\gamma\left[(\mathbb{E}\mathbb{E})_{ll}-\mathbb{E}^\infty_1\cdot\mathbb{E}^\infty_2\right]\\&\quad+e^{i|\alpha\nabla|t}\mathcal{A}^+_\gamma\left[(\mathbb{E}\mathbb{E})_{ll}(0)-\mathbb{E}^\infty_1\cdot\mathbb{E}^\infty_2(0)\right]+i\mathcal{A}^+_\gamma S_n e^{i\alpha\gamma t}\partial_t(\mathbb{E}\mathbb{E})_{ll}\\&=:N^1+N^2+N^3+N^4.\end{aligned}\tag{6.24}$$

For the $\mathcal{Y}$ norm, first we have

$$\begin{aligned}&\left\|N^3_X\right\|_{\mathcal{Y}}\lesssim\alpha^{-1}\left\|\mathbb{E}_1\cdot\mathbb{E}_2-\mathbb{E}^\infty_1\cdot\mathbb{E}^\infty_2\right\|_{L^\infty L^2}\lesssim\alpha^{-1},\\&\left\|N^1\right\|_{\mathcal{Y}}\lesssim\kappa\|\mathbb{E}_{>c^{1/3}}\|^2_{\mathrm{Str}^E}\lesssim\kappa\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}+o(1),\end{aligned}\tag{6.25}$$

by using Proposition 5.1 in the second line. For the time derivative term, we use

$$\|\dot{\mathbb{E}}\|_{L^2}\lesssim c\|\mathbb{E}\|_{H^1}+c\|n\mathbb{E}\|_{H^{-1}}\lesssim c,\tag{6.26}$$

and also $\mathrm{Str}^E\subset L^3(L^\infty+c^{-1/3}L^{18})$ to deduce

$$\left\|\partial_t(\mathbb{E}\mathbb{E})\right\|_{L^2L^2}\lesssim\kappa\|\mathbb{E}\|_{\mathrm{Str}^E}\|\dot{\mathbb{E}}\|_{L^\infty L^2}\lesssim c\kappa.\tag{6.27}$$

By the same product estimate, we have

$$\|\mathbb{E}\mathbb{E}\|_{L^2H^1}\lesssim\kappa\|\mathbb{E}\|^2_{\mathrm{Str}^E}.\tag{6.28}$$

Using these bounds, we get

$$\begin{aligned}&\left\|N^2_X\right\|_{\mathcal{Y}}\lesssim\alpha^{-1}\left\|\left(i\partial_t+|\alpha\nabla|\right)N^2_X\right\|_{L^2L^2}\lesssim o(1)+\kappa\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E},\\&\left\|N^4_X\right\|_{\mathcal{Y}}\lesssim\alpha^{-1}\left\|\partial_t(\mathbb{E}\mathbb{E})\right\|_{L^2L^2}\lesssim c^{-1}.\end{aligned}\tag{6.29}$$

Thus by adding them up,

$$\left\|\left(N^+-N^{+\omega}\right)_X\right\|_{\mathcal{Y}}\lesssim\kappa\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}+o(1).\tag{6.30}$$

For the Strichartz-type norms, by using Propositions 5.3 and 5.6, we have

$$\begin{aligned}\left\|N^1\right\|_{\mathrm{Str}^n+L^\infty B^{1/2}_{2,\infty}}&\lesssim\|\mathbb{E}_{>c^{1/3}}\|_{\mathrm{Str}^E\cap\mathcal{X}}\\&\lesssim\left\|\mathbb{E}-\mathbb{E}^\omega\right\|_{\mathrm{Str}^E}+\left\|\mathbb{E}-\mathbb{E}^\omega-E^4-E^5\right\|_{\mathcal{X}}+o(1),\end{aligned}\tag{6.31}$$

where $\mathbb{E}^\omega$, E^4 and E^5 are negligible for the higher frequency $>c^{1/3}$. Decomposing into the resonant frequency and the rest, we have

$$\begin{aligned}\|N^2+N^3\|_{L^\infty(\mathcal{W}^{k,p}+H^{1/2})} &\lesssim \|(\mathbb{E}\mathbb{E})_{ll}-\mathbb{E}_1^\infty\cdot\mathbb{E}_2^\infty\|_{L^\infty(L^1\cap H^{1/2})}\\ &\lesssim \|\mathbb{E}-\mathbb{E}^\omega\|_{\mathrm{Str}^E}+o(1),\end{aligned}\tag{6.32}$$

where $\mathcal{W}^{k,p}$ and L^1 were used for N_Y^j, and $H^{1/2}$ for N_X^j. Using the α gain in the wave Strichartz, we have

$$\begin{aligned}\|N_X^4\|_{\mathrm{Str}^n} &\lesssim \|\partial_t(\mathbb{E}\mathbb{E})_{ll}\|_{\alpha^{1/3}L^{3/2}B^{2/3}_{6/5}} \lesssim \alpha^{-1/3}c^{5/9}T\|\partial_t(\mathbb{E}\mathbb{E})_{ll}\|_{L^\infty B^{-1}_{6/5}}\\ &\lesssim c^{-1/9}T\|\dot{\mathbb{E}}_{\leqslant c^{1/3}}\|_{L^\infty H^{-1}}\|\mathbb{E}_{\leqslant c^{1/3}}\|_{L^\infty H^1}\lesssim c^{-1/9}\kappa.\end{aligned}\tag{6.33}$$

Also, by Lemma 4.3 with $1/q=2/3+1/p$, we have

$$\|\mathcal{A}_\gamma^+ e^{i|\alpha\nabla|t}\varphi_\gamma\|_{\mathcal{W}^{k,p}}\lesssim \|\varphi_\gamma\|_{L^1}^{1/3+2/p}\|\varphi_\gamma\|_{L^2}^{2/3-2/p}.\tag{6.34}$$

Hence combining it with the same Strichartz estimate as above, we get

$$\begin{aligned}\|N_\gamma^4\|_{L^\infty\mathcal{W}^{k,p}} &\lesssim \|\partial_t(\mathbb{E}\mathbb{E})_{llR}\|_{L^1L^1}^{1/3+2/p}\|\partial_t(\mathbb{E}\mathbb{E})_{llR}\|_{\alpha^{1/3}L^{3/2}B^{2/3}_{6/5}}^{2/3-2/p}\\ &\lesssim T\alpha^{-1/3(2/3-2/p)}=o(1).\end{aligned}\tag{6.35}$$

Gathering the above estimates, we get

$$\|N^+-N^{+\omega}\|_{\mathrm{Str}^n+L^\infty(\mathcal{W}^{k,p}+B^{1/2}_{2,\infty})}\lesssim \|\mathbb{E}-\mathbb{E}^\omega\|_{\mathrm{Str}^E}+\|\mathbb{E}-\mathbb{E}^\omega-E^4-E^5\|_{\mathcal{X}}+o(1).\tag{6.36}$$

Using the estimates (6.22), (6.30) and (6.36), as well as their counterparts for N^0 and N^-, we deduce that (6.6) holds. This ends the proof of convergence. (3.7) is a direct consequence of Lemma 4.3 and (3.6).

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